1 Power of point

The power of point P with respect to circle with centre O and radius r is defined as

$$p = PO^2 - r^2$$

Theorem 1: For any line through point P which intersects the circle at points A and B

$$p = PA \times PB$$

Theorem 2: For tangent PC where C is tangent point

$$p = PC^2$$

1. Square $ABCD$ of side length a has a circle inscribed in it. Let M be the midpoint of AB Find the length of that portion of the segment MC that lies outside of the circle.

2. Line OA is tangent to a circle at point A and chord BC is parallel to OA. Lines OB and OC intersect the circle for the second time at points K and L, respectively. Prove that line KL divides segment OA in halves.

3. We have a triangle ABC. Points K, L and M are chosen on the sides BC, AC and AB, such that AK, BL and CM intersect in one point. We know that $ALKB$ and $BMLC$ are cyclic quadrilaterals. Show that $AMKC$ is also a cyclic quadrilateral.

4. On the longer diagonal AC of parallelogram $ABCD$ point M is chosen, such that $BCDM$ is a cyclic quadrilateral. Show that BD is tangent to circumcircles of triangles AMD and AMB.

5. Let ABC be a triangle with $\angle B > \angle C$. Let P and Q be two different points on line AC such that $\angle PBA = \angle QBA = \angle ACB$ and A is located between P and C. Suppose that there exists an interior point D of segment BQ for which $PD = PB$. Let the ray AD intersect the circle ABC at $R \neq A$. Prove that $QB = QR$.

2 Radical axis

Radical axis is the locus of points at which tangents drawn to both circles are equal.

Theorem 1: The power of points on radical axis is equal with respect to both circles.

Theorem 2: Radical axis is a line.

Theorem 3: The three radical axes for three circles intersect in one point called the radical centre.

1. Prove that the midpoints of the four common tangents to two non-intersecting circles lie on one line.

2. Prove that the diagonals AD, BE and CF of circumscribed hexagon $ABCDEF$ intersect in one point. (Brianchon theorem)

3. Circles k_1 and k_2 intersect at points M and N. Line l intersects circle k_1 at points A and C and circle k_2 at points B and D, such that points A, B, C and D lie on the line l in that order. Let X be such point on line MN that M lies between X and N. Rays AX and BM intersect at point P, rays DX and CM at point Q. Prove that $PQ \parallel l$.

4. Point E is chosen on the median CD of triangle ABC. Line AB is tangent to circle c_1 at point A and to circle c_2 at point B such that both circles go through point E. The second intersection of c_1 and AC is M. The second intersection of c_1 and BC is N. Prove that tangent lines to circles c_1 and c_2 at points M and N respectively intersect on line CD.

5. Three circles intersect pairwise at points A_1 and A_2, B_1 and B_2, C_1 and C_2. Prove that $A_1B_2 \times B_1C_2 \times C_1A_2 = A_2B_1 \times B_2C_1 \times C_2A_1$.

6. The extensions of sides AB and CD of quadrilateral $ABCD$ meet at point F and the extensions of sides BC and AD meet at point E. Prove that the circles with diameters AC, BD and EF have a common radical axis and the orthocenters of triangles ABE, CDE, ADF and BCF lie on it.

Hints

Power of point

1. Write down power of point C.
2. Identify similar triangles and write down power of the intersection of KL and AB
3. Write down the power of the intersection for each circle
4. Write down the power of the intersection of the diagonals
5. Prove that quadrilateral $DRCQ$ is cyclic

Radical axis

1.
2. Find three circles for which the diagonals are radical lines for.
3. What's the power of X with respect to each circle? What other points lie on the circle PQM?
4. Prove that quadrilateral $ABNM$ is cyclic.
5. Where's the radical centre? Find 3 pairs of similar triangles.
6. For each triangle, what's the power of orthocentre with respect to each circles?