关系的运算

School of Computer Wuhan University

- 1/145 -

- 1 关系的合成
 - 关系的合成
 - 关系的幂
 - 关系的闭包
 - 传递闭包的求解算法

关系上的运算

Remark

由于关系就是集合,因此集合上的运算也是关系的运算.

- " \leq " " $\mathbb{1}_A$ " = "<";
- ℝ上有: "≤"∩"≥"= "=";
- "≤"」"≥"= ℝ 上的全域关系:
- a "□"」"□" → 人瑞光系
- 由于关系的对象是n重组,因此还有些一般集合不具有的运算.

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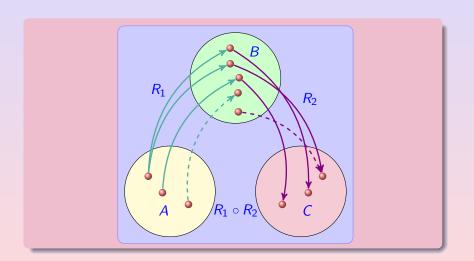
关系上的运算

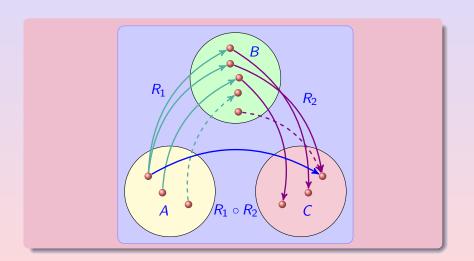
Remark

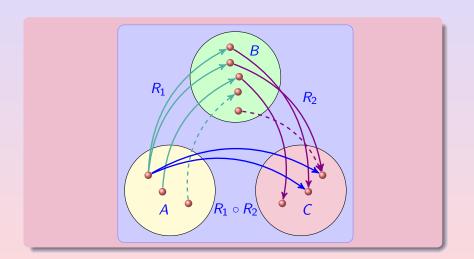
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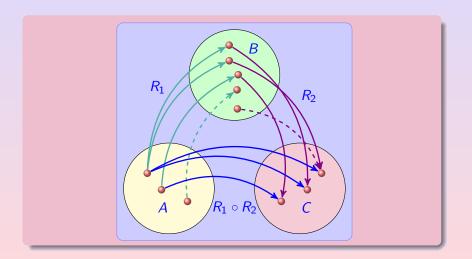
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由于关系的对象是n重组,因此还有些一般集合不具有的运算.









合成的定义

Definition (合成关系, Composite Relation)

设 $\mathcal{R}_1 \subseteq A \times B$, $\mathcal{R}_2 \subseteq B \times C$, \mathcal{R}_1 和 \mathcal{R}_2 的合成记为 $\mathcal{R}_1 \circ \mathcal{R}_2$ ($\mathcal{R}_1 \mathcal{R}_2$)定义为:

 $\mathcal{R}_1 \mathcal{R}_2 \triangleq \{ \langle a, c \rangle \mid a \in A, c \in C \land \exists b \in B \land a \mathcal{R}_1 b \land b \mathcal{R}_2 c \}$ 是A到C上的关系.

Remark

合成的条件:第一个关系的陪域(codomain)和第二个关系的域(domain)是相同的集合.

Example

R1是兄弟关系; Ro父子关系; R1 Ro是叔侄关系;

 $\mathcal{R} = \{\langle a, b \rangle \mid \text{anbinf直航航线}\}, \mathcal{R}\mathcal{R}$ 是城市之间经过一个城市转机的间接航线(记为 \mathcal{R}^2);

- $\bullet \ (=_4)^2 = =_4;$
- $\emptyset \mathcal{R} = \mathcal{R} \emptyset = \emptyset$;
- 合成对应的SQL语句: SELECT R₁ .first, R₂ .second FROM R₃ .JOIN R₃ ON R₄ .second = R₂ .first.

