MATH442/551 Lecture 1

Wednesday September 5

1 Review

In this course, I assume that you are familiar with the material covered in out MATH 342, Intermediate Ordinary Differential Equations. Specifically, you will need to know

- The existence and uniqueness of the solutions.
- Linear systems of ordinary differential equations (ODEs)
  - Their solutions. For a system

    $$\frac{dx}{dt} = Ax,$$

    the solution is $x(t) = x(0)e^{At}$, where $x \in \mathbb{R}^n$, and A is an $n \times n$ real constant matrix.
  - Origin as an equilibrium, and its classification. Depending on the eigenvalues of A, the orignin can be a node, a spiral node, a center, or a saddle. Saddle: all eigenvalues are real, some are positive and some are negative. Node: either all eigenvalues are positive, or all eigenvalues are negative. Spiral node: some eigenvalues have imaginary part, and either all eigenvalues have positive real part, or all have negative real part. Center: all eigenvalues are pure imaginary.
  - The stability of the origin, which is a measure of how the system resists small perturbations to its equuilibrium (origin). If the system moves away from origin with small perturbations, then the origin is unstable. If it stays close to the origin, then the origin is stable. If in addition, it moves back to the origin, then the origin is asymptotically stable. The stability of the origin is determined by the real parts of the eigenvalues of A. if there is at least one eigenvalue with positive real part, then the origin is unstable, otherwise it is stable. If all eigenvalues have negative real parts, then the origin is asymptotically stable.

- Autonomous nonlinear planer systems of ODEs. Consider a system $\frac{dx}{dt} = f(x)$ where $x \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$.
  - Trajectories: the parametric curve $x(t)$ of solutions.
  - Vector field: f, which gives the direction of the trajectory at every point x in the phase space $\mathbb{R}^n$.
  - Equilibria: points satisfying $f(x) = 0$, which are constant solutions.
    * Linear stability: The behavior of the trajectories near an equilkibrium is approximately the same as the trajectories of the linearization $\frac{dx}{dt} = Jx$ where $J = Df$ is the Jacobian of the vector field f at the equilibrium. However, this is only applicable if J has no eigenvalue with zero real part.
    * Direct method (Lyapunov functions): Like an energy function in physics. If there is a positive definite function $F(x)$ such that $\nabla F \cdot \frac{dx}{dt}$ is negative definite, then the equilibrium at $x = 0$ is asymptotically stable.
  - Nullclines and phase plane analysis.
  - Close orbits are the trajectories of periodic solutions. Attracting or expelling closed orbits are called limit cycles. There are theorems determining the existence or non-existence of such solutions.
    * Bendixon Theorem: if the divergence of he vector field ∇f does not change sign in a region D then there is no closed orbit in the region.
    * Poincare-Bendixon Theorem: if the trajectories do not leave a compact region D and there is no equilibrium in D, then there is an asymptotically stable limit cycle in D.

If you are not familiar with these concepts, it might be a good time for you to pick up the MATH 342 textbook and review them in the first couple weeks of this course.