DSA5105 Self Practice Questions

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Disclaimer: Questions are partially generated / inspired by ChatGPT. The answer is my own anwser which could be incorrect. Please raise an issue if you spot any errors.

Question 1: Spectral Clustering

Given the matrix representing an undirected graph with three nodes:

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 7 \\ 2 & 7 & 0 \end{bmatrix}$$

Cluster the nodes into two clusters using spectral clustering. Show your working.

Question 2

Question 3

Question 4

Answers

Question 1

First we compute the Graph Laplacian and the degree matrix:

$$L_G = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 9 & -7 \\ -2 & -7 & 9 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Then the normalized Laplacian is:

$$\mathcal{L}_{\mathcal{G}} = \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{7}{9} \\ -\frac{1}{3} & -\frac{7}{9} & 1 \end{pmatrix}$$

Compute the determinant $det(L_G - \lambda I)$ manually:

$$\begin{aligned}
\det(L - \lambda I) &= \begin{vmatrix} 1-\lambda & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1-\lambda & -\frac{7}{9} \\ -\frac{1}{3} & -\frac{7}{9} & 1-\lambda \end{vmatrix} \\
&= (1-\lambda) \begin{vmatrix} 1-\lambda & -\frac{7}{9} \\ -\frac{7}{9} & 1-\lambda \end{vmatrix} - \left(-\frac{1}{3}\right) \begin{vmatrix} -\frac{1}{3} & -\frac{7}{9} \\ -\frac{1}{3} & 1-\lambda \end{vmatrix} + \left(-\frac{1}{3}\right) \begin{vmatrix} -\frac{1}{3} & 1-\lambda \\ -\frac{1}{3} & -\frac{7}{9} \end{vmatrix} \\
&= (1-\lambda)\left[(1-\lambda)^2 - \frac{49}{81}\right] + \frac{1}{3}\left[-\frac{1}{3}(1-\lambda) - \frac{7}{27}\right] + \left(-\frac{1}{3}\right)\left[\frac{7}{27} - \frac{1}{3}(1-\lambda)\right] \\
&= (1-\lambda)\left[(1-\lambda)^2 - \frac{49}{81}\right] + \frac{2}{3}\left[-\frac{1}{3}(1-\lambda) - \frac{7}{27}\right] \\
&= (1-\lambda)^3 - \frac{49}{81} + \frac{49}{81}\lambda - \frac{32}{81} + \frac{2}{9}\lambda \\
&= -\lambda^3 + 3\lambda^2 - 3\lambda + 1 + \frac{67}{81}\lambda - 1 \\
&= -\lambda^3 + 3\lambda^2 - \frac{176}{81}\lambda
\end{aligned}$$

Solving using GC, $\lambda = \frac{11}{9}, \frac{16}{9}, 0$

Find the eigenvector corresponding to the second largest eigenvector, which is $\frac{11}{9}$:

$$\left(\begin{array}{ccc|c} -\frac{2}{9} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{2}{9} & -\frac{7}{9} & 0 \\ -\frac{1}{3} & -\frac{7}{9} & -\frac{2}{9} & 0 \end{array}\right)$$

We have the eigenvector:

$$v_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

By thresholding with 0 (multiplying by $D^{-\frac{1}{2}}$ is equivalent to scaling and there is no sign change in this case), we can put node 1 into cluster 1 and node 2 and 3 into cluster 2, which also make sense since the weight between node 2 and 3 is 7 as compared to weight 2 between node 1 and node 2 and node 3 and node 2: