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Thomson problem

The objective of the **Thomson problem** is to determine the minimum electrostatic potential energy configuration of *N* electrons constrained to the surface of a unit sphere that repel each other with a force given by Coulomb's law. The physicist J. J. Thomson posed the problem in 1904[1] after proposing an atomic model, later called the plum pudding model, based on his knowledge of the existence of negatively charged electrons within neutrally-charged atoms.

Related problems include the study of the geometry of the minimum energy configuration and the study of the large *N* behavior of the minimum energy.

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Mathematical statement

The physical system embodied by the Thomson problem is a special case of one of eighteen unsolved mathematics problems proposed by the mathematician Steve Smale — "Distribution of points on the 2-sphere".[2] The solution of each *N*-electron problem is obtained when the *N*-electron configuration constrained to the surface of a sphere of unit radius, $r = 1$, yields a global electrostatic potential energy minimum, $U(N)$.

The electrostatic interaction energy occurring between each pair of electrons of equal charges ($e_i = e_j = e$, with e the elementary charge of an electron) is given by Coulomb's Law,

$$U_{ij}(N) = k_e \frac{e_i e_j}{r_{ij}}.$$

Here, k_e is Coulomb's constant and $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ is the distance between each pair of electrons located at points on the sphere defined by vectors $\mathbf{r}_i$ and $\mathbf{r}_j$, respectively.

Simplified units of $e = 1$ and $k_e = 1$ are used without loss of generality. Then,

$$U_{ij}(N) = \frac{1}{r_{ij}}.$$

The total electrostatic potential energy of each *N*-electron configuration may then be expressed as the sum of all pair-wise interactions

$$U(N) = \sum_{i<j} \frac{1}{r_{ij}}.$$

The global minimization of $U(N)$ over all possible collections of *N* distinct points is typically found by numerical minimization algorithms.

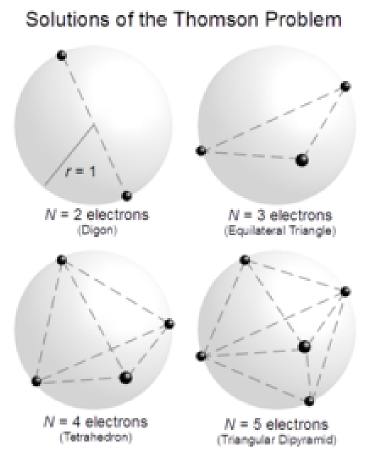
Example

The solution of the Thomson problem for two electrons is obtained when both electrons are as far apart as possible on opposite sides of the origin, $r_{ij} = 2r = 2$, or

$$U(2) = \frac{1}{2}.$$

Known solutions

Minimum energy configurations have been rigorously identified in only a handful of cases.



Schematic geometric solutions of the mathematical Thomson Problem for up to *N* = 5 electrons.

- For *N* = 1, the solution is trivial as the electron may reside at any point on the surface of the unit sphere. The total energy of the configuration is defined as zero as the electron is not subject to the electric field due to any other sources of charge.
- For *N* = 2, the optimal configuration consists of electrons at antipodal points.
- For *N* = 3, electrons reside at the vertices of an equilateral triangle about a great circle.[3]
- For *N* = 4, electrons reside at the vertices of a regular tetrahedron.
- For *N* = 5, a mathematically rigorous computer-aided solution was reported in 2010 with electrons residing at vertices of a triangular dipyramid.[4]
- For *N* = 6, electrons reside at vertices of a regular octahedron.[5]
- For *N* = 12, electrons reside at the vertices of a regular icosahedron.[6]

Notably, the geometric solutions of the Thomson problem for *N* = 4, 6, and 12 electrons are known as Platonic solids whose faces are all congruent equilateral triangles. Numerical solutions for *N* = 8 and 20 are not the regular convex polyhedral configurations of the remaining two Platonic solids, whose faces are square and pentagonal, respectively.

