Linear Model

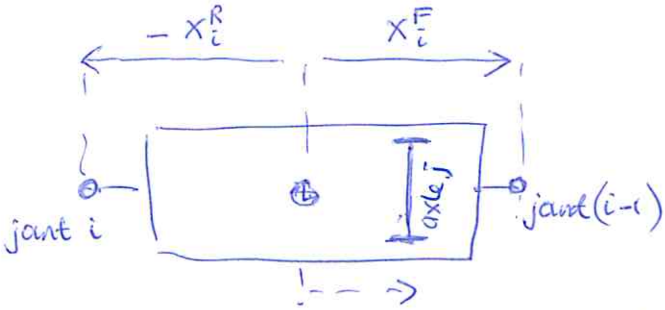
linear05.m

units $1, 2, \ldots n$ (label i)

axles $1, 2, \ldots, m_i$ (label j)

may have different number of axles on each unit, so $m = m_i$ axles.

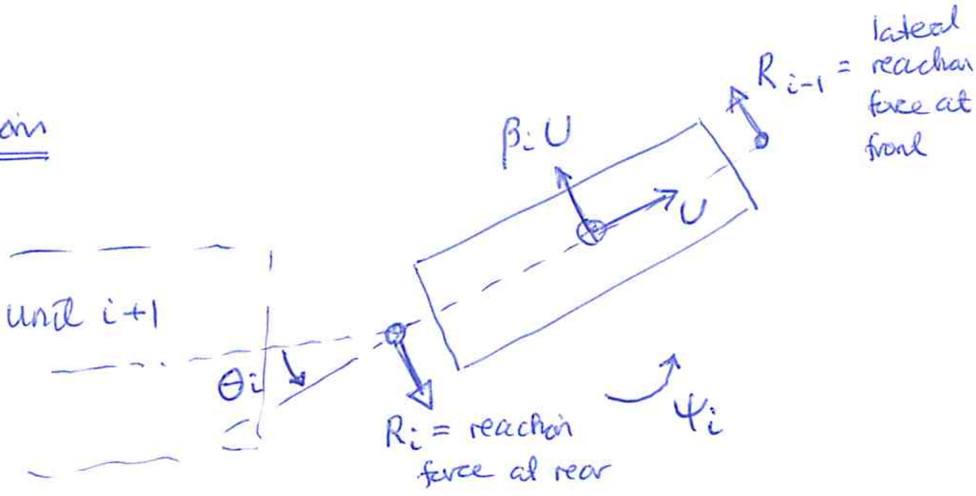
geometry



$x_i^{(F,R)}$ = x coord of joint, L_{ij} = x coord of axle j

joints labelled $1, \ldots, n-1$, corresponds to rear of unit i

kinematics and reaction



articulation angle

$\theta_i = \psi_i - \psi_{i+1}$

$r_i = \dot{\psi}_i$ (yaw rate)

assume small angles $\Rightarrow$ long velocity $\approx U$ on all units

states $\underline{X} = [\beta_1\ \beta_2 - \beta_n, r_1, r_2, \ldots, r_n, \theta_1, \theta_2, \ldots \theta_{n-1}]^T$

$3n-1$ states + $(n-1)$ reaction forces $[R_1, \ldots, R_{n-1}]^T$

axles $N = \sum_i m_i$

[**note** later define a reduced set of independent states

$\underline{X}_a = [\beta_1, r_1, \ldots r_n, \theta_1 \ldots \theta_{n-1}]^T$ with $\underline{x}_b = [\beta_2, \ldots, \beta_n]$ eliminated]

EOM $$M_i U(\dot{\beta}_i + r_i) = \sum Y_a + R_{i-1} - R_i \quad (1)$$

where Y_a = F_y at axle a of unit i due to tyre forces

$$I_{zi} \dot{r}_i = \sum L_a Y_a + X_i^F R_{i-1} - X_i^R R_i \quad (2)$$

(note: X_i^R is usually negative)

$$\dot{\theta}_i = r_i - r_{i+1} \quad (3)$$

CONSTRAINTS lateral velocity at rear joint of unit i

$$\beta_{i+1} U + X_{i+1}^F r_{i+1} = \beta_i U + X_i^R r_i + U\theta_i \quad (*)$$

differentiate $\Rightarrow$

$$U(\dot{\beta}_{i+1} - \dot{\beta}_i) + X_{i+1}^F \dot{r}_{i+1} - X_i^R \dot{r}_i - U\dot{\theta}_i = 0 \quad (4)$$

