Assignment 4 Solutions

1. a) $g(x)$ is an increasing fn so we can apply the transform method.

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$$y = 1 - \frac{\sqrt{4-x^2}}{2}$$

$$4(y-1)^2 = 4 - x^2$$

$$4y^2 - 8y + 4 - 4 = x^2 \Rightarrow x = 2\sqrt{y(2-y)} = g^{-1}(y)$$

$$\frac{d}{dy} g^{-1}(y) = \frac{d}{dy} 2\left(y(2-y)\right)^{\frac{1}{2}}$$

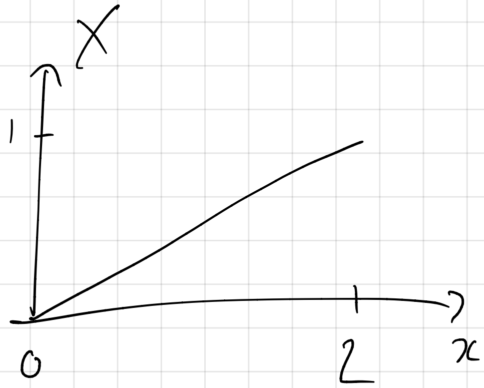
$$= 2 \times \frac{1}{2} \times \left(y(2-y)\right)^{-\frac{1}{2}} \times (2-2y) = \frac{2(1-y)}{\sqrt{y(2-y)}}$$

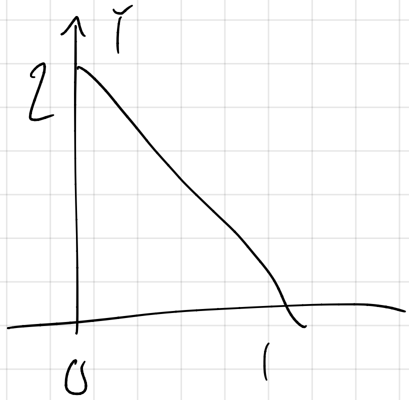
$$f_Y(y) = f_X\left(g^{-1}(y)\right) \left| \frac{dg^{-1}(y)}{dy} \right|$$

$$= \frac{\cancel{2}\cancel{\sqrt{y(1-y)}}}{\cancel{2}} \times \left| \frac{2(1-y)}{\cancel{\sqrt{y(2-y)}}} \right| = 2(1-y)$$

Hence $f_Y(y) = \begin{cases} 2(1-y) & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$

b)





2. pdf of X_i is $f(x) = \lambda e^{-\lambda x}$ with CDF $F(x) = 1 - e^{-\lambda x}$.

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From the lecture notes,

$$f_{X_{(1)}}(x) = n\left(1 - F(x)\right)^{n} f(x).$$

$$= n\left(e^{-\lambda x}\right)^{n-1} \times \lambda e^{-\lambda x}$$

$$= n\lambda e^{-n\lambda x}$$

Therefore $X_{(1)} \sim Exp(n\lambda)$

3. a) $P(\text{no defects}) = 1 - p_1 - p_2$

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b) Want marginal, let this be binomial.

Success = works are *[unclear]* prob $1-p_1-p_2$.

Let W denote the # of works, then $W \sim Bino(5, 1-p_1-p_2)$

$$Pr(W \geq 1) = 1 - P(W=0) = \underline{1-(1-p_1-p_2)^5}$$

c) $C = Y_1 + 3Y_2$ $\quad E[C] = E[Y_1] + 3E[Y_2]$

$$= np_1 + 3np_2 = \underline{n(p_1+3p_2)}$$

$$Var(C) = E[C^2] - E[C]^2$$

$$C^2 = Y_1^2 + 6Y_1Y_2 + 9Y_2^2$$

$$Var(C) = E[Y_1^2] + 6E[Y_1Y_2] + 9E[Y_2^2] - \underbrace{(E(Y_1) + 3E[Y_2])^2}_{E[Y_1]^2 + 6E[Y_1]E[Y_2] + 9E[Y_2]^2}$$

$$Var(C) = \overbrace{E[Y_1^2] - E[Y_1]^2}^{Var(Y_1)} + 9\overbrace{(E[Y_2^2] - E[Y_2]^2)}^{Var(Y_2)} + 6\overbrace{(E[Y_1Y_2] - E[Y_1]E[Y_2])}^{Cov(Y_1,Y_2)}.$$

$$\mathrm{Var}(c) = \mathrm{Var}(Y_1) + 9\mathrm{Var}(Y_2) + 6\,\mathrm{cov}(Y_1, Y_2).$$

$$= np_1(1-p_1) + 9np_2(1-p_2) - 6np_1p_2$$

4. a)

