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Parameter estimation for fractional Ornstein–Uhlenbeck processes of general Hurst parameter

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Abstract This paper studies the least squares estimator (LSE) for the drift parameter of an Ornstein–Uhlenbeck process driven by fractional Brownian motion, whose observations can be made either continuously or at discrete time instants. A central limit theorem is proved when the Hurst parameter $H \in (0, 3/4]$ and a noncentral limit theorem is proved for $H \in (3/4, 1)$. Thus, the open problem left in the previous paper (Hu and Nualart in Stat Probab Lett 80(11–12):1030–1038, 2010) is completely solved, where a central limit theorem for the least squares estimator is proved for $H \in [1/2, 3/4)$. The LSE is then used to study the asymptotics for other alternative estimators, such as the ergodic type estimator.

Keywords Fractional Brownian motion · Fractional Ornstein–Uhlenbeck processes · Parameter estimation · Fourth moment theorem · Central limit theorem · Noncentral limit theorem

1 Introduction

Consider the fractional Ornstein–Uhlenbeck process defined as the unique pathwise solution to the stochastic differential equation

$$X_t = X_0 - \theta \int_0^t X_s ds + \sigma dB_t^H, \tag{1.1}$$

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with initial condition $X_0 \in \mathbb{R}$, where $B^H = \{B_t^H, t \geq 0\}$ is a fractional Brownian motion (fBm) of Hurst parameter $H \in (0, 1)$, and θ and σ are positive parameters.

Assume that the process can be observed continuously or at discrete time instants. The Hurst parameter H and the volatility parameter σ can be estimated by quadratic variation methods (for example, see Istas and Lang 1997; Kubilius and Mishura 2012; Berzin et al. 2015 and the references therein), or using regression methods (see Berzin and León 2008). In this paper, we will focus on the estimation of the drift parameter θ for any $H \in (0, 1)$. Several estimators for θ have been proposed previously for some particular Hurst parameter ranges. A summary of some relevant results are presented below.

(i) In the case of continuous observations, Kleptsyna and Le Breton studied the maximum likelihood estimator (MLE) for $H > \frac{1}{2}$ in the paper (Kleptsyna and Le Breton 2002), which is defined by

$$\hat{\theta}_{MLE} = -\left\{\int_0^T Q^2(s)dw_s^H\right\}^{-1} \int_0^T Q(s)dZ_s,$$

where

$$Q(t) = \frac{d}{dw_t^H}\int_0^t k_H(t,s)X_s ds, \quad Z_t = \int_0^t k_H(t,s)dX_s,$$

$k_H(t,s) = \kappa_H^{-1} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}$ and $w_t^H = \lambda_H^{-1} t^{2-2H}$ with constants κ_H and λ_H depending on H. They proved the almost sure convergence of $\hat{\theta}_{MLE}$ to θ as T tends to infinity. Later on, Brouste and Kleptsyna investigated MLE in the paper (Brouste and Kleptsyna 2010) and proved the following central limit thoerem.

$$\sqrt{T}(\hat{\theta}_{MLE} - \theta) \xrightarrow[T\to\infty]{\mathcal{L}} N(0, 2\theta).$$

Bercu, Courtin and Savy used a different approach to prove the above central limit theorem for the MLE in the case of $H > \frac{1}{2}$ (see Bercu et al. 2011). The two papers claimed without proof that the above convergence is also valid for $H \in (0, \frac{1}{2})$. It is worth noting that Tudor and Viens have also obtained the almost sure convergence of both the MLE and a version of the MLE using discrete observations for all $H \in (0, 1)$ (see Tudor and Viens 2007).

(ii) On the other hand, the least squares estimator was proposed in the paper (Hu and Nualart 2010), which is defined by

$$\hat{\theta}_T = -\frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} = \theta - \sigma\frac{\int_0^T X_t dB_t^H}{\int_0^T X_t^2 dt}, \tag{1.2}$$

where the integral with respect to B^H is interpreted in the Skorohod sense. They also introduced another estimator $\widetilde{\theta}_T$ based on the ergodic theorem given by

$$\widetilde{\theta}_T = \left(\frac{1}{\sigma^2 H\Gamma(2H)T}\int_0^T X_t^2 dt\right)^{-\frac{1}{2H}}. \tag{1.3}$$

Almost sure convergence and central limit theorems for these two estimators have been proved for $H \in [\frac{1}{2}, \frac{3}{4})$. Sottinen and Viitasaari derived a central limit theorem and a Berry–Esseen bound for $\widetilde{\theta}_T$ when $H \in (0, 1)$ in a recently published paper (Sottinen and Viitasaari 2017). However, they did not give an explicit expression for the limiting variance. In this paper, the asymptotic behavior of $\widetilde{\theta}_T$ will be covered using a different approach.

When $H \in (0, \frac{1}{2}) \cup [\frac{3}{4}, 1)$, the central limit theorems for the least squares estimator $\hat{\theta}_T$ have not been known yet. The first objective of Sect. 3 is to prove the asymptotic consistency of $\hat{\theta}_T$ by using a new method, different from that in Hu and Nualart (2010), which is valid for all $H \in (0, 1)$. This method involves the relationship between the divergence and Stratonovich integrals and the integration by parts technique and it is based on the pathwise properties of the fractional Ornstein–Uhlenbeck process established in the paper (Cheridito et al. 2003). The next and the main objective of this paper is to establish a central limit theorem for the least squares estimator $\hat{\theta}_T$ for $H \in (0, \frac{3}{4}]$ and a noncentral limit theorem for $H \in (\frac{3}{4}, 1)$. In the later case, we can identify the limit as a Rosenblatt random variable. In Sect. 4, we will use the results of MLE to obtain the asymptotics of the ergodic type estimator $\widetilde{\theta}_T$, and make a comparison of the asymptotic variance for these three estimators and show that the least squares estimator performs better than the maximum likelihood estimator when $H \in (0, \frac{1}{2})$. Since the ergodic-type estimator $\widetilde{\theta}_T$ is a function of a pathwise Riemann integral that appears simpler than the other two estimators, we will use $\widetilde{\theta}_T$ to construct a consistent estimator $\bar{\theta}_n$ for high frequency data (if only discrete observations are available). The asymptotic behavior of $\bar{\theta}_n$ and the numerical simulation will be studied in Sect. 5 of this paper. The proofs of our results are highly technical and rely on some sophisticated computation, which we shall put in the "Appendix". The main tool we use is Malliavin calculus which is recalled in Sect. 2. We use C to denote a generic constant that may vary according to the context.

2 Preliminaries

In this section, we briefly recall some notions and results on fractional Brownian motion, and Malliavin calculus. The fractional Brownian motion (fBm) $B^H = \{B_t^H, t \in \mathbb{R}\}$ with Hurst parameter $H \in (0, 1)$ is a zero mean Gaussian process, defined on a complete probability space $(\Omega, \mathcal{F}, P)$, with the following covariance function

$$\mathbb{E}(B_t^H B_s^H) = R_H(t, s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}). \tag{2.1}$$

This process is self-similar of order $H > 0$, that is, for any $a > 0$ the processes $\{B_{at}^H, t \in \mathbb{R}\}$ and $\{a^H B_t^H, t \in \mathbb{R}\}$ are the same in law. From (2.1), it is easy to see that $\mathbb{E}|B_t^H - B_s^H|^2 = |t - s|^{2H}$. Then it follows from Kolmogorov's continuity criterion that on any finite interval, almost surely all paths of fBm are α-Hölder continuous with $\alpha < H$. Denote by η_T the α-Hölder coefficient of fBm on the interval $[0, T]$, i.e.,

$$\eta_T = \sup_{t \neq s \in [0,T]} \frac{\left|B_t^H - B_s^H\right|}{|t - s|^\alpha}. \tag{2.2}$$

Clearly, $\mathbb{E}|\eta_T|^q = T^{q(H-\alpha)}\mathbb{E}|\eta_1|^q$ for any $q > 1$, by the self-similarity property of fBm.

Let $\mathcal{F}$ denote the σ-field obtained from the completion of the σ-field generated by B^H. Let $\mathcal{E}$ denote the space of all real valued step functions on $\mathbb{R}$. The Hilbert space $\mathfrak{H}$ is defined as the closure of $\mathcal{E}$ endowed with the inner product

$$\langle \mathbf{1}_{[a,b]}, \mathbf{1}_{[c,d]} \rangle_{\mathfrak{H}} = \mathbb{E}\left((B_b^H - B_a^H)(B_d^H - B_c^H) \right).$$

Under the convention that $\mathbf{1}_{[0,t]} = -\mathbf{1}_{[t,0]}$ if $t < 0$, the mapping $\mathbf{1}_{[0,t]} \mapsto B_t^H$ can be extended to a linear isometry between $\mathfrak{H}$ and the Gaussian space $\mathcal{H}_1$ spanned by B^H. We denote this isometry by $\mathfrak{H} \ni \varphi \mapsto B^H(\varphi)$. If $f, g \in \mathfrak{H}$ and g is a continuously differentiable function with compact support, we can use step functions in $\mathcal{E}$ to approximate f and g and by a limiting argument we deduce

$$\langle f, g\rangle_{\mathfrak{H}} = \int_{\mathbb{R}^2} f(t)g'(s)\frac{\partial R_H(t,s)}{\partial t}dtds \tag{2.3}$$

(see Hu et al. 2013). We can also use Fourier transform to compute $\langle f, g\rangle_{\mathfrak{H}}$, namely,

$$\langle f, g\rangle_{\mathfrak{H}} = \frac{1}{c_H^2}\int_{\mathbb{R}} \mathcal{F}f(\xi)\overline{\mathcal{F}g(\xi)}|\xi|^{1-2H}d\xi, \tag{2.4}$$

where $c_H = \left(\frac{2\pi}{\Gamma(2H+1)\sin(\pi H)}\right)^{\frac{1}{2}}$ (see Pipiras and Taqqu 2000). When $H > 1/2$, for any $f, g \in L^{1/H}([0,T])$, if we extend f and g to be zero on $\mathbb{R} \cap [0,T]^c$, then $f, g \in \mathfrak{H}$ and we have the following simple identity

$$\langle f, g\rangle_{\mathfrak{H}} = \alpha_H \int_{[0,T]^2} f(u)g(v)|u-v|^{2H-2}dudv, \tag{2.5}$$

where $\alpha_H = H(2H-1)$.

Next we define two types of stochastic integrals: Stratonovich integral and divergence integral. Given a stochastic process $\{v(t), t \geq 0\}$ such that $\int_0^t |v(s)|ds < \infty$ a.s. for all $t > 0$, the Stratonovich integral $\int_0^t v(s) \circ dB_s^H$ is defined as the following limit in probability if it exists

$$\lim_{\epsilon\to 0}\int_0^t v(s)\dot{B}_s^{H,\varepsilon}ds,$$

where $\dot{B}_s^{H,\varepsilon}$ is a symmetric approximation of $\dot{B}_s^H$:

$$\dot{B}_s^{H,\varepsilon} = \frac{1}{2\epsilon}(B_{s+\epsilon}^H - B_{s-\epsilon}^H).$$

Before we define the divergence integral, we present some background of Malliavin calculus. For a smooth and cylindrical random variable $F = f(B^H(\varphi_1), \ldots, B^H(\varphi_n))$, with $\varphi_i \in \mathfrak{H}$ and $f \in C_b^\infty(\mathbb{R}^n)$ (f and all of its partial derivatives are bounded), we define its Malliavin derivative as the $\mathfrak{H}$-valued random variable given by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B^H(\varphi_1), \ldots, B^H(\varphi_n))\varphi_i.$$

By iteration, one can define the k-th derivative D^kF as an element of $L^2(\Omega; \mathfrak{H}^{\otimes k})$. For any natural number k and any real number $p \geq 1$, we define the Sobolev space $\mathbb{D}^{k,p}$ as the closure of the space of smooth and cylindrical random variables with respect to the norm $||\cdot||_{k,p}$ defined by

$$||F||_{k,p}^p = \mathbb{E}(|F|^p) + \sum_{i=1}^k \mathbb{E}\left(||D^iF||_{\mathfrak{H}^{\otimes i}}^p\right).$$

The divergence operator δ is defined as the adjoint of the derivative operator D in the following manner. An element $u \in L^2(\Omega; \mathfrak{H})$ belongs to the domain of δ, denoted by Dom δ, if there is a constant c_u depending on u such that

$$|\mathbb{E}(\langle DF, u\rangle_{\mathfrak{H}})| \leq c_u||F||_{L^2(\Omega)}$$

for any $F \in \mathbb{D}^{1,2}$. If $u \in \operatorname{Dom}\delta$, then the random variable $\delta(u)$ is defined by the duality relationship

$$\mathbb{E}(F\delta(u)) = \mathbb{E}(\langle DF, u\rangle_{\mathfrak{H}}),$$

which holds for any $F \in \mathbb{D}^{1,2}$. If $u = \{u_t, t \in [0,T]\}$ is a stochastic process, whose trajectories belong to $\mathfrak{H}$ almost surely (with the convention $u_t = 0$ if $t \notin [0,T]$) and $u \in \operatorname{Dom}\delta$, we make use of the notation $\int_0^T u_t dB_t^H = \delta(u)$ and call $\delta(u)$ the divergence integral of u with respect to the fractional Brownian motion B^H on $[0,T]$. It is worth noting that the divergence integral of fBm with respect to itself does not exist if $H \in (0, \frac{1}{4})$ because the paths of the fBm are too irregular (see Cheridito and Nualart 2005). For this reason, in Cheridito and Nualart (2005) the authors introduce an extended divergence integral δ^* such that $\operatorname{Dom}\delta^* \cap L^2(\Omega;\mathfrak{H}) = \operatorname{Dom}\delta$ and the extended divergence operator δ^* restricted to $\operatorname{Dom}\delta$ coincides with the divergence operator. In a similar way we can introduce the iterated divergence operator δ^k for each integer $k \geq 2$, defined by the duality relationship

$$\mathbb{E}(F\delta^k(u)) = \mathbb{E}\left(\langle D^k F, u\rangle_{\mathfrak{H}^{\otimes k}}\right),$$

for any $F \in \mathbb{D}^{k,2}$, where $u \in \operatorname{Dom}\delta^k \subset L^2(\Omega;\mathfrak{H}^{\otimes k})$.

