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Root Finding Methods

Fish 559; Lecture 10a

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What is Root Finding-I?

- Find the value for $\underline{x}$ such that the following system of equations is satisfied:

$$f_i(\underline{x}) = \underline{0}; \quad i = 1, 2, \ldots .n$$

- This general problem emerges very frequently in stock assessment and management.
- We will first consider the case i=1 as it is the most common case encountered.

What is Root Finding-II?

- Typical examples in fisheries assessment and management include:
  - Find *K* for a Schaefer model so that if the Schaefer model is projected from *K* in year 0 to year *m*, the biomass in year *m* equals *Z*.
  - Find the catch limit so that the probability of recovery equals a pre-specified value.
  - Find $F_{0.1}$ so that:

$$\left.\frac{dY}{dF}\right|_{F=F_{0.1}} = 0.1\left.\frac{dY}{dF}\right|_{F=0} \quad \rightarrow \quad \left.\frac{dY}{dF}\right|_{F=F_{0.1}} - 0.1\left.\frac{dY}{dF}\right|_{F=0} = 0$$

Methods for Root Finding

- There are several methods for finding roots, the choice of among these depends on:
  - The cost of evaluating the function.
  - Whether the function is differentiable (it must be continuous and monotonic for most methods).
  - Whether the derivative of the function is easily computable.
  - The cost of programming the algorithm.

The Example

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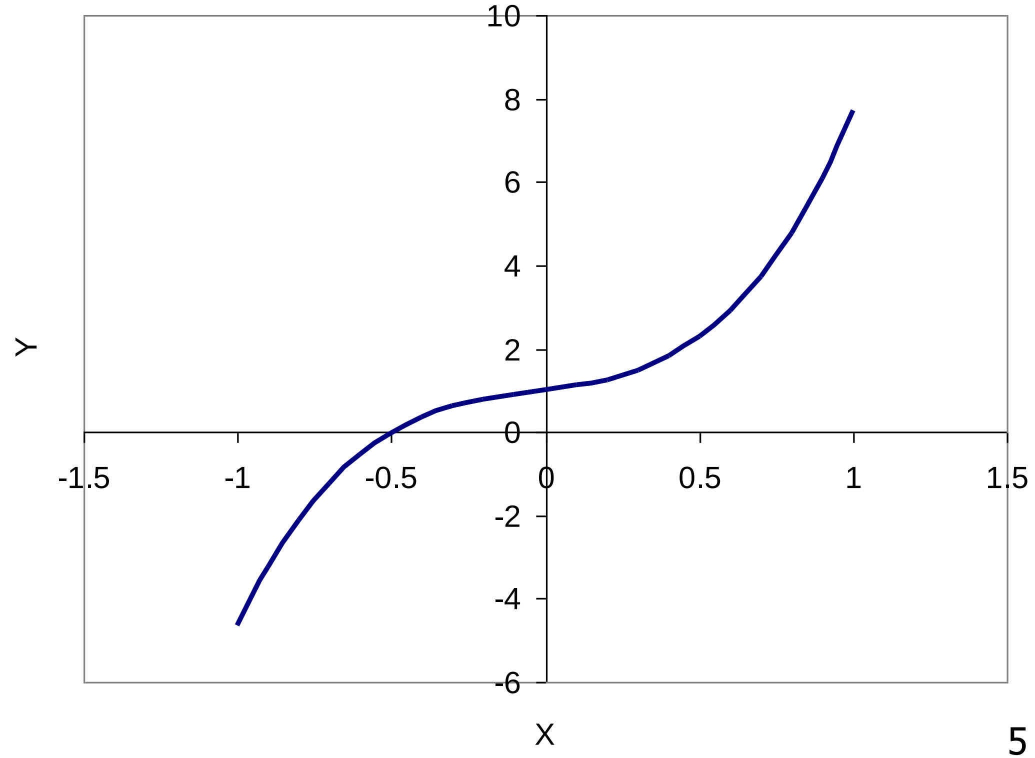
- We wish to find the value of x which satisfies the equation:

$$e^{x} + 5x^{3} = 0$$



Derivative-free methods

The Bisection Method-I

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1. Find x_1 and x_2 such that $f(x_1) < 0$ and $f(x_2) > 0$.
2. Set $x = (x_1 + x_2)/2$ and compute $f(x)$.
3. If $f(x) < 0$, replace x_1 by x.
4. If $f(x) > 0$, replace x_2 by x.
5. Repeat steps $2-4$ until $f(x) \approx 0$.

