Name________________________________

Student Number________________________

Computer Engineering 3SK3

DAY CLASS **Dr. D. Zhao**
DURATION OF EXAMINATION: 3 Hours
MCMASTER UNIVERSITY FINAL EXAMINATION **April, 2012**

THIS EXAMINATION PAPER INCLUDES **2** PAGES AND **7** QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

Use of Casio FX-991 calculator only is allowed.
No other aids are allowed.

Answer **all** questions.
Show all your work. Partial credit will be given.

1. Consider the integration of the function $f(x) = 0.5+x^2+2x^3+5x^4$ over the interval $x = -1$ to $x = 1$.

(a) (3 marks) Use the Simpson's 1/3 rule with a step size $h = 0.5$ to approximate the above integral.

(b) (2 marks) Find the truncation error in (a).

(c) (3 marks) If the **relative** error (in absolute value) is to be less than 10^{-5}, what is the minimum number of segments when using the Simpson's 1/3 rule to estimate the integral?

2. Consider the differential equation $dy/dx = 2\text{-}3x\text{-}4y$ with the initial condition $y(0) = 1$.

(a) (3 marks) Find the analytical solution to the above equation.

(b) (3 marks) Use the Euler's method with a step size of 0.2 to estimate $y(0.4)$.

(c) (3 marks) Find the truncation error in (b) and briefly explain reasons that cause the error.

3. Consider the function $f(x) = 3\cos(3x-2)$.

(a) (3 marks) Use the Newton's method to find the local optimum with relative error less than 1%, starting from $x_0=1$.

(b) (2 marks) Is the solution in (a) a local minimum or local maximum, and why?

(c) (3 marks) If the golden section search method is used with initial points $x_l=0$ and $x_u=1$, estimate the number of iterations required to ensure an **absolute** error less than 0.0001.

Continued on page 2

4. Consider fitting the function $y = \alpha \sin(2x)$ to the data (x_i, y_i), $i = 1, 2, \dots, n$.

(a) (4 marks) Using the Least-Squares criterion, derive an expression to find α from (x_i, y_i)'s.

(b) (3 marks) Given the following data, find the value of α using the expression derived in (a).

i	1	2	3
x_i	0	$\pi/3$	$2\pi/3$
y_i	0.02	2.61	-2.50

(c) (2 marks) Given that y =2.85 when $x = \pi/5$, find the approximation error in the above evaluation.

5. Given the matrix

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 3 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix}.$$

The singular value decomposition of the matrix can be written as $A = USV^T$, where U, S and V are given by

$$U = \begin{bmatrix} u_{11} & 0.6109 & 0.4520 \\ -0.7283 & -0.3309 & -0.6001 \\ u_{31} & -0.7193 & 0.6600 \end{bmatrix}, \quad S = \begin{bmatrix} s_{11} & 0 & 0 \\ 0 & 2.5537 & 0 \\ 0 & 0 & 0.0667 \end{bmatrix}, \quad V = \begin{bmatrix} -0.8149 & 0.5682 & 0.1142 \\ -0.1610 & -0.4112 & 0.8972 \\ -0.5568 & -0.7128 & -0.4266 \end{bmatrix},$$

and V^T is the transpose of V.

(a) (3 marks) Find u_{11} and u_{31} in matrix U.

(b) (2 marks) Find s_{11} in matrix S.

6. (5 marks) Use Gauss elimination to find solution to the system $AX=b$ where

$$A = \begin{bmatrix} 0 & 1/2 & 2 \\ 2 & -3 & 3/2 \\ 1/2 & 2 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1/2 \\ 31/4 \\ -1/2 \end{bmatrix}.$$

7. Consider a 12-bit computer that has 1 sign bit for s, 5 bits for c (c is between 0 and 31, and $c = e + 15$), and 6 bits for f. In terms of s, e, and f, the base 10 numbers are given by $x = (-1)^s 2^e (1 + f)$.

(a) (3 marks) What is the machine precision on this computer? Consider both chopping and rounding.

(b) (3 marks) What are the smallest and largest **positive** numbers that can be represented accurately on this computer? Do not consider any special reservations.

End of questions.

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(b) (2 marks) Find the truncation error in (a).

