Inequalities

1. **General mean inequality.** The mean of order p of positive real numbers $x_1, \ldots, x_n$ is defined as:

$$M_p = \begin{cases} \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{1/p} & \text{for } p \neq 0 \\ \sqrt[n]{x_1 \ldots x_n} & \text{for } p = 0 \end{cases}$$

In particular

Smallest element	$\min\{x_i\}$	$M_{-\infty}$
Harmonic mean	HM	M_{-1}
Geometric mean	GM	M_0
Arithmetic mean	AM	M_1
Quadratic mean	QM	M_2
Largest element	$\max\{x_i\}$	M_∞

Then for any real p and q

$$M_p \leq M_q \iff p \leq q$$

2. **Cauchy inequality.** For real numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$

$$\left(\sum_{i=1}^n x_i y_i\right)^2 \leq \sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2$$

3. **Chebyshev inequality.** For real numbers $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$

$$\frac{1}{n}\sum_{i=1}^n x_i y_i \geq \left(\frac{1}{n}\sum_{i=1}^n x_i\right)\left(\frac{1}{n}\sum_{i=1}^n y_i\right) \geq \frac{1}{n}\sum_{i=1}^n x_i y_{n+1-i}$$

4. **Jensen inequality.** Given positive real numbers $\lambda_1, \ldots, \lambda_n$ for which $\lambda_1 + \ldots + \lambda_n = 1$ and a convex function $f(x)$ the following holds:

$$f(\lambda_1 x_1 + \ldots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \ldots + \lambda_n f(x_n)$$

Similarly, when $f(x)$ is a concave function, then

$$f(\lambda_1 x_1 + \ldots + \lambda_n x_n) \geq \lambda_1 f(x_1) + \ldots + \lambda_n f(x_n)$$

Problems

1. Let $a_1, \dots, a_n$ be positive real numbers such that $a_1 \dots a_n = 1$ Prove that

$$(1 + a_1) \dots (1 + a_n) \geq 2^n$$

2. For real numbers $x_1, \dots, x_n, y_1, \dots, y_n$ the following holds

$$x_1 + \dots + x_n \geq x_1 y_1 + \dots + x_n y_n$$

Prove that

$$x_1 + \dots + x_n \leq \frac{x_1}{y_1} + \dots + \frac{x_n}{y_n}$$

3. Let n be an integer ($n \geq 2$) and $a_1, \dots, a_n$ be positive real numbers such that $a_1 + \dots + a_n = 1$. Prove the following inequality for any positive real numbers $x_1, \dots, x_n$ for which $x_1 + \dots + x_n = 1$

$$2 \sum_{i<j} x_i x_j \leq \frac{n-2}{n-1} + \sum_{i=1}^{n} \frac{a_i x_i^2}{1 - a_i}$$

When does the equality hold?

4. Prove that for any positive real numbers $a_1, \dots, a_n$

$$\frac{1}{\frac{1}{1+a_1} + \dots + \frac{1}{1+a_n}} - \frac{1}{\frac{1}{a_1} + \dots + \frac{1}{a_n}} \geq n$$

5. Let $x_1, \dots, x_n$ be positive real numbers such that

$$\frac{1}{1 + x_1} + \dots + \frac{1}{1 + x_n} = 1$$

Prove that

$$x_1 \dots x_n \geq (n-1)^n$$

6. Prove for real numbers $x_1, \dots, x_5$

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \geq \frac{2}{\sqrt{3}}(x_1 x_2 + x_2 x_3 + x_3 x_4 + x_4 x_5)$$

7. Let a, b, c be positive real numbers. Prove that

$$\left(1 + \frac{a}{b}\right)\left(1 + \frac{b}{c}\right)\left(1 + \frac{c}{a}\right) \geq 2\left(1 + \frac{a+b+c}{\sqrt[3]{abc}}\right)$$

8. Given positive real numbers $x_1, \dots, x_n$ for which $x_1^2 + \dots + x_n^2 = 1$, find the minimal value of the expression

$$\frac{x_1^5}{x_2 + x_3 + \dots + x_n} + \frac{x_2^5}{x_1 + x_3 + \dots + x_n} + \dots + \frac{x_n^5}{x_1 + x_2 + \dots + x_{n-1}}$$

9. Let $x_1, \ldots, x_n$ be positive real numbers for which $x_1 + \cdots + x_n = 1$. Prove that

$$\frac{x_1}{\sqrt{1-x_1}} + \cdots + \frac{x_n}{\sqrt{1-x_n}} \geq \frac{\sqrt{x_1} + \cdots + \sqrt{x_n}}{\sqrt{n-1}}$$

10. Let a, b, c be positive real numbers. Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3$$