Week 2 Discussion Solutions

Properties of integers:

Assume a, b, and c are integers.

1. Even / odd
   - If a and b are both ***even***, then $a + b$ is **<u>even</u>** and $a \cdot b$ is **<u>even</u>**

     To see why, if a and b are both even, then $a = 2 \cdot k$ and $b = 2 \cdot j$ for some integers k and j. Then $a + b = (2 \cdot k) + (2 \cdot j) = 2 \cdot (k + j)$ and $a \cdot b = (2 \cdot k) \cdot (2 \cdot j) = 2 \cdot (k \cdot 2 \text{-} j)$.

   - If a and b are both ***odd***, then $a + b$ is **<u>even</u>** and $a \cdot b$ is **<u>odd</u>**

     To see why, if a and b are both odd, then $a = 2 \cdot k + 1$ and $b = 2 \cdot j + 1$ for some integers k and j. Then $a + b = (2 \cdot k + 1) + (2 \cdot j + 1) = 2 \cdot (k + j + 1)$ and $a \cdot b = (2 \cdot k + 1) \cdot (2 \cdot j + 1) = 4 \cdot k \cdot j + 2 \cdot k + 2 \cdot j + 1 = 2 \cdot (2 \cdot k \cdot j + k + j) + 1$.

2. 71 mod 6 is 5. If a mod b is 0 then b is a *divisor* of a

3. Justification if statement is true, counter-example if statement is false.
   a. If a divides $b \cdot c$, then a divides b or a divides c

      This statement is false. Let $a = 12$, $b = 3$, and $c = 8$. Then 12 divides 24 but 12 does not divide 3 and 12 does not divide 8.

   b. If a divides b and a divides c, then a divides $(b - c)$

      This statement is true. If a divides b, then $b = a \cdot k$ for some integer k. If a divides c, then $c = a \cdot m$ for some integer m. Then, $b - c = a \cdot k - a \cdot m = a \cdot (k - m) = a \cdot n$ for some integer $n = k - m$. So, by definition, a divides $(b - c)$.

   c. If $\gcd(a, b) = 1$, then a and b are prime

      This statement is false. Let $a = 3$ and $b = 4$. Then $\gcd(3, 4) = 1$ but 4 is not prime. However, if $\gcd(a, b) = 1$, then we say that a and b are *co-prime* (or *relatively prime*).

   d. If a and b are prime, then $\gcd(a, b) = 1$

      (Note that c. and d. are *converses* of each other.) This statement is false. Let $a = 7$ and $b = 7$, then $\gcd(7, 7) = 7$. Note that if we add in the restriction that $a \neq b$, then the statement "If a and b are prime and $a \neq b$, then $\gcd(a, b) = 1$" is true.

Propositional logic

1. 2^N Recall that a truth table specifies the truth value of a propositional formula for every possible combination of truth values for the N variables in the formula.

2. Show that $\neg P \Rightarrow Q$ is equivalent to $P \lor Q$:

P	Q	$\neg P \Rightarrow Q$	$P \vee Q$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

3. Show that $(P \Rightarrow Q) \wedge (\neg P \Rightarrow \neg Q)$ is equivalent to $P \Leftrightarrow Q$: One way to show this is to use a truth table (as was done for 2. above). Another way is to note that the *contrapositive* of $\neg P \Rightarrow \neg Q$ is $\neg(\neg Q) \Rightarrow \neg(\neg P)$ which can be rewritten as $Q \Rightarrow P$. Thus, $(P \Rightarrow Q) \wedge (\neg P \Rightarrow \neg Q)$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$, which is equivalent to $P \Leftrightarrow Q$

4.

   a. XOR is the negation of $\Leftrightarrow$

   b. XOR is equivalent to $(A \wedge \neg B) \vee (\neg A \wedge B)$:

A	B	A XOR B	**(i)** $A \wedge \neg B$	**(ii)** $\neg A \wedge B$	**(i)** $\vee$ **(ii)**
T	T	F	F	F	F
T	F	T	T	F	T
F	T	T	F	T	T
F	F	F	F	F	F