- R_1 是兄弟关系; R_2 父子关系; R_1 R_2 是叔侄关系;
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 - $(=_4)^2 = =_4;$
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- 。 R1是兄弟关系; R2父子关系; R1 R2是叔侄关系;
- - $(=_4)^2 = =_4;$
 - $\mathcal{R} \subseteq A \times B$; \mathbb{N} , $\mathbb{1}_A \mathcal{R} = \mathcal{R} \mathbb{1}_B = \mathcal{R}$;
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合成对应的SQL语句: $SELECT \mathcal{R}_1$.first, \mathcal{R}_2 .second FROM \mathcal{R}_1 JOIN \mathcal{R}_2 ON \mathcal{R}_1 .second = \mathcal{R}_2 .first.



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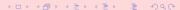
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Theorem

设 $\mathcal{R}_1 \subseteq A \times B$, $\mathcal{R}_2, \mathcal{R}_3 \subseteq B \times C$, $\mathcal{R}_4 \subseteq C \times D$:

- $\mathcal{R}_1(\mathcal{R}_2 \cup \mathcal{R}_3) = \mathcal{R}_1 \mathcal{R}_2 \cup \mathcal{R}_1 \mathcal{R}_3 \ (\circ \mathsf{M} \cup \ \mathsf{hohm});$
- ③ $(R_2 \cup R_3) R_4 = R_2 R_4 \cup R_3 R_4$ (○对∪的分配律);



Proof.



Proof.

- $\exists b(\langle a,b\rangle \in \mathcal{R}_1 \land \langle b,c\rangle \in \mathcal{R}_2) \land \exists b(\langle a,b\rangle \in \mathcal{R}_1 \land \langle b,c\rangle \in \mathcal{R}_3)$



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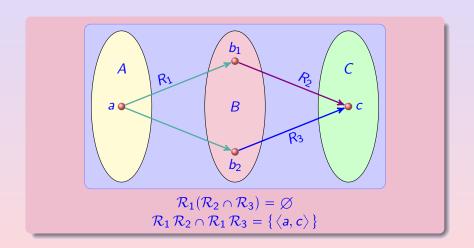


Proof.





②的反例



Example

求所有可以间接通航的城市:



a和b可间接通航, iff:

$$\exists a_1, a_2, \ldots, a_{n-1} \ (a \mathcal{R} a_1 \land a_1 \mathcal{R} a_2 \land \ldots \land a_{n-1} \mathcal{R} b)$$

则: $\langle a, a_1 \rangle \in \mathcal{R}$;

$$\langle a,a_2
angle\in\mathcal{R}^2;$$

.

$$\langle a, a_{n-1} \rangle \in (\mathcal{R}^{n-2}) \mathcal{R} \triangleq \mathcal{R}^{n-1};$$

$$\langle a,b\rangle\in(\mathcal{R}^{n-1})\,\mathcal{R}\triangleq\mathcal{R}^n$$
;

...a和b可间接通航。iff、∃ $n\langle a,b\rangle \in \mathbb{R}^n$.

Example

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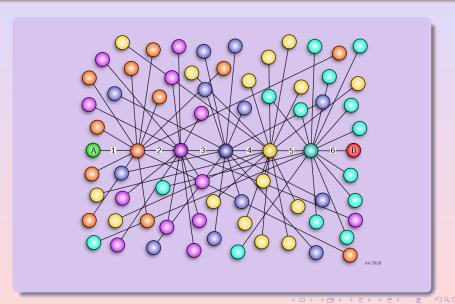
$$\langle a, a_2 \rangle \in \mathcal{R}^2$$
;

$$\langle a, a_{n-1} \rangle \in (\mathcal{R}^{n-2}) \mathcal{R} \triangleq \mathcal{R}^{n-1};$$

$$\langle a,b\rangle\in(\mathcal{R}^{n-1})\,\mathcal{R}\triangleq\mathcal{R}^n$$
;

∴ a n b可间接通航, iff, $\exists n \langle a, b \rangle \in \mathbb{R}^n$.

