Fish 559 - Fall 2018

Numerical Computing for Fisheries Assessment and Management

Basic Information

- Instructor:
  - Andre Punt (FISH 116; aepunt@u)
- Class web-site
  - http://courses.washington.edu/fish507/index.htm
- Prerequisites for this course
  - Fish 458, some programming experience (or talk to me)

Class Structure

- The ADMB Workshop
- Lectures (Room 136): F (11.30-12:20)
- Computer laboratory sessions (Room 136): M, W (11:30-13:20)
- Grading
  - Participation in the workshop (20%)
  - Four homework assignments (40%).
  - Project (40%).

The Web-site

- The powerpoint slides for each lecture will be posted the day before the lecture concerned.
- Homeworks:
  - Solutions will be posted the day after all homeworks have been submitted.
  - Discussion of questions among students is encouraged; cheating is not.
- The ‘Readings and Resources’ page includes useful links – if there are any other links you find useful, let me know and I will update the resources page.

Course Overview

- Four focus areas:
  - how to use R to perform numerical analyses;
  - how to use Template Model Builder to fit models to data;
  - how to use STAN to fit Bayesian hierarchical models; and
  - standard numerical techniques.
- The lecture topics are available on the web-site.

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Mixed Effects Modeling-I

Fish 559; Lecture 1

What are Fixed and Random Effects?

- Fixed Effects:
  - Parameters associated with the entire population or with certain *repeatable* levels of experimental units.
    - The value for the vessel factor for the *SS Titantic*.
- Random Effects:
  - Parameters associated with individual experimental units *drawn at random* from a population.
    - The value of the vessel factor for the 23rd vessel in a database.

Mixed Effects Models

- A mixed effects model is a model that has both *fixed effects* and *random effects*.
- Often the same random effect is assigned to observations sharing a common classification factor:
  - Consider length-at-age data for several (randomly selected) stocks of a species. The random factor is stock and the observations are length-at-age data.
  - Random effects can the thought of as ways of modelling the covariance structure of the data.

Estimation Techniques

- Classical (frequentist) and Bayesian methods can be used to estimate the parameters of mixed effects models.
- Lectures 1 and 7 focus on the use of frequentist methods. We will cover Bayesian mixed effects models when we cover STAN AND TMB.

Linear Mixed Effects Models

Vector of fixed effects

$$\underline{y}_i = \mathbf{X}_i \underline{\beta} + \mathbf{Z}_i \underline{b}_i + \underline{\varepsilon}_i, \qquad i = 1, 2, \ldots., M$$

Observations for group *i*

Vector of random effects

The traditional linear modeling framework is a special case of of this model in which there are no random effects.

A Simple Example
(*sensu* Pinherio and Bates, Chapter 1)