COUNT the number of variables and equations

$$\underline{X} = \begin{array}{c} \begin{bmatrix} \beta \\ r \\ \theta \end{bmatrix} \end{array} \begin{array}{l} n \\ n \\ n-1 \end{array} \Rightarrow n_x = 3n-1 \quad ; \; [\underline{R}] \; n_R = n-1$$

$$\text{total} = n_x + n_R = \underline{\underline{4n-2}}$$

$$\text{Eqns} \quad (1)+(2)+(3)+(4) = n + n + (n-1) + (n-1) = \underset{\equiv n_v}{\underline{\underline{4n-2}}} \quad \checkmark$$

INPUTS

$$Y_{ij} = \underbrace{-\frac{C_{\alpha ij}}{U}(\beta_i U + L_{ij} r_i)}_{= K_1 X} + \underbrace{C_{\alpha ij} \delta_{ij}}_{= K_2 \delta}$$

$n_v = 4n-2$ = no. of variable (and equations)

$n_x = 3n-1$ = no. of states

$n_R = n-1$ = no. of reaction forces (and no. of constraints)

$N = \Sigma m_i$ = no. of axles

$$\boxed{\underbrace{S}_{n_v \times n_v} \begin{pmatrix} \dot{X} \\ R \end{pmatrix} + \underbrace{P}_{n_v \times n_x} X = \underbrace{B_y}_{n_v \times N} Y}$$

equation sequence $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, PX gives all the state variables directly written in EOM, eg. $M_i U r_i$ in 1

$\Rightarrow$

$$\begin{pmatrix} \dot{X} \\ R \end{pmatrix} = S^{-1}(B_y Y - PX)$$

$$= S^{-1}(B_y K_1 - P) X + S^{-1} B_y K_2 \delta$$

$\rightarrow$ ~~the~~ upper n_x rows $\Rightarrow$

$$A = \left[S^{-1}(B_y K_1 - P) \right]_{\text{rows 1 to } n_x}$$

$$B = \left[S^{-1} B_y K_2 \right]_{\text{rows 1 to } n_x}$$

$$\Rightarrow \boxed{\dot{X} = AX + B\delta}$$

Eliminate Constraints

Full set of states was convenient for setting up the EOM

For simulation it is better to reduce the number of states as on P1. In matrix form

$\underline{x}_a = T_1 \underline{x}$ where / e.g. for $n=3$, $T_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & & & 0 \\ 0 & \underbrace{0 \quad 0}_{\text{miss out } \beta_2 \text{ and } \beta_3} & 0 & 1 & 0 & \cdots & 0 \\ & & & \cdots & & & \\ 0 & 0 & \cdots & & & 0 & 1 \end{bmatrix}$

$\underline{x}_b = T_2 \underline{x}$

T_2 gives the "missing rows" from T_1

$\underline{x}_b$ has $n_R = n-1$ els

$\underline{x}_a$ has $2n$ els.

$$\begin{bmatrix} \underline{x}_a \\ \underline{x}_b \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \underline{x} \equiv T \underline{x} \Rightarrow \underline{x} = T^{-1}\begin{pmatrix} \underline{x}_a \\ \underline{x}_b \end{pmatrix}$$

write $T^{-1} = \bar{T} = (\bar{T}_1 \,\vdots\, \bar{T}_2)$

$\Rightarrow x = \bar{T}_1 \underline{x}_a + \bar{T}_2 \underline{x}_b$.

state equations: $\dot{x} = Ax + Bu \qquad (\underline{u} = \underline{\delta})$

$\Rightarrow T_1 \dot{x} = T_1 A x + T_1 B u$

$\dot{x}_a = T_1 A \underbrace{x} + (T_1 B) u$

$x = \bar{T}_1 \underline{x}_a + \bar{T}_2 x_b$

$= (\bar{T}_1 + \bar{T}_2 E) \underline{x}_a$

$\equiv U \underline{x}_a$

$$\boxed{\dot{\underline{x}}_a = \underbrace{(T_1 A U)}_{A_m} \underline{x}_a + \underbrace{(T_1 B)}_{B_m} u}$$

CONSTRAINTS

$Q x = 0 \quad (n_R \text{ equs})$

$\underbrace{Q \bar{T}} \, \underbrace{T \underline{x}}_{= \begin{pmatrix} x_a \\ x_b \end{pmatrix}} = 0$

$\rightarrow Q\bar{T} = (Q\bar{T}_1 \,\vdots\, Q\bar{T}_2)$

$\therefore Q \bar{T}_1 \underline{x}_a + Q \bar{T}_2 \underline{x}_b = 0$

$\Rightarrow \underline{x}_b = \underbrace{-(Q\bar{T}_2)^{-1}(Q\bar{T}_1)}_{E} \underline{x}_a$

$\underline{x}_b = E \underline{x}_a$ to eliminate $\underline{x}_b$