3 Optimisation and Convex Sets

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Optimisation Example A cereal manufacturer makes two kinds of muesli, *nutty special* and *fruity extra*. Each muesli consists of raisins and nuts.

- 0.2 boxes of raisins and 0.4 boxes of nuts make 1 box of *nutty special*.
- 0.4 boxes of raisins and 0.2 boxes of nuts make 1 box of *fruity extra*.
- 14 boxes of raisins and 10 boxes of nuts are available each day.
- The profit on each box of *nutty special* is $8.
- The profit on each box of *fruity extra* is $10.

Assuming that all the muesli that is made will be sold, how many boxes of *nutty special* and *fruity extra* should be made each day, in order to maximise the profit?

◊ 3.0: Mathematical Formulation . . .

3.1 Convex sets

Optimisation problems typically involve a set of *linear constraints* which restrict the possible solutions.

For example, $x_1, x_2, \ldots, x_n$ measure the quantities of inputs (perhaps to a manufacturing process). These are subject to constraints (such as availability). Hence

$$\begin{array}{ccccc} a_{11}x_1 & + \cdots + & a_{1n}x_n & \leq & b_1 \\ a_{21}x_1 & + \cdots + & a_{2n}x_n & \leq & b_2 \\ & \vdots & & & \\ a_{m1}x_1 & + \cdots + & a_{mn}x_n & \leq & b_m \end{array}$$

or

$$A\mathbf{x} \leq \mathbf{v}.$$

If, in addition, each $x_i \geq 0$, then we write $\mathbf{x} \geq 0$.

Subject to these constraints we may wish to *maximise* profit, given by

$$f(x_1, \ldots, x_n) = c_1x_1 + \cdots + c_nx_n.$$

Alternatively $f(x_1, \ldots, x_n)$ may represent a cost which we would want to *minimise*. $f(x_1, \ldots, x_n)$ is the *objective function*.

The constraints define the *feasible region* which is a *convex set*. The maximum or minimum value of the objective function occurs at a vertex. Hence we need only find the value of the objective function at the vertices of the feasible region.

Definition 3.1. A convex set C is a set of points (or region) in $\mathbb{R}^n$ such that the line segment joining any two points in C lies completely in C.

◊ 3.1: Examples ...

Often we consider sets defined by systems of inequalities.

◊ 3.2: Examples ...

We now give a more precise (mathematical) formulation of convex sets and vertices, and also show that a set defined by linear inequalities is a convex set. We use vectors in $\mathbb{R}^n$. Recall that we extended vectors to n-dimensional space $\mathbb{R}^n$:

$$\mathbb{R}^n = \{(v_1, \ldots, v_n) \mid v_i \in \mathbb{R}\}$$

Recall that a set C is convex if for any two points P and Q in C, the line joining P and Q lies in C.

We look for a mathematical expression for the line segment PQ. This is done with vectors.

◊ 3.3: Derivation of formula for line segment ...

Thus a set C (of $\mathbb{R}^n$) is *convex* if for any two points P and Q in C, with position vectors $\mathbf{u}$ and $\mathbf{v}$ respectively ($\overrightarrow{OP} = \mathbf{u}, \overrightarrow{OQ} = \mathbf{v}$), then for all t, $0 \le t \le 1$,

$$(1-t)\mathbf{u} + t\mathbf{v} \in C.$$

We now show the set of points (vectors) satisfying a set of inequalities such as $A\mathbf{x} \le \mathbf{b}$ is convex.

