Assignment 3

Question 1: a) $E[t] = \int_0^{15\pi} t \times f(t)\,dt$

$= \int_0^{15\pi} t \times \frac{1}{30} \times \sin\left(\frac{t}{30}\right) dt$ → move the constant $\frac{1}{30}$ out.

$= \frac{1}{30} \int_0^{15\pi} t \times \sin\left(\frac{t}{30}\right) dt$ → use integration by parts.

$u = t$

$\frac{du}{dt} = 1$

$du = dt$

$v = \int \sin\left(\frac{t}{30}\right) dt$ → let $u = \frac{t}{30}$ → use U-substitution.

$\frac{du}{dt} = \frac{1}{30}$

$30\,du = dt$

$v = 30 \int \sin u \, du$

$v = 30(-\cos u) + c$

$v = -30\cos\left(\frac{t}{30}\right) + c$ #

⇒ Put into this form $\int u\,dv = uv - \int v\,du$

$= \frac{1}{30} \times \left[t \times \left(-30\cos\left(\frac{t}{30}\right)\right) - \int -30\cos\left(\frac{t}{30}\right) dt\right]$

$= \frac{1}{30} \times \left[-30t\cos\left(\frac{t}{30}\right) + 30\int \cos\left(\frac{t}{30}\right) dt\right]$ → let $x = \frac{t}{30}$ → Use U-substitution

$\frac{dx}{dt} = \frac{1}{30}$

$30dx = dt$

$= \frac{1}{30}\left[-30t\cos\left(\frac{t}{30}\right) + 30\int \cos x\,(30\,dx)\right]$

$= \frac{1}{30} \times \left[-30t\cos\left(\frac{t}{30}\right) + 900(\sin x)\right]$

$= \frac{1}{30}\left[-30t\cos\left(\frac{t}{30}\right) + 900\sin\left(\frac{t}{30}\right)\Big|_0^{15\pi}\right]$ → take into account of the definite integral

$= \frac{1}{30}\left\{-30(15\pi)\cos\left(\frac{15\pi}{30}\right) + 900\sin\left(\frac{15\pi}{30}\right) - \left[-30(0)\left(\cos\left(\frac{0}{30}\right) + 900\sin\left(\frac{0}{30}\right)\right)\right]\right\}$

$= \frac{1}{30} \times \left\{\left[-450\pi\cos\left(\frac{\pi}{2}\right) + 900\sin\left(\frac{\pi}{2}\right)\right] - \left[900\sin(0)\right]\right\}$

$= \frac{1}{30} \times \left\{\left[-450\pi(0) + 900(1)\right] - \left[900(0)\right]\right\}$

$= 30$

b) $CDF = F_T(t) = \int_0^{15\pi} f(t)\,dt = \int_0^{15\pi} \frac{1}{30}\sin\left(\frac{t}{30}\right) dt$

$= \frac{1}{30}\int_0^{15\pi} \sin\left(\frac{t}{30}\right) dt$ → let $u = \frac{t}{30}$

$\frac{du}{dt} = \frac{1}{30}$

$30du = dt$

$= \frac{1}{30}(30)\int \sin u \, du$

$= 1(-\cos u) + c$

$= -\cos\left(\frac{t}{30}\right)\Big|_0^{15\pi}$

$= -\cos\left(\frac{15\pi}{30}\right) - \left(-\cos\left(\frac{0}{30}\right)\right)$

$= 0 + 1 = 1$ #

c) $P(15 \leq t \leq 30) = F(30) - F(15)$

$$= \int_{15}^{30} \frac{1}{30} \sin\left(\frac{t}{30}\right) dt$$

Based on the resulting CDF from b)

$$= -\cos\left(\frac{t}{30}\right) \Big|_{15}^{30}$$

$$= -\cos\left(\frac{30}{30}\right) - \left(-\cos\left(\frac{15}{30}\right)\right)$$

$$= 0.3373 \#$$

d) $P(t \geq 15)$ to complete the test $P(t \leq 30)$?

