Probability and Statistics 2021 — Assignment 5

1. You are out in a thunder storm (again). Lightening strikes occur as a Poisson process with rate λ per hour. Let X, Y and Z be the number of strikes between the times 9–11, 11–3 and 3–7pm respectively.

   (a) What is the joint pmf of X, Y, Z.

   (b) What is the conditional joint pmf of X, Y, Z given that $X + Y + Z = 20$. What type of distribution is this?

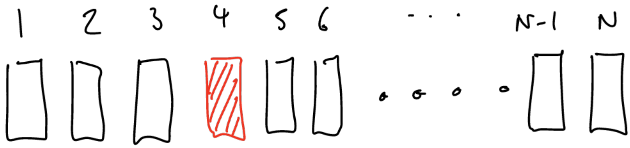
[8 marks]

2. An eight-sided die is rolled n times. Let X_1 be the number of 1s that are observed and let X_4 be the number of 4s. Find the covariance, $\mathrm{cov}(X_1, X_4)$, and correlation $\mathrm{corr}(X_1, X_4)$.

[6 marks]

3. Consider a collection of N books arranged in a line along a bookshelf. At successive units of time, a book is selected randomly from the collection. After the book has been consulted, it is replaced on the shelf one position to the left of its original position, with the book in that position moved to the right by one. That is, the selected book and its neighbour to the left swap positions. If the selected book is already in the leftmost position, it is returned there. All but one of the books have plain covers and are equally likely to be selected. The other book has a red cover. At each time unit, the red book will be selected with probability p, where $0 < p < 1$. Each other book will be selected with probability $(1 - p)/(N - 1)$. Successive choices of book are independent.

   Number the positions on the shelf from 1 (at the left) to N (at the right). Write X_n for the position of the red book after n units of time.



   (a) Show that $\{X_n\}_{n \in \mathbb{N}}$ is a Markov chain, with non-zero transition probabilities given by:

$$
\begin{aligned}
p_{i,i-1} &= p && \text{for } i = 2, 3, \ldots, N, \\
p_{i,i+1} &= \frac{1-p}{N-1} && \text{for } i = 1, 2, \ldots, N-1, \\
p_{i,i} &= 1 - p - \frac{1-p}{N-1} && \text{for } i = 2, 3, \ldots, N-1, \\
p_{1,1} &= 1 - \frac{1-p}{N-1}, \\
p_{N,N} &= 1 - p.
\end{aligned}
$$

Note you must explain why these are the correct probabilities for the system described.

(b) If π_i is the limiting probability of the system being in state i, show that

$$\pi_2 = \frac{1-p}{p(N-1)}\pi_1, \quad \pi_3 = \frac{1-p}{p(N-1)}\pi_2.$$

(c) Find the general solution for π_i, for $1 \leq i \leq N$.

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[~~15~~ marks]