CS 577: Introduction to Algorithms	**Midterm 2 Solutions**
Out: 03/30/21	**Due: 04/06/21**

Name: ______________________ Wisc ID: ______________________

Problem 1 [70 points]

Before Thanos gathered all the infinity stones, there was a time when he had to travel the galaxies via his spaceship. Being the very meticulous person he is, Thanos plans to visit planets $1, 2, \ldots, n$ in order and wants to minimize the cost he has to spend on fuel. Being cursed with knowledge, he knows the price of fuel at each planet to plan his itinerary with minimal cost.

Formally, let c_i be the cost of fuel (per ton) at planet i and let d_i be the distance in light-years from planet 1 to planet i. You may assume all the planets are on a linear path such that the distance between planet i and j is equal to $|d_i - d_j|$ light-years. The spaceship can hold up to T tons of fuel and can travel one light-year for one ton of fuel. Unfortunately, there is a flat convenience fee p_i one has to pay to buy fuel. So if Thanos were to buy 5 tons on planet i, it would cost $p_i + 5c_i$, but if he decides to skip on getting fuel here, it would cost nothing.

A. [40 points] Give a dynamic programming recurrence that returns the minimum cost Thanos has to spend on his trip. Thanos starts his trip at planet 1 with an empty fuel tank. He shouldn't run out of fuel between traveling planets but he doesn't have to fill up all T tons at once and he may even skip on getting fuel on a planet. You may assume T and d_i's are positive integers and T is large enough to travel between any two adjacent planets.

Solution:

Note that Thanos does not need to pump any non-integer amount of fuel at any planet because the distances between planets are always integers and one ton can travel one light-year. Then we can devise an recurrence relation as follows. Let $\mathsf{OPT}(i, t)$ be defined as the minimum cost to travel from planet i to n, starting with t tons of fuel at planet i where $t \in \{0, 1, \ldots, T\}$. Let t' be the amount of fuel bought at planet i plus t so that the money spent at planet i is $c_i(t' - t) + p_i$ if $t' - t > 0$ and 0 if not. Then the recurrence for OPT is

$$\mathsf{OPT}(i, t) = \min_{t \le t' \le T} \begin{cases} p_i + c_i(t' - t) + \mathsf{OPT}(t' - |d_{i+1} - d_i|, i + 1) \text{ if } t' - t > 0 \\ \mathsf{OPT}(t - |d_{i+1} - d_i|, i + 1) \text{ if } t' - t = 0 \end{cases} \tag{1}$$

The base cases are $\forall t, \mathsf{OPT}(n, t) = 0$ and $\forall i, \mathsf{OPT}(t, i) = \infty$ if $t < 0$. The last set of base cases may not be necessary if we only considered $\max(t, |d_{i+1} - d_i|) \le t' \le T$ in the recurrence. The final answer is $\mathsf{OPT}(1, 0)$. (Assuming that p_i and c_i can possibly be negative, for completeness, the base case can be $\mathsf{OPT}(n, t) = p_n + c_n(t' - t)$ if the value were negative. This was not expected from students since costs are usually positive.)

B. [30 points] Prove the correctness of the recurrence and analyze its runtime in terms of n and T.

Solution:

Proof of Correctness We use mathematical induction by inducting on the variable i from n to 1.

Base case ($i = n$): Thanos is already at planet n so there is no need to spend more. Thus $\mathsf{OPT}(n, t) = 0$.

Inductice Hypothesis ($i \geq k$): Assume $\mathsf{OPT}(i, t)$ to be the minimum cost to travel from planet i to n, starting with t tons of fuel at planet i.

Inductive step ($i = k-1$): Suppose we are to compute $\mathsf{OPT}(i, t)$. At planet i, Thanos can either purchase fuel or not. If he does not buy fuel, then there is no money to spend at planet i and he will travel to planet $i + 1$ with $t - |d_{i+1} - d_i|$ tons left in the tank. Then the minimum cost is equal to $\mathsf{OPT}(i+1.t - |d_{i+1} - d_i|)$ by the inductive hypothesis. If he does buy fuel, the amount he buys determines how much fuel he starts with at planet $i + 1$. So if he fills up to t' tons, this costs $p_i + c_i(t' - t)$. Then Thanos will arrive at planet $i + 1$ with $t' - |d_{i+1} - d_i|$ tons left. Thus, the minimum cost in this case is equal to $p_i + c_i(t' - t) + \mathsf{OPT}(i + 1, t' - |d_{i+1} - d_i|$. Since we consider all the choices that could be made (skipping or buying $t' - t$ tons of fuel for $t \leq t' \leq T$), we minimize over all cases. Therefore, this recurrence relation provides the minimum cost from planet i to n with t tons of fuel at planet i.

