1 Divisibility

1.1 Rules

For integers a, b, c, d:

1. If $a \mid b$ and $c \mid d$ then $ac \mid bd$
2. If $a \mid b$ and $a \mid c$, then $a \mid b + c$
3. **Euclid's algorithm.**
   $\gcd(a, b) = \gcd(a, b - a)$
4. **Corollary of Euclid's algorithm.**
   $ax + by = n$ has solution (x, y) in integers if and only if $gcd(a, b) \mid n$

1.2 Problems

1. Prove that if $m - p \mid mn + pq$, then $m - p \mid mq + np$.
2. Prove that $17 \mid 2a + 3b \iff 17 \mid 9a + 5b$
3. Prove that it is not possible to find positive integers n and $m > 1$, such that $102^{2017} + 103^{2017} = n^m$
4. Find all positive integers d, such that d divides both: $n^2 + 1$ and $(n + 1)^2 + 1$
5. Find all prime numbers p for which $p^2 + 2543$ has less than 16 positive divisors.
6. Prove that $\gcd(a^m - 1, a^n - 1) = a^{\gcd(m,n)} - 1$
7. Prove that any two non-equal integers in form $2^{2^n} + 1$, where n is a positive integer are coprime-
8. $a_1, a_2, \ldots, a_{2n}$ are mutually distinct integers. Find all integers x satisfying
   $$(x - a_1) \ldots (x - a_{2n}) = (-1)^n (n!)^2$$
9. Let $a_0, \ldots, a_n \geq -1$ be integers such that at least one of them is non-zero. It is known that $a_0 + 2a_1 + 2^2 a_2 + \cdots + 2^n a_n = 0$. Show that $a_0 + a_1 + \cdots + a_n > 0$.

2 Congruences

2.1 Rules

For integers a, b, c, d, m, n and prime p.

1. $a \equiv b \mod m \iff m \mid a - b$
2. $a \equiv b \mod m$ and $c \equiv d \mod m \implies a + c \equiv b + d \mod m$
3. $a \equiv b \mod m \iff an \equiv bn \mod mn$
4. $a \equiv b \mod m \implies an \equiv bn \mod m$
5. **Fermat's little theorem.**
   $a^p \equiv a \mod p$
6. **Wilson's theorem.**
   $(p-1)! \equiv -1 \mod p$

2.2 Problems

1. Prove that $2018!! + 2017!!$ is divisible by 2019
2. Find all primes p, such that:
   (a) $p + 4$, $p + 14$ are primes;
   (b) $8p^2 + 1$ is a prime;
   (c) $p + 10$, $p + 1$ is a prime;
   (d) $4p^2 + 1$, $6p^2 + 1$ are primes;
   (e) $p^2 - 6$, $p^2 + 6$ are primes;
   (f) $p^4 - 6$ is a prime;
   (g) $p^3 + 6$, $p^3 - 6$ are primes;
   (h) $p^2 - 2$, $2p^2 - 1$, $3p^2 + 4$ are primes;
   (i) $2^p + 1$, $2^p - 1$ are primes;
   (j) p, q, $p^q + q^p$ are primes.
3. In a series of integers, the next element is obtained by concatenating the element's order number to previous element by using carry. The first elements of the series are therefore:

   $$1, 12, 123, 1234, \ldots, 123456789, 1234567900, 12345679011$$

   Find all elements in the series which are divisible by 7.

4. Prove that there exist no integers $n > 1$, such that $n \mid 3^n - 2^n$.
5. Find all positive integers n which satisfy the following condition: For all integers a and b which are coprime with n

$$a \equiv b \mod n \iff ab \equiv 1 \mod n$$

6. Prove that it is not possible to separate 18 consecutive integers into 2 subsets with 9 elements in such way that the product of each subset is equal.