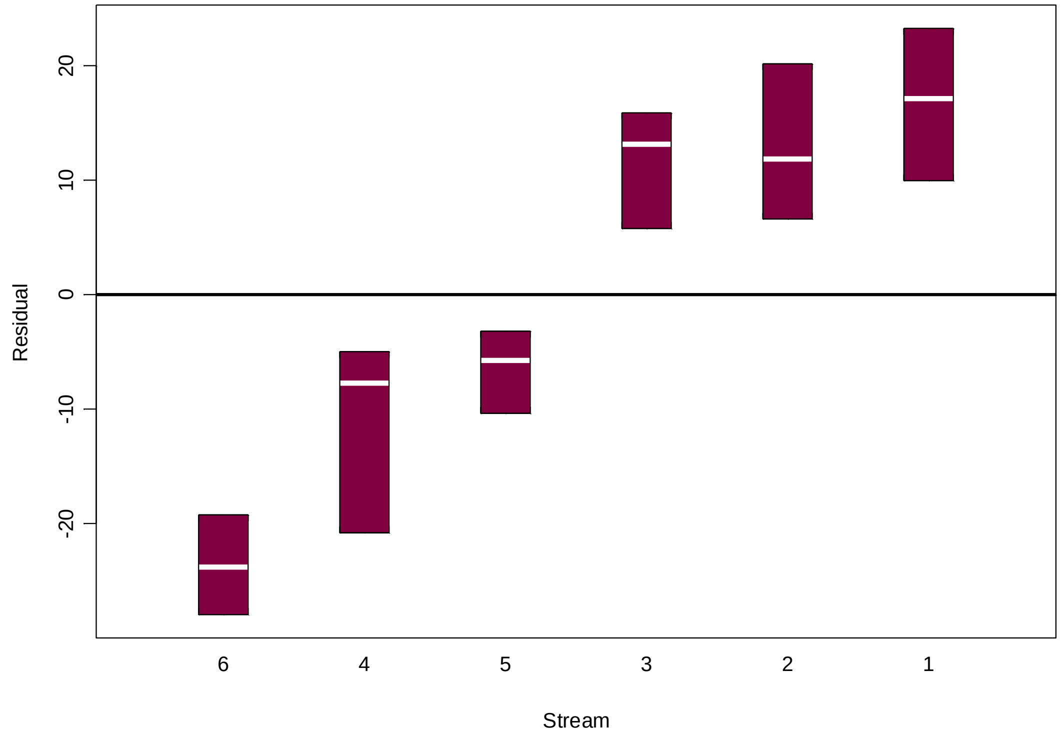
- We select six streams at random and determine the density at each stream three times.
- The questions:
  - What is the density in a typical stream?
  - What is the variation in density among streams?
  - What is the variation in density estimates within a stream?

Fit of the Standard Linear Model

$$y_{i,j} = \beta + \varepsilon_{i,j}$$

$$i = 1, 2, .., 6; \; j = 1, 2, 3$$

$$\hat{\beta} = 78.63; \hat{\sigma} = 16.03$$



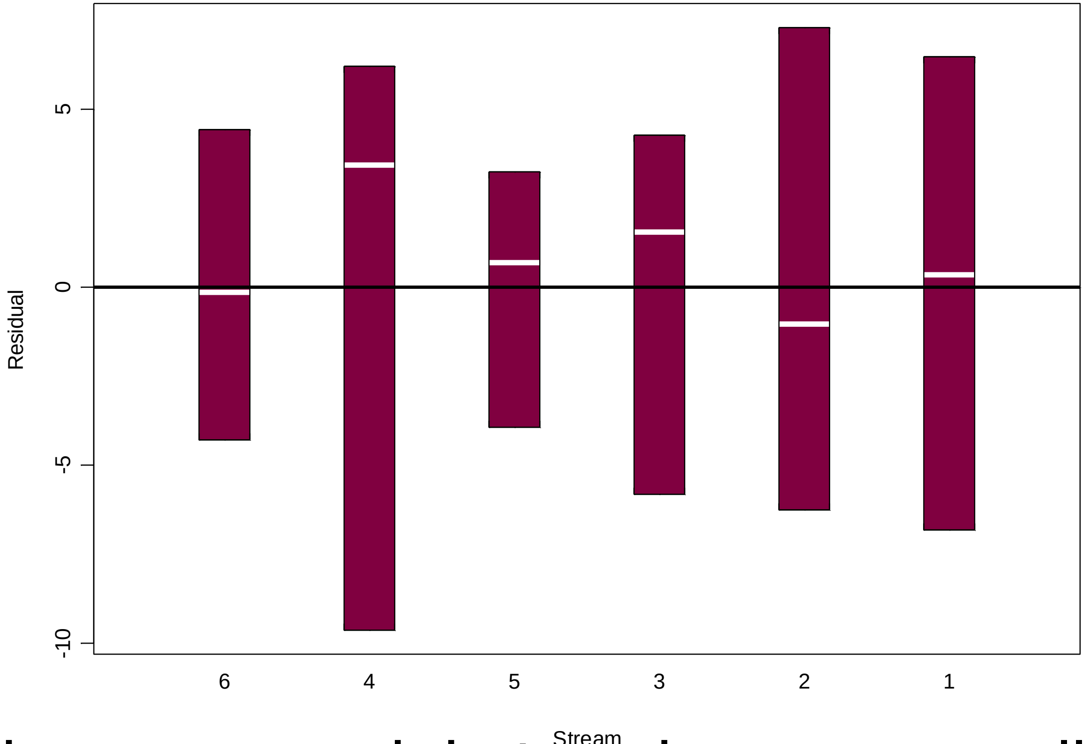
This analysis attributed all the error to within-stream variation. However, it is clear that there is between-stream variation in density but we have no way to comment on it.