To do so we extend the dot (scalar) product by the definition that if $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ then

$$\mathbf{u} \cdot \mathbf{v} = (u_1, \ldots, u_n) \cdot (v_1, \ldots, v_n) = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \,.$$

All the properties of the dot product carry over from $\mathbb{R}^2$ or $\mathbb{R}^3$ to $\mathbb{R}^n$: for three examples,

$$\begin{aligned} (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} &= \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} \\ (t\mathbf{u}) \cdot \mathbf{v} &= t(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (t\mathbf{v}) \\ \mathbf{u} \cdot \mathbf{v} &= \mathbf{v} \cdot \mathbf{u} \,. \end{aligned}$$

Consider the system of inequalities $A\mathbf{x} \le \mathbf{b}$; that is,

$$\begin{array}{ccccc} a_{11}x_1 & + \cdots + & a_{1n}x_n & \le & b_1 \\ a_{21}x_1 & + \cdots + & a_{2n}x_n & \le & b_2 \\ & & \vdots & & \\ a_{m1}x_1 & + \cdots + & a_{mn}x_n & \le & b_m \end{array}$$

The ith inequality $(i = 1, \ldots, m)$ $a_{i1}x_1 + \cdots + a_{in}x_n \le b_i$ can be written in the form $\mathbf{a}_i \cdot \mathbf{x} \le b_i$ where $\mathbf{a}_i = (a_{i1}, \ldots, a_{in})$ and $\mathbf{x} = (x_i, \ldots, x_n)$.

Consider the set $C_i = \{\mathbf{x} \mid \mathbf{a}_i \cdot \mathbf{x} \leq b_i\} = \{\mathbf{x} = (x_1, \ldots, x_n) \mid (a_{i1}, a_{i2}, \ldots, a_{in}) \cdot \mathbf{x} \leq b_i\}$. That is, C_i is the set consisting of all vectors $\mathbf{x} = (x_i, \ldots, x_n)$ in $\mathbb{R}^n$ satisfying the ith inequality $a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i$ or $\mathbf{a}_i \cdot \mathbf{x} \leq b_i$. We show that C_i is a convex set in $\mathbb{R}^n$.

◊ 3.4: Showing that C_i is convex ...

Theorem 3.1. *The intersection of two convex sets is convex.*

◊ 3.5: proof ...

We have shown that each C_i is convex, and the theorem just proved shows that the intersection of convex sets is convex.

Thus $A\mathbf{x} \leq \mathbf{b}$ defines a convex set in $\mathbb{R}^n$, as does the combination $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

Finally, as a necessary preliminary to introducing our general optimisation theorem, we give precise (mathematical) definitions of a *vertex* of a convex set and a bounded convex set.

Vertices A vertex $\mathbf{v}$ of C is point of C that does not lie on a straight line between two other points in C.

Definition 3.2. A point P with position vector $\overrightarrow{OP} = \mathbf{v}$ is a *vertex* of the convex set C if $\mathbf{v}$ cannot be written in the form

$$(1-t)\mathbf{u} + t\mathbf{w}, \quad 0 \leq t \leq 1,$$

for any two distinct points $\mathbf{u}, \mathbf{w} \in C$ except when $\mathbf{v} = \mathbf{u}$ (and $t = 0$) or $\mathbf{v} = \mathbf{w}$ (and $t = 1$).

Definition 3.3. A convex set C is *bounded* if there is some positive number M such that for any point P in C, the length

$$\|\overrightarrow{OP}\| < M.$$

(That is, the length of $\overrightarrow{OP}$ cannot be arbitrarily large for points P in C.)

We now state our main result concerning optimisation problems.

Theorem 3.2. *Let C be a* convex set *defined by a set of linear inequalities $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. If C is bounded, then the linear function*

$$f(x_1, \ldots, x_n) = c_1 x_1 + \cdots + c_n x_n$$

takes its maximum (or minimum) value on C at a vertex of C. If C is an unbounded convex set, and if f takes a maximum (or minimum) value on C, then this maximum (or minimum) occurs at a vertex of C.

◊ **3.6: Proof** ...

3.2 Methods for solving optimisation problems

Theorem 3.2 tells us how to solve these optimisation problems. Determine the vertices, evaluate the linear function f at each vertex, and check which is the maximum (or minimum) value. The only problem that remains to be solved is to find the vertices. In two dimensions we can sketch the region and determine the vertices, which are the intersections of certain boundary lines.

