Assignment 3

Q1: a) $P(\mathcal{X} \le x) = P(\frac{1}{\theta} Y_{(n)} \le x)$

$= P(Y_{(n)} \le \theta x)$ → maximum distribution

$= P(Y_1 \le \theta x \cap Y_2 \le \theta x \cap Y_3 \le \theta x \cap \cdots \cap Y_n \le \theta x)$

$= P(Y_1 \le \theta x) P(Y_2 \le \theta x) P(Y_3 \le \theta x) \cdots P(Y_n \le \theta x)$ by independence.

$= [F_Y(\theta x)]^n$

Given that $Y \overset{iid}{\sim} \text{Uniform}(0, \theta)$, its CDF $F_Y(\theta x)$ is

$$[F_Y(\theta x)]^n = \left[\frac{\theta x - 0}{\theta - \theta}\right]^n$$

$$= \left[\frac{\theta x}{\theta}\right]^n$$

$$= [x]^n$$

To derive its pdf, we differentiate the CDF

$$f_{\mathcal{X}}(x) = \frac{dF_Y(\theta x)}{dx} = n x^{n-1}$$

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Given the pdf of a Beta distribution of Beta$(n, 1)$

$$f(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \times x^{\alpha-1}(1-x)^{\beta-1}$$ given $\alpha = n$ and $\beta = 1$

$$f(x; n, 1) = \frac{\Gamma(n+1)}{\Gamma(n)\Gamma(1)} \times x^{n-1}(1-x)^{1-1}$$

using the property of ~~g~~ gamma function

$$= \frac{n\Gamma(n)}{(n-1)!\, 1} \times x^{n-1}$$

$$= \frac{n\cancel{(n-1)!}}{\cancel{(n-1)!}\, 1} \times x^{n-1}$$

$$= n x^{n-1}$$

Since, the pdf of the pivotal quantity $\mathcal{X}$ is the as the pdf of the Beta distribution of Beta$(n, 1)$, it is shown that $\mathcal{X}$ has a Beta distribution of Beta$(n, 1)$.

$$\left\{\begin{array}{lll} f_{\mathcal{X}}(x) = n x^{n-1}, & x \in (0, \theta) \\ f(x; n, 1) = n x^{n-1}, & \text{Beta}(n, 1) \end{array}\right\}$$ same pdf.

b) $0.95 = P(L \leq \theta \leq U)$

$= P\left(\frac{1}{U} \leq \frac{1}{\theta} \leq \frac{1}{L}\right)$ ✓ given $x = \frac{1}{\theta} Y_n$

$= P\left(\frac{Y_n}{U} \leq \frac{Y_n}{\theta} \leq \frac{Y_n}{L}\right)$

$= P\left(\frac{Y_n}{U} \leq x \leq \frac{Y_n}{L}\right)$ where $x \sim Beta(n,1)$

for symmetric confidence interval, we have $P(\beta_{n,1,0.975} \leq x \leq \beta_{n,1,0.025}) = 0.95$

This implies :

$\frac{Y_n}{U} = \beta_{n,1,0.975}$ and $\frac{Y_n}{L} = \beta_{n,1,0.025}$

$U = \frac{Y_n}{\beta_{n,1,0.975}}$ $L = \frac{Y_n}{\beta_{n,1,0.025}}$

The 95% symmetric confidence interval for θ is $\left(\frac{Y_n}{\beta_{n,1,0.025}}, \frac{Y_n}{\beta_{n,1,0.975}}\right)$.

c) given $n = 15$ and $Y_n = \max_i(Y_i) = 15$

The critical values are

$\beta_{15,1,0.025} \approx 0.9983$ using R qbeta(0.975, 15, 1)

$\beta_{15,1,0.975} \approx 0.7820$ using R qbeta(0.025, 15, 1)

Therefore, the 95% symmetric confidence interval for θ is

$\left(\frac{15}{0.9983}, \frac{15}{0.7820}\right)$

$= (15.0255, 19.1816)$ #

Q2: a) i) $X^T X = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$

$2 \times n \qquad n \times 2$

$$= \begin{bmatrix} (1)(1)+(1)(1)+\cdots+(1)(1) & (1)(x_1)+(1)(x_2)+\cdots+(1)(x_n) \\ x_1(1)+x_2(1)+\cdots+(x_n)(1) & (x_1)(x_1)+(x_2)(x_2)+\cdots+(x_n)(x_n) \end{bmatrix}$$

2×2

→ The dot product of all the elements in the first row of X^T and first column of X is just the summation of n number of 1s which is just n.

→ The dot product of all the elements in the first row of X^T and second column of X is the summation of n number of x_i elements which is just $n\bar{x}$

↳ the same goes for the dot product of second row of X^T and first column of X.