For any integer $m \geq 1$, we use $\mathfrak{H}^{\otimes m}$ and $\mathfrak{H}^{\odot m}$ to denote the m-th tensor product and the m-th symmetric tensor product of the Hilbert space $\mathfrak{H}$, respectively. We denote by $\mathcal{H}_m$ the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_m(B^H(\varphi)) : \varphi \in \mathfrak{H}, ||\varphi||_{\mathfrak{H}} = 1\}$, where H_m is the m-th Hermite polynomial defined by

$$H_m(x) = (-1)^m e^{\frac{x^2}{2}} \frac{d^m}{dx^m} e^{-\frac{x^2}{2}}, \quad m \geq 1,$$

and $H_0(x) = 1$. The space $\mathcal{H}_m$ is called the Wiener chaos of order m. The m-th multiple integral of $\varphi \in \mathfrak{H}^{\odot m}$ is defined by the identity $I_m(\varphi) = \delta^m(\varphi)$, and in particular, $I_m(\phi^{\otimes m}) = H_m(B^H(\phi))$ for any $\phi \in \mathfrak{H}$. The map I_m provides a linear isometry between $\mathfrak{H}^{\odot m}$(equipped with the norm $\sqrt{m!}||\cdot||_{\mathfrak{H}^{\otimes m}}$) and $\mathcal{H}_m$ (equipped with $L^2(\Omega)$ norm) (see Nourdin and Peccati 2012, Theorem 2.7.7). By convention, $\mathcal{H}_0 = \mathbb{R}$ and $I_0(x) = x$.

The space $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_m$, which is known as the Wiener chaos expansion. Thus, any square integrable random variable $F \in L^2(\Omega)$ has the following expansion,

$$F = \sum_{m=0}^{\infty} I_m(f_m),$$

where $f_0 = \mathbb{E}(F)$, and $f_m \in \mathfrak{H}^{\odot m}$ are uniquely determined by F. We denote by J_m the orthogonal projection onto the m-th Wiener chaos $\mathcal{H}_m$. This means that $I_m(f_m) = J_m(F)$ for every $m \geq 0$.

Let $\{e_k, k \geq 1\}$ be a complete orthonormal system in the Hilbert space $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot n}$, $g \in \mathfrak{H}^{\odot m}$, and $p = 1, \ldots, n \wedge m$, the p-th contraction between f and g is the element of $\mathfrak{H}^{\otimes(m+n-2p)}$ defined by

$$f \otimes_p g = \sum_{i_1,\ldots,i_p=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_p}\rangle_{\mathfrak{H}^{\otimes p}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_p}\rangle_{\mathfrak{H}^{\otimes p}}.$$

The following result (known as the fourth moment theorem) provides necessary and sufficient conditions for the convergence of some random variables to a normal distribution (see Nualart and Ortiz-Latorre 2008; Nualart and Peccati 2005; Nourdin and Peccati 2012).

Theorem 1 *Let $n \geq 2$ be a fixed integer. Consider a collection of elements $\{f_T, T > 0\}$ such that $f_T \in \mathfrak{H}^{\odot n}$ for every $T > 0$. Assume further that*

$$\lim_{T\to\infty} \mathbb{E}[I_n(f_T)^2] = \lim_{T\to\infty} n!\|f_T\|^2_{\mathfrak{H}^{\otimes n}} = \sigma^2.$$

Then the following conditions are equivalent:

1. $\lim_{T\to\infty} \mathbb{E}[I_n(f_T)^4] = 3\sigma^2$.
2. *For every $p = 1, \ldots, n-1$, $\lim_{T\to\infty} ||f_T \otimes_p f_T||_{\mathfrak{H}^{\otimes 2(n-p)}} = 0$.*
3. *As T tends to infinity, the n-th multiple integrals $\{I_n(f_T), T \geq 0\}$ converge in distribution to a standard Gaussian random variable $N(0, \sigma^2)$.*
4. $\|D(I_n(f_T))\|^2_{\mathfrak{H}} \xrightarrow[T\to\infty]{L^2(\Omega)} n\sigma^2$.
5. $\lim_{T\to\infty} \mathrm{Var}(\|D(I_n(f_T))\|^2_{\mathfrak{H}}) = 0$.

Remark 2 The multidimensional version of the above theorem is also stated and proved in Nualart and Ortiz-Latorre (2008), Nourdin and Peccati (2012), Peccati and Tudor (2004).

For the two real-valued random variables F and G, the total variation distance between the laws of F and G is defined by the quantity

$$d_{\mathrm{TV}}(F, G) = \sup_{B\in\mathscr{B}(\mathbb{R})} |P(F \in B) - P(G \in B)|,$$

where the supremum is taken over Borel sets B of $\mathbb{R}$. Then we have the following bounds on normal approximation inside a Wiener chaos (see Nourdin and Peccati 2012).

Proposition 3 *Let $n \geq 2$ be an integer, and $F_T = I_n(f)$ be a multiple integral of order n with $\mathbb{E}(F_T^2) = 1$. Let N be a random variable with the standard normal distribution. Then the total variation distance between F_T and N is bounded as follows.*

$$d_{\mathrm{TV}}(F_T, N) \leq 2\sqrt{\mathrm{Var}\left(\frac{1}{n}\|DF_T\|^2_{\mathfrak{H}}\right)}.$$

Finally, let us recall the definition of the Rosenblatt process that will appear in the limit theorems of Sect. 3. Fix $H > 3/4$ and $t \in [0, 1]$. Consider the sequence of functions of two variables

$$\xi_{n,t} = 2^n \sum_{i=1}^{[2^n t]} \mathbf{1}^{\otimes 2}_{((i-1)2^{-n}, i2^{-n}]}.$$

Through a direct computation using (2.5) one can show that this sequence is Cauchy in $\mathfrak{H}^{\otimes 2}$ and converges to the distribution denoted by $\delta_{0,t}$ and defined by

$$\langle \delta_{0,t}, f\rangle = \int_0^t f(s, s)ds, \tag{2.6}$$

for any test function f on $\mathbb{R}^2$. It turns out (see Nourdin et al. 2010 for the proofs) that the sequence $I_2(\xi_{n,t})$ converges in L^2 as n tends to infinity to the Rosenblatt random variable $R_t = I_2(\delta_{0,t})$. For any $f \in L^{1/H}([0, 1]^2)$, we have the following formula, letting f equal to zero on $\mathbb{R}^2 \cap [0, 1]^c$,

$$\mathbb{E}(R_t I_2(f)) = 2\langle \delta_{0,t}, f\rangle_{\mathfrak{H}^{\otimes 2}} = 2\alpha_H^2 \int_0^t dv \int_{[0,1]^2} f(u_1, u_2)|u_1-v|^{2H-2}|u_2-v|^{2H-2}du_1du_2. \tag{2.7}$$

3 Asymptotics of the least squares estimator

The maximum likelihood estimator and the least squares estimator coincide in the Brownian motion case, but for the fractional Ornstein–Uhlenbeck processes they are different (see Hu and Nualart 2010; Kleptsyna and Le Breton 2002). We shall focus on the least squares estimator given as

$$\hat{\theta}_T = -\frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} = \theta - \sigma \frac{\int_0^T X_t dB_t^H}{\int_0^T X_t^2 dt}, \tag{3.1}$$

where dB_t^H denotes the divergence integral. In the paper (Hu and Nualart 2010), the almost sure convergence of $\hat{\theta}_T$ to θ is proved for $H \geq \frac{1}{2}$ and the central limit theorem is obtained for $H \in [\frac{1}{2}, \frac{3}{4})$. Here we shall extend these results for a general Hurst parameter $H \in (0, 1)$. To simplify notation, we assume $X_0 = 0$. In this case the solution to (1.1) is given by

$$X_t = \sigma \int_0^t e^{-\theta(t-s)} dB_s^H. \tag{3.2}$$

Theorem 4 *For $H \in (0, 1)$, $\hat{\theta}_T \to \theta$ a.s. as $T \to \infty$.*

Proof Using integration by parts, we can write

$$X_t = \sigma \int_0^t e^{-\theta(t-s)} dB_s^H = \sigma \left(B_t^H - \theta \int_0^t B_s^H e^{-\theta(t-s)} ds \right). \tag{3.3}$$

Since X_t is in the first Wiener chaos, we have the relationship between the divergence integral and the Stratonovich integral as

$$\int_0^T X_t dB_t^H = \int_0^T X_t \circ dB_t^H - \ell(T), \tag{3.4}$$

where $\ell(T) = \mathbb{E} \int_0^T X_t \circ dB_t^H$. Using (3.3), $\ell(T)$ can be computed as follows

$$\begin{aligned} \ell(T) &= \sigma \mathbb{E} \int_0^T \left(B_t^H - \theta \int_0^t B_s^H e^{-\theta(t-s)} ds \right) \circ dB_t^H \\ &= \sigma \left[\frac{1}{2} T^{2H} - \theta \int_0^T \int_0^t e^{-\theta(t-s)} \frac{\partial \mathbb{E}(B_s^H B_t^H)}{\partial t} ds dt \right] \\ &= \frac{\sigma}{2} T^{2H} - m(T), \end{aligned} \tag{3.5}$$

where

$$m(T) := H\theta\sigma \int_0^T \int_0^t e^{-\theta(t-s)} (t^{2H-1} - (t-s)^{2H-1}) ds dt.$$

Making the substitutions $t - s \to u$, $s \to v$ and then integrating first in the variable v yield

$$m(T) = \frac{\sigma}{2} \gamma_{\theta T}^1 T^{2H} + \sigma \theta^{-2H} \gamma_{\theta T}^{2H+1} \left(H - \frac{1}{2} \right) - T H \sigma \theta^{1-2H} \gamma_{\theta T}^{2H}. \tag{3.6}$$

In the above equation, we use the notation $\gamma_T^\alpha = \int_0^T e^{-x} x^{\alpha-1} dx$. Observe that γ_T^α converges to $\Gamma(\alpha)$ exponentially fast as $T \to \infty$. Then clearly we have

$$\lim_{T\to\infty} T^{-1} \ell(T) = \lim_{T\to\infty} T^{-1} \left(\frac{\sigma}{2} T^{2H} - m(T) \right) = H\sigma\theta^{1-2H} \Gamma(2H). \tag{3.7}$$

On the other hand, we have

$$\sigma \int_0^T X_t \circ dB_t^H = \int_0^T X_t \circ (dX_t + \theta X_t dt) = \frac{X_T^2}{2} + \theta \int_0^T X_t^2 dt. \tag{3.8}$$

Combining (3.4) and (3.8) we obtain

$$\sigma \int_0^T X_t dB_t^H = \frac{X_T^2}{2} + \theta \int_0^T X_t^2 dt - \sigma \ell(T). \tag{3.9}$$

From Lemma 18, we see $\lim_{T\to\infty} \frac{X_T^2}{T} = 0$. Therefore, by Lemma 19, (3.7), and (3.9), we have

$$\lim_{T\to\infty} T^{-1}\sigma \int_0^T X_t dB_t^H = 0.$$

As a consequence,

$$\lim_{T\to\infty} \hat{\theta}_T = \lim_{T\to\infty} \left(\theta - \frac{\sigma \int_0^T X_t dB_t^H}{\int_0^T X_t^2 dt}\right) = \theta.$$

□

The next theorem shows the asymptotic laws for the least squares estimator $\hat{\theta}_T$.

Theorem 5 *As $T \to \infty$, the following convergence results hold true.*

(i) *For $H \in (0, \frac{3}{4})$, $\sqrt{T}(\hat{\theta}_T - \theta) \xrightarrow{\mathcal{L}} N(0, \theta\sigma_H^2)$, where*

$$\sigma_H^2 = \begin{cases} (4H-1) + \frac{2\Gamma(2-4H)\Gamma(4H)}{\Gamma(2H)\Gamma(1-2H)} & \text{when } H \in (0, \frac{1}{2}), \\ (4H-1)\left(1 + \frac{\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2-2H)\Gamma(2H)}\right) & \text{when } H \in [\frac{1}{2}, \frac{3}{4}). \end{cases}$$

(ii) *For $H = \frac{3}{4}$, $\frac{\sqrt{T}}{\sqrt{\log(T)}}(\hat{\theta}_T - \theta) \xrightarrow{\mathcal{L}} N(0, 4\pi^{-1}\theta)$.*

(iii) *For $H \in (\frac{3}{4}, 1)$, $T^{2-2H}(\hat{\theta}_T - \theta) \xrightarrow{\mathcal{L}} \frac{-\theta^{2H-1}}{H\Gamma(2H)} R_1$, where $R_1 = I_2(\delta_{0,1})$ is the Rosenblatt random variable and $\delta_{0,1}$ is the Dirac-type distribution defined in (2.6).*

Remark 6 It is interesting to note that when $H \in (0, \frac{1}{2})$, by the fact $\lim_{z\to 0} z\Gamma(z) = 1$, we have

$$\lim_{H\to \frac{1}{2}^-} \sigma_H^2 = 2$$

which is consistent with $\sigma_H^2 = 2$ if $H = \frac{1}{2}$. Moreover, we also see that $\lim_{H\to 0} \sigma_H^2 = 0$.

Proof The case $H \in [\frac{1}{2}, \frac{3}{4})$ was proved in Hu and Nualart (2010). We shall use Malliavin calculus to prove the theorem for $H \in (0, \frac{1}{2}) \cup [\frac{3}{4}, 1)$.

