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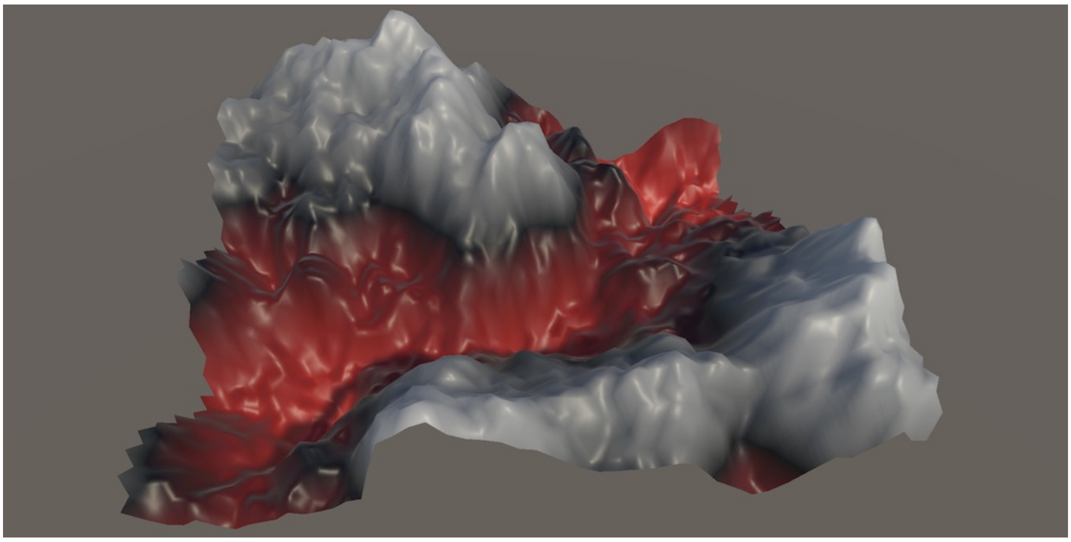
Perlin Derivatives Derivatives of Interpolation

Find derivatives for linear, bilinear, and trilinear interpolation.
Calculate derivatives for value and Perlin noise.

This is the third tutorial in a series about pseudorandom surfaces. In it we will calculate derivatives of value and Perlin noise.

This tutorial is made with Unity 2020.3.38f1.



2D Perlin noise with analytical normal vectors.

1 Value Noise

As we did for simplex noise, we begin with value noise, as that means that we can treat the gradients as constant values, simplifying the derivatives. So add options for 1D, 2D, and 3D value noise to `ProceduralSurface`.

```
static SurfaceJobScheduleDelegate[,] surfaceJobs = {
    {
        SurfaceJob<Lattice1D<LatticeNormal, Value>>.ScheduleParallel,
        SurfaceJob<Lattice2D<LatticeNormal, Value>>.ScheduleParallel,
        SurfaceJob<Lattice3D<LatticeNormal, Value>>.ScheduleParallel
    },
    …
};

public enum NoiseType {
    PerlinValue, Simplex, SimplexSmoothTurbulence, SimplexValue
}
```

1.1 Lattice Interpolator

We use the function $t(x) = 6x^5 - 15x^4 + 10x^3$ to interpolate between lattice points. We'll need the derivative of that, so add a field for it to `LatticeSpan4` named `dt`.

```
public struct LatticeSpan4 {
    public int4 p0, p1;
    public float4 g0, g1;
    public float4 t, dt;
}
```

In section 2.6 of Pseudorandom Noise / Value Noise we already determined that $t'(x) = 30x^4 - 60x^3 + 30x^2$. Calculate it to both in `LatticeNormal` and `LatticeTiling`.

```
    public LatticeSpan4 GetLatticeSpan4 (float4 coordinates, int frequency) {
        …
        float4 t = coordinates - points;
        span.t = t * t * t * (t * (t * 6f - 15f) + 10f);
        span.dt = t * t * (t * (t * 30f - 60f) + 30f);
        return span;
    }
```

Does tiling the noise affect the derivatives?

No, how the gradients are selected doesn't matter for the calculation.

1.2 Linear Interpolation

We use the linear interpolation function to transition between lattice points. This function can be mathematically defined as $l(a, b, t) = a(1 - t) + bt = a + (b - a)t$ where t is the variable interpolator. The rate of change of that interpolation is $l'_t(a, b, t) = b - a$.

