1 Functions in number theory

1.1 Rules

1. **Fermat's little theorem.**
   For prime p and integer a
   $$a^p \equiv a \mod p$$

2. **Wilson's theorem.**
   $$(p-1)! \equiv -1 \mod p$$
   if and only if when p is prime number.

3. **Number of factors**
   The number of positive factors of $n = p_1^{\alpha_1} \dots p_k^{a_k}$
   $$d(n) = (\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_k + 1)$$

4. **Sum of factors**
   The sum of positive factors of $n = p_1^{\alpha_1} \dots p_k^{a_k}$
   $$\sigma(n) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \dots \frac{p_k^{\alpha_k+1} - 1}{p_k - 1}$$

5. **Euler's function**
   Euler's function or totient function $\varphi(n)$ is defined for $n = p_1^{\alpha_1} \dots p_k^{a_k}$ as the number of positive integers less than n and coprime to n. Then
   $$\varphi(n) = n\left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_k}\right)$$

6. **Euler's theorem** (Generalisation of Fermat's theorem)
   Let n be a natural number and a an integer such that $\gcd(a, n) = 1$. Then
   $$a^{\varphi(n)} \equiv 1 \mod n$$

1.2 Problems

1. Find all primes p, for which the sum of all positive factors of p^4 is a perfect square.

2. Prove that for positive integer n
   $$\sum_{d|n} \varphi(d) = n$$

3. Prove that for positive integers a and b
   $$\varphi(ab) = \varphi(a)\varphi(b)\frac{\gcd(a, b)}{\varphi(\gcd(a, b))}$$

4. One of Euler's conjectures was disproved in the 1960s by three American mathematicians when they showed there was a positive integer such that $133^5 + 110^5 + 84^5 + 27^5 = n^5$. Find the value of n.

5. How many prime numbers p are there, such that $29^p + 1$ is a multiple of p?

6. Prove that if $\gcd(a, n) = 1$, then

$$a^b \equiv a^{b \mod \varphi(n)} \mod n$$

7. Find the last three digits of $2008^{2007^{2006^{\cdots^{2^1}}}}$.

8. Prove that there exists no positive integer for which $n! + 19^{93}$ is a perfect square.

9. Find all pairs of positive prime numbers (p_1, p_2) for which the equation

$$\phi(n^2) = n + p_1 p_2$$

has a solution for n in positive integers.

10. Let p be a prime number and let n be a positive integer. Let q be a positive divisor of $(n+1)^p - n^p$. Show that $q - 1$ is divisible by p.