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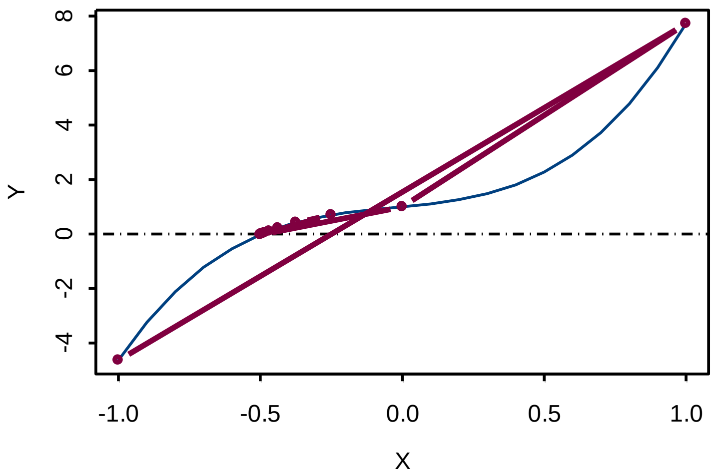
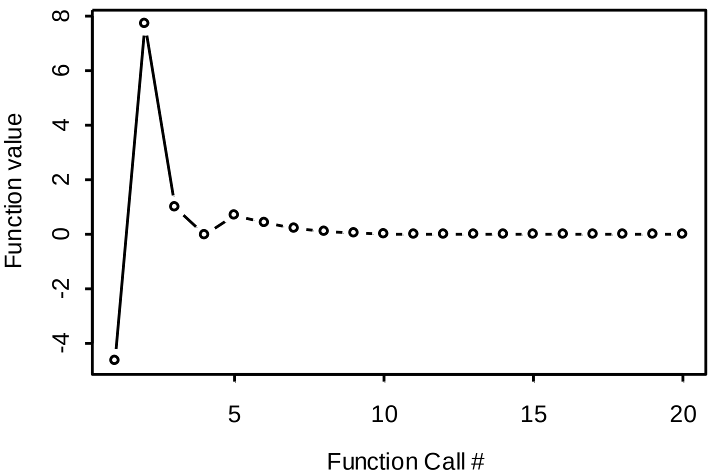
The Bisection Method-II





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The False Positive Method-I

1. Find x_1 and x_2 such that $f(x_1) < 0$ and $f(x_2) > 0$.
2. Set $x = x_1 + (x_2 - x_1) f(x_1)/(f(x_1) - f(x_2))$ and compute $f(x)$.
3. If $f(x) < 0$, replace x_1 by x.
4. If $f(x) > 0$, replace x_2 by x.
5. Repeat steps $2-4$ until $f(x) \approx 0$.