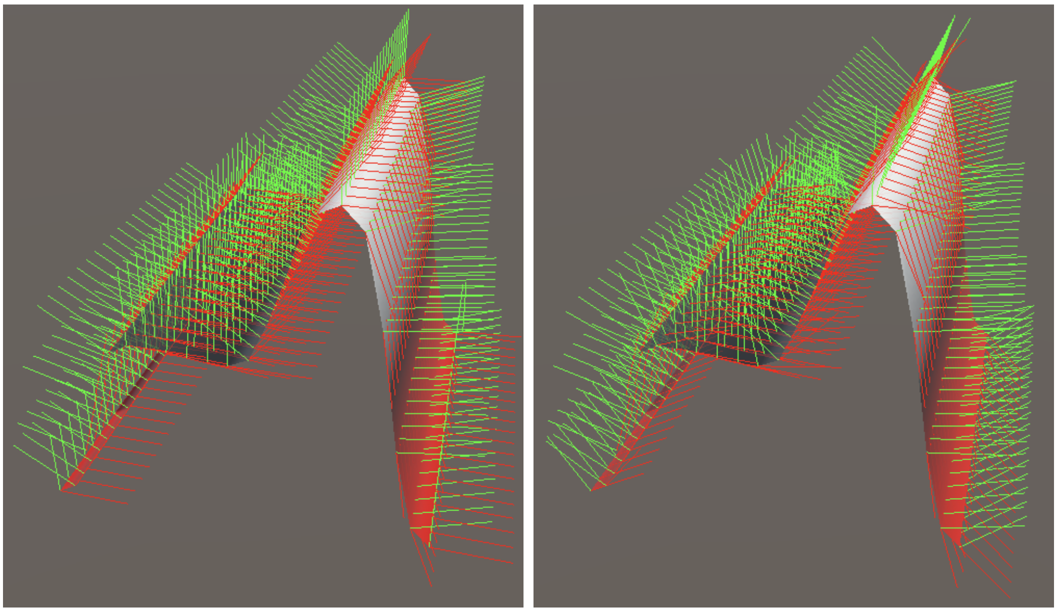
Our 1D value noise function is $v(x) = l(a, b, t(x))$ so by applying the chain rule we find its derivative $v'(x) = (b - a)t'(x)$.

Adjust `Lattice1D.GetNoise4` so it stores its gradients in variables and returns an explicit `Sample4`.

```
	public Sample4 GetNoise4(float4x3 positions, SmallXXHash4 hash, int frequency) {
		LatticeSpan4 x = default(L).GetLatticeSpan4(positions.c0, frequency);

		var g = default(G);
		//return g.EvaluateCombined(lerp(
		//	g.Evaluate(hash.Eat(x.p0), x.g0).v,
		//	g.Evaluate(hash.Eat(x.p1), x.g1).v, x.t
		//));
		Sample4
			a = g.Evaluate(hash.Eat(x.p0), x.g0),
			b = g.Evaluate(hash.Eat(x.p1), x.g1);

		return g.EvaluateCombined(new Sample4 {
			v = lerp(a.v, b.v, x.t)
		});
	}
```

Then include the derivative calculation.

```
		return g.EvaluateCombined(new Sample4 {
			v = lerp(a.v, b.v, x.t),
			dx = frequency * (b.v - a.v) * x.dt
		});
```



1D value noise; analytical and recalculated; frequency 4.

1.3 Bilinear Interpolation

For 2D value noise we use bilinear interpolation, so an interpolation of two interpolations. Begin by adjusting `Lattice2D.GetNoise4` so it also stores its gradients in variables and returns an explict `Sample4`.

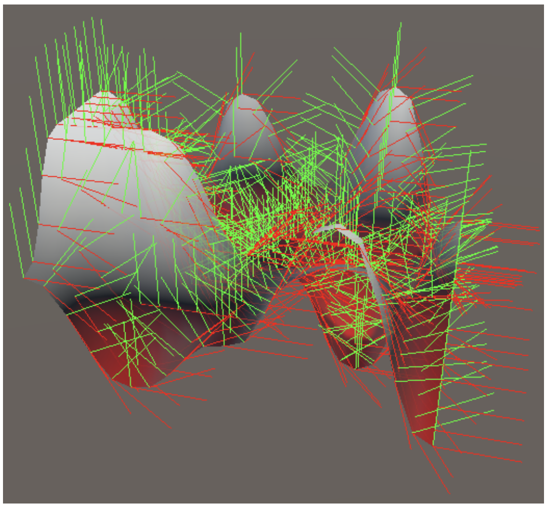
```
		var g = default(G);
		Sample4
			a = g.Evaluate(h0.Eat(z.p0), x.g0, z.g0),
			b = g.Evaluate(h0.Eat(z.p1), x.g0, z.g1),
			c = g.Evaluate(h1.Eat(z.p0), x.g1, z.g0),
			d = g.Evaluate(h1.Eat(z.p1), x.g1, z.g1);

		return g.EvaluateCombined(new Sample4 {
			v = lerp(lerp(a.v, b.v, z.t), lerp(c.v, d.v, z.t), x.t)
		});
```

The function for 2D value noise is $v(x, z) = l(l(a, b, t(z)), l(c, d, t(z)), t(x))$. As this is hard to read let's introduce a variant notation for the interpolation function: $l(a, b, t) \equiv a \xrightarrow{t} b$. The arrow operator indicates interpolation from its left to its right operand, with the interpolator written above it. It has priority over addition and subtraction.

