1 Geometry

1.1 Inscribed angles

1. Inscribed angles subtending the same arc are equal to each other
2. Inscribed angle is equal to half of the central angle subtending the same arc.
3. **Alternate segment theorem**: Angle between chord and tangent is equal to inscribed angle which subtends to that chord.

1.2 Power of point

1. The **power of point** P with respect to circle with centre O and radius r is defined as
$$p = PO^2 - r^2$$
2. For any line through point P which intersects the circle at points A and B
$$p = PA \times PB$$

1.3 Trigonometry

1. Basic trigonometric identities
$$\frac{\sin x}{\cos x} = \tan x$$
$$\sin^2 x + \cos^2 x = 1$$
2. Double angle formulae
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\sin 2x = 2 \sin x \cos x$$
3. Sine law
$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R$$
4. Cosine law
$$a^2 + b^2 - c^2 - 2ab \cos \gamma = 0$$

2 Algebra

2.1 Polynomials

1. **Bezout's theorem**: A polynomial $P(x)$ is divisible by the binomial $(x-a)$ if and only if $P(a)=0$.

2. **The fundamental theorem of algebra**: Every non-constant polynomial has a complex root.

3. **The rational root theorem**: If $x=p/q$ is a rational zero of a polynomial $P(x)=a_n x^n+\ldots+a_0$ with integer coefficients and $p,q=1$, then $p|a_0$ and $q|a_n$.

4. **Vieta's formulae**: If the solutions polynomial of degree n are $x_1,x_2,\ldots,x_n$ and $a_n=1$, then the following holds:

$$\begin{aligned}
x_1+x_2+\ldots+x_n &= -a_{n-1},\\
x_1x_2+x_1x_3+\ldots+x_{n-1}x_n &= a_{n-2},\\
x_1x_2x_3+x_1x_2x_4+\ldots+x_{n-2}x_{n-1}x_n &= -a_{n-3},\\
&\ldots\\
x_1x_2\ldots x_n &= (-1)^n a_0.
\end{aligned}$$

2.2 Inequalities

1. **General mean inequality**: The mean of order p of positive real numbers $x_1,\ldots,x_n$ is defined as:

$$M_p=\begin{cases}\left(\frac{x_1^p+\cdots+x_n^p}{n}\right)^{1/p} & \text{for } p\neq 0\\ \sqrt[n]{x_1\ldots x_n} & \text{for } p=0\end{cases}$$

   In particular

Smallest element	$\min\{x_i\}$	$M_{-\infty}$
Harmonic mean	HM	M_{-1}
Geometric mean	GM	M_0
Arithmetic mean	AM	M_1
Quadratic mean	QM	M_2
Largest element	$\max\{x_i\}$	M_∞

   Then for any real p and q

$$M_p\le M_q \iff p\le q$$

2. **Cauchy inequality**: For real numbers $x_1,\ldots,x_n,y_1,\ldots,y_n$

$$\left(\sum_{i=1}^n x_iy_i\right)^2\le\sum_{i=1}^n x_i^2\sum_{i=1}^n y_i^2$$

3. **Chebyshev inequality**: For real numbers $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$

$$\frac{1}{n}\sum_{i=1}^{n} x_i y_i \geq \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) \geq \frac{1}{n}\sum_{i=1}^{n} x_i y_{n+1-i}$$

4. **Jensen inequality.** Given positive real numbers $\lambda_1, \ldots, \lambda_n$ for which $\lambda_1 + \ldots + \lambda_n = 1$ and a convex function $f(x)$ the following holds:

$$f(\lambda_1 x_1 + \ldots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \ldots + \lambda_n f(x_n)$$

Similarly, when $f(x)$ is a concave function, then

$$f(\lambda_1 x_1 + \ldots + \lambda_n x_n) \geq \lambda_1 f(x_1) + \ldots + \lambda_n f(x_n)$$

2.3 Functional equations

1. If $f(x) = f(y)$ implies $x = y$, then f is **injective**
2. If for each element y in function codomain, there exists x for which $f(x) = y$, then f is **surjective**.
3. If f is both injective and surjective then f is **bijective** (one-to-one).
4. **Cauchy functions**: If any of the following is satisfied