Cereal example A cereal manufacturer makes two kinds of muesli, Nutty Special and Fruity Extra. Each muesli consists of raisons and nuts. 0.2 boxes of raisins and 0.4 boxes of nuts make 1 box of Nutty Special. 0.4 boxes of raisins and 0.2 boxes of nuts make 1 box of Fruity Extra. Suppose that 14 boxes of raisins and 10 boxes of nuts are available each day. The profit on each box of Nutty Special is \$8 and on each box of Fruity Extra is \$10. Assuming all the muesli that is made is sold how many boxes of Nutty Special and Fruity Extra should be made each day in order to maximise the profit?

Graphical Solution Recall the mathematical formulation. We wish to *maximise* the profit

$$P(x, y) = 8x + 10y,$$

subject to the four *constraints*

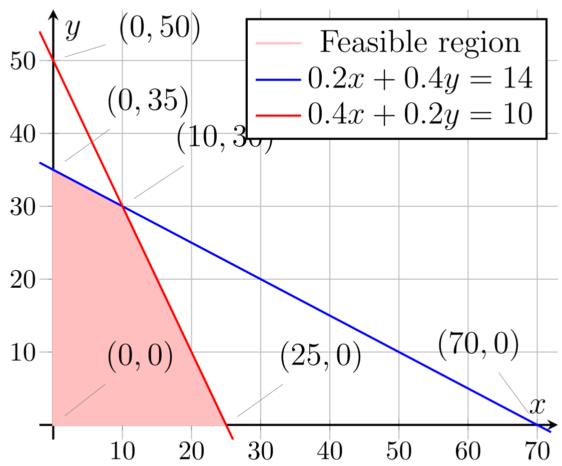
$$0.2x + 0.4y \leq 14 \tag{3.1}$$

$$0.4x + 0.2y \leq 10 \tag{3.2}$$

$$x \geq 0 \tag{3.3}$$

$$y \geq 0 \tag{3.4}$$

The set of points $(x, y) \in \mathbb{R}^2$ satisfying the constraints is the *feasible region*. We plot this region and find the solution.



◊ 3.7: Finding the solutions . . .

This graphical method is only practical if there are just two variables in the problem.

We now give an algebraic method of solution of optimisation problems which is applicable in general.

First note that if we have an inequality of the form

$$a_{j1}x_1 + \cdots + a_{jn}x_n \geq b_j$$

we can multiply by (-1) and put our inequality in the form

$$-a_{j1}x_1 + (-a_{j2}x_2) + \cdots + (-a_{jn}x_n) \leq (-b_j).$$

Thus we may suppose all our inequalities are of the form

$$a_{j1}x_1 + \cdots + a_{jn}x_n \leq b_j.$$

To solve an Optimisation Problem we have to find the vertices of the convex set (which is the feasible region). The values of the function f are then be computed at each vertex and the maximum or minimum of f determined.

Method For each inequality, add a *slack variable* to produce a linear equation (equality):

$$\left.\begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1 \\ \qquad\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m \end{array}\right\} m \text{ inequalities}$$

$$\left.\begin{array}{llllll} a_{11}x_1 + \cdots + a_{1n}x_n & + x_{n+1} & & & = b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n & & + x_{n+2} & & = b_2 \\ \quad\vdots & & & & \\ a_{m1}x_1 + \cdots + a_{mn}x_n & & & + x_{n+m} & = b_m \end{array}\right\} \begin{array}{l} m \text{ different} \\ \text{slack} \\ \text{variables} \\ \text{added} \end{array}$$

Note each variable $x_j \geq 0$ (including the slack variables). (If one of the constraints is actually an equality, then there is no need to add a slack variable.)

As there are m equations there are m possible pivots or basic variables; here by a *pivot column* we mean a column with a pivot, in other words one of the columns of the $m \times m$ identity matrix I_m. There are $\binom{m+n}{m}$ possible choices for m pivots or basic variables. For *each choice*, use Gauss–Jordan elimination to obtain a row equivalent matrix with the chosen columns as pivot columns.

Put the free variables $= 0$ and solve for the basic variables. This solution is called a basic solution.

There at most $\binom{m+n}{m}$ basic solutions; however for some of the choices of basic variables it may not be possible to get a basic solution; for the last (or second to last or ...) choice of pivot it may not be possible to pivot as there may be a zero in the pivot position.