Now we can write $v(x, z) = \left(a \xrightarrow{t(z)} b\right) \xrightarrow{t(x)} \left(c \xrightarrow{t(z)} d\right)$.

The partial X derivative of that is $v'_x(x, z) = \left(c \xrightarrow{t(z)} d - a \xrightarrow{t(z)} b\right) t'(x)$, because the interpolations in the Z dimension are constant in this case. Add it to the noise result.

```
			v = lerp(lerp(a.v, b.v, z.t), lerp(c.v, d.v, z.t), x.t),
			dx = frequency * (lerp(c.v, d.v, z.t) - lerp(a.v, b.v, z.t)) * x.dt
```



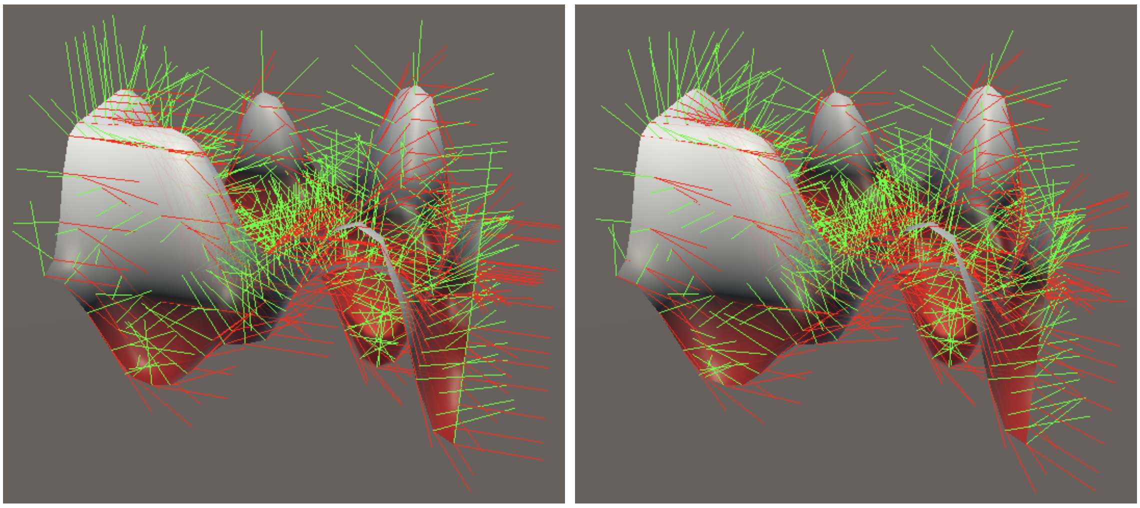
2D value noise; analytical X derivatives only.

For the partial Z derivative we have two separate Z interpolations of which we take the derivative, and then X interpolate those: $v'_z(x, z) = (b - a)t'(z) \xrightarrow{t(x)} (d - c)t'(z)$.

Can we interpolate between derivatives?

Yes, because the outer interpolation only applies addition, subtraction, and multiplication with a constant: $\left(a(x) \xrightarrow{t} b(x)\right)' = a'(x) + (b'(x) - a'(x))t$.

```
v = lerp(lerp(a.v, b.v, z.t), lerp(c.v, d.v, z.t), x.t),
dx = frequency * (lerp(c.v, d.v, z.t) - lerp(a.v, b.v, z.t)) * x.dt,
dz = frequency * lerp((b.v - a.v) * z.dt, (d.v - c.v) * z.dt, x.t)
```



2D value noise; analytical and recalculated.

1.4 Trilinear Interpolation

For 3D value noise we have to extend this approach to three dimensions. Again begin by adjusting the `Lattice3D.GetNoise4` method, as before.

```
var gradient = default(G);
Sample4
    a = gradient.Evaluate(h00.Eat(z.p0), x.g0, y.g0, z.g0),
    b = gradient.Evaluate(h00.Eat(z.p1), x.g0, y.g0, z.g1),
    c = gradient.Evaluate(h01.Eat(z.p0), x.g0, y.g1, z.g0),
    d = gradient.Evaluate(h01.Eat(z.p1), x.g0, y.g1, z.g1),
    e = gradient.Evaluate(h10.Eat(z.p0), x.g1, y.g0, z.g0),
    f = gradient.Evaluate(h10.Eat(z.p1), x.g1, y.g0, z.g1),
    g = gradient.Evaluate(h11.Eat(z.p0), x.g1, y.g1, z.g0),
    h = gradient.Evaluate(h11.Eat(z.p1), x.g1, y.g1, z.g1);

return gradient.EvaluateCombined(new Sample4 {
    v = lerp(
        lerp(lerp(a.v, b.v, z.t), lerp(c.v, d.v, z.t), y.t),
        lerp(lerp(e.v, f.v, z.t), lerp(g.v, h.v, z.t), y.t),
        x.t
    )
});
```

