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Some unique properties of eigenvector centrality

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Abstract

Eigenvectors, and the related centrality measure Bonacich's $c(\beta)$, have advantages over graph-theoretic measures like degree, betweenness, and closeness centrality: they can be used in signed and valued graphs and the beta parameter in $c(\beta)$ permits the calculation of power measures for a wider variety of types of exchange. Degree, betweenness, and closeness centralities are defined only for classically simple graphs—those with strictly binary relations between vertices. Looking only at these classical graphs, where eigenvectors and graph–theoretic measures are competitors, eigenvector centrality is designed to be distinctively different from mere degree centrality when there are some high degree positions connected to many low degree others or some low degree positions are connected to a few high degree others. Therefore, it will not be distinctively different from degree when positions are all equal in degree (regular graphs) or in core-periphery structures in which high degree positions tend to be connected to each other.

Keywords: Centrality; Eigenvector

1. Introduction

Bonacich (1972) suggested that the eigenvector of the largest eigenvalue of an adjacency matrix could make a good network centrality measure. Unlike degree, which weights every contact equally, the eigenvector weights contacts according to their centralities. Eigenvector centrality can also be seen as a weighted sum of not only direct connections but indirect connections of every length. Thus it takes into account the entire pattern in the network. Important extensions are beta-centrality $c(\beta)$,which permits an assessment of power in negatively connected bargaining networks – networks in which one's own power is reduced by connection to others with many alternative exchange partners (Bonacich, 1987) – and to networks with negative as well as positive ties, where a hostile connection to a high status other reduces ones status but a negative connection to a disliked other raises ones status (Bonacich and Lloyd, 2004). Many others have used eigenvector

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centrality and it continues to be refined and developed (Brundes and Cornelsen, 2003; Estrada and Rodrìguez-Velázquez, 2005; Ruhnau, 2000; Richards and Seary, 2000).

The first part of this paper shows that in ordinary graphs, in which relationships either exist or fail to exist between vertices, the eigenvector and $c(\beta)$ measures are designed to assess certain varieties of centrality but not others. The second part will give some circumstances in which the measures give unexpected but not necessarily undesirable results.

1.1. Eigenvector centrality, beta-centrality, and their usual relationship

First, let us look at some mathematical preliminaries. Let $G(E,V)$ be a graph, consisting of vertices V and edges E. Let A be the adjacency matrix for this graph; $a_{ij} = 1$ if vertices i and j are connected by an edge and $a_{ij} = 0$ if they are not. Later the complication of negatively valued relationships will be introduced. Because A is symmetric all its eigenvalues are real, its eigenvectors are orthogonal, and it is diagonalizable (Golub and Van Loon, 1983).

Eq. (1) describes eigenvector centrality x in two equivalent ways, as a matrix equation and as a sum. The centrality of a vertex is proportional to the sum of the centralities of the vertices to which it is connected. λ is the largest eigenvalue of A and n is the number of vertices:

$$Ax = \lambda x, \qquad \lambda x_i = \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, \ldots, n \tag{1}$$

A second measure of centrality, beta-centrality or $c(\beta)$, is defined as a weighted sum of paths connecting other vertices to each position, where longer paths are weighted less. This measure permits varying the degree to which status is transmitted from one vertex to another and it also permits the assessment of power in *negatively connected* (Cook et al., 1983) exchange networks when being connected to exploitable isolates increases ones power:

$$c(\beta) = \sum_{k=1}^{\infty} \beta^{k-1} A^k 1 \tag{2}$$

where $|\beta| < 1/\lambda$ and 1 is a vector of ones.

Eq. (2) is an infinite sum that converges only if $|\beta| < 1/\lambda$, where λ is the largest eigenvalue. There is an important relation between the two measures. $c(\beta)$ usually approaches x as beta approaches $1/\beta$ from below. To show this we must first remind ourselves that for symmetric matrices A and any power of A is the weighted sum of the products of the eigenvectors with their own transposes each product $x_i x_i^t$ weighted by its eigenvalue λ_i (Golub and Van Loon, 1983):

$$A = \sum_{i=1}^{n} \lambda_i x_i x_i^t, \qquad A^k = \sum_{i=1}^{n} \lambda_i^k x_i x_i^t \tag{3}$$

