DSA5105 Self Practice Questions

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Disclaimer: Questions are partially generated / inspired by ChatGPT. The answers are my own answers which could be incorrect. Please raise an issue if you spot any errors. Informal citations are made in *italic* when credits are owed to external sources.

Question 1: Spectral Clustering

Given the matrix representing an undirected graph with three nodes:

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 7 \\ 2 & 7 & 0 \end{bmatrix}$$

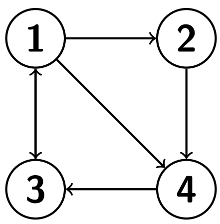
Cluster the nodes into two clusters using spectral clustering. Show your working.

Question 2: Page Rank

Given the following adjacency matrix representing a directed graph for four nodes:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

The corresponding graph can be drawn as follows:



Run the page rank algorithm to determine the page importance ranks for the four pages.

Question 3: Value Iteration

Adapted from CS4246 tutorial question for MDP

Consider the following MDP with two states and actions, and with discount factor $\gamma = 0.9$.

$P(s_1\|s_1, a_1)$	$P(s_2\|s_1, a_1)$	$P(s_1\|s_2, a_1)$	$P(s_2\|s_2, a_1)$
0.7	0.3	0	1

$P(s_1\|s_1, a_2)$	$P(s_2\|s_1, a_2)$	$P(s_1\|s_2, a_2)$	$P(s_2\|s_2, a_2)$
0.3	0.7	0	1

$R(s_1, a_1)$	$R(s_1, a_2)$	$R(s_2, a_1)$	$R(s_2, a_2)$
2	1	5	5

Derive the value functions and the optimal actions in each state after the first and second iteration, assuming the initial value for value functions is all 0. Also determine the optimal policy and the optimal value for value functions at infinite horizon.

Answers

Question 1

First we compute the Graph Laplacian and the degree matrix:

$$L_G = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 9 & -7 \\ -2 & -7 & 9 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Then the normalized Laplacian is:

$$\mathcal{L}_{\mathcal{G}} = \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{7}{9} \\ -\frac{1}{3} & -\frac{7}{9} & 1 \end{pmatrix}$$

Compute the determinant $det(\mathcal{L}_{\mathcal{G}} - \lambda I)$ manually:

$$\begin{aligned}
\det(\mathcal{L}_{\mathcal{G}} - \lambda I) &= \begin{vmatrix} 1-\lambda & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1-\lambda & -\frac{7}{9} \\ -\frac{1}{3} & -\frac{7}{9} & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} 1-\lambda & -\frac{7}{9} \\ -\frac{7}{9} & 1-\lambda \end{vmatrix} - \left(-\frac{1}{3}\right)\begin{vmatrix} -\frac{1}{3} & -\frac{7}{9} \\ -\frac{1}{3} & 1-\lambda \end{vmatrix} + \left(-\frac{1}{3}\right)\begin{vmatrix} -\frac{1}{3} & 1-\lambda \\ -\frac{1}{3} & -\frac{7}{9} \end{vmatrix} \\
&= (1-\lambda)\left[(1-\lambda)^2 - \frac{49}{81}\right] + \frac{1}{3}\left[-\frac{1}{3}(1-\lambda) - \frac{7}{27}\right] + \left(-\frac{1}{3}\right)\left[\frac{7}{27} - \frac{1}{3}(1-\lambda)\right] \\
&= (1-\lambda)\left[(1-\lambda)^2 - \frac{49}{81}\right] + \frac{2}{3}\left[-\frac{1}{3}(1-\lambda) - \frac{7}{27}\right] \\
&= (1-\lambda)^3 - \frac{49}{81} + \frac{49}{81}\lambda - \frac{32}{81} + \frac{2}{9}\lambda \\
&= -\lambda^3 + 3\lambda^2 - 3\lambda + 1 + \frac{67}{81}\lambda - 1 \\
&= -\lambda^3 + 3\lambda^2 - \frac{176}{81}\lambda
\end{aligned}$$

