Assignment 4 Written Solution

Problem 1 | **Problem 2** | **Problem3** | **Problem 4**

Problem 1:

Proof by contradiction: Suppose that 49 jelly beans are picked at random from a bowl containing red, white, purple, yellow, orange, pink, black, and green jelly beans. Assume that it is not the case that at least 7 jelly beans must be the same color.**(*)**

If it is not the case that at least 7 jelly beans of the same color, that means there are 6 or fewer jelly beans of each color. Since there are 8 different colors, the maximum number of jelly beans is $8 \times 6 = 48$.

This contradicts the statement that there are 49 jelly beans. Therefore, the assumption **(*)** must be false and it must be the case that at least 7 jelly beans must be the same color. ■

Problem 2:

Proof: To prove that the three statements are equivalent, we need to prove that (i) ⇔ (ii) and (ii) ⇔ (iii).

To prove (i) ⇔ (ii), we break it into two parts:

- Prove (i) ⇒ (ii): if $3n - 8$ is odd, then n is odd.
- Prove (ii) ⇒ (i): if n is odd, then $3n - 8$ is odd.

Proof (i) ⇒ (ii): To prove if $3n - 8$ is odd, then n is odd, we prove the contrapositive: if n is even, then $3n - 8$ is even.

> Suppose n is even. Then $n = 2k$ for some integer k.
> Then, $3n - 8 = 3(2k) - 8 = 6k - 8 = 2(3k - 4)$
> Since k is an integer, so is $3k - 4$ (by closure).
> Thus, $3n - 8 = 2m$ for integer $m = 3k - 4$ so, by definition, $3n - 8$ is even.

Proof (ii) ⇒ (i): We use a direct proof.

> Suppose n is odd. Then $n = 2k + 1$ for some integer k.
> Then, $3n - 8 = 3(2k + 1) - 8 = 6k - 5 = 6k - 6 + 1 = 2(3k - 3) + 1$
> Since k is an integer, so is $3k - 3$ (by closure).
> Thus, $3n - 8 = 2m + 1$ for integer $m = 3k - 3$ so, by definition, $3n - 8$ is odd.

Having proved both directions of the biconditional, we have shown that $3n - 8$ is odd if and only if n is odd.

To prove (ii) ⇔ (iii), we break it into two parts:

- Prove (ii) ⇒ (iii): if n is odd, then $n^2 + 3$ is even.
- Prove (iii) ⇒ (ii): if $n^2 + 3$ is even, then n is odd.

Proof (ii) ⇒ (iii): We use a direct proof.

> Suppose n is odd. Then $n = 2k + 1$ for some integer k.
> Then, $n^2 + 3 = (2k + 1)^2 + 3 = 4k^2 + 4k + 1 + 3 = 4k^2 + 4k + 4 = 2(2k^2 + 2k + 2)$
> Since k is an integer, so is $2k^2 + 2k + 2$ (by closure).
> Thus, $n^2 + 3 = 2m$ for integer m = so, by definition, $n^2 + 3$ is even.

Proof (iii) ⇒ (ii): To prove if $n^2 + 3$ is even, then n is odd, we prove the contrapositive: if n is even, then $n^2 + 3$ is odd.

> Suppose n is even. Then $n = 2k$ for some integer k.
> Then, $n^2 + 3 = (2k)^2 + 3 = 4k^2 + 3 = 4k^2 + 2 + 1 = 2(2k^2 + 1) + 1$
> Since k is an integer, so is $2k^2 + 1$ (by closure).
> Thus, $n^2 + 3 = 2m + 1$ for integer $m = 2k^2 + 1$ so, by definition, $n^2 + 3$ is odd.

Having proved both directions of the biconditional, we have shown that n is odd if and only if $n^2 + 3$ is even.

We have proven that (i) ⇔ (ii) and (ii) ⇔ (iii) and thus (i), (ii), and (iii) are equivalent. ■

Problem 3:

Proof by cases: We break the problem up into cases based on who is the truth-teller and which path is the correct path.

