Assignment 3

Question 1

a) i) $\log(\lambda_{11})$ means $x_i = 0$ $x_j = 0$

$\log(\lambda_{11}) = \gamma_0 + \gamma_1(0) + \gamma_2(0) + \gamma_3(0)(0)$

$\log(\lambda_{11}) = \gamma_0$

ii) $\log(\lambda_{12})$ means $x_i = 0$ $x_j = 1$

$\log(\lambda_{12}) = \gamma_0 + \gamma_1(0) + \gamma_2(1) + \gamma_3(0)(1)$

$\log(\lambda_{12}) = \gamma_0 + \gamma_2$

iii) $\log(\lambda_{21})$ means $x_i = 1$ $x_j = 0$

$\log(\lambda_{21}) = \gamma_0 + \gamma_1(1) + \gamma_2(0) + \gamma_3(1)(0)$

$\log(\lambda_{21}) = \gamma_0 + \gamma_1$

iv) $\log(\lambda_{22})$ means $x_i = 1$, $x_j = 1$

$\log(\lambda_{22}) = \gamma_0 + \gamma_1(1) + \gamma_2(1) + \gamma_3(1)(1)$

$\log(\lambda_{22}) = \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3$

b) $$P(Y_{i2} \mid Y_{i1} + Y_{i2} = n_i) = \frac{P(Y_{i2} \cap Y_{i1} + Y_{i2} = n_i)}{P(Y_{i1} + Y_{i2} = n_i)}$$

$$= \frac{① \; P(Y_{i2})\,P(Y_{i1} = n_i - Y_{i2})}{② \; P(Y_{i1} + Y_{i2} = n_i)} \quad \text{Independence}$$

given that the poisson distribution fuction (pdf) is $P(Y \mid \lambda) = \dfrac{e^{-\lambda}\lambda^{Y}}{Y!}$

① $P(Y_{i2})\,P(Y_{i1} = n_i - Y_{i2})$ substitute the y terms into the poisson pdf.

$$\frac{e^{-\lambda_{12}}\lambda_{12}^{Y_{i2}}}{Y_{i2}!} \times \frac{e^{-\lambda_{i1}}\lambda_{i1}^{Y_{i1}}}{Y_{i1}!} \quad \text{expanding } Y_{i1} = n_i - Y_{i2} \rightsquigarrow \frac{e^{-\lambda_{12}}\lambda_{12}^{Y_{i2}}}{Y_{i2}!} \times \frac{e^{-\lambda_{i1}}\lambda_{i1}^{(n_i - Y_{i2})}}{(n_i - Y_{i2})!}$$

② $P(Y_{i1} + Y_{i2} = n_i)$ substitute the y terms into the poisson pdf.

$$\Downarrow$$

$$\frac{e^{-(\lambda_{i1}+\lambda_{i2})}(\lambda_{i1}+\lambda_{i2})^{(Y_{i1}+Y_{i2})}}{(Y_{i1}+Y_{i2})!}$$ simplify to $$\frac{e^{-\lambda_{i1}-\lambda_{i2}}\,\lambda_{i1}+\lambda_{i2}^{\,n_i}}{n_i!}$$

Put them back together :

$$P(Y_{i2} \mid Y_{i1}+Y_{i2}=n_i) = \frac{\left[\frac{e^{-\lambda_{i2}}\lambda_{i2}^{Y_{i2}}}{Y_{i2}!} \times \frac{e^{-\lambda_{i1}}\lambda_{i1}^{(n_i-Y_{i2})}}{(n_i-Y_{i2})!}\right]}{\left[\frac{e^{-\lambda_{i1}-\lambda_{i2}}(\lambda_{i1}+\lambda_{i2})^{n_i}}{n_i!}\right]} \quad ②$$

