Probability and Statistics Assignment 2.

1. a) pmf $= f(N) = (1-p)^{n-1}(p)$, $n = 1, 2, \ldots, N$

b) $P(N=3) = (1-p)^{3-1}(p)$
$= (1-p)^2(p)$
$= (1-p)(1-p)(p)$
$= (1-p-p+p^2)(p)$
$= (1-2p+p^2)(p)$
$= p^3-2p^2+p$

c) $P(N \leq m) = 1 - P(N > m)$
$= 1 - (1-p)^{m-1}(p)$

d) $E[N] = \frac{1}{p}$

2. a) It is a binomial distribution $f(y) = \binom{3}{y} p^y (1-p)^{3-y}$

n = number of tickets.

b) $P(Y \geq 1) = 1 - P(Y=0)$
$= 1 - \binom{3}{0}(p)^0(1-p)^{3-0}$
$= 1 - \left(\frac{3!}{0!3!}\right)(1)(1-p)^3$
$= 1 - (1)(1)(1-p)^3$
$= 1 - (1-p)(1-p)(1-p)$
$= 1 - (1-p)(1-p-p+p^2)$
$= 1 - (1-p)(1-2p+p^2)$
$= 1 - [1-2p+p^2-p+2p^2-p^3]$
$= 1 - [1-3p+3p^2-p^3]$
$= 1 - 1 + 3p - 3p^2 + p^3$
$= p^3 - 3p^2 + 3p$
$\quad\ \ \|$
$3p - 3p^2 + p^3$ # (proven)

c) → As n increases, the binomial distribution will eventually converge to n×p.
Given that n=3 is compare to n=1, the probability does more or less tripple.

Ex: $p = \frac{1}{3}$ n = 1
$1 - \left(\frac{3!}{0!3!}\right)\left(\frac{1}{3}\right)^0\left(1-\frac{1}{3}\right)^{1-0} = 0.3333$

when n = 3
$3\left(\frac{1}{3}\right) - 3\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 = 0.7037$

→ if p is small then $p \to 0$, $p^2 \to 0$, $p^3 \to 0$ even faster meaning that the probability will likely be small as well.

3. I'm assuming that $X+Y=a$ ← random constants

$X \sim \text{Poisson}(\lambda)$

$Y \sim \text{Poisson}(\mu)$

Based on the Definition of Total law of Probability

$Y = a - X$

$$P(X+Y=a) = \sum_{i=0}^{a} P(X=i \cap Y=a-i)$$

By the definition of Independence.

$$= \sum_{i=0}^{a} P(X=i)\,P(Y=a-i).$$

$$= \sum_{i=0}^{a} \frac{e^{-\lambda}\lambda^{i}}{i!} \cdot \frac{e^{-\lambda}\mu^{a-i}}{(a-i)!}$$

$$= \left(e^{-\lambda} \sum_{i=0}^{a} \frac{\lambda^{i}\mu^{a-i}}{i!\,(a-i)!}\right)$$ multiply $a!$

$$= e^{-\lambda} \sum_{i=0}^{a} \frac{a!\,\lambda^{i}\mu^{a-i}}{i!\,(a-i)!} \Rightarrow e^{-\lambda} \sum_{i=0}^{a} \frac{a!}{i!\,(a-i)!}\left(\lambda^{i}\mu^{a-i}\right)$$

this is just the binomial equation.

$$= e^{-\lambda} \sum_{i=0}^{a} \binom{a}{i}\left(\lambda^{i}\mu^{a-i}\right)$$ divide by $a!$

$$= \frac{e^{-\lambda}}{a!} \sum_{i=0}^{a} \binom{a}{i}\left(\lambda^{i}\mu^{a-i}\right)$$

By the definition of binomial expansion. in reverse.

