Assignment 5 Solutions

30 marks total

1. Lightening strikes are Poisson, so numbers in non-overlapping intervals are independent.

8

$X \sim \text{Poisson}(2\lambda)$, $Y \sim \text{Poisson}(4\lambda)$, $Z \sim \text{Poisson}(4\lambda)$.

a) $$p(x,y,z) = \frac{e^{-2\lambda}(2\lambda)^x}{x!} \cdot \frac{e^{-4\lambda}(4\lambda)^y}{y!} \cdot \frac{e^{-4\lambda}(4\lambda)^z}{z!}$$

for $x, y, z \geq 0$.

b) let $T = X + Y + Z \sim \text{Poisson}(10\lambda)$

$$P(X=x, Y=y, Z=z \mid T=t) = \frac{P(T=t \mid x,y,z) \cdot P(x,y,z)}{P(t)}.$$

$$P(t \mid x,y,z) = \begin{cases} 0 & \text{if } x+y+z \neq t \\ 1 & \text{if } x+y+z = t \end{cases}$$

$$P(x,y,z \mid t) = \frac{\dfrac{e^{-2\lambda}(2\lambda)^x}{x!} \cdot \dfrac{e^{-4\lambda}(4\lambda)^y}{y!} \cdot \dfrac{e^{-4\lambda}(4\lambda)^z}{z!}}{\dfrac{e^{-15\lambda}(10\lambda)^t}{t!}}$$

$$= \frac{t!}{x!\,y!\,z!} \left(\frac{2}{10}\right)^x \left(\frac{4}{10}\right)^y \left(\frac{4}{10}\right)^z$$

So X, Y, Z is conditionally multinomial with $n = 10$, $p_1 = \frac{1}{5}$, $p_2 = \frac{2}{5}$, $p_3 = \frac{2}{5}$.

2. $X_1 \sim \text{Bino}\left(n, \frac{1}{8}\right)$

$\boxed{6}$ $X_1 + X_4 \sim \text{Bino}\left(n, \frac{2}{8}\right)$

From the notes we have $\text{Var}(X_1 + X_4) = n(1-p)p = n\frac{2}{8} \times \frac{6}{8} = \frac{12n}{64}$

Also have $\text{Var}(X_1 + X_4) = \text{Var}(X_1) + \text{var}(X_4) + 2\,\text{Cov}(X_1, X_4)$

Hence $\frac{12n}{64} = n \cdot \frac{1}{8} \times \frac{7}{8} + n\frac{1}{8} \times \frac{7}{8} + 2\,\text{cov}(X_1, X_4)$.

$\Rightarrow \text{cov}(X_1, X_4) = \frac{n}{2}\left(\frac{12}{64} - \frac{2 \times 7}{64}\right) = \frac{-n}{64}$

$$\text{Corr}(X_1, X_4) = \frac{\text{cov}(X_1, X_4)}{\sqrt{\text{Var}(X_1) \cdot \text{Var}(X_4)}} = \frac{\frac{-n}{64}}{\sqrt{\frac{7n}{64} \times \frac{7n}{64}}}$$

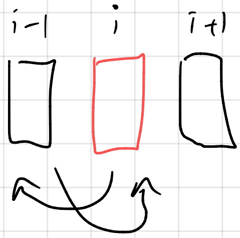
$$= \frac{-\cancel{n}}{\cancel{64}} \cdot \frac{\cancel{64}}{7\cancel{n}} = \frac{-1}{7}$$

3. a) That the process is Markov follows from the fact that books are picked at random and successive choices are independent. Thus the process will be memoryless.

8

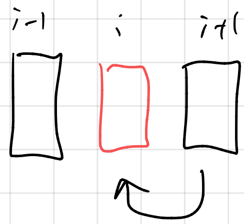
$P_{i,i-1} = p$

The red book is selected with probability p and moves to the right if $i = 2, 3, \dots N$. If $i = 1$, it doesn't move.



$P_{i,i+1} = \frac{1-p}{N-1}$

A blank book is selected with probability $1-p$, and it is to the right of the red one, which occurs with probability $\frac{1}{N-1}$ as there are $N-1$ blank books and they are equally likely to be picked. If this happens the red book moves to the left.



If red book is at $i = N$, then there is no book to the right, hence this only works for $i = 1, \dots, N-1$.