Therefore:

$$\begin{aligned} c(\beta) &= \sum_{k=1}^{\infty} \beta^{k-1} A^k 1 = \sum_{k=1}^{\infty} \beta^{k-1} \left(\sum_{i=1}^{n} \lambda_i^k x_i x_i^t \right) 1 = \frac{1}{\beta} \left(\sum_{i-1}^{n} \left(\sum_{k=1}^{\infty} (\beta \lambda_i)^k \right) x_i x_i^t \right) 1 \\ &= \frac{1}{\beta} \left(\sum_{i=1}^{n} \frac{\beta \lambda_i}{1 - \beta \lambda_i} x_i x_i^t \right) 1 \text{ when } \beta \lambda_i < 1 \text{ for every } i \end{aligned} \tag{4}$$

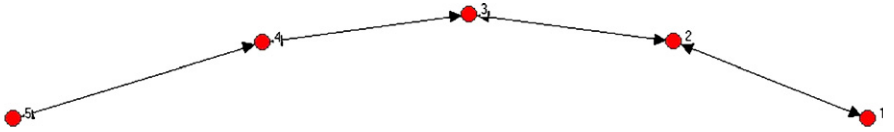

Fig. 1. An illustrative network.

(Nothing requires that A be an adjacency matrix. The edge values can be other than 0 and 1). Note that if there is a distinctively greatest eigenvalue, as beta approaches the reciprocal of this eigenvalue from below the coefficient for the eigenvector of this eigenvalue comes to predominate over the others, so $c(\beta)$ approaches x. The following will usually be true:

$$\lim_{\beta \to 1/\lambda -} c(\beta) = x \tag{5}$$

1.1.1. Uses of $c(\beta)$ and x in negatively connected networks

Eigenvector centrality x and $c(\beta)$ are complementary. When $\beta > 0$, variations in beta from its minimum of zero to its maximum of $1/\lambda$ allows for the centralities of vertices to be calculated under different assumptions about the degree to which status or centrality is transmitted between vertices. The greater the value of beta, the greater the transmission. If the popularity of one individual "rubs off" on his associates, then beta should be large (but less than $1/\lambda$). Similarly, to the extent that individuals transmit to their contacts information that they have learned from their own contacts, beta should be large. On the other hand, as Bonacich (1987) shows, a negative beta successfully predicts the results of experiments on bargaining in networks where powerful positions have many potential partners each with few options. A vertex is more powerful to the degree its potential bargaining partners themselves have few alternatives. A negative beta ensures that status is *reduced* to the degree that ones contacts have many contacts.

For example, consider the network in Fig. 1, which illustrated the usefulness of both positive and negative values for beta.

Fig. 2 shows the plot of the beta-centralities of positions 2 and 3 for varying values of beta. When beta is 0, $c(\beta)$ is proportional to degree. When beta is positive position 2 is less central

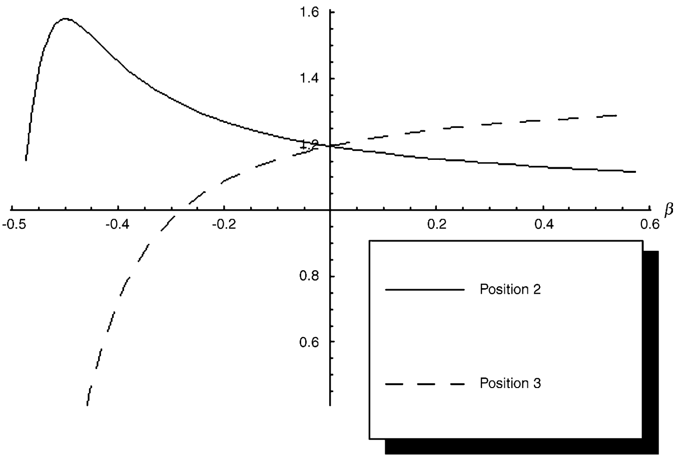

Fig. 2. The effects of positive and negative values of β on centrality scores for the network in Fig. 1.

Table 1
Eigenvector centrality for two networks in Fig. 7

Position	Network	
	3A	3B
1	0.366	0.568
2	0.798	1.024
3	1.367	1.276
4	−1.088	−1.276
5	−1.000	−1.024
6	−1.088	−0.568

than 3 because position 2 is connected to a peripheral actor (1). On the other hand, when beta is negative position 2 is more powerful than position 3 because position 2 has the good fortune to be connected to a exploitable position 1, an actor without alternatives. Experimental evidence supports these inferences.