Example 2: Six Degrees of Separation (六度分隔)



Definition (关系的幂, Power of relation)

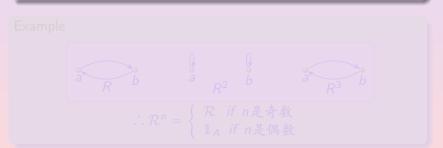
设 \mathcal{R} 是A上的关系, $n \in \mathbb{N}$, \mathcal{R} 的乘幂递归定义如下:

- $\mathbf{0} \ \mathcal{R}^0 = \mathbb{1}_A;$

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- $\mathbf{0} \ \mathcal{R}^0 = \mathbb{1}_A;$

Example



$$\therefore \mathcal{R}^n = \begin{cases} \mathcal{R} & \text{if } n \neq 3 \\ \mathbb{1}_A & \text{if } n \neq 3 \end{cases}$$

- $(\mathcal{R}^m)^n = \mathcal{R}^{mn};$

①的证明.

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Theorem

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①的证明.

- ① n = 0时, $\mathcal{R}^m \mathcal{R}^0 = \mathcal{R}^m \mathbb{1}_A = \mathcal{R}^m = \mathcal{R}^{m+0}$;
- ② $\mathfrak{F}_n = k \mathfrak{H}, \, \mathcal{R}^m \mathcal{R}^k = \mathcal{R}^{m+k};$
- ③ n = k + 1时:

$$\mathcal{R}^{m}\mathcal{R}^{k+1}$$

$$=\mathcal{R}^{m}(\mathcal{R}^{k}\mathcal{R}) \quad \text{(def)}$$

$$=(\mathcal{R}^{m}\mathcal{R}^{k})\mathcal{R} \quad \text{(结合律)}$$

$$=\mathcal{R}^{m+k+1} \quad \text{(归纳假设)}$$

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① 设|A| = n, 则存在 $i, j \in \{1, 2\}$ 0 $\{1, 2\}$ 0 $\{1, 2\}$ 0 $\{2\}$ 1 $\{2\}$ 2 $\{2\}$ 2 $\{2\}$ 3 $\{2\}$ 4 $\{2\}$ 3 $\{2\}$ 4 $\{2\}$ 5 $\{2\}$ 6 $\{2\}$ 9 $\{$

Proof.

Corollary

- ① 设|A| = n, 则存在i, j $0 \le i < j \le 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.
- Proof.

- Corollary

Theorem

① 设|A| = n, 则存在 $i, j \in \{1, 2\}$ 0 《 $i < j \in \{2\}^n$ ", 使得: $\mathbb{R}^i = \mathbb{R}^j$.

Proof.

- \bullet 而 \mathcal{R}^0 , \mathcal{R}^1 , ..., $\mathcal{R}^{2''}$ 共有 $2^{n'}$ + 1项;
- 根据抽屉原则, $\exists i, j \ 0 \leq i < j \leq 2^{n^{\epsilon}}$, 使得: $\mathcal{R}' = \mathcal{R}'$.

Corollary

Theorem

① 设|A| = n, 则存在 $i, j \in \{0\}$ 0 $\{i < j \le 2^{n^2}\}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

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① 设|A| = n, 则存在 $i, j \in \{0\}$ 0 《 $i < j \in \{2\}^n$ 》,使得: $\mathbb{R}^i = \mathbb{R}^j$.

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Proof.

- $2 : |\mathscr{P}(A \times A)| = 2^{n^2};$
- ③ 而 $\mathcal{R}^0, \mathcal{R}^1, \dots, \mathcal{R}^{2^{n^2}}$ 共有 2^{n^2} +1项;
- ④ 根据抽屉原则, $\exists i, j \ 0 \leq i < j \leq 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

Corollary

闭包

Description (闭包, Closure)

········

数学上把包含某个给定的集合,并且具有某个性质的最小集合称 为闭包.