Generalizations

One can also ask for ground states of particles interacting with arbitrary potentials. To be mathematically precise, let *f* be a decreasing real-valued function, and define the energy functional $\sum_{i<j} f(|x_i - x_j|)$

Traditionally, one considers $f(x) = x^{-\alpha}$ also known as Riesz α-kernels. For integrable Riesz kernels see;[7] for non-integrable Riesz kernels, the Poppy-seed bagel theorem holds, see.[8] Notable cases include *α* = ∞, the Tammes problem (packing); *α* = 1, the Thomson problem; *α* = 0, Whyte's problem (to maximize the product of distances).

One may also consider configurations of *N* points on a sphere of higher dimension. See spherical design.

Relations to other scientific problems

The Thomson problem is a natural consequence of Thomson's plum pudding model in the absence of its uniform positive background charge.[9]

> "No fact discovered about the atom can be trivial, nor fail to accelerate the progress of physical science, for the greater part of natural philosophy is the outcome of the structure and mechanism of the atom."
>
> —Sir J. J. Thomson[10]

Though experimental evidence led to the abandonment of Thomson's plum pudding model as a complete atomic model, irregularities observed in numerical energy solutions of the Thomson problem have been found to correspond with electron shell-filling in naturally occurring atoms throughout the periodic table of elements.[11]

The Thomson problem also plays a role in the study of other physical models including multi-electron bubbles and the surface ordering of liquid metal drops confined in Paul traps.

The generalized Thomson problem arises, for example, in determining the arrangements of the protein subunits which comprise the shells of spherical viruses. The "particles" in this application are clusters of protein subunits arranged on a shell. Other realizations include regular arrangements of colloid particles in *colloidosomes*, proposed for encapsulation of active ingredients such as drugs, nutrients or living cells, fullerene patterns of carbon atoms, and VSEPR theory. An example with long-range logarithmic interactions is provided by the Abrikosov vortices which would form at low temperatures in a superconducting metal shell with a large monopole at the center.

Configurations of smallest known energy

In the following table N is the number of points (charges) in a configuration, E_1 is the energy, the symmetry type is given in Schönflies notation (see Point groups in three dimensions), and r_i are the positions of the charges. Most symmetry types require the vector sum of the positions (and thus the electric dipole moment) to be zero.

It is customary to also consider the polyhedron formed by the convex hull of the points. Thus, v_i is the number of vertices where the given number of edges meet, 'e is the total number of edges, f_3 is the number of triangular faces, f_4 is the number of quadrilateral faces, and θ_1 is the smallest angle subtended by vectors associated with the nearest charge pair. Note that the edge lengths are generally not equal; thus (except in the cases *N* = 4, 6, 12, 24) the convex hull is only topologically equivalent to the uniform polyhedron or Johnson solid listed in the last column.[12]

