Assignment 2 – Solutions

26 marks total

1. a) The ith trial can be thought of as whether or not the ith offer is above \$400. Assuming the offers are iid, N is geometric with parameter p.

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$$f_N(x) = P(N=x) = (1-p)^{x-1}p \quad \text{for } x = 1, 2, 3, \ldots$$

b) $P(N=3) = (1-p)^2 p$

c) For this we require the CDF.

$$F(m) = P(N \le m) = \sum_{i=1}^{m} (1-p)^{i-1} p.$$

$$= p \, \frac{1-(1-p)^m}{1-(1-p)} = 1-(1-p)^m$$

OR Use the result from the tutorial that $P(N>m) = (1-p)^m$

Hence $P(N \le m) = 1-(1-p)^m$

d) As $N \sim Geo(p)$, $E[N] = 1/p$.

2. a) $\text{Bino}(3, p)$

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b) $$P(X \geq 1) = 1 - P(X=0)$$
$$= 1 - \binom{3}{0}(1-p)^3$$
$$= 1 - (1-p)(1-2p+p^2)$$
$$= 1 - [1 - 2p + p^2 - p + 3p^2 - p^3]$$
$$= 3p - 3p^2 + p^3$$

c) No as $3p - \underbrace{p(3p - p^2)}$

This term is always positive,
hence $P(X \geq 1) < 3p$

If p is small the second term will be much smaller than the first

More technically $P(X \geq 1) = 3p + O(p)$.

3. $\boxed{5}$

$$P(X+Y=k) = \sum_{j=0}^{k} P(X+Y=k \mid X=j)\,P(X=j)$$

$$= \sum_{j=0}^{k} P(Y=k-j)\,P(X=j)$$

$$= \sum_{j=0}^{k} \frac{e^{-\lambda}\lambda^{k-j}}{(k-j)!} \cdot \frac{e^{-\mu}\mu^{j}}{j!}$$

$$= \frac{e^{-(\lambda+\mu)}}{k!} \underbrace{\sum_{j=0}^{k} \binom{k}{j} \mu^{j} \lambda^{k-j}}_{\text{Use binomial thm}}$$

$$= \frac{e^{-(\lambda+\mu)}(\lambda+\mu)^{k}}{k!}$$

Hence $X+Y \sim \text{Poiss}(\lambda+\mu)$.

4. $\boxed{5}$

$$m(t) = \left(\frac{1-p}{1-pe^t}\right)^r, \quad m'(t) = (1-p)^r \cdot \frac{d}{dt}(1-pe^t)^{-r}$$

$$= (1-p)^r \cdot -r \times -pe^t \times (1-pe^t)^{-r-1}$$

$$= rpe^t(1-p)^r \cdot (1-pe^t)^{-r-1}$$

$$m'(0) = rp(1-p)^r(1-p)^{-r-1}$$

$$= \underline{\frac{rp}{1-p}}$$

5) $$g(t) = \left(\frac{1-p}{1-pt}\right)^r$$

$$\underline{g(0) = (1-p)^r}$$

$$g'(t) = (1-p)^r \cdot -r \cdot -p \cdot (1-pt)^{-r-1}$$

$$= rp(1-p)^r \cdot (1-pt)^{-r-1}$$

$$\underline{g'(0) = rp(1-p)^r.}$$

$$P(X>1) = 1 - P(X=0) - P(X=1)$$

$$\underline{= 1 - (1-p)^r - rp(1-p)^r}$$

5. $\boxed{R}\ \boxed{B}\ \boxed{R}\ \boxed{R}\ \boxed{B}\ \ldots\ldots\ \boxed{B}$ 52 cards

$\boxed{5}$

So 51 pairs.

Let $\mathbb{I}_{X_i}$ = indicator that cards i and $i+1$ are both red...

$$X = \sum_{i=1}^{51} \mathbb{I}_{X_i}$$

$$E[X] = 51\, E[\mathbb{I}_{X_i}] = 51 \times \underbrace{P(\text{pair})}_{\frac{\binom{26}{2}}{\binom{52}{2}}} = 12.5$$

$$= \frac{26}{52} \times \frac{25}{51}$$