Coursework (2) for *Introductory Lectures on Optimization*

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Excercise 1. univariate functions are in the set of $\mathcal{F}^1(\mathbb{R})$:

$$\begin{aligned}
&f(x) = e^x \\
&f(x) = |x|^p, p > 1 \\
&f(x) = \frac{x^2}{1+|x|} \\
&f(x) = |x| - \ln(1+|x|)
\end{aligned}$$

Proof of Excercise 1: □

Excercise 2. Prove that

$$0 \le f(\boldsymbol{y}) - f(\boldsymbol{x}) - \langle \nabla f(\boldsymbol{x}),\ \boldsymbol{y} - \boldsymbol{x} \rangle \le \frac{L}{2}\|\boldsymbol{x} - \boldsymbol{y}\|^2$$

holds if we have

$$0 \le \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}) - f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \le +\alpha(1-\alpha)\frac{L}{2}\|\boldsymbol{x} - \boldsymbol{y}\|^2.$$

Proof of Excercise 2: Since we have

$$\begin{aligned}
&0 \le \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}) - f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}), \\
\Longleftrightarrow\ & f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \le \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}).
\end{aligned} \tag{1}$$

According to Eq. 1, we can obtain that $f(\cdot)$ is a convex function. Based on the properties of convex function, we can get

$$\begin{aligned}
& f(\boldsymbol{y}) \ge f(\boldsymbol{x}) + \langle \nabla f(\boldsymbol{x}),\ \boldsymbol{y} - \boldsymbol{x} \rangle \\
\Longleftrightarrow\ & 0 \le f(\boldsymbol{y}) - f(\boldsymbol{x}) - \langle \nabla f(\boldsymbol{x}),\ \boldsymbol{y} - \boldsymbol{x} \rangle
\end{aligned} \tag{2}$$

Since we have

$$\begin{aligned}
&\alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}) - f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \\
&= \alpha(f(\boldsymbol{x}) - f(\boldsymbol{y})) + f(\boldsymbol{y}) - f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y})
\end{aligned} \tag{3}$$

□

Excercise 3. Prove that

$$f(\boldsymbol{x}) + \langle \nabla f(\boldsymbol{x}),\ \boldsymbol{y} - \boldsymbol{x} \rangle + \frac{1}{2L}\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\|^2 \le f(\boldsymbol{y})$$

holds if we have

$$\alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}) \ge f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) + \frac{\alpha(1-\alpha)}{2L}\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\|^2,$$

Proof of Excercise 3: bla.bla... bla bla.. bla. $\square$

Excercise 4. Let f be continuously differentiable. Prove that both conditions below, holding for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\alpha \in [0, 1]$, are equivalent to inclusion $\mathcal{S}^1_\mu(\mathbb{R}^n)$

$$\langle \nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y}),\ \boldsymbol{x} - \boldsymbol{y} \rangle \geq \mu \|\boldsymbol{x} - \boldsymbol{y}\|^2, \tag{4}$$

$$\begin{aligned} \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}) \geq\ & f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \\ & + \alpha(1-\alpha)\frac{\mu}{2}\|\boldsymbol{x} - \boldsymbol{y}\|^2. \end{aligned} \tag{5}$$

Proof of Excercise 4: bla.bla... bla bla.. bla. $\square$