N	E_1	Symmetry	$\left\|\sum \mathbf{r}_i\right\|$	v_3	v_4	v_5	v_6	v_7	v_8	e	f_3	f_4	θ_1	Equivalent polyhedron
2	0.500000000	$D_{\infty h}$	0	–	–	–	–	–	–	1	–	–	180.000°	digon
3	1.732050808	D_{3h}	0	–	–	–	–	–	–	3	1	–	120.000°	triangle
4	3.674234614	T_d	0	4	0	0	0	0	0	6	4	0	109.471°	tetrahedron
5	6.474691495	D_{3h}	0	2	3	0	0	0	0	9	6	0	90.000°	triangular dipyramid
6	9.985281374	O_h	0	0	6	0	0	0	0	12	8	0	90.000°	octahedron
7	14.452977414	D_{5h}	0	0	5	2	0	0	0	15	10	0	72.000°	pentagonal dipyramid
8	19.675287861	D_{4d}	0	0	8	0	0	0	0	16	8	2	71.694°	square antiprism
9	25.759986531	D_{3h}	0	0	3	6	0	0	0	21	14	0	69.190°	triaugmented triangular prism
10	32.716949460	D_{4d}	0	0	2	8	0	0	0	24	16	0	64.996°	gyroelongated square dipyramid
11	40.596450510	C_{2v}	0.013219635	0	2	8	1	0	0	27	18	0	58.540°	edge-contracted icosahedron
12	49.165253058	I_h	0	0	0	12	0	0	0	30	20	0	63.435°	icosahedron
13	58.853230612	C_{2v}	0.008820367	0	1	10	2	0	0	33	22	0	52.317°	
14	69.306363297	D_{6d}	0	0	0	12	2	0	0	36	24	0	52.866°	gyroelongated hexagonal dipyramid
15	80.670244114	D_3	0	0	0	12	3	0	0	39	26	0	49.225°	
16	92.911655302	T	0	0	0	12	4	0	0	42	28	0	48.936°	
17	106.050404829	D_{5h}	0	0	0	12	5	0	0	45	30	0	50.108°	
18	120.084467447	D_{4d}	0	0	2	8	8	0	0	48	32	0	47.534°	
19	135.089467557	C_{2v}	0.000135163	0	0	14	5	0	0	50	32	1	44.910°	
20	150.881568334	D_{3h}	0	0	0	12	8	0	0	54	36	0	46.093°	
21	167.641622399	C_{2v}	0.001406124	0	1	10	10	0	0	57	38	0	44.321°	
22	185.287536149	T_d	0	0	0	12	10	0	0	60	40	0	43.302°	
23	203.930190663	D_3	0	0	0	12	11	0	0	63	42	0	41.481°	
24	223.347074052	O	0	0	0	24	0	0	0	60	32	6	42.065°	snub cube
25	243.812760299	C_s	0.001021305	0	0	14	11	0	0	68	44	1	39.610°	
26	265.133326317	C_2	0.001919065	0	0	12	14	0	0	72	48	0	38.842°	
27	287.302615033	D_{5h}	0	0	0	12	15	0	0	75	50	0	39.940°	
28	310.491542358	T	0	0	0	12	16	0	0	78	52	0	37.824°	
29	334.634439920	D_3	0	0	0	12	17	0	0	81	54	0	36.391°	
30	359.603945904	D_2	0	0	0	12	18	0	0	84	56	0	36.942°	
31	385.530838063	C_{3v}	0.003204712	0	0	12	19	0	0	87	58	0	36.373°	
32	412.261274651	I_h	0	0	0	12	20	0	0	90	60	0	37.377°	
33	440.204057448	C_s	0.004356481	0	0	15	17	1	0	92	60	1	33.700°	
34	468.904853281	D_2	0	0	0	12	22	0	0	96	64	0	33.273°	
35	498.569872491	C_2	0.000419208	0	0	12	23	0	0	99	66	0	33.100°	
36	529.122408375	D_2	0	0	0	12	24	0	0	102	68	0	33.229°	
37	560.618887731	D_{5h}	0	0	0	12	25	0	0	105	70	0	32.332°	

