Subject

Q2: a) $\text{bias}_{T_1}(A) = E[T_1] - A$

$$= E\left[\frac{(X_1+X_2)}{2} \times \frac{(Y_1+Y_2)}{2} \times \frac{1}{2}\right] - bh\frac{1}{2}$$

$$= E\left[\frac{(X_1+X_2)}{2}\right] E\left[\frac{(Y_1+Y_2)}{2}\right] \times \frac{1}{2} - bh\frac{1}{2}$$

$$= \left(\left[\frac{1}{2} \times (E[X_1] + E[X_2])\right] \times \left[\frac{1}{2}(E[Y_1] + E[Y_2])\right]\right)\frac{1}{2} - bh\frac{1}{2}$$

$$= \left(\left[\frac{1}{2} \times (b+b)\right] \times \left[\frac{1}{2}(h+h)\right]\right)\frac{1}{2} - bh\frac{1}{2}$$

$$= \left(\frac{2b}{2} \times \frac{2h}{2}\right)\frac{1}{2} - bh\frac{1}{2}$$

$$= bh\frac{1}{2} - bh\frac{1}{2}$$

$$= 0$$

shown.

$\text{bias}_{T_2}(A) = E[T_2] - A$

$$= E\left[\frac{(X_1Y_1\frac{1}{2}) + (X_2Y_2\frac{1}{2})}{2}\right] - bh\frac{1}{2}$$

$$= \frac{1}{2}\left[E\left[\frac{(X_1Y_1)}{2} + \frac{(X_2Y_2)}{2}\right]\right] - bh\frac{1}{2}$$

$$= \frac{1}{2}\left[\frac{1}{2}E[X_1Y_1] + \frac{1}{2}E[X_2Y_2]\right] - bh\frac{1}{2}$$

$$= \frac{1}{2}\left[\frac{1}{2}E[X_1]E[Y_1] + \frac{1}{2}E[X_2]E[Y_2]\right] - bh\frac{1}{2}$$

$$= \frac{1}{2}\left[\frac{1}{2}bh + \frac{1}{2}bh\right] - bh\frac{1}{2}$$

$$= \frac{1}{2}bh - bh\frac{1}{2}$$

$$= 0$$

shown.

b) $\text{var}(X_1 Y_1) = E[(X_1 Y_1)^2] - E[X_1 Y_1]^2$

$= E[X_1^2 Y_1^2] - (E[X_1]E[Y_1])^2$

$= E[X_1^2]E[Y_1^2] - (E[X_1]E[Y_1])^2$

$\Downarrow \qquad \Downarrow$

$\text{var}(X_1) = E[X_1^2] - E[X_1]^2$ $\qquad$ $\text{var}(Y_1) = E[Y_1^2] - E[Y_1]^2$

$E[X_1^2] = \text{var}(X_1) + E[X_1]^2$ $\qquad$ $E[Y_1^2] = \text{var}(Y_1) + E[Y_1]^2$

$= [\text{var}(X_1) + E[X_1]^2] \times [\text{var}(Y_1) + E[Y_1]^2] - (bh)^2$

$= (\sigma^2 + b^2)(\sigma^2 + h^2) - b^2h^2$

$= \sigma^4 + \sigma^2 h^2 + \sigma^2 b^2 + b^2h^2 - b^2h^2$

$= \sigma^4 + \sigma^2 h^2 + \sigma^2 b^2$

$= \sigma^2(\sigma^2 + h^2 + b^2)$

$= \sigma^2(\sigma^2 + b^2 + h^2)$ # shown.

c) $\operatorname{cov}(X_1Y_1, X_1Y_2) = E[(X_1Y_1 - E[X_1Y_1])(X_1Y_2 - E[X_1Y_2])]$

$= E[\; X_1Y_1 \times X_1Y_2 - (X_1Y_1)E[X_1Y_2] - E[X_1Y_1](X_1Y_2) + E[X_1Y_1]E[X_1Y_2]]$

$= E[\; X_1^2 Y_1 Y_2 - (X_1Y_1)E[X_1]E[Y_2] - E[X_1]E[Y_1](X_1Y_2) + E[X_1]E[Y_1] \times$
$E[X_1]E[Y_2]]$

using
$E[x^2] = \operatorname{var}(x) + E[x]^2$
$= \sigma^2 + \mu^2$

$= E[X_1^2]E[Y_1]E[Y_2] - E[X_1]E[Y_1]E[X_1]E[Y_2] - E[X_1]E[Y_1]E[X_1]E[Y_2]$
$+ E[X_1]E[Y_1]E[X_1]E[Y_2]$

$= (\sigma^2 + \mu^2)(\mu \times \mu) - \mu \times \mu \times \mu \times \mu$

$= (\sigma^2 + \mu^2)\mu^2 - \mu^2\mu^2$

$= \sigma^2\mu^2 + \mu^2\mu^2 - \mu^2\mu^2$

$= \sigma^2\mu^2$ # shown.

