Mathematical Formulation of Image Deblur

CoE 3SK3

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1 Background

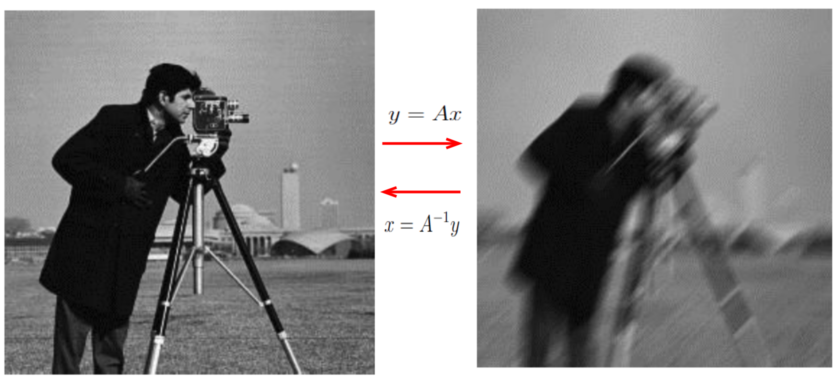

Figure 1: Examples

Blur process The formation of the blurred image y can be model as convolving the original image x with a convolutional kernel:

1. For pixel (i, j) in the original image x, its neighboring pixels are characterized by a window centered at pixel (i, j).

2. The new pixel value at location (i, j) is the weighted sum of original pixel values over this local region, i.e.,

$$y_{ij} = \sum_p \sum_q w_{p,q} x_{i+p,j+q}. \tag{1}$$

The window size and the weights depend on the blur kernel. For example,

- when using a 3×3 blur kernel, the range of subscript p and q are both $\{-1, 0, 1\}$ and the layout of this blur kernel is

$$\begin{bmatrix} w_{-1,-1} & w_{-1,0} & w_{-1,1} \\ w_{0,-1} & w_{0,0} & w_{0,1} \\ w_{1,-1} & w_{1,0} & w_{1,1} \end{bmatrix}; \tag{2}$$

- when using a 3×1 blur kernel, the range of subscript p and q are $\{-1, 0, 1\}$ and $\{0\}$ respectively;
- when using a 1×3 blur kernel, the range of subscript p and q are $\{0\}$ and $\{-1, 0, 1\}$ respectively.

In this context, each pixel value in the blurred image is determined by a few pixel values in the original image x. Suppose that the size of the image is $m \times n$, if we treat both y and x as mn-dimensional vectors, the dependency between them can be modeled by $y = Ax$, i.e.,

$$\begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{m1} \\ y_{12} \\ y_{22} \\ \vdots \\ y_{m2} \\ \vdots \\ y_{1n} \\ y_{2n} \\ \vdots \\ y_{mn} \end{bmatrix} = A \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{m1} \\ x_{12} \\ x_{22} \\ \vdots \\ x_{m2} \\ \vdots \\ x_{1n} \\ x_{2n} \\ \vdots \\ x_{mn} \end{bmatrix}. \tag{3}$$

- A is a sparse matrix and the size of A is $mn \times mn$.
- Each row of matrix A only contains a limited number of non-zero elements, reflecting the importance of pixel values from the original image x in influencing a pixel value of the blurred image y.
- The locations of non-zero elements in the row of A corresponding to pixel y_{ij} can be determined by analyzing $y_{ij} = \sum_p \sum_q w_{p,q} x_{i+p,j+q}$. Each non-zero element's location corresponds to a pixel value in the original image x that participates in the formation of y_{ij}.
- Don't forget the range of subscripts of $x_{i+p,j+q}$, i.e., $1 \leq i + p \leq m$ and $1 \leq j + q \leq n$. Therefore, when y_{ij} is located at the border or corner, fewer pixels from the original image x contribute to its formation. Your code should identify these special pixel locations and assign fewer non-zero elements to their corresponding rows in matrix A.