Step 1 We use Theorem 1 to prove the central limit theorem when $H \in (0, \frac{1}{2})$. By (3.1) and (3.2), we can write our target quantity as

$$\sqrt{T}(\hat{\theta}_T - \theta) = -\frac{\frac{\sigma^2}{\sqrt{T}} \int_0^T (\int_0^t e^{-\theta(t-s)} dB_s^H) dB_t^H}{\int_0^T X_t^2 dt/T} = \frac{-\frac{\sigma^2}{2\sqrt{T}} F_T}{\int_0^T X_t^2 dt/T}, \tag{3.10}$$

where

$$F_T = \int_0^T \int_0^T e^{-\theta|t-s|} dB_s^H dB_t^H. \tag{3.11}$$

We introduce the function

$$f(s,t) = \frac{1}{\sqrt{T}} e^{-\theta|s-t|} \mathbf{1}_{[0,T]^2}. \tag{3.12}$$

Then $\frac{1}{\sqrt{T}} F_T = I_2(f)$ is in the second Wiener chaos. Our main objective is to use Theorem 1 to obtain the central limit theorem for the term $\frac{1}{\sqrt{T}} F_T$ and then we apply Lemma 19 and Slutsky's theorem for (3.10) to obtain the central limit theorem of $\hat{\theta}_T$. First of all, let us check the variance assumption in Theorem 1. By the isometry between the Hilbert space $\mathfrak{H}^{\otimes 2}$ and the second chaos $\mathcal{H}_2$, we have

$$\mathbb{E}\left(\frac{1}{T} F_T^2\right) = \frac{2}{T} \langle e^{-\theta|s_1-t_1|}, e^{-\theta|s_2-t_2|} \rangle_{\mathfrak{H}\otimes\mathfrak{H}}.$$

To compute the above norm, we shall use the definition of the tensor product space where the norm in the Hilbert space $\mathfrak{H}$ is defined by (2.3), namely,

$$\mathbb{E}\left(\frac{1}{T} F_T^2\right) = \frac{2}{T} \int_{[0,T]^4} \frac{\partial e^{-\theta|s_1-t_1|}}{\partial t_1} \frac{\partial e^{-\theta|s_2-t_2|}}{\partial s_2} \frac{\partial R_H(s_1,s_2)}{\partial s_1} \frac{\partial R_H(t_1,t_2)}{\partial t_2} ds_1 ds_2 dt_1 dt_2. \tag{3.13}$$

By Eq. (6.26) in Lemma 17, we have

$$\lim_{T\to\infty} \mathbb{E}\left(\frac{1}{T} F_T^2\right) = 4H^2\theta^{1-4H}\Gamma(2H)^2\Big((4H-1) + \frac{2\Gamma(2-4H)\Gamma(4H)}{\Gamma(2H)\Gamma(1-2H)}\Big). \tag{3.14}$$

Next, let us check the second condition in Theorem 1. Without loss of generality we can assume $\theta = 1$. The first contraction of the kernel f is

$$(f \otimes_1 f)(s,t) = \frac{1}{T} \langle e^{-|\cdot-s|} \mathbf{1}_{[0,T]}(\cdot), e^{-|\cdot-t|} \mathbf{1}_{[0,T]}(\cdot) \rangle_{\mathfrak{H}}. \tag{3.15}$$

We want to prove that the norm of the above function in the Hilbert space $\mathfrak{H}^{\otimes 2}$ goes to 0 as $T \to \infty$. Using the identity (2.4), we rewrite

$$\begin{aligned}
(f \otimes_1 f)(s,t) &= \frac{1}{Tc_H^2} \int_{\mathbb{R}} \mathcal{F}(e^{-|\cdot-s|}\mathbf{1}_{[0,T]}(\cdot))(\xi) \overline{\mathcal{F}(e^{-|\cdot-t|}\mathbf{1}_{[0,T]}(\cdot))(\xi)} |\xi|^{1-2H} d\xi \\
&= \frac{4}{Tc_H^2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{e^{-is\eta}}{1+\eta^2} \cdot \frac{1-e^{-iT(\xi-\eta)}}{i(\xi-\eta)} d\eta \right) \\
&\qquad \left(\int_{\mathbb{R}} \frac{e^{it\eta'}}{1+\eta'^2} \cdot \frac{1-e^{iT(\xi-\eta')}}{-i(\xi-\eta')} d\eta' \right) |\xi|^{1-2H} d\xi.
\end{aligned}$$

Observe that the function $f \otimes_1 f$ is the inverse Fourier transformation of the following function

$$h(s,t) = \frac{4}{Tc_H^2} \int_{\mathbb{R}} \left(\frac{1}{1+s^2} \cdot \frac{1-e^{-iT(\xi+s)}}{i(\xi+s)} \right) \left(\frac{1}{1+t^2} \cdot \frac{1-e^{iT(\xi-t)}}{-i(\xi-t)} \right) |\xi|^{1-2H} d\xi.$$

By the Parseval's identity, the norm of the function $f \otimes_1 f$ in the space $\mathfrak{H}^{\otimes 2}$ can be computed as

$$\begin{aligned}\|f \otimes_1 f\|^2_{\mathfrak{H}^{\otimes 2}} &= \frac{1}{c_H^2}\int_{\mathbb{R}^2} |h(\eta,\eta')|^2 |\eta|^{1-2H}|\eta'|^{1-2H} d\eta d\eta' \\ &\le \frac{C}{T^2}\int_{\mathbb{R}^2} \frac{|\eta|^{1-2H}}{1+\eta^4}\frac{|\eta'|^{1-2H}}{1+\eta'^4} \\ &\quad \left(\int_{\mathbb{R}} \frac{|e^{iT(\xi-\eta)}-1|}{|\xi-\eta|}\frac{|e^{iT(\xi-\eta')}-1|}{|\xi-\eta'|}|\xi|^{1-2H} d\xi\right)^2 d\eta d\eta'. \end{aligned} \tag{3.16}$$

Now our task is to show the right-hand side of the above inequality goes to 0 as $T \to \infty$. This can be achieved by studying the asymptotic behavior of the multiple integral in (3.16), which is denoted by I. Making a change of variable $\xi \to x + \eta$ yields

$$\begin{aligned} I &= \int_{\mathbb{R}^2} \frac{|\eta|^{1-2H}}{1+\eta^4}\frac{|\eta'|^{1-2H}}{1+\eta'^4}\left(\int_{\mathbb{R}} \frac{|e^{iTx}-1|}{|x|}\frac{|e^{iT(x+\eta-\eta')}-1|}{|x+\eta-\eta'|}|x+\eta|^{1-2H} dx\right)^2 d\eta d\eta' \\ &\le 2\int_{\mathbb{R}^2} \frac{|\eta|^{1-2H}}{1+\eta^4}\frac{|\eta'|^{1-2H}}{1+\eta'^4}\left(\int_{\mathbb{R}} \frac{|e^{iTx}-1|}{|x|^{2H}}\frac{|e^{iT(x+\eta-\eta')}-1|}{|x+\eta-\eta'|} dx\right)^2 d\eta d\eta' \\ &\quad + 2\int_{\mathbb{R}^2} \frac{|\eta|^{3-6H}}{1+\eta^4}\frac{|\eta'|^{1-2H}}{1+\eta'^4}\left(\int_{\mathbb{R}} \frac{|e^{iTx}-1|}{|x|}\frac{|e^{iT(x+\eta-\eta')}-1|}{|x+\eta-\eta'|} dx\right)^2 d\eta d\eta'. \end{aligned}$$

Making another change of variable $\eta' \to \eta - y$, we can write

$$\begin{aligned} I &\le 2\int_{\mathbb{R}^2} \frac{|\eta|^{1-2H}}{1+\eta^4}\frac{|\eta-y|^{1-2H}}{1+(\eta-y)^4}\left(\int_{\mathbb{R}} \frac{|e^{iTx}-1|}{|x|^{2H}}\frac{|e^{iT(x+y)}-1|}{|x+y|} dx\right)^2 d\eta dy \\ &\quad + 2\int_{\mathbb{R}^2} \frac{|\eta|^{3-6H}}{1+\eta^4}\frac{|\eta-y|^{1-2H}}{1+(\eta-y)^4}\left(\int_{\mathbb{R}} \frac{|e^{iTx}-1|}{|x|}\frac{|e^{iT(x+y)}-1|}{|x+y|} dx\right)^2 d\eta dy \\ &=: 2(I_1 + I_2). \end{aligned}$$

For the term I_2, taking into account that

$$M := \sup_{y\in\mathbb{R}} \int_{\mathbb{R}} \frac{|\eta|^{3-6H}}{1+\eta^4}\frac{|\eta-y|^{1-2H}}{1+(\eta-y)^4} d\eta < \infty,$$

we see

$$I_2 \le M \int_{\mathbb{R}}\left(\int_{\mathbb{R}} \frac{|e^{iTx}-1|}{|x|}\frac{|e^{iT(x+y)}-1|}{|x+y|} dx\right)^2 dy = M\|f * f\|^2_{L^2(\mathbb{R})},$$

where $f(x) = \frac{|e^{iTx}-1|}{|x|}$. By Young's inequality

$$I_2 \le M\|f\|^4_{L^{4/3}(\mathbb{R})} = M\left(\int_{\mathbb{R}} \frac{|e^{iTx}-1|^{4/3}}{|x|^{4/3}} dx\right)^3 = MT\left(\int_{\mathbb{R}} \frac{|e^{ix}-1|^{4/3}}{|x|^{4/3}} dx\right)^3 = CT.$$

Now we consider the term I_1. The measure $\mu(dy) = \int_{\mathbb{R}} \frac{|\eta|^{1-2H}}{1+\eta^4}\frac{|\eta-y|^{1-2H}}{1+(\eta-y)^4} d\eta$ is finite and has a bounded density with respect to the Lebesgue measure. Consider the function

$g(x)=\frac{|e^{iTx}-1|}{|x|^{2H}}$. For any $p\geq 2$,

$$I_1=\|f*g\|^2_{L^2(\mathbb{R},\mu)}\leq C_1\|f*g\|^2_{L^p(\mathbb{R},\mu)}\leq C_2\|f*g\|^2_{L^p(\mathbb{R})}.$$

Let p also satisfy $p>\frac{1}{2H}$ and for such p we can choose α and β such that $\alpha>1, 2H\beta>1$ and $\frac{1}{\alpha}+\frac{1}{\beta}=1+\frac{1}{p}$. Then by Young's inequality

$$\begin{aligned}I_1&\leq C_2\|f\|^2_{L^\alpha(\mathbb{R})}\|g\|^2_{L^\beta(\mathbb{R})}\\&=C_2\left(\int_{\mathbb{R}}\frac{|e^{iTx}-1|^\alpha}{|x|^\alpha}dx\right)^{\frac{2}{\alpha}}\left(\int_{\mathbb{R}}\frac{|e^{iTx}-1|^\beta}{|x|^{2H\beta}}dx\right)^{\frac{2}{\beta}}.\end{aligned}$$

A change of variable $x\to y/T$ tells us that $I_1\leq CT^{4H-\frac{2}{p}}$. From (3.16), we obtain

$$\|f\otimes_1 f\|^2_{\mathfrak{H}^{\otimes 2}}\leq CT^{(-1)\vee(4H-2-\frac{2}{p})},\tag{3.17}$$

and this goes to 0 as T tends to infinity. By Theorem 1, as T goes to infinity, the term $\frac{1}{\sqrt{T}}F_T$ converges in distribution to a centered Gaussian random variable with variance given by (3.14). Applying Slutsky's theorem and Lemma 19 from "Appendix" to the Eq. (3.10), we finish the proof of the theorem for $H\in(0,\frac{1}{2})$.

Step 2 Case $H=\frac{3}{4}$. First note that Lemma 17 in the "Appendix" gives the limiting variance of $\frac{F_T}{\sqrt{T\log T}}$. To obtain the central limit theorem, we need to check one of the equivalent conditions in Theorem 1. This can be dealt with in a similar way as in the proof of Theorem 3.4 in Hu and Nualart (2010) by verifying condition 5 of Theorem 1. However, it is worth noting that it also suffices to verify the equivalent condition $\|f\otimes_1 f\|^2_{\mathfrak{H}^{\otimes 2}}\to 0$, and the arguments used in the case of $H\in(0,\frac{1}{2})$ can be extended to the case $H\in(0,\frac{3}{4}]$.

Step 3 In this step we will prove the theorem when $H\in(\frac{3}{4},1)$. Recall that the term F_T is given by (3.11). By (3.1) and (3.2), we write

$$T^{2-2H}(\hat{\theta}_T-\theta)=\frac{-\frac{\sigma^2}{2}T^{1-2H}F_T}{\int_0^T X_t^2dt/T}.$$

Denote

$$\widetilde{F}_T=T^{2H}\int_{[0,1]^2}e^{-\theta T|t-s|}dB_s^H dB_t^H.\tag{3.18}$$

By the self-similarity property of the fBm, the process $\{F_T, T>0\}$ has the same law as $\{\widetilde{F}_T, T>0\}$. To prove part (iii) of the theorem, we need to show $T^{1-2H}F_T\xrightarrow{\mathcal{L}}2\theta^{-1}R_1$. It suffices to prove

$$\lim_{T\to\infty}\mathbb{E}(T^{1-2H}\widetilde{F}_T-2\theta^{-1}R_1)^2=0.\tag{3.19}$$

By Eqs. (6.28) and (6.29), we see immediately that

$$\begin{aligned}\lim_{T\to\infty}\mathbb{E}\left(T^{2-4H}\widetilde{F}_T^2\right)=\lim_{T\to\infty}\mathbb{E}\left(T^{2-4H}F_T^2\right)&=\frac{16\alpha_H^2\theta^{-2}}{(4H-2)(4H-3)},\\\lim_{T\to\infty}\mathbb{E}\left[2\theta^{-1}R_1(T^{1-2H}\widetilde{F}_T)\right]&=\frac{16\alpha_H^2\theta^{-2}}{(4H-2)(4H-3)},\end{aligned}$$

where $\alpha_H = H(2H-1)$. On the other hand, we have

$$\mathbb{E}(2\theta^{-1}R_1)^2 = 8\theta^{-2}\alpha_H^2 \int_{[0,1]^4} \delta_{0,1}(s-t)\delta_{0,1}(s'-t')|s-s'|^{2H-2}|t-t'|^{2H-2}dsdtds'dt'$$
$$= 8\theta^{-2}\alpha_H^2 \int_{[0,1]^2} |t-s|^{4H-4}dsdt = \frac{16\theta^{-2}\alpha_H^2}{(4H-3)(4H-2)}.$$

This shows (3.19) and hence completes the proof of the theorem. □

As an immediate consequence of the proof of the central limit theorem for $\frac{1}{\sqrt{T}}F_T$ when $H \in (0, \frac{3}{4})$, we can derive the total variation distance between $\frac{1}{\sqrt{T}}F_T$ and its limiting distribution. The case $H = \frac{3}{4}$ is similar. This is summarized in the following proposition.

