Assignment 5

1. a)

$$\begin{array}{ccccccc} & x & & Y & & Z & \\ | & & | & & | & & | \\ 9 & & 11 & & 3 & & 7 \end{array}$$

$$\left.\begin{aligned} P(N(9,11)=x) &= P(N(0,2)=x) = P(N(2)=x) \\ P(N(11,3)=Y) &= P(N(0,4)=Y) = P(N(4)=Y) \\ P(N(3,7)=Z) &= P(N(0,4)=Z) = P(N(4)=Z) \end{aligned}\right\}$$

By using the properties of the independent and stationary increments

Joint pmf of x, Y and Z:

$$\left.\begin{aligned} P(x,Y,Z) &= P(N(2)=x \cap N(4)=Y \cap N(4)=Z) \\ &= P(N(2)=x) \times P(N(4)=Y) \times P(N(4)=Z) \end{aligned}\right\}$$

By independence

$$= \left[\frac{(\lambda 2)^x}{x!} e^{-\lambda 2}\right] \times \left[\frac{(\lambda 4)^Y}{Y!} e^{-\lambda 4}\right] \times \left[\frac{(\lambda 4)^Z}{Z!} e^{-\lambda 4}\right]$$

$$= \left[\frac{(\lambda 2)^x}{x!} \times \frac{(\lambda 4)^Y}{Y!} \times \frac{(\lambda 4)^Z}{Z!}\right] e^{-\lambda 2 - \lambda 4 - \lambda 4}$$

$$= \left[\frac{2^x \lambda^x}{x!} \times \frac{4^Y \lambda^Y}{Y!} \times \frac{4^Z \lambda^Z}{Z!}\right] e^{-10\lambda}$$

$$= \left[\frac{2^x 4^{Y+Z} \lambda^{x+Y+Z}}{x!\,Y!\,Z!}\right] e^{-10\lambda}$$

$$= \frac{2^x (2)^{2(Y+Z)} \lambda^{x+Y+Z}}{x!\,Y!\,Z!} e^{-10\lambda}$$

$$= \frac{2^{x+2Y+2Z} \lambda^{x+Y+Z}}{x!\,Y!\,Z!} e^{-10\lambda}$$

b) $P(x,y,z \mid X+Y+Z=20) = \dfrac{P(x+y+z=20 \mid x,y,z)\,P(x,y,z)}{P(x+y+z=20)}$

Find $P(x+y+z=20) = P(N(9,7)=20) = P(N(0,10)=20) = P(N(10)=20)$ #

By the independent and stationary increments

$$P(N(10)=20) = \frac{(10\lambda)^{20}}{20!}e^{-10\lambda}$$ #

Find $P(x+y+z=20 \mid x,y,z)$ = 0 or 1. → This equates to saying that either the lightening strikes occur or it doesn't because $x+y+z=20$ is the total number of lightening strikes that can possibly occur in the given time frame, since it is conditioned on x, y and z, if either x, y or z does not occur then $P(x+y+z=20 \mid x,y,z)=0$. Only in the case where all x, y and z occurs then $P(x+y+z=20 \mid x,y,z)=1$ occurs.

$$P(x,y,z \mid x+y+z=20) = \frac{1 \times P(x,y,z)}{P(x+y+z=20)}$$

$$= \frac{\left(\frac{2^x\lambda^x}{x!} \times \frac{4^y\lambda^y}{y!} \times \frac{4^z\lambda^z}{z!}\times \cancel{e^{-10\lambda}}\right)}{\left(\frac{(10\lambda)^{20}}{20!}\cancel{e^{-10\lambda}}\right)}$$

$$= \left[\frac{2^x4^y4^z\lambda^{x+y+z}}{x!y!z!} \times \frac{20!}{(10\lambda)^{20}}\right]$$

sub in $x+y+z=20$

$$= \left[\frac{2^x4^y4^z\lambda^{x+y+z}}{(10\lambda)^{20}} \times \frac{20!}{x!y!z!}\right]$$