Objective Given any blur kernel and an image with size $m \times n$, represent the blur kernel into the form of matrix A in Eq. (3). Then, the blurred image y can be simulated by $y = Ax$ and the deblurred $\hat{x}$ can be obtained by solving an inverse problem, i.e., $\hat{x} = A^{-1}y$.

Requirements

- Do not directly use the *inv* function to solve this problem.
- Use the LU decomposition of A to calculate A^{-1}.
- Don't use the built-in function such as *lu*. Implement LU decomposition by yourself, referring to lecture notes, part 3, page 17-26.

2 Appendix

Matlab ref code The original layout of a 2D image is a matrix. We can convert it to a column vector with the Matlab function *reshape*. An example of the usage of *reshape* function is shown in Fig. 2.

```
>> M = randi(10, 2, 3)

M =

     7    10     5
     4     1     4
```

```
>> Mc = reshape(M, [], 1)

Mc =

     7
     4
    10
     1
     5
     4
```

```
>> Mr = reshape(Mc, 2, 3)

Mr =

     7    10     5
     4     1     4
```

Figure 2: Examples about using reshape function.

2.1 Examples

2.1.1 2D Kernel

Use 2D blur kernel

$$\begin{bmatrix} 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 \end{bmatrix} \tag{4}$$

to smooth the 2D image x whose size is 4×4. The blurred image is denoted as y. The relationship between elements in y and x is defined as:

sliding window

- $$\begin{bmatrix} 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 \end{bmatrix}, \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \tag{5}$$

  $$y_{11} = \frac{1}{2}x_{11} + \frac{1}{8}x_{21} + \frac{1}{8}x_{12} \tag{6}$$

- $$\begin{bmatrix} 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 \end{bmatrix}, \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \tag{7}$$

  $$y_{21} = \frac{1}{8}x_{11} + \frac{1}{2}x_{21} + \frac{1}{8}x_{31} + \frac{1}{8}x_{22} \tag{8}$$

- $$\begin{bmatrix} 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 \end{bmatrix}, \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \tag{9}$$
$$y_{31} = \frac{1}{8}x_{21} + \frac{1}{2}x_{31} + \frac{1}{8}x_{41} + \frac{1}{8}x_{32} \tag{10}$$

- $$\begin{bmatrix} 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 \end{bmatrix}, \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \tag{11}$$
$$y_{41} = \frac{1}{8}x_{31} + \frac{1}{2}x_{41} + \frac{1}{8}x_{42} \tag{12}$$

- $$\begin{bmatrix} 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 \end{bmatrix}, \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \tag{13}$$
$$y_{12} = \frac{1}{8}x_{11} + \frac{1}{2}x_{12} + \frac{1}{8}x_{22} + \frac{1}{8}x_{13} \tag{14}$$

- $$\begin{bmatrix} 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 \end{bmatrix}, \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \tag{15}$$
$$y_{22} = \frac{1}{8}x_{21} + \frac{1}{8}x_{12} + \frac{1}{2}x_{22} + \frac{1}{8}x_{32} + \frac{1}{8}x_{23} \tag{16}$$

- $$\begin{bmatrix} 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 \end{bmatrix}, \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \tag{17}$$
$$y_{32} = \frac{1}{8}x_{31} + \frac{1}{8}x_{22} + \frac{1}{2}x_{32} + \frac{1}{8}x_{42} + \frac{1}{8}x_{33} \tag{18}$$

- $$\begin{bmatrix} 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 \end{bmatrix}, \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \tag{19}$$
$$y_{42} = \frac{1}{8}x_{41} + \frac{1}{8}x_{32} + \frac{1}{2}x_{42} + \frac{1}{8}x_{43} \tag{20}$$

- $$\begin{bmatrix} 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 \end{bmatrix}, \quad \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \tag{21}$$
  $$y_{13} = \frac{1}{8}x_{12} + \frac{1}{2}x_{13} + \frac{1}{8}x_{23} + \frac{1}{8}x_{14} \tag{22}$$

