Geometric inequalities

1. Prove the inequalities for $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$$|\tan x| \geq |x| \geq |\sin x| \geq \frac{2}{\pi}|x|$$
2. Prove the inequalities for $0 < \alpha < \beta < \frac{\pi}{2}$
$$\frac{\sin\beta}{\sin\alpha} < \frac{\beta}{\alpha} < \frac{\tan\alpha}{\tan\beta}$$
3. Prove the inequalities where a, b, c are lengths of triangles sides
   (a) $a^2 + b^2 + c^2 < 2(ab + bc + ca)$
   (b) $a^4 + b^4 + c^4 < 2(a^2b^2 + b^2c^2 + c^2a^2)$
4. Prove that $R \geq 2r$ given that R is the radius of circumcircle and r is the radius of incircle of a triangle.
5. *(Ptolemy's inequality).* Prove the inequality for any four points A, B, C, D on a plane
$$AC \cdot BD \leq AB \cdot CD + AD \cdot BC$$
   When does the equality hold?
6. *(Erdős-Mordell inequality).* Let P be a point in the interior of $\triangle ABC$ and X, Y,Z projections of P to BC,AC,AB respectively. Prove that
$$PA + PB + PC \geq 2(PX + PY + PZ)$$
   When does the equality hold?
7. Points A, B, C, X, Y, Z, P are defined as before. Prove that
$$AP \cdot BC + BP \cdot AC + CP \cdot AB \geq 4S_{\triangle ABC}$$
8. Points A, B, C, X, Y, Z, P are defined as before. Prove that
$$AP \cdot XP + BP \cdot YP + CP \cdot ZP \geq 2(XP \cdot YP + YP \cdot ZP + ZP \cdot XP)$$
9. Given a convex hexagon $ABCDEF$ such that $AB \parallel DE$, $BC \parallel EF$, $CD \parallel FA$ and with perimeter P. Let R_A, R_C, R_E be the radii of circles FAB, BCD, DEF respectively. Prove that
$$R_A + R_C + R_C \geq \frac{P}{2}$$