1.1.2. Uses of $c(\beta)$ and x in signed graphs

In a signed graph relations can have values of −1 as well as +1 and 0. Positive and negative relations may correspond to liking and disliking relationships, respectively. Balance theory has been a powerful approach to networks with positive and negative relations. Balance theory is based on the assumption that friends of friends are friends, enemies of friends and friends of enemies are enemies, and enemies of enemies are friends. In its most general formulation a balanced valued network is one in which the products of the relationship values of all cycles are positive. Moreover, in any balanced structure there are, implicitly, two cliques with all positive relationships between members of the same clique and all negative relations between members of different cliques (Cartwright and Harary, 1956).

In balanced graphs the eigenvector of the largest eigenvalue of a connected graph reveals the two cliques: one clique will have all positive values and the other all negative values (Bonacich and Lloyd, 2004). Moreover, from (4) it is evident that (5) holds true in signed graphs as well as ordinary graphs. When beta is small $c(\beta)$ assesses simply the balance of positive and negative relations of each vertex but as beta increase, $c(\beta)$ reveals the macro-level balanced patterns of cliques.

For example, consider the following networks.

Negative relations in these networks are indicated by broken lines. Both of these networks are balanced, with two cliques such that all positive choices are within cliques and all negative choices are between two cliques, the cliques consisting of vertices one, two and three on the one hand and vertices four, five, and six on the other. The eigenvectors associated with the largest eigenvalues in both these networks reveals this balanced pattern.

In both networks the two cliques are easily identified, with one clique having all positive values and the other all negative values. Moreover, in both networks positions achieve status not only through their positive connections to members of their own clique but through negative connections to members of the other clique. Thus, in 3A position 3 and in 3B positions 4 and 6 achieve higher status within their own cliques because of their negative connections to members of the other clique (Fig. 3).

In network 3A the plot in Fig. 4 shows that $c(\beta)$ approaches the eigenvector given in first column of Table 1 as beta approaches $1/\lambda$, its maximum value. One could explore the effect in this

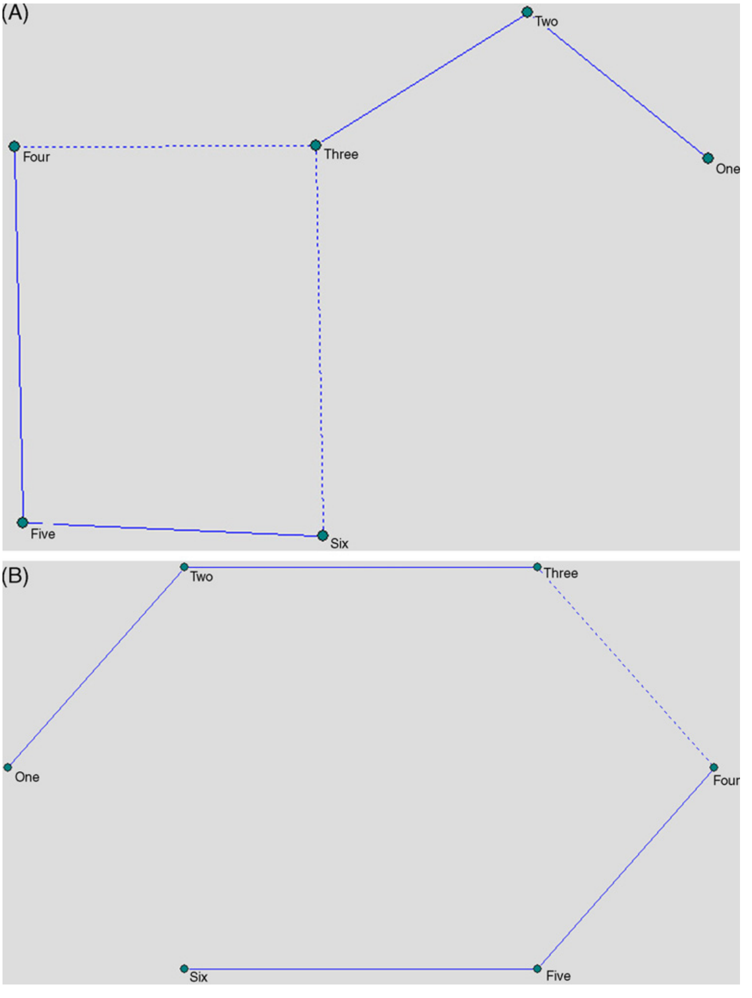

Fig. 3. Two networks with both positive and negative relations.

