Assignment 1.

Date.

Q1: a) $E[x_i x_j] = E[x_i^2]$ (if $i=j$) case ①

$var(x_i) = E[x_i^2] - E[x_i]^2$

$E[x_i^2] = var(x_i) + E[x_i]^2$

$= \sigma^2 + u^2$ # shown.

$= u^2 + \sigma^2$

if $i \neq j$ case ②

$Cov(x_i x_j) = E[x_i x_j] - E[x_i]E[x_j]$

$E[x_i x_j] = cov(x_i x_j) + E[x_i]E[x_j]$ – given that x_i and x_j are independent, the $cov(x_i, x_j)$ must be 0.

$= 0 + u.u$

$= u^2$ # shown.

b) $E[x_i \bar{x}]$ if $i=j$ assume

$E[x_i \bar{x}] = E[x_i \times \frac{1}{n}\sum_{i=1}^{n} x_i]$

$= E[\frac{1}{n}\sum_{i=1}^{n} x_i x_i]$ variance ~~formula~~

$= E[\frac{1}{n}\sum_{i=1}^{n} x_i^2]$ → using $E[\frac{1}{n}\sum_{i=1}^{n} x_i^2] = var(\frac{1}{n}\sum_{i=1}^{n} x_i) + E[\frac{1}{n}\sum_{i=1}^{n} x_i]^2$

$= var(\frac{1}{n}\sum_{i=1}^{n} x_i) + E[\frac{1}{n}\sum_{i=1}^{n} x_i]^2$

By independence $= (\frac{1}{n})^2 \sum_{i=1}^{n} var(x_i) + \frac{1}{n}\sum_{i=1}^{n} E[x_i]^2 = \frac{1}{n^2} \times n\sigma^2 + \frac{1}{n} \times n \times u^2$

$= \frac{\sigma^2}{n} + u^2$

$= u^2 + \frac{\sigma^2}{n}$ # shown.

c) $E[S_{xx}] = E[\sum_{i=1}^{n} (x_i - \bar{x})^2]$

$= E[\sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})]$

$E[x_i^2] = Var(x_i) + E[x_i]^2$

$= E[\sum_{i=1}^{n} x_i^2 - x_i\bar{x} - \bar{x}x_i + \bar{x}^2]$

$= \sum_{i=1}^{n} E[x_i^2 - 2x_i\bar{x} + \bar{x}^2]$

$= \sum_{i=1}^{n} [E[x_i^2] - 2E[x_i\bar{x}] + E[\bar{x}^2]]$ → substitute $(E[x_i^2] = u^2 + \sigma^2)$

$E[\bar{x}^2] = u^2 + \frac{\sigma^2}{n}$ → since $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

$E[x_i\bar{x}] = u^2 + \frac{\sigma^2}{n}$

from b)

$= \sum_{i=1}^{n} (u^2 + \sigma^2 - 2(u^2 + \frac{\sigma^2}{n}) + u^2 + \frac{\sigma^2}{n})$

$= \sum_{i=1}^{n} [2u^2 + \sigma^2 - 2u^2 - \frac{2\sigma^2}{n} + \frac{\sigma^2}{n}]$

$= \sum_{i=1}^{n} [\sigma^2 - \frac{\sigma^2}{n}]$

$= n(\sigma^2 - \frac{\sigma^2}{n}) = n\sigma^2 - \sigma^2 = (n-1)\sigma^2$ # shown.

Subject:

d) $$\text{bias}_{S^2}(S) = E[S^2] - \sigma^2$$

$$= E\left[\frac{\sum (x_i - \bar{x})^2}{n-1}\right] - \sigma^2$$

$$= \frac{1}{n-1} E\left[\sum (x_i - \bar{x})^2\right] - \sigma^2$$

from c) $\leftarrow$ $\Downarrow$

$$= \frac{1}{n-1} E[S_{xx}] - \sigma^2$$

$$= \frac{1}{\cancel{n-1}} \cancel{(n-1)} \sigma^2 - \sigma^2$$

$$= \sigma^2 - \sigma^2$$

$$= 0$$ shown #