   (a) The function is continuous at one point,

   (b) The function is monotonic on any interval,

   (c) The function is bounded on any interval.

   then all of the following functional equations have the respective solutions.

$$f(x+y) = f(x) + f(x) \quad \Rightarrow \quad f(x) = cx \tag{1}$$

$$f(xy) = f(x)f(y) \quad \Rightarrow \quad f(x) = x^c \tag{2}$$

$$f(xy) = f(x) + f(y) \quad \Rightarrow \quad f(x) = c\log|x| \tag{3}$$

$$f(x+y) = f(x)f(y) \quad \Rightarrow \quad f(x) = e^{cx} \tag{4}$$

3 Number Theory

3.1 Divisibility

1. If $a \mid b$ and $c \mid d$ then $ac \mid bd$
2. If $a \mid b$ and $a \mid c$, then $a \mid b + c$
3. **Euclid's algorithm.**
   $\gcd(a, b) = \gcd(a, b - a)$
4. **Corollary of Euclid's algorithm.**
   $ax + by = n$ has solution (x, y) in integers if and only if $gcd(a, b) \mid n$

3.2 Congruences

For integers a, b, c, d, m, n and prime p.

1. $a \equiv b \mod m \iff m \mid a - b$
2. $a \equiv b \mod m$ and $c \equiv d \mod m \implies a + c \equiv b + d \mod m$
3. $a \equiv b \mod m \iff an \equiv bn \mod mn$
4. $a \equiv b \mod m \implies an \equiv bn \mod m$

3.3 Exponential congruences

1. **Fermat's little theorem.**
   For prime p and integer a
   $$a^p \equiv a \mod p$$
2. **Wilson's theorem.**
   $$(p-1)! \equiv -1 \mod p$$
   if and only if when p is prime number.
3. **Number of factors**
   The number of positive factors of $n = p_1^{\alpha_1} \dots p_k^{a_k}$
   $$d(n) = (\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_k + 1)$$
4. **Sum of factors**
   The sum of positive factors of $n = p_1^{\alpha_1} \dots p_k^{a_k}$
   $$\sigma(n) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} + \dots \frac{p_k^{\alpha_k+1} - 1}{p_k - 1}$$

5. **Euler's function**
   Euler's function or totient function $\varphi(n)$ is defined for $n = p_1^{\alpha_1} \dots p_k^{a_k}$ as the number of positive integers less than n and coprime to n. Then

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_k}\right)$$

6. **Euler's theorem** (Generalisation of Fermat's theorem)
   Let n be a natural number and a an integer such that $\gcd(a, n) = 1$. Then

$$a^{\varphi(n)} \equiv 1 \mod n$$

4 Combinatorics

4.1 Counting of objects

1. **Permutations**: $P_n = n!$
2. **Variations**: $V_n^k = \frac{n!}{(n-k)!}$
3. **Combinations**: $C_n^k = \frac{n!}{(n-k)!}$

4.2 Pigeonhole principle

1. If a set of $nk + 1$ different elements is partitioned into n mutually disjoint subsets, then at least one subset will contain at least $k + 1$ elements.

4.3 Graph Theory

1. **Tree** is a connected graph with no circuits. A connected graph with n vertices is a tree if and only if it has $n - 1$ edges.
2. **Euler path** is a path in which every edge of the graph appears exactly once. Likewise **Euler circuit** is a circuit in which every edge appears exactly once.
3. If each vertex in a connected graph has even degree, then the graph contains an Euler circuit.
4. If a connected graph has exactly two vertices with odd degree, it contains an Euler path.
5. A **Hamilton circuit** is a circuit in which each vertex appears exactly once.
6. A **planar graph** can be embedded in a plane with edges corresponding to non-intersecting lines (not necessarily straight). A planar graph with n vertices has at most $3n - 6$ edges.
7. **Dirac's theorem**: A graph with n vertices contains a Hamilton cycle if the degree of each vertex is at least $n/2$.
8. **Euler's formula**: $E + 2 = F + V$, where E, F, V are the numbers of edges, faces and vertices of a polyhedron.