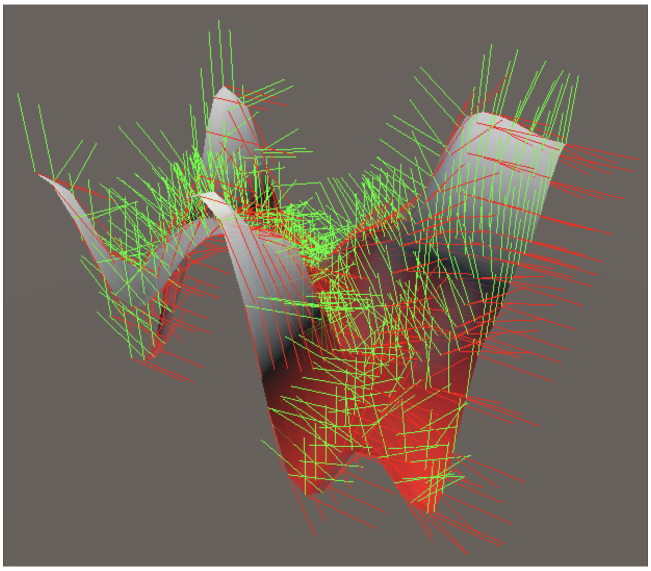
The entire function can be written as

$$v(x, y, z) = \left(\left(a \xrightarrow{t(z)} b\right) \xrightarrow{t(y)} \left(c \xrightarrow{t(z)} d\right)\right) \xrightarrow{t(x)} \left(\left(e \xrightarrow{t(z)} f\right) \xrightarrow{t(y)} \left(g \xrightarrow{t(z)} h\right)\right).$$

Its partial X derivative works the same as for 2D noise, except that it is larger:

$$v'_x(x, y, z) = \left(\left(e \xrightarrow{t(z)} f\right) \xrightarrow{t(y)} \left(g \xrightarrow{t(z)} h\right) - \left(a \xrightarrow{t(z)} b\right) \xrightarrow{t(y)} \left(c \xrightarrow{t(z)} d\right)\right) t'(x).$$

```
    v = lerp(
        lerp(lerp(a.v, b.v, z.t), lerp(c.v, d.v, z.t), y.t),
        lerp(lerp(e.v, f.v, z.t), lerp(g.v, h.v, z.t), y.t),
        x.t
    ),
    dx = frequency * (
        lerp(lerp(e.v, f.v, z.t), lerp(g.v, h.v, z.t), y.t) -
        lerp(lerp(a.v, b.v, z.t), lerp(c.v, d.v, z.t), y.t)
    ) * x.dt
```



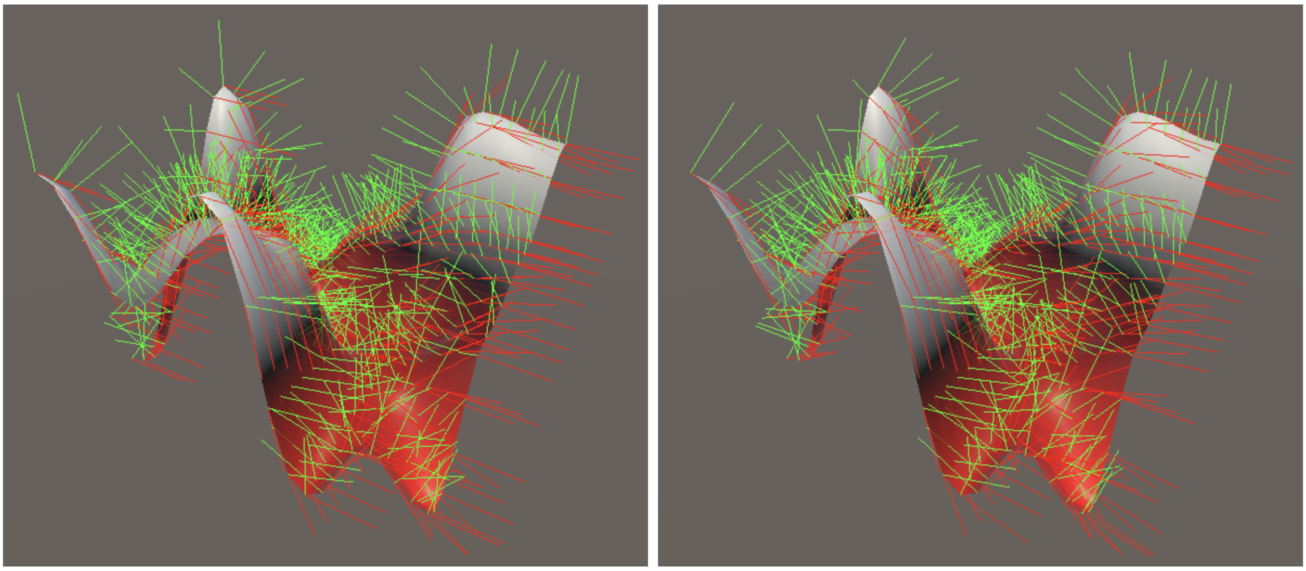
3D value noise; analytical X derivatives only.