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a)

x_0	x_1	x_2	x_3	x_4
-1	-0.5	0	0.5	1

$h = \frac{b-a}{n}$

$0.5 = \frac{1-(-1)}{n}, n = 4$

$$I = (b-a)\frac{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n}$$

$I = (1-(-1)) \times \frac{f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)}{3 \times 4}$

$f(x_0) = 4.5$ $\quad f(x_2) = 0.5$ $\quad f(x_4) = 8.5$

$f(x_1) = 0.8125$ $\quad f(x_3) = 1.3125$

$I = 2 \times \frac{4.5 + 4 \times 0.8125 + 2 \times 0.5 + 4 \times 1.3125 + 8.5}{12}$

$= 3.75$

$$E_t = -\frac{1}{90}h^5 f^{(4)}(\xi)$$

b) $f(x) = 0.5+x^2+2x^3+5x^4$

$f'(x) = 2x + 6x^2 + 20x^3$

$f''(x) = 2 + 12x + 60x^2$

$f^3(x) = 12 + 120x$

$f^4(x) = 120$

Truncation error

$E_t = 2 \cdot -\frac{1}{90} h^5 f^{(4)}(\xi)$

$= -2 \times \frac{1}{90} \times 0.5^5 \times 120$

$= -0.0833$

c) $E_t < 10^{-5}$

$E_t = \frac{n}{2} \cdot -\frac{1}{90} h^5 f^4(\xi) < 10^{-5}$

$h = \frac{1-(-1)}{n}, n = \frac{2}{h} \rightarrow \frac{n}{2} = \frac{2}{h} \times \frac{1}{2} = \frac{1}{h}$

$E_t = -\frac{1}{90} \times \frac{1}{h} \cdot h^5 \cdot 120 = -\frac{4}{3}h^4$

relative error : $\frac{E_t}{\text{true value}} = \frac{-\frac{4}{3}h^4}{3.75 - 0.0833} < 10^{-5}$

true − estim = E_t

$h < 0.0724$

$n = \frac{2}{h}, \; n > 27.62$

$n = 28$

Simpson's 1/3规则是一种数值积分方法，用于近似计算定积分。在使用这个规则时，我们将积分区间分为若干等分，每个小区间都用一个二次多项式来近似原函数。这样做的目的是为了用简单的多项式积分来近似复杂函数的积分。

当你提到“segment等于4”时，这意味着整个积分区间被分成了4个小段。但是，Simpson's 1/3规则是每次处理两个小段。这是因为在每两个小段上，我们用一个二次多项式来近似原函数，这需要三个点（每个小段的两端点和中间点）。

所以，如果整个区间被分为4个小段，实际上我们需要两次应用Simpson's 1/3规则。第一次应用处理前两个小段，第二次应用处理后两个小段。每次应用都会产生一个近似积分值，这两个值相加就给出了整个区间的近似积分值。

至于“truncation error”（截断误差），它指的是因为使用多项式近似而产生的误差。在Simpson's 1/3规则中，误差与小段数（也就是分割数量）有关。如果分割得更细，误差通常会减少。这个误差的具体计算需要考虑函数的高阶导数及分割的细致程度。在实际应用中，理解这个误差的来源和大小对于评估积分结果的准确性非常重要。

2. Consider the differential equation $dy/dx = 2\text{-}3x\text{-}4y$ with the initial condition $y(0) = 1$.

(a) (3 marks) Find the analytical solution to the above equation.

(b) (3 marks) Use the Euler's method with a step size of 0.2 to estimate $y(0.4)$.

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a)

$f(x) = 3\cos(3x-2)$.

$f'(x) = -9\sin(3x-2)$

$$x_{k+1} = x_k + p = x_k - \frac{F'(x_k)}{F''(x_k)}$$

$f''(x) = -27\cos(3x-2)$

a) $x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)} = 1 - \frac{-9\sin(3x-2)}{-27\cos(3x-2)} = 1 - \frac{\sin(3x-2)}{3\cos(3x-2)} = 0.4809$ $\quad \epsilon = \frac{|x_1 - x_0|}{x_1} = 107.96\%$

$x_2 = x_1 - \frac{f'(x_1)}{f''(x_1)} = 0.4809 - \frac{\sin(3x-2)}{3\cos(3x-2)} = 0.6886$ $\quad \epsilon = \frac{|x_2 - x_1|}{x_2} = 30.166\%$