Example

● 所有的可以间接通航的城市之间的关系, 是直接通航城市的传递闭包;

设 $\mathcal{R} \subseteq A \times B$, 关系 \mathcal{R} 的逆关系, 记为 $\widetilde{\mathcal{R}}$ (读作tilde), 定义如下: $\widetilde{\mathcal{R}} = \{\langle y, x \rangle \mid \langle x, y \rangle \in \mathcal{R} \} \subseteq B \times A$

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Example

 $\widetilde{\leqslant} = \geqslant$; $\widetilde{\mathbb{1}}_A = \mathbb{1}_A$; $\widetilde{\subseteq} = \supseteq$;

- 关系的逆是关系的对偶概念;如果R具有五性,则R也相应的具有;
- 关系的逆与关系的补是不同的概念:
 - $\overline{\mathcal{R}} = \{\langle x, y \rangle \mid \langle x, y \rangle \notin \mathcal{R}\} \subseteq A \times B$

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设 $\mathcal{R} \subseteq A \times B$, 关系 \mathcal{R} 的逆关系, 记为 $\widetilde{\mathcal{R}}$ (读作tilde), 定义如下: $\widetilde{\mathcal{R}} = \{\langle y, x \rangle \mid \langle x, y \rangle \in \mathcal{R}\} \subseteq B \times A$

Example

- $\stackrel{\sim}{\leqslant} = \geqslant; \widetilde{\mathbb{1}_A} = \mathbb{1}_A; \stackrel{\sim}{\subseteq} = \supseteq;$
- 关系的逆是关系的对偶概念;如果R具有五性,则 \widetilde{R} 也相应的具有;
- 关系的逆与关系的补是不同的概念: $\overline{\mathcal{R}} = \{\langle x, y \rangle | \langle x, y \rangle \notin \mathcal{R} \} \subseteq A \times B$

 \mathcal{R} 是对称关系, iff, $\mathcal{R} = \tilde{\mathcal{R}}$.



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Proof.

- ⇒ $\forall \langle x, y \rangle \in \mathcal{R}, \therefore \langle y, x \rangle \in \mathcal{R}; So \langle x, y \rangle \in \mathcal{R}$ ∴ $\mathcal{R} \subseteq \widetilde{\mathcal{R}}, but \stackrel{\widetilde{\mathcal{R}}}{\mathcal{R}} = \mathcal{R};$ $So, \stackrel{\widetilde{\mathcal{R}}}{\mathcal{R}} \subseteq \stackrel{\widetilde{\mathcal{R}}}{\mathcal{R}} = \mathcal{R} (\cdots \widetilde{\mathcal{R}} d \mathcal{L} d \mathcal{L$
- ∀⟨x,y⟩∈ R ∴ ⟨x,y⟩∈ R; So⟨y,x⟩∈ R
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Definition

设 $\mathcal{R} \subseteq A^2$, \mathcal{R} 的自反(对称、传递)闭包 \mathcal{R}' 是满足下述三条件的关系:

- ② R'是自反的(对称的、传递的);
- ③ 设 \mathbb{R}'' 是满足上述两条件的关系,则 $\mathbb{R}' \subseteq \mathbb{R}''$.

分别记R的自反、对称和传递闭包为: r(R), s(R)和t(R).

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R是自反的(对称的、传递的), iff, R = r(R) (s(R)), t(R)).

