1 Vectors

1.1 Rules

1. The angle between vectors $\mathbf{a}$ and $\mathbf{b}$ is the angle one should rotate vector $\mathbf{a}$ to align it with $\mathbf{b}$. Then:

   (a) $\angle(\mathbf{a},\mathbf{b}) = -\angle(\mathbf{b},\mathbf{a})$

   (b) $\angle(\mathbf{a},\mathbf{b}) + \angle(\mathbf{b},\mathbf{c}) = \angle(\mathbf{a},\mathbf{c})$

   (c) $\angle(\mathbf{a},\mathbf{b}) = \angle(-\mathbf{a},\mathbf{b}) - 180°$

2. The dot product (also inner product or scalar product) of vectors $\mathbf{a}$ and $\mathbf{b}$ is:

   $$\mathbf{a}\cdot\mathbf{b} \equiv (\mathbf{a},\mathbf{b}) = |\mathbf{a}||\mathbf{b}|\cos\angle(\mathbf{a},\mathbf{b})$$

   Then:

   (a) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$

   (b) $|\mathbf{a}\cdot\mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$

   (c) $(\lambda\mathbf{a} + \mu\mathbf{b})\cdot\mathbf{c} = \lambda\mathbf{a}\cdot\mathbf{c} + \mu\mathbf{b}\cdot\mathbf{c}$

   (d) If $|\mathbf{a}|, |\mathbf{b}| \neq 0$,then $\mathbf{a}\cdot\mathbf{b} = 0$ if and only if when $\mathbf{a} \perp \mathbf{b}$.

3. The cross product (also vector product) of vectors $\mathbf{a}$ and $\mathbf{b}$ has direction for which $\mathbf{c} \perp \mathbf{a}$ and $\mathbf{c} \perp \mathbf{b}$ and magnitude of:

   $$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\angle(\mathbf{a},\mathbf{b})$$

   Alternatively:

   $$\mathbf{a}\times\mathbf{b} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

   where $\mathbf{i},\mathbf{j},\mathbf{k}$ are unit vectors. Then:

   (a) $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$

   (b) $(\lambda\mathbf{a} + \mu\mathbf{b})\times\mathbf{c} = \lambda\mathbf{a}\times\mathbf{c} + \mu\mathbf{b}\times\mathbf{c}$

   (c) If $|\mathbf{a}|, |\mathbf{b}| \neq 0$,then $\mathbf{a}\times\mathbf{b} = 0$ if and only if when $\mathbf{a} \parallel \mathbf{b}$.

1.2 Problems

1. Prove that $|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2)$.

2. Prove that if $(\mathbf{a}+\mathbf{b}) \perp (\mathbf{a}-\mathbf{b})$, then $|\mathbf{a}| = |\mathbf{b}|$.

3. Let $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 0$ and $|OA| = |OB| = |OC|$. Prove that ABC is an equilateral triangle

4. Prove that it is possible to make another triangle $A_1B_1C_1$ from the medians AA_1,BB_1,CC_1 of a triangle ABC. Prove that $\triangle ABC \sim \triangle A_1B_1C_1$ and find their similarity coefficient.

5. From a point inside a convex n-gon, the rays are drawn perpendicular to the sides and intersecting the sides (or their continuations). On these rays the vectors $\mathbf{a_1}, \ldots, \mathbf{a_n}$ whose lengths are equal to the lengths of the corresponding sides are drawn. Prove that $\mathbf{a_1} + \cdots + \mathbf{a_n} = 0$.

6. Consider n pairwise non-codirected vectors ($n \geq 3$) whose sum is equal to zero. Prove that there exists a convex n-gon such that the set of vectors formed by its sides coincides with the given set of vectors.

7. (*Napoleon's theorem*)Three equilateral triangles ABD, BCE, CAF are constructed outside the triangle ABC. Prove that the centres of the constructed triangles form an equilateral triangle.

8. (a) Let A, B, C and D be arbitrary points on a plane. Prove that

$$\overrightarrow{AB} \cdot \overrightarrow{CD} + \overrightarrow{BC} \cdot \overrightarrow{AD} + \overrightarrow{CA} \cdot \overrightarrow{BD} = 0$$

   (b) Prove that the heights of a triangle intersect at one point.

9. Let O be the centre of the circumcircle of triangle ABC and let point H satisfy $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$. Prove that H is the intersection point of heights of triangle.

10. Given points A, B, C and D. Prove that $AB^2 + BC^2 + CD^2 + DA^2 \geq AC^2 + BD^2$, where the equality is attained only if $ABCD$ is a parallelogram.

11. Points $A_1, \ldots, A_n$ lie on a circle with center O and $\overrightarrow{OA_1} + \cdots + \overrightarrow{OA_n} = 0$. Prove that for any point X we have $XA_1 + \cdots + XA_n \geq nR$, where R is the radius of the circle.

12. Prove Ceva's theorem: X, Y and Z are points on the sides BC, CA and AB of a triangle ABC respectively. Then, lines AX, BY,CZ intersect at a single point if and only if

$$\frac{\overrightarrow{BX}}{\overrightarrow{XC}} \cdot \frac{\overrightarrow{CY}}{\overrightarrow{YA}} \cdot \frac{\overrightarrow{AZ}}{\overrightarrow{ZB}} = 1$$

13. Prove Menelaus's theorem: X, Y and Z are points on the sides (or their elongations) BC, CA and AB of a triangle ABC respectively. Then, X,Y and Z lie on the same line if and only if

$$\frac{\overrightarrow{BX}}{\overrightarrow{XC}} \cdot \frac{\overrightarrow{CY}}{\overrightarrow{YA}} \cdot \frac{\overrightarrow{AZ}}{\overrightarrow{ZB}} = -1$$

14. Let ABC be an acute-angled triangle with orthocentre H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

15. Is it possible to construct a triangle for which both coordinates of each vertex are integers?

16. The angles of a triangle are 22.5°, 45° and 112.5°. Prove that inside this triangle there exists a point that is located on the median through one vertex, the angle bisector through another vertex and the altitude through the third vertex.