1 Chebyshev's inequality

For real numbers $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$

$$\frac{1}{n}\sum_{i=1}^{n} a_i b_i \geq \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right) \geq \frac{1}{n}\sum_{i=1}^{n} a_i b_{n+1-i}$$

1. Prove Chebyshev's inequality by proving the following equation.

   Prove that
   $$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i - a_j)(b_i - b_j) = n\sum_{k=1}^{n} a_k b_k - \sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k$$
   When does the equality hold?

2. For triangle with angles α, β, γ and opposite sides with lengths a, b, c respectively prove that
   $$\frac{\pi}{3} \leq \frac{\alpha a + \beta b + \gamma c}{a+b+c} \leq \frac{\pi}{2}$$

3. Prove that for positive real numbers
   $$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$$

4. Prove that for positive real numbers
   $$\frac{(b+c)(b^2+c^2)}{a} + \frac{(a+c)(a^2+c^2)}{b} + \frac{(a+b)(a^2+b^2)}{c} \geq 4(a^2+b^2+c^2)$$

5. Prove that for positive real numbers
   $$\frac{x}{x^2+yz} + \frac{y}{y^2+xz} + \frac{z}{z^2+xy} \leq \frac{x^2+y^2+z^2}{2xyz}$$

6. Triangle has angles α, β, γ, perimeter $2p$ and radius of circumcircle R. Prove that
   $$\cot^2\alpha + \cot^2\beta + \cot^2\gamma \geq 3\left(\frac{9R^2}{p^2} - 1\right)$$
   When does the equality hold?

2 Rearrangement inequality

For real numbers $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$ and a permutation $a'_1, \ldots, a'_n$ of $a_1, \ldots, a_n$

$$a_1 b_1 + \cdots + a_n b_n \geq a'_1 b_1 + \cdots + a'_n b_n \geq a_1 b_n + \cdots + a_n b_1$$

7. Prove rearrangement inequality. When does the equality hold?
8. For real numbers $a_1, \ldots, a_n$ and for its permutation $a'_1, \ldots, a'_n$, prove that

$$\sum_{i=1}^{n} \frac{a_i}{a'_i} \geq n$$

9. Prove the mean inequalities by using rearrangement inequality

$$SM \geq AM \geq GM \geq HM$$

10. Prove Cauchy inequality using rearrangement inequality.
11. Prove Chebyshev's inequality using rearrangement inequality.