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闭包的构造(1/2)

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 $\exists m, n \ \langle x, y \rangle \in \mathcal{R}^m, \ \langle y, z \rangle \in \mathcal{R}^n; \ \therefore \langle x, z \rangle \in \mathcal{R}^m \mathcal{R}^n = \mathcal{R}^{m+n} \subseteq \bigcup_{i=1}^m \mathcal{R}^i;$

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闭包的构造(1/2)

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③ 设传递关系 $\mathbb{R}' \supseteq \mathbb{R}$,则要证明: $\bigcup_{i=1} \mathbb{R}' \subseteq \mathbb{R}'$;
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- n = 1 时, $\mathcal{R} \subseteq \mathcal{R}'$;
- ② $\dot{\mathbf{Q}}_n = k$ 时结论成立,n = k + 1 时:
 - $\therefore \exists v \langle x, v \rangle \in \mathcal{R}^k \land \langle v, z \rangle \in \mathcal{R}^k$
 - S_{α}/x \sqrt{x} C_{α}/x \sqrt{x} C_{α}/x \sqrt{x}
 - So $\langle x, y \rangle \in \mathcal{R}' \land \langle y, z \rangle \in \mathcal{R}';$
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Example

$$r(<) = <; s(<) = \neq;$$

 $s(\leq) =$ 全域关系: $r(\neq) =$ 全域关系:

• 设尺是城市之间有直接航线的关系,则城市之间有间接航线的关系等于 $| \mathcal{R}^i |$

Examples

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Theorem

$$\mathcal{L}[A]=\mathbf{n},\;\mathcal{R}\subseteq A^2,\;\mathbf{M}\colon\;\mathbf{t}(\mathcal{R})=igcup_{i=1}^n\mathcal{R}^i;$$

Theorem

$$\mathcal{E}|A|=n,\ \mathcal{R}\subseteq A^2,\ M:\ t(\mathcal{R})=\bigcup_{i=1}^n\mathcal{R}^i;$$

- ① 设 $\langle x_0, x_{n+1} \rangle \in \mathcal{R}^{n+1}$;
- $\exists x_1, x_2, \ldots, x_n x_0 \mathcal{R} x_1 \land x_1 \mathcal{R} x_2 \land \ldots x_n \mathcal{R} x_{n+1};$
- ③ 即尺关系图中有从x₀到xn+1长度为n+1的有向路径;
- ⑥ 而X1, X2,...,Xn+1 n+ 1个元素只能在|A| = n个元素中选取;
- ⑥ 所以根据抽屉原则, $\exists 1 \leq i < j \leq n+1 \times_i = x_j$;
- $\bigcirc \therefore x_0 \mathcal{R} x_1 \wedge x_1 \mathcal{R} x_2 \wedge \cdots \wedge x_i \mathcal{R} x_{j+1} \wedge \cdots \wedge x_n \mathcal{R} x_{n+1};$
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$$\bigcirc \ \, \therefore \langle x_0, x_{n+1} \rangle \in \mathcal{R}^{n+1-(j-i)} \subseteq \bigcup_{i=1}^n \mathcal{R}^i.$$



Examples

$$\mathcal{R} = \{\langle a,b \rangle, \langle b,c \rangle, \langle c,a \rangle \}$$
的传递闭包
$$t(R) = R \cup R^2 \cup R^3$$

Propostion

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- ② 而 $t(\mathcal{R}) = \bigcup_{i=1} \mathcal{R}^i \supseteq \mathbb{1}_A;$
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Propostion

$$rt(\mathcal{R}) = tr(\mathcal{R});$$

Proof.

$$rt(\mathcal{R}) = \bigcup_{i=0}^{\infty} \mathcal{R}^i; \quad tr(\mathcal{R}) = \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i$$

用归纳法证明下述等式即可:

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i \in \mathcal{R}} \mathcal{R}^i;$$

由此:

$$\forall n \ (\mathbb{1}_A \cup \mathcal{R})^n \subseteq \bigcup \mathcal{R}^i \subseteq \bigcup \mathcal{R}^i;$$

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 $rt(\mathcal{R}) = tr(\mathcal{R}).$

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$$rt(\mathcal{R}) = tr(\mathcal{R}).$$

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

- n = 0时上述等式成立;
- ② 设n = k时上述等式成立,则n = k + 1时:

(c) hfwang

闭包之间的关系(3/3)

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

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- ② 设n = k时上述等式成立,则n = k + 1时:

闭包之间的关系(3/3)