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Correct version of 2(d)

d) $MSE_{T_1}(A) = Var(T_1) - \cancel{bias_{T_1}(A)} = 0$ because T_1 is unbiased. (from (a))

$$= var\left(\frac{(X_1+X_2)}{2} \times \frac{(Y_1+Y_2)}{2} \times \frac{1}{2}\right)$$

$$= \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 var\left((X_1+X_2)(Y_1+Y_2)\right) \to \text{by independence}$$

$$= \frac{1}{64} var(X_1Y_1 + X_1Y_2 + X_2Y_1 + X_2Y_2)$$

$$= \frac{1}{64}\Big(var(X_1Y_1) + var(X_1Y_2) + var(X_2Y_1) + var(X_2Y_2) + 2cov(X_1Y_1, X_1Y_2) \text{ ①}$$
$$+ 2cov(X_1Y_1, X_2Y_1) \text{ ②} + 2cov(X_1Y_1, X_2Y_2) \text{ ③} + 2cov(X_1Y_2, X_2Y_1) \text{ ④}$$
$$+ 2cov(X_1Y_2, X_2Y_2) \text{ ⑤} + 2cov(X_2Y_1, X_2Y_2) \text{ ⑥}$$

In the case of ③ and ④ the covariance is 0 !

③ $2cov(X_1Y_1, X_2Y_2) = 2\left[E[(X_1Y_1 - E[X_1Y_1]) \times (X_2Y_2 - E[X_2Y_2])]\right]$

$$= 2\left[E\{(X_1Y_1)(X_2Y_2) - (X_1Y_1)E[X_2Y_2] - E[X_1Y_1](X_2Y_2) + E[X_1Y_1]E[X_2Y_2]\}\right]$$

$$= 2\left[E[X_1]\cancel{E[Y_1]E[X_2]}E[Y_2] - \cancel{E[X_1]E[Y_1]}E[X_2]E[Y_2] - E[X_1]E[Y_1]E[X_2]E[Y_2] + \cancel{E[X_1]}E[Y_1]E[X_2]E[Y_2]\right]$$

$$= 0$$

④ $2cov(X_1Y_2, X_2Y_1) = 2\left(E[(X_1Y_2 - E[X_1]E[Y_2]) \times (X_2Y_1 - E[X_2]E[Y_1])]\right)$

$$= 2\left(E[(X_1Y_2)(X_2Y_1) - (X_1Y_2)E[X_2]E[Y_1] - E[X_1]E[Y_2](X_2Y_1) + E[X_1]E[Y_2]E[X_2]E[Y_1]]\right)$$

$$= 2\left(E[X_1]E[Y_2]\cancel{E[X_2]E[Y_1]} - E[X_1]E[Y_2]E[X_2]E[Y_1] - E[X_1]E[Y_2]E[X_2]E[Y_1] + \cancel{E[X_1]E[Y_2]}E[X_2]E[Y_1]\right)$$

$$= 0$$

In the case of ①②⑤⑥ the covariance is $\sigma^2 h^2$ (from (c)).

Example: ⑥ $2cov(X_2Y_1, X_2Y_2) = 2\left(E[((X_2Y_1) - E[X_2]E[Y_1]) \times ((X_2Y_2) - E[X_2]E[Y_2])]\right)$

$$= 2\left(E[X_2Y_1X_2Y_2 - X_2Y_1E[X_2]E[Y_2] - E[X_2]E[Y_1]X_2Y_2 + E[X_2]E[Y_1]E[X_2]E[Y_2]]\right)$$

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$$= 2\left(E\left[x_2^2 Y_1 Y_2 - x_2 Y_1 E[x_2]E[Y_2] - E[x_2]E[Y_1]\, x_2 Y_2 + E[x_2]E[Y_1]E[x_2]E[Y_2] \right) \right.$$

$$= 2\left(E[x_2^2]E[Y_1]E[Y_2] - E[x_2]^2 E[Y_1]E[Y_2] - E[x_2]^2 E[Y_1]E[Y_2] + E[x_2]^2 E[Y_1]E[Y_2] \right)$$

using variance

$$E[x_2^2] = Var(x_2) + E[x_2]^2 \rightarrow = 2\left(\left(var(x_2) + E[x_2]^2 \right) E[Y_1]E[Y_2] - b^2h^2 \right)$$

$$= 2\left((\sigma^2 + b^2) h^2 - b^2h^2 \right)$$

$$= 2\left(\sigma^2 h^2 + \cancel{b^2h^2} - \cancel{b^2h^2} \right)$$

$$= 2\sigma^2 h^2 \ \#$$

⑤ $$2\,cov(x_1Y_2, x_2Y_2) = 2\left(\left(E[x_1Y_2 - E[x_1]E[Y_2]] \right) \times \left(x_2Y_2 - E[x_2]E[Y_2] \right) \right)$$