Case 1 : Beck tells the truth and is by the path that leads to the restroom; Heck lies and is by the path that leads to the swamp.

- If Beck is asked "Will the other person tell me that your path leads to the restroom?" Beck will answer "No".
- If Heck is asked "Will the other person tell me that your path leads to the restroom?" Heck will answer "Yes".

Case 2 : Beck tells the truth and is by the path that leads to the swamp; Heck lies and is by the path that leads to the restroom.

- If Beck is asked "Will the other person tell me that your path leads to the restroom?" Beck will answer "Yes".
- If Heck is asked "Will the other person tell me that your path leads to the restroom?" Heck will answer "No".

Case 3 : Heck tells the truth and is by the path that leads to the restroom; Beck lies and is by the path that leads to the swamp. (note this is the same as case 1 with Beck and Heck reversed)

- If Beck is asked "Will the other person tell me that your path leads to the restroom?" Beck will answer "Yes".
- If Heck is asked "Will the other person tell me that your path leads to the restroom?" Heck will answer "No".

Case 4 : Heck tells the truth and is by the path that leads to the swamp; Beck lies and is by the path that leads to the restroom. (note this is the same as case 2 with Beck and Heck reversed)

- If Beck is asked "Will the other person tell me that your path leads to the restroom?" Beck will answer "No".
- If Heck is asked "Will the other person tell me that your path leads to the restroom?" Heck will answer "Yes".

In all cases, the person who is on the path to the restroom answers "No" and the person who is on the path to swamp answers "Yes".

So, if the person you ask answers "Yes", you should take the other path (i.e., the path they are not next to); if the person you ask answers "No", you should take the path they are next to. ■

Problem 4:

Proof by induction:

Let **P(*n*)** : $\frac{0}{1!}+\frac{1}{2!}+\frac{2}{3!}+\ldots+\frac{n}{(n+1)!}=1-\frac{1}{(n+1)!}$

We will show that P(*n*) holds for every $n \in \mathbb{N}$ using induction.

Base case: Show P(0) holds.

On the left side of the equation, we have $\frac{0}{1!}=0$ and on the right side of the equation, we have $1-\frac{1}{(0+1)!}=1-\frac{1}{1!}=0$

Since both sides are equal, P(0) holds.

Inductive step: Show P(*k*) ⇒ P(*k* + 1)

Induction hypothesis: P(*k*) holds, that is, $\frac{0}{1!}+\frac{1}{2!}+\frac{2}{3!}+\ldots+\frac{k}{(k+1)!}=1-\frac{1}{(k+1)!}$

Starting with the left side of the equation for P(*k* + 1):

$$\frac{0}{1!}+\frac{1}{2!}+\frac{2}{3!}+\ldots+\frac{k}{(k+1)!}+\frac{k+1}{((k+1)+1)!}$$

$$=\left(\frac{0}{1!}+\frac{1}{2!}+\frac{2}{3!}+\ldots+\frac{k}{(k+1)!}\right)+\frac{k+1}{(k+2)!}$$

$$= \left(1 - \frac{1}{(k+1)!}\right) + \frac{k+1}{(k+2)!} \quad \text{(by the induction hypothesis)}$$

$$= 1 - \left(\frac{k+2}{k+2} \cdot \frac{1}{(k+1)!}\right) + \frac{k+1}{(k+2)!}$$

$$= 1 + \frac{-(k+2)}{(k+2)!} + \frac{k+1}{(k+2)!}$$

$$= 1 + \frac{-k-2+k+1}{(k+2)!}$$

$$= 1 - \frac{1}{(k+2)!}$$

$$= 1 - \frac{1}{((k+1)+1)!}$$

We get to the right side of the equation and thus, P(k + 1) holds.

Therefore, by induction, $\frac{0}{1!} + \frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$ for every $n \in \mathbb{N}$. ■
