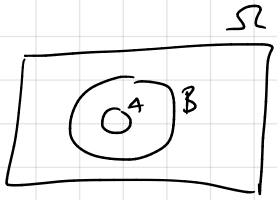
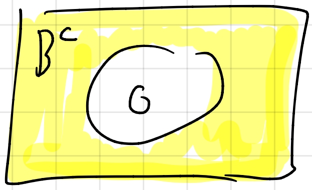
Assignment 1 Solutions

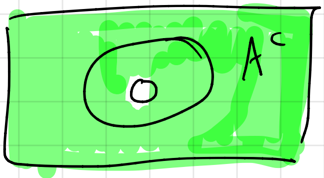
1. a) Intuitively if $P(B|A)=1$ then we have that $A \subset B$.



So,



$B^c \subset A^c$ and $P(A^c|B^c) = 1$.



$$P(A\cap B) = 1 - P((A\cap B)^c) = 1 - \underbrace{P(A^c \cup B^c)}_{P(A^c)+P(B^c)-P(A^c\cap B^c)}.$$

$$P(A\cap B) = P(B|A)\,P(A).$$

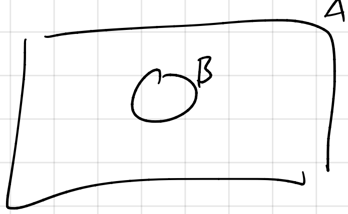
$$P(B|A)\,P(A) = 1 - P(A^c) - P(B^c) + P(A^c|B^c)\,P(B^c).$$

$$P(A) + P(A^c) = 1 - P(B^c) + P(A^c|B^c)\,P(B^c).$$

$$0 = -P(B^c) + P(A^c|B^c)\,P(B^c).$$

$$1 = P(A^c|B^c)$$ as $P(B^c) > 0$ so can divide.

b) If $P(A) = 1$ then $A = \Omega$



$P(A^c) = 0$.

$$P(B) = P(B \cap A) + P(B \cap A^c)$$

$$= P(A|B)\,P(B)$$

$$P(B|A^c)\,P(A^c)$$

$$1 = P(A|B)$$

as $P(B) > 0$.

$$P(B|A^c)\,\big(1 - P(A)\big) = 0$$

2. This is the sort of question that becomes easier if we think of distributing the cards to P1, then 2 etc.. rather than giving each one a card in turn.

a) $\binom{52}{13}$ Choose 13 cards from 52 without repetition.

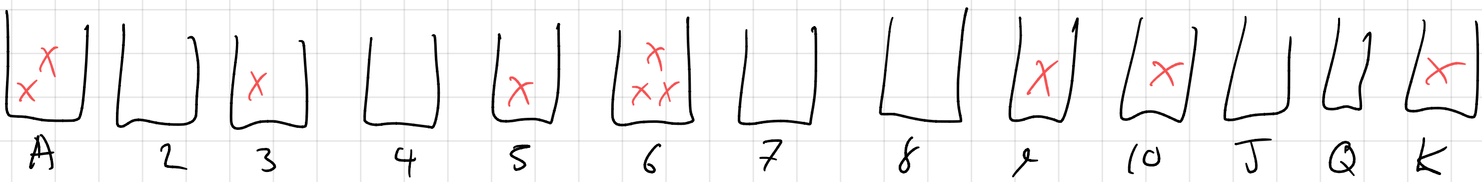
↑ order doesn't matter.

b) Distribution to player 1 → then 2 → then 3 → rest go to 4

$$\binom{52}{13} \times \binom{39}{13} \times \binom{26}{13} \underbrace{\binom{13}{13}}_{=1}$$

3. This is different to before as we now have 10 multiples of each card, so its now possible to be dealt 10 Aces, etc.....

Think of this as distributing 10 balls into 13 urns where each urn represents a card type.



This represents the hand with: A, A, 3, 5, 6, 6, 6, 9, 10, K

Total number of ways $= \binom{13+10-1}{10} = \binom{22}{10}$.

4. List all possible orderings

6 possible arrangements of ages.

