

DSA5105 Self Practice Questions

Zhao Peiduo

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Disclaimer: Questions are partially generated / inspired by ChatGPT. The answers are my own answers which could be incorrect. Please raise an issue if you spot any errors. Informal citations are made in *italic* when credits are owed to external sources.

Question 1: Spectral Clustering

Given the matrix representing an undirected graph with three nodes:

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 7 \\ 2 & 7 & 0 \end{bmatrix}$$

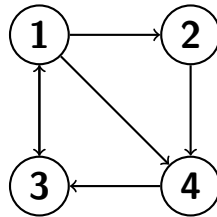
Cluster the nodes into two clusters using spectral clustering. Show your working.

Question 2: Page Rank

Given the following adjacency matrix representing a directed graph for four nodes:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

The corresponding graph can be drawn as follows:



Run the page rank algorithm to determine the page importance ranks for the four pages.

Answers

Question 1

First we compute the Graph Laplacian and the degree matrix:

$$L_G = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 9 & -7 \\ -2 & -7 & 9 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Then the normalized Laplacian is:

$$\mathcal{L}_G = \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{7}{9} \\ -\frac{1}{3} & -\frac{7}{9} & 1 \end{pmatrix}$$

Compute the determinant $\det(\mathcal{L}_G - \lambda I)$ manually:

$$\begin{aligned} \det(\mathcal{L}_G - \lambda I) &= \begin{vmatrix} 1-\lambda & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1-\lambda & -\frac{7}{9} \\ -\frac{1}{3} & -\frac{7}{9} & 1-\lambda \end{vmatrix} \\ &= (1-\lambda) \begin{vmatrix} 1-\lambda & -\frac{7}{9} \\ -\frac{7}{9} & 1-\lambda \end{vmatrix} - \left(-\frac{1}{3}\right) \begin{vmatrix} -\frac{1}{3} & -\frac{7}{9} \\ -\frac{1}{3} & 1-\lambda \end{vmatrix} + \left(-\frac{1}{3}\right) \begin{vmatrix} -\frac{1}{3} & 1-\lambda \\ -\frac{7}{9} & -\frac{7}{9} \end{vmatrix} \\ &= (1-\lambda) \left[(1-\lambda)^2 - \frac{49}{81} \right] + \frac{1}{3} \left[-\frac{1}{3}(1-\lambda) - \frac{7}{27} \right] + \left(-\frac{1}{3}\right) \left[\frac{7}{27} - \frac{1}{3}(1-\lambda) \right] \\ &= (1-\lambda) \left[(1-\lambda)^2 - \frac{49}{81} \right] + \frac{2}{3} \left[-\frac{1}{3}(1-\lambda) - \frac{7}{27} \right] \\ &= (1-\lambda)^3 - \frac{49}{81} + \frac{49}{81}\lambda - \frac{32}{81} + \frac{2}{9}\lambda \\ &= -\lambda^3 + 3\lambda^2 - 3\lambda + 1 + \frac{67}{81}\lambda - 1 \\ &= -\lambda^3 + 3\lambda^2 - \frac{176}{81}\lambda \end{aligned}$$

Solving using GC, $\lambda = \frac{11}{9}, \frac{16}{9}, 0$

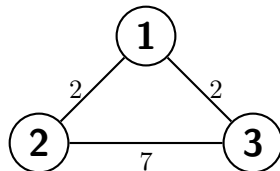
Find the eigenvector corresponding to the second smallest eigenvector, which is $\frac{11}{9}$:

$$\left(\begin{array}{ccc|c} -\frac{2}{9} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{2}{9} & -\frac{7}{9} & 0 \\ -\frac{1}{3} & -\frac{7}{9} & -\frac{2}{9} & 0 \end{array} \right)$$

We have the eigenvector:

$$v_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

By thresholding with 0 (multiplying by $D^{-\frac{1}{2}}$ is equivalent to scaling and there is no sign change in this case), we can put node 1 into cluster 1 and node 2 and 3 into cluster 2, which also make sense since the weight between node 2 and 3 is 7 as compared to weight 2 between node 1 and node 2 and node 3 and node 2:



Question 2

Applying the page rank scheme for the adjacency matrix:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & 1 & 0 & 0 \end{bmatrix}$$

and solve for the eigenvector corresponding to the eigenvalue of 1:

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 0 \\ \frac{1}{3} & -1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & -1 & 1 & 0 \\ \frac{1}{3} & 1 & 0 & -1 & 0 \end{array} \right)$$

We have the following eigenvector(not normalized to unit vector, but scaling does not affect ranking):

$$v = \begin{pmatrix} 1.5 \\ 0.5 \\ 1.5 \\ 1 \end{pmatrix}$$

From this vector we conclude that page 1 and 3 are of equal importance, followed by page 4 and lastly page 2.