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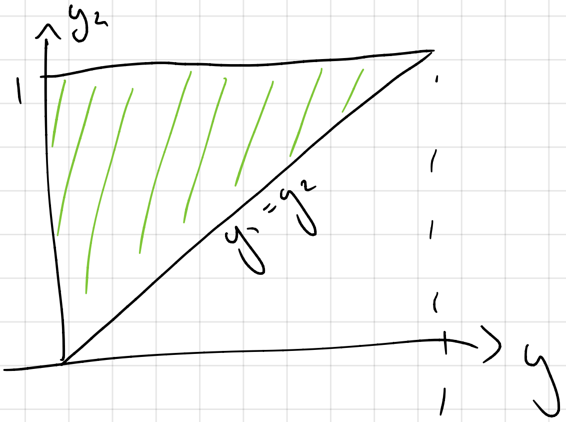


b) f must satisfy two conditions:

$$f(y_1, y_2) \geq 0 \quad \text{for all } y_1, y_2 \in \mathbb{R}$$

and $$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(y_1, y_2)\, dy_1\, dy_2 = 1$$

Note that this was originally omitted from the solutions and hence was not awarded marks

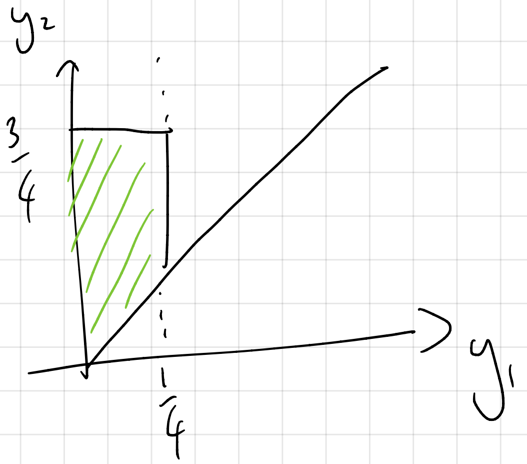
$$\int_0^1 \int_{y_2=y_1}^{1} k(1-y_2)\, dy_2\, dy_1 = k \int_0^1 \left[y_2 - \frac{1}{2} y_2^2 \right]_{y_2=y_1}^{1} dy_1$$

$$= k \int_0^1 \left(\frac{1}{2} - y_1 + \frac{1}{2} y_1^2 \right) dy_1$$

$$= k \left(\frac{1}{2}(y_1 - y_1^2) + \frac{1}{6} y_1^3 \right) \Big|_0^1$$

$$= \frac{k}{6}$$

So if $k = 6$, both conditions above are satisfied.

c) 

$$\int_0^{\frac{1}{4}} \int_{y_2=y_1}^{\frac{3}{4}} 6(1-y_2)\, dy_2\, dy_1$$

$$= 6 \int_0^{\frac{1}{4}} \left[y_2 - \frac{1}{2} y_2^2 \right]_{y_2=y_1}^{\frac{3}{4}} dy_1$$

$$= 6 \int_0^{\frac{1}{4}} \left(\frac{15}{32} - y_1 + \frac{1}{2} y_1^2 \right) dy_1$$

$$= 6 \left[\frac{15}{32} y_1 \right]_0^{\frac{1}{4}} = \frac{17}{32} \simeq 0.531$$

d) $f_1(y_1) = \int_{y_2=y_1}^{1} 6(1-y_2)\, dy_2 = 3(1-2y_1+y_1^2)$ for $0 \leq y_1 \leq 1$

and $f_1(y_1) = 0$ otherwise

e) $f_2(y_2) = \int_{y_1=0}^{y_1=y_2} 6(1-y_2)\, dy_1 = 6(y_2 - y_2^2)$ for $0 \leq y_2 \leq 1$

and $f_2(y_2) = 0$ otherwise