38	593.038503566	D_{6d}	0	0	0	12	26	0	0	108	72	0	33.236°
39	626.389009017	D_{3h}	0	0	0	12	27	0	0	111	74	0	32.053°
40	660.675278835	T_d	0	0	0	12	28	0	0	114	76	0	31.916°
41	695.916744342	D_{3h}	0	0	0	12	29	0	0	117	78	0	31.528°
42	732.078107544	D_{5h}	0	0	0	12	30	0	0	120	80	0	31.245°
43	769.190846459	C_{2v}	0.000399668	0	0	12	31	0	0	123	82	0	30.867°
44	807.174263085	O_h	0	0	0	24	20	0	0	120	72	6	31.258°
45	846.188401061	D_3	0	0	0	12	33	0	0	129	86	0	30.207°
46	886.167113639	T	0	0	0	12	34	0	0	132	88	0	29.790°
47	927.059270680	C_s	0.002482914	0	0	14	33	0	0	134	88	1	28.787°
48	968.713455344	O	0	0	0	24	24	0	0	132	80	6	29.690°
49	1011.557182654	C_3	0.001529341	0	0	12	37	0	0	141	94	0	28.387°
50	1055.182314726	D_{6d}	0	0	0	12	38	0	0	144	96	0	29.231°
51	1099.819290319	D_3	0	0	0	12	39	0	0	147	98	0	28.165°
52	1145.418964319	C_3	0.000457327	0	0	12	40	0	0	150	100	0	27.670°
53	1191.922290416	C_{2v}	0.000278469	0	0	18	35	0	0	150	96	3	27.137°
54	1239.361474729	C_2	0.000137870	0	0	12	42	0	0	156	104	0	27.030°
55	1287.772720783	C_2	0.000391696	0	0	12	43	0	0	159	106	0	26.615°
56	1337.094945276	D_2	0	0	0	12	44	0	0	162	108	0	26.683°
57	1387.383229253	D_3	0	0	0	12	45	0	0	165	110	0	26.702°
58	1438.618250640	D_2	0	0	0	12	46	0	0	168	112	0	26.155°
59	1490.773335279	C_2	0.000154286	0	0	14	43	2	0	171	114	0	26.170°
60	1543.830400976	D_3	0	0	0	12	48	0	0	174	116	0	25.958°
61	1597.941830199	C_1	0.001091717	0	0	12	49	0	0	177	118	0	25.392°
62	1652.909409898	D_5	0	0	0	12	50	0	0	180	120	0	25.880°
63	1708.879681503	D_3	0	0	0	12	51	0	0	183	122	0	25.257°
64	1765.802577927	D_2	0	0	0	12	52	0	0	186	124	0	24.920°
65	1823.667960264	C_2	0.000399515	0	0	12	53	0	0	189	126	0	24.527°
66	1882.441525304	C_2	0.000776245	0	0	12	54	0	0	192	128	0	24.765°
67	1942.122700406	D_5	0	0	0	12	55	0	0	195	130	0	24.727°
68	2002.874701749	D_2	0	0	0	12	56	0	0	198	132	0	24.433°
69	2064.533483235	D_3	0	0	0	12	57	0	0	201	134	0	24.137°
70	2127.100901551	D_{2d}	0	0	0	12	50	0	0	200	128	4	24.291°
71	2190.649906425	C_2	0.001256769	0	0	14	55	2	0	207	138	0	23.803°
72	2255.001190975	I	0	0	0	12	60	0	0	210	140	0	24.492°
73	2320.633883745	C_2	0.001572959	0	0	12	61	0	0	213	142	0	22.810°
74	2387.072981838	C_2	0.000641539	0	0	12	62	0	0	216	144	0	22.966°
75	2454.369689040	D_3	0	0	0	12	63	0	0	219	146	0	22.736°
76	2522.674871841	C_2	0.000943474	0	0	12	64	0	0	222	148	0	22.886°
77	2591.850152354	D_5	0	0	0	12	65	0	0	225	150	0	23.286°
78	2662.046474566	T_h	0	0	0	12	66	0	0	228	152	0	23.426°
79	2733.248357479	C_s	0.000702921	0	0	12	63	1	0	230	152	1	22.636°
80	2805.355875981	D_{4d}	0	0	0	16	64	0	0	232	152	2	22.778°
81	2878.522829664	C_2	0.000194289	0	0	12	69	0	0	237	158	0	21.892°