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

- □ n = 0时上述等式成立;
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$$\begin{aligned} & (\mathbb{1}_A \cup \mathcal{R})^{k+1} \\ &= (\mathbb{1}_A \cup \mathcal{R})^k (\mathbb{1}_A \cup \mathcal{R}) \\ &= \left(\bigcup_{i=0}^k \mathcal{R}^i\right) (\mathbb{1}_A \cup \mathcal{R}) \\ &= \left(\left(\bigcup_{i=0}^k \mathcal{R}^i\right) \mathcal{R}\right) \left(\bigcup_{i=0}^k \left(\bigcup_{i=0}^k \mathcal{R}\right) \mathcal{R}\right) \left(\bigcup_{i=0}^k \left(\bigcup_{i=0}^k \mathcal{R}\right) \mathcal{R}\right) \left(\bigcup_{i=0}^k \left(\bigcup_{i=0}^k \mathcal{R}\right) \mathcal{R}\right) \left(\bigcup_{i=0}^k \mathcal{R}\right) \left(\bigcup_{i=0}^k \left(\bigcup_{i=0}^k \mathcal{R}\right) \mathcal{R}\right) \left(\bigcup_{i=0}^k \left(\bigcup_{i=0}^k \mathcal{R}\right) \mathcal{R}\right) \left(\bigcup_{i=0}^k \left(\bigcup_{i=0}^k \mathcal{R}\right) \mathcal{R}\right) \left(\bigcup_{i=0}^k \mathcal{R}\right) \left(\bigcup_{i=0}^k$$

$$\int_{i=0}^{\infty} \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^{i} \right) \mathbf{1}_{A} \right) dt$$

闭包之间的关系(3/3)

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

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- ② 设n = k时上述等式成立,则n = k + 1时:

$$\begin{array}{l} & \textbf{(1}_A \cup \mathcal{R})^{k+1} \\ = (\mathbb{1}_A \cup \mathcal{R})^k (\mathbb{1}_A \cup \mathcal{R}) & \text{(by 乘幂的定义)} \\ = \begin{pmatrix} \binom{k}{i=0} & \mathcal{R}^i \end{pmatrix} (\mathbb{1}_A \cup \mathcal{R}) & \text{(by 归纳假设)} \\ = \begin{pmatrix} \binom{k}{i=0} & \mathcal{R}^i \end{pmatrix} \mathcal{R} \end{pmatrix} \bigcup \begin{pmatrix} \binom{k}{i=0} & \mathcal{R}^i \end{pmatrix} \mathbb{1}_A \end{pmatrix} & \text{(by 合成对并的分配率)} \\ = \begin{pmatrix} \binom{k+1}{i=0} & \mathcal{R}^i \end{pmatrix} \bigcup \begin{pmatrix} \binom{k}{i=0} & \mathcal{R}^i \end{pmatrix} & \text{(by 合成对并的分配率)} \\ = \begin{pmatrix} \binom{k+1}{i=0} & \mathcal{R}^i \end{pmatrix} \bigcup \begin{pmatrix} \binom{k}{i=0} & \mathcal{R}^i \end{pmatrix} & \text{(by 合成对并的分配率)} \\ \end{pmatrix}$$

闭包之间的关系(3/3)

Proof(continued).

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

- □ n = 0时上述等式成立;
- ② 设n = k时上述等式成立,则n = k + 1时:

$$(\mathbb{1}_A \cup \mathcal{R})^{k+1}$$

= $(\mathbb{1}_A \cup \mathcal{R})^k (\mathbb{1}_A \cup \mathcal{R})$

(by 乘幂的定义)

$$= \left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) \mathcal{R} \cup \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) \mathbb{1}_{A}\right)$$

(by 合成对并的分配率)

$$=\bigcup \mathcal{R}^i$$

Chfware

闭包之间的关系(3/3)

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

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闭包之间的关系(3/3)

