Problem 1

Parts (a) and (b)

First, we present the reduction from SUBSETSUM to PARTITION. Recall that the decision version of the SUBSETSUM problem is defined as follows: given a collection U of positive integers and a target value t, determine whether there is a subset $S \subseteq U$ such that the sum of the integers in S is exactly t, i.e., determine whether there is a subset S s.t. $\sum_{s \in S} s = t$.

The reduction is as follows: let U, t be an instance of SUBSETSUM and let W be sum of elements in U. Note that if this is a true instance, then we can partition the elements of U into the sets S and $U \setminus S$ such that

$$\sum_{s \in S} = t, \text{ and } \sum_{s \in U \setminus S} = W - t.$$

Since in the PARTITION problem we want the partitions to have the same sum, it makes sense to add the element $2t - W$ to S, obtaining U'. In that case, both sets S and $U' \setminus S$ would have sum equal to t. Indeed, it turns out that this idea is sufficient for the reduction, which we present in more detail in the following discussion.

The instance of PARTITION we construct from U, t is

$$U' = U \cup \{2t - W\}.$$

After constructing U', we call the black-box for PARTITION with input U' and accept if and only if it accepts.

To argue correctness, we need to show that there is a subset of U that sums to t if and only if there is a subset S' of U' such that

$$\sum_{s \in S'} s = \sum_{s \in U' \setminus S'} s = 1/2 \cdot \sum_{u \in U'} u = (W + 2t - W)/2 = t.$$

Suppose there is a subset S of U that sums to t, then the same subset S serves as a solution for PARTITION, as it has total sum exactly half the sum of all elements in U'.

For the other direction, suppose that there is a subset S' of U' such that $\sum_{s \in S'} s = \sum_{s \in U' \setminus S'} = t$. One of the sets S' or $U' \setminus S'$ must not contain the element $2t - W$ that we added to U to obtain U'. In that case, the same set serves as a solution for SUBSETSUM, as it only contains elements from U and adds up to t.

Parts (c) and (d)

Recall the definition of PARTITION: given a collection U' of positive integers that sum to W, determine whether there is a subset S such that

$$\sum_{s \in S} s = \sum_{s \in U \setminus S} s = 1/2 \cdot \sum_{u \in U} u = W/2.$$

Note that even though the SQUAREDSUM problem allows for other values of k, it makes sense to try coming up with a reduction from PARTITION to SQUAREDSUM using $k = 2$, because then the objective for both problems becomes more similar. With this in mind, consider maintaining the input U we get from PARTITION. If we have a true instance of PARTITION, then there is a set $S \subseteq U$ such that

$$\sum_{s \in S} s = \sum_{s \in U \setminus S} s = W/2.$$

If we square both sums, each is then equal to $(W/2)^2 = W^2/4$ and their sum is equal to $W^2/2$. Now, recall that if we have two numbers P and Q that add up to W, we minimize $P^2 + Q^2$ exactly when $P = Q = W/2$. This property follows from the following inequality:

$$2(P^2 + Q^2) \geq (P + Q)^2,$$

with equality if and only if $P = Q$.

Note that this is exactly what we need, so we could just set the target to $W^2/2$, and then both problems would be checking for the exact same condition. We now formalize this reduction: given an instance U of PARTITION, we construct the instance $(U, k = 2, B = W^2/2)$ of SQUAREDSUM. Then, we call the black-box for SQUAREDSUM with input $(U, 2, W^2/2)$ and return yes if and only if it returns yes.

By the inequality above, S_1, S_2 is a solution for this instance of SQUAREDSUM if and only if $\sum_{s \in S_1} s = \sum_{s \in S_2} s = W/2$. The latter holds if and only if S_1 is a solution to the initial instance of the PARTITION. We have thus given a reduction from PARTITION to SQUAREDSUM.

Problem 2

Part (a)

To prove that 577-CYCLE is in NP, we show that there is a polynomial-time verifier algorithm for the problem. Note that if an instance (G, w) of 577-CYCLE is true, then there exists a simple cycle C in G with weight exactly 577. With this in mind, our verifier algorithm could just receive as input the instance (G, w) as well a cycle C in the form of a sequence of vertices of G, and then check that C is indeed a simple cycle and that the weights of its edges add up to exactly 577. As this runs in polynomial time, we established that 577-CYCLE is in NP.

Parts (b) and (c)

There are two relatively straightforward reductions to the problem. One is from SUBSETSUM and the other is from HAMILTONIANCYCLE. For completeness, we present both problems' definitions.

SUBSETSUM is the problem of deciding, on input a set of integers U and an integer t, whether there exists $S \subseteq U$ such that $\sum_{s \in S} s = t$.

HAMILTONIANCYCLE is the problem of deciding, on input a directed graph G, whether there exists a simple cycle in G that visits every vertex.