Solving using GC, $\lambda = \frac{11}{9}, \frac{16}{9}, 0$

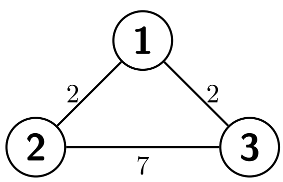
Find the eigenvector corresponding to the second smallest eigenvector, which is $\frac{11}{9}$:

$$\left(\begin{array}{ccc|c} -\frac{2}{9} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{2}{9} & -\frac{7}{9} & 0 \\ -\frac{1}{3} & -\frac{7}{9} & -\frac{2}{9} & 0 \end{array}\right)$$

We have the eigenvector:

$$v_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

By thresholding with 0 (multiplying by $D^{-\frac{1}{2}}$ is equivalent to scaling and there is no sign change in this case), we can put node 1 into cluster 1 and node 2 and 3 into cluster 2, which also make sense since the weight between node 2 and 3 is 7 as compared to weight 2 between node 1 and node 2 and node 3 and node 2:



Question 2

Applying the page rank scheme for the adjacency matrix:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & 1 & 0 & 0 \end{bmatrix}$$

and solve for the eigenvector corresponding to the eigenvalue of 1:

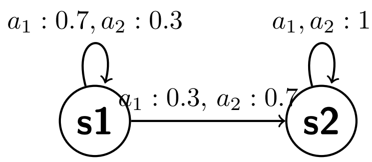
$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 0 \\ \frac{1}{3} & -1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & -1 & 1 & 0 \\ \frac{1}{3} & 1 & 0 & -1 & 0 \end{array}\right)$$

We have the following eigenvector(not normalized to unit vector, but scaling does not affect ranking):

$$v = \begin{pmatrix} 1.5 \\ 0.5 \\ 1.5 \\ 1 \end{pmatrix}$$

From this vector we conclude that page 1 and 3 are of equal importance, followed by page 4 and lastly page 2.

Question 3



For the first iteration:

$$J_1(s_1) = 2, \quad J_1(s_2) = 5, \quad a^*(s_1) = a_1, \quad a^*(s_2) = a_1 \text{ or } a_2.$$

For the second iteration:
At s_1,

$$J_2(s_1, a_1) = 2 + 0.9 * (0.7 * 2 + 0.3 * 5) = 4.61$$
$$J_2(s_1, a_2) = 1 + 0.9 * (0.3 * 2 + 0.7 * 5) = 4.69$$

So the optimal action, $a^*(s_1) = a_2$, and $J_2(s_1) = 4.69$.
At s_2,

$$J_2(s_2, a_1) = 5 + 0.9 * (0 * 2 + 1 * 5) = 9.5$$
$$J_2(s_2, a_2) = 5 + 0.9 * (0 * 2 + 1 * 5) = 9.5$$

So the optimal action, $a^*(s_1) = a_1$ or a_2, and $J_2(s_1) = 9.5$.

To compute the optimal policy and value function at infinite horizon, notice that once entering s_2, the subsequent actions will result in the same state and we are "trapped" in s_2. Using geometric progression formula we get:

$$J^*(s_2) = 5 + \gamma(5) + \gamma^2(5) + \cdots = \frac{5}{1-\gamma} = 50$$

Now we compute $J^*(s_1)$ at infinite horizon, we have:

$$
\begin{aligned}
J^*(s_1, a_1) &= 2 + 0.9(0.7J^*(s_1, a_1) + 0.3J^*(s_2)) \\
J^*(s_1, a_1) &= 41.89 \\
\\
J^*(s_1, a_2) &= 1 + 0.9(0.3J^*(s_1, a_2) + 0.7J^*(s_2)) \\
J^*(s_1, a_2) &= 44.89
\end{aligned}
$$

Therefore, the optimal action at s_1 is a_2, which makes sense since we want to get s_2 as quickly as possible so that we can get higher rewards constantly thereafter. To summarize, the results are:

$$
\begin{aligned}
J^*(s_1) = 44.89,\ \ a^*(s_1) = a_1 \\
J^*(s_2) = 50,\ \ a^*(s_2) = a_1 \text{ or } a_2
\end{aligned}
$$