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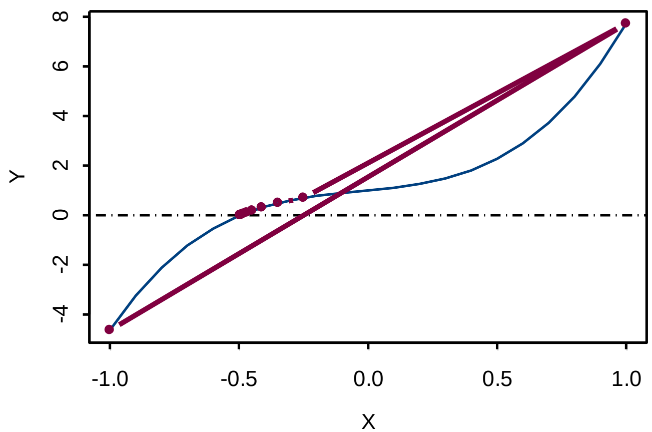
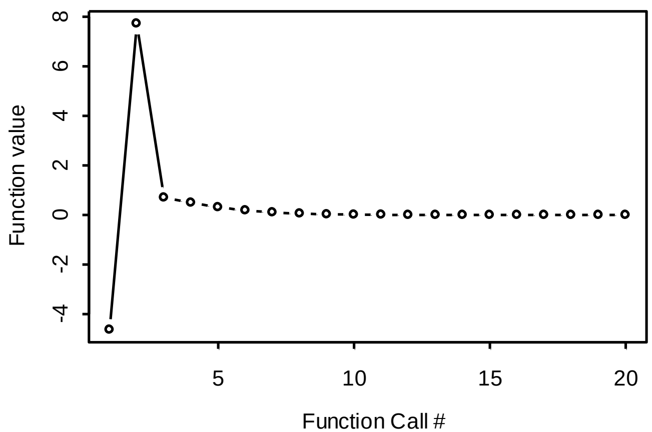
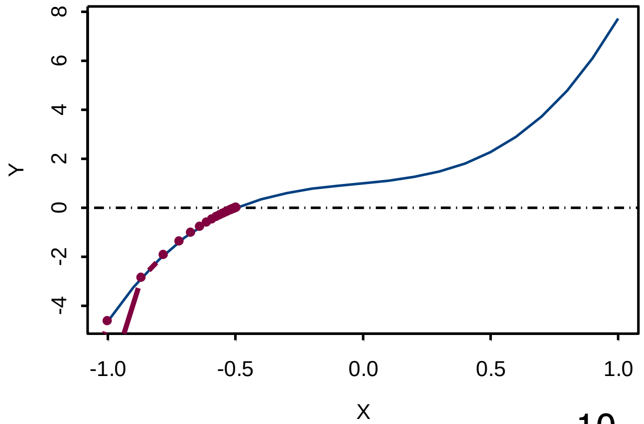
The False Positive Method-II





The initial vectors need not bound the solution



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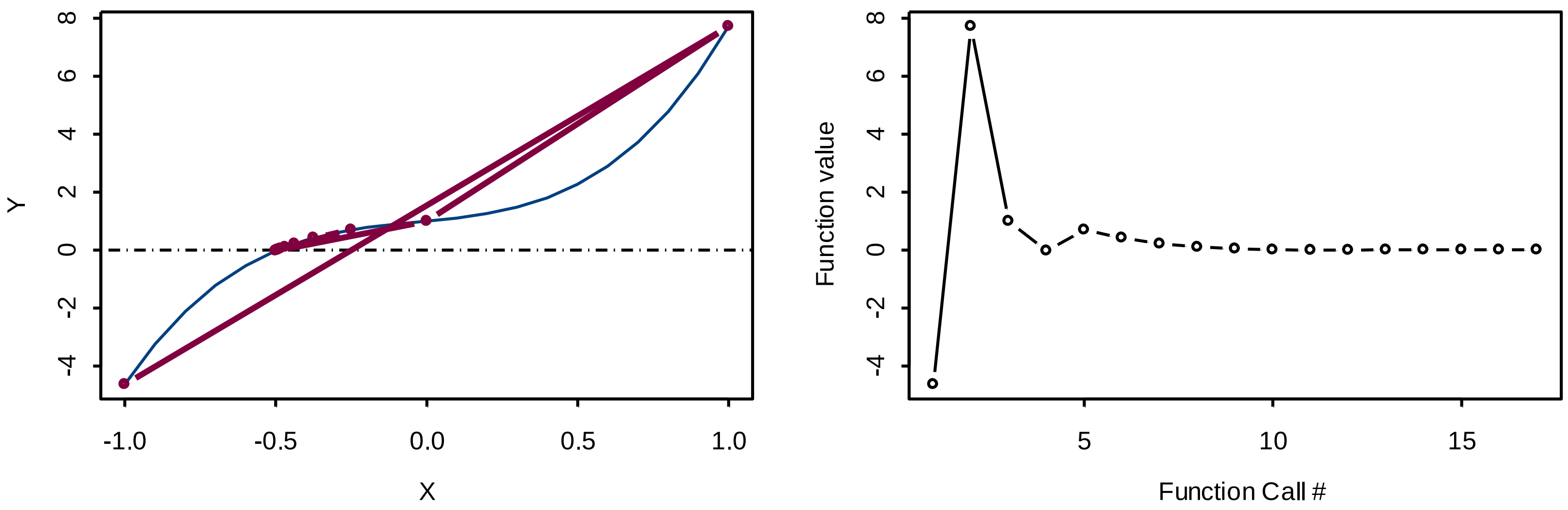
Brent's Method (The method of choice)

