Smooth Convex Optimization

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1 Minimization of Smooth Functions

1.1 Smooth Convex Functions

In this section, we consider the unconstrained minimization problem

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} f(\boldsymbol{x}), \tag{1.1}$$

where the objective function $f(\cdot)$ is smooth enough. $\mathscr{F}$ represents *differentiable functions*.

Assumption 1.1 *For any $f \in \mathscr{F}$, the first-order optimality condition is sufficient for a point to be a global solution to 1.1* ♣

Assumption 1.2 *If $f_1,\ f_2 \in \mathscr{F}$ and $\alpha,\ \beta \geq 0$, then $\alpha f_1 + \beta f_2 \in \mathscr{F}$.* ♣

Assumption 1.3 *Any linear function $l(\boldsymbol{x}) = \alpha + \langle \boldsymbol{a}, \boldsymbol{x} \rangle$ belongs to $\mathscr{F}$.* ♣

Note that the linear function $l(\cdot)$ perfectly fits Assumption 1.1. Clearly, $\nabla l(\boldsymbol{x}) = 0$ implies that this function is constant, and any point $\mathbb{R}^n$ is its global minimum.

Definition 1.1 (Convex Set)
A set $Q \subseteq \mathbb{R}^n$ is called convex *if we have*

$$\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y} \in Q \quad \forall \boldsymbol{x},\ \boldsymbol{y} \in Q \text{ and } \forall \alpha \in [0,1].$$ ♣

Definition 1.2 (Convex Function)
A continuously differentiable *function $f(\cdot)$ is called* convex *on a convex set Q (notation $f \in \mathscr{F}^1(Q)$) if we have*

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \langle \nabla f(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle\,. \tag{1.2}$$

If $-f(\cdot)$ is convex, we call $f(\cdot)$ concave. ♣

Theorem 1.1
If $f \in \mathscr{F}^1(\mathbb{R}^n)$ and $\nabla f(\boldsymbol{x}^*) = 0$ then $\boldsymbol{x}^*$ is the global minimum of $f(\cdot)$ on $\mathbb{R}^n$. ♢

PROOF (OF THEOREM 1.1) In the inequality 1.2, for any $\boldsymbol{x} \in \mathbb{R}^n$ we have

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*) + \langle \nabla f(\boldsymbol{x}^*), \boldsymbol{x} - \boldsymbol{x}^* \rangle = f(\boldsymbol{x}^*). \qquad \blacksquare$$