SMIII

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1a) $\log(\lambda_{ij}) = \gamma_0 + \gamma_1 x_i + \gamma_2 x_j + \gamma_3 x_i x_j$

i) Treatment A & Controls

$x_i = 0, x_j = 0$

$$\log(\lambda_{11}) = \gamma_0 + 0 + 0 + 0$$
$$= \gamma_0$$

ii) Treatment A, Cases

$x_i = 0, x_j = 1$

$$\log(\lambda_{12}) = \gamma_0 + 0 + \gamma_2 + 0$$
$$= \gamma_0 + \gamma_2$$

iii) Treatment B, controls

$x_i = 1, x_j = 0$

$$\log(\lambda_{21}) = \gamma_0 + \gamma_1$$

iv) Treatment B, cases

$x_i = 1, x_j = 1$

$$\log(\lambda_{22}) = \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3$$

1b) $P(Y_{i2} \mid Y_{i1} + Y_{i2} = n_i) = P(Y_{i2} = y_{i2} \mid Y_{i1} + Y_{i2} = n_i)$

using conditional prob: $= \dfrac{P(Y_{i2} = y_{i2} \cap Y_{i1} + Y_{i2} = n_i)}{P(Y_{i1} + Y_{i2} = n_i)}$

due to independence $= \dfrac{P(Y_{i2} = y_{i2}) \cdot P(Y_{i1} + Y_{i2} = n_i)}{P(Y_{i1} + Y_{i2} = n_i)}$ $\}$ (1) $\}$ (2)

from (2)

$P(Y_{i1} + Y_{i2} = n_i)$

$Y_{i1} + Y_{i2} \sim Po(\lambda_{i1} + \lambda_{i2})$

therefore $P(Y_{i1} + Y_{i2} = n_i) = \dfrac{(\lambda_{i1} + \lambda_{i2})^{n_i} e^{-(\lambda_{i1} + \lambda_{i2})}}{(n_i)!}$

from (1)

$$P(Y_{i2} = y_{i2}) \cdot P(Y_{i1} + Y_{i2} = n_i) = P(Y_{i2} = y_{i2}) \cdot P(Y_{i1} = n_i - y_{i2})$$
$$= \frac{(\lambda_{i2})^{y_{i2}} e^{-\lambda_{i2}}}{(y_{i2})!} \cdot \frac{(\lambda_{i1})^{n_i - y_{i2}} e^{-\lambda_{i1}}}{(n_i - y_{i2})!}$$

Now $^{(1)}/_{(2)}$

$$= \frac{(\lambda_{i2})^{y_{i2}} e^{-\lambda_{i2}}}{(y_{i2})!} \cdot \frac{(\lambda_{i1})^{n_i - y_{i2}} e^{-\lambda_{i1}}}{(n_i - y_{i2})!} \cdot \frac{(n_i)!}{(\lambda_{i1} + \lambda_{i2})^{n_i} e^{-(\lambda_{i1} + \lambda_{i2})}}$$

$$= \frac{(n_i)!}{(y_{i2})!(n_i - y_{i2})!} \cdot \frac{(\lambda_{i2})^{y_{i2}} \cancel{e^{-\lambda_{i2} - \lambda_{i1}}} \cdot (\lambda_{i1})^{n_i - y_{i2}}}{(\lambda_{i1} + \lambda_{i2})^{n_i} \cdot \cancel{e^{-(\lambda_{i1} + \lambda_{i2})}}}$$

$$= \binom{n_i}{y_{i2}} \cdot \frac{(\lambda_{i2})^{y_{i2}} \cdot (\lambda_{i1})^{n_i - y_{i2}}}{(\lambda_{i1} + \lambda_{i2})^{n_i - y_{i2} + y_{i2}}}$$

$$= \binom{n_i}{y_{i2}} \cdot \frac{(\lambda_{i2})^{y_{i2}} \cdot (\lambda_{i1})^{n_i - y_{i2}}}{(\lambda_{i1} + \lambda_{i2})^{n_i - y_{i2}} (\lambda_{i1} + \lambda_{i2})^{y_{i2}}}$$

$$= \underbrace{\binom{n_i}{y_{i2}} \cdot \left(\frac{\lambda_{i2}}{\lambda_{i1} + \lambda_{i2}}\right)^{y_{i2}} \cdot \left(\frac{\lambda_{i1}}{\lambda_{i1} + \lambda_{i2}}\right)^{n_i - y_{i2}}}$$

This is the Binomial PMF for $Y \sim B(n_i, \frac{\lambda_{i2}}{\lambda_{i1} + \lambda_{i2}})$

$\therefore \pi_i = \frac{\lambda_{i2}}{\lambda_{i1} + \lambda_{i2}}$

c) $\log\left(\frac{\pi_i}{1 - \pi_i}\right) = \log\left(\frac{\lambda_{i2}}{\lambda_{i1} + \lambda_{i2}} \cdot \frac{1}{1 - \left(\frac{\lambda_{i2}}{\lambda_{i1} + \lambda_{i2}}\right)}\right)$

$$= \log\left(\frac{\lambda_{i2}}{\lambda_{i1} + \lambda_{i2}} \cdot \frac{1}{\frac{\lambda_{i1}}{\lambda_{i1} + \lambda_{i2}}}\right)$$

$$= \log\left(\frac{\lambda_{i2} \cancel{(\lambda_{i1} + \lambda_{i2})}}{\lambda_{i1} \cancel{(\lambda_{i1} + \lambda_{i2})}}\right)$$

$$= \log\left(\frac{\lambda_{i2}}{\lambda_{i1}}\right)$$

$$= \log(\lambda_{i2}) - \log(\lambda_{i1})$$

$$\beta_0 + \beta_1 x_i = \log(\lambda_{i2}) - \log(\lambda_{i1})$$

If Treatment A,

$\beta_0 + \beta_1 x_1 = \log(\lambda_{12}) - \log(\lambda_{11})$

using solutions from Q1a)

$\beta_0 + \beta_1(0) = \gamma_2 \Rightarrow \beta_0$ (*)

If Treatment B,

$\beta_0 + \beta_1 x_2 = \log(\lambda_{22}) - \log(\lambda_{21})$

using solutions from Q1a)

$\beta_0 + \beta_1 = \gamma_2 + \gamma_3$ (**)

substituting (*) into (**)

$$\cancel{\gamma_2} + \beta_1 = \cancel{\gamma_2} + \gamma_3$$
$$\beta_1 = \gamma_3$$

d) Testing that treatment has no effect on the probability of being a cases is implying that:

$$\log\left(\frac{\pi_i}{1 - \pi_i}\right) = \beta_0$$

This implies that $\beta_1 = 0$, which also implies $\gamma_3 = 0$. Therefore when plugging $\gamma_3 = 0$ into $\log(\lambda_{ij})$ we get:

$$\log(\lambda_{ij}) = \gamma_0 + \gamma_1 x_i + \gamma_2 x_j + \underbrace{(0) x_i x_j}_{\text{interaction term}}$$

$$\log(\lambda_{ij}) = \gamma_0 + \gamma_1 x_i + \gamma_2 x_j$$

As shown above, the Poisson model has no interaction term anymore.

Therefore we conclude that, testing that treatment has no effect on the probability of being cases, is the same as testing for no interaction in the Poisson Model.