The Next Step: Fit a Model with a "Stream-Effect"

$$y_{i,j} = \beta_i + \varepsilon_{i,j}$$

$$i = 1, 2, .., 6;\ j = 1, 2, 3$$

$$\hat{\sigma} = 6.1$$



The residual pattern has been removed, but we have no overall mean and no way to comment on between-stream variance, i.e. we cannot say anything about *the population* of streams.

Finally: A Mixed Effects Model

- We construct a random effects model as follows:

$$y_{i,j} = \beta + b_i + \varepsilon_{i,j}$$

- β is the mean density across the *population*,
- b_i is a random variable representing the deviation from the population mean:

$$b_i \sim N(0;\sigma_b^2)$$

- $\varepsilon_{i,j}$ is a random variable representing the deviation for observation *j* from mean density for stream *i*, i.e. $\varepsilon_{i,j} \sim N(0;\sigma^2)$

Finally: A Mixed Effects Model (Some notes-I)

- The residuals are assumed to be independent, normally distributed random variables with constant variance.
- The random effects are assumed to be normally distributed.
- This only works if the streams have been selected at random from the population of streams.
- Mixed effects models are also called *hierarchical models*.
- The number of parameters in this model is three (β, σ and σ_b) because stream is just another level of random variation (we never count the residuals as parameters).

Finally: A Mixed Effects Model (Some notes-II)

- The model we have is a one-way linear classification model.
- Introducing random effects can be thought of (in this case) as allowing for correlation of $\sigma_b^2/(\sigma^2+\sigma_b^2)$ among the density estimates for a given stream.
- There are ways to estimate the random effects (we will often be interested in their values).

Finally: A Mixed Effects Model

- The results from fitting the mixed effects model are:

$$\beta = 78.63; \sigma_b = 15.79; \sigma = 6.1$$

- These estimate are similar to the linear model with separate "stream effects" (but would not be if this was an unbalanced design).

Time for some R
(first create a data framework)

```
FileN <- "LEC1A.TXT"
TheData <- scan(FileN,what=list(Stream=0,NULL,Density=0),n=3*18)
Streams <- data.frame(TheData)
```

Time for some R-II

- The call to LME first gives the fixed effects (the intercept in this case), followed by the data set, and finally the random component of the model.

- Fitting the fixed effects models:
  - lm1 <- lm(Density~1,data=Streams)
  - lm2 <- lm(Density~Stream-1,data=Streams)
- Fitting the mixed effects model:
  - lm3 <- lme(Density ~ 1,data=Streams, random = ~ 1 | Stream)

The LME Function

- The full call is:

  lme(fixed, data, random, correlation, weights, subset, method, na.action, control)

- fixed – formula for the fixed effects;
- data – must be a data frame;
- random – the random effects (random effects can be nested);
- correlation – correlation structure of the residuals (optional);
- weights – variance structure of the residuals (optional);
- subset – which rows of the data set to include (vector of boolean)
- method – ML or REML
- control – optional control specifications (some of these are pretty useful).

Linear Mixed Models (Notes)

- The linear mixed model can be fitted using Maximum Likelihood (ML) or Restricted Maximum Likelihood (REML). If you have many experimental units, this hardly matters, but if you have few replicates, this can make a major difference (I always use REML).

