

Functional equations

1 Injective and surjective functions

- If $f(x) = f(y) \Rightarrow x = y$, then f is injective
- If for each element y in function codomain, there exists x for which $f(x) = y$, then f is surjective.
- If f is both injective and surjective then f is bijective.

Problems

- 1.1. Let $f : X \rightarrow Y$ and $g : Y \rightarrow X$ and $g(f(x)) = x$. Prove that f is injective and g is surjective.
- 1.2. Prove that for any function $f : X \rightarrow Y$, there exists a set Z and functions $g : X \rightarrow Z$ and $h : Z \rightarrow Y$, such that g is injective and h is surjective.
- 1.3. Find all strictly monotonic functions $f : \mathbb{R} \rightarrow \mathbb{R}$ which satisfy

$$f(x + f(y)) = f(x) + y$$

2 Cauchy functional equations

- 2.1. Find all functions $f : \mathbb{Q} \rightarrow \mathbb{Q}$ for which $f(x) + f(y) = f(x + y)$.
- 2.2. If the above functional equation satisfies any one of the following conditions, find the solutions to the functional equation if $f : \mathbb{R} \rightarrow \mathbb{R}$.
 - The function is continuous at one point,
 - The function is monotonic on any interval,
 - The function is bounded on any interval.
- 2.3. Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ for which $f(x)f(y) = f(xy)$, given it satisfies any of the constraints defined in question 2.2.
- 2.4. Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ for which $f(x) + f(y) = f(xy)$, given it satisfies any of the constraints defined in question 2.2.
- 2.5. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}^+$ for which $f(x)f(y) = f(x + y)$, given it satisfies any of the constraints defined in question 2.2.
- 2.6. In what domains are the solutions of the functional equations defined in questions 2.3-2.5 the only solutions if they are not subject to any of the constraints defined in 2.2?
- 2.7. Find all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ for which

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

2.8. Find all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ for which

$$f(x+y) + f(x-y) = 2f(y)$$

2.9. Find all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ for which

$$f(x+y) + f(x-y) = 2f(x)$$

2.10. Find all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ for which

$$f(x+y) + f(x-y) = 2f(x)f(y)$$

3 Recurrence relations

- A recurrence relation is a relation that determines the elements of a sequence $x_n, n \in \mathbb{N}_0$, as a function of previous elements. A recurrence relation of the form

$$(\forall n \geq k) \quad x_n + a_1 x_{n-1} + \dots + a_k x_{n-k} = 0$$

for constants a_1, \dots, a_k is called a linear homogeneous recurrence relation of order k .

- We define the characteristic polynomial of the relation as

$$P(x) = x^k + a_1 x^{k-1} + \dots + a_k$$

- Let $P(x)$ factorize as

$$P(x) = (x - \alpha_1)^{k_1} (x - \alpha_2)^{k_2} \dots (x - \alpha_r)^{k_r}$$

where $\alpha_1, \dots, \alpha_r$ are distinct complex numbers and k_1, \dots, k_r are positive integers.

- The general solution of this recurrence relation is in this case given by

$$x_n = p_1(n)\alpha_1^n + p_2(n)\alpha_2^n + \dots + p_r(n)\alpha_r^n$$

where p_i is a polynomial of degree less than k_i .

- In particular, if $P(x)$ has k distinct roots, then all p_i are constant.
- If x_0, \dots, x_{k-1} are set, then the coefficients of the polynomials are uniquely determined.

Problems

- 3.1. Find the closed form expression for n -th term of the Fibonacci sequence.
- 3.2. Let S_n denote the number of ternary sequences (consisting of 0,1, and 2s) of length n , such that they do not contain a substring of "10", "01", or "11". Find a closed form expression for S_n .
- 3.3. Let r be a real number, and let x_n be a sequence such that $x_0 = 0, x_1 = 1$, and $x_{n+2} = rx_{n+1} - x_n$ for $n \geq 0$. For which values of r does $x_1 + x_3 + \dots + x_{2m-1} = x_m^2$ for all positive integers m ?
- 3.4. Let a_n, b_n , and c_n be geometric sequences with different common ratios and let $a_n + b_n + c_n = d_n$ for all integers n . If $d_1 = 1, d_2 = 2, d_3 = 3, d_4 = -7, d_5 = 13$, and $d_6 = -16$, find d_7 .