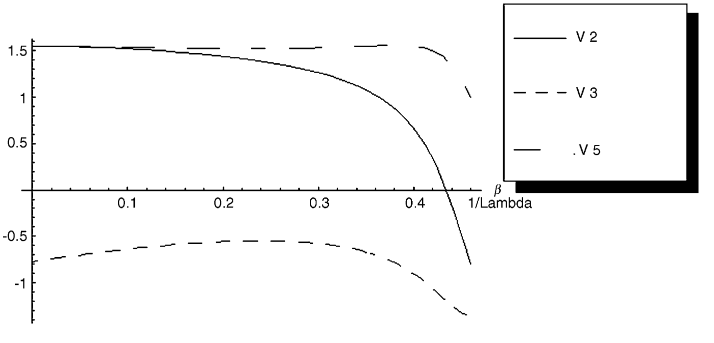

Fig. 4. $c(\beta)$ as a function of β for network 3A.

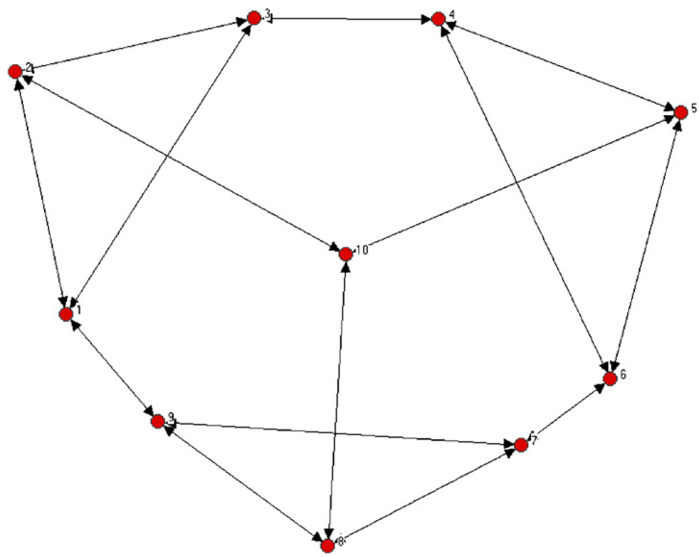

Fig. 5. A regular network.

network of varying beta. One can see, not surprisingly, that increasing the importance of balanced ties raises position 3's status relative to his clique members because most of 3's ties are negative connections to members of the other group.[1]

2. Singular features of eigenvector centrality

After having discussed the standard advantages of the eigenvector and $c(\beta)$ as measures of centrality we are prepared to discuss some of their unusual (but not necessarily undesirable) features as measures of centrality.

2.1. There are classes of networks in which the eigenvector x and c(β) are no different from degree

There are networks in which x and $c(\beta)$ give results equal to degree even when other measures of centrality do not. Among these are *regular* networks, networks in which all positions have the same degree. In a regular network all the values of A^k1 for any k will be the same for all vertices; if all positions are of degree m then all positions will be the origin of m^2 paths of length 2, m^3 paths of length 3, and so on. Thus, using Eqs. (2) and (5), all positions will have equal scores for x and $c(\beta)$. Consider, for example, the regular networks in Figs. 5 and 6.

Position 10 in Fig. 5 is distinctive; it is the only position none of whose contacts are related to each other. Position 10 scores higher than any other vertex on betweenness centrality, closeness centrality and information centrality, but all positions (including position 10) have equal scores in the eigenvector centrality measure and on $c(\beta)$ for all values of β.

In Fig. 6 vertices in the set 7 through 12 are distinctively distant from one another; every vertex in this set of a distance of 4 from some other vertex. This graph–theoretic distinctiveness shows up

[1] The direction of an eigenvector of a balanced graph is irrelevant; if x is an eigenvector so is $-x$.

