Miscelleanous revision problems for BMO

1. (x, y, z) is a point on unit sphere. Find the maximal value of $x + 2y + 3z$.
2. Prove for $x \geq 0$ that $x^5 + 1 \geq x^3 + x^2$.
3. Prove for $x, y, z \geq 0$
$$3(x^3 + y^3 + z^3)^2 \geq (x^2 + y^2 + z^2)^3$$
4. Prove for $a, b, c \geq 0$
$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3$$
5. Given positive real numbers $x_1, \ldots, x_n$ for which $x_1^2 + \cdots + x_n^2 = 1$, find the minimal value of the expression
$$\frac{x_1^5}{x_2 + x_3 + \cdots + x_n} + \frac{x_2^5}{x_1 + x_3 + \cdots + x_n} + \cdots + \frac{x_n^5}{x_1 + x_2 + \cdots + x_{n-1}}$$
6. Let $x_1, \ldots, x_n$ be positive real numbers for which $x_1 + \cdots + x_n = 1$. Prove that
$$\frac{x_1}{\sqrt{1 - x_1}} + \cdots + \frac{x_n}{\sqrt{1 - x_n}} \geq \frac{\sqrt{x_1} + \cdots + \sqrt{x_n}}{\sqrt{n - 1}}$$
7. Find all factors of $10^{2013} - 1$ which are smaller than 100.
8. Find the number of possible values for positive integer k if it is known that $\operatorname{lcm}(6^6, 8^8, k) = 12^{12}$