The partial Z derivate also works like the 2D version, except that it goes one level deeper:

$$v'_z(x, y, z) = \left((b - a)t'(z) \xrightarrow{t(y)} (d - c)t'(z)\right) \xrightarrow{t(x)} \left((f - e)t'(z) \xrightarrow{t(y)} (h - g)t'(z)\right)$$

.

```
dx = frequency * (
    lerp(lerp(e.v, f.v, z.t), lerp(g.v, h.v, z.t), y.t) -
    lerp(lerp(a.v, b.v, z.t), lerp(c.v, d.v, z.t), y.t)
) * x.dt,
dz = frequency * lerp(
    lerp((b.v - a.v) * z.dt, (d.v - c.v) * z.dt, y.t),
    lerp((f.v - e.v) * z.dt, (h.v - g.v) * z.dt, y.t),
    x.t
)
```

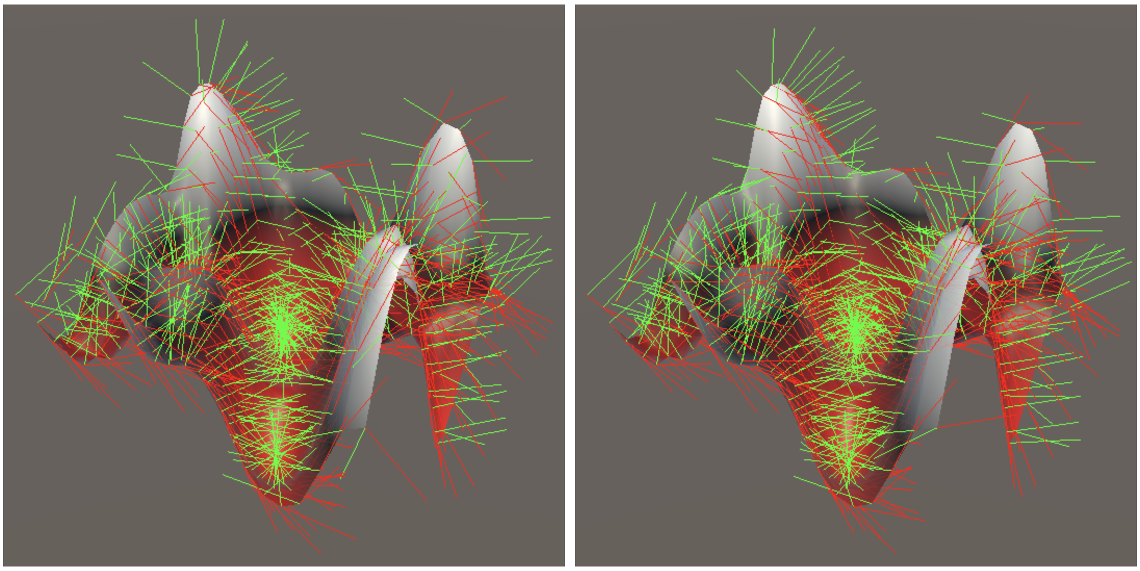


3D value noise; analytical XZ derivatives and recalculated.

Finally, the partial Y derivative sits in the middle and looks like a mix of the other two derivatives:

$$v'_y(x, y, z) = \left(c \xrightarrow{t(z)} d - a \xrightarrow{t(z)} b\right) t'(y) \xrightarrow{t(x)} \left(g \xrightarrow{t(z)} h - e \xrightarrow{t(z)} f\right) t'(y).$$

```
dx = frequency * (
    lerp(lerp(e.v, f.v, z.t), lerp(g.v, h.v, z.t), y.t) -
    lerp(lerp(a.v, b.v, z.t), lerp(c.v, d.v, z.t), y.t)
) * x.dt,
dy = frequency * lerp(
    (lerp(c.v, d.v, z.t) - lerp(a.v, b.v, z.t)) * y.dt,
    (lerp(g.v, h.v, z.t) - lerp(e.v, f.v, z.t)) * y.dt,
    x.t
),
dz = frequency * lerp(
    lerp((b.v - a.v) * z.dt, (d.v - c.v) * z.dt, y.t),
    lerp((f.v - e.v) * z.dt, (h.v - g.v) * z.dt, y.t),
    x.t
)
```



3D value noise rotated 45° on all axes; analytical and recalculated.