Proposition 7 *Let F_T be given by (3.11) and let $\sigma_T^2 = \mathbb{E}((f_T F_T)^2)$ be its variance, with the normalizing factor $f_T = \frac{1}{\sqrt{T}}\mathbf{1}_{\{H\in(0,\frac{3}{4})\}} + \frac{1}{\sqrt{T\log(T)}}\mathbf{1}_{\{H=\frac{3}{4}\}}$. Let N denote a random variable with the standard normal distribution. Then*

$$d_{\mathrm{TV}}\Big(\frac{f_T F_T}{\sigma_T}, N\Big) \le \begin{cases} \frac{C}{\sqrt{T}} & \text{when } H \in (0, \frac{1}{2}) \\ \frac{C}{\sqrt{T^{3-4H}}} & \text{when } H \in [\frac{1}{2}, \frac{3}{4}) \\ \frac{C}{\sqrt{\log(T)}} & \text{when } H = \frac{3}{4}. \end{cases}$$

Proof It suffices to consider the case $H \in (0, \frac{3}{4})$. The case $H = \frac{3}{4}$ can be treated in a similar way. Recall that $\frac{1}{\sqrt{T}}F_T = I_2(f)$, where the kernel f is given by (3.12). Applying Proposition 3 yields

$$d_{\mathrm{TV}}\Big(\frac{F_T}{\sqrt{T}\sigma_T}, N\Big) \le 2\sqrt{\mathrm{Var}\left(\frac{1}{2}\|\frac{1}{\sigma_T\sqrt{T}}DF_T\|_{\mathfrak{H}}^2\right)}$$
$$= \frac{1}{\sigma_T^2}\|f\otimes_1 f\|_{\mathfrak{H}^{\otimes 2}} \le C\sqrt{T^{(-1)\vee(4H-2-\frac{2}{p})}},$$

for any $p \ge 2$, where for the above identity we used Lemma 5.2.4 from Nourdin and Peccati (2012), and for the last inequality we used the inequality (3.17). Clearly, when $H \in (0, \frac{1}{2})$, the bound is $C/\sqrt{T}$. When $H \in [\frac{1}{2}, \frac{3}{4})$, $p = 2$ is chosen to derive the bound. □

Remark 8 We make some comments on the distance between the normalized F_T, $\hat{\theta}_T$, and their limiting distributions.

1. Recall that $\sqrt{T}(\hat{\theta}_T - \theta) = \frac{-\frac{\sigma^2}{2}F_T/\sqrt{T}}{\frac{1}{T}\int_0^T X_t^2 dt}$. We have obtained the asymptotic behavior for the numerator in the preceding Proposition 7. By Lemma 19, The denominator converges to a constant almost surely, and the convergence rate is of $\sqrt{T}$ (see Sottinen and Viitasaari 2017). It is challenging to study the total variation distance between $\sqrt{T}(\hat{\theta}_T - \theta)$ with its limiting normal distribution, since it involves the quotient of two dependent random variables. This is left as an open problem.
2. For $H \in (\frac{3}{4}, 1)$, we can get a convergence rate for (3.19) by examining the proof of (6.28) and (6.29) in Lemma 17. In this way we find that $\mathbb{E}\left(T^{1-2H}\tilde{F}_T - 2\theta^{-1}R_1\right)^2 = O(T^{3-4H})$. This implies that $T^{1-2H}F_T$ (which has the same law as $T^{1-2H}\tilde{F}_T$ defined by (3.18)) converges to the Rosenblatt random variable in law at the rate of $\sqrt{T^{4H-3}}$.

4 Ergodic type estimator

In this section, we shall use the results of the last section to consider a simulation friendly estimator: ergodic type estimator.

Theorem 9 *Define an ergodic-type estimator for the drift parameter θ by*

$$\widetilde{\theta}_T = \left(\frac{1}{\sigma^2 H\Gamma(2H)T} \int_0^T X_t^2 dt \right)^{-\frac{1}{2H}}. \tag{4.1}$$

Then $\widetilde{\theta}_T \to \theta$ almost surely as $T \to \infty$. Furthermore, we have the following central limit theorem ($H \leq 3/4$) and noncentral limit theorem ($H > 3/4$).

(1) *When $H \in (0, \frac{3}{4})$, we have $\sqrt{T}(\widetilde{\theta}_T - \theta) \xrightarrow{\mathcal{L}} N(0, \frac{\theta}{(2H)^2}\sigma_H^2)$ as $T \to \infty$, where σ_H^2 is defined in Theorem 5.*

(2) *When $H = \frac{3}{4}$, we have $\frac{\sqrt{T}}{\sqrt{\log(T)}}(\widetilde{\theta}_T - \theta) \xrightarrow{\mathcal{L}} N(0, \frac{16\theta}{9\pi})$ as $T \to \infty$.*

(3) *When $H \in (\frac{3}{4}, 1)$, we have $T^{2-2H}(\widetilde{\theta}_T - \theta) \xrightarrow{\mathcal{L}} \frac{-\theta^{2H-1}}{H\Gamma(2H+1)} R_1$, where $R_1 = I_2(\delta_{0,1})$ is the Rosenblatt random variable, and $\delta_{0,1}$ is the Dirac-type function defined in (2.6).*

Proof The paper Hu and Nualart (2010) provides a proof of the theorem when $H \in (\frac{1}{2}, \frac{3}{4})$. Here we present a proof valid for all $H \in (0, 1)$. By Lemma 19, it is easy to see $\widetilde{\theta}_T \to \theta$ almost surely as $T \to \infty$.

We prove the central limit theorem when $H \in (0, \frac{3}{4})$. For $H \in [\frac{3}{4}, 1)$, the proof is similar. By (3.1) and (3.9), we can derive an expression for $\int_0^T X_t^2 dt$, and then express $\widetilde{\theta}_T$ as a function of $\hat{\theta}_T$. In this way, we obtain

$$\sqrt{T}(\widetilde{\theta}_T - \theta) = \sqrt{T} \left[\left(\frac{\sigma^2 H\Gamma(2H)\hat{\theta}_T}{-\frac{X_T^2}{2T} + \sigma T^{-1}\ell(T)} \right)^{\frac{1}{2H}} - \theta \right].$$

By Lemmas 18 and (3.7) we have

$$\begin{aligned}\sqrt{T}(\widetilde{\theta}_T - \theta) &= \sqrt{T} \left[\left(\frac{1}{\theta^{1-2H} + o(T^{-1/2})} \right)^{\frac{1}{2H}} \hat{\theta}_T^{\frac{1}{2H}} - \theta \right] \\ &= \sqrt{T}\theta^{1-\frac{1}{2H}}(\hat{\theta}_T^{\frac{1}{2H}} - \theta^{\frac{1}{2H}}) + \sqrt{T}\, o(T^{-1/2})\hat{\theta}_T^{\frac{1}{2H}}.\end{aligned}$$

Meanwhile, we can write

$$\sqrt{T}\left[\hat{\theta}_T^{\frac{1}{2H}} - \theta^{\frac{1}{2H}}\right] = \sqrt{T}\left[\frac{1}{2H}\theta^{\frac{1}{2H}-1}(\hat{\theta}_T - \theta) + \frac{1-2H}{8H^2}(\hat{\theta}_T - \theta)^2(\theta_T^*)^{\frac{1}{2H}-2}\right]$$

for some θ_T^* between θ and $\hat{\theta}_T$. Now the theorem follows from Theorem 5. □

Remark 10 By the property for gamma function: $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$ for $z \notin \mathbb{Z}$, we see $\lim_{H\to 0} \frac{\theta}{(2H)^2}\sigma_H^2 = \frac{\pi^2}{2}\theta$.

Now we have obtained the asymptotic law of the least squares estimator (LSE) $\hat{\theta}_T$ and the ergodic type estimator (ETE) $\widetilde{\theta}_T$. Next, we compare these two estimators with the maximum likelihood estimator by computing their asymptotic variance. For convenience, we assume

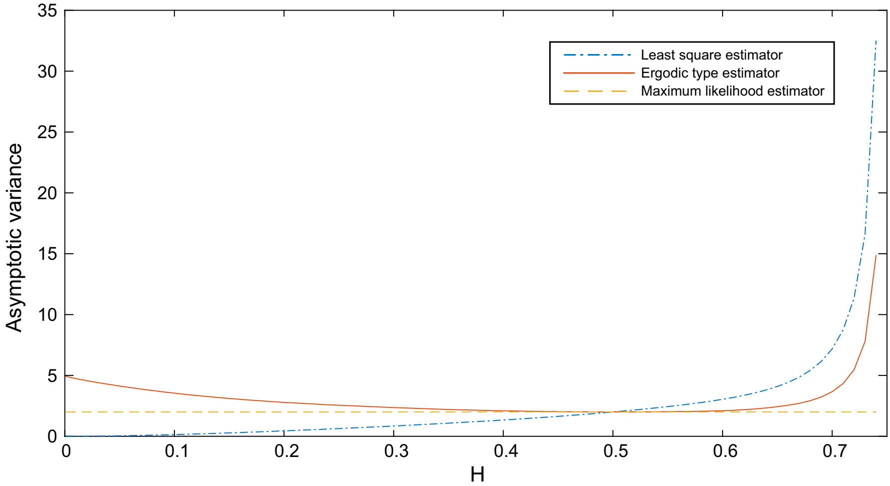

Fig. 1 Asymptotic variance of the three estimators

$\theta = 1$. As it can be seen from Fig. 1, the asymptotic variance of LSE increases as H increases. When $H \in (0, \frac{1}{2})$, the asymptotic variance of LSE is less than that of MLE, where the converse is true for $H \in (\frac{1}{2}, \frac{3}{4})$. The asymptotic variance of ETE decreases on $H \in (0, \frac{1}{2})$ and then increases on $H \in (\frac{1}{2}, \frac{3}{4})$; however, it does not blow up as fast as LSE does when H is close to $\frac{3}{4}$. If we justify these three estimators only based on asymptotic variance, LSE performs best when $H \in (0, \frac{1}{2})$ and MLE performs best when $H \in (\frac{1}{2}, \frac{3}{4})$. At $H = \frac{1}{2}$, these three estimators have the same asymptotic variance.

5 Discrete case and simulation

The estimators $\hat{\theta}_T$ and $\widetilde{\theta}_T$ are based on continuous time data. In practice the process can only be observed at discrete time instants. This motivates us to construct an estimator based on discrete observations. We assume that the fractional Ornstein–Uhlenbeck process X given by (3.2) can be observed at discrete time points $\{t_k = kh, k = 0, 1, \dots, n\}$. We shall use nh instead of T for the time period of the observation. Here h represents the observation frequency and it depends on n. We will only consider the high frequency observation case, namely, we shall assume that $h \to 0$ as $n \to \infty$. We shall use ergodic type estimator since it can be expressed as a pathwise Riemann integral with respect to time. The following theorem shows its asymptotic consistency and some results on its asymptotic law.

Theorem 11 *Assume the fractional Ornstein–Uhlenbeck process X given by (3.2) is observed at discrete time points $\{t_k = kh, k = 0, 1, \dots, n\}$. Suppose that h depends on n and as $n \to \infty$, h goes to 0 and nh converges to ∞. In addition, we make the following assumptions on h and n:*

(1) *When $H \in (0, \frac{3}{4})$, $nh^p \to 0$ for some $p \in (1, \frac{3+2H}{1+2H} \wedge (1+2H))$ as $n \to \infty$.*
(2) *When $H = \frac{3}{4}$, $\frac{nh^p}{\sqrt{\log(nh)}} \to 0$ for some $p \in (1, \frac{9}{5})$ as $n \to \infty$.*
(3) *When $H \in (\frac{3}{4}, 1)$, $nh^p \to 0$ for some $p \in (1, \frac{3-H}{2-H})$ as $n \to \infty$.*

Set

$$\bar{\theta}_n = \left(\frac{1}{n\sigma^2 H\Gamma(2H)} \sum_{k=1}^{n} X_{kh}^2 \right)^{-\frac{1}{2H}}. \tag{5.1}$$

Then $\bar{\theta}_n$ converges to θ almost surely as $n \to \infty$. Moreover, as n tends to infinity, we have the following central and noncentral limit theorems.

(1) *When $H \in (0, \frac{3}{4})$, $\sqrt{nh}(\bar{\theta}_n - \theta) \xrightarrow{\mathcal{L}} N(0, \frac{\theta}{(2H)^2}\sigma_H^2)$, where σ_H^2 is given in Theorem 5.*

(2) *When $H = \frac{3}{4}$, $\frac{\sqrt{nh}}{\sqrt{\log(nh)}}(\bar{\theta}_n - \theta) \xrightarrow{\mathcal{L}} N(0, \frac{16\theta}{9\pi})$.*

(3) *When $H \in (\frac{3}{4}, 1)$, $(nh)^{2-2H}(\bar{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \frac{-\theta^{2H-1}}{H\Gamma(2H+1)} R_1$, where $R_1 = I_2(\delta_{0,1})$ is the Rosenblatt random variable and $\delta_{0,1}$ is the Dirac-type function defined in (2.6).*

Before we prove Theorem 11, we state and prove an auxillary result in the following lemma about the regularity of sample paths of the fractional Ornstein–Uhlenbeck process X.

Lemma 12 *Let X_t be given by (3.2). Then for every interval $[0, T]$ and any $0 < \epsilon < H$,*

$$|X_t - X_s| \le V_1|t-s|^{H-\epsilon} + V_2|t-s| \quad \text{a.s.}, \tag{5.2}$$

where the random variables V_i are defined as follows: $V_1 = \sigma\eta_T$ where η_T is given by (2.2) with $\alpha = H - \epsilon$, $V_2 = 2\sigma\theta \sup_{u\in[0,T]} |B_u^H|$.