The *vertices* of the convex set are the non-negative basic solutions. (Non-negative as each $x_j \geq 0$.)

In the previous example the basic solutions correspond to the intersections of the defining (bounding) lines. The negative basic solutions correspond to those intersections which are not vertices (that is, lie outside the feasible region).

Example Find the maximum value of the function

$$f(x_1, x_2, x_3) = 6x_1 + 4x_2 - 7x_3$$

subject to the constraints

$$\begin{aligned} 2x_1 + x_2 - 2x_3 &\leq 10, \\ x_1 + 4x_2 - x_3 &\leq 12, \end{aligned}$$

and $x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$

Solution Add *slack variables* x_4 and x_5 so that the first two inequalities become equalities:

$$\begin{aligned} 2x_1 + x_2 - 2x_3 + x_4 &= 10, \\ x_1 + 4x_2 - x_3 + x_5 &= 12 \end{aligned}$$

Note that $x_4 \geq 0$ and $x_5 \geq 0$ also.

The vertices of the feasible region are the non-negative basic solutions. To find the basic solutions we consider all $\binom{5}{2} = 10$ pairs of variables. The free variables in each case are put equal to zero and the solution is then a basic solution. Take the non-negative basic solutions and determine the maximum value of f for each of these solutions.

Note

- It is usually possible to go from one basic solution to another by a single pivot operation.
- Not every pair of variables may give rise to a basic solution. The second pivot operation is not be possible when there is a zero in the position in which you must perform this pivoting operation.

	Pivots (basic var's)	Equations						Basic solution
1.	x_4, x_5	2	1	−2	1	0	10	
		1	4	−1	0	1	12	$(0,0,0,10,12)$
2.	pivot (1,3)							
	x_3, x_5	−1	−1/2	1	−1/2	0	−5	
		0	7/2	0	−1/2	1	7	$(0,0,-5,0,7)$
3.	pivot (1,2)							
	x_2, x_5	2	1	−2	1	0	10	
		−7	0	7	−4	1	−28	$(0,10,0,0,-28)$
4.	pivot (1,1)							
	x_1, x_5	1	1/2	−1	1/2	0	5	
		0	7/2	0	−1/2	1	7	$(5,0,0,0,7)$
5.	pivot (2,2)							
	x_1, x_2	1	0	−1	4/7	−1/7	4	
		0	1	0	−1/7	2/7	2	$(4,2,0,0,0)$
6.	pivot (2,3)							
	x_1, x_3	Is not possible as the (2,3) entry is zero. Thus x_1, x_3 *cannot* be a pair of basic variables.						
7.	pivot (2,4)							
	x_1, x_4	1	4	−1	0	1	12	
		0	−7	0	1	−2	−14	$(12,0,0,-14,0)$
8.	pivot (1,2)							
	x_2, x_4	1/4	1	−1/4	0	1/4	3	
		7/4	0	−7/4	1	−1/4	7	$(0,3,0,7,0)$
9.	pivot (2,3)							
	x_2, x_3	0	1	0	−1/7	2/7	2	
		−1	0	1	−4/7	1/7	−4	$(0,2,-4,0,0)$
10.	pivot (1,4): $R1, R2 = R2, R1$							
	x_3, x_4	−1	−4	1	0	−1	−12	
		0	−7	0	1	−2	−14	$(0,0,-12,-14,0)$

◊ **3.8: Finding the solution . . .**

3.3 Formulation of optimisation problems

There are two parts to any optimisation problem. The first is to formulate the problem as a mathematical problem and the second is to solve the mathematical problem. In the next examples we particularly consider the first part of the problem.

Example 1: A typical optimisation problem

A fruit dealer can transport up to 800 boxes of fruit from Renmark to Adelaide on a truck. He must transport *at least* 200 boxes of oranges, at *at least* 100 boxes of grapefruit and *at most* 200 boxes of tangerines. The profit per box is \$2 for oranges, \$1 for grapefruit, and \$3 for tangerines. *How many boxes* of *each* kind of fruit should be loaded onto the truck in order to maximise profit?