82	2952.569675286	D_2	0	0	0	12	70	0	0	240	160	0	22.206°
83	3027.528488921	C_2	0.000339815	0	0	14	67	2	0	243	162	0	21.646°
84	3103.465124431	C_2	0.000401973	0	0	12	72	0	0	246	164	0	21.513°
85	3180.361442939	C_2	0.000416581	0	0	12	73	0	0	249	166	0	21.498°
86	3258.211605713	C_2	0.001378932	0	0	12	74	0	0	252	168	0	21.522°
87	3337.000750014	C_2	0.000754863	0	0	12	75	0	0	255	170	0	21.456°
88	3416.720196758	D_2	0	0	0	12	76	0	0	258	172	0	21.486°
89	3497.439018625	C_2	0.000070891	0	0	12	77	0	0	261	174	0	21.182°
90	3579.091222723	D_3	0	0	0	12	78	0	0	264	176	0	21.230°
91	3661.713699320	C_2	0.000033221	0	0	12	79	0	0	267	178	0	21.105°
92	3745.291636241	D_2	0	0	0	12	80	0	0	270	180	0	21.026°
93	3829.844338421	C_2	0.000213246	0	0	12	81	0	0	273	182	0	20.751°
94	3915.309269620	D_2	0	0	0	12	82	0	0	276	184	0	20.952°
95	4001.771675565	C_2	0.000116638	0	0	12	83	0	0	279	186	0	20.711°
96	4089.154010060	C_2	0.000036310	0	0	12	84	0	0	282	188	0	20.687°
97	4177.533599622	C_2	0.000096437	0	0	12	85	0	0	285	190	0	20.450°
98	4266.822464156	C_2	0.000112916	0	0	12	86	0	0	288	192	0	20.422°
99	4357.139163132	C_2	0.000156508	0	0	12	87	0	0	291	194	0	20.284°
100	4448.350634331	T	0	0	0	12	88	0	0	294	196	0	20.297°
101	4540.590051694	D_3	0	0	0	12	89	0	0	297	198	0	20.011°
102	4633.736565899	D_3	0	0	0	12	90	0	0	300	200	0	20.040°
103	4727.836616833	C_2	0.000201245	0	0	12	91	0	0	303	202	0	19.907°
104	4822.876522746	D_6	0	0	0	12	92	0	0	306	204	0	19.957°
105	4919.000637616	D_3	0	0	0	12	93	0	0	309	206	0	19.842°
106	5015.984595705	D_2	0	0	0	12	94	0	0	312	208	0	19.658°
107	5113.953547724	C_2	0.000064137	0	0	12	95	0	0	315	210	0	19.327°
108	5212.813507831	C_2	0.000432525	0	0	12	96	0	0	318	212	0	19.327°
109	5312.735079920	C_2	0.000647299	0	0	14	93	2	0	321	214	0	19.103°
110	5413.549294192	D_6	0	0	0	12	98	0	0	324	216	0	19.476°
111	5515.293214587	D_3	0	0	0	12	99	0	0	327	218	0	19.255°
112	5618.044882327	D_5	0	0	0	12	100	0	0	330	220	0	19.351°
113	5721.824978027	D_3	0	0	0	12	101	0	0	333	222	0	18.978°
114	5826.521572163	C_2	0.000149772	0	0	12	102	0	0	336	224	0	18.836°
115	5932.181285777	C_3	0.000049972	0	0	12	103	0	0	339	226	0	18.458°
116	6038.815593579	C_2	0.000259726	0	0	12	104	0	0	342	228	0	18.386°
117	6146.342446579	C_2	0.000127609	0	0	12	105	0	0	345	230	0	18.566°
118	6254.877027790	C_2	0.000332475	0	0	12	106	0	0	348	232	0	18.455°
119	6364.347317479	C_2	0.000685590	0	0	12	107	0	0	351	234	0	18.336°
120	6474.756324980	C_s	0.001373062	0	0	12	108	0	0	354	236	0	18.418°
121	6586.121949584	C_3	0.000838863	0	0	12	109	0	0	357	238	0	18.199°
122	6698.374499261	I_h	0	0	0	12	110	0	0	360	240	0	18.612°
123	6811.827228174	C_{2v}	0.001939754	0	0	14	107	2	0	363	242	0	17.840°
124	6926.169974193	D_2	0	0	0	12	112	0	0	366	244	0	18.111°
125	7041.473264023	C_2	0.000088274	0	0	12	113	0	0	369	246	0	17.867°

