Graph theory

Terminology

1. Graph $G(V, E)$ is set of *vertices* V and *edges* E (pairs of vertices). If there is an edge $(x, y) \in E$, then x and y are said to be *connected*.
2. A *proper graph* is a graph with no more than one edge between each pair of vertices and in which no vertex is connected to itself.
3. Proper graph in which each pair of vertices is connected is called a *complete graph*.
4. A *complement* graph $\overline{G}$ has vertices equal to vertices of G and its vertices x and y are connected if and only if x and y are not connected in G.
5. An *oriented graph* is one in which the pairs in the graph are ordered.
6. The vertices of a *k-partite graph* can be partitioned into k non-empty disjoint sets in such way that there are no connections between vertices in each set.
7. The *degree* of a vertex x is the number of times x is the endpoint of an edge. Then
$$\sum_{x \in V} d(x) = 2|E|$$
8. A *trajectory (or path)* is a sequence of vertices for which subsequent vertices are connected.
9. A *circuit* is a path that ends and starts in the same vertex.
10. A *cycle* is a circuit in which no vertex appears more than once (except the initial/final vertex).
11. In a *connected graph* there exists a path between any two points of the graph.
12. *Tree* is a connected graph with no circuits. A connected graph with n vertices is a tree if and only if it has $n - 1$ edges.
13. *Euler path* is a path in which every edge of the graph appears exactly once. Likewise *Euler circuit* is a circuit in which every edge appears exactly once.
14. If each vertex in a connected graph has even degree, then the graph contains an Euler circuit.
15. If a connected graph has exactly two vertices with odd degree, it contains an Euler path.
16. A *Hamilton circuit* is a circuit in which each vertex appears exactly once.
17. A *planar graph* can be embedded in a plane with edges corresponding to non-intersecting lines (not necessarily straight). A planar graph with n vertices has at most $3n - 6$ edges.

Theorems

1. **Ramsey's theorem:** For positive integers $r_1, \ldots, r_k$, there exists a finite Ramsey number.
   (a) case $k = 2$
   Ramsey number $R(r_1, r_2)$ is a minimal number of people at the party such that there are either r_1 people who know each other or r_2 people who do not.
   (b) General case for k
   Ramsey's number of $R(r_1, r_2, \ldots, r_k)$ is the minimal number of points in a complete graph whose edges are coloured with k different colours, such that there exists a colour i for which we can find r_i vertices that are all connected with edges of colour i.
2. **Kuratowski's theorem**
   (a) K_n is a complete graph with n vertices and Kn, m is a complete bipartite graph with n and m vertices in the subsets.
   (b) Graphs K_5 and $K_{3,3}$ are non-planar.
   (c) Every non-planar graph contains a subgraph which can be obtained from one of these graphs by a subdivision of its edges.
3. **Hall's marriage theorem**
   (a) In a set of n women and n men, a woman a_i is happy to be married to any man in set A_i and a man is happy to marry a woman who wants to marry him. Hall's marriage theorem states that each person can be happily married if and only if the sets $A_1, A_2, \ldots, A_n$ meets the marriage condition.
   (b) The marriage condition states that for any subset of women I, the number of men whom at least one woman would be happy to marry $\left|\bigcup_{i \in I} A_i\right| \geq |I|$

Problems

1. Prove that at least one of G and $\overline{G}$ is connected.
2. *(Dirac's theorem).* Prove that a graph with n vertices contains a Hamilton cycle if the degree of each vertex is at least $n/2$.
3. *(Euler's formula).* A convex polyhedron has E edges, F faces and V vertices. Prove that $E + 2 = F + V$.
4. In a forest each of n animals ($n \leq 3$) lives in its own cave, and there is exactly one separate path between any two of these caves. Before the election for King of the Forest some of the animals make an election campaign. Each campaign-making animal visits each of the other caves exactly once, uses only the paths for moving from cave to cave, never turns from one path to another between the caves and returns to its own cave in the end of its campaign. It is also known that no path between two caves is used by more than one campaign-making animal.

(a) Prove that for any prime n, the maximum possible number of campaign-making animals is $\frac{n-1}{2}$

(b) Find the maximum number of campaign-making animals for $n = 9$.

5. Certain squares of an $n \times n$ board are coloured black and the rest white. Every white square shares a side with a black square. Every pair of black squares can be joined by chain of black squares, so that consecutive members of the chain share a side. Show that there are at least $\frac{n^2-2}{3}$ black squares.

6. Consider a round-robin tournament with $2n + 1$ teams, where each team plays each other team exactly once. We say that three teams X, Y and Z, form a *cycle triplet* if X beats Y , Y beats Z, and Z beats X. There are no ties.

   (a) Determine the minimum number of cycle triplets possible.

   (b) Determine the maximum number of cycle triplets possible.

7. Define a k-clique to be a set of k people such that every pair of them are acquainted with each other. At a certain party, every pair of 3-cliques has at least one person in common, and there are no 5-cliques. Prove that there are two or fewer people at the party whose departure leaves no 3-clique remaining.

8. For integers r, n where $r < n$ a grid $r \times n$ is populated with numbers $1, 2, \ldots, n$, such that no number occurs twice in any row or column. Prove that it is possible to append $n - r$ rows, so that the condition is still met in the new $n \times n$ grid.

9. On some planet, there are 2^N countries ($N \geq 4$). Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag. We say that a set of N flags is *diverse* if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same colour. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

10. Let k be a positive integer. Find all positive integers n for which it is possible to choose n points on the sides of a triangle (different from its vertices) and connect some of them with a line such that

    (a) There is at least 1 point on each side

    (b) For each pair of points X and Y which are on different sides of the triangle, there exists exactly k points on the third side which are all connected to both X and Y, and exactly k points which are all connected to neither of X or Y.

11. A country has $n \geq 3$ airports. Two-way flights operate between between some pairs of the airports. It is known that for each pair of airports, there exists a third one which is not directly connected to either of them. Find the maximum number of two-way flights.