Proof Consider the process $Q_t = \sigma\theta \int_0^t B_v^H e^{-\theta(t-v)} dv$. Using (3.3), for any $s, t \in [0, T]$ and $s < t$, we have

$$|X_t - X_s| = \left| \sigma(B_t^H - B_s^H) - (Q_t - Q_s) \right| \le \sigma \left| B_t^H - B_s^H \right| + |Q_t - Q_s|.$$

Note that

$$\begin{aligned}
|Q_t - Q_s| &\le \sigma\theta \left| \int_s^t B_v^H e^{-\theta(t-v)} dv \right| + \sigma\theta \left| \int_0^s B_v^H (e^{-\theta(t-v)} - e^{-\theta(s-v)}) dv \right| \\
&\le \sigma\theta \sup_{v\in[s,t]} |B_v^H| \int_s^t e^{-\theta(t-v)} dv + \sigma\theta \sup_{v\in[0,s]} |B_v^H| \left(1 - e^{-\theta(t-s)}\right) \int_0^s e^{-\theta(s-v)} dv \\
&\le 2\sigma\theta \sup_{v\in[0,t]} |B_v^H||t-s|.
\end{aligned}$$

Using the above inequality for $|Q_t - Q_s|$ and applying (2.2), with $\alpha = H - \epsilon$, for $B_t^H - B_s^H$ yield

$$|X_t - X_s| \le \sigma\eta_T|t-s|^{H-\epsilon} + 2\sigma\theta \sup_{u\in[0,t]} |B_u^H||t-s|.$$

□

Proof of Theorem 11 Let $T = nh$, $Z_n = \frac{1}{nh} \int_0^{nh} X_t^2 dt$, and $\psi_n = \frac{1}{n} \sum_{k=1}^n X_{kh}^2$. Consider the function

$$f(x) = \sqrt{x}\mathbf{1}_{\{0<H<3/4\}} + \sqrt{x}/\sqrt{\log(x)}\mathbf{1}_{\{H=3/4\}} + x^{2-2H}\mathbf{1}_{\{3/4<H<1\}}.$$

Step 1 We claim that $f(nh)\,|Z_n - \psi_n| \to 0$ almost surely as $n \to \infty$. Applying Markov's inequality for $\delta > 0, q > 1$ yields

$$P(f(nh)\,|Z_n - \psi_n| > \delta) \le \delta^{-q} f(nh)^q \mathbb{E}\,|Z_n - \psi_n|^q. \tag{5.3}$$

We apply Minkowski's inequality to obtain

$$\begin{aligned}\mathbb{E}\,|Z_n-\psi_n|^q &= (nh)^{-q}\mathbb{E}\left|\sum_{j=1}^{n}\int_{(j-1)h}^{jh}(X_t+X_{jh})(X_t-X_{jh})dt\right|^q\\ &\le (nh)^{-q}\left(\sum_{j=1}^{n}\int_{(j-1)h}^{jh}\left(\mathbb{E}(|X_t+X_{jh}||X_t-X_{jh}|)^q\right)^{1/q}dt\right)^q.\end{aligned}$$

Taking into account of Lemma 12, we have

$$\begin{aligned}&\mathbb{E}\,|Z_n-\psi_n|^q\\ &\le (nh)^{-q}\left(\sum_{j=1}^{n}\int_{(j-1)h}^{jh}\|V_1(X_t+X_{jh})\|_{L^q}|t-jh|^{H-\epsilon}+\|V_2(X_t+X_{jh})\|_{L^q}|t-jh|dt\right)^q,\end{aligned}$$

where the V_i's are defined in Lemma 12. By Hölder's inequality and the fact $\|X_t\|_{L^q}=(\mathbb{E}|X_t|^q)^{1/q}\le M_q$ for all $t>0$, $q>1$, we can write

$$\|V_i(X_t+X_{jh})\|_{L^q}\le 2M_{qr_i}\|V_i\|_{qs_i},$$

where $1/r_i+1/s_i=1$. Therefore,

$$\mathbb{E}\,|Z_n-\psi_n|^q\le C\left(M_{qr_1}^q\|V_1\|_{qs_1}^q h^{q(H-\epsilon)}+M_{qr_2}^q\|V_2\|_{qs_2}^q h^q\right),$$

where C denotes a generic constant.

By (2.2), $\|V_1\|_{qs_1}^q=CT^{q\epsilon}$ for $\epsilon\in(0,H)$. By the self-similarity property of fBm, $\|V_2\|_{qs_2}^q=CT^{qH}$. Using these observations, we obtain

$$\mathbb{E}\,|Z_n-\psi_n|^q\le C\left((nh)^{q\epsilon}h^{q(H-\epsilon)}+(nh)^{qH}h^q\right),$$

and plugging this inequality to (5.3), we get

$$P(f(nh)\,|Z_n-\psi_n|>\delta)\le C\delta^{-q}f(nh)^q\left((nh)^{q\epsilon}h^{q(H-\epsilon)}+(nh)^{qH}h^q\right). \tag{5.4}$$

If the right-hand side of the above inequality is summable with respect to n, then $f(nh)\,|Z_n-\psi_n|\to 0$ almost surely by the Borel–Cantelli Lemma. We show this summability when $H\in(0,1/2)$ and the other cases are similar. The right-hand side of (5.4) can be written as

$$Cn^{-1-\lambda}\left((nh^{\beta_1})^{\gamma_1}+(nh^{\beta_2})^{\gamma_2}\right),$$

where

$$\beta_1=\frac{q/2+q\epsilon+q(H-\epsilon)}{1+\lambda+q\epsilon+q/2},\quad \beta_2=\frac{3/2q+qH}{1+\lambda+q/2+qH},$$

and γ_i's are the denominator of β_i's. Note that the positive variables ϵ and λ can be arbitrarily small and q can be arbitrarily large. In this way, we have $\beta_1\in(1,1+2H)$ and $\beta_2\in(1,\frac{3+2H}{1+2H})$. If $nh^p\to 0$ for some $p\in(1,\min(\frac{3+2H}{1+2H},1+2H))$, then $nh^{\beta_i}\to 0$ by carefully choosing these free variables.

Step 2 We prove the almost sure convergence of $\bar{\theta}_n$. Denote $\rho=\sigma^2H\Gamma(2H)$. Recall that $\widetilde{\theta}_T$ is given in Theorem 9. By the mean value theorem, we can write

$$\bar{\theta}_n - \theta = \left(\frac{\psi_n - Z_n}{\rho} + \widetilde{\theta}_T^{-2H} \right)^{-\frac{1}{2H}} - \theta = \widetilde{\theta}_T - \theta + \int_0^1 g_n(\lambda) d\lambda, \tag{5.5}$$

where $g_n(\lambda) = -\frac{1}{2H} \frac{\psi_n - Z_n}{\rho} \left(\lambda \frac{\psi_n - Z_n}{\rho} + \widetilde{\theta}_T^{-2H} \right)^{-\frac{1}{2H}-1}$.

The result in Step 1 also implies $Z_n - \psi_n \to 0$ almost surely as $n \to \infty$, so $\lim_{n\to\infty} g_n(\lambda) = 0$ a.s. for all $\lambda \in [0, 1]$. Meanwhile, for almost all ω, there exists $N := N(\omega) \in \mathbb{N}$ such that for $n > N$,

$$\left| \frac{\psi_n - Z_n}{\rho} \right| < \frac{1}{3} \theta^{-2H}, \quad \left| \widetilde{\theta}_T^{-2H} - \theta^{-2H} \right| < \frac{1}{3} \theta^{-2H}.$$

Then for $n > N$, $|g_n(\lambda)| \leq C\theta$. By the dominated convergence theorem,

$$\lim_{n\to\infty} \int_0^1 g_n(\lambda) d\lambda = 0 \quad \text{a.s.}.$$

Then it is clear that $\bar{\theta}_n$ converges to θ almost surely.

Step 3 We prove the asymptotic laws of $\bar{\theta}_n$. Equation (5.5) yields

$$f(nh)(\bar{\theta}_n - \theta) = f(T)(\widetilde{\theta}_T - \theta) + f(nh) \int_0^1 g_n(\lambda) d\lambda.$$

Using the result of Step 1 and the similar arguments in step 2, we obtain

$$\lim_{n\to\infty} \int_0^1 f(nh) g_n(\lambda) d\lambda = 0 \quad \text{a.s.}.$$

Then it is clear that $f(nh)(\bar{\theta}_n - \theta)$ converges in law to the same random variable as $f(T)(\widetilde{\theta}_T - \theta)$ when T tends to infinity. By Theorem 9, we finish the proof. □

Next, we use the R package Yuima to do some Monte Carlo simulations. The Wood-Chan simulation method is used to generate fractional Gaussian noise, and the Euler–Maruyama scheme is used to produce sample observations of the stochastic differential equation (1.1) (we take $\sigma = 1$).

First we choose $\theta = 1$. For each H value, only one trajectory is generated and $\bar{\theta}_n$ is calculated along this trajectory. The values of $\bar{\theta}_n$ are plotted in Fig. 2 as T increases. As it can be seen, $\bar{\theta}_n$ converges to the true value $\theta = 1$ as sufficient number of observations are obtained.

Next we choose $\theta = 0.5$. For each H value, we perform 5000 Monte Carlo simulations to generate 5000 trajectories. For each trajectory, the quantity $\sqrt{nh}(\bar{\theta}_n - \theta)$ is calculated, and the density plot of these 5000 estimators is obtained, which is displayed in Fig. 3. The graphs show that the density plot of the simulation results is close to the kernel of the limiting distribution of $\sqrt{nh}(\bar{\theta}_n - \theta)$ when $H = 0.25, 0.5, 0.6$. For $H > \frac{3}{4}$, the limiting distribution, known as Rosenblatt distribution, is not known to have a closed form. Readers who are interested in the density plot of Rosenblatt random variable are referred to the paper (Veillette and Taqqu 2013) and the references therein.

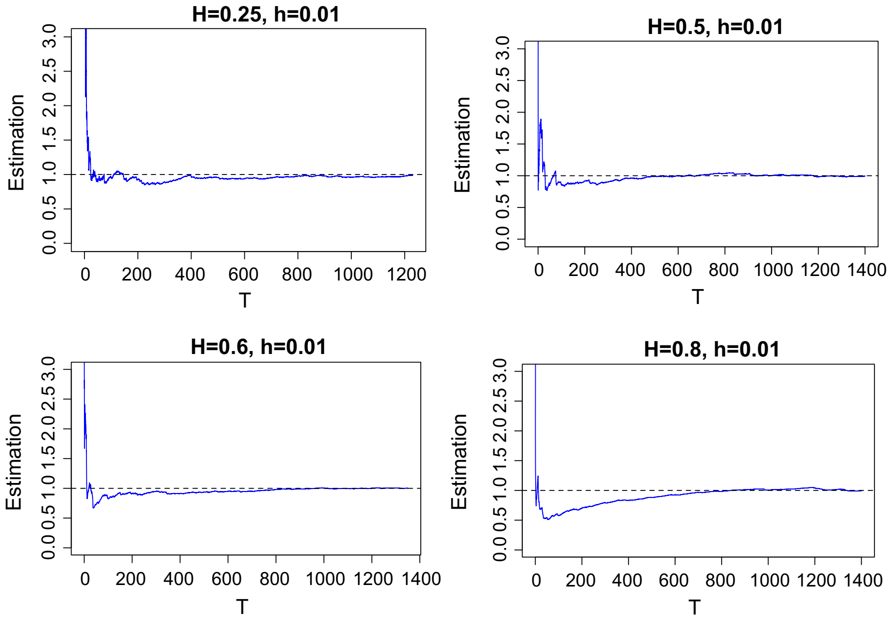

Fig. 2 The one-trajectory simulation results of $\bar{\theta}_n$ for different H values, with $\theta = 1$ and $h = 0.01$

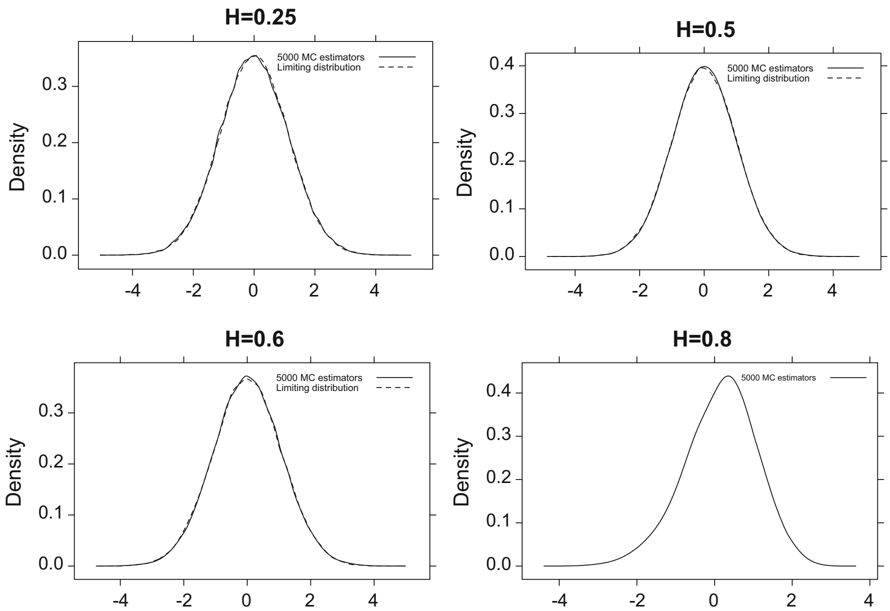

Fig. 3 Density plots for 5000 simulation results of $\sqrt{nh}(\bar{\theta}_n - \theta)$ and its limiting distribution, with $\theta = 0.5$, $h = 0.01$, $n = 100{,}000$

6 Appendix

This section contains some technical results needed in the proofs of the main theorems of the paper. First we need to identify the limits of some multiple integrals. Denote

$$\psi(x,u) = \psi_T(x,u) = T^{4H+1}e^{-\theta T(u+x)}, \tag{6.1}$$

$$\varphi_1(x) := \int_x^1 [(t-x)^{2H-1}-1][(1-t)^{2H-1}-(1-t+x)^{2H-1}]dt, \tag{6.2}$$

$$\varphi_2(x) := \int_x^1 [(t-x)^{2H-1}-t^{2H-1}](1-t+x)^{2H-1}dt, \tag{6.3}$$

$$\varphi_3(x,u) := \int_x^1 \operatorname{sgn}(u-t)|u-t|^{2H-1} - \operatorname{sgn}(x+u-t)|x+u-t|^{2H-1}dt, \tag{6.4}$$

$$\begin{aligned}\varphi_4(x,u) := \int_x^1 & t^{2H-1}\operatorname{sgn}(x+u-t)|x+u-t|^{2H-1}\\ &-(t-x)^{2H-1}\operatorname{sgn}(u-t)|u-t|^{2H-1}dt,\end{aligned} \tag{6.5}$$

$$\begin{aligned}\varphi_5(x,u) := \int_x^1 & \operatorname{sgn}(x+u-t)|x+u-t|^{2H-1}(1-t)^{2H-1}\\ &-\operatorname{sgn}(u-t)|u-t|^{2H-1}(1-t+x)^{2H-1}dt.\end{aligned} \tag{6.6}$$

Fix an $\epsilon \in (0, \frac{1}{4})$. Denote $[0,1]^2 = \mathcal{I}_1 \cup \mathcal{I}_2$ where $\mathcal{I}_1 = [0,\epsilon]^2$ and $\mathcal{I}_2 = [0,1]^2 \backslash [0,\epsilon]^2$.