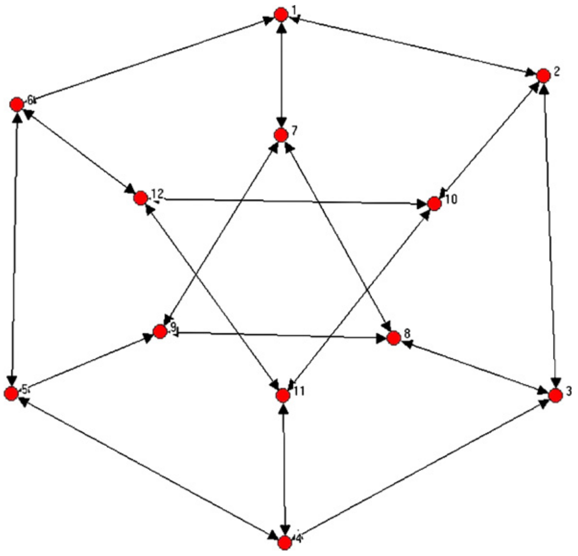

Fig. 6. A regular network.

in other measures of centrality; according to betweenness, closeness, and information centrality positions 1 through 6 are more central. Yet all positions are equal in x and $c(\beta)$. Eigenvector centrality capitalizes on how differences in degree can propagate through a network. It will not show differences if all vertices have the same degree.

This is a characteristic, not a defect, of eigenvector centrality. If one believes that differences in degree drive centrality, status, or power, then eigenvector centrality is called for. For example, sociometric popularity may be ultimately be based on being chosen by popular others. If each individual is limited to a fixed number of relationships (each individual's two best friends, etc.), one should not expect to find differences in eigenvector popularity.

This same issue shows up in some non-regular networks as well. Consider the network of Fig. 7.

In this network the centrality measures x and $c(\beta)$, for any value of beta, are proportional to degree, while neither closeness or betweenness centralities are proportional to degree. The peripheral positions have no betweenness centrality and the middle position is disproportionately high in betweenness centrality.[2] The reason for this is a peculiar characteristic of the network in Fig. 7: $A\mathbf{1}$ and $A^2\mathbf{1}$ are proportional to each other; the number of paths of length one (the degree) and of length two are proportional. Whenever this occurs, $A^k\mathbf{1}$, the number of paths of length k, will also be proportional to degree. Eq. (2) shows that, as a consequence, the scores in x and $c(\beta)$ are also proportional to degree. As an example, in a core-periphery structure in which central individuals are very likely to be connected to one another, eigenvector centrality will be close to degree centrality. Eigenvector centrality is sensitive to situations in which a low-degree individual is connected to high degree others (his centrality will be relatively high) or in which a high-degree other is only connected to low degree others (his centrality will be relatively low).

[2] The unstandardized betweenness centrality scores for the seven positions in Fig. 7 are 12, 5, 5, 5, 0, 0, 0.

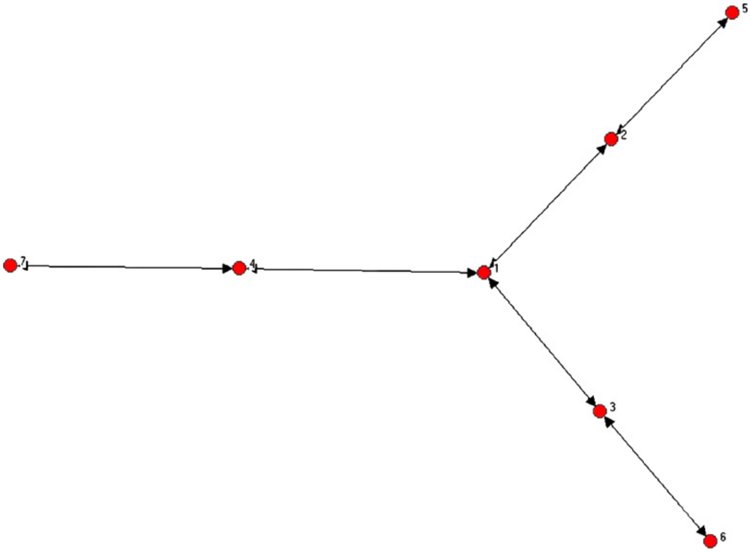

Fig. 7. A non-regular network in which eigenvector centrality equals degree.

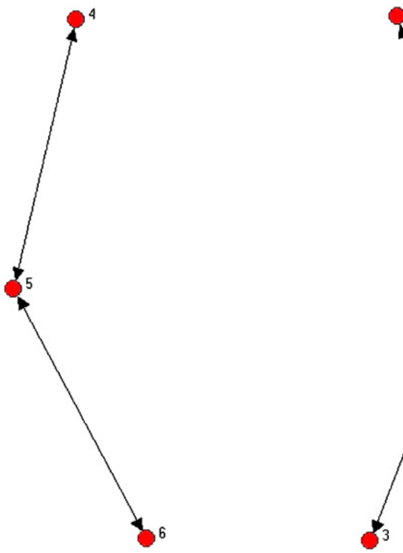

Fig. 8. A network in which $c(\beta)$ does not approach an eigenvector.

