Assignment 4.

1\. a) The function $g(x)$ is strictly increasing on $[0,2]$, so

Find the $g^{-1}(y)$ (Inverse) $\to$ $Y = g(x) = 1 - \frac{\sqrt{4-x^2}}{2}$

$$\left(y = 1 - \frac{\sqrt{4-x^2}}{2} \right) \times 2$$

$$2y = 2 - \sqrt{4-x^2}$$

$$2y - 2 = -\sqrt{4-x^2}$$

$$\left(2 - 2y = \sqrt{4-x^2} \right)^2$$

$$(2-2y)^2 = 4 - x^2$$

$$(2-2y)^2 - 4 = -x^2$$

$$\left(4 - (2-2y)^2 = x^2 \right)^{\frac{1}{2}}$$

$$x = \pm\sqrt{4-(2-2y)^2}$$

$$x = \pm\sqrt{4-(4-8y+4y^2)}$$

$$x = \pm\sqrt{8y-4y^2}$$

$$x = \pm\sqrt{4}\sqrt{2y-y^2}$$

$$g^{-1}(y) = x = 2\sqrt{2y-y^2} \ \#$$

$\to$ because the support for $g(x)$ is only from $[0,2]$, we only consider the positive

Find $\frac{dg^{-1}(y)}{dy}$ $\to$

$$\frac{dg^{-1}(y)}{dy} = 2\sqrt{2y-y^2}$$

$\to$ use chain rule

$$= 2u^{\frac{1}{2}}$$

$u = 2y - y^2$

$\frac{du}{dy} = 2 - 2y$

$$= 2\left(\tfrac{1}{2}\right)(u)^{-\frac{1}{2}} \times 2-2y$$

$$= \frac{2-2y}{\sqrt{u}}$$

$$= \frac{2-2y}{\sqrt{2y-y^2}} \ \#$$

Find $f_Y(y)$ $\to$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$$

$$= \frac{2\sqrt{2y-y^2}}{2} \times \frac{2-2y}{\sqrt{2y-y^2}}$$

$$= 2 - 2y \ \#$$

The support for X is

$$0 \leq x \leq 2$$

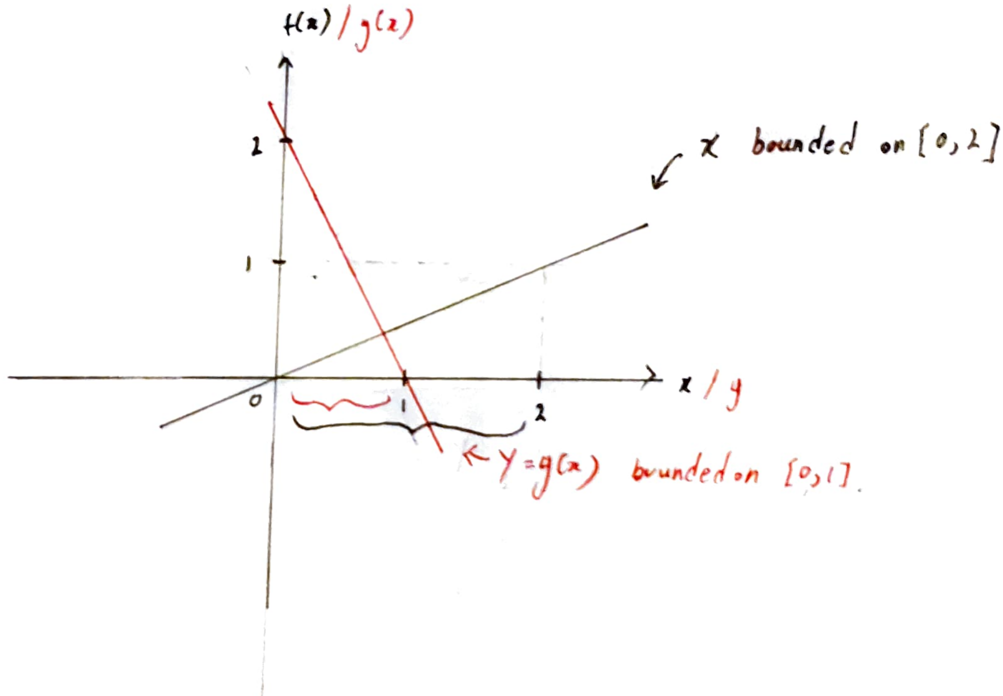
The support for Y is

$$0 \leq y \leq 1$$

PDF for $Y = g(x)$ is

$$f_Y(y) \begin{cases} 2-2y, & 0 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

b)



→ Consider that X_1 is the minimum of the observations.

$$F_{X_1}(x) = P(X_1 \leq x) = 1 - P(X_1 > y)$$

→ if X_1 is smaller than some value x then all the observations that are greater than x_1 must be greater than x_1.