$$= \left[\frac{2^x4^y4^z\lambda^{x+y+z}}{(10\lambda)^{x+y+z}} \times \frac{20!}{x!y!z!}\right]$$

$$= \left[\frac{2^x4^y4^z\lambda^{x+y+z}}{10^{x+y+z}\lambda^{x+y+z}} \times \frac{20!}{x!y!z!}\right]$$

$$= \left[\frac{(2\lambda)^x(4\lambda)^y(4\lambda)^z}{(10\lambda)^x(10\lambda)^y(10\lambda)^z} \times \frac{20!}{x!y!z!}\right]$$

$$= \left[\left(\frac{2\lambda}{10\lambda}\right)^{x} \left(\frac{4\lambda}{10\lambda}\right)^{y} \left(\frac{4\lambda}{10\lambda}\right)^{z} \right] \times \frac{20!}{x!\,y!\,z!}$$

$$= \left(\frac{1}{5}\right)^{x} \left(\frac{2}{5}\right)^{y} \left(\frac{2}{5}\right)^{z} \times \frac{20!}{x!\,y!\,z!}$$

$\rightarrow$ This is a multinomial distibution.

3 a)

2. This is a multinomial distribution because it satisfies the following 4 properties:

1) n identical trials
2) There are 8 number of outcomes to each trial.
3) The probability of each outcome is the same for each trial. $= \left(\frac{1}{8}\right)$
4) The trials are independent.

x_1 = number of 1's

x_4 = number of 4's

$$\begin{aligned}\operatorname{cov}(x_1, x_4) &= -n p_1 p_4, \text{ if } 1 \neq 4.\\ &= -n\left(\tfrac{1}{8}\right)\left(\tfrac{1}{8}\right)\\ &= -\frac{n}{64} \ \#\end{aligned}$$

$$\operatorname{corr}(x_1, x_4) = \frac{\operatorname{cov}(x_1, x_4)}{\underbrace{sd(x_1)\, sd(x_4)}_{}}$$

[To get the standard deviation:]

$$\begin{aligned}\operatorname{Var}(x_1) &= np_1(1-p_1)\\ &= n\left(\tfrac{1}{8}\right)\left(1-\tfrac{1}{8}\right)\\ &= \frac{7n}{64}\end{aligned}$$

$$\begin{aligned}sd(x_1) &= \sqrt{\frac{7n}{64}}\\ &= \frac{\sqrt{7n}}{8}\end{aligned}$$

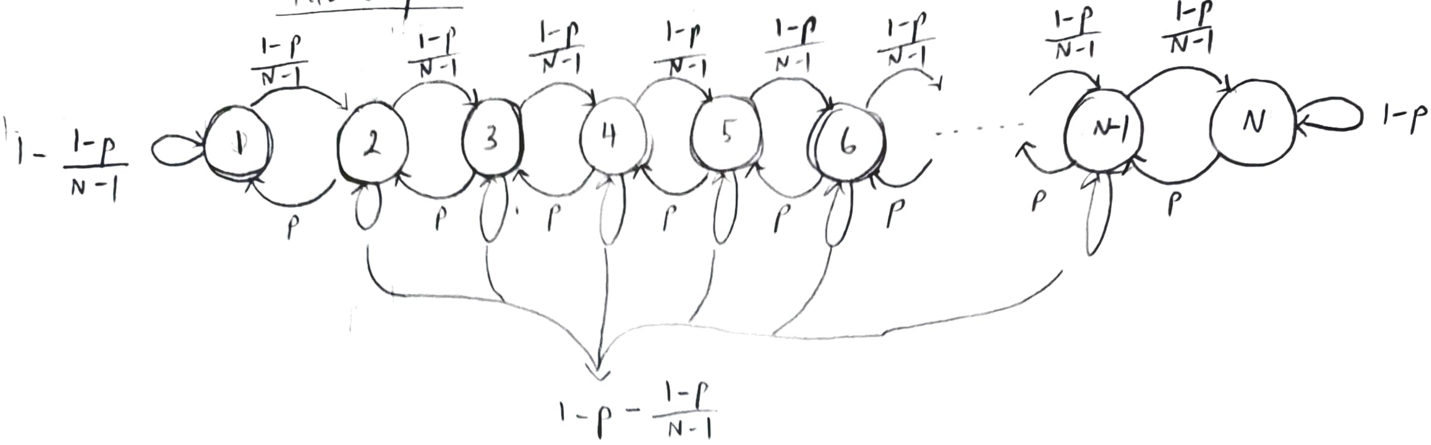
$$\begin{aligned}\operatorname{Var}(x_4) &= np_4(1-p_4)\\ &= n\left(\tfrac{1}{8}\right)\left(1-\tfrac{1}{8}\right)\\ &= \frac{7n}{64}\end{aligned}$$