Lemma 13 *Let $H \in (0, \frac{1}{2})$. When $(x,u) \in \mathcal{I}_1$, we have the following estimates.*

(i)
$$|\varphi_1(x)| \le x^{2H}, \tag{6.7}$$

(ii)
$$|\varphi_3(x,u)| \le C(x^{2H}+u^{2H}+|u-x|^{2H}), \tag{6.8}$$

(iii)
$$|\varphi_5(x,u)| \le C(x^{2H}+u^{2H}+|u-x|^{2H}), \tag{6.9}$$

where C is a constant independent of x, u.

Proof First we prove (6.7). Observe that

$$0 \le \varphi_1(x) \le \int_x^1 f(x,t)dt, \tag{6.10}$$

where

$$f(x,t) = (t-x)^{2H-1}[(1-t)^{2H-1}-(1-t+x)^{2H-1}].$$

It is clear that

$$f(x,t) \le \Big(\frac{1-x}{2}\Big)^{2H-1}[(1-t)^{2H-1}-(1-t+x)^{2H-1}],$$

for $\frac{1+x}{2} \le t \le 1$. For $x \le t \le \frac{1+x}{2}$, applying the mean value theorem for the second factor of $f(x,t)$ yields

$$f(x,t) \le (1-2H)(t-x)^{2H-1}\Big(\frac{1-x}{2}\Big)^{2H-2}x.$$

Integrating the right-hand side of the above two inequalities with respect to t, we obtain

$$\begin{aligned}\int_x^1 f(x,t)dt &\leq \frac{1}{2H}\left(\frac{1-x}{2}\right)^{2H-1}\left[\left(\frac{1-x}{2}\right)^{2H}-\left(\frac{1+x}{2}\right)^{2H}+x^{2H}\right]\\ &\quad+\frac{1-2H}{2H}\left(\frac{1-x}{2}\right)^{4H-2}x\\ &\leq \frac{1}{2H}\left(\frac{1-\epsilon}{2}\right)^{2H-1}x^{2H}+\frac{1-2H}{2H}\left(\frac{1-\epsilon}{2}\right)^{4H-2}x^{2H},\end{aligned}$$

where we have used the inequality $x < x^{2H}$ on $\mathcal{I}_1$ (i.e., $x \in (0,\epsilon)$). Thus, (6.7) follows from the above inequality and (6.10).

Next we prove (6.8). Note that the antiderivative of the function $\operatorname{sgn}(x)|x|^{2H-1}$ is $(2H)^{-1}|x|^{2H}$, so we can compute $\varphi_3(x,u)$ as follows

$$\varphi_3(x,u) = \frac{1}{2H}(|u-x|^{2H}-(1-u)^{2H}+(1-x-u)^{2H}-u^{2H}). \tag{6.11}$$

Applying the inequality

$$\left|(1-x-u)^{2H}-(1-u)^{2H}\right| \leq 2H(1-x-u)^{2H-1}x \leq 2H(1-2\epsilon)^{2H-1}x^{2H},$$

and the triangular inequality to (6.11) yields

$$\begin{aligned}|\varphi_3(x,u)| &\leq (2H)^{-1}(|u-x|^{2H}+u^{2H}+2H(1-2\epsilon)^{2H-1}x^{2H})\\ &\leq C(|u-x|^{2H}+u^{2H}+x^{2H}) \quad \forall\, x,u \in \mathcal{I}_1.\end{aligned}$$

Finally, we prove (6.9). Denote

$$\begin{aligned}\zeta_{x,u}(t) &= \operatorname{sgn}(x+u-t)|x+u-t|^{2H-1}(1-t)^{2H-1}\\ &\quad-\operatorname{sgn}(u-t)|u-t|^{2H-1}(1-t+x)^{2H-1}.\end{aligned}$$

Let $\delta \in (\frac{1}{2},1)$. Since $\epsilon \in (0,\frac{1}{4})$ and $(x,u) \in (0,\epsilon)^2$, the interval $(x,1)$ can be decomposed into the following three intervals, where

$$J_1 = (x,\ u+x), \quad J_2 = (u+x,\ \delta), \quad J_3 = (\delta,\ 1).$$

Then $\varphi_5(x,u) = \sum_{k=1}^3 \int_{J_k} \zeta_{x,u}(t)dt$. We consider the above three integrals separately.

Case 1 When $t \in J_1$, we have

$$(1-t)^{2H-1} \leq (1-u-x)^{2H-1} \leq (1-2\epsilon)^{2H-1}. \tag{6.12}$$

When t falls in different subintervals of J_1, we bound $(1-t+x)^{2H-1}$ in different ways. Namely, if $t \in (x,u)$ and $u \geq x$,

$$(1-t+x)^{2H-1} \leq (1+x-u)^{2H-1} \leq (1-\epsilon)^{2H-1}. \tag{6.13}$$

If $t \in (x \vee u, x+u)$,

$$(1-t+x)^{2H-1} \leq (1-u)^{2H-1} \leq (1-\epsilon)^{2H-1}. \tag{6.14}$$

Applying (6.12) for the first summand in $\zeta_{x,u}(t)$, (6.13) and (6.14) for the second summand, we can bound the integration of $\zeta_{x,u}(t)$ on J_1 as follows

$$\Big| \int_{J_1} \zeta_{x,u}(t)dt \Big| \le (1-2\epsilon)^{2H-1} \int_x^{u+x} (x+u-t)^{2H-1} dt$$
$$+ (1-\epsilon)^{2H-1} \left(\int_x^u (u-t)^{2H-1} 1_{\{u \ge x\}} dt + \int_{x \vee u}^{x+u} (t-u)^{2H-1} dt \right).$$

Integrating with respect to t yields

$$\Big| \int_{J_1} \zeta_{x,u}(t)dt \Big| \le C(u^{2H} + (u-x)^{2H} 1_{\{u \ge x\}} + x^{2H}).$$

Case 2 For $t \in J_2$, we rewrite

$$-\int_{J_2} \zeta_{x,u}(t)dt = \int_{u+x}^{\delta} (1-t)^{2H-1} \big((t-u-x)^{2H-1} - (t-u)^{2H-1} \big)$$
$$+ (t-u)^{2H-1} \big((1-t)^{2H-1} - (1-t+x)^{2H-1} \big) dt,$$

which is nonnegative. In the above integrand, we bound $(1-t)^{2H-1}$ by $(1-\delta)^{2H-1}$ for the first summand. For the second summand, we apply the mean value theorem for the difference part and bound $(t-u)^{2H-1}$ by x^{2H-1}. Then integrating t yields

$$0 \le -\int_{J_2} \zeta_{x,u}(t)dt \le \frac{(1-\delta)^{2H-1}}{2H} \big((\delta-u-x)^{2H} - (\delta-u)^{2H} + x^{2H} \big)$$
$$+ x^{2H} \big((1-\delta)^{2H-1} - (1-u-x)^{2H-1} \big)$$
$$\le \frac{(1-\delta)^{2H-1}}{2H} x^{2H} + (1-\delta)^{2H-1} x^{2H} \le C x^{2H}.$$

Case 3 For $t \in J_3$, we rewrite

$$-\int_{J_3} \zeta_{x,u}(t)dt = \int_{\delta}^{1} (t-u-x)^{2H-1} ((1-t)^{2H-1} - (1-t+x)^{2H-1})$$
$$+ (1-t+x)^{2H-1} ((t-u-x)^{2H-1} - (t-u)^{2H-1}) dt,$$

which is nonnegative. In the above integrand, we bound $(t-u-x)^{2H-1}$ by $(\delta-2\epsilon)^{2H-1}$ for the first summand. For the second summand, apply the mean value theorem for the difference part and bound $(1-t+x)^{2H-1}$ by x^{2H-1}. Then integrating t yields

$$0 \le -\int_{J_3} \zeta_{x,u}(t)dt \le \frac{(\delta-2\epsilon)^{2H-1}}{2H} \big((1-\delta)^{2H} - (1-\delta+x)^{2H} + x^{2H} \big)$$
$$+ x^{2H} \big((\delta-u-x)^{2H-1} - (1-u-x)^{2H-1} \big)$$
$$\le \frac{(\delta-2\epsilon)^{2H-1}}{2H} x^{2H} + x^{2H} (\delta-u-x)^{2H-1} \le C x^{2H}.$$

In the last step we have applied the inequality $\delta - u - x \ge \delta - 2\epsilon$.

Lemma 14 *Suppose* $H \in (0, \frac{1}{2})$. *Let* $\psi(x,u)$ *and* $\varphi_4(x,u)$ *defined by* (6.1) *and* (6.5), *respectively. Fix* $\epsilon \in (0, 1/4)$. *Then*

$$\lim_{T \to \infty} \int_{[0,\epsilon]^2} \psi(x,u)(x^{2H} + u^{2H} + |x-u|^{2H}) dx du = 0, \tag{6.15}$$

and

$$\lim_{T\to\infty}\int_{[0,1]^2}\psi(x,u)\varphi_4(x,u)dxdu$$
$$=\theta^{-1-4H}\Big(\Gamma(2H)^2(2H-2^{-1})+\frac{\Gamma(2-4H)\Gamma(2H)\Gamma(4H)}{\Gamma(1-2H)}\Big). \qquad (6.16)$$

Proof We first prove (6.15). For the first summand, making the change of variables $Tx\to x_1$ and $Tu\to x_2$ yields

$$\int_{[0,\epsilon]^2}T^{4H+1}e^{-\theta T(x+u)}x^{2H}dxdu=T^{2H-1}\int_{[0,T\epsilon]^2}e^{-\theta(x_1+x_2)}x_1^{2H}dx_1dx_2, \qquad (6.17)$$

which goes to 0 as $T\to\infty$. A similar argument could be applied to the second summand. For the third summand, by symmetry it suffices to consider the integral on the region $\{u>x\}$. Making the change of variables $T(u-x)\to x_1$, $Tx\to x_2$ yields

$$\int_{[0,\epsilon]^2}T^{4H+1}e^{-\theta T(x+u)}|u-x|^{2H}dxdu=2T^{2H-1}\int_{[0,T\epsilon]^2,x_1+x_2\le T\epsilon}e^{-\theta(x_1+2x_2)}x_1^{2H}dx_1dx_2, \qquad (6.18)$$

which goes to 0 as $T\to\infty$.

Next we show (6.16). Set

$$\Theta:=\lim_{T\to\infty}\int_{[0,1]^2}\psi(x,u)\varphi_4(x,u)dxdu.$$

Making change of variables, $\theta Tx\to x$, $\theta Tu\to u$, $\theta Tt\to t$, we can write

$$\Theta=\theta^{-1-4H}\int_{[0,\infty)^2}e^{-(u+x)}dxdu$$
$$\times\int_x^\infty[t^{2H-1}\mathrm{sgn}(x+u-t)|x+u-t|^{2H-1}-(t-x)^{2H-1}\mathrm{sgn}(u-t)|u-t|^{2H-1}]dt.$$

The above integral can be decomposed as follows

$$\Theta=\theta^{-1-4H}(L_1-L_2+L_3),$$

where

$$L_1:=\int_{[0,\infty)^2}e^{-(x+u)}dxdu\int_x^{x+u}t^{2H-1}(x+u-t)^{2H-1}dt,$$
$$L_2:=\int_{[0,\infty)^2,u>x}e^{-(x+u)}dxdu\int_x^u(t-x)^{2H-1}(u-t)^{2H-1}dt,$$
$$L_3:=\int_{[0,\infty)^2}e^{-(x+u)}dxdu\left(\int_{u\vee x}^\infty(t-x)^{2H-1}(t-u)^{2H-1}dt-\int_{x+u}^\infty t^{2H-1}(t-x-u)^{2H-1}dt\right).$$

Making the change of variables $t-x\to s$ and integrating u, we obtain

$$L_1=\Gamma(2H)\int_{[0,\infty)^2}e^{-(x+s)}(x+s)^{2H-1}dxds=\Gamma(2H)^2 2H.$$

Denote by $\mathrm{B}(\alpha,\beta)$ the Beta function. Then

$$L_2=\mathrm{B}(2H,2H)\int_{[0,\infty)^2,u>x}e^{-(x+u)}(u-x)^{4H-1}dxdu.$$

By setting $u - x \to v$ and integrating in x first, we deduce $L_2 = \Gamma(2H)^2/2$. To compute L_3, by symmetry it suffices to integrate on the region $\{u < x\}$. For the second integral, we make the change of variables $t - u \to y$. In this way, we obtain

$$L_3 = 2\int_{0<u<x<y<\infty} e^{-(u+x)}((y-u)^{2H-1} - (y+u)^{2H-1})(y-x)^{2H-1}dydxdu.$$

The change of variables $x - u \to a$, $y - x \to b$ yields

$$\begin{aligned} L_3 &= 2\int_{\mathbb{R}_+^3} e^{-(a+2u)}b^{2H-1}\big[(a+b)^{2H-1} - (a+b+2u)^{2H-1}\big]dudadb \\ &= 2\int_{\mathbb{R}_+^3} e^{-(a+2u)}b^{2H-1}dudadb\int_a^{2u+a}(1-2H)(b+z)^{2H-2}dz \\ &= 2(1-2H)\int_{\mathbb{R}_+^2} e^{-(a+2u)}duda\int_a^{2u+a}(\int_{\mathbb{R}_+} b^{2H-1}(b+z)^{2H-2}db)dz. \end{aligned}$$

Setting $z/(b+z) \to v$ and integrating v on $[0, 1]$, we obtain

$$\begin{aligned} L_3 &= 2(1-2H)\mathrm{B}(2-4H, 2H)\int_{\mathbb{R}_+^2} e^{-(a+2u)}duda\int_a^{2u+a} z^{4H-2}dz \\ &= \frac{\Gamma(2-4H)\Gamma(2H)\Gamma(4H)}{\Gamma(1-2H)}. \end{aligned}$$

Then, the lemma follows from the above computations of L_1, L_2 and L_3.