126	7157.669224867	D_4	0	0	2	16	100	8	0	372	248	0	17.920°
127	7274.819504675	D_5	0	0	0	12	115	0	0	375	250	0	17.877°
128	7393.007443068	C_2	0.000054132	0	0	12	116	0	0	378	252	0	17.814°
129	7512.107319268	C_2	0.000030099	0	0	12	117	0	0	381	254	0	17.743°
130	7632.167378912	C_2	0.000025622	0	0	12	118	0	0	384	256	0	17.683°
131	7753.205166941	C_2	0.000305133	0	0	12	119	0	0	387	258	0	17.511°
132	7875.045342797	I	0	0	0	12	120	0	0	390	260	0	17.958°
133	7998.179212898	C_3	0.000591438	0	0	12	121	0	0	393	262	0	17.133°
134	8122.089721194	C_2	0.000470268	0	0	12	122	0	0	396	264	0	17.214°
135	8246.909486992	D_3	0	0	0	12	123	0	0	399	266	0	17.431°
136	8372.743302539	T	0	0	0	12	124	0	0	402	268	0	17.485°
137	8499.534494782	D_5	0	0	0	12	125	0	0	405	270	0	17.560°
138	8627.406389880	C_2	0.000473576	0	0	12	126	0	0	408	272	0	16.924°
139	8756.227056057	C_2	0.000404228	0	0	12	127	0	0	411	274	0	16.673°
140	8885.980609041	C_1	0.000630351	0	0	13	126	1	0	414	276	0	16.773°
141	9016.615349190	C_{2v}	0.000376365	0	0	14	126	0	1	417	278	0	16.962°
142	9148.271579993	C_2	0.000550138	0	0	12	130	0	0	420	280	0	16.840°
143	9280.839851192	C_2	0.000255449	0	0	12	131	0	0	423	282	0	16.782°
144	9414.371794460	D_2	0	0	0	12	132	0	0	426	284	0	16.953°
145	9548.928837232	C_s	0.000094938	0	0	12	133	0	0	429	286	0	16.841°
146	9684.381825575	D_2	0	0	0	12	134	0	0	432	288	0	16.905°
147	9820.932378373	C_2	0.000636651	0	0	12	135	0	0	435	290	0	16.458°
148	9958.406004270	C_2	0.000203701	0	0	12	136	0	0	438	292	0	16.627°
149	10096.859907397	C_1	0.000638186	0	0	14	133	2	0	441	294	0	16.344°
150	10236.196436701	T	0	0	0	12	138	0	0	444	296	0	16.405°
151	10376.571469275	C_2	0.000153836	0	0	12	139	0	0	447	298	0	16.163°
152	10517.867592878	D_2	0	0	0	12	140	0	0	450	300	0	16.117°
153	10660.082748237	D_3	0	0	0	12	141	0	0	453	302	0	16.390°
154	10803.372421141	C_2	0.000735800	0	0	12	142	0	0	456	304	0	16.078°
155	10947.574692279	C_2	0.000603670	0	0	12	143	0	0	459	306	0	15.990°
156	11092.798311456	C_2	0.000508534	0	0	12	144	0	0	462	308	0	15.822°
157	11238.903041156	C_2	0.000357679	0	0	12	145	0	0	465	310	0	15.948°
158	11385.990186197	C_2	0.000921918	0	0	12	146	0	0	468	312	0	15.987°
159	11534.023960956	C_2	0.000381457	0	0	12	147	0	0	471	314	0	15.960°
160	11683.054805549	D_2	0	0	0	12	148	0	0	474	316	0	15.961°
161	11833.084739465	C_2	0.000056447	0	0	12	149	0	0	477	318	0	15.810°
162	11984.050335814	D_3	0	0	0	12	150	0	0	480	320	0	15.813°
163	12136.013053220	C_2	0.000120798	0	0	12	151	0	0	483	322	0	15.675°
164	12288.930105320	D_2	0	0	0	12	152	0	0	486	324	0	15.655°
165	12442.804451373	C_2	0.000091119	0	0	12	153	0	0	489	326	0	15.651°
166	12597.649071323	D_{2d}	0	0	0	16	146	4	0	492	328	0	15.607°
167	12753.469429750	C_2	0.000097382	0	0	12	155	0	0	495	330	0	15.600°
168	12910.212672268	D_3	0	0	0	12	156	0	0	498	332	0	15.655°
169	13068.006451127	C_s	0.000068102	0	0	13	155	1	0	501	334	0	15.537°

