CS 577 - Randomized Algorithms

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WISCONSIN
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QuickSort

Recall: Linear Time Selection

Problem

Find the kth value in an unsorted array A of n numbers if A were sorted.

Recall: Linear Time Selection

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Find the kth value in an unsorted array A of n numbers if A were sorted.

Algorithm: QuickSelect

Input : A array $A[1..n]$ and an int k.
Output: The kth element of A if A were sorted.

if $n = 1$ **then return** $A[1]$
Choose a pivot $A[p]$
$r := \text{Partition}(A[1..n], p)$
if $k < r$ **then**
    return *QuickSelect*$(A[1..r-1], k)$
else if $k > r$ **then**
    return *QuickSelect*$(A[r+1..n], k-r)$
else
    return $A[r]$
end

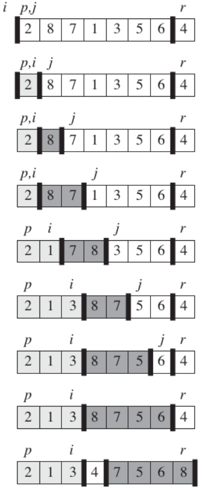
QuickSort

Algorithm: QuickSort

Input : An array $A[1..n]$.
Output: A sorted from 1 to n.
Choose a pivot $A[p]$
$r := \textsc{Partition}(A[1..n], p)$
$\textsc{QuickSort}(A[1..r-1])$
$\textsc{QuickSort}(A[r+1..n])$
return A

QuickSort

QuickSort partition step:



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Why no combine step?

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Why no combine step?

Because QuickSort sorts in-place.

QuickSort

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TopHat 1: What is the complexity of the partition step?

QuickSort

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TopHat 1: What is the complexity of the partition step? $O(n)$.

QuickSort Analysis

Worst Case

Algorithm: QuickSort

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TopHat 2: What is the worst-case recurrence?

QuickSort Analysis

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Worst-case recurrence

$$
\begin{aligned}
T(n) &\leq T(n-1) + T(0) + O(n) \\
&\leq T(n-2) + 2T(0) + 2O(n) \\
&\leq n(T(0) + O(n)) \\
&= O\left(n^2\right)
\end{aligned}
$$

QuickSort Analysis

Best Case

Algorithm: QuickSort

Input : An array $A[1..n]$.
Output: A sorted from 1 to n.
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TopHat 3: What is the best-case recurrence?

QuickSort Analysis

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Best-case recurrence

$$T(n) \leq 2T(n/2) + O(n)$$
$$= O(n \log n)$$

QuickSort Analysis

Average Case

Observation 1

For $0 < \varepsilon < 1$,

$$\begin{aligned} T(n) &= T(\varepsilon n) + T((1-\varepsilon)n) + \Theta(n) \\ &= \Theta(n \log n) \end{aligned}$$

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Probabilistic Argument

Expected Runtime:

$$T(n) \leq \Pr[\Theta(n) \text{ split}] \cdot \Theta(n \log n) + \Pr[o(n) \text{ split}] \cdot \Theta(n^2)$$

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QuickSort Analysis

Average Case

Average Case Recurrence (uniform dist on orderings)

$$T(n) \leq \frac{1}{n}\sum_{i=1}^{n}(T(i-1) + T(n-i)) + O(n)$$

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- Improve QuickSort by more complicated pivot choice.

QuickSort with MomPivot

Algorithm: QuickSort

Input : An array $A[1..n]$.
Output: A sorted from 1 to n.
Choose a pivot $A[p]$ using MomPivot
$r := \textsc{Partition}(A[1..n], p)$
$\textsc{QuickSort}(A[1..r-1])$
$\textsc{QuickSort}(A[r+1..n])$
return A

MomPivot Recurrence Worst-Case

$$\begin{aligned} T(n) &\leq T(7n/10) + T(3n/10) + O(n) \\ &= O(n \log n) \end{aligned}$$

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- Improve QuickSort by more complicated pivot choice.
- What would be an easy way to get this average case performance? UAR choose the pivot.

Randomized Algorithms

Randomization and Algorithms

Random Input

- Average Case analysis:
  - Input is drawn from some distribution π.
  - Under distribution π, average run-time, memory, etc...
- We saw an example when we analyzed QuickSort for a uniform distribution.

Randomization and Algorithms

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Randomized Algorithms

- Algorithm flips a coin to make some decisions.

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Randomized Algorithms

- Algorithm flips a coin to make some decisions.
- Non-Deterministic: simultaneously considers multiple algorithms weighted by the probability distribution.