By using the Bayes Rule :

$$P(t \leq 30 \mid t \geq 15) = \frac{P(t \geq 15 \cap t \leq 30)}{P(t \geq 15)}$$

$$= \frac{P(15 \leq t \leq 30)}{P(t \geq 15)}$$

$$= \frac{0.3373}{1 - P(0 \leq t \leq 15)}$$ → based on the result obtained from c)

$$= \frac{0.3373}{1 - \left[\int_0^{15} \frac{1}{30} \sin\left(\frac{t}{30}\right) dt\right]}$$ → based on the resulting CDF from b)

$$= \frac{0.3373}{1 - \left[-\cos\left(\frac{t}{30}\right) \Big|_0^{15}\right]}$$

$$= \frac{0.3373}{1 - \left[-\cos\left(\frac{15}{30}\right) - \left(-\cos(0)\right)\right]}$$

$$= \frac{0.3373}{0.8776}$$

$$= 0.3844 \#$$

2. a) $\sum_{z=1}^{\infty} f(z) = \sum_{z=1}^{\infty} c2^{-z}$ → i) we know that Geometric Series is in the form of $\sum_{n=c}^{\infty} ar^n$

ii) convert the $f(z)$ into $ar^n \#$ $= c2^{-z}$ → by law of exponents

$$= c(2^{-1})^z$$

↓ But the z starts from 0 instead of 1

so, we have to minus off the first term.

$$= \sum_{z=0}^{\infty} c\left(\frac{1}{2}\right)^z - c\left(\frac{1}{2}\right)^0$$

we know that the sum of geometric series is $\frac{a}{1-r}$. where $a = c$, $r = \frac{1}{2}$ →

$$= \frac{c}{1-\frac{1}{2}} - c$$

$$= \frac{c}{\frac{1}{2}} - c$$

$$= 2c - c$$

$$= c$$

→ we know that a valid pmf is $\sum_{n=1}^{\infty} f(z) = 1$

$$c = 1 \#$$

2.

b) $\sum_{z=1}^{\infty} e^{tz} \times f(z) = \sum_{z=1}^{\infty} e^{tz} \times 2^{-z}$

$= \sum_{z=1}^{\infty} e^{tz} \times \frac{1}{2^z}$

$= \sum_{z=1}^{\infty} \frac{e^{tz}}{2^z}$

using the result of the geometric series

$= \sum_{z=1}^{\infty} \left(\frac{e^t}{2}\right)^z$

$= \sum_{z=0}^{\infty} \left(\frac{e^t}{2}\right)^z - \left(\frac{e^t}{2}\right)^0$

$= \frac{1}{1-\frac{e^t}{2}} - 1$

$= \frac{1}{\frac{2-e^t}{2}} - 1$

$= \frac{2}{2-e^t} - 1 = \frac{2}{2-e^t} - \frac{2-e^t}{2-e^t} = \frac{2-2+e^t}{2-e^t} = \frac{e^t}{2-e^t}$ # proven.

c) $E[Z] = \frac{dm(t)}{dt} = \frac{e^t}{2-e^t} \begin{matrix} u \\ v \end{matrix} \Rightarrow$ use the Quotient Rule $\left(\frac{v\frac{du}{dz} - u\frac{dv}{dz}}{v^2}\right)$

$u = e^t \qquad v = 2-e^t$

$\frac{du}{dt} = e^t \qquad \frac{dv}{dt} = -e^t$

$= \frac{(2-e^t)(e^t) - (e^t)(-e^t)}{(2-e^t)^2}\Bigg|_{t=0}$

$= \frac{2e^t - e^{2t} + e^{2t}}{(2-e^t)^2}\Bigg|_{t=0}$

$= \frac{2e^t}{(2-e^t)(2-e^t)}\Bigg|_{t=0}$

$= \frac{2e^t}{4-4e^t+e^{2t}}\Bigg|_{t=0}$ # $= \frac{2e^0}{4-4e^0+e^{2(0)}} = \frac{2}{4-4+1}$