170	13226.681078541	D_{2d}	0	0	0	12	158	0	0	504	336	0	15.569°
171	13386.355930717	D_3	0	0	0	12	159	0	0	507	338	0	15.497°
172	13547.018108787	C_{2v}	0.000547291	0	0	14	156	2	0	510	340	0	15.292°
173	13708.635243034	C_s	0.000286544	0	0	12	161	0	0	513	342	0	15.225°
174	13871.187092292	D_2	0	0	0	12	162	0	0	516	344	0	15.366°
175	14034.781306929	C_2	0.000026686	0	0	12	163	0	0	519	346	0	15.252°
176	14199.354775632	C_1	0.000283978	0	0	12	164	0	0	522	348	0	15.101°
177	14364.837545298	D_5	0	0	0	12	165	0	0	525	350	0	15.269°
178	14531.309552587	D_2	0	0	0	12	166	0	0	528	352	0	15.145°
179	14698.754594220	C_1	0.000125113	0	0	13	165	1	0	531	354	0	14.968°
180	14867.099927525	D_2	0	0	0	12	168	0	0	534	356	0	15.067°
181	15036.467239769	C_2	0.000304193	0	0	12	169	0	0	537	358	0	15.002°
182	15206.730610906	D_5	0	0	0	12	170	0	0	540	360	0	15.155°
183	15378.166571028	C_1	0.000467899	0	0	12	171	0	0	543	362	0	14.747°
184	15550.421450311	T	0	0	0	12	172	0	0	546	364	0	14.932°
185	15723.720074072	C_2	0.000389762	0	0	12	173	0	0	549	366	0	14.775°
186	15897.897437048	C_1	0.000389762	0	0	12	174	0	0	552	368	0	14.739°
187	16072.975186320	D_5	0	0	0	12	175	0	0	555	370	0	14.848°
188	16249.222678879	D_2	0	0	0	12	176	0	0	558	372	0	14.740°
189	16426.371938862	C_2	0.000020732	0	0	12	177	0	0	561	374	0	14.671°
190	16604.428338501	C_3	0.000586804	0	0	12	178	0	0	564	376	0	14.501°
191	16783.452219362	C_1	0.001129202	0	0	13	177	1	0	567	378	0	14.195°
192	16963.338386460	I	0	0	0	12	180	0	0	570	380	0	14.819°
193	17144.564740880	C_2	0.000985192	0	0	12	181	0	0	573	382	0	14.144°
194	17326.616136471	C_1	0.000322358	0	0	12	182	0	0	576	384	0	14.350°
195	17509.489303930	D_3	0	0	0	12	183	0	0	579	386	0	14.375°
196	17693.460548082	C_2	0.000315907	0	0	12	184	0	0	582	388	0	14.251°
197	17878.340162571	D_5	0	0	0	12	185	0	0	585	390	0	14.147°
198	18064.262177195	C_2	0.000011149	0	0	12	186	0	0	588	392	0	14.237°
199	18251.082495640	C_1	0.000534779	0	0	12	187	0	0	591	394	0	14.153°
200	18438.842717530	D_2	0	0	0	12	188	0	0	594	396	0	14.222°
201	18627.591226244	C_1	0.001048859	0	0	13	187	1	0	597	398	0	13.830°
202	18817.204718262	D_5	0	0	0	12	190	0	0	600	400	0	14.189°
203	19007.981204580	C_s	0.000600343	0	0	12	191	0	0	603	402	0	13.977°
204	19199.540775603	T_h	0	0	0	12	192	0	0	606	404	0	14.291°
212	20768.053085964	I	0	0	0	12	200	0	0	630	420	0	14.118°
214	21169.910410375	T	0	0	0	12	202	0	0	636	424	0	13.771°
216	21575.596377869	D_3	0	0	0	12	204	0	0	642	428	0	13.735°
217	21779.856080418	D_5	0	0	0	12	205	0	0	645	430	0	13.902°
232	24961.252318934	T	0	0	0	12	220	0	0	690	460	0	13.260°
255	30264.424251281	D_3	0	0	0	12	243	0	0	759	506	0	12.565°
256	30506.687515847	T	0	0	0	12	244	0	0	762	508	0	12.572°
257	30749.941417346	D_5	0	0	0	12	245	0	0	765	510	0	12.672°
272	34515.193292681	I_h	0	0	0	12	260	0	0	810	540	0	12.335°