Linear Mixed Models (Understanding the output)

```
Linear mixed-effects model fit by REML
 Data: Streams
       AIC      BIC    logLik
  133.7973 136.2969 -63.89865

Random effects:
 Formula:  ~ 1 | Stream
        (Intercept) Residual
StdDev:    15.78877 6.083881

Fixed effects: Density ~ 1
               Value Std.Error DF  t-value p-value
(Intercept) 78.62872   6.60332 12 11.90745   0

Standardized Within-Group Residuals:
       Min         Q1       Med         Q3      Max
 -1.670431 -0.8239512 0.1256738 0.5295573 1.298799

Number of Observations: 18
Number of Groups: 6
```

AIC and BIC for model selection

The among-stream SD (variance of re)

The within-stream SD

The fixed effects

Fixed and Random Effects Together

- Often, we will have cases in which there are some known sources of variability (the random effects) and some factors which may have different impacts.
- These models will have both *fixed* and *random* effects.

Model selection with fixed and random effects (Zuur et al)

1. Set the fixed effects based on “as many explanatory variables as possible”.
2. Select a random effects structure using AIC (using REML).
3. Fix the random effects structure and compare fixed effects options (using ML estimation). [Likelihood tests do not work with REML]
4. Compute the final model using REML.

Fixed and Random Effects Together (Length-weight regressions)

The Problem:

- We select 10 individuals and measure the relationships between their lengths and weights as they grow. The relationship between length and weight is:

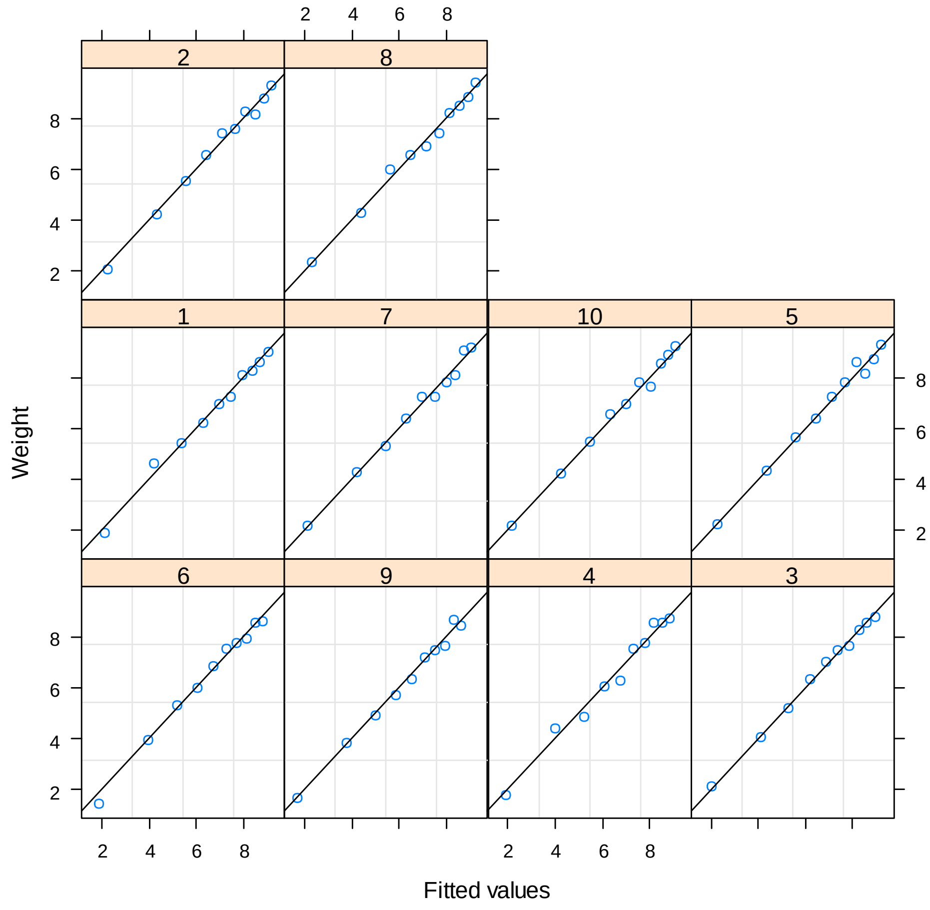
$$W = aL^b; \qquad \ell\mathrm{n}W_{i,j} = \ell\mathrm{n}a_i + b_i \ell\mathrm{n}L_{i,j} + \varepsilon_{i,j}$$

- We are interested in whether *b* differs among individuals (*a* is individual-specific and individuals were selected randomly)
- The formulae:
  - TheData <- scan("LEC1B.TXT",what=list(Subject=0,TST1=0,TST2=0,Length=0,Weight=0,TST3=0),n=6*100)
  - LenW <- data.frame(TheData)
  - lm1 <- lme(Weight~Length,data=LenW,random = ~1 |Subject,method="REML")

Fixed and Random Effects Together (Length-weight regressions)

Always plot your data!

