Constrained Optimization
Chapter 15

LINEAR PROGRAMMING

- An optimization approach that deals with meeting a desired objective such as maximizing profit or minimizing cost in presence of constraints such as limited resources
- Mathematical functions representing both the objective and the constraints are linear.

Standard Form/

- Basic linear programming problem consists of two major parts:
  - The objective function
  - A set of constraints
- For maximization problem, the objective function is generally expressed as

$$\text{Maximize } Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

c_j= payoff of each unit of the *j*th activity that is undertaken

x_j= magnitude of the *j*th activity

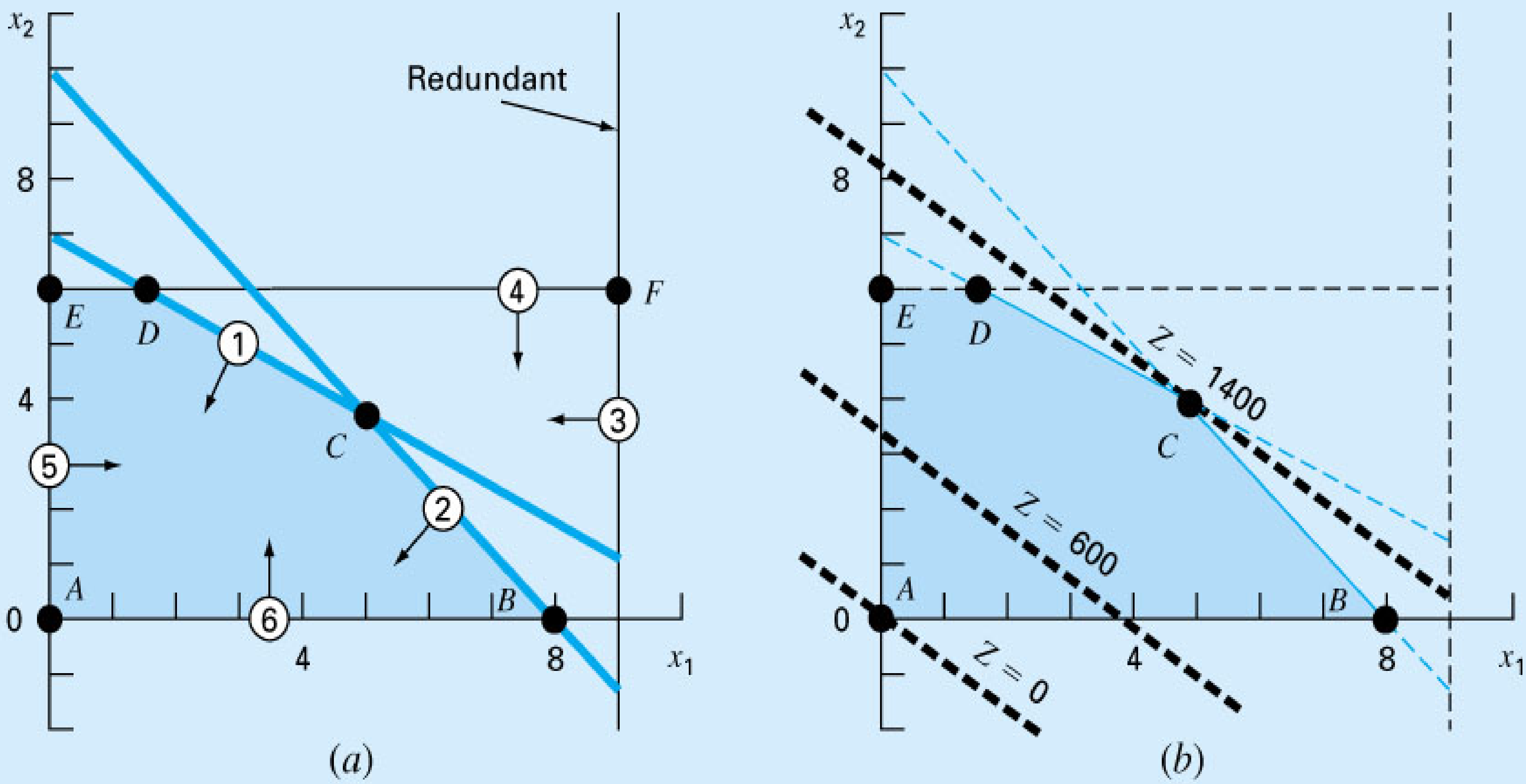
Z= total payoff due to the total number of activities