282	37147.294418462	I	0	0	0	12	270	0	0	840	560	0	12.166°
292	39877.008012909	D_5	0	0	0	12	280	0	0	870	580	0	11.857°
306	43862.569780797	T_h	0	0	0	12	294	0	0	912	608	0	11.628°
312	45629.313804002	C_2	0.000306163	0	0	12	300	0	0	930	620	0	11.299°
315	46525.825643432	D_3	0	0	0	12	303	0	0	939	626	0	11.337°
317	47128.310344520	D_5	0	0	0	12	305	0	0	945	630	0	11.423°
318	47431.056020043	D_3	0	0	0	12	306	0	0	948	632	0	11.219°
334	52407.728127822	T	0	0	0	12	322	0	0	996	664	0	11.058°
348	56967.472454334	T_h	0	0	0	12	336	0	0	1038	692	0	10.721°
357	59999.922939598	D_5	0	0	0	12	345	0	0	1065	710	0	10.728°
358	60341.830924588	T	0	0	0	12	346	0	0	1068	712	0	10.647°
372	65230.027122557	I	0	0	0	12	360	0	0	1110	740	0	10.531°
382	68839.426839215	D_5	0	0	0	12	370	0	0	1140	760	0	10.379°
390	71797.035335953	T_h	0	0	0	12	378	0	0	1164	776	0	10.222°
392	72546.258370889	C_1	0	0	0	12	380	0	0	1170	780	0	10.278°
400	75582.448512213	T	0	0	0	12	388	0	0	1194	796	0	10.068°
402	76351.192432673	D_5	0	0	0	12	390	0	0	1200	800	0	10.099°
432	88353.709681956	D_3	0	0	0	24	396	12	0	1290	860	0	9.556°
448	95115.546986209	T	0	0	0	24	412	12	0	1338	892	0	9.322°
460	100351.763108673	T	0	0	0	24	424	12	0	1374	916	0	9.297°
468	103920.871715127	S_6	0	0	0	24	432	12	0	1398	932	0	9.120°
470	104822.886324279	S_6	0	0	0	24	434	12	0	1404	936	0	9.059°

According to a conjecture, if $m = n + 2$, p is the polyhedron formed by the convex hull of m points, q is the number of quadrilateral faces of p, then the solution for m electrons is $f(m)$: $f(m) = 0^n + 3n - q$.[13]

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