Randomized Algorithms

Types of Randomized Algorithms:

Monte Carlo

- With probability p returns the correct answer:

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Atlantic City

- Probabilistic run-time and correctness.

Randomization and Approximation

Guarantee in Expectation

Returns a solution that has a r approximation ratio in expectation:

$$\forall I, \mathbb{E}[\textsc{alg}(I)] \leq r \cdot \textsc{opt}(I) + \eta$$

Probability Review / Primer

Probability Space

- *Sample space* Ω of all possible outcomes.
    - Can be infinite, but we will focus on finite.
    - Ex: 4-sided die (D4): $\Omega = \{1, 2, 3, 4\}$.

Probability Review / Primer

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TH: Pr[Roll 2, 3, or 4 on a fair D4] = 3/4

Conditional Probability and Independence

Conditional Probability

Probability of ε given $\mathcal{F}$.

$$\Pr[\varepsilon|\mathcal{F}] = \frac{\Pr[\varepsilon \cap \mathcal{F}]}{\Pr[\mathcal{F}]}$$

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- Events ε and $\mathcal{F}$ are independent if $\Pr[\varepsilon|\mathcal{F}] = \Pr[\varepsilon]$ and $\Pr[\mathcal{F}|\varepsilon] = \Pr[\mathcal{F}]$.

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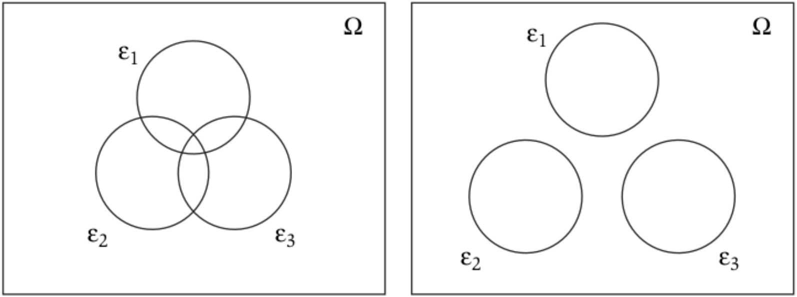
- This implies $\Pr[\varepsilon \cap \mathcal{F}] = \Pr[\varepsilon] \cdot \Pr[\mathcal{F}]$.
- Generalization: Say $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are independent.

$$\Pr\left[\bigcap_{i=1}^{n} \varepsilon_i\right] = \prod_{i=1}^{n} \Pr[\varepsilon_i]$$

Union Bound



Union Bound

$$\Pr\left[\bigcup_{i=1}^{n} \varepsilon_i\right] \leq \sum_{i=1}^{n} \Pr[\varepsilon_i],$$

where equality *only* if events are mutually exclusive.

Random Variables and Expectation

Random Variables

- Technical: Given a probability space, a random variable X is a function from the sample space to the natural (finite – real if infinite) numbers, such that, for number j, $X^-1(j)$ is the set of all sample points taking the value j is an event.

Ex: $\Pr[X = 1] = 1/4$, where X is a toss of a 4-sided die.

Random Variables and Expectation

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- Informally: A random variable X takes on a value that depends on a random process.

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Expectation Properties

Let X and Y be random variables, and a be a constant.

- Linearity of expectation:
  - $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
  - $\mathbb{E}[aX] = a\,\mathbb{E}[X]$

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- If X and Y are independent, $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$.

Random QuickSort

QuickSort with Random Pivot

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Input : An array $A[1..n]$.
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Expected Runtime (Pivot UAR)

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\begin{aligned}
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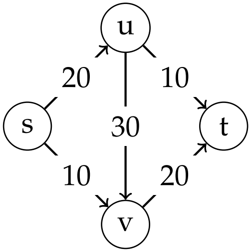
Min-Cut

Random Min-Cut



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- We will see a polynomial time algorithm (flows).

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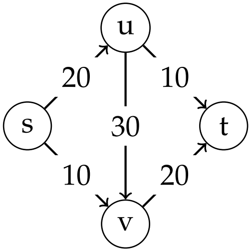

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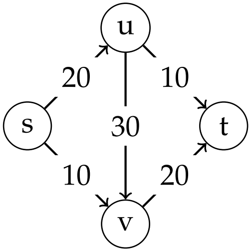
- We will see a polynomial time algorithm (flows).
- Because:
  - Nice example of a Monte Carlo algorithm.
  - Has a good run-time for dense graphs.