Note the order in which the data are plotted



Fixed and Random Effects Together (Length-weight regressions)

```
Linear mixed-effects model fit by REML
 Data: LenW
      AIC      BIC  logLik
  6.72662 17.06649 0.63669

Random effects:
 Formula:  ~ 1 | Subject
        (Intercept) Residual
StdDev:   0.2113517 0.205754

Fixed effects: Weight ~ Length
                Value  Std.Error DF   t-value p-value
(Intercept) -0.026674 0.09560888 89   -0.2790  0.7809
     Length  3.017151 0.02958709 89  101.9753  <.0001

Standardized Within-Group Residuals:
       Min         Q1         Med        Q3      Max
 -2.419116 -0.5799786 -0.05204277 0.5719096 2.320844

Number of Observations: 100
Number of Groups: 10
```

Theoretical Aspects-I

Vector of fixed effects

Vector of random effects

$$\underline{y}_i = \mathbf{X}_i \underline{\beta} + \mathbf{Z}_i \underline{b}_i + \underline{\varepsilon}_i, \qquad i = 1, 2, ..., M$$

Observations for group *i*

$$\underline{b}_i \sim N(0, \mathbf{\Psi}); \underline{\varepsilon}_i \sim N(0, \sigma^2 \mathbf{I}) \qquad \sigma^2 \mathbf{\Psi}^{-1} = \mathbf{\Delta}^T \mathbf{\Delta}$$

This matrix will be a function of parameters, Θ

Tricks and Traps

- If you use REML estimation, you can only compare models with the same fixed-effects structure.
- The standard output does not provide:
  - the "best estimates" of the random effects - use *coef(lmout)* to find them.
  - the confidence intervals for the variance parameters – use *intervals(lmout, 0.95)*

Theoretical Aspects-II

$$L(\underline{y}_i \mid \underline{b}_i, \underline{\beta}, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{n_i/2}} \exp\left(-\frac{\left\| \underline{y}_i - \mathbf{X}_i \underline{\beta} - \mathbf{Z}_i \underline{b}_i \right\|^2}{2\sigma^2} \right)$$

$$p(\underline{b}_i \mid \underline{\theta}, \sigma^2) = \frac{1}{(2\pi)^{q/2} \sqrt{|\,\mathbf{\Psi}\,|}} \exp\left(-\frac{\underline{b}_i^T \mathbf{\Psi}^{-1} \underline{b}_i}{2} \right) = \frac{1}{(2\pi)^{q/2} \sigma^q \operatorname{abs} |\,\mathbf{\Delta}\,|} \exp\left(-\frac{\left\| \mathbf{\Delta} \underline{b}_i \right\|^2}{2\sigma^2} \right)$$

$$L(y \mid \underline{\beta}, \underline{\theta}, \sigma^2) = \prod_{i=1}^{M} \frac{\operatorname{abs} |\,\mathbf{\Delta}\,|}{(2\pi\sigma^2)^{n_i/2}} \int \frac{\exp\left[-\left(\left\| \underline{y}_i - \mathbf{X}_i \underline{\beta} - \mathbf{Z}_i \underline{b}_i \right\|^2 + \left\| \mathbf{\Delta} \underline{b}_i \right\|^2 \right) / 2\sigma^2 \right]}{(2\pi\sigma^2)^{q/2}} \, d\underline{b}_i$$