- The false positive method assumes approximate linear behavior between the root estimates; Brent's method assumes quadratic behavior, i.e.:

$$x = \frac{f(a)\,f(b)\,c}{[f(c)-f(a)][f(c)-f(b)]} + \frac{f(a)\,f(c)\,b}{[f(b)-f(a)][f(b)-f(c)]} + \frac{f(b)\,f(c)\,a}{[f(a)-f(b)][f(a)-f(c)]}$$

- The number of function calls can be much less than for the bisection and false positive methods (at the cost of a more complicated computer program).
- Brent's method underlies the R function *uniroot*.

Brent's Method



Derivative-based methods

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Newton's Method-I (Single-dimension case)

- Consider the Taylor series expansion of the function *f*:

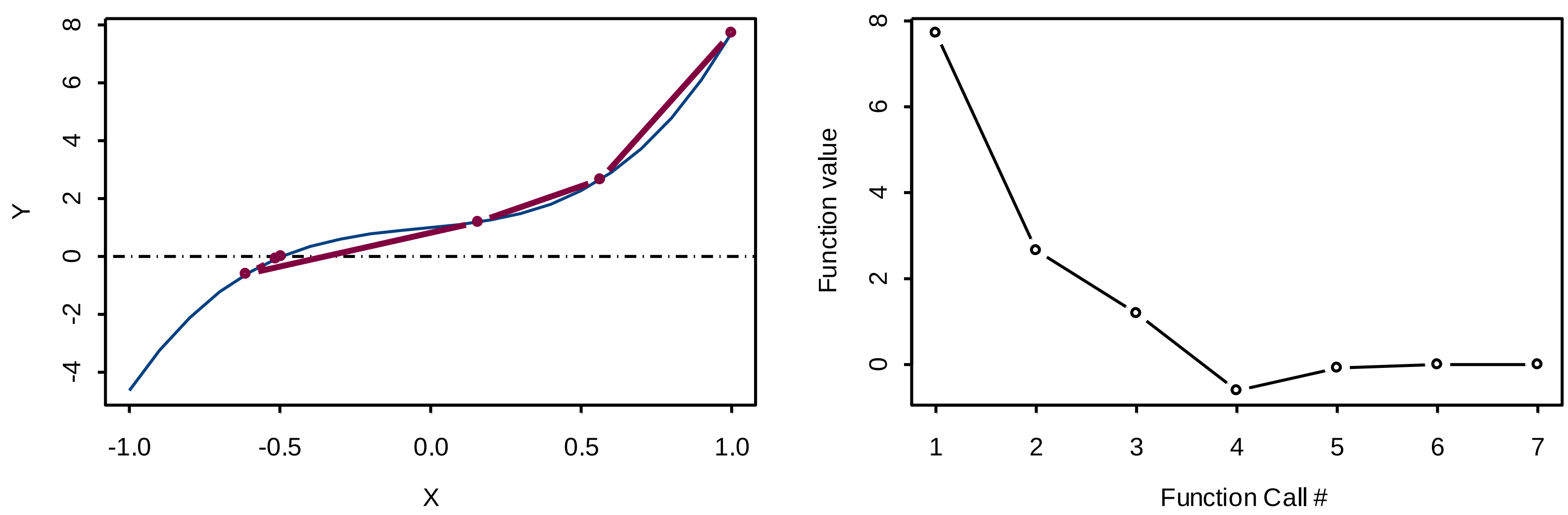
$$f(x+\delta) \approx f(x) + f'(x)\delta + f''(x)\frac{\delta^2}{2} + \ldots$$