Random Min-Cut



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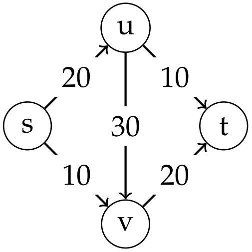

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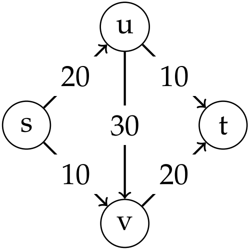

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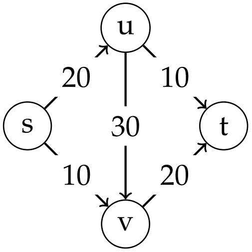

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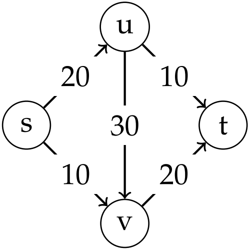

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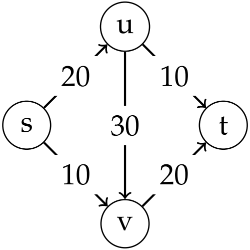

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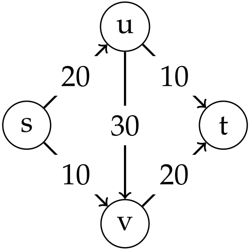

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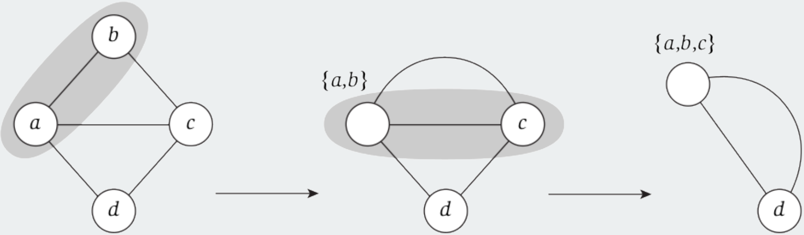
- (u, v) edge contraction:
  - create a supernode $\{u, v\}$



Karger's Algorithm

Algorithm: Contraction Algorithm

Input : Multigraph $G = (V, E)$
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if *G has exactly 2 nodes u and v* **then**
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Theorem 1

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- There are $n-2$ steps in Contraction Algorithm.

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Multiple Runs

- With $\binom{n}{2}$ runs, we get:

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Definition

A function that converts some input value into a hash value.

- Input: A large universe of values U. Typically, assume $|U| \gg n$.
- Output: A hash value for $u \in U$ to $\{0, 1, 2, \ldots, n-1\}$.

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Why?

Typically used to generate keys for a dictionary data structure.

Dictionary Data Structure

Dictionary

- Storage of a subset of values from U.
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- $u \mod n$: Risk of collision can be large especially if say n is a power of 2.
- $u \mod p$, where p is a prime: Less risk than n especially if p is not tiny, but $p \approx n$.

Random Hash Function

$h(x)$: Return a value from 0 to $n-1$ UAR.

Lemma 2

Given $h(x)$, the probability that $h(u) = h(v)$ for any $u, v \in U$ is [TopHat]

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What is the problem with this random hash function?
For a dictionary, Delete(u) and Lookup(u) won't work since $h(u)$ returns a random value!

Universal Class of Hash Functions

Randomly choosing a hash function

Definition

Let $\mathcal{H}$ be a class of functions such that:

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Let $\mathcal{H}$ be a universal class of hash functions mapping U to $[0..n-1]$. Let $S \subseteq U$ be of size $\leq n$. The expected number of elements $s \in S$ where $h(s) = h(u)$ for any $u \in U$ when h is chosen UAR from $\mathcal{H}$ is ≤ 1.

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- Let $X = \sum_{s \in S} X_s$.

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Randomly choosing a hash function

Theorem 3

Let $\mathcal{H}$ be a universal class of hash functions mapping U to $[0..n-1]$. Let $S \subseteq U$ be of size $\leq n$. The expected number of elements $s \in S$ where $h(s) = h(u)$ for any $u \in U$ when h is chosen UAR from $\mathcal{H}$ is ≤ 1.

Proof.

- Fix $u \in U$. Let X_s be a random variable that is 1 if $h(s) = h(u)$; 0 otherwise.
- Let $X = \sum_{s \in S} X_s$.