2.2. *Beta-centrality does not always converge to an eigenvector; Eq. (5) is sometimes not correct*

2.2.1. *Eq. (5) is not true when the largest eigenvalue of a symmetric matrix has multiplicity greater than 1*

If there are two or more components in a graph each component will be described by its own eigenvector (Bonacich, 1972). For example, consider the network in Fig. 8.

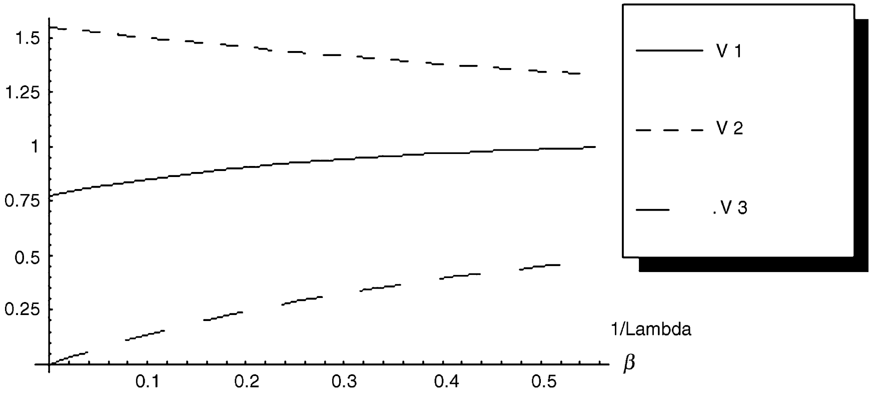

Fig. 9. $c(\beta)$ as a function of β for network 3B.

The eigenvalues of the matrix for this network are 1.41, 1.41, 0, 0, −1.41, and −1.41. Two eigenvectors corresponding to the largest eigenvalue are (0.5, 0.71, 0.5, 0, 0, 0) and (0, 0, 0, 0.50, 0.71, 0.50). But, $c(\beta)$ converges to neither of these: (0.87, 1.23, 0.87, 0.87, 1.23, 0.87).[3] The reason why this happens is evident from Eq. (4); there are two coefficients $\beta\lambda_i/(1-\beta\lambda_i)$ for two different eigenvalue products that remain equal. This anomaly will occur whenever there are two or more largest components that are isomorphic images of one another. If there are two or more non-identical components the eigenvector of the adjacency matrix will describe only one of the components. In such circumstances the correct solution is to analyze each component separately.

2.2.2. Eq. (5) is not true for signed adjacency matrices when the eigenvector of the largest eigenvalue sums to zero

Fig. 4 showed that in network 3A as beta increases the clique structure became more and more evident. Fig. 9 shows the effect of varying beta on $c(\beta)$ for network 3B. Here $c(\beta)$ does not approach the eigenvector shown in the second column of Table 1: position 3 does not become the most central as beta increases. The research strategy of examining the structural effects of varying beta would not work.

The difference between these two networks in Fig. 3 is small but important. The elements of the eigenvector of the largest eigenvalues for network 3B sum to zero because of the symmetry of the two cliques. This summing to zero means that $c(\beta)$ approaches not the eigenvector associated with the largest eigenvalue but rather the eigenvector associated with the second largest eigenvalue. The last expression of (4) shows that as β approaches $1/\lambda_1$, if $x_1^t 1 = 0$, the first term drops out and the limiting value of $c(\beta)$ is a function of the remaining eigenvectors. What this suggests is that using varying beta may be hazardous for signed networks.

3. Conclusions

Eigenvectors and beta-centrality measures have a number of advantages over conventional graph–theory based measures of centrality: they can be used with valued or signed graphs; they can be used for negatively connected exchange networks; they allow for variations in the degree

[3] In networks with multiple largest eigenvalues any orthogonal transformation of a set of eigenvectors will also be a set of eigenvectors.

to which status is transmitted from position to position. When applied to standard binary-valued graphs these measures are especially sensitive to situations in which a high degree position is connected to many low degree positions or a low degree position is connected to a few high degree positions. The measure is, therefore, distinctively appropriate when centrality is ultimately driven by differences in degree. Moreover, there are some highly symmetric situations in which $c(\beta)$ does not approach the eigenvector measure and in which other approaches are called for.

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