Theoretical Aspects-III

$$L(y\,|\,\underline{\beta},\underline{\theta},\sigma^2)=\prod_{i=1}^{M}\frac{\text{abs}\,|\,\boldsymbol{\Delta}\,|}{(2\pi\sigma^2)^{n_i/2}}\int\frac{\exp\left[-\left(\left\|\underline{y}_i-\mathbf{X}_i\underline{\beta}-\mathbf{Z}_i\underline{b}_i\right\|^2+\left\|\boldsymbol{\Delta}\underline{b}_i\right\|^2\right)/2\sigma^2\right]}{(2\pi\sigma^2)^{q/2}}d\underline{b}_i$$

We now add the marginal distribution of the random effects into extra rows.

$$\underline{\tilde{y}}_i=\begin{bmatrix}\underline{y}_i\\0\end{bmatrix};\qquad \tilde{\mathbf{X}}_i=\begin{bmatrix}\mathbf{X}_i\\0\end{bmatrix};\qquad \tilde{\mathbf{Z}}_i=\begin{bmatrix}\mathbf{Z}_i\\\Delta\end{bmatrix};$$

$$L(y\,|\,\underline{\beta},\underline{\theta},\sigma^2)=\prod_{i=1}^{M}\frac{\text{abs}\,|\,\boldsymbol{\Delta}\,|}{(2\pi\sigma^2)^{n_i/2}}\int\frac{\exp\left[-\left\|\underline{\tilde{y}}_i-\tilde{\mathbf{X}}_i\underline{\beta}-\tilde{\mathbf{Z}}_i\underline{b}_i\right\|^2/2\sigma^2\right]}{(2\pi\sigma^2)^{q/2}}d\underline{b}_i$$

Theoretical Aspects-IV

$$L(y \mid \underline{\beta}, \underline{\theta}, \sigma^2) = \prod_{i=1}^{M} \frac{\text{abs}\,|\,\boldsymbol{\Delta}\,|}{(2\pi\sigma^2)^{n_i/2}} \int \frac{\exp\left[-\left\| \underline{\tilde{y}}_i - \tilde{\mathbf{X}}_i \underline{\beta} - \tilde{\mathbf{Z}}_i \underline{b}_i \right\|^2 / 2\sigma^2 \right]}{(2\pi\sigma^2)^{q/2}} \, d\underline{b}_i$$

We can determine the conditional modes of the random effects given the data by minimizing the numerator.

$$\hat{\underline{b}}_i = (\tilde{\mathbf{Z}}_i^T \, \tilde{\mathbf{Z}}_i)^{-1} \, \tilde{\mathbf{Z}}_i^T \, (\underline{y}_i - \tilde{\mathbf{X}}_i \, \underline{\beta})$$

Substituting back:

$$\begin{aligned} \left\| \underline{\tilde{y}}_i - \tilde{\mathbf{X}}_i \underline{\beta} - \tilde{\mathbf{Z}}_i \underline{b}_i \right\|^2 &= \left\| \underline{\tilde{y}}_i - \tilde{\mathbf{X}}_i \underline{\beta} - \tilde{\mathbf{Z}}_i \widehat{\underline{b}}_i \right\|^2 + \left\| \tilde{\mathbf{Z}}_i (\underline{b}_i - \widehat{\underline{b}}_i) \right\|^2 \\ &= \left\| \underline{\tilde{y}}_i - \tilde{\mathbf{X}}_i \underline{\beta} - \tilde{\mathbf{Z}}_i \widehat{\underline{b}}_i \right\|^2 + (\underline{b}_i - \widehat{\underline{b}}_i)^T \, \tilde{\mathbf{Z}}_i^T \, \tilde{\mathbf{Z}}_i \, (\underline{b}_i - \widehat{\underline{b}}_i) \end{aligned}$$