$P_{i,i} = 1 - p - \frac{1-p}{N-1}$.

Prob of swapping two blank books, hence the red book doesn't move.

$P_{1,1} = 1 - \frac{1-p}{N-1}$ If red ball is at $i=1$, it can only be moved by selecting the blank ball to the right. This happens with probability $\frac{1-p}{N-1}$, hence the probability it doesn't move is $1 - \frac{1-p}{N-1}$.

$P_{N,N} = 1-p$. Similar to last case, the red ball can only be moved by selecting it with prob p. So the probability it doesn't move is $1-p$.

The last three can also be obtained by 1 − the first two for the correct values of i.

b). Stationary distribution satisfies $\pi = \pi \mathbb{P}$ where

4

$$\mathbb{P} = \begin{pmatrix} 1-\left(\frac{1-p}{N-1}\right), & \frac{1-p}{N-1}, & 0, & 0, & 0 & \cdots \\ p, & 1-p-\left(\frac{1-p}{N-1}\right), & \frac{1-p}{N-1}, & 0, & 0 & \cdots \\ 0, & \boxed{p, \quad 1-p-\left(\frac{1-p}{N-1}\right), \quad \frac{1-p}{N-1},} & & & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

→ this pattern repeats down the diagonals.

$$(\pi_1, \pi_2, \pi_3, \ldots \pi_N) = (\pi_1, \pi_2, \pi_3, \ldots \pi_N)\mathbb{P}.$$

So $\pi_1 = \pi_1\left(1 - \frac{1-p}{N-1}\right) + p\pi_2$

$\pi_1 = \left(\cancel{1} - \cancel{1} + \frac{1-p}{N-1}\right) = p\pi_2 \quad \Rightarrow \quad \boxed{\pi_2 = \frac{1-p}{(N-1)p}\pi_1}$

Doing the next row x col multiplication,

$$\pi_2 = \left(\frac{1-p}{N-1}\right)\pi_1 + \left[1-p-\left(\frac{1-p}{N-1}\right)\right]\pi_2 + p\pi_3$$

$$= \cancel{\left(\frac{1-p}{N-1}\right)} \cdot \frac{\cancel{(N-1)}p}{\cancel{1-p}}\,\pi_2 + \left[1-p-\left(\frac{1-p}{N-1}\right)\right]\pi_2 + p\pi_3$$

$$= \left(1-\left(\frac{1-p}{N-1}\right)\right)\pi_2 + p\pi_3$$

This has the same form as the last eq^n hence

$$\boxed{\pi_3 = \frac{1-p}{p(N-1)}\pi_2}$$

c) Following the pattern from part (b) we see

4 $$\pi_{i+1} = \frac{1-p}{p(N-1)}\pi_i \quad \text{for } i = 2, 3, \ldots, N-1$$

Checking the $i=N$ case, the lower-right part of $\mathbb{P}$ is,

$$\begin{pmatrix} \cdots, & 0, & \frac{1-p}{N-1}, & 0 \\ \cdots & p, & 1-p-\left(\frac{1-p}{N-1}\right), & \frac{1-p}{N-1} \\ \cdots & 0, & p, & 1-p \end{pmatrix}$$

Hence, $\pi_N = \left(\frac{1-p}{N-1}\right)\pi_{N-1} + (1-p)\pi_N \Rightarrow \pi_N = \frac{1-p}{(N-1)p}\pi_{N-1}$

So the pattern is the same.

Let $A = \frac{1-p}{(N-1)p}$ then

$$\pi_2 = A\pi_1$$

$$\pi_3 = A^2\pi_1 \quad \text{etc...}$$

$$\pi_i = A^{i-1}\pi_1 \quad \text{for } i = 1, \ldots, N$$

We also have $\sum_{i=1}^{N} \pi_i = 1$, so

$$\pi_1 \sum_{i=1}^{N} A^{i-1} = 1$$

$$\pi_1 \underbrace{\sum_{j=0}^{N-1} A^j}_{\text{geometric series}} = 1$$

$$\pi_1 \left(\frac{A^N - 1}{A - 1}\right) = 1$$

$$\Rightarrow \pi_i = \left(\frac{A-1}{A^N-1}\right) \cdot A^{i-1} \quad \text{for } i = 1, \ldots, N$$