- Now for "small" values of δ and for "well-behaved" functions we can ignore the 2nd and higher order terms. We wish to find $f(x+\delta)=0$ so:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Newton's method involves the iterative use of the above equation.

Newton’s Method-II



Note that Newton’s method may diverge rather than converge. This makes it of questionable value for general application.

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Ujevic et al's method

1. Set $x_1, \eta,$ and $k = 0$

2. Set $k = k + 1$

3. Set $z_k = x_k - f(x_k) / f'(x_k)$

3. If $|f(x_k)| > \eta$ then $\gamma_k = -f'(x_k)(z_k - x_k)/(f(z_k) - f(x_k))$

4. If $|f(x_k)| \le \eta$ then $\gamma_k = -1$

5. Set $x_{k+1} = x_k + (z_k - x_k) f(x_k)/[f(x_k) - \gamma_k f(z_k)]$

6. Repeat steps $2 - 5$ until $f(x) \approx 0.$

Multi-dimensional problems-I

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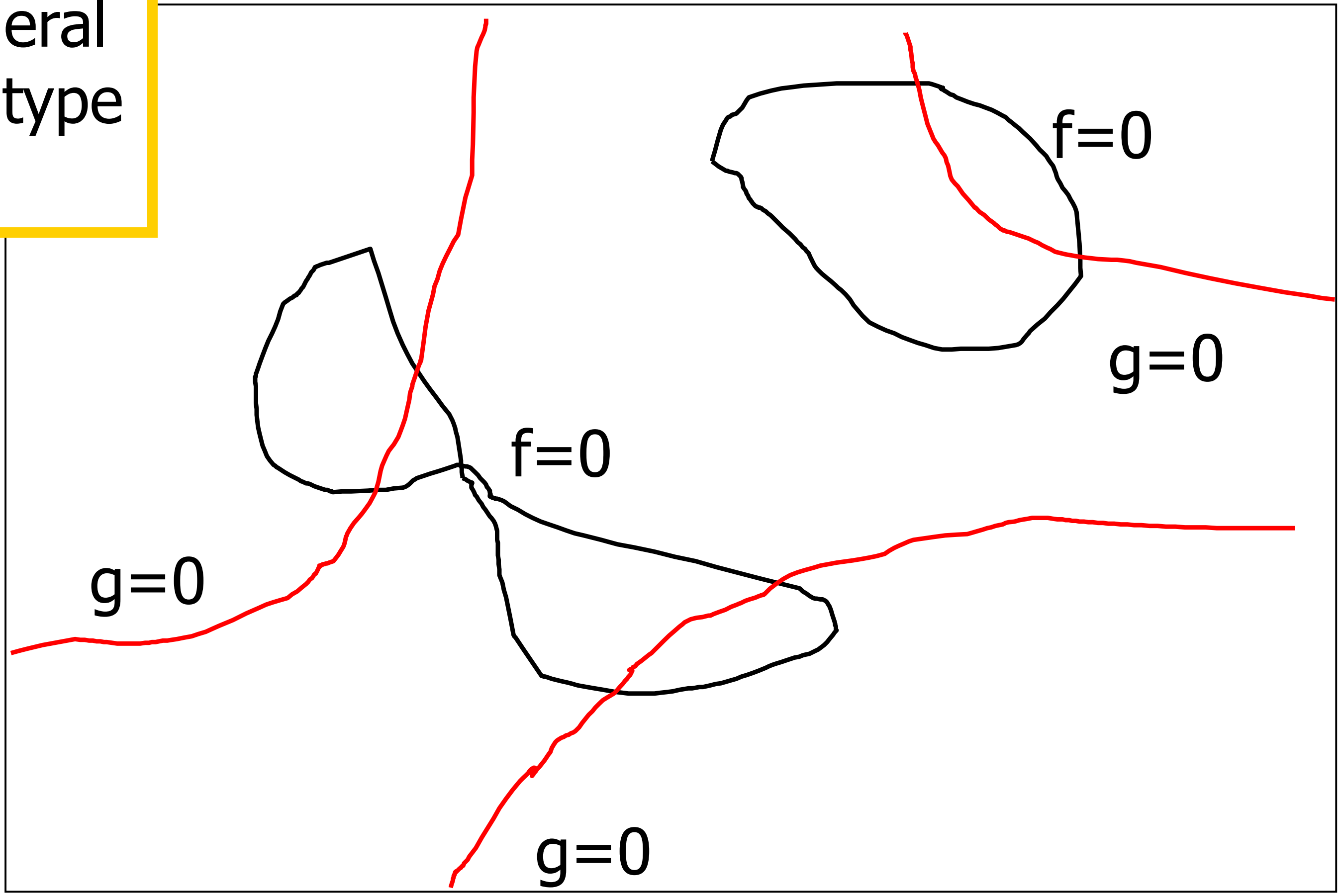
There is no general solution to this type of problem