$$= 1 - P(X_1 \geq x \cap X_2 \geq x \cap X_3 \geq x \cap \cdots \cap X_n \geq x)$$ → by independence

$$= 1 - P(X_1 \geq x) P(X_2 \geq x) P(X_3 \geq x) \cdots P(X_n \geq x)$$

$$= 1 - (1 - F_X(x))^n \equiv F_{X_1}(x)$$

$$f_{X_1}(x) = \frac{d}{dx} F_{X_{(1)}}(x) = n(1 - F_X(x))^{n-1} f_X(x)$$

→ given the pdf of exponential is $\lambda e^{-\lambda x}$

$$= n(1 - (1 - e^{-\lambda x}))^{n-1} \lambda e^{-\lambda x}$$

$$= n(1 - 1 + e^{-\lambda x})^{n-1} \lambda e^{-\lambda x}$$

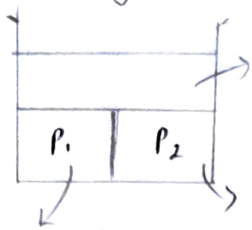
$$= \lambda n(e^{-\lambda x(n-1)}) e^{-\lambda x}$$

$$= \lambda n(e^{-\lambda x n + \lambda x - \lambda x})$$

$$= \lambda n(e^{-\lambda x n}) \sim \exp(n\lambda)$$

proven #

3. a) the box of old Game boy consoles.



have no defects.

have more than 1 defect. $P(x>1)$

have exactly 1 defect $P(x=1)$

P(choosing a working console with no defects) $= 1-(p_1+p_2)$

= the whole box $-$[(Probability of exactly 1 defect) + (Probability of more than 1 defects)]

b) Gameboy Consoles

1 2 3 4 5

x = number of consoles with no defects.

$x \sim \text{Binomial}(5, 1-p_1-p_2)$

- 5 fixed trials
- each trial is either is a defect or not a defect
- assume the trials are independent.
- The probability of not getting defect is constant $= 1-p_1-p_2$.

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To find the probability of getting at least 1 with no defects → $P(x \geq 1) = 1 - P(x=0)$

$$= 1 - \binom{5}{0}(1-p_1-p_2)^0(1-(1-p_1-p_2))^{5-0}$$

$$= 1 - \frac{5!}{0!\,5!}(1)(1-1+p_1+p_2)^5$$

$$= 1 - (1)(1)(p_1+p_2)^5$$

$$= 1 - (p_1+p_2)^5$$

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3. c)

Given Y_1 = number of consoles with 1 defect

Given Y_2 = number of consoles with more than 1 defects.

$Y_1 \sim Bin(n, p_1)$

$Y_2 \sim Bin(n, p_2)$

Given that the cost of repairing defective consoles is $C = Y_1 + 3Y_2$

the Expected value of C is

$Ex(C = Y_1 + 3Y_2)$

$$E(C) = E(Y_1) + 3E[Y_2]$$

$$= np_1 + 3np_2 \#$$

Variance of C is

$$Var(C) = var(Y_1 + 3Y_2)$$

$$= var(Y_1) + 3var(Y_2) + 2 \times 3 \, cov(Y_1, Y_2)$$

$$= np_1(1-p_1) + 3(np_2(1-p_2)) + 6(-np_1p_2)$$

$$= np_1 - np_1^2 + 3np_2 - 3np_2^2 - 6np_1p_2 \#$$

we have 3 outcomes $i = 1, 2, 3$ to each trial

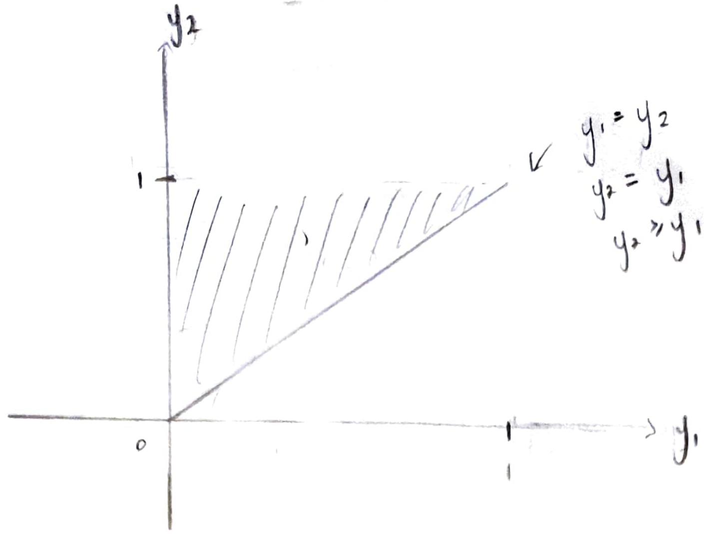
Y_0 = no defects $\quad Y_1$ = 1 defect $\quad Y_2$ = more than 1 defects.

$\quad \| \quad\quad \| \quad\quad \|$

$p_0 = 1 - p_1 - p_2 \quad p_1 \quad p_2$

multinomial distributions. because each trial has 3 outcomes.