$$\begin{aligned}sd(x_4) &= \sqrt{\frac{7n}{64}}\\ &= \frac{\sqrt{7n}}{8}\end{aligned}$$

$$\begin{aligned}\operatorname{corr}(x_1, x_4) &= \frac{-\frac{n}{64}}{\frac{\sqrt{7n}}{8} \times \frac{\sqrt{7n}}{8}}\\ &= \frac{-\frac{n}{64}}{\frac{(7n)^{\frac{1}{2}+\frac{1}{2}}}{8 \times 8}}\\ &= -\frac{\cancel{n}}{\cancel{64}} \times \frac{\cancel{64}}{7\cancel{n}}\\ &= -\frac{1}{7} \ \#\end{aligned}$$

3. a)

State Diagram



Transition Matrix

$$
\begin{array}{c|ccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots & N-1 & N \\
\hline
1 & 1-\frac{1-p}{N-1} & \frac{1-p}{N-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & p & 1-p-\frac{1-p}{N-1} & \frac{1-p}{N-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & p & 1-p-\frac{1-p}{N-1} & \frac{1-p}{N-1} & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & p & 1-p-\frac{1-p}{N-1} & \frac{1-p}{N-1} & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & p & 1-p-\frac{1-p}{N-1} & \frac{1-p}{N-1} & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & p & 1-p-\frac{1-p}{N-1} & \frac{1-p}{N-1} & 0 & 0 \\
\vdots & & & & & & & & & \\
N-1 & 0 & 0 & 0 & 0 & 0 & 0 & p & 1-p-\frac{1-p}{N-1} & \frac{1-p}{N-1} \\
N & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p & 1-p
\end{array}
$$

Property of transition Matrix

$$\sum_{j\in N}[\mathbb{P}]_{ij} = \sum_{j\in N} p_{ij} = 1$$

For row 1: $\rightarrow$ $p_{11} + p_{12} + \cdots + 0 = \left(1-\frac{1-p}{N-1}\right) + \left(\frac{1-p}{N-1}\right)$
$= 1$ #

For row N: $\rightarrow$ $0 + \cdots + p_{NN-1} + p_{NN} + \cdots + 0 = p + (1-p)$
$= 1$ #

The sum of all the states in every row is equal to 1, thus showing why these non-zero transition probabilities for the system is correct.

b) Using the equilibrium Equations:

$$\pi_1 = \pi_2 p + \pi_1\left(1-\frac{1-p}{N-1}\right)$$

$$\pi_1 = \pi_2 p + \pi_1 - \frac{1-p}{N-1}\pi_1$$

$$\pi_1 - \pi_1 = \pi_2 p - \frac{1-p}{N-1}\pi_1$$

$$0 = \pi_2 p - \frac{1-p}{N-1}\pi_1$$

$$\left(\frac{1-p}{N-1}\pi_1 = \pi_2 p\right) \times \frac{1}{p}$$

$$\left(\frac{1}{p}\right)\left(\frac{1-p}{N-1}\right)\pi_1 = \pi_2$$

$$\pi_2 = \frac{1-p}{p(N-1)}\pi_1 \quad \text{\# shown.}$$

$$\pi_2 = \pi_3 p + \pi_2\left(1-p-\frac{1-p}{N-1}\right) + \pi_1\left(\frac{1-p}{N-1}\right)$$