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=1}^n \mathcal{R}^i;$$

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$$\begin{array}{l} \left(\mathbb{1}_{A} \cup \mathcal{R}\right)^{k+1} \\ = \left(\mathbb{1}_{A} \cup \mathcal{R}\right)^{k} (\mathbb{1}_{A} \cup \mathcal{R}) \\ = \left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) (\mathbb{1}_{A} \cup \mathcal{R}) \\ = \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) \mathcal{R}\right) \bigcup \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) \mathbb{1}_{A}\right) \\ = \left(\bigcup_{i=1}^{k+1} \mathcal{R}^{i}\right) \bigcup \left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) \mathbb{1}_{A} \end{array} \right) \text{ (by 合成对并的分配率)} \\ = \left(\bigcup_{i=1}^{k+1} \mathcal{R}^{i}\right) \bigcup \left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) \\ = \left(\bigcup_{i=1}^{k+1} \mathcal{R}^{i}\right) \bigcup \left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) \\ = \left(\bigcup_{i=0}^{k+1} \mathcal{R}^{i}\right) \bigcup \left(\bigcup_{i=0}^{k+1} \mathcal{R}^{i}\right) \\ = \left(\bigcup_{i=0}^{k+1} \mathcal{R}^{i}\right) \bigcup \left(\bigcup_{i=0}^{k+$$

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Theorem

设
$$\mathcal{R} \subseteq A \times B$$
, $\mathcal{S} \subseteq B \times C$; $|A| = m$, $|B| = n$ 和 $|C| = p$, 则: $M_{\mathcal{R} \mathcal{S}} = M_{\mathcal{R}} \cdot M_{\mathcal{S}}$; 其中: $M_{\mathcal{R}} = (a_{ij})_{m \times n}$; $M_{\mathcal{S}} = (b_{ij})_{n \times p}$ $M_{\mathcal{R}} \cdot M_{\mathcal{S}} = (c_{ij})_{m \times p}$; $c_{ij} \triangleq \bigvee_{i} a_{ik} \wedge b_{kj}$;

设
$$A = \{x_1, x_2, \dots, x_m\}, B = \{y_1, y_2, \dots, y_n\}, C = \{z_1, z_2, \dots, z_p\}$$

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- $c_{ij}=1$

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- $2 \iff \exists k \ a_{ik} = 1 \land b_{ki} = 1$

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$$c_{ij} = 1$$

设:
$$M_{\mathcal{R}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
; $M_{\mathcal{S}} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$;

则:

$$\mathbf{M}_{\mathcal{R}\mathcal{S}} = \mathbf{M}_{\mathcal{R}} \cdot \mathbf{M}_{\mathcal{S}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$



Description

$$M_{t(\mathcal{R})} = \sum_{i=1}^{n} M_{\mathcal{R}}^{i}$$

其中: MR是n阶方阵;

• 计算
$$M \cdot M$$
的每个元素 $c_{ij} = \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj} \dots O(n)$

• 计算
$$\sum M_{\mathcal{R}}^{i}$$
 $O(n^4)$

传递闭包的求解算法

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 $O(n^{4})$.

Definition

设 $A = \{a_1, a_2, \dots, a_n\}, \mathcal{R} \subseteq A^2, M$ 是 \mathcal{R} 的关系矩阵; n阶方阵 W_k 递归定义如下:

- $\mathbf{0} \ W_0 = M;$
- ② $W_k = (w_{ij}^k)_{n \times n}$, 其中: $w_{ij}^k = 1$, iff, $A_i = 1$, iff, $A_i = 1$, A_i

Propostion

 $W_n = M_{t(\mathcal{R})}$

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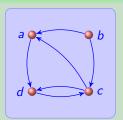
- **1** $W_0 = M$;
- ② $W_k = (w_{ii}^k)_{n \times n}$, 其中: $w_{ii}^k = 1$, iff, A_i 到 a_i 有一条仅经 过a1, a2,..., ak的有向路径.

