Functional equations

1 Injective and surjective functions

- If $f(x) = f(y) \Rightarrow x = y$, then f is injective
- If for each element y in function codomain, there exists x for which $f(x) = y$, then f is surjective.
- If f is both injective and surjective then f is bijective.

Problems

1.1. Let $f : X \to Y$ and $g : Y \to X$ and $g(f(x)) = x$. Prove that f is injective and g is surjective.

1.2. Prove that for any function $f : X \to Y$, there exists a set Z and functions $g : X \to Z$ and $h : Z \to Y$, such that g is injective and h is surjective.

1.3. Find all strictly monotonic functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$f(x + f(y)) = f(x) + y$$

2 Cauchy functional equations

2.1. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ for which $f(x) + f(y) = f(x + y)$.

2.2. If the above functional equation satisfies any one of the following conditions, find the solutions to the functional equation if $f : \mathbb{R} \to \mathbb{R}$.

- The function is continuous at one point,
- The function is monotonic on any interval,
- The function is bounded on any interval.

2.3. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ for which $f(x)f(y) = f(xy)$, given it satisfies any of the constraints defined in question 2.2.

2.4. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}$ for which $f(x) + f(y) = f(xy)$, given it satisfies any of the constraints defined in question 2.2.

2.5. Find all functions $f : \mathbb{R} \to \mathbb{R}^+$ for which $f(x)f(y) = f(x + y)$, given it satisfies any of the constraints defined in question 2.2.

2.6. In what domains are the solutions of the functional equations defined in questions 2.3-2.5 the only solutions if they are not subject to any of the constraints defined in 2.2?

2.7. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

2.8. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$f(x+y) + f(x-y) = 2f(y)$$

2.9. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$f(x+y) + f(x-y) = 2f(x)$$

2.10. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$f(x+y) + f(x-y) = 2f(x)f(y)$$

3 Recurrence relations

- A recurrence relation is a relation that determines the elements of a sequence $x_n, n \in \mathbb{N}_{\not\vdash}$, as a function of previous elements. A recurrence relation of the form

$$(\forall n \geq k) \qquad x_n + a_1 x_n - 1 + \ldots + a_k x_{n-k} = 0$$

  for constants $a_1, \ldots, a_k$ is called a linear homogeneous recurrence relation of order k.
- We define the characteristic polynomial of the relation as

$$P(x) = x^k + a_1 x^{k-1} + \ldots + a_k$$

- Let $P(x)$ factorize as

$$P(x) = (x - \alpha_1)^{k_1} (x - \alpha_2)^{k_2} \ldots (x - \alpha_r)^{k_r}$$

  where $\alpha_1, \ldots, \alpha_r$ are distinct complex numbers and $k_1, \ldots, k_r$ are positive integers.
- The general solution of this recurrence relation is in this case given by

$$x_n = p_1(n)\alpha_1^n + p_2(n)\alpha_2^n + \ldots + p_r(n)\alpha_r^n$$

  where p_i is a polynomial of degree less than k_i.
- In particular, if $P(x)$ has k distinct roots, then all p_i are constant.
- If $x_0, \ldots, x_{k-1}$ are set, then the coefficients of the polynomials are uniquely determined.

Problems

3.1. Find the closed form expression for n-th term of the Fibonacci sequence.

3.2. Let S_n denote the number of ternary sequences (consisting of 0,1, and 2s) of length n, such that they do not contain a substring of "10", "01", or "11". Find a closed form expression for S_n.

3.3. Let r be a real number, and let x_n be a sequence such that $x_0 = 0, x_1 = 1$, and $x_{n+2} = rx_{n+1} - x_n$ for $n \geq 0$. For which values of r does $x_1 + x_3 + \cdots + x_{2m-1} = x_m^2$ for all positive integers m?

3.4. Let a_n, b_n, and c_n be geometric sequences with different common ratios and let $a_n + b_n + c_n = d_n$ for all integers n. If $d_1 = 1$, $d_2 = 2$, $d_3 = 3$, $d_4 = -7$, $d_5 = 13$, and $d_6 = -16$, find d_7.