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{s \in S} X_s\right] = \sum_{s \in S} \mathbb{E}[X_s]$$

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Designing a Universal Class of Hash Functions

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- Let $\mathcal{A}$ be the set of all vectors of the form $a = (a_1, a_2, \ldots, a_r)$, where $0 \leq a_i < p$.
- $\mathcal{H}$ contains $h_a(x) = \left(\sum_{i=1}^{r} a_i x_i\right) \mod p$ for all $a \in \mathcal{A}$.

Analyze our definition of $\mathcal{H}$

Lemma 4 (Technical Lemma)

For any prime p and any integer $z \not\equiv 0 \mod p$, and any two integers α, β, if $\alpha z \equiv \beta z \mod p$, then $\alpha \equiv \beta \mod p$.

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- Lemma 4 shows there is a single value for a_j to satisfy (1).
- So, $\Pr[h_a(x) = h_a(y)] \leq \frac{1}{p}$.

□

MAX SAT

Satisfiability Problem (SAT)

$$(x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_2 \vee \overline{x_3})$$

Preliminaries

- A set of boolean terms/literals: $X : x_1, \ldots, x_n$.

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Satisfiability Problem (SAT)

TH: What values will satisfy the example?

$$(x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_2 \vee \overline{x_3})$$

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3SAT Problem

Given a set of literals: $X : x_1, \ldots, x_n$, and a collection of clauses $\mathcal{C} : C_1 \wedge C_2 \wedge \cdot \wedge C_k$, does there exist a satisfying assignment?

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Given a set of literals: $X : x_1, \ldots, x_n$, and a collection of clauses $\mathcal{C} : C_1 \wedge C_2 \wedge \cdot \wedge C_k$, each of length 3, does there exist a satisfying assignment?

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Given a 3SAT problem satisfying as many clauses as possible.

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TH: Suggest a randomized algorithm.

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Random Assignment

For each x_i, independently assign a value of 0 or 1 with probability $\frac{1}{2}$ each.

Analyze Random Assignment

Clause C_i

- Let Z_i be a random variable: 1 if clause is satisfied, 0 otherwise.

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- Each clause has 3 variables x_i each with $\Pr[x_i = 0] = \frac{1}{2}$:

$$\Pr[Z_i = 1] = 1 - \Pr[Z_i = 0] = 1 - \left(\frac{1}{2}\right)^3 = \frac{7}{8}$$

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Overall

Let $Z = \sum_{i=1}^{k} Z_i$:

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Overall

Let $Z = \sum_{i=1}^{k} Z_i$:

$$\begin{aligned}\mathbb{E}[Z] &= \mathbb{E}\left[\sum_{i=1}^{k} Z_i\right] \\ &= \mathbb{E}[Z_1] + \mathbb{E}[Z_2] + \cdots + \mathbb{E}[Z_k] \text{ , by Linearity of Expectation,} \\ &= \frac{7}{8}k\end{aligned}$$

Interesting Corollaries

Theorem 5

Random Assign satisfies 7/8 of the clauses in expectation.

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Proof.

Since the expectation is a weighted average, its value is between the maximum and minimum possible values. $\square$

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Every 3-SAT with ≤ 7 clauses is satisfiable.

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Every 3-SAT with ≤ 7 clauses is satisfiable.

Proof.

For $k \leq 7$, $\frac{7}{8}k > k - 1$. $\square$

Waiting For a Good Assignment

Theorem 8

There exists a randomized algorithm with a polynomial expectation running time that is guaranteed to produce a truth assignment satisfying at least 7/8 of all k clauses.

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- Let p_j be the probability that j clauses are satisfied.
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- By Definition of expectation:

$$\frac{7}{8}k = \sum_{j=0}^{k} jp_j = \sum_{j<\frac{7}{8}k} jp_j + \sum_{j \geq \frac{7}{8}k} jp_j$$

$$\leq \left(\frac{7k}{8} - \frac{1}{8}\right) \sum_{j<\frac{7}{8}k} p_j + k \sum_{j \geq \frac{7}{8}k} p_j$$

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$$\iff \frac{7}{8}k \leq \left(\frac{7k}{8} - \frac{1}{8}\right)(1-p) + kp \leq \left(\frac{7k}{8} - \frac{1}{8}\right) + kp$$

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- We need to calculate $p = \sum_{j \geq \frac{7}{8}k} p_j$.
- With $p = \frac{1}{8k}$, we have a Bernoulli trial:
  Within $8k$ tries, we expect an assignment that satisfies $\frac{7}{8}$ of the clauses.

$\square$

Appendix

References

Image Sources I

WISCONSIN UNIVERSITY OF WISCONSIN–MADISON https://brand.wisc.edu/web/logos/