Assignment 3 Solutions

1. a) $$E[T] = \frac{1}{30}\int_0^{15\pi} \underset{f}{t}\ \underset{g'}{\sin(t/30)}$$

integrate by parts $f = t$, $g' = \sin\left(\frac{t}{30}\right)$

$g = -30\cos(t/30)$.

$$= \underbrace{\Big[-t\cos(t/30)\Big]_0^{15\pi}}_{=0} - \int_0^{15\pi} -\cos(t/30)\ dt$$

$$= \int_0^{15\pi} \cos(t/30)\ dt = \Big[30\sin(t/30)\Big]_0^{15\pi}$$

$$= 30 \text{ mins}$$

b). $$F(t) = \int_0^t f(y)\,dy = \frac{1}{30}\int_0^t \sin(y/30)\ dy.$$

$$= \frac{1}{\cancel{30}}\Big[-\cancel{30}\cos(y/30)\Big]_0^t$$

$$= 1 - \cos(t/30)$$

Hence $$F(t) = \begin{cases} 0 & t < 0 \\ 1 - \cos(t/30) & 0 \leq t \leq 15\pi \\ 1 & t > 1 \end{cases}$$

c) $P(15 < T < 30) = F(30) - F(15)$

$= \underline{0.337}$

d) $P(T < 30 \mid T > 15) = \dfrac{P(T<30,\ T>15)}{P(T>15)} = \dfrac{F(30)-F(15)}{1-F(15)}$

$= \underline{0.384}$

2. a) f must satisfy $f(x) \geq 0 \ \forall x = 1, 2, 3, \ldots$ and $\sum_{x=1}^{\infty} f(x) = 1$

Hence $c \sum_{z=1}^{\infty} 2^{-z} = 1$.

$$c \sum_{z=1}^{\infty} \left(\frac{1}{2}\right)^z = c\left(\sum_{z=1}^{\infty} \left(\frac{1}{2}\right)^z + \left(\frac{1}{2}\right)^0 - \left(\frac{1}{2}\right)^0\right)$$

$$= c\left(\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^x - 1\right) = c\left(\frac{1}{1-\frac{1}{2}} - 1\right) = c.$$

Thus $c=1$ to be valid pmf.

b) $m(t) = E[e^{tZ}] = \sum_{i=1}^{\infty} e^{ti} 2^{-i} = \sum_{i=1}^{\infty} \left(\frac{e^t}{2}\right)^i$

Using the same trick as before $m(t) = \sum_{i=0}^{\infty} \left(\frac{e^t}{2}\right) - 1$

hence $m(t) = \dfrac{1}{1-e^t/2} - 1 = \dfrac{e^t}{2-e^t}$

c) $E[Z] = \dfrac{dm(t)}{dt}\Big|_{t=0} = \dfrac{2e^t}{(2-e^t)^2}\Big|_{t=0} = 2$

$\text{Var}(Z) = E[Z^2] - E[Z]^2 = E[Z^2] - 4$

$E[Z^2] = \dfrac{d^2m(t)}{dt^2}\Big|_{t=0} = \dfrac{2e^t(2-e)^2 - 2e^t(2)(2-e^{-t})(-e^t)}{(2-e^t)^4}\Bigg|_{t=0}$

$= \dfrac{2e^t(e^t+2)}{(2-e^t)^3}\Bigg|_{t=0} = 6.$

Hence $\text{var}(Z) = 6-4 = \underline{2}$

3. $c = 4y + 1$ is increasing, so we can use a transform method.

$$c = 4y + 1 \Rightarrow y = \frac{c-1}{4}$$

$$\frac{dy}{dc} = \frac{1}{4}. \quad f_C(c) = f_Y(h^{-1}(c)) \times \frac{1}{4}.$$

$$f_C(c) = \frac{1}{2} \exp\left(-\frac{1}{2}\left(\frac{c-1}{4}\right)\right) \times \frac{1}{4}$$

$$= \frac{1}{8} \exp\left(\frac{1-c}{8}\right).$$

$$f_C(c) = \begin{cases} \frac{1}{8} \exp\left(\frac{1-c}{8}\right) & 1 \le c \le \infty \\ 0, & \text{otherwise} \end{cases}$$

4. a) Exp is memoryless so waiting time don't matter.

$P(T \leq 3 \mid T > 2) = 1 - e^{-(x)} = \underline{0.63}$

b) Poisson process: $P(N(3) = 3) = \frac{3^3}{3!} \times e^{-3} = 0.22$

c) Mean = λh = $\underline{60 \text{ strikes}}$

d). Time to this event has gamma dist^n, expected value = 5 mins.