Problem 1

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Proof.

Proof by contradiction: assume that there are 49 jelly beans picked at random from a bowl containing red, white, purple, yellow, orange, pink, black and green such that no seven jelly beans have the same color. We introduce eight variables: $x_1 \dots x_8$. The variable x_i is the number of jelly beans that have color red or white ... or green. x_1 is the number of jelly beans that have color red, x_8 is the number of jelly beans that have color greens, etc.

Since every jelly bean has one specific color, the number of jelly beans picked from the bowl must be $x_1+x_2+x_3+\dots+x_8$. By our assumption, there are no seven jelly beans have the same color, so each x_i is at most 6. Therefore the number of jelly beans picked from the bowl is at most 48:

$$
\begin{aligned}
& x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8 \\
& \leq 6+6+6+6+6+6+6+6 \\
& = 6*8 = 48.
\end{aligned}
$$

The fact that the number of jelly beans is at most 48 contradicts the fact that there are 49 jelly beans picked from the bowl. Therefore, there must be at least 7 jelly beans must be the same color. ∎

Problem 2

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i. $3n-8$ is odd
ii. n is odd
iii. n^2+3 is even

Proof: We assume n is an integer and then prove the equivalence:

n is odd if and only if $3n-8$ is odd

Proof: ~~then~~ n is odd implies that $3n-8$ is odd.

Since n is odd, by definition $n=2k+1$, for some integer k.

Plug $n=2k+1$ into $3n-8$ to get:

$$3n-8 = 3(2k+1)-8 = 6k+3-8 = 6k-5$$
$$= 2(3k)-5$$
$$= \cancel{2(3k-3)-5}$$
$$2(3k-3)-5+6$$
$$= 2(3k-3)+1$$

Since k is an integer, $3k-3$ is also an integer.

Since $3n-8 = 2m+1$,

where $m=3k-3$ is an integer, $3n-8$ is odd.

Proof: ~~that~~ $3n-8$ is odd implies n is odd. (using proof by contrapositive)

Assume: n is even.

So $n=2k$ for some integer k.

by definition Plugging $n=2k$ into $3n-8$ gives

$$3n-8 = \cancel{3(2k-8)}\,3(2k)-8$$
$$= 2(3k-4)$$

Since k is an integer, $3k-4$ is also an integer.

~~3k~~ $3n-8$ is equal to 2 times an integer.

Therefore, $3n-8$ is an even integer

so, the contrapositive, $3n-8$ is odd implies n is odd is also true.

Since (n is odd) $\Rightarrow$ ($3n-8$ is odd) and ($3n-8$ is odd) $\Rightarrow$ (n is odd), (n is odd) $\Leftrightarrow$ ($3n-8$ is odd)

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Problem 2

i. $3n-8$ is odd
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Proof: we assume n is an integer and then prove the equivalence:

n is odd if and only if n^2+3 is even

Proof: n is odd implies that n^2+3 is even

~~Since n is odd, $n=2k+1$, for some integer k.~~

Since n is odd, by definition $n=2k+1$, for some integer k, Plug $n=2k+1$ into n^2+3 to get:

$$n^2+3 = (2k+1)^2+3 = 4k^2+4k+4$$
$$=2(2k^2+2k+2)$$

Since k is an integer, $2k^2+2k+2$ is also an integer.

Since $n^2+3=2m$, where $m=2k^2+2k+2$ is an integer, n^2+3 is even. ■

Proof: n^2+3 is even implies n is odd. (using proof by contrapositive)

Assume n is even. So, $n=2k$ for some integer k.

Plug $n=2k$ into n^2+3 gives by definition $n^2+3=(2k)^2+3$
$$= 4k^2+3$$
$$=2(2k^2+1)+1$$

Since k is an integer, $2k^2+1$ is also an integer.

Since $n^2+3 = 2m+1$, where $m=2k^2+1$ is an integer, n^2+3 is odd.