- $$\begin{bmatrix} 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 \end{bmatrix}, \quad \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \tag{23}$$
  $$y_{23} = \frac{1}{8}x_{22} + \frac{1}{8}x_{13} + \frac{1}{2}x_{23} + \frac{1}{8}x_{33} + \frac{1}{8}x_{24} \tag{24}$$

- $$\begin{bmatrix} 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 \end{bmatrix}, \quad \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \tag{25}$$
  $$y_{33} = \frac{1}{8}x_{32} + \frac{1}{8}x_{23} + \frac{1}{2}x_{33} + \frac{1}{8}x_{43} + \frac{1}{8}x_{34} \tag{26}$$

- $$\begin{bmatrix} 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 \end{bmatrix}, \quad \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \tag{27}$$
  $$y_{43} = \frac{1}{8}x_{42} + \frac{1}{8}x_{33} + \frac{1}{2}x_{43} + \frac{1}{8}x_{44} \tag{28}$$

- $$\begin{bmatrix} 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 \end{bmatrix}, \quad \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \tag{29}$$
  $$y_{14} = \frac{1}{8}x_{13} + \frac{1}{2}x_{14} + \frac{1}{8}x_{24} \tag{30}$$

- $$\begin{bmatrix} 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 \end{bmatrix}, \quad \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \tag{31}$$
  $$y_{24} = \frac{1}{8}x_{23} + \frac{1}{8}x_{14} + \frac{1}{2}x_{24} + \frac{1}{8}x_{34} \tag{32}$$

- $$\begin{bmatrix} 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 \end{bmatrix}, \quad \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \tag{33}$$

  $$y_{34} = \frac{1}{8}x_{33} + \frac{1}{8}x_{24} + \frac{1}{2}x_{34} + \frac{1}{8}x_{44} \tag{34}$$

- $$\begin{bmatrix} 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 \end{bmatrix}, \quad \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \tag{35}$$

  $$y_{44} = \frac{1}{8}x_{43} + \frac{1}{8}x_{34} + \frac{1}{2}x_{44} \tag{36}$$

$$\begin{bmatrix} y_{11} \\ y_{21} \\ y_{31} \\ y_{41} \\ y_{12} \\ y_{22} \\ y_{32} \\ y_{42} \\ y_{13} \\ y_{23} \\ y_{33} \\ y_{43} \\ y_{14} \\ y_{24} \\ y_{34} \\ y_{44} \end{bmatrix} = \begin{bmatrix}
\frac{1}{2} & \frac{1}{8} & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{8} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{8} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{8} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{8} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{8} & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{8} & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & \frac{1}{8} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{8} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{1}{2} & \frac{1}{8} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{1}{2}
\end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{41} \\ x_{12} \\ x_{22} \\ x_{32} \\ x_{42} \\ x_{13} \\ x_{23} \\ x_{33} \\ x_{43} \\ x_{14} \\ x_{24} \\ x_{34} \\ x_{44} \end{bmatrix} \tag{37}$$

- The diagonal elements in the matrix A are highlighted in purple. For each row of the matrix A, the distance between the diagonal element and the furthest non-zero element depends on the image size.
- Blue elements represent pixels at corners. Pink elements indicate pixels located at the border but not at the corners. These special pixel locations have fewer non-zero elements in their corresponding rows of the matrix A.
- The above derivation process demonstrates how each pixel in the blurred image y is computed from pixels of x. You don't need to simulate the blurred image y by iteratively placing the sliding window at every pixel. You just need to transform the blur kernel into matrix form A and employ the equation $y = Ax$ to directly generate the blurred image y.
- Avoid directly replicating the matrix A from Eq. (37) as the solution, as varying the weights of the blur kernel and the image size will result in a distinct matrix A. Your implementation should be general to any 5-pixel blur kernels and images of varying sizes.