→ The dot product of all the elements in the second row of X^T and second column of X is the summation of the square of x_i^2.

$$= \begin{bmatrix} 1+1+1+\cdots+1 & x_1+x_2+\cdots+x_n \\ x_1+x_2+\cdots+x_n & x_1^2+x_2^2+x_3^2+\cdots+x_n^2 \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{n} 1 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} = \begin{bmatrix} n & n\bar{x} \\ n\bar{x} & \sum_{i=1}^{n} x_i^2 \end{bmatrix}$$ # shown.

ii) $\det(X^T X) = \begin{vmatrix} n & n\bar{x} \\ n\bar{x} & \sum_{i=1}^{n} x_i^2 \end{vmatrix} = n\sum_{i=1}^{n} x_i^2 - (n\bar{x})(n\bar{x})$

$$= n\sum_{i=1}^{n} x_i^2 - n^2\bar{x}^2$$

$$= n\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right)$$

$$= n S_{xx}$$ # shown.

b) For $X^T X$ to be invertible then the $\det(X^T X) \neq 0$

This means $nS_{xx} \neq 0$ which also means $S_{xx} \neq 0$.

If that is the case then $S_{xx} = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 \neq 0$

which means $x_i \neq \bar{x}$ $\forall i = 1, 2, \ldots, n$ then $\sum_{i=1}^{n} (x_i - \bar{x})^2 = 0$.

In conclusion, $x_i \neq x_j$ for all $i, j = 1, 2, \ldots, n$ for $X^T X$ to be invertible.

c) $$(X^TX)^{-1}X^Ty = \left(\begin{bmatrix} n & n\bar{x} \\ n\bar{x} & \sum x_i^2 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \frac{1}{n\sum x_i^2 - (n\bar{x})(n\bar{x})} \begin{bmatrix} \sum x_i^2 & -n\bar{x} \\ -n\bar{x} & n \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix}}_{2\times n} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{n\times 1}$$

based on the result from b)

$$= \frac{1}{n\sum_{i=1}^n x_i^2 - n^2\bar{x}^2} \begin{bmatrix} \sum_{i=1}^n x_i^2 & -\sum_{i=1}^n x_i \\ -\sum_{i=1}^n x_i & n \end{bmatrix} \begin{bmatrix} (1)(y_1)+(1)(y_2)+\cdots+(1)(y_n) \\ x_1y_1+x_2y_2+\cdots+x_ny_n \end{bmatrix}$$

$$= \frac{1}{n\sum_{i=1}^n x_i^2 - n^2\bar{x}^2} \begin{bmatrix} \sum_{i=1}^n x_i^2 & -\sum_{i=1}^n x_i \\ -\sum_{i=1}^n x_i & n \end{bmatrix} \begin{bmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_iy_i \end{bmatrix}$$

$$= \frac{1}{n\sum_{i=1}^n x_i^2 - n^2\bar{x}^2} \begin{bmatrix} (\sum_{i=1}^n x_i^2)(\sum_{i=1}^n y_i) + (-\sum_{i=1}^n x_i)(\sum_{i=1}^n x_iy_i) \\ (-\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i) + (n)(\sum_{i=1}^n x_iy_i) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \dfrac{\sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i - \sum_{i=1}^n x_i \sum_{i=1}^n x_iy_i}{n\sum_{i=1}^n x_i^2 - n^2\bar{x}^2} \\[2ex] \dfrac{-\sum_{i=1}^n x_i \sum_{i=1}^n y_i + n\sum_{i=1}^n x_iy_i}{n\sum_{i=1}^n x_i^2 - n^2\bar{x}^2} \end{bmatrix}}_{2\times 1}$$

$\rightarrow$ take $\sum_{i=1}^n x_i^2 = S_{xx} + n\bar{x}^2$

$\sum_{i=1}^n x_iy_i = S_{xy} + n\bar{x}\bar{y}$

$$= \begin{bmatrix} \dfrac{(S_{xx}+n\bar{x}^2)(n\bar{y}) - [(n\bar{x})(S_{xy}+n\bar{x}\bar{y})]}{nS_{xx}} \\[2ex] \dfrac{-(n\bar{x}\times n\bar{y}) + n(S_{xy}+n\bar{x}\bar{y})}{nS_{xx}} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{n\bar{y}S_{xx} + \cancel{n^2\bar{x}^2\bar{y}} - n\bar{x}S_{xy} - \cancel{n^2\bar{x}^2\bar{y}}}{nS_{xx}} \\[2ex] \dfrac{-\cancel{n^2\bar{x}\bar{y}} + nS_{xy} + \cancel{n^2\bar{x}\bar{y}}}{nS_{xx}} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\cancel{n}\bar{y}\cancel{Sxx}}{\cancel{nSxx}} - \dfrac{\cancel{n}\bar{x}Sxy}{\cancel{n}Sxx} \\ \\ \dfrac{\cancel{n}Sxy}{\cancel{n}Sxx} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{y} - \dfrac{Sxy}{Sxx}\bar{x} \\ \\ \dfrac{Sxy}{Sxx} \end{bmatrix}$$

shown.