$= 2$ #

$Var(Z) = E[Y^2] - [E[Y]]^2$

$= \frac{d^2m(t)}{dt^2}\Bigg|_{t=0} - (2)^2$ $\quad u = 2e^t \quad v = 4-4e^t+e^{2t} \Rightarrow$ let $u = 2t$

$\frac{du}{dt} = 2e^t \quad \frac{dv}{dt} = -4e^t + 2e^{2t} \quad \frac{du}{dt} = 2$

$= \frac{2e^t}{4-4e^t+e^{2t}} \begin{matrix} u \\ v \end{matrix}\Bigg|_{t=0}$

$= \frac{(4-4e^t+e^{2t})(2e^t) - (2e^t)(-4e^t+2e^{2t})}{(4-4e^t+e^{2t})^2}\Bigg|_{t=0} - 4$

$= \frac{8e^t - \cancel{8e^{2t}} + 2e^{3t} + \cancel{8e^{2t}} - 4e^{3t}}{(4-4e^t+e^{2t})^2}\Bigg|_{t=0} - 4$

$= \frac{8e^t - 2e^{3t}}{(4-4e^t+e^{2t})^2}\Bigg|_{t=0} - 4 = \frac{8(e^0) - 2(e^{3(0)})}{(4-4e^0+e^{2(0)})^2} - 4 = \frac{6}{1} - 4$

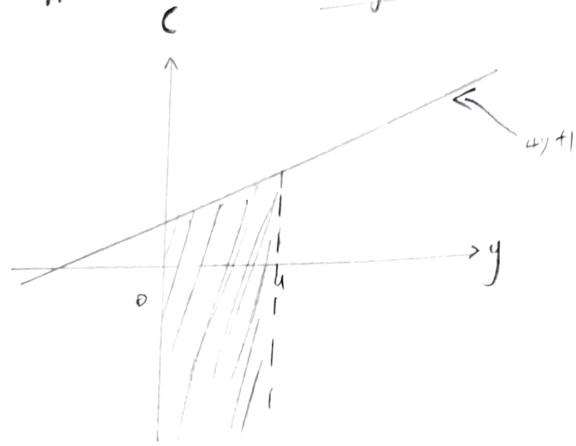
$= 6-4$

$= 2$ #

3. $Y \sim \exp\left(\frac{1}{2}\right)$ $\quad C = 4Y+1$

The support for Y is $\underline{0 \leq y < \infty}$

The support for C is $\underline{1 \leq y < \infty}$



$$F_C(c) = P(C \leq c)$$
$$= P(4Y+1 \leq c)$$

$4Y+1 = c$

$Y = \frac{c-1}{4}$

$$= P\left(Y \leq \frac{c-1}{4}\right)$$

$$\int_0^{\frac{c-1}{4}} \lambda e^{-\lambda y} dy = \lambda \int_0^{\frac{c-1}{4}} e^{-\lambda y}\, dy$$

let $u = -\lambda y$

$\frac{du}{dy} = -\lambda$

$\frac{du}{-\lambda} = dy$

$$= \lambda \int_0^{\frac{c-1}{4}} e^u \frac{du}{-\lambda}$$

$$= \frac{\lambda}{-\lambda} \int_0^{\frac{c-1}{4}} e^u\, du$$

$$= -1\left(e^u\right)\Big|_0^{\frac{c-1}{4}}$$

$$= -e^{-\lambda y}\Big|_0^{\frac{c-1}{4}}$$

given $\lambda = \frac{1}{2}$

$$= -e^{-\left(\frac{1}{2}\right)\left(\frac{c-1}{4}\right)} + e^{-\left(\frac{1}{2}\right)(0)}$$

$$= -e^{-\left(\frac{c-1}{8}\right)} + \frac{1}{e^0}$$

$$= -e^{-\left(\frac{c-1}{8}\right)} + 1$$

$$= -e^{-\left(\frac{c-1}{8}\right)} = -e^{-\left(\frac{c-1}{8}\right)} + 1$$

CDF is

$$F_C(c) = \begin{cases} 0 & c < 1 \\ -e^{-\left(\frac{c-1}{8}\right)} + 1 & 1 \leq c < \infty \end{cases}$$