2 Perlin Noise

Just like with simplex noise, to support Perlin noise we must treat the gradients as functions and thus incorporate them into the derivatives. Add 1D, 2D, and 3D Perlin noise options to `ProceduralSurface` so that we can see then.

2.1 Interpolation with Gradients

```
static SurfaceJobScheduleDelegate[,] surfaceJobs = {
	{
		SurfaceJob<Lattice1D<LatticeNormal, Perlin>>.ScheduleParallel,
		SurfaceJob<Lattice2D<LatticeNormal, Perlin>>.ScheduleParallel,
		SurfaceJob<Lattice3D<LatticeNormal, Perlin>>.ScheduleParallel
	},
	…
};

public enum NoiseType {
	Perlin, PerlinValue, Simplex, SimplexSmoothTurbulence, SimplexValue
}
```

2.2 1D Gradients

For 1D Perlin noise we have to use the function $p(x) = a(x) \xrightarrow{t(x)} b(x)$. We find its derivative by applying the product rule:
$p'(x) = a'(x) \xrightarrow{t(x)} b'(x) + (b(x) - a(x))t'(x)$.

How did you find that derivative?

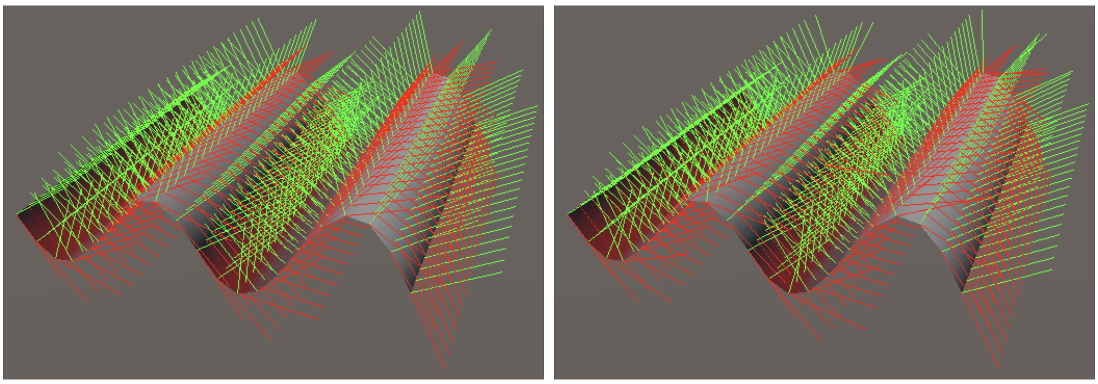
$p(x) = a(x) \xrightarrow{t(x)} b(x) = a(x) + (b(x) - a(x))t(x)$.

Apply the product rule:
$p'(x) = a'(x) + (b'(x) - a'(x))t(x) + (b(x) - a(x))t'(x)$.

Note that the first portion is an interpolation between the gradient derivatives based on $t(x)$.

Adjust `Lattice1D.GetNoise4` by adding the new part to its derivative. Note that if the gradients are constant their derivatives are zero, so the new interpolation would add nothing.

```
			dx = frequency * (lerp(a.dx, b.dx, x.t) + (b.v - a.v) * x.dt)
```



1D Perlin noise; analytical and recalculated; frequency 2.

2.3 2D Gradients

For 2D noise the partial X derivative changes in the same way, also gaining a interpolation of gradient derivatives:
$p'_x(x, z) = \left(a'_x \xleftrightarrow{t(z)} b'_x\right) \xleftrightarrow{t(x)} \left(c'_x \xleftrightarrow{t(z)} d'_x\right) + \left(c \xleftrightarrow{t(z)} d - a \xleftrightarrow{t(z)} b\right) t'(x)$. Here I omitted the (x, z) arguments that all gradients should have. Add the new portion of the derivative to `Lattice2D.GetNoise4`.