$$= \left[\frac{e^{-\lambda_{i2}}\lambda_{i2}^{Y_{i2}}}{Y_{i2}!} \times \frac{e^{-\lambda_{i1}}\lambda_{i1}^{(n_i-Y_{i2})}}{(n_i-Y_{i2})!}\right] \times \left[\frac{n_i!}{e^{-\lambda_{i1}-\lambda_{i2}}(\lambda_{i1}+\lambda_{i2})^{n_i}}\right]$$

$$= \left[\frac{(e^{-\lambda_{i2}}\lambda_{i2}^{Y_{i2}}) \times (e^{-\lambda_{i1}}\lambda_{i1}^{(n_i-Y_{i2})})}{Y_{i2}!\,(n_i-Y_{i2})!}\right] \times \left[\frac{n_i!}{e^{-\lambda_{i1}-\lambda_{i2}}(\lambda_{i1}+\lambda_{i2})^{n_i}}\right]$$

$$= \left[\frac{\cancel{e^{-\lambda_{i2}-\lambda_{i1}}}\,\lambda_{i2}^{Y_{i2}}\,\lambda_{i1}^{(n_i-Y_{i2})}}{Y_{i2}!\,(n_i-Y_{i2})!}\right] \times \left[\frac{n_i!}{\cancel{e^{-\lambda_{i1}-\lambda_{i2}}}(\lambda_{i1}+\lambda_{i2})^{n_i}}\right]$$

$$= \frac{\lambda_{i2}^{Y_{i2}}\,\lambda_{i1}^{(n_i-Y_{i2})}\,n_i!}{Y_{i2}!\,(n_i-Y_{i2})!\,(\lambda_{i1}+\lambda_{i2})^{n_i}}$$

set aside the combination terms

$$= \left[\frac{n_i!}{Y_{i2}!\,(n_i-Y_{i2})!}\right] \times \frac{\lambda_{i2}^{Y_{i2}}\,\lambda_{i1}^{(n_i-Y_{i2})}}{(\lambda_{i1}+\lambda_{i2})^{n_i}}$$ ↳ introduce a 0 $-Y_{i2}+Y_{i2}$

$$= \binom{n_i}{Y_{i2}} \times \frac{\lambda_{i2}^{Y_{i2}}\,\lambda_{i1}^{(n_i-Y_{i2})}}{(\lambda_{i1}+\lambda_{i2})^{(n_i-Y_{i2}+Y_{i2})}}$$

$$= \binom{n_i}{Y_{i2}} \frac{\lambda_{i2}^{Y_{i2}}\,\lambda_{i1}^{(n_i-Y_{i2})}}{(\lambda_{i1}+\lambda_{i2})^{Y_{i2}}(\lambda_{i1}+\lambda_{i2})^{n_i-Y_{i2}}}$$

$$= \binom{n_i}{Y_{i2}} \frac{\lambda_{i2}^{Y_{i2}}}{(\lambda_{i1}+\lambda_{i2})^{Y_{i2}}} \times \frac{\lambda_{i1}^{n_i-Y_{i2}}}{(\lambda_{i1}+\lambda_{i2})^{n_i-Y_{i2}}}$$ factor out the common exponent.

$$= \binom{n_i}{Y_{i2}} \left(\frac{\lambda_{i2}}{\lambda_{i1}+\lambda_{i2}}\right)^{Y_{i2}} \times \left(\frac{\lambda_{i1}}{\lambda_{i1}+\lambda_{i2}}\right)^{n_i-Y_{i2}}$$

$$\Downarrow$$

$$\binom{n_i}{Y_{i2}}\left(\frac{\lambda_{i2}}{\lambda_{i1}+\lambda_{i2}}\right)^{Y_{i2}}\left(\frac{\lambda_{i1}}{\lambda_{i1}+\lambda_{i2}}\right)^{n-Y_{i2}}$$

equivalent to the Binomial distribution formula $= \binom{n}{x} p^{x} q^{n-x} \sim B(n,p)$

