Tutorial 3SK3

LU Decomposition

LU Decomposition

$A\underline{x}=\underline{b}$,

Step 1: do gaussian elimination to get

U and L matrix , $LU=A$ / $LU=PA$

Step 2: $L\underline{d}=\underline{b}$ to get $\underline{d}$,

use forward substitution, $d_1 \to d_2 \to \cdots d_n$

(since L is lower triangular)

Step 3: $U\underline{x}=\underline{d}$ to get $\underline{x}$,

use backward substitution, $x_n \to x_{n-1} \to \cdots x_1$

(since U is upper triangular)

Note that if we do row exchange in the gaussian elimination like we mentioned before, a permutation matrix P is required,

Example 1

- Solve the system using LU decomposition

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 9 \\ 4x_1 + 5x_2 + 6x_3 &= 24 \\ 3x_1 + x_2 - 2x_3 &= 4 \end{aligned}$$

Solution

①

(1) If we want to write these equ. in matrix

$$\overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \overset{b}{\begin{bmatrix} 9 \\ 24 \\ 4 \end{bmatrix}}$$

In LU decomposition:

$$A = LU \;/\; PA = LU \quad —(1)$$

$$= \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}}$$

To find upper triangular matrix U:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -5 & -11 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\therefore U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & -1 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to R_3 - \frac{5}{3}R_2 \end{matrix}$$

From equ. (i): $LU = A$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{bmatrix}$$

$\boxed{l_{21} = 4}$

$\boxed{l_{31} = 3}$

$2l_{31} - 3l_{32} = 1$

$6 - 3l_{32} = 1$

$\boxed{l_{32} = +5/3}$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & +5/3 & 0 \end{bmatrix}$$

From step 2: $L\underline{d} = \underline{b}$ (to get $\underline{d}$)

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 5/3 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 24 \\ 4 \end{bmatrix}$$

$\boxed{d_1 = 9}$

$4d_1 + d_2 = 24$

$36 + d_2 = 24$

$\boxed{d_2 = -12}$

$3d_1 + \frac{5}{3}d_2 + d_3 = 4$

$27 + \frac{5}{3}(-12) + d_3 = 4$

$\boxed{d_3 = -3}$

so, we got $\underline{d} = \begin{bmatrix} 9 \\ -12 \\ -3 \end{bmatrix}$

step 3: $U\underline{x} = \underline{d}$ (to get $\underline{x}$)

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -12 \\ -3 \end{bmatrix}$$

$-x_3 = -3$

$\boxed{x_3 = 3}$

$-3x_2 - 6x_3 = -12$

$-3x_2 - 18 = -12$

$\boxed{x_2 = -2}$

$x_1 + 2x_2 + 3x_3 = 9$

$x_1 + 2\cdot(-2) + 3\cdot 3 = 9$

$x_1 = 4$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$$

Example 2

- Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & 8 & 6 \end{bmatrix}$, Use LU decomposition to compute A^{-1}, $LU = PA$, if permutation is required.

Solution

2 Given that,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & 8 & 6 \end{bmatrix}$$

$$P.A = LU \quad \text{—(i)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

To find, U:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & 8 & 6 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -2 \\ 0 & 2 & -3 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\therefore U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -3 \\ 0 & 0 & -2 \end{bmatrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{Row exchange from } R_3 \text{ to } R_2$$

$$\therefore P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

To find L:

From equ. (i)

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -3 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & 8 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -3 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 8 & 6 \\ 2 & 4 & 4 \end{bmatrix}$$

$$\boxed{l_{21} = 3} \qquad 2l_{31} + 2l_{32} = 4$$

$$4 + 2l_{32} = 4$$

$$\boxed{l_{31} = 2} \qquad \boxed{l_{32} = 0}$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

To calculate A^{-1}, let's consider $A^{-1} = B$

$$AA^{-1} = I$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & 8 & 6 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Which can be written as :

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & 8 & 6 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & 8 & 6 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad A\underline{b} = \underline{e}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & 8 & 6 \end{bmatrix} \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Now we know, $LU = PA$

$$LU\underline{b} = PA\underline{b}$$

$$LU\underline{b} = P\underline{e} \quad \text{— (11)}$$

let,

$$U\underline{b} = \underline{d}$$

$$(11) \rightarrow \quad L\underline{d} = P\underline{e}$$

From step 2 :

$$L\underline{d} = P\underline{e}$$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_{11} \\ d_{21} \\ d_{31} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{d_{11} = 1}$ $\quad 3d_{11} + d_{21} = 0$ $\quad 2d_{11} + d_{31} = 0$

$\boxed{d_{21} = -3}$ $\quad \boxed{d_{31} = -2}$

$$d_1 = \begin{bmatrix} d_{11} \\ d_{21} \\ d_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$$

From step 3 :

$$U\underline{b} = \underline{d}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$$

$\Rightarrow \quad -2b_{31} = -2$ $\quad 2b_{21} - 3b_{31} = -3$

$\boxed{b_{31} = 1}$ $\quad \boxed{b_{21} = 0}$

$b_{11} + 2b_{21} + 3b_{31} = 1$

$\boxed{b_{11} = -2}$

$$\therefore \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = b_1$$

In a same manner

From step 2:

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_{12} \\ d_{22} \\ d_{32} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$d_2 = \begin{bmatrix} d_{12} \\ d_{22} \\ d_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

From step 3:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$b_2 = \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -3 \\ -3/4 \\ -1/2 \end{bmatrix}$$

Again from step 2:

$$d_3 = \begin{bmatrix} d_{13} \\ d_{23} \\ d_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

From step 3: $b_3 = \begin{bmatrix} -1 \\ 1/2 \\ 0 \end{bmatrix}$

$$\therefore B = A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ 0 & -3/4 & 1/2 \\ 1 & -1/2 & 0 \end{bmatrix}$$