We first present the reduction from SUBSETSUM. Let $(U = \{a_1, a_2, \ldots a_n\}, t)$ be an instance of SUBSETSUM. We create an instance (G, w) of 577-CYCLE as follows:

1. Start with a vertex s.
2. For each $1 \leq i \leq n$, create two vertices u_i and v_i. Add the edges (u_i, u_{i+1}) with weight a_i, (u_i, v_{i+1}) with weight 0, (v_i, u_{i+1}) with weight a_i and (v_i, v_{i+1}) with weight 0.
3. Add the edges (s, u_1) and (s, v_1), both with weight $577 - t$.
4. Add the edges (u_n, s) and (v_n, s), both with weight 0.

Once (G, w) is constructed, call the black-box for 577-CYCLE with it as input and output whatever it does.

The idea is that choosing a vertex u_i to be part of the cycle is equivalent to including the item a_i in the set S, while choosing the vertex v_i is equivalent to not including the item a_i in S. To establish correctness for this reduction, we prove the statement "$(U = \{a_1, a_2, \ldots a_n\}, t)$ is a true instance of SUBSETSUM *if and only if* (G, w) is a true instance of 577-CYCLE".

Assume $(U = \{a_1, a_2, \dots a_n\}, t)$ is a true instance of SUBSETSUM and let $S \subseteq U$ be such that $\sum_{s \in S} s = t$. We use S to construct a solution $C = (x_0, x_1, \dots, x_n, x_0)$ for the instance (G, w) of 577-CYCLE as follows:

1. Let $x_0 = s$.
2. For every i, if $a_i \in S$, then let $x_i = u_i$, and if $a_i \notin S$, then let $x_i = v_i$.

Note that C is indeed a simple cycle and that the total weight of C is equal to $577 - t + \sum_{a_i \in S} a_i = 577 - t + t = 577$. We conclude that (G, w) is a true instance of 577-CYCLE.

For the other direction, assume that (G, w) is a true instance of 577-CYCLE and let C be a solution for it. Note that if we remove the vertex s from G, then it is a directed acyclic graph (DAG). Because of this, we may assume that C starts and ends at s. Not only that, because of the way G is set up we can assume that C is of the form $(s, x_1, x_2, \dots, x_n, s)$ where $x_i \in \{u_i, v_i\}$ for all i. Again, we use C to construct a solution S for the instance $(U = \{a_1, a_2, \dots a_n\}, t)$ of SUBSETSUM. For every i, if $x_i = u_i$, then include a_i in S, and if $x_i = v_i$, then do not include a_i in S. Note that because C starts at s and the first edge it takes has weight $577 - t$, its total weight is exactly

$$577 = 577 - t + \sum_{a_i \in S} a_i.$$

Which means that $\sum_{a_i \in S} a_i = t$ and thus $(U = \{a_1, a_2, \dots a_n\}, t)$ is a true instance of SUBSETSUM.

The reduction from HAMILTONIANCYCLE is as follows: let G be an instance of (directed) HAMILTONIANCYCLE and let n be the number of vertices in G. We create an instance (G', w) of 577-CYCLE as follows:

1. Initially, let G' be a copy of G.
2. Select an arbitrary vertex s in G', set the weight of edges leaving s to $577 + n - 1$ and the weight of edges entering s to -1.
3. Set the weight of the remaining edges in G to -1.

Once (G', w) is constructed, call the black-box for 577-CYCLE with it as input and output whatever it does.

The idea is that any solution for (G', w) needs to go through vertex s and accumulate weight $577 + n - 1$ since this is the only one with a positive value. In that case, the solution would need to visit every other vertex to make up for the extra weight and get back to 577. As the solution for (G', w) must be a simple cycle, this guarantees that it is indeed a Hamiltonian cycle in the original graph G. To establish correctness for this reduction, we prove the statement "G is a true instance of HAMILTONIANCYCLE *if and only if* (G', w) is a true instance of 577-CYCLE".

Assume G is a true instance of HAMILTONIANCYCLE and let C be a simple cycle in G that visits every vertex. We claim that C itself is also a solution for the instance (G', w) of 577-CYCLE. Indeed, C must visit the vertex s, so it has weight $577 + n - 1 -$ |# of other vertices in C|. As C visits every vertex in G, this adds up to exactly 577, and we conclude that (G', w) is a true instance of 577-CYCLE.

For the other direction, assume that (G', w) is a true instance of 577-CYCLE and let C be a simple cycle in G' with total weight exactly 577. Because s is the only vertex with positive weight, we can assume that C starts and ends at s. Again, we argue that C itself is also a solution for the instance G of HAMILTONIANCYCLE. As before, the total weight of C is $577 = 577 + n - 1 -$ |# of other vertices in C|. Thus, we conclude that C visits every vertex in G, and that G is a true instance of HAMILTONIANCYCLE.