$$Y \sim B\left(n_i, \frac{\lambda_{i2}}{\lambda_{i1}+\lambda_{i2}}\right) \#$$

c) $\log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 x_i$ given that from b) $\pi_i = \frac{\lambda_{i2}}{\lambda_{i1}+\lambda_{i2}}$

$$\log\left[\frac{\left(\frac{\lambda_{i2}}{\lambda_{i1}+\lambda_{i2}}\right)}{\left(1-\frac{\lambda_{i2}}{\lambda_{i1}+\lambda_{i2}}\right)}\right] = \beta_0 + \beta_1 x_i$$

$$\log\left[\frac{\left(\frac{\lambda_{i2}}{\lambda_{i1}+\lambda_{i2}}\right)}{\frac{(\lambda_{i1}+\lambda_{i2})}{(\lambda_{i1}+\lambda_{i2})} - \frac{\lambda_{i2}}{(\lambda_{i1}+\lambda_{i2})}}\right] = \log\left[\frac{\lambda_{i2}}{\cancel{\lambda_{i1}+\lambda_{i2}}} \times \frac{\cancel{(\lambda_{i1}+\lambda_{i2})}}{\lambda_{i1}+\cancel{\lambda_{i2}}-\cancel{\lambda_{i2}}}\right]$$

$$= \log\left(\frac{\lambda_{i2}}{\lambda_{i1}}\right)$$

$$= \log \lambda_{i2} - \log \lambda_{i1}$$

$$\log \lambda_{i2} - \log \lambda_{i1} = \beta_0 + \beta_1 x_i$$

If $i=1$, both λ_{12} and λ_{11} refers to treatment A $\rightarrow$ This implies that $x_i = 0$

$$\beta_0 + \beta_1 x_i = \log \lambda_{12} - \log \lambda_{11} \quad \rightarrow \text{from the result in a)}$$

$$\beta_0 + \beta_1(0) = \gamma_0 + \gamma_2 - \gamma_0$$

$$\beta_0 = \gamma_2 \#$$

If $i=2$, both λ_{22} and λ_{21} refers to treatment B $\rightarrow$ This implies that $x_i = 1$

$$\beta_0 + \beta_1 x_i = \log \lambda_{22} - \log \lambda_{21} \quad \rightarrow \text{from the result in b)}$$

$$\beta_0 + \beta_1(1) = \cancel{\gamma_0} + \cancel{\gamma_1} + \gamma_2 + \gamma_3 - (\cancel{\gamma_0} + \cancel{\gamma_1})$$

$$\beta_0 + \beta_1 = \gamma_2 + \gamma_3 \qquad \text{substitute } \beta_0 = \gamma_2$$

$$\gamma_2 + \beta_1 = \gamma_2 + \gamma_3$$

$$\beta_1 = \gamma_2 - \gamma_2 + \gamma_3$$

$$\beta_1 = \gamma_3 \ \#$$

d) The idea behind testing treatment has no effect on the probability of being a case means that the predictor term x_i for treatment has 0 coefficient.

in the case of

$$\log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 x_i \qquad \beta_1 = 0$$

$$\log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0$$

from c) we know that $\beta_1 = \gamma_3$, this means $\beta_1 = 0 = \gamma_3$

substituting $\gamma_3 = 0$ into the poisson model

$$\log(\lambda_{ij}) = \gamma_0 + \gamma_1 x_i + \gamma_2 x_j + \gamma_3 x_i x_j$$

$$\log(\lambda_{ij}) = \gamma_0 + \gamma_1 x_i + \gamma_2 x_j + \underbrace{(0) x_i x_j}$$

$$\log(\lambda_{ij}) = \gamma_0 + \gamma_1 x_i + \gamma_2 x_j$$

as you can see the interaction terms in the Poisson model dissapears

Hence, showing that testing that treatment has no effect on the probability of being a case, is equivalent for test for no interaction in the Poisson model.