$$\pi_2 = \pi_3 p + \pi_2 - p\pi_2 - \frac{1-p}{N-1}\pi_2 + \left(\frac{1-p}{N-1}\right)\pi_1$$

$$\pi_2 - \pi_2 = \pi_3 p - p\pi_2 - \frac{1-p}{N-1}\pi_2 + \frac{1-p}{N-1}\pi_1$$

substitute $\pi_2 = \frac{1-p}{p(N-1)}\pi_1$ to $p\pi_2$

$$0 = \pi_3 p - \left[\frac{1-p}{\cancel{p}(N-1)} \times \cancel{p}\right]\pi_1 - \frac{1-p}{N-1}\pi_2 + \frac{1-p}{N-1}\pi_1$$

$$0 = \pi_3 p - \cancel{\frac{1-p}{N-1}\pi_1} - \frac{1-p}{N-1}\pi_2 + \cancel{\frac{1-p}{N-1}\pi_1}$$

$$0 = \pi_3 p - \frac{1-p}{N-1}\pi_2$$

$$\left(\frac{1-p}{N-1}\pi_2 = \pi_3 p\right) \times \frac{1}{p}$$

$$\left(\frac{1}{p}\right)\left(\frac{1-p}{N-1}\right)\pi_2 = \pi_3$$

$$\pi_3 = \frac{1-p}{p(N-1)}\pi_2 \quad \text{\# shown\#}$$

c) Given the boundary equations:

$$\pi_2 = \frac{1-p}{p(N-1)}\pi_1 \quad \text{and} \quad \pi_3 = \frac{1-p}{p(N-1)}\pi_2$$

① Find the general equation:

$$\pi_3 = \left(\frac{1-p}{p(N-1)}\right)\left(\frac{1-p}{p(N-1)}\right)\pi_1$$

→ substitute π_2 into π_3

$$\pi_3 = \left(\frac{1-p}{p(N-1)}\right)^2\pi_1$$

$$\Downarrow$$

$$\pi_i = \left(\frac{1-p}{p(N-1)}\right)^{i-1}\pi_1 \quad \#$$

② To find the π_i, we normalise the equation:

$$\sum_{i=1}^{N}\pi_i = 1$$

→ substitute π_i into

$$\sum_{i=1}^{N}\left(\frac{1-p}{p(N-1)}\right)^{i-1}\pi_1 = 1$$

$$\pi_1\sum_{i=1}^{N}\underline{\left(\frac{1-p}{p(N-1)}\right)^{i-1}} = 1$$

↳ is exactly a geometric series.

c) continue ....

we can convert $\pi_1 \sum_{i=1}^{N} \left(\frac{1-p}{p(N-1)}\right)^{i-1} = 1$ to $\pi_1 \left(\frac{1-\left(\frac{1-p}{p(N-1)}\right)^N}{1-\left(\frac{1-p}{p(N-1)}\right)} \right) = 1$

$$\pi_1 = \frac{1}{\left(\frac{1-\left(\frac{1-p}{p(N-1)}\right)^N}{1-\left(\frac{1-p}{p(N-1)}\right)}\right)}$$

$$\pi_1 = \frac{1-\left(\frac{1-p}{p(N-1)}\right)}{1-\left(\frac{1-p}{p(N-1)}\right)^N}$$

In conclusion:

$$\pi_i = \left[\frac{1-\left(\frac{1-p}{p(N-1)}\right)}{1-\left(\frac{1-p}{p(N-1)}\right)^N}\right]\left(\frac{1-p}{p(N-1)}\right)^{i-1} \text{ for } 1 \le i \le N \ \#$$