$$\cancel{= 2\left(E[x_1x_2Y_2^2 - x_1Y_2E[x_2]E[Y_2] - E[x_1]E[Y_2]\,x_2Y_2 + E[x_1]E[Y_2]E[x_2]E[Y_2] \right)}$$

using variance

$$E[Y_2^2] = var(Y_2) + E[Y_2]^2 \quad \rightarrow = 2\left(E[x_1]E[x_2]E[Y_2^2] - E[x_1]E[Y_2]^2E[x_2] - E[x_1]E[x_2]E[Y_2]^2 + E[x_1]E[x_2]E[Y_2]^2 \right)$$

$$\cancel{= 2\left(b^2\left(var(Y_2) + E[Y_2]^2 \right) - b^2h^2 \right)}$$

$$= 2\left(b^2(\sigma + h^2) - b^2h^2 \right)$$

$$= 2\left(\sigma b^2 + \cancel{b^2h^2} - \cancel{b^2h^2} \right)$$

$$= 2\sigma b^2$$

② $$2\,cov(x_1Y_1, x_2Y_1) = 2\left(E[(x_1Y_1 - E[x_1]E[Y_1]) \times (x_2Y_1 - E[x_2]E[Y_1])] \right)$$

$$= 2\left(E[x_1x_2Y_1^2 - x_1Y_1E[x_2]E[Y_1] - x_2Y_1E[x_1]E[Y_1] + E[x_1]E[Y_1]E[x_2]E[Y_1]] \right)$$

$$= 2\left(E[x_1]E[x_2]E[Y_1^2] - E[x_1]E[x_2]E[Y_1]^2 - E[x_2]E[x_1]E[Y_1]^2 + E[x_1]E[x_2]E[Y_1]^2 \right)$$

$$= 2\left(b^2\left(Var(Y_1) + E[Y_1]^2 \right) - b^2h^2 \right)$$

$$= 2\left(b^2(\sigma^2 + h^2) - b^2h^2 \right)$$

$$= 2\left(\sigma^2b^2 + b^2h^2 - b^2h^2 \right)$$

$$= 2\sigma^2 b^2$$

★ where case 1 is the same as c).

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case 1 6 ↑

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$$= \frac{1}{64}\left(4\sigma^2(\sigma^2+b^2+h^2) + 2(2\sigma^2h^2) + 2(2\sigma^2b^2)\right)$$

$$= \frac{1}{64}\left(4\sigma^4 + 4\sigma^2b^2 + 4\sigma^2h^2 + 4\sigma^2h^2 + 4\sigma^2b^2\right)$$

$$= \frac{1}{64}\left(4\sigma^4 + 8\sigma^2b^2 + 8\sigma^2h^2\right)$$

$$= \frac{1}{16}\sigma^4 + \frac{1}{8}\sigma^2b^2 + \frac{1}{8}\sigma^2h^2 \quad \#$$

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$MSE_{T_2}(A) = var(T_2) - \cancel{bias_{T_2}(A)} = 0$ because T_2 is unbiased (from (a))

$$= var\left(\frac{(x_1Y_1\frac{1}{2}) + (x_2Y_2\frac{1}{2})}{2}\right)$$

$$= \left(\frac{1}{2}\right)^2 var\left(\frac{x_1Y_1}{2} + \frac{x_2Y_2}{2}\right)$$

using the proof in T_1 for case (3) covariance is 0.

$$= \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 var(x_1Y_1 + x_2Y_2)$$

$$= \frac{1}{16}\left(var(x_1Y_1) + var(x_2Y_2) + \cancel{2\,cov(x_1Y_1, x_2Y_2)}^{\,0}\right)$$

using from b) $var(x_iY_i) = 6^2(6^2+b^2+h^2)$

$$= \frac{1}{16}\left(2\sigma^2(\sigma^2+b^2+h^2)\right)$$

$$= \frac{1}{16}\left(2\sigma^4 + 2\sigma^2b^2 + 2\sigma^2h^2\right)$$

$$= \frac{1}{8}\sigma^4 + \frac{1}{8}\sigma^2b^2 + \frac{1}{8}\sigma^2h^2 \ \#$$

In conclusion, T_1 is preferred over T_2 because its MSE is smaller as $T_1 = \frac{1}{16}\sigma^4 + \frac{1}{8}\sigma^2b^2 + \frac{1}{8}\sigma^2h^2 < T_2 = \frac{1}{8}\sigma^4 + \frac{1}{8}\sigma^2b^2 + \frac{1}{8}\sigma^2h^2$, as the first term of T_1 is smaller.