$$= \frac{e^{-\lambda}}{a!} \sum_{i=0}^{a} (\lambda+\mu)^{a}$$

$\lambda+\mu = \beta$ some random constant

$$= \frac{e^{-\lambda}\beta^{a}}{a!}$$

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a) $E[X]$ for $m(t) = E[e^{tX}]$

$$m'(t) = \frac{dm(t)}{dt} = \frac{d}{dt}E[e^{tX}] \quad \text{let } u = \frac{1-p}{1-pe^t}$$

$$= \frac{d}{dt}\left(\frac{1-p}{1-pe^t}\right)^r$$

$$= r(u)^{r-1} \cdot \frac{du}{dt}\left(\frac{1-p}{1-pe^t}\right)^{u}_{v}$$

$$= r(u)^{r-1} \cdot \left[\frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}\right]$$

$$= r(u)^{r-1} \cdot \frac{(1-pe^t)(0) - (1-p)(-pe^t)}{(1-pe^t)^2}$$

$$= r(u)^{r-1} \cdot \left(\frac{-(1-p)(-pe^t)}{(1-pe^t)^2}\right)$$

$$= r\left(\frac{1-p}{1-pe^t}\right)^{r-1} \cdot \left(-\frac{(1-p)(-pe^t)}{(1-pe^t)^2}\right)$$

let $t=0$, $m'(0) = r\left(\frac{1-p}{1-pe^0}\right)^{r-1} \cdot \left(-\frac{(1-p)(-pe^0)}{(1-pe^0)^2}\right)$

$$= r\left(\frac{1-p}{1-p}\right)^{r-1} \cdot \left(-\frac{(1-p)(-p)}{(1-p)^2}\right)$$

$$= r(1)^{r-1} \cdot \frac{\cancel{(1-p)}(p)}{\cancel{(1-p)}(1-p)}$$

$$= r \cdot \left(\frac{p}{1-p}\right)$$

$$= \frac{rp}{1-p}$$

b) $E[X]$ for $g(t) = E[t^X]$

$$g'(t) = \frac{dg(t)}{dt} = \frac{d}{dt}\left(\frac{1-p}{1-pt}\right)^r \quad \text{let } s = \frac{1-p}{1-pt}$$

$$= r(s)^{r-1} \cdot \frac{du}{dt}\left(\frac{1-p}{1-pt}\right)^{u}_{v}$$

$$= r(s)^{r-1} \cdot \left(\frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}\right)$$

$$= r(s)^{r-1} \cdot \left[\frac{(1-pt)(0) - (1-p)(-p)}{(1-pt)^2}\right]$$

$$= r(s)^{r-1} \cdot \left(\frac{-(1-p)(-p)}{(1-pt)^2}\right)$$

$$g'(t) = r\left(\frac{1-p}{1-pt}\right)^{r-1} \cdot \left(\frac{-(1-p)(-p)}{(1-pt)^2}\right)$$

To calculate $P(X>1) = 1 - P(X=0) - P(X=1)$

To find $P(X=0)$, use the properties of probability generating function where $g(0) = P(X=0)$

$$g(0) = P(X=0) = \left(\frac{1-p}{1-p(0)}\right)^r = \left(\frac{1-p}{1}\right)^r = (1-p)^r$$

To find $P(X=1)$, find $g'(0) = P(X=1) =$

$$g'(0) = P(X=1) = r\left(\frac{1-p}{1-p(0)}\right)^{r-1} \cdot \left(\frac{(1-p)(p)}{(1-p(0))^2}\right)$$

$$= r\left(\frac{1-p}{1}\right)^{r-1} \cdot \left(\frac{(1-p)(p)}{1}\right)$$

$$= r(1-p)^{r-1} \cdot (1-p)(p)$$

$$= rp(1-p)^r$$

Result $= 1 - \left[(1-p)^r + rp(1-p)^r\right]$

$$= 1 - \left[(1+rp)(1-p)^r\right]$$

5. $I_1 \begin{cases} 1 & \text{where card1 \underline{and} card2 are red} \\ 0 & \underline{\text{otherwise}} \end{cases}$

$I_2 \begin{cases} 1 & \text{where card2 \underline{and} card3 are red} \\ 0 & \underline{\text{otherwise}} \end{cases}$

$\vdots$

$I_{51} \begin{cases} 1 & \text{where card51 \underline{and} card52 are red} \\ 0 & \underline{\text{otherwise}} \end{cases}$

$\downarrow$

$$E[I_A] = E[I_1] + E[I_2] + \cdots + E[I_{51}]$$

$$= 52 \text{ cards} \times \left[\text{probability that the first card is red}\right] \times \left[\text{probability that the adjacent card is red}\right]$$

$$= 52 \times \frac{26}{52} \times \frac{25}{51}$$

$$= \frac{26 \times 25}{51} = 12.7451 \#$$

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