Geometric inequalities

1. Prove the inequalities for $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$$|\tan x| \geq |x| \geq |\sin x| \geq \frac{2}{\pi}|x|$$

2. Prove the inequalities for $0 < \alpha < \beta < \frac{\pi}{2}$

$$\frac{\sin\beta}{\sin\alpha} < \frac{\beta}{\alpha} < \frac{\tan\alpha}{\tan\beta}$$

3. Prove the inequalities where a, b, c are lengths of triangles sides
   (a) $a^2 + b^2 + c^2 < 2(ab + bc + ca)$
   (b) $a^4 + b^4 + c^4 < 2(a^2b^2 + b^2c^2 + c^2a^2)$

4. Prove that $R \geq 2r$ given that R is the radius of circumcircle and r is the radius of incircle of a triangle.

5. *(Ptolemy's inequality).* Prove the inequality for any four points A, B, C, D on a plane

$$AC \cdot BD \leq AB \cdot CD + AD \cdot BC$$

   When does the equality hold?

6. *(Erdős-Mordell inequality).* Let P be a point in the interior of $\triangle ABC$ and X, Y,Z projections of P to BC,AC,AB respectively. Prove that

$$PA + PB + PC \geq 2(PX + PY + PZ)$$

   When does the equality hold?

7. Points A, B, C, X, Y, Z, P are defined as before. Prove that

$$AP \cdot BC + BP \cdot AC + CP \cdot AB \geq 4S_{\triangle ABC}$$

8. Points A, B, C, X, Y, Z, P are defined as before. Prove that

$$AP \cdot XP + BP \cdot YP + CP \cdot ZP \geq 2(XP \cdot YP + YP \cdot ZP + ZP \cdot XP)$$