Before we can solve this problem we need to *formulate* it mathematically. This means doing three things:

1. Identify the unknowns in the problem, that is, define the variables;
2. Write down what is to be maximised or minimised (objective function);
3. Write down any constraints, including non-negativity constraints.

Thus a typical set-up (and the sort we want to see!) is

In general

1. Let $x_1 = \cdots$, $x_2 = \cdots$, ...
2. Maximise $z = \cdots$ (or minimise)
3. subject to

$$a_1x_1 + a_2x_2 + \cdots \leq b_1$$
etc.
$$x_i \geq 0$$

This problem

1. Let $x_1 =$ the no. of boxes of oranges, $x_2 =$ the no. of boxes of grapefruit, $x_3 =$ the no. of boxes of tangerines loaded onto the truck.
2. Maximise (profit) $z = 2x_1 + x_2 + 3x_3$ (in dollars)
3. subject to

$$\begin{aligned} &x_1 + x_2 + x_3 \leq 800, \\ &x_1 \geq 200, \\ &x_2 \geq 100, \\ &x_3 \leq 200, \\ &x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0. \end{aligned}$$

Now, how should *you* approach the problem to get it in this form?

1. Read the problem right through before you start.

2. Focus on the question asked in the problem.

   Clue: It normally begins "How" (How many, how much, etc.) and ends with "?". What follows after the "how" should indicate to you the quantities you wish to find, that is, the unknowns. Thus "How many boxes of each kind of fruit" suggests (since there are three kinds of fruit in our problem) that you need three variables, each one representing the number of boxes of a particular type of fruit. Consequently, you can use this to define your variables.

3. To get the objective function, look for the *task* or *objective* the question sets.

   Clue: This normally includes the words 'maximise' or 'minimise'. Thus in the fruit dealer problem the phrase 'maximise profit' gives the objective, and hence the objective function.

   Note that profits, costs, etc., are given on a unit basis, for example, 2 dollars per box of oranges. Thus if we use x_1 of these units, that is x_1 boxes, we get a profit from oranges of $2x_1$ dollars. Similarly for grapefruits and tangerines. Thus total profit will be a sum of the three individual contributions: from oranges, grapefruit, and tangerines.

4. Look for *constraints* on the variables in the problem. These might be recorded in sentence form, or tabulated. A correct choice of variables in Step 2 would generally lead to an easy identification of constraints in Step 4. The fruit dealer problem has four constraints, not including the non-negativity conditions. These are based on the total number of boxes the truck can carry, and the individual quotas on the three different kinds of fruit. Don't forget non-negativity constraints, if your variables in Step 2 must be greater than or equal to zero.

If you follow these steps, you should be well on the way to formulating an optimisation problem mathematically and remember, the more practice you do, the better you'll get at it.

◊ **3.9: Solving this problem** ...

Example 2: chemical plant

A manufacturer of a certain chemical product has two plants where the product is made. Plant X can make at most 30 tons per week and plant Y can make at most 40 tons per week. The manufacturer wants to make a total of at least 50 tons per week. The amount of particulate matter found weekly in the atmosphere over a nearby town is measured and found to be 20 pounds for each ton of the product made by plant X and 30 pounds for each ton of the product made at plant Y. How many tons should be made weekly at each plant to minimise the total amount of particulate matter in the atmosphere?

◊ **3.10: Mathematical Formulation** ...

Example 3: nutrition

A nutritionist is planning a meal that includes foods A and B as its main staples. Suppose that each gram of food A contains 60 units of protein, 30 units of iron, and 30 units of thiamine; each gram of food B contains 30 units of protein, 30 units of iron, and 90 units of thiamine. Suppose that each gram of A costs 2 cents, while each gram of B costs 3 cents. The nutritionist wants the meal to provide at least 360 units of protein, at least 270 units of iron, and at least 450 units of thiamine. How many grams of each of the foods should be used to minimise the cost of the meal?

◊ **3.11: Mathematical Formulation** ...