Theoretical Aspects-V

Now:

$$\int \frac{\exp\left[-\left\|\underline{\tilde{y}}_i - \tilde{\mathbf{X}}_i \underline{\beta} - \tilde{\mathbf{Z}}_i \underline{b}_i\right\|^2 / 2\sigma^2\right]}{(2\pi\sigma^2)^{q/2}} d\underline{b}_i = \int \frac{\exp\left[-\left(\left\|\underline{\tilde{y}}_i - \tilde{\mathbf{X}}_i \underline{\beta} - \tilde{\mathbf{Z}}_i \underline{\hat{b}}_i\right\|^2 + (\underline{b}_i - \underline{\hat{b}}_i)^T \tilde{\mathbf{Z}}_i^T \tilde{\mathbf{Z}}_i (\underline{b}_i - \underline{\hat{b}}_i)\right) / 2\sigma^2\right]}{(2\pi\sigma^2)^{q/2}} d\underline{b}_i$$

$$= e^{-\frac{1}{2\sigma^2}\left\|\underline{\tilde{y}}_i - \tilde{\mathbf{X}}_i \underline{\beta} - \tilde{\mathbf{Z}}_i \underline{\hat{b}}_i\right\|^2} \int \frac{\exp\left[-(\underline{b}_i - \underline{\hat{b}}_i)^T \tilde{\mathbf{Z}}_i^T \tilde{\mathbf{Z}}_i (\underline{b}_i - \underline{\hat{b}}_i) / 2\sigma^2\right]}{(2\pi\sigma^2)^{q/2}} d\underline{b}_i$$

$$\int \frac{\exp\left[-(\underline{b}_i - \underline{\hat{b}}_i)^T \tilde{\mathbf{Z}}_i^T \tilde{\mathbf{Z}}_i (\underline{b}_i - \underline{\hat{b}}_i) / 2\sigma^2\right]}{(2\pi\sigma^2)^{q/2}} d\underline{b}_i = \frac{1}{\sqrt{|\tilde{\mathbf{Z}}_i^T \tilde{\mathbf{Z}}_i|}} \int \frac{\exp\left[-(\underline{b}_i - \underline{\hat{b}}_i)^T \tilde{\mathbf{Z}}_i^T \tilde{\mathbf{Z}}_i (\underline{b}_i - \underline{\hat{b}}_i) / 2\sigma^2\right]}{(2\pi\sigma^2)^{q/2} / \sqrt{|\tilde{\mathbf{Z}}_i^T \tilde{\mathbf{Z}}_i|}} d\underline{b}_i$$

$$= \frac{1}{\sqrt{|\tilde{\mathbf{Z}}_i^T \tilde{\mathbf{Z}}_i|}}$$

Theoretical Aspects-VI

Substituting back:

$$L(y\,|\,\underline{\beta},\underline{\theta},\sigma^2)=\prod_{i=1}^{M}\frac{\text{abs}\,|\,\mathbf{\Delta}\,|}{(2\pi\sigma^2)^{n_i/2}}\int\frac{\exp\left[-\left(\left\|\underline{\tilde{y}}_i-\tilde{\mathbf{X}}_i\underline{\beta}-\tilde{\mathbf{Z}}_i\underline{b}_i\right\|^2\right)/2\sigma^2\right]}{(2\pi\sigma^2)^{q/2}}\,d\underline{b}_i$$

$$=\prod_{i=1}^{M}\frac{\text{abs}\,|\,\mathbf{\Delta}\,|}{(2\pi\sigma^2)^{n_i/2}\sqrt{|\,\tilde{\mathbf{Z}}_i^T\,\tilde{\mathbf{Z}}_i\,|}}\exp\left[-\left(\left\|\underline{\tilde{y}}_i-\tilde{\mathbf{X}}_i\underline{\beta}-\tilde{\mathbf{Z}}_i\underline{\hat{b}}_i\right\|^2\right)/2\sigma^2\right]$$

$$=\frac{1}{(2\pi\sigma^2)^{N/2}}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{M}\left\|\underline{\tilde{y}}_i-\tilde{\mathbf{X}}_i\underline{\beta}-\tilde{\mathbf{Z}}_i\underline{\hat{b}}_i\right\|^2\right)\prod_{i=1}^{M}\frac{\text{abs}\,|\,\mathbf{\Delta}\,|}{\sqrt{|\,\tilde{\mathbf{Z}}_i^T\,\tilde{\mathbf{Z}}_i\,|}}$$