Lemma 15 *Denote* $\mathcal{I}_1 = [0,\epsilon]^2$ *and* $\mathcal{I}_2 = [0,1]^2\backslash[0,\epsilon]^2$. *The functions* ψ *and* φ_i *are given by* (6.1) *to* (6.6). *Suppose* $H \in (0, \frac{1}{2})$. *For* $j = 1, 2$ *and* $i = 1, 2, 3, 5$, *we have the following result.*

$$\lim_{T\to\infty}\int_{\mathcal{I}_j} \psi\varphi_i dxdu = 0. \tag{6.19}$$

Proof The proof of (6.19) is divided into the cases $j = 2$ and $j = 1$.

Case $j = 2$: Clearly, for $(x, u) \in \mathcal{I}_2$,

$$\psi(x,u) \le T^{4H+1}e^{-\theta T\epsilon}, \tag{6.20}$$

which implies

$$\int_{\mathcal{I}_2} \psi\varphi_i dxdu \ \to 0 \quad \text{for } i = 1, 2, 3, 5 \tag{6.21}$$

as $T \to \infty$. Thus, (6.19) holds true for $j = 2$.

Case $j = 1$: For $i = 2$, we evaluate the integral of $\psi\varphi_2$ on $\mathcal{I}_1$ by making change of variables $Tx \to x$, $Tu \to u$ and $Tt \to t$. In this way, we obtain

$$\int_{\mathcal{I}_1} \psi\varphi_2 dxdu = \int_{[0,T\epsilon]^2} e^{-\theta(u+x)}dxdu\int_x^T[(t-x)^{2H-1} - t^{2H-1}](T-t+x)^{2H-1}dt. \tag{6.22}$$

Clearly $(T-t+x)^{2H-1} \le x^{2H-1}$, so the integrand of the above triple integral is bounded by the function $e^{-\theta(u+x)}((t-x)^{2H-1} - t^{2H-1})\mathbf{1}_{\{t\ge x\}}x^{2H-1}$ which is integrable on $[0,\infty)^3$.

As $T \to \infty$, $(T - t + x)^{2H-1} \to 0$. Applying the dominated convergence theorem, we have

$$\lim_{T\to\infty} \int_{\mathcal{I}_1} \psi\varphi_2 dx du = 0. \tag{6.23}$$

The cases $i = 1, 3, 5$ follow from (6.7), (6.8) and (6.9) and Lemma 14.

Lemma 16 *For $n \geq 0$, and $H \in [\frac{3}{4}, 1)$, set*

$$A_{1,H}(T) = T^{3-4H} \int_0^T \int_0^{T-t} s^n e^{-\theta s} t^{2H-2} (s+t)^{2H-2} ds dt,$$

and

$$A_{2,H}(T) = T^{3-4H} \int_0^T \int_0^T s^n e^{-\theta s} t^{2H-2} (s+t)^{2H-2} ds dt.$$

Then

(i) For $H \in (\frac{3}{4}, 1)$, $\lim_{T\to\infty} A_{1,H}(T) = \lim_{T\to\infty} A_{2,H}(T) = \dfrac{\theta^{-(n+1)}\Gamma(n+1)}{4H-3}$;

(ii) For $H = \frac{3}{4}$, $\lim_{T\to\infty} \dfrac{A_{1,H}(T)}{\log T} = \lim_{T\to\infty} \dfrac{A_{2,H}(T)}{\log T} = \Gamma(n+1)\theta^{-(n+1)}$.

Proof (i) For $H \in (\frac{3}{4}, 1)$, we have

$$A_{2,H}(T) \leq T^{3-4H} \int_0^T \int_0^T s^n e^{-\theta s} t^{4H-4} ds dt,$$

and

$$A_{1,H}(T) \geq T^{3-4H} \int_0^T \int_0^{T-t} s^n e^{-\theta s} (s+t)^{4H-4} ds dt.$$

For the right-hand sides of the above two inequalities, we integrate first in t to obtain

$$\frac{1}{4H-3} \left(\int_0^T s^n e^{-\theta s} ds - T^{3-4H} \int_0^T s^{n+4H-3} e^{-\theta s} ds \right)$$
$$\leq A_{1,H}(T) \leq A_{2,H}(T) \leq \frac{1}{4H-3} \int_0^T s^n e^{-\theta s} ds.$$

This yields (i) by letting $T \to \infty$.

(ii) For $H = \frac{3}{4}$, by the L'Hopital rule, we have

$$\lim_{T\to\infty} \frac{A_{2,H}(T)}{\log T} = \lim_{T\to\infty} T \left[\int_0^T s^n e^{-\theta s} T^{-\frac{1}{2}} (s+T)^{-\frac{1}{2}} ds + \int_0^T T^n e^{-\theta T} t^{-\frac{1}{2}} (T+t)^{-\frac{1}{2}} dt \right].$$

The second summand on the right-hand side of the above equation goes to 0 as $T \to \infty$, so

$$\lim_{T\to\infty} \frac{A_{2,H}(T)}{\log T} \leq \int_0^\infty s^n e^{-\theta s} ds. \tag{6.24}$$

On the other hand, by the inequality $t \le s+t$,

$$\frac{A_{1,H}(T)}{\log T} \ge \frac{1}{\log T}\int_0^T\int_0^{T-t} s^n e^{-\theta s}(s+t)^{-1}dsdt$$
$$= \frac{1}{\log T}\left[\log T\int_0^T s^n e^{-\theta s}ds - \int_0^T s^n e^{-\theta s}\log s ds\right].$$

The function $s^n e^{-\theta s}\log s$ is integrable on $[0,\infty)$. Thus,

$$\lim_{T\to\infty}\frac{A_{1,H}(T)}{\log T} \ge \int_0^\infty s^n e^{-\theta s}ds. \tag{6.25}$$

By (6.24) and (6.25), we conclude the proof of (ii).

Lemma 17 *Let F_T, $\widetilde{F}_T$ be defined by (3.11) and (3.18), respectively. Moreover, let R_1 be defined in Part (iii) of Theorem 5. Then we have the following convergence results.*

(i) *When $0 < H < \frac{1}{2}$ we have*

$$\lim_{T\to\infty}\mathbb{E}\left(\frac{1}{T}F_T^2\right) = 4H^2\theta^{1-4H}\Gamma(2H)^2\left((4H-1)+\frac{2\Gamma(2-4H)\Gamma(4H)}{\Gamma(2H)\Gamma(1-2H)}\right). \tag{6.26}$$

(ii) *When $H=\frac{3}{4}$, we have*

$$\lim_{T\to\infty}\frac{\mathbb{E}\left(F_T^2\right)}{T\log(T)} = 9/4\theta^{-2}. \tag{6.27}$$

(iii) *When $H>\frac{3}{4}$, we have*

$$\lim_{T\to\infty}\mathbb{E}\left(T^{2-4H}F_T^2\right) = \frac{16\alpha_H^2\theta^{-2}}{(4H-2)(4H-3)}, \tag{6.28}$$

$$\lim_{T\to\infty}\mathbb{E}[T^{1-2H}R_1\widetilde{F}_T] = \frac{8\alpha_H^2\theta^{-1}}{(4H-2)(4H-3)}, \tag{6.29}$$

where $\alpha_H = H(2H-1)$.

In the above lemma, we do not give a statement when $H\in[\frac{1}{2},\frac{3}{4})$, because this case has been studied in Hu and Nualart (2010).

Proof **Part (i):** Assume $H\in(0,\frac{1}{2})$. Applying L'Hopital's rule to (3.13) yields

$$\lim_{T\to\infty}\mathbb{E}\left(\frac{1}{T}F_T^2\right) = \lim_{T\to\infty}4H^2\theta^2(I_1+I_2), \tag{6.30}$$

where

$$I_1 = (H\theta)^{-1}\int_{[0,T]^3} e^{-\theta(T-t_1)}\frac{\partial e^{-\theta|s_2-t_2|}}{\partial s_2}[T^{2H-1}-(T-s_2)^{2H-1}]\frac{\partial R_H(t_1,t_2)}{\partial t_2}ds_2dt_1dt_2,$$
$$I_2 = -(H\theta)^{-1}\int_{[0,T]^3} e^{-\theta(T-s_1)}\frac{\partial e^{-\theta|s_2-t_2|}}{\partial s_2}\frac{\partial R_H(s_1,s_2)}{\partial s_1}$$
$$\times[t_2^{2H-1}+(T-t_2)^{2H-1}]ds_1ds_2dt_2. \tag{6.31}$$

To compute the limit of $\mathbb{E}(\frac{1}{T}F_T^2)$ we will consider that of I_1 and I_2.

Computation of $\lim_{T\to\infty} I_1$: We first compute explicitly the partial derivatives in the integrand of I_1. On the region $\{t_2 > s_2\}$, we make change of variables $1-\frac{t_1}{T}\to u$, $\frac{t_2}{T}-\frac{s_2}{T}\to x$ and $1-\frac{s_2}{T}\to t$, and on the region $\{t_2 < s_2\}$, we make change of variables $1-\frac{t_1}{T}\to u$, $\frac{s_2}{T}-\frac{t_2}{T}\to x$ and $1-\frac{t_2}{T}\to t$. In this way, I_1 can be written as

$$\begin{aligned}I_1 = \int_{[0,1]^3, x\le t} & T^{4H+1}e^{-\theta T(u+x)}(1-t^{2H-1})\\ & \Big((1-t+x)^{2H-1} - \mathrm{sgn}(x+u-t)|x+u-t|^{2H-1}\Big)\,dudxdt\\ & - \int_{[0,1]^3, x\le t} T^{4H+1}e^{-\theta T(u+x)}\Big(1-(t-x)^{2H-1}\Big)\\ & \Big((1-t)^{2H-1} - \mathrm{sgn}(u-t)|u-t|^{2H-1}\Big)\,dudxdt.\end{aligned} \tag{6.32}$$

Reorganize the terms in the above integrals we have

$$I_1 = \int_{[0,1]^2} \psi(x,u)\sum_{i=1}^{4}\varphi_i\,dxdu, \tag{6.33}$$

where the functions ψ, φ_i are given by (6.1) to (6.5).

By (6.19), we see

$$\lim_{T\to\infty} I_1 = \lim_{T\to\infty}\int_{[0,1]^2} \psi(x,u)\varphi_4(x,u)\,dxdu, \tag{6.34}$$

whose value is computed in (6.16) of Lemma 14.

Computation of $\lim_{T\to\infty} I_2$: We first compute explicitly the partial derivatives in the integrand of (6.31). On the region $\{s_2 > t_2\}$, we make change of variables $T-s_1\to Tu$, $s_2-t_2\to Tx$ and $T-t_2\to Tt$, and on the region $\{t_2 > s_2\}$, we make change of variables $T-s_1\to Tu$, $t_2-s_2\to Tx$ and $T-s_2\to Tt$. In this way,

$$\begin{aligned}I_2 = \int_{[0,1]^3, t\ge x} & T^{4H+1}e^{-\theta T(u+x)}\Big((1-u)^{2H-1} + \mathrm{sgn}(x+u-t)|x+u-t|^{2H-1}\Big)\\ & \Big(t^{2H-1}+(1-t)^{2H-1}\Big)\,dudxdt - \int_{[0,1]^3, t\ge x} T^{4H+1}e^{-\theta T(u+x)}\\ & \Big((1-u)^{2H-1} + \mathrm{sgn}(u-t)|u-t|^{2H-1}\Big)\Big((1-t+x)^{2H-1}+(t-x)^{2H-1}\Big)\,dudxdt.\end{aligned} \tag{6.35}$$

Note that

$$\int_x^1 \Big(t^{2H-1}+(1-t)^{2H-1}\Big) - \Big((1-t+x)^{2H-1}+(t-x)^{2H-1}\Big)\,dt = 0,$$

so I_2 can be simplified and rewritten as

$$I_2 = \int_{[0,1]^2} \psi(x,u)\big(\varphi_4(x,u)+\varphi_5(x,u)\big)\,dxdu, \tag{6.36}$$

where $\psi(x,u)$, $\varphi_4(x,u)$ and $\varphi_5(x,u)$ are given by (6.1), (6.5) and (6.6) respectively.

By (6.34) and the result of (6.19) for $i = 5$, we have

$$\lim_{T\to\infty} I_2 = \lim_{T\to\infty} I_1. \tag{6.37}$$

Then part **(i)** follows from (6.30), (6.34), (6.37) and (6.16).

Part (ii) and (iii): Assume $H \geq 3/4$. Using (2.5), we have

$$\mathbb{E}(F_T^2) = 2\alpha_H^2 I_T, \tag{6.38}$$

where $\alpha_H = H(2H-1)$, and

$$I_T = \int_{[0,T]^4} e^{-\theta|s_2-u_2|-\theta|s_1-u_1|}|s_2-s_1|^{2H-2}|u_2-u_1|^{2H-2}du_1du_2ds_1ds_2. \tag{6.39}$$

Applying L'Hopital rule yields

$$\begin{cases} \lim\limits_{T\to\infty} \mathbb{E}(T^{2-4H}F_T^2) = \dfrac{8\alpha_H^2}{4H-2}\lim\limits_{T\to\infty} T^{3-4H}J_T & \text{when } H\in\left(\dfrac{3}{4},1\right) \quad (6.40)\\ \lim\limits_{T\to\infty}\dfrac{\mathbb{E}F_T^2}{T\log T} = \dfrac{9}{8}\lim\limits_{T\to\infty}\dfrac{J_T}{\log T} & \text{when } H=\dfrac{3}{4}, \quad (6.41)\end{cases}$$

where

$$J_T = \int_{[0,T]^3} e^{-\theta|T-u_2|-\theta|s_1-u_1|}(T-s_1)^{2H-2}|u_2-u_1|^{2H-2}du_1du_2ds_1.$$

Denote

$$h(T) = T^{3-4H}\mathbf{1}_{\{H\in(\frac{3}{4},1)\}} + (\log T)^{-1}\mathbf{1}_{\{H=\frac{3}{4}\}}.$$

Then, finding the limits (6.40) and (6.41) is reduced to the computation of $\lim_{T\to\infty} h(T)J_T$.