$x_3 = x_2 - \frac{f'(x_2)}{f''(x_2)} = 0.6886 - \frac{\sin(3x-2)}{3\cos(3x-2)} = 0.6666$ $\quad \epsilon = \frac{|x_3 - x_2|}{x_3} = 3.295\%$

$x_4 = x_3 - \frac{f'(x_3)}{f''(x_3)} = 0.6666 - \frac{\sin(3x-2)}{3\cos(3x-2)} = 0.6667$ $\quad \epsilon = \frac{|x_4 - x_3|}{x_4} = 0.01\%$

b) $f''(x) = -27\cos(3x-2) = -27\cos(3\times 0.6667-2) = -27 < 0$, Local maximum

C) $|e| > (x_u - x_L) \cdot 0.618^n$

$0.0001 > 1 \cdot 0.618^n$

$n > \frac{\log 0.0001}{\log 0.618} = 19.14$

$n > 20$

4. Consider fitting the function $y = \alpha \sin(2x)$ to the data (x_i, y_i), $i = 1, 2, \dots, n$.

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(b) (3 marks) Given the following data, find the value of α using the expression derived in (a).

i	1	2	3
x_i	0	$\pi/3$	$2\pi/3$
y_i	0.02	2.61	-2.50

(c) (2 marks) Given that $y = 2.85$ when $x = \pi/5$, find the approximation error in the above evaluation.

a) $a\sin 2x - y = 0$

$$\min S_r = \min \sum_{i=1}^{n} e_i^2 = \min \sum_{i=1}^{n} (a\sin(2x_i) - y_i)^2$$

$$\frac{\partial S_r}{\partial a} = 2\sum_{i=1}^{n} (a\sin 2x_i - y_i)\sin 2x_i = 0$$

$$\sum_{i=1}^{n} a(\sin 2x_i)^2 - \sum_{i=1}^{n} \sin 2x_i \cdot y_i = 0$$

$$a = \frac{\sum_{i=1}^{n} \sin 2x_i \cdot y_i}{\sum_{i=1}^{n} \sin^2 2x_i}$$

b)

$$a = \frac{\sum_{i=1}^{n} \sin 2x_i \cdot y_i}{\sum_{i=1}^{n} (\sin 2x_i)^2} = \frac{4.425}{1.5} = 2.95$$

c) based on b)

$$a = \frac{\sin 2x_i \cdot y_i}{(\sin 2x_i)^2}, \quad x = \frac{\pi}{5}$$

$$2.95 = \frac{\sin 2\cdot\frac{\pi}{5} \cdot y}{[\sin(2\times\frac{\pi}{5})]^2}, \quad y = 2.806$$

$$E_a = \frac{|cal - est.|}{est} = \frac{|2.806 - 2.85|}{2.85} = 1.56\%$$

5. Given the matrix

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The singular value decomposition of the matrix can be written as $A = USV^T$, where U, S and V are given by

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and V^T is the transpose of V.

(a) (3 marks) Find u_{11} and u_{31} in matrix U.

(b) (2 marks) Find s_{11} in matrix S.

$A = USV^T$

$u_{11}s_{11}$ 1.56 0.03

$-0.7283 V_{11}$ −0.84 −0.04

$s_{11}V_{31}$ −1.837 0.044

$-4.78 u_{11} + 0.8898 = 4$

$$A^T \begin{bmatrix} 4 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \cdot A \begin{bmatrix} 4 & 0 & 1 \\ 3 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 17 & 15 & 2 \\ 15 & 19 & 7 \\ 2 & 7 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 17-\lambda & 15 & 2 \\ 15 & 19-\lambda & 7 \\ 2 & 7 & 5-\lambda \end{bmatrix}$$

$\lambda_1 = 34.47$ $\quad \sqrt{34.47} = 5.871 \rightarrow s_{11}$

$\lambda_2 = 4.448\times 10^{-3}$ $\quad \sqrt{\ } = 0.0667$

$\lambda_3 = 6.521$ $\quad \sqrt{\ } = 2.5536$

$17-\lambda\,[(19-\lambda)(5-\lambda) - 7\times 7]$ $\quad (17-\lambda)(\lambda^2 - 24\lambda + 46)$