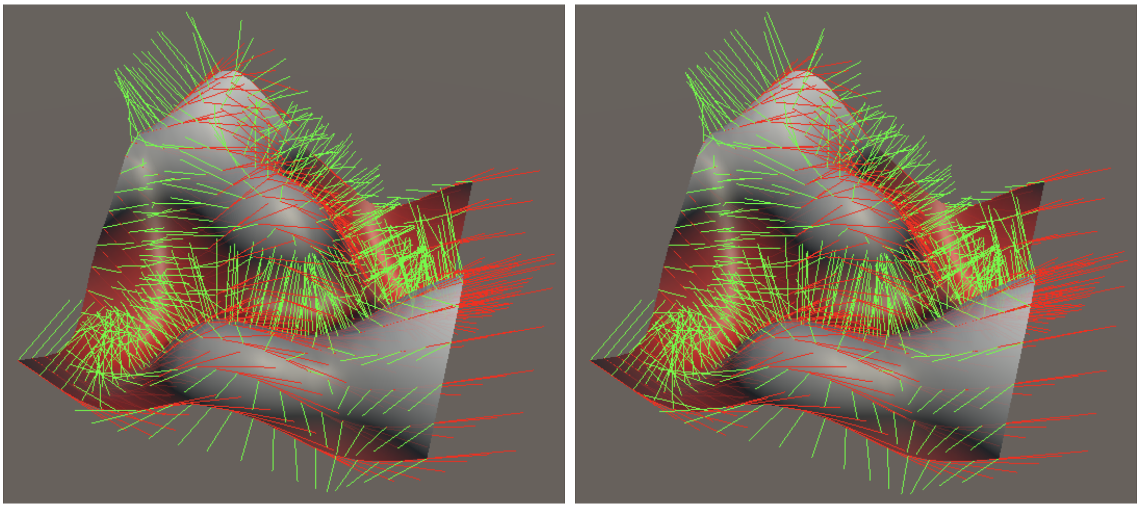
```
v = lerp(lerp(a.v, b.v, z.t), lerp(c.v, d.v, z.t), x.t),
dx = frequency * (
    lerp(lerp(a.dx, b.dx, z.t), lerp(c.dx, d.dx, z.t), x.t) +
    (lerp(c.v, d.v, z.t) - lerp(a.v, b.v, z.t)) * x.dt
),
```

And the partial Z derivative changes likewise: $p'_z(x, z) = \left(a \overset{t(z)}{\longleftrightarrow} b\right)'_z \overset{t(x)}{\longleftrightarrow} \left(c \overset{t(z)}{\longleftrightarrow} d\right)'_z$. Here I didn't write out the derivatives of the inner interpolations.

```
dz = frequency * lerp(
    lerp(a.dz, b.dz, z.t) + (b.v - a.v) * z.dt,
    lerp(c.dz, d.dz, z.t) + (d.v - c.v) * z.dt,
    x.t
)
```

To complete the 2D derivatives we must also adjust `BaseGradients.Square` so it returns its derivatives, like `BaseGradients.Circle`.

```
public static Sample4 Square (SmallXXHash4 hash, float4 x, float4 y) {
    float4x2 v = SquareVectors(hash);
    return new Sample4 {
        v = v.c0 * x + v.c1 * y,
        dx = v.c0,
        dz = v.c1
    };
}
```



2D Perlin noise; analytical and recalculated.

2.4 3D Gradients

The changes to 3D noise again follow the same pattern as for 2D. Interpolations of gradient derivatives are added where applicable. First for the partial X derivative.

```
			dx = frequency * (
				lerp(
					lerp(lerp(a.dx, b.dx, z.t), lerp(c.dx, d.dx, z.t), y.t),
					lerp(lerp(e.dx, f.dx, z.t), lerp(g.dx, h.dx, z.t), y.t),
					x.t
				) + (
					lerp(lerp(e.v, f.v, z.t), lerp(g.v, h.v, z.t), y.t) -
					lerp(lerp(a.v, b.v, z.t), lerp(c.v, d.v, z.t), y.t)
				) * x.dt
			),
```

Second for the partial Y derivative.

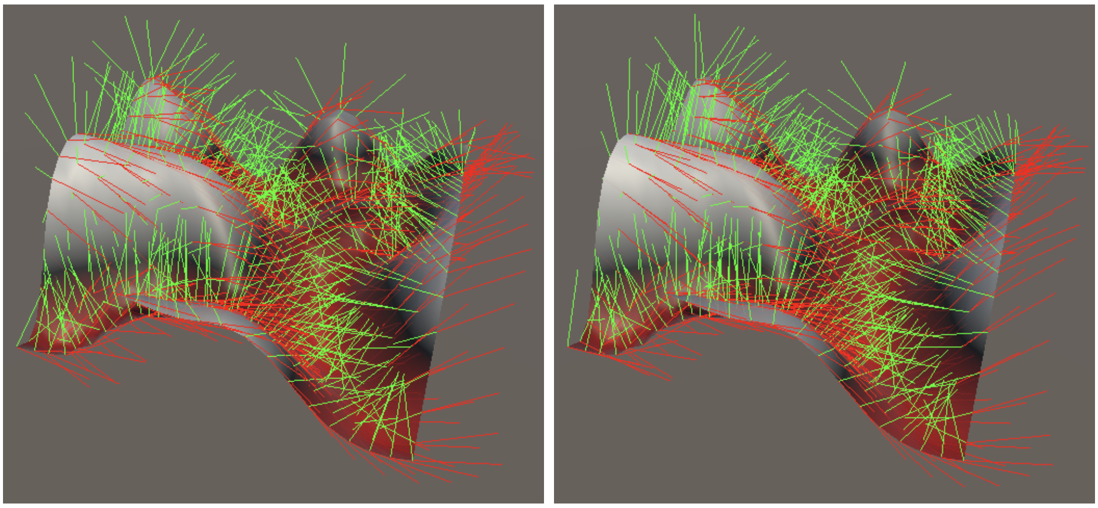
```
			dy = frequency * lerp(
				lerp(lerp(a.dy, b.dy, z.t), lerp(c.dy, d.dy, z.t), y.t) +
				(lerp(c.v, d.v, z.t) - lerp(a.v, b.v, z.t)) * y.dt,
				lerp(lerp(e.dy, f.dy, z.t), lerp(g.dy, h.dy, z.t), y.t) +
				(lerp(g.v, h.v, z.t) - lerp(e.v, f.v, z.t)) * y.dt,
				x.t
			),
```

