Surface Fitting of LiDAR Point Clouds

Course ID: CoE 3SK3
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Background

Light Detection and Ranging (LiDAR) technology can be used to acquire detailed 3D point cloud data, capturing the spatial form of surfaces. The objective of surface fitting in the context of LiDAR point clouds is to model the 3D data in a more accessible and manageable form, allowing for enhanced visualization, analysis, and interpretation of the scanned environments. In this project, we will use the least square method to fit 3D data points to a surface for the volume measurement of bulk materials and for shape recognition.

Topic 1: automatic volume measurement

Measuring volumes of stored bulk materials is required in many applications, such as mining, construction, transportation and etc. Manual volume measurement of a large body is not only inefficient but also costly and inaccurate. In stark contrast, employing the LiDAR technique for automatic volume measurement not only ensures accuracy but also operates in a non-contact manner, enabling real-time volume monitoring.

As illustrated in Fig.1, LiDAR emits laser pulses to the object and records the reflected waves. As such, a set of points $\{\mathbf{p}_i = (x_i, y_i, z_i) | i = 1, 2, \cdots, N\}$ is sampled on the target surface. Given the captured LiDAR point clouds, a straightforward method for volume estimation is to sum up the captured height z_i ($z_i \geq 0$) at location (x_i, y_i). This naive approach is effective only when the LiDAR scanning is sufficiently dense and noise-free. However, current LiDAR sensors can only provide sparse and noisy observations. To deal with the data inadequacy, one can fit a continuous surface model to the degraded and incomplete observation set $\{\mathbf{p}_i = (x_i, y_i, z_i) | i = 1, 2, \cdots, N\}$. The surface of stored bulk materials can be explicitly represented by a function $z = f(x, y)$ that maps the coordinate (x, y) to the corresponding height z. Once the fitted function $z = f(x, y)$ is obtained, the volume can be estimated on-the-fly by evaluating the integral $\int_{x_{\min} \leq x \leq x_{\max}} \int_{y_{\min} \leq y \leq y_{\max}} f(x, y) \mathrm{d}x \mathrm{d}y$.

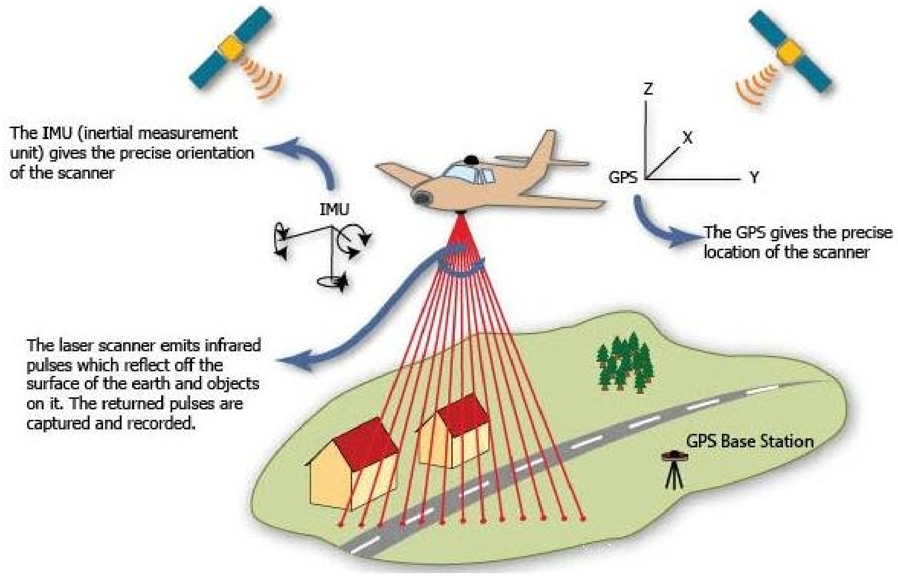

Figure 1: Illustration for the working principle of LiDAR

Case 1.1: quadratic model

For simplicity, we start with simple quadratical surfaces of the general formulation:

