1 Cauchy-(Bunyakovsky)-Schwarz inequality

For $a_1, \dots, a_n, b_1, \dots, b_n \in \mathcal{R}$

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$$

1. Prove Cauchy-Schwarz inequality. When does the equality hold?
2. Prove that
$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \sum_{i<j}(a_i b_j - a_j b_i)^2.$$
3. Prove that for positive real numbers $a_1, \dots, a_n, b_1, \dots, b_n$
$$\left(\sum_{i=1}^{n} \frac{1}{x_i y_i}\right)\left(\sum_{i=1}^{n}(x_i + y_i)^2\right) \geq 4n^2$$
4. Prove that if the real numbers a, b and c satisfy $a^2 + b^2 + c^2 = 3$ then
$$\frac{a^2}{2 + b + c^2} + \frac{b^2}{2 + c + a^2} + \frac{c^2}{2 + a + b^2} \geq \frac{(a + b + c)^2}{12}.$$
5. Prove that for all positive real numbers a, b, c
$$\frac{a^3}{bc} + \frac{b^3}{ac} + \frac{c^3}{ab} \geq a + b + c$$
and determine when equality occurs
6. Given real numbers $x_1, \dots, x_n$ for which $\sum_{i=1}^{n} x_i^2 = 1$. Prove that for each integer $k \geq 2$ we can find integers $a_1, \dots, a_n$ from which at least one is non-zero and for which $|a_i| \leq k - 1$ and
$$|a_1 x_1 + a_2 x_2 + \ldots + a_n x_n| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}.$$
7. Let ABC be a triangle such that
$$\cot^2 \frac{A}{2} + 2^2 \cot^2 \frac{B}{2} + 3^2 \cot^2 \frac{C}{2} = \left(\frac{6p}{7r}\right)^2$$
where p is semiperimeter and r is radius of incircle. Find the sidelenghts of ABC if it is known that they are all integers and their gcd is 1.
8. Prove that for positive real numbers a, b, c for which $abc = 1$
$$\frac{1}{a^3(b + c)} + \frac{1}{b^3(a + c)} + \frac{1}{c^3(a + b)} \geq \frac{3}{2}$$

9. For positive real numbers a, b, c prove that

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{a^2+c^2}{a+c} \geq a+b+c$$

10. For positive real number a, b, c prove that

$$\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1} \geq \sqrt{6(x+y+z)}$$