To find the PDF:

$$f_c(c) = \frac{dF_c(c)}{dc} = -e^{-\frac{c-1}{8}} + 1$$

let $u = -\frac{c-1}{8}$ $\quad = -e^u + 1$

$\frac{du}{dc} = -\frac{1}{8}(c-1)$ $\quad = -e^u \cdot \left(-\frac{1}{8}\right)$

$= -\frac{1}{8}$

$$= \left(\frac{1}{8}\right)\left(e^{-\frac{c-1}{8}}\right) \#$$

PDF:

$$f_c(c) = \begin{cases} \left(\frac{1}{8}\right)\left(e^{-\frac{c-1}{8}}\right) & 1 \leq c < \infty \\ 0 & \text{otherwise} \end{cases}$$

4. a) Probability since the last strike

Probability for the next strike in the next minute

$P(N(0,2)=1) \cap P(N(2,3)=1)$

↳ By using the independence of non-overlapping times of the lightning strike and the time invariance

$= P(N(2)=1) \times P(N(0,1)=1)$

$= P(N(2)=1) \times P(N(1)=1)$

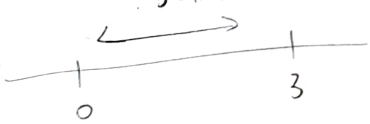
given the $\lambda = 1$

$= \frac{(1)(2)^1}{1!} e^{-(1)(2)} \times \frac{(1)(1)^1}{1!} e^{-(1)(1)}$

$= (2 \times e^{-2}) \times (e^{-1}) \rightarrow \frac{2}{e^2} \times \frac{1}{e}$

$= \frac{2}{e^3} = 0.09957$

b) 3 strikes.



$\{S_3 = 3\} = \{N(3) = 3\}$

$P(N(0,3)=3) = P(N(3)=3)$ → By using the time invariance

$= \frac{(1)(3)^3}{3!} e^{-(1)(3)}$

$= \frac{27}{3!} e^{-(3)}$

$= \frac{27}{6} \times \frac{1}{e^3}$

$= \frac{9}{2} \times \frac{1}{e^3} = \frac{9}{2e^3} = 0.2240$

c)

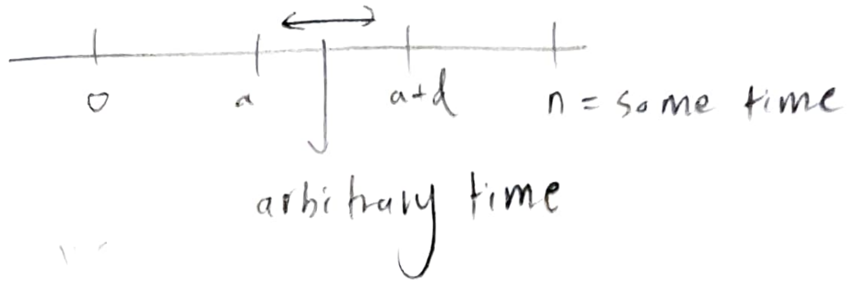
Since we know that the rate of strikes per min $= \lambda = 1$, the number of expected lightning strikes in the next 60 minutes is 60.

$M = \lambda t$

$= (1)(60)$

$= 60$ number of strikes

d)



By using the stationary increments and time invariance,

Given that we start counting at some time a and the time until the 5th strike is some k.

$N(a+k) - N(a)$ is just dependent on the time k which is

$$5 = \lambda t$$
$$5 = (1)(t)$$
$$t = 5 \text{ minutes.}$$

where $\lambda = 1$ strike per minute