- The constraints can be represented generally as

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \le b_i$$

- Where a_{ij}=amount of the ith resource that is consumed for each unit of the jth activity and b_i=amount of the ith resource that is available
- The general second type of constraint specifies that all activities must have a positive value, $x_i>0$.
- Together, the objective function and the constraints specify the linear programming problem.

Figure 15.1



x_2
8
4
0
Redundant
E
D
1
4
F
C
3
5
2
6
A
B
4
8
x_1
(a)
x_2
8
E
D
Z = 1400
C
Z = 600
A
B
0
Z = 0
4
8
x_1
(b)

LP formulation example

- Consider drugs 1, 2, 3, with per unit benefit b1, b2, b3; costs c1, c2, c3; side effects s1, s2, s3.
- For a patient, find dosages x1, x2, x3 of the three drugs to maximize the benefit, while controlling the total cost below C, and total side effect below S.

Max Z=b1*x1 + b2*x2 + b3*x3 objective function

c1*x1 + c2*x2 + c3*x3 < C

s1*x1 + s2*x2 + s3*x3 < S

LP formulation example

$$\text{Max } Z = 2*x1 + x2$$

$$x1 + 2*x2 < 12;$$

$$2*x1 + x2 < 10;$$

$b_1 = 2 \qquad b_2 = 1$

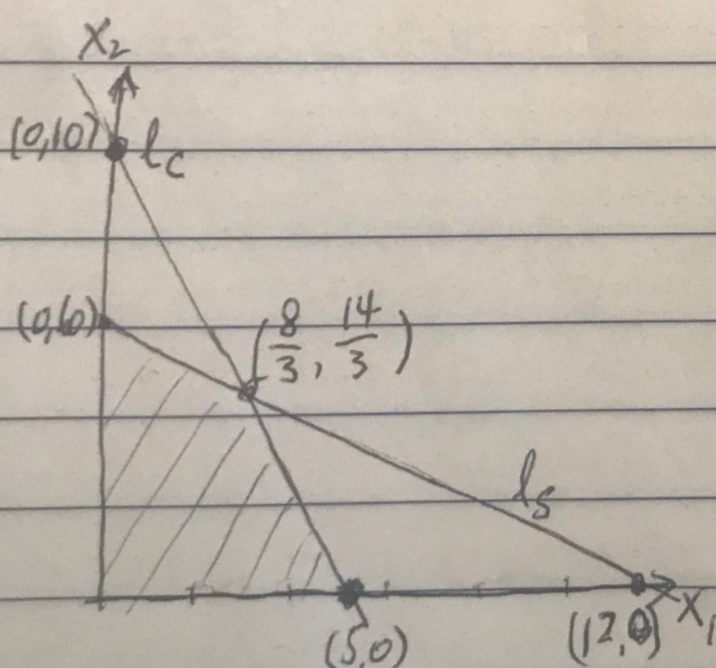
$c_1 = 2 \qquad c_2 = 1 \qquad C = 10$

$S_1 = 1 \qquad S_2 = 2 \qquad S = 12$

$$\max z = 2x_1 + x_2$$

$$l_c : 2x_1 + x_2 < 10$$

$$l_s : x_1 + 2x_2 < 12$$



$$z(\tfrac{8}{3}, \tfrac{14}{3}) = \tfrac{1}{3}(2\times 8 + 14) = 10$$

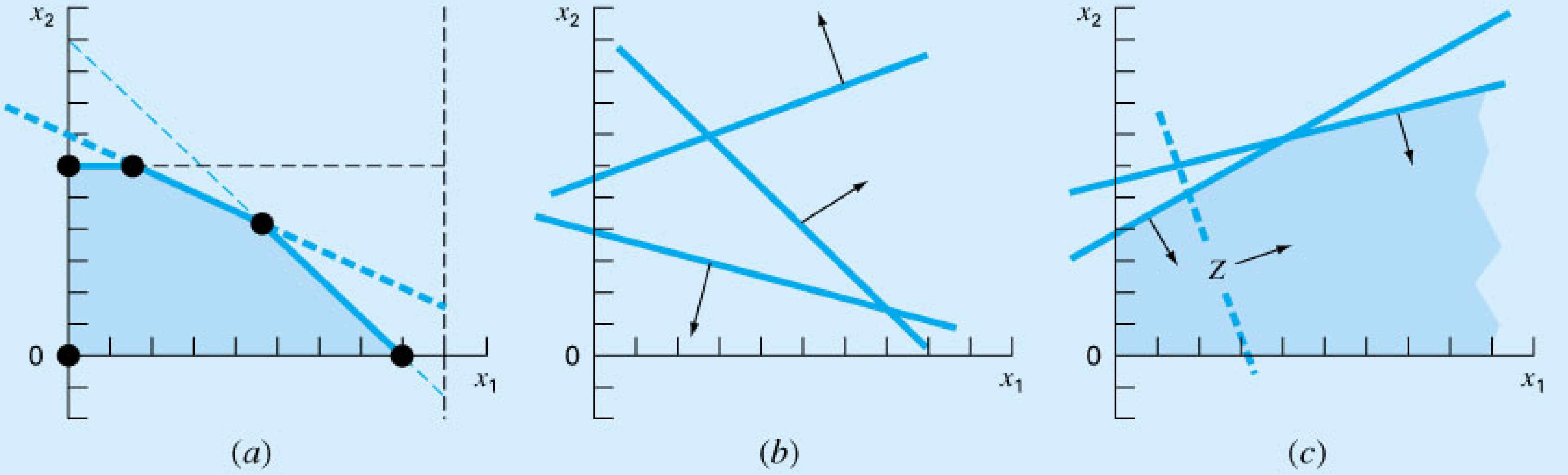
$$z(5,0) = 10$$

$$z(0,6) = 6$$

Possible outcomes that can be generally obtained in a linear programming problem/

1. *Unique solution*. The maximum objective function intersects a single point.
2. *Alternate solutions*. Problem has an infinite number of optima corresponding to a line segment.