4. a)



The triangular region given by $\{(y_1, y_2) \mid 0 \leq y_1 \leq y_2 \leq 1\}$

→ the $f(y_1, y_2)$ is only non-zero over the trianglear region.

b) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1\, dy_2 = 1$

$\left[\int_0^1 \left[\int_0^{y_2} k(1-y_2)\, dy_1\right] dy_2\right]$, given $k = 6$

$= \int_0^1 \int_0^{y_2} 6(1-y_2)\, dy_1\, dy_2$

$= \int_0^1 \int_0^{y_2} 6 - 6y_2\, dy_1\, dy_2$

$= \int_0^1 \left(6y_1 - 6y_2 y_1\right)\Big|_0^{y_2}$

$= \int_0^1 \left[\left(6y_2 - 6y_2 y_2\right) - \left(6(0) - 6y_2(0)\right)\right] dy_2$

$= \int_0^1 \left(6y_2 - 6y_2^2\right) dy_2$

$= \dfrac{6y_2^2}{2} - \dfrac{6y_2^3}{3}\Big|_0^1$

$= \left[\dfrac{6(1)^2}{2} - \dfrac{6(1)}{3}\right] - \left[\dfrac{6(0)}{2} - \dfrac{6(0)}{3}\right]$

$= 3 - 2$

$= 1$

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4. c) $P(Y \le \frac{1}{4}, Y_2 \le \frac{3}{4}) = \int_0^{\frac{1}{4}} \int_{y_1}^{\frac{3}{4}} 6(1-y_2)\, dy_2\, dy_1$

$= \int_0^{\frac{1}{4}} \int_{y_1}^{\frac{3}{4}} 6 - 6y_2 \; dy_2\, dy_1$

$= \int_0^{\frac{1}{4}} \left(6y_2 - \frac{\cancel{6}^{3} y_2^2}{\cancel{2}} \right) \Big|_{y_1}^{\frac{3}{4}} dy_1$

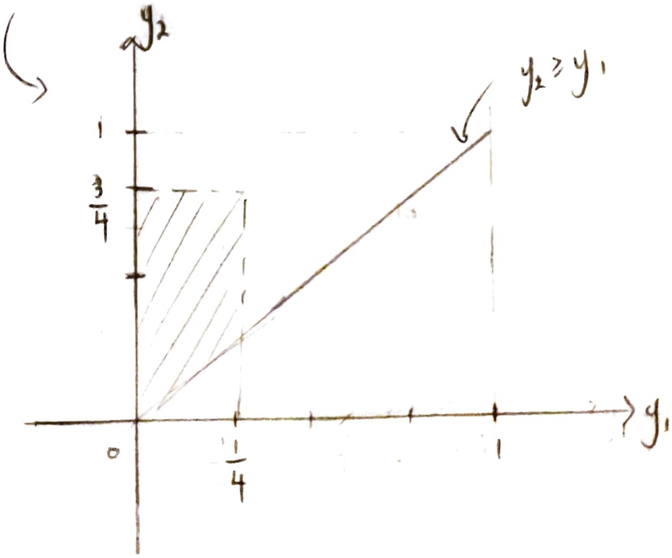
$= \int_0^{\frac{1}{4}} \left[\left(6\left(\frac{3}{4}\right) - 3\left(\frac{3}{4}\right)^2 \right) - \left(6y_1 - 3y_1^2 \right) \right] dy_1$

$= \int_0^{\frac{1}{4}} \frac{45}{16} - 6y_1 + 3y_1^2 \; dy_1$

$= \left(\frac{45}{16} y_1 - \frac{\cancel{6}^{3} y_1^2}{\cancel{2}_{1}} + \frac{\cancel{3}^{1} y_1^3}{\cancel{3}} \right) \Big|_0^{\frac{1}{4}}$

$= \left[\frac{45}{16}\left(\frac{1}{4}\right) - 3\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 \right] - (0)$

$= \frac{17}{32} = 0.53125$



4. d) $f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2$

$$= \int_0^{\frac{1}{4}} 6(1-y_2)\, dy_2$$

$$= \int_0^{\frac{1}{4}} 6 - 6y_2\, dy_2$$

$$= 6y_2 - \frac{\cancel{6}^{3} y_2^2}{\cancel{2}_{1}} \Big|_0^{\frac{1}{4}}$$

$$= \left[6\left(\frac{1}{4}\right) - 3\left(\frac{1}{4}\right)^2\right] - [0]$$

$$= \frac{21}{16} = 1.3125 \qquad \underline{0 \le y_1 \le \frac{1}{4}}$$

e) $f_2(y_2) = \int_{-\infty}^{\infty} \underline{f(y_1, y_2)\, dy_1}$

$$= \int_{y_1}^{\frac{3}{4}} 6(1-y_2)\, dy_1$$

$$= 6y_1 - 6y_2 y_1 \Big|_{y_1}^{\frac{3}{4}}$$

$$= 6\left(\frac{3}{4}\right) - 6y_2\left(\frac{3}{4}\right) - [6y_1 - 6y_2 y_1]$$

$$= \frac{18}{4} - \frac{\cancel{18}^{9}}{\cancel{4}_{2}} y_2 - 6y_1 + 6y_2 y_1$$

$$= 6y_2 y_1 - 6y_1 - \frac{9}{2} y_2 + \frac{9}{2} \qquad y_1 \le y_2 \le \frac{3}{4}$$