Runtime Analysis OPT creates a n by $T + 1$ matrix so there are $O(nT)$ subproblems to solve, while each subproblem must consider at most $T + 1$ cases in its recurrence relation. Since it takes $O(T)$ time to compute $\mathsf{OPT}(i, t)$ according to the Bellman equation, the runtime is $O(nT^2)$.

Problem 2 [30 points]

This question is about a reality TV show Random Idol, which runs as follows. The show has two rounds and n contestants. In each of the rounds, the contestants participate in a lottery contest that ranks them from best to worst uniformly at random.[1] A contestant i is declared a random idol if for **every other** contestant j, i beats j (i.e. has a better rank than j) in **at least one** of the two rounds.

Any number of participants can be declared random idols. For example, if $n = 4$ and the first round ordering over participants is $1, 2, 3, 4$ in order from best to worst, and the second round ordering is $3, 1, 4, 2$, then participants 1 and 3 are declared random idols.[2] If the orderings in the two rounds are $1, 2, 3, 4$ and $4, 3, 2, 1$, then all of the participants are declared random idols.

For each of the following parts, choose *one* of the given options.

1. What is the probability that a particular contestant i is ranked best in *both* of the first two rounds?

   ◯ $1/(\log n)$ ◯ $1/i$ ◯ $1/n$ ◯ $1/n^2$ ◯ $1/(n!)^2$

   Provide a short proof justifying your answer.

   Solution:

   The correct answer is $1/n^2$. Since we choose a uniformly random permutation of the players the probability that someone is ranked first in the first round is $1/n$. The probability that the same player wins the second round is again $1/n$. Since the rounds are independent we have that the probability that some player wins in both rounds is $1/n^2$.

[1] That is, every ranking/permutation is equally likely.

[2] 2 does not beat 1 in any of the rounds, and 4 does not beat 1 or 3.

2. For some r in $\{1, \cdots, n\}$ what is the probability that the contestant ranked rth best in the first round is declared a random idol at the end of the second round?

   ◯ $1/n$ ◯ $1/r$ ◯ r/n ◯ $1/(r!)$ ◯ $(r!)/(n!)$

   Provide a short proof justifying your answer.

   Solution:

   The correct answer is $1/r$. A player that was the r-th best in the first round has to win the r players the were in front of them in the second round. When we condition on the positions of $n - r$ the players that were behind our player we obtain that the ranking of the remaining r players is again a uniformly random permutation. Therefore, the probability that our player is ranked best among the r players is $1/r$ and does not depend on the ordering of the remaining $n - r$ players.

3. What is the expected number of contestants that are declared random idols?

   ◯ $\Theta(1)$ ◯ $\Theta(\log n)$ ◯ $\Theta(\sqrt{n})$ ◯ $\Theta(n)$

   Provide a short proof justifying your answer.

   Solution:

   The correct answer is $\Theta(\log n)$. Define indicator random variables X_i to be 1 if player i is declared random idol and 0 otherwise. The total number of random idols is $X = \sum_{i=1}^{n} X_i$. From linearity of expectation we have $E[X] = \sum_{i=1}^{n} P[X_i = 1]$. From part 2 we know that

$$
\begin{aligned}
P[X_i = 1] &= \sum_{r=1}^{n} P[X_i = 1 | \text{player } i \text{ is } r\text{-th in first round}] \cdot P[\text{player } i \text{ is } r\text{-th in first round}] \\
&= \sum_{r=1}^{n} \frac{1}{r}\frac{1}{n}.
\end{aligned}
$$

   Therefore, $E[X] = \sum_{i=1}^{n} P[X_i = 1] = \sum_{r=1}^{n} \frac{1}{r} = \Theta(\log(n))$.