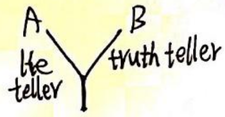
So, the contrapositive, n^2+3 is even implies n is odd is also true.

Since $(n \text{ is odd}) \Rightarrow ((n^2+3) \text{ is even})$ and $((n^2+3) \text{ is even}) \Rightarrow (n \text{ is odd})$,

$(n \text{ is odd}) \Leftrightarrow ((n^2+3) \text{ is even})$ ■

Since $(i) \Leftrightarrow (ii)$ and $(ii) \Leftrightarrow (iii)$, these three statements are equivalent.

Problem 3:



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Case 1: I ask the lie teller

① Answer: "Yes"

Since I ask the lie teller, it is false that the truth teller will tell me A leads to ~~restaurant~~ restroom, so path A is not the way to ~~restauran~~ restroom. Thus, path B leads to restroom, which is the opposite way that ~~stand stan~~ the person I asked stand.

② Answer: "No"

Since I ask the lie teller, it is false that the truth person will tell that path A, which is the path lie teller stands, can lead to restroom. Thus, truth teller will tell me that A will lead to restroom. Therefore, this time the way to restroom is the path that the person I asked stand.

Case 2: I ask the truth teller

① Answer: "Yes"

Since I ask the truth teller, so it's true that lie teller will tell me that the path B, which is the road truth teller stands is the way to restroom. However, since the lie teller tells lie, so the way to restroom is path A, which is the opposite way that the person I asked stands.

② Answer: "No"

Since I ask the truth teller, so ~~the~~ it's true that lie teller will tell me that path B is not the way to restroom. Since lie teller tells lie, so, path B is the way to restroom. Thus, I take the way that the person I asked stands.

Therefore, if the person answers "Yes", I should take the opposite way the asked person stands.

Problem 4

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Proof: ~~Theorem~~: For every $n \in N$

Let $P(n)$: $\frac{0}{1!} + \frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

~~Proof~~. $\sum_{j=0}^{n} \frac{j}{(j+1)!} = 1 - \frac{1}{(n+1)!}$

By induction on n.

~~Let $P(n)$: $\frac{0}{1!} + \frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{n}{(n+1)!}$~~

Base case: $n = 0$. We want to prove $\sum_{j=0}^{0} \frac{j}{(j+1)!} = 1 - \frac{1}{(0+1)!}$

when $n=0$ the left side of the equation is $\sum_{j=0}^{0} \frac{j}{(j+0)!} = 0$

when $n=0$, the right side of the equation is $1 - \frac{1}{(0+1)!} = 0$

Therefore, $\sum_{j=0}^{1} \frac{j}{(j+1)!} = 1 - \frac{1}{(1+1)!}$

Inductive step: suppose that for positive integer k, $\sum_{j=0}^{k} \frac{j}{(j+1)!} = 1 - \frac{1}{(k+1)!}$

then ~~for~~ we will show that $\sum_{j=0}^{k+1} \frac{j}{(j+1)!} = 1 - \frac{1}{(k+2)!}$

Proof:

Starting with the left side of the equation to be proven:

$$\sum_{j=0}^{k+1} \frac{j}{(j+1)!} = \sum_{j=0}^{k} \frac{j}{(j+1)!} + \frac{k+1}{(k+2)!} \quad \text{by separating out the last item}$$

$$= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} \quad \text{by the inductive hypothesis}$$

$$= \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+2)!}$$

$$= \frac{((k+1)! - 1)(k+2) + k+1}{(k+2)!}$$

$$= \frac{(k+2)! - 1}{(k+2)!} = 1 - \frac{1}{(k+2)!} \quad \text{by algebra}$$

Conclusion: Therefore, by induction, $P(n)$ holds for every $n \in N$

$$\frac{0}{1!} + \frac{1}{2!} + \frac{2}{3!} + \ldots \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$ ∎