

Smooth Convex Optimization

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1 Minimization of Smooth Functions

1.1 Smooth Convex Functions

In this section, we consider the unconstrained minimization problem

$$\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}), \quad (1.1)$$

where the objective function $f(\cdot)$ is smooth enough. \mathcal{F} represents *differentiable functions*.

Assumption 1.1

For any $f \in \mathcal{F}$, the first-order optimality condition is sufficient for a point to be a global solution to 1.1

Assumption 1.2

If $f_1, f_2 \in \mathcal{F}$ and $\alpha, \beta \geq 0$, then $\alpha f_1 + \beta f_2 \in \mathcal{F}$.

Assumption 1.3

Any linear function $l(\mathbf{x}) = \alpha + \langle \mathbf{a}, \mathbf{x} \rangle$ belongs to \mathcal{F} .

Note that the linear function $l(\cdot)$ perfectly fits Assumption 1.1. Clearly, $\nabla l(\mathbf{x}) = 0$ implies that this function is constant, and any point \mathbb{R}^n is its global minimum.

Definition 1.1 (Convex Set)

A set $Q \subseteq \mathbb{R}^n$ is called convex if we have

$$\alpha \mathbf{x} + (1 - \alpha) \mathbf{y} \in Q \quad \forall \mathbf{x}, \mathbf{y} \in Q \text{ and } \forall \alpha \in [0, 1].$$

Definition 1.2 (Convex Function)

A continuously differentiable function $f(\cdot)$ is called convex on a convex set Q (notation $f \in \mathcal{F}^1(Q)$) if we have

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle. \quad (1.2)$$

If $-f(\cdot)$ is convex, we call $f(\cdot)$ concave.

Theorem 1.1

If $f \in \mathcal{F}^1(\mathbb{R}^n)$ and $\nabla f(\mathbf{x}^*) = 0$ then \mathbf{x}^* is the global minimum of $f(\cdot)$ on \mathbb{R}^n .

Proof of Theorem 1.1: In the inequality 1.2, for any $\mathbf{x} \in \mathbb{R}^n$ we have

$$f(\mathbf{x}) \geq f(\mathbf{x}^*) + \langle \nabla f(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^* \rangle = f(\mathbf{x}^*).$$

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