Probability and Statistics 2021 — Assignment 4

Where appropriate, remember to always state the full pdf with support or you will loose marks. For example, in Q4 (d) and (e).

1. Consider the random variable X with pdf

$$f(x) = \begin{cases} x/2, & \text{if } 0 \leq x \leq 2 \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the pdf of $Y = g(X) = 1 - \sqrt{4 - X^2}/2$.

   (b) Sketch the probability density functions of X and Y.

   Hint: is g an increasing function?

   [8 marks]

2. Suppose $X_1, \ldots, X_n$ are i.i.d. $\exp(\lambda)$ for some $\lambda > 0$ and some integer $n > 0$. Prove that $X_{(1)}$, the first order statistic, is $\exp(n\lambda)$.

   [3 marks]

3. An antiques shop has a large box of old Gameboy consoles. A proportion p_1 of these have exactly one defect and a proportion p_2 has more than one defect, where $p_1 + p_2 < 1$.

   (a) What is the probability of randomly choosing a working console, i.e. one with no defects?

   (b) If 5 are chosen at random, what is the probability of getting at least one with no defects? Hint: this requires thinking, not a big calculation.

   (c) Keith buys n Gameboys selected at random from the box. The cost of repairing the broken Gameboys is $C = Y_1 + 3Y_2$, where Y_1 is the number with one defect and Y_2 is the number with more than one defect. Find the expected value and variance of C.

   [9 marks]

4. Consider two random variables Y_1 and Y_2 with the joint probability density function

$$f(y_1, y_2) = \begin{cases} k(1 - y_2) & 0 \leq y_1 \leq y_2 \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$$

   (a) Draw a sketch of the area where the probability density function is nonzero.

   (b) What are the conditions for f to be a valid pdf? Show that for these to be true, we need $k = 6$.

   (c) Draw the appropriate integration region and find $P(Y_1 \leq 1/4, Y_2 \leq 3/4)$.

   (d) Find the marginal density function of Y_1.

   (e) Find the marginal density function of Y_2.

   [12 marks]