$$z = f(x, y) = Ax^2 + By^2 + Cxy + Dx + Ey + F. \tag{1}$$

The objective is to estimate the unknown parameters A, B, C, D, E and F using the observed points in the provided file 'quadratic_surface.mat'. The fitting error is

$$\mathcal{L} = \frac{1}{N}\sum_{i=1}^{N}(Ax_i^2 + By_i^2 + Cx_iy_i + Dx_i + Ey_i + F - z_i)^2 \tag{2}$$

which is minimized when $\frac{\partial \mathcal{L}}{\partial A} = 0$, $\frac{\partial \mathcal{L}}{\partial B} = 0$, $\frac{\partial \mathcal{L}}{\partial C} = 0$, $\frac{\partial \mathcal{L}}{\partial D} = 0$, $\frac{\partial \mathcal{L}}{\partial E} = 0$ and $\frac{\partial \mathcal{L}}{\partial F} = 0$. Refer to the lecture notes[1], use linear regression to estimate the unknown parameters A, B, C, D, E and F.

Task 1.1.1 Derive six linear equations involving the unknown parameters A, B, C, D, E and F based on the error minimization condition. Then, reformulate these linear equations into the form of Eq.(3). Namely, derive the matrix $\mathbf{X} \in \mathbb{R}^{6\times 6}$ and vector $\mathbf{Y} \in \mathbb{R}^6$ in Eq.(3).

$$\begin{bmatrix} X_{00} & X_{01} & X_{02} & X_{03} & X_{04} & X_{05} \\ X_{10} & X_{11} & X_{12} & X_{13} & X_{14} & X_{15} \\ X_{20} & X_{21} & X_{22} & X_{23} & X_{24} & X_{25} \\ X_{30} & X_{31} & X_{32} & X_{33} & X_{34} & X_{35} \\ X_{40} & X_{41} & X_{42} & X_{43} & X_{44} & X_{45} \\ X_{50} & X_{51} & X_{52} & X_{53} & X_{54} & X_{55} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{bmatrix} \tag{3}$$

Task 1.1.2 Estimate the unknown parameters A, B, C, D, E and F by solving Eq.(3).

Task 1.1.3 Report the volume bounded by the estimated surface and the ground plane ($z = 0$) by evaluating the integral $\int_{x_{\min}\le x\le x_{\max}}\int_{y_{\min}\le y\le y_{\max}} f(x, y)\mathrm{d}x\mathrm{d}y$.

Case 1.2: cubic model

The quadratic surface model assumes smooth and simple shapes. A bivariate cubic function can model more complex shapes of a bulk. In this context, the sensed surface is approximated by the following cubic function,

$$z = f(x, y) = Ax^3 + By^3 + Cx^2y + Dxy^2 + Ex^2 + Fy^2 + Gxy + Hx + Iy + J. \tag{4}$$

The objective is to estimate the unknown parameters $A, B, C, D, E, F, G, H, I$ and J with the observed points in the provided file 'cubic_surface.mat'. The fitting error is

$$\mathcal{L} = \frac{1}{N}\sum_{i=1}^{N}(Ax_i^3 + By_i^3 + Cx_i^2y_i + Dx_iy_i^2 + Ex_i^2 + Fy_i^2 + Gx_iy_i + Hx_i + Iy_i + J - z_i)^2 \tag{5}$$

which is minimized when $\frac{\partial \mathcal{L}}{\partial A} = 0$, $\frac{\partial \mathcal{L}}{\partial B} = 0$, $\frac{\partial \mathcal{L}}{\partial C} = 0$, $\frac{\partial \mathcal{L}}{\partial D} = 0$, $\frac{\partial \mathcal{L}}{\partial E} = 0$, $\frac{\partial \mathcal{L}}{\partial F} = 0$, $\frac{\partial \mathcal{L}}{\partial G} = 0$, $\frac{\partial \mathcal{L}}{\partial H} = 0$, $\frac{\partial \mathcal{L}}{\partial I} = 0$ and $\frac{\partial \mathcal{L}}{\partial J} = 0$.