Propostion

$$W_n = M_{t(\mathcal{R})}$$

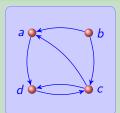


Example



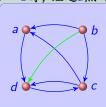
$$W_0 = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Example



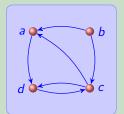
$$W_0 = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_d^a$$

k = 1时,经过a点的路径:



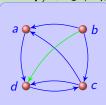
$$W_1 = \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Example



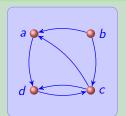
$$W_0 = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

k=2时,经过a,b点的路径:b没有引入的边,所以没有经过b的路径



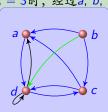
$$W_2 = \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Example

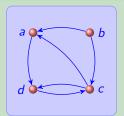


$$W_0 = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_{d}^{a}$$

k = 3时, 经过a, b, c点的路径:

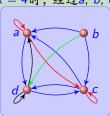


$$W_3 = \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right)$$



$$W_0 = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

k = 4时, 经过a, b, c, d点的路径:



$$W_4 = \left(\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right)$$

Description (W_k 和 W_{k+1} 的关系)

$$w_{ii}^{k+1} = 1$$
, iff, 下述两条件之一成立:

- w^k_{ii} = 1, 即从a_i到a_j有一条仅经过a₁, a₂,..., a_k的有向路径;
- ② 有一条仅经过a₁, a₂,..., a_{k+1}, 并且仅经过a_{k+1}一次的路径: 如:

 $a_i, x_1, x_2, \dots, x_p, a_{k+1}, y_1, y_2, \dots, y_q, a_j$ 其中: x_1, x_2, \dots, x_p 和 y_1, y_2, \dots, y_q 都在 $\{a_1, a_2, \dots, a_k\}$ 中; $w_{(k+1)}^k = 1 \land w_{(k+1)i}^k = 1;$

故:

$$w_{ij}^{k+1} = w_{ij}^k \vee (w_{i(k+1)}^k \wedge w_{(k+1)j}^k)$$

Description (W_k 和 W_{k+1} 的关系)

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- $w_{ij}^k = 1$, $p_{ij} M_{ai} = 1$, $p_{ij} M_{$
- ② 有一条仅经过 $a_1, a_2, ..., a_{k+1}$, 并且仅经过 a_{k+1} 一次的路径: 如:

 $a_i, x_1, x_2, \dots, x_p, a_{k+1}, y_1, y_2, \dots, y_q, a_j$ 其中: x_1, x_2, \dots, x_p 和 y_1, y_2, \dots, y_q 都在 $\{a_1, a_2, \dots, a_k\}$ 中; \vdots $w_{(k+1)}^k = 1 \land w_{(k+1)}^k = 1;$

故:

$$w_{ij}^{k+1} = w_{ij}^k \vee (w_{i(k+1)}^k \wedge w_{(k+1)j}^k)$$

Wk的计算

Description (W_k 和 W_{k+1} 的关系)

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$$a_i, x_1, x_2, \dots, x_p, a_{k+1}, y_1, y_2, \dots, y_q, a_j$$

其中: x_1, x_2, \dots, x_p 和 y_1, y_2, \dots, y_q 都在 $\{a_1, a_2, \dots, a_k\}$ 中;
 $w_{i(k+1)}^k = 1 \land w_{(k+1)j}^k = 1;$

故:

$$w_{ij}^{k+1} = w_{ij}^k \vee (w_{i(k+1)}^k \wedge w_{(k+1)j}^k)$$

```
Warshall算法.
procedure warshall (Matrix M_R)
  W := M_{\mathcal{R}};
  for k := 1 to n do {
     for i := 1 to n do {
       for j := 1 to n do {
          w_{ii} := w_{ii} \vee (w_{ik} \wedge w_{ki});
```

- ① 关系的合成
 - 关系的合成
 - 关系的幂
 - 关系的闭包
 - 传递闭包的求解算法