Third for the partial Z derivative.

```
			dz = frequency * lerp(
				lerp(
					lerp(a.dz, b.dz, z.t) + (b.v - a.v) * z.dt,
					lerp(c.dz, d.dz, z.t) + (d.v - c.v) * z.dt,
					y.t
				),
				lerp(
					lerp(e.dz, f.dz, z.t) + (f.v - e.v) * z.dt,
					lerp(g.dz, h.dz, z.t) + (h.v - g.v) * z.dt,
					y.t
				),
				x.t
			)
```

And finally the inclusion of derivatives in `BaseGradients.Octahedron`.

```
	public static Sample4 Octahedron (
		SmallXXHash4 hash, float4 x, float4 y, float4 z
	) {
		float4x3 v = OctahedronVectors(hash);
		return new Sample4 {
			v = v.c0 * x + v.c1 * y + v.c2 * z,
			dx = v.c0,
			dy = v.c1,
			dz = v.c2
		};
	}
```

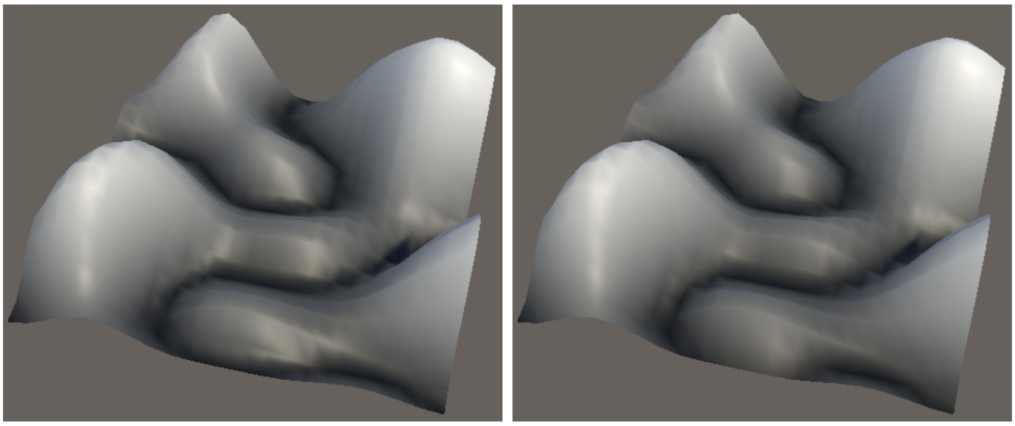


3D Perlin noise; rotated 45° on all axes; frequency 3; analytical and recalculated.

2.5 Turbulence

We wrap up this tutorial with the inclusion of smooth turbulence variants of Perlin noise.

```
static SurfaceJobScheduleDelegate[,] surfaceJobs = {
	{
		SurfaceJob<Lattice1D<LatticeNormal, Perlin>>.ScheduleParallel,
		SurfaceJob<Lattice2D<LatticeNormal, Perlin>>.ScheduleParallel,
		SurfaceJob<Lattice3D<LatticeNormal, Perlin>>.ScheduleParallel
	},
	{
		SurfaceJob<Lattice1D<LatticeNormal, Smoothstep<Turbulence<Perlin>>>>
			.ScheduleParallel,
		SurfaceJob<Lattice2D<LatticeNormal, Smoothstep<Turbulence<Perlin>>>>
			.ScheduleParallel,
		SurfaceJob<Lattice3D<LatticeNormal, Smoothstep<Turbulence<Perlin>>>>
			.ScheduleParallel
	},
	…
};

public enum NoiseType {
	Perlin, PerlinSmoothTurbulence, PerlinValue,
	Simplex, SimplexSmoothTurbulence, SimplexValue
}
```



2D Perlin smooth turbulence noise; frequency 2; analytical and recalculated.

The next tutorial is Voronoi Derivatives.

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