$-15\,(15\times(5-\lambda) - 7\times 2)$ $\quad -15(61 - 15\lambda)$

$+2\,(15\times 7 - (19-\lambda)2)$ $\quad +2(67 + 2\lambda)$

$17\lambda^2 - 408\lambda + 782 - \lambda^3 + 24\lambda^2 - 46\lambda$

$-915 + 225\lambda + 134 + 4\lambda$

$-\lambda^3 + 41\lambda^2 - 225\lambda + 1 = 0$

6. (5 marks) Use Gauss elimination to find solution to the system $AX=b$ where

$$A=\begin{bmatrix} 0 & 1/2 & 2 \\ 2 & -3 & 3/2 \\ 1/2 & 2 & 1 \end{bmatrix} \text{ and } b=\begin{bmatrix} 1/2 \\ 31/4 \\ -1/2 \end{bmatrix}.$$

7. Consider a 12-bit computer that has 1 sign bit for s, 5 bits for c (c is between 0 and 31, and $c = e + 15$), and 6 bits for f. In terms of s, e, and f, the base 10 numbers are given by $x=(-1)^s 2^e (1+f)$.

(a) (3 marks) What is the machine precision on this computer? Consider both chopping and rounding.

(b) (3 marks) What are the smallest and largest **positive** numbers that can be represented accurately on this computer? Do not consider any special reservations.

6)

$$\begin{cases} \frac{1}{2}x_2 + 2x_3 = \frac{1}{2} & 1) \\ 2x_1 - 3x_2 + \frac{3}{2}x_3 = \frac{31}{4} & 2) \\ \frac{1}{2}x_1 + 2x_2 + x_3 = -\frac{1}{2} & 3) \end{cases}$$

pivoting, change 1) & 2)

$$\begin{cases} 2x_1 - 3x_2 + \frac{3}{2}x_3 = \frac{31}{4} & 1) \\ \frac{1}{2}x_2 + 2x_3 = \frac{1}{2} & 2) \\ \frac{1}{2}x_1 + 2x_2 + x_3 = -\frac{1}{2} & 3) \end{cases}$$

$3) - \frac{1}{4}\cdot 1)$

$$\begin{aligned} 2x_1 - 3x_2 + \tfrac{3}{2}x_3 &= \tfrac{31}{4} & (1) \\ \tfrac{1}{2}x_2 + 2x_3 &= \tfrac{1}{2} & (2) \\ 2.75x_2 + \tfrac{5}{8}x_3 &= -2.4375 & (3') \end{aligned}$$

$3' - 5.5\cdot(2)$

$$\begin{aligned} 2x_1 - 3x_2 + \tfrac{3}{2}x_3 &= \tfrac{31}{4} & (1) \\ \tfrac{1}{2}x_2 + 2x_3 &= \tfrac{1}{2} & (2) \\ -10.375x_3 &= -5.1875 & (3') \end{aligned}$$

$x_3 = \frac{1}{2}$

$x_2 = -1 \quad x_1 = 2$

7)

$\underline{s}\ \underbrace{\_\,\_\,\_\,\_\,\_}_{c}\ \underbrace{\_\,\_\,\_\,\_\,\_\,\_}_{f}$

$c: 0-31$

$c = e+15 \quad e\begin{cases} c=0 \to e=-15 \\ c=31 \to e=16 \end{cases}$

a) $\epsilon_{mach} = \begin{cases} 2^{1-t}, & \text{for chopping} \\ \frac{2^{1-t}}{2} = 2^{-t}, & \text{for rounding} \end{cases}$

$t-1 = f$ (# of bits)

$t = f+1 = 6+1 = 7$

chopping : $2^{1-7} = 2^{-6}$ $(0.000001)_2$

rounding : 2^{-7}

b) not consider special case

$s=0 \to$ +ve number

$e = c-15 \begin{cases} \text{smallest}: c=1 \to e=-14 \\ \text{largest}: c=30 \to e=15 \end{cases}$

$f: \begin{cases} \text{smallest} = 0 \\ \text{largest} = 0.111111 = 0.984375 \end{cases}$

smallest

$(-1)^0 \cdot 2^{-14} \times 1 = 2^{-14}$

largest

$(-1)^0 \cdot 2^{15} \times 1.984375 = 65024$