1 Polynomials

Definition 1. Polynomial function $P\colon \mathbb{R} \to \mathbb{R}$ can be presented in the form of

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, \ldots, a_n$ are real numbers called the polynomial coefficients. Largest n for which $a_n \neq 0$ is called the degree of polynomial.

Theorem 2 (Bezout's theorem). A polynomial $P(x)$ is divisible by the binomial $(x - a)$ if and only if $P(a) = 0$.

Theorem 3 (The fundamental theorem of algebra). Every non-constant polynomial has a complex root.

Theorem 4 (The rational root theorem). If $x = p/q$ is a rational zero of a polynomial $P(x) = a_n x^n + \ldots + a_0$ with integer coefficients and $(p, q) = 1$, then $p|a_0$ and $q|a_n$.

Theorem 5 (Vieta's formulae). If the solutions polynomial of degree n are $x_1, x_2, \ldots, x_n$ and $a_n = 1$, then the following holds:

$$\begin{aligned}
x_1 + x_2 + \ldots + x_n &= -a_{n-1}, \\
x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n &= a_{n-2}, \\
x_1x_2x_3 + x_1x_2x_4 + \ldots + x_{n-2}x_{n-1}x_n &= -a_{n-3}, \\
&\ldots \\
x_1x_2\ldots x_n &= (-1)^n a_0.
\end{aligned}$$

1. Find the roots of polynomial $P(x) = x^5 - x^4 - 13x^3 + x^2 + 12x$.
2. Find the value of $x^5 + 2x^2 - 4x + 2010$ given that $x^3 + 2x + 2 = 0$.
3. How many points do we need to uniquely define a polynomial of degree n?
4. The roots for polynomial $P_2(x) = ax^2 + bx + c$ are x_1 and x_2. Find the coefficients for third order polynomial which has roots x_1^2, x_2^2 and x_1x_2.
5. In $x^3 + px^2 + qx + r$ one root is the sum of the two others. Find the relationship between p, q and r.
6. Find the roots of the polynomial $P(x) = ax^4 + bx^3 + cx^2 + bx + a$
7. Polynomial with integer coefficients $ax^3 + bx^2 + cx + d$ has ad odd and bc even. Show that at least one zero of the polynomial is irrational.
8. Polynomial of degree n with non-negative coefficients and leading coefficient 1 has n real roots. Prove that $P(2) \geq 3^n$.