Task 1.2.1 Derive ten linear equations involving the unknown parameters $A, B, C, D, E, F, G, H, I$ and J based on the error minimization condition. Then, reformulate these linear equations into the form of Eq.(6). Namely, derive the matrix $\mathbf{X} \in \mathbb{R}^{10\times 10}$ and vector $\mathbf{Y} \in \mathbb{R}^{10}$ in Eq.(6).

$$\begin{bmatrix} X_{00} & X_{01} & X_{02} & X_{03} & X_{04} & X_{05} & X_{06} & X_{07} & X_{08} & X_{09} \\ X_{10} & X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} & X_{17} & X_{18} & X_{19} \\ X_{20} & X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} & X_{27} & X_{28} & X_{29} \\ X_{30} & X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} & X_{37} & X_{38} & X_{39} \\ X_{40} & X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} & X_{17} & X_{18} & X_{19} \\ X_{50} & X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} & X_{17} & X_{18} & X_{19} \\ X_{60} & X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} & X_{67} & X_{68} & X_{69} \\ X_{70} & X_{71} & X_{72} & X_{73} & X_{74} & X_{75} & X_{76} & X_{77} & X_{78} & X_{79} \\ X_{80} & X_{81} & X_{82} & X_{83} & X_{84} & X_{85} & X_{86} & X_{87} & X_{88} & X_{89} \\ X_{90} & X_{91} & X_{92} & X_{93} & X_{94} & X_{95} & X_{96} & X_{97} & X_{98} & X_{99} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \\ I \\ J \end{bmatrix} = \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \\ Y_7 \\ Y_8 \\ Y_9 \end{bmatrix} \tag{6}$$

[1] Multiple Linear Regression, page 10-11

Task 1.2.2 Estimate the unknown parameters $A, B, C, D, E, F, G, H, I$ and J by solving Eq.(6).

Task 1.2.3 Report the volume bounded by the estimated surface and the ground plane ($z = 0$) by evaluating the integral $\int_{x_{\min} \le x \le x_{\max}} \int_{y_{\min} \le y \le y_{\max}} f(x, y) \mathrm{d}x \mathrm{d}y$.

Topic 2: automatic quadratic surface recognition

The explicit representation $z = f(x, y)$ introduced in Topic 1 has a significant limitation in efficiently representing arbitrary surfaces, particularly when a location (x, y) corresponds to multiple height z. In contrast, implicit representation can easily represent arbitrary topological structures. In this method, the surface $\mathcal{S}$ is implicitly defined by the zero-level set of a function $f(x, y, z)$, denoted as $\mathcal{S} = \{(x, y, z) | f(x, y, z) = 0\}$. For simplicity, here we constrain the form of $f(x, y, z)$ to a quadratic function: $f(x, y, z) = Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J$. The error on fitting an implicit representation is

$$\mathcal{L} = \frac{1}{N} \sum_{i=1}^{N} (Ax_i^2 + By_i^2 + Cz_i^2 + Dx_iy_i + Ey_iz_i + Fx_iz_i + Gx_i + Hy_i + Iz_i + J - 0)^2 \tag{7}$$

which is minimized when $\frac{\partial \mathcal{L}}{\partial A} = 0$, $\frac{\partial \mathcal{L}}{\partial B} = 0$, $\frac{\partial \mathcal{L}}{\partial C} = 0$, $\frac{\partial \mathcal{L}}{\partial D} = 0$, $\frac{\partial \mathcal{L}}{\partial E} = 0$, $\frac{\partial \mathcal{L}}{\partial F} = 0$, $\frac{\partial \mathcal{L}}{\partial G} = 0$, $\frac{\partial \mathcal{L}}{\partial H} = 0$, $\frac{\partial \mathcal{L}}{\partial I} = 0$ and $\frac{\partial \mathcal{L}}{\partial J} = 0$.