$$f(x, y) = 0$$
$$g(x, y) = 0$$



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Multi-dimensional problems-II

- There are two “solutions” to the problem: find the vector $\underline{x}$ so that the following system of equations is satisfied:

$$f_i(\underline{x}) = 0; \quad i = 1, 2, 3 \ldots N$$

  - Use a multiple-dimension version of the Newton-Raphson method;
  - Treat the problem as a non-linear minimization problem.

Multi-dimensional problems-III

(the multi-dimensional Newton-Raphson method)

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- The Taylor series expansion about $\underline{x}$ is:

$$f_i(\underline{x}+\underline{\delta x}) \approx f_i(\underline{x}) + \sum_j \frac{\partial f_i}{\partial x_j}\delta x_j = 0$$

- This can be written as a series of linear equations:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & & \frac{\partial f_1}{\partial x_N} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & & \frac{\partial f_2}{\partial x_N} \\ & & & \\ \frac{\partial f_N}{\partial x_1} & \frac{\partial f_N}{\partial x_2} & & \frac{\partial f_N}{\partial x_N} \end{pmatrix} \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \\ \delta x_N \end{pmatrix} = \begin{pmatrix} -f_1(\underline{x}) \\ -f_2(\underline{x}) \\ \\ -f_N(\underline{x}) \end{pmatrix}$$

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Multi-dimensional problems-IV

(the multi-dimensional Newton-Raphson method)

- Given a current vector $\underline{x}^{old}$, it can be updated according to the equation:

$$
\begin{pmatrix} x_1^{new} \\ x_2^{new} \\ \\ x_N^{new} \end{pmatrix} = \begin{pmatrix} x_1^{old} \\ x_1^{old} \\ \\ x_N^{old} \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & & \frac{\partial f_1}{\partial x_N} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & & \frac{\partial f_2}{\partial x_N} \\ & & & \\ \frac{\partial f_N}{\partial x_1} & \frac{\partial f_N}{\partial x_2} & & \frac{\partial f_N}{\partial x_N} \end{pmatrix}^{-1} \begin{pmatrix} -f_1(\underline{x}) \\ -f_2(\underline{x}) \\ \\ -f_N(\underline{x}) \end{pmatrix}
$$

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Multi-dimensional problems-V
(use of optimization methods)

- Rather than attempting to solve the system of equations using, say, Newton's method, it is often more efficient to apply an optimization method to minimize the quantity:

$$SS = \sum_i f_i(\underline{x})^2$$