Making the change of variables $x = T-u_2$, $y = u_1-s_1$ and $z = T-s_1$ in the region $\{u_1 > s_1\}$ and the change of variables $x = T-u_2$, $y = s_1-u_1$, $z = T-s_1$ in the region $\{u_1 < s_1\}$, we can write J_T as follows

$$\begin{aligned} J_T = &\int_{[0,T]^3,y<z} e^{-\theta(x+y)}z^{2H-2}|x+y-z|^{2H-2}dxdydz\\ &+\int_{[0,T]^3,y+z<T} e^{-\theta(x+y)}z^{2H-2}|y+z-x|^{2H-2}dxdydz. \end{aligned}\tag{6.42}$$

Consider the functions

$$f_1(x,y,z) = e^{-\theta(x+y)}z^{2H-2}|x+y-z|^{2H-2}, \qquad f_2(x,y,z) = e^{-\theta(x+y)}z^{2H-2}|y+z-x|^{2H-2}.$$

For the first integral of (6.42), we split the integration interval $\{y < z\}$ into $\{x+y < z\}\cup\{x+y\geq z, y<z\}$. For the second integral of (6.42), we write the integration interval as $\{y+z < T\} = \{x+y < T, x\leq y\}\cup\{x+y < T, 0 < x-y < z\}\cup\{x+y < T, x-y\geq z\}\cup\{x+y\geq T\}\backslash\{y+z\geq T\}$. In this way, we can split J_T into seven integrals. It turns out that some of them are bounded by a constant independent of T and they do not contribute to the limit, because $h(T)\to 0$. More precisely, we can derive the following bounds:

$$\int_{[0,T]^3, x+y\geq z, y<z} f_1(x,y,z)dxdydz \leq \int_{[0,T]^3, x+y\geq z} f_1(x,y,z)dxdydz$$
$$= C_1 \int_{[0,T]^2} e^{-\theta(x+y)}(x+y)^{4H-3}dxdy \leq C,$$

where in the second step we integrated in z and the last step follows from the inequality $x + y \geq 2\sqrt{xy}$. It is trivial to show that

$$\int_{[0,T]^3, x+y\geq T} f_2(x,y,z)dxdydz \leq e^{-\theta T} \int_{[0,T]^3} z^{2H-2}|y+z-x|^{2H-2}dxdydz \leq C,$$

and

$$\int_{[0,T]^3, x+y<T, x-y\geq z} f_2(x,y,z)dxdydz \leq \int_{[0,T]^3, x-y\geq z} e^{-\theta(x+y)}z^{2H-2}(x-y-z)^{2H-2}dxdydz$$
$$= C_1 \int_{[0,T]^2} e^{-\theta(x+y)}(x-y)^{4H-3}dxdy \leq C.$$

The last bounded integral is

$$\int_{[0,T]^3, y+z\geq T} f_2(x,y,z)dxdydz \leq \int_{[0,T]^3, y+z\geq T} e^{-\theta(x+y)}z^{2H-2}(T-x)^{2H-2}dxdydz$$
$$\leq \left(\int_0^T e^{-\theta x}(T-x)^{2H-2}dx\right)^2 \leq C,$$

where in the second step we have used the inequality $z^{2H-2} \leq (T-y)^{2H-2}$ and the last step follows from the following inequality

$$\int_0^T e^{-\theta x}(T-x)^{2H-2}dx \leq \int_0^{T/2} e^{-\theta x}x^{2H-2}dx + \int_{T/2}^T e^{-\theta(T-x)}(T-x)^{2H-2}dx$$
$$\leq 2\int_0^\infty e^{-\theta x}x^{2H-2}dx.$$

With these observations,

$$\lim_{T\to\infty} h(T)J_T = \lim_{T\to\infty} h(T)\int_{x+y<z} f_1(x,y,z)dxdydz$$
$$+ \lim_{T\to\infty} h(T)\int_{\{x+y<T, x\leq y\}} f_2(x,y,z)dxdydz$$
$$+ \lim_{T\to\infty} h(T)\int_{\{x+y<T, 0<x-y<z\}} f_2(x,y,z)dxdydz.$$

We make change of variables $z-(x+y) \to u, x+y \to v, y \to y$ for the first term, $y-x \to u, z \to v, y \to y$ for the second term, and $x-y \to u, z-x+y \to v, y \to y$ for the third term. In this way, we obtain

$$\lim_{T\to\infty} h(T)J_T = \lim_{T\to\infty} h(T)\int_{[0,T]^3,u+v<T,y<v} e^{-\theta v}(u+v)^{2H-2}u^{2H-2}dydudv$$
$$+\lim_{T\to\infty} h(T)\int_{[0,T]^3,u<y<(T+u)/2} e^{-\theta(-u+2y)}v^{2H-2}(u+v)^{2H-2}dydudv$$
$$+\lim_{T\to\infty} h(T)\int_{[0,T]^3,u+v<T,y<(T-u)/2} e^{-\theta(u+2y)}(u+v)^{2H-2}v^{2H-2}dydudv.$$

Finally, the limits (6.27) and (6.28) follow from integrating in the variable y and an application of Lemma 16.

We proceed now to the proof of (6.29). Assume $H > 3/4$. Recall that $R_1 = I_2(\delta_{0,1})$ is given in Theorem 5 and $\widetilde{F}_T$ is given by (3.18). By (2.7), we can write

$$\mathbb{E}(R_1(T^{1-2H}\widetilde{F}_T)) = 2\alpha_H^2 T\int_{[0,1]^3} e^{-\theta T|t-s|}|t-t'|^{2H-2}|s-t'|^{2H-2}dsdtdt'.$$

We make the change of variables $Tt \to x$, $Ts \to y$, $Tt' \to z$ to rewrite the above equation as

$$\mathbb{E}(T^{1-2H}R_1\widetilde{F}_T) = \frac{2\alpha_H^2}{T^{4H-2}}\int_{[0,T]^3} e^{-\theta|x-y|}|x-z|^{2H-2}|y-z|^{2H-2}dxdydz.$$

By the symmetry of x, y in the above equation, applying L'Hopital's rule yields

$$\lim_{T\to\infty}\mathbb{E}(T^{1-2H}R_1\widetilde{F}_T) = \frac{\alpha_H^2}{2H-1}\lim_{T\to\infty} T^{3-4H}\Big(2\int_{[0,T]^2} e^{-\theta(T-y)}(T-z)^{2H-2}|y-z|^{2H-2}dydz$$
$$+\int_{[0,T]^2} e^{-\theta|x-y|}(T-x)^{2H-2}(T-y)^{2H-2}dxdy\Big) \tag{6.43}$$
$$=: \frac{\alpha_H^2}{2H-1}\lim_{T\to\infty} T^{3-4H}(2L_1+L_2). \tag{6.44}$$

To compute L_1, on the region $\{y > z\}$ we make the change of variables $y-z \to t,\ T-y \to s$ and on the region $\{y < z\}$ we make the change of variables $z-y \to s,\ T-z \to t$. In this way we obtain

$$L_1 = \int_{[0,T]^2,s+t<T} e^{-\theta s}(s+t)^{2H-2}t^{2H-2}dsdt + \int_{[0,T]^2,s+t<T} e^{-\theta(s+t)}t^{2H-2}s^{2H-2}dsdt$$

For the term L_2, by symmetry it is sufficient to consider the region $\{x > y\}$ and making the change of variables $T-x \to t,\ x-y \to s$, we obtain

$$L_2 = 2\int_{[0,T]^2,s+t<T} e^{-\theta s}t^{2H-2}(s+t)^{2H-2}dsdt.$$

Notice that the second summand of L_1 is bounded by $\int_{[0,\infty)^2} e^{-\theta(s+t)}t^{2H-2}s^{2H-2}dsdt$. Therefore,

$$\lim_{T\to\infty}\mathbb{E}(T^{1-2H}R_1\widetilde{F}_T) = \frac{4\alpha_H^2}{2H-1}\lim_{T\to\infty} T^{3-4H}\int_{[0,T]^2,s+t<T} e^{-\theta s}(s+t)^{2H-2}t^{2H-2}dsdt$$
$$= \frac{4\alpha_H^2\theta^{-1}}{(2H-1)(4H-3)},$$

where the last step is due to Lemma 16. This finishes the proof of Lemma 17. □

Lemma 18 *Let Y_T be defined by*

$$Y_t = \sigma \int_{-\infty}^{t} e^{-\theta(t-s)} dB_s^H = X_t + e^{-\theta t}\xi, \tag{6.45}$$

where

$$\xi = \sigma \int_{-\infty}^{0} e^{\theta s} dB_s^H. \tag{6.46}$$

For any $\alpha > 0$, $\frac{Y_T}{T^\alpha}$ converges almost surely to zero as T tends to infinity.

Proof The case $H \geq \frac{1}{2}$ was proved in Hu and Nualart (2010). Here, we present a different proof valid for all $H \in (0, 1)$. We denote $\beta := \mathbb{E}\xi^2 = \sigma^2\theta^{-2H} H\Gamma(2H)$, which is computed in Lemma 19. Notice that the covariance of the process Y_t for $t > 0$ is computed as

$$\begin{aligned}
\mathrm{Cov}(Y_0, Y_t) &= e^{-\theta t}\mathbb{E}\Big(\xi\Big[\xi + \sigma \int_0^t e^{\theta u} dB_u^H\Big]\Big) \\
&= e^{-\theta t}\beta + e^{-\theta t}\sigma^2\mathbb{E}\Big(\int_{-\infty}^{0} e^{\theta s} dB_s^H \int_0^t e^{\theta u} dB_u^H\Big).
\end{aligned}$$

We use integration by parts for both integrals in the above equation to rewrite

$$\mathrm{Cov}(Y_0, Y_t) = e^{-\theta t}\beta + g_1(t) - g_2(t),$$

where

$$g_1(t) = e^{-\theta t}\sigma^2\theta^2\mathbb{E}\Big(\int_{-\infty}^{0}\int_0^t B_s^H B_u^H e^{\theta(u+s)} du ds\Big), \quad g_2(t) = \sigma^2\theta\mathbb{E}\Big(\int_{-\infty}^{0} B_s^H B_t^H e^{\theta s} ds\Big).$$

By Fubini theorem and the explicit form of the covariance of fBm,

$$\begin{aligned}
g_1(t) &= \frac{1}{2}e^{-\theta t}\sigma^2\theta^2 \int_{-\infty}^{0}\int_0^t (|s|^{2H} + u^{2H} - (u-s)^{2H})e^{\theta(u+s)} du ds \\
&= \beta(1 - e^{-\theta t}) + \frac{1}{2}e^{-\theta t}\sigma^2\theta \int_0^t e^{\theta u}u^{2H} du - \frac{\beta}{2}(e^{\theta t} - e^{-\theta t}).
\end{aligned}$$

When we compute the above double integral, we write the integrand as three items by distributing $e^{\theta(u+s)}$ and then integrate the terms one by one. For the term involving $(u-s)^{2H}$, we make the change of variables $u - s \to x, s \to y$ and integrate in the variable y first. Similarly,

$$\begin{aligned}
g_2(t) &= \frac{1}{2}\sigma^2\theta \int_{-\infty}^{0} (|s|^{2H} + t^{2H} - (t-s)^{2H})e^{\theta s} ds \\
&= \beta + \frac{1}{2}\sigma^2 t^{2H} - \beta e^{\theta t} + \frac{1}{2}\sigma^2\theta e^{\theta t} \int_0^t e^{-\theta s} s^{2H} ds.
\end{aligned}$$

Denote $a_t = o(b_t)$ if $\lim_{t\to 0} \frac{a_t}{b_t} = 0$. Notice that $\int_0^t e^{\theta(u-t)} u^{2H} du - \int_0^t e^{\theta(t-s)} s^{2H} ds = o(t^{2H})$. Based on the above computations, for t small, we have

$$\mathrm{Cov}(Y_0, Y_t) = \beta\left[1 - \frac{\theta^{2H}}{\Gamma(2H+1)} t^{2H} + o(t^{2H})\right].$$

The lemma now follows from Theorem 3.1 of Pickands (1969). □

Lemma 19 *Let the stochastic process X_t satisfy (1.1) (with $\sigma_t = \sigma$). Then $\frac{1}{T}\int_0^T X_t^2 dt \to \sigma^2\theta^{-2H} H\Gamma(2H)$ a.s. and in L^2, as $T \to \infty$.*

Proof When $H \geq \frac{1}{2}$, the Lemma is proved in Hu and Nualart (2010). We shall handle the case of general Hurst parameter in a similar way. The process $\{Y_t, t \geq 0\}$ defined by (6.45) is Gaussian, stationary and ergodic for all $H \in (0, 1)$. By the ergodic theorem,

$$\frac{1}{T}\int_0^T Y_t^2 dt \to \mathbb{E}(Y_0^2), \quad \text{as } T \text{ goes to infinity,}$$

almost surely and in L^2. This implies

$$\frac{1}{T}\int_0^T X_t^2 dt \to \mathbb{E}(Y_0^2),$$

as T goes to infinity, almost surely and in L^2. Moreover, integrating by parts yields

$$\begin{aligned}\mathbb{E}(Y_0^2) = \mathbb{E}(\xi^2) = \sigma^2\mathbb{E}\left(\int_{-\infty}^0 e^{\theta s} dB_s^H\right)^2 &= \theta^2\sigma^2\mathbb{E}\int_{-\infty}^0\int_{-\infty}^0 B_s^H B_r^H e^{\theta(s+r)} ds dr \\ &= \theta^2\sigma^2\int_0^\infty\int_0^\infty e^{-\theta(s+r)} R_H(s,r) ds dr = \sigma^2\theta^{-2H} H\Gamma(2H).\end{aligned}$$

In the last step of the above computation, we use the same idea as near the end of the proof for Lemma 18. Namely, one writes out the explicit form of $R_H(s, r)$, split the integrand into three items by distributing $e^{-\theta(s+r)}$ to the summands of $R_H(s, r)$, and then integrate the three items one by one. For the item involving $|s - r|^{2H}$, noticing the symmetry of s, r, one can make change of variables $s - r \to u$, $r \to v$, and then integrate in the variable v first. □

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Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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