Task 2.1 Derive ten linear equations involving the unknown parameters $A, B, C, D, E, F, G, H, I$ and J based on the error minimization condition. Then, reformulate these linear equations into the form of Eq.(8). Namely, derive the matrix $\mathbf{X} \in \mathbb{R}^{10 \times 10}$ in Eq.(8).

$$\begin{bmatrix} X_{00} & X_{01} & X_{02} & X_{03} & X_{04} & X_{05} & X_{06} & X_{07} & X_{08} & X_{09} \\ X_{10} & X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} & X_{17} & X_{18} & X_{19} \\ X_{20} & X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} & X_{27} & X_{28} & X_{29} \\ X_{30} & X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} & X_{37} & X_{38} & X_{39} \\ X_{40} & X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} & X_{17} & X_{18} & X_{19} \\ X_{50} & X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} & X_{17} & X_{18} & X_{19} \\ X_{60} & X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} & X_{67} & X_{68} & X_{69} \\ X_{70} & X_{71} & X_{72} & X_{73} & X_{74} & X_{75} & X_{76} & X_{77} & X_{78} & X_{79} \\ X_{80} & X_{81} & X_{82} & X_{83} & X_{84} & X_{85} & X_{86} & X_{87} & X_{88} & X_{89} \\ X_{90} & X_{91} & X_{92} & X_{93} & X_{94} & X_{95} & X_{96} & X_{97} & X_{98} & X_{99} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \\ I \\ J \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

- **Hint**: directly solving Eq.(8) will result in a solution whose elements are all zero. To address this problem, we can use the eigenvector of $\mathbf{X}$ corresponding to the smallest eigenvalue of $\mathbf{X}$ as an estimate of $[A, B, C, D, E, F, G, H, I, J]^T$. The relationship between this eigenvector $\mathbf{v}$ and the corresponding ground truth parameters $\theta = [A, B, C, D, E, F, G, H, I, J]^T$ is $\mathbf{v} = \alpha\theta + \mathbf{n}$ where α is a scale factor and $\mathbf{n}$ denotes the model noise.

Task 2.2

- For the observed points in 'implicit_1.mat', estimate their corresponding parameters $A, B, C, D, E, F, G, H, I$ and J. Then, based on the value and signs of these parameters, classify the type of underlying surface.
- For the observed points in file 'implicit_2.mat', estimate their corresponding parameters $A, B, C, D, E, F, G, H, I$ and J. Then, based on the value and signs of these parameters, classify the type of underlying surface.
- **Note**: Refer to Fig.2, recognize the surface type by analyzing the values and signs of surface parameters. You must not draw a conclusion via eyeballing.

Requirements

- Independent work.
- Write a report and include the derivations and experimental results.
- Use Matlab to write your program. Python is also acceptable.
- Submit your code together with the written report.

- Do not use any built-in functions in Matlab. Use only addition, subtraction, multiplication, division, square root, power, matrix inverse, matrix transpose, summation, mean, min, max, integral2, eig, load(), size().

Appendix

- We provide three sets of observations with known ground truth parameters[2,3,4], which can be used to check whether your implementation of least square estimation is correct.
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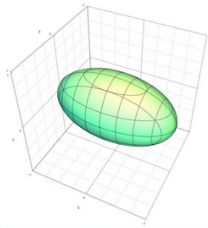
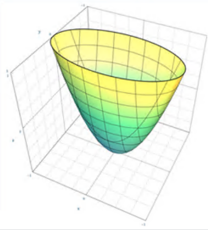
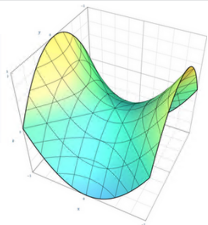
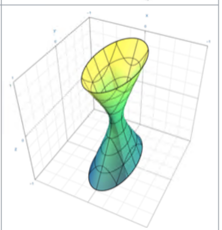
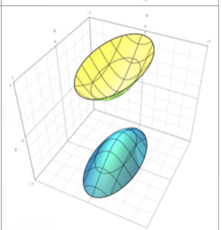
Non-degenerate real quadric surfaces		
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	
Elliptic paraboloid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0$	
Hyperbolic paraboloid	$\frac{x^2}{a^2} - \frac{y^2}{b^2} - z = 0$	
Hyperboloid of one sheet or Hyperbolic hyperboloid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	
Hyperboloid of two sheets or Elliptic hyperboloid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$	

Figure 2: The representation of some typical quadratic surfaces.

[2] quadratic_surface_self_check.mat
[3] cubic_surface_self_check.mat
[4] implicit_surface_self_check.mat