3. *No feasible solution*.
4. *Unbounded problems*. Problem is under-constrained and therefore open-ended.

Figure 15.2



The Simplex Method/

- Assumes that the optimal solution will be an extreme point.
- The approach must discern whether during problem solution an extreme point occurs.
- To do this, the constraint inequalities are reformulated as equalities by introducing slack variables.

- A slack variable measures how much of a constrained resource is available, e.g.,

  $7x_1 + 11\,x_2 \le 77$

  If we define a slack variable S_1 as the amount of raw gas that is not used for a particular production level (x_1, x_2) and add it to the left side of the constraint, it makes the relationship exact.

  $7x_1 + 11\,x_2 + S_1 = 77$

- If slack variable is positive, it means that we have some slack that is we have some surplus that is not being used.
- If it is negative, it tells us that we have exceeded the constraint.
- If it is zero, we have exactly met the constraint. We have used up all the allowable resource.

Maximize

$$Z = 150x_1 + 175x_2$$

$$\begin{aligned}
7x_1 + 11x_2 + S_1 \qquad\qquad\qquad &= 77 \\
10x_1 + 8x_2 + \qquad + S_2 \qquad\qquad &= 80 \\
x_1 \qquad\qquad\qquad + S_3 \qquad &= 9 \\
x_2 \qquad\qquad\qquad\qquad + S_4 &= 6
\end{aligned}$$

$$x_1, x_2, S_1, S_2, S_3, S_4 \geq 0$$

- We now have a system of linear algebraic equations.
- For even moderately sized problems, the approach can involve solving a great number of equations. For *m* constraints and *n* unknowns, the number of simultaneous equations to be solved are:

$$C_n^{n+m} = \frac{(n+m)!}{n!\,m!}$$

Figure 15.3