BCL
CBL
BLC
CLB
LCB
LBC

arrangements where
$L > C$

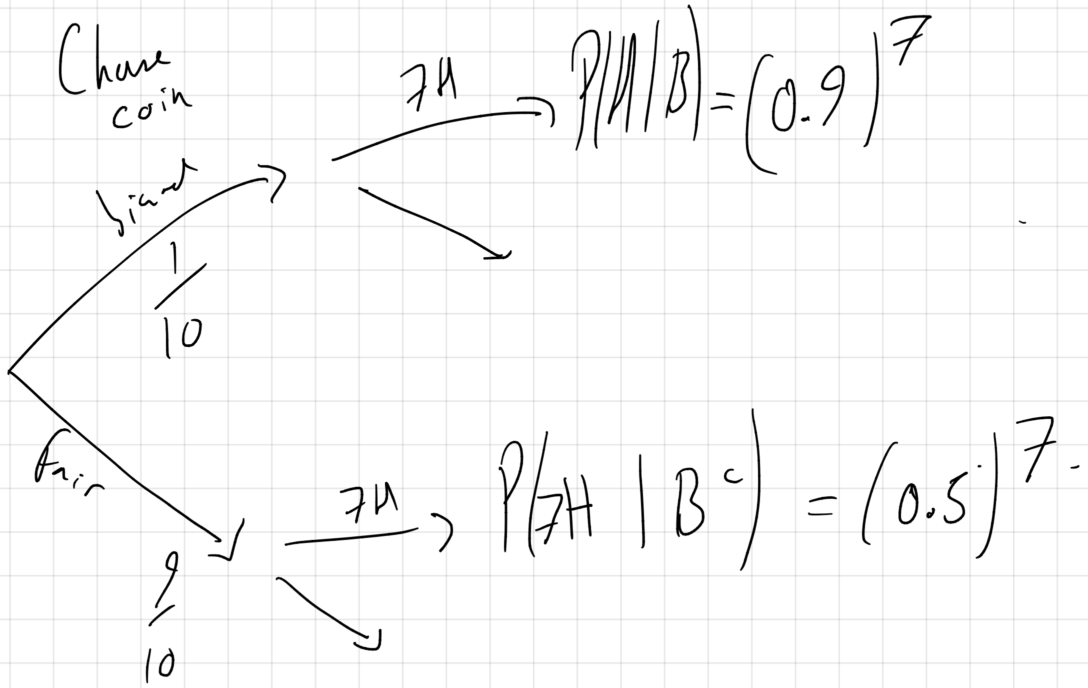
Given L is older than C restricts to:

BCL
CBL
CLB

Now two where L is older than B

so $P(L > B) = \frac{2}{3}$

5.



$$P(B \mid 7H) = \frac{P(7H \mid B) \cdot P(B)}{P(7H)}$$

$$P(7H) = P(7H \mid B)P(B) + P(7H \mid B^c)\,P(B^c)$$

$$= (0.9)^7 \times \frac{1}{10} + (0.5)^7 \times \frac{9}{10} = 0.05486$$

$$P(B \mid 7H) = \frac{(0.9)^7 \cdot \frac{1}{10}}{0.05486} = 0.87$$

6. a) $R = A$ iff all people in A is a friend of Ringo and all the people in S/A is not a friend.

$$P(R=A) = \underbrace{\left(\frac{1}{2}\right)^{|A|}}_{\text{friends}} \times \underbrace{\left(1-\frac{1}{2}\right)^{50-|A|}}_{\text{not friends}} = \left(\frac{1}{2}\right)^{50}$$

b) $P(R \subseteq J)$ What is the probability that all R's friends are a subset of J's friends?

$R \subseteq J$ iff each of R's friends are also friends with J.

Note, that if a person is not friends with R, it doesn't matter if they are friends with J or not.

$$P(R \subseteq J) = \prod_{i=1}^{50} P(i \text{ is friends with } R \text{ and } i \text{ is friends with } J \text{ or } i \text{ is not friends with } R).$$

$$= \prod_{j=1}^{50} P(i \in R \text{ and } i \in J) + P(i \notin R)$$

$$= \prod_{i=1}^{50} P(i \in R)\,P(i \in J) + P(i \notin R).$$

$$= \prod_{i=1}^{10} \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} = \left(\frac{3}{4}\right)^{50}$$

c) $P(R \cup J = S)$ is the probability that everyone in S is friends with either R, J or both.

$$P(i \text{ is friends with neither}) = P(i \notin R)\, P(i \notin J) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

$$P(R \cup J = S) = \left(1 - \frac{1}{4}\right)^{50} = \left(\frac{3}{4}\right)^{50}.$$