Functional equations

Injective and surjective functions

1. If $f(x) = f(y) \Rightarrow x = y$, then f is injective
2. If for each element y in function codomain, there exists x for which $f(x) = y$, then f is surjective.
3. If f is both injective and surjective then f is bijective.

Problems

1. Let $f : X \to Y$ and $g : Y \to X$ and $g(f(x)) = x$.
2. Prove that for any function $f : X \to Y$, there exists a set Z and functions $g : X \to Z$ and $h : Z \to Y$, such that g is injective and f is surjective.
3. Find all strictly monotonous functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(x + f(y)) = f(x) + y$$

Cauchy functional equations

1. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ for which $f(x) + f(y) = f(x + y)$.
2. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+\mathbb{PM}$ for which $f(x)f(y) = f(xy)$.
3. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}$ for which $f(x) + f(y) = f(xy)$.
4. Find all functions $f : \mathbb{Q} \to \mathbb{Q}^+$ for which $f(x)f(y) = f(x + y)$.
5. If in questions 1-4 any of the following conditions is satisfied, prove that the solutions are the same for $f : \mathbb{R} \to \mathbb{R}$.
   (a) The function is continuous at one point,
   (b) The function is monotonic on any interval,
   (c) The function is bounded on any interval.

Problems

6. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ for which

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

Recurrence relations

- A recurrence relation is a relation that determines the elements of a sequence $x_n, n \in \mathbb{N}_{\not k}$, as a function of previous elements. A recurrence relation of the form

$$(\forall n \geq k) \qquad x_n + a_1 x_n - 1 + \ldots + a_k x_{n-k} = 0$$

  for constants $a_1, \ldots, a_k$ is called a linear homogeneous recurrence relation of order k.

- We define the characteristic polynomial of the relation as

$$P(x) = x^k + a_1 x^{k-1} + \ldots + a_k$$

- Let $P(x)$ factorize as

$$P(x) = (x - \alpha_1)^{k_1} (x - \alpha_2)^{k_2} \ldots (x - \alpha_r)^{k_r}$$

  where $\alpha_1, \ldots, \alpha_r$ are distinct complex numbers and $k_1, \ldots, k_r$ are positive integers.

- The general solution of this recurrence relation is in this case given by

$$x_n = p_1(n)\alpha_1^n + p_2(n)\alpha_2^n + \ldots + p_r(n)\alpha_r^n$$

  where p_i is a polynomial of degree less than k_i.

- In particular, if $P(x)$ has k distinct roots, then all p_i are constant.

- If $x_0, \ldots, x_{k-1}$ are set, then the coefficients of the polynomials are uniquely determined.