Chapter 1

Nonlinear Optimization

In this chapter, we mainly focus on:

- Main notations and concepts used in *Continuous Optimization*.
- Complexity Analysis of the problems of Global Optimization.
- Local Minimization and 2 main methods: *the Gradient Method* and *the Newton Method*.
- Some standard methods in General Nonlinear Optimization.

1.1 The World of Nonlinear Optimization

1.1.1 General Formulation of the Problem

Let $\boldsymbol{x}$ be an n-dimensional *real vector*

$$\boldsymbol{x} = \left(x^{(1)}, \ldots, x^{(n)}\right)^\top \in \mathbb{R}^n$$

and $f_0(\cdot), \ldots, f_m(\cdot)$ be some *real-valued* functions defined on a set $Q \subset \mathbb{R}^n$. In this way, let's consider the general minimization problem:

$$\begin{aligned} &\min\ f_0(\boldsymbol{x}), \\ &\text{s.t. } f_j(\boldsymbol{x})\ \&\ 0, \quad j = 1, \ldots, m, \\ &\boldsymbol{x} \in Q, \end{aligned} \tag{1.1}$$

where the sign & can be $\leq$, $\geq$ or $=$.

Notations:

- f_0 is called *objective* function.
- The vector function $\boldsymbol{f}(\boldsymbol{x}) = \left(f_1(\boldsymbol{x}), \ldots, f_m(\boldsymbol{x})^\top\right)$ is called the vector of *functional constraints*.
- The set Q is called the *basic feasible set*.
- The set $\mathscr{F} = \{x \in Q \mid f_j(\boldsymbol{x}),\ j = 1, \ldots, m\}$ is called *the entire feasible set* of problem 1.1.

Classification

Natural Classification:

- *Constrained problems*: $\mathscr{F} \subset \mathbb{R}^n$.
- *Unconstrained problems*: $\mathscr{F} \equiv \mathbb{R}^n$.
- *Smooth problems*: all $f_j(\cdot)$ are differentiable.
- *Nonsmooth problems*: there are several nondifferentiable components $f_k(\cdot)$.
- *Linearly constrained problems*: the functional constraints are affine:

$$f_j(x) = \langle \boldsymbol{a}_j, \boldsymbol{x} \rangle + b_j$$

  - *Linear optimization Problem*: $f_0(\cdot)$ is also affine.
  - *Quadratic optimization problem*: $f_0(\cdot)$ is Quadratic.
  - *Quadratic constrained quadratic problem*: $f_0(\cdot), \ldots, f_m(\cdot)$ are all quadratic.

Classification Based on the Feasible Set:

- Problem 1.1 is called *feasible* if $\mathscr{F} \neq \emptyset$.
- Problem 1.1 is called *strictly* feasible if there exists an $\boldsymbol{x} \in Q$ such that $f_j(\boldsymbol{x}) < 0$ for all inequality constraints and $f_j(\boldsymbol{x}) = 0$ for all equality constraints. (*Slater condition.*)

Classification Based on Solution:

- A point $\boldsymbol{x}^* \in \mathscr{F}$ is called the optimal *global solution* to problem 1.1 if $f_0(\boldsymbol{x}^*) \leq f_0(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathscr{F}$ (*global minimum*). $f_0(\boldsymbol{x}^*)$ is called the global *optimal value* of the problem.
- A point $\boldsymbol{x}^* \in \mathscr{F}$ is called a *local solution* to problem 1.1 if there exists a set $\hat{\mathscr{F}} \subset \mathscr{F}$ such that $\forall \boldsymbol{x} \in \operatorname{int}\hat{\mathscr{F}},\ f_0(\boldsymbol{x}^*) \leq f_0(\boldsymbol{x})$. If $\forall \boldsymbol{x} \in \hat{\mathscr{F}} \setminus \{\boldsymbol{x}^*\},\ f_0(\boldsymbol{x}^*) \leq f_0(\boldsymbol{x})$, then $\boldsymbol{x}^*$ is called *strict* (or *isolated*) local minimum.