

# MATH 442/551 Assignment #2 Solutions

October 12, 2018

1. Consider the following system

$$\frac{dx}{dt} = \begin{bmatrix} e^{-t} & \frac{t^2+1}{t^2} \\ (2-a)\frac{1-t}{t} & a\frac{1-t}{t} \end{bmatrix} x,$$

where  $x \in \mathbb{R}^3$ . Determine the range of  $a$  that makes the origin unstable (ignore that case where eigenvalues has zero real parts).

Solution: Note that the system is an asymptotically autonomous system, i.e.,

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -(2-a) & -a \end{bmatrix} x + \begin{bmatrix} e^{-t} & \frac{1}{t^2} \\ \frac{2-a}{t} & \frac{a}{t} \end{bmatrix} x,$$

The origin is unstable if and only if the origin of the limit system

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -(2-a) & -a \end{bmatrix} x$$

is unstable, i.e., one of its eigenvalues  $\lambda_1$  and  $\lambda_2$  having a positive real part (ignoring the case with eigenvalues having zero real part). In the case where one eigenvalue is positive and the other is negative,  $\lambda_1\lambda_2 = 2 - a < 0$ , i.e.  $a > 2$ ; in the case where both are positive, or are complex conjugates with positive real part,  $\lambda_1\lambda_2 = 2 - a > 0$  and  $\lambda_1 + \lambda_2 = -a > 0$ , i.e.,  $a < 0$ . So the origin is unstable is  $a > 2$  or  $a < 0$ .

2. Consider the following linear system,

$$\frac{dx}{dt} = Ax + B(t)x,$$

where  $x \in \mathbb{R}^n$ ,  $A$  is an  $n \times n$  matrix which eigenvalues all have negative real parts.  $B(t)$  is a continuous  $n \times n$  matrix. Show that, there exists a constant  $b > 0$ , such that the origin is asymptotically stable as long as  $\|B(t)\| < b$  for all  $t \geq 0$ .

Proof: Rewrite the system as an integral equation by treating  $B(t)x$  as a forcing term and solving it.

$$\frac{d}{dt}(e^{-At}x) = e^{-At}\frac{dx}{dt} + e^{-At}Ax = e^{-At}B(t)x.$$

Integrate from 0 to  $t$  on both sides,

$$e^{-At}x - x(0) = \int_0^t e^{-As}B(s)x(s)ds,$$

i.e.,

$$x = e^{At}x(0) + \int_0^t e^{A(t-s)}B(s)x(s)ds.$$

Thus,

$$\|x(t)\| \leq \|e^{At}\| \|x(0)\| + \int_0^t \|e^{A(t-s)}\| \|B(s)\| \|x\| ds.$$

Since eigenvalues of  $A$  all have negative real part, there exists positive constants  $C$  and  $\mu_0$  such that  $\|e^{At}\| \leq Ce^{-\mu_0 t}$  for all  $t$ . Together with the fact  $\|B(t)\| \leq b$  uniformly in  $t$ ,

$$\|x(t)\| \leq C\|x(0)\|e^{-\mu_0 t} + \int_0^t Cbe^{-\mu_0(t-s)}\|x(s)\|ds.$$

That is,

$$e^{\mu_0 t}\|x(t)\| \leq C\|x(0)\| + \int_0^t Cbe^{\mu_0 s}\|x(s)\|ds.$$

Apply the Gronwall's inequality,

$$e^{\mu_0 t}\|x(t)\| \leq C\|x(0)\|e^{Cbt},$$

i.e.,

$$\|x(t)\| \leq \|x(0)\|Ce^{-(\mu_0 - Cb)t}.$$

Thus, pick  $b < \mu_0/C$ , then the origin is stable because for all  $\varepsilon > 0$ , pick  $\delta = \varepsilon/C$ , then for all  $\|x(0)\| < \delta$ ,  $\|x(t)\| \leq \varepsilon$ . In addition, the origin is asymptotically stable, because  $\|x(t)\| \rightarrow 0$  as  $t \rightarrow \infty$ .

3. Consider the system

$$\begin{aligned}\frac{dx}{dt} &= 1 + y - x^2 - y^2, \\ \frac{dy}{dt} &= 1 - x - x^2 - y^2.\end{aligned}$$

- (a) Find the equilibria and classify them (a saddle, or an unstable spiral node, etc).
- (b) Show that there is a periodic solution (using polar coordinates).
- (c) Linearize about the periodic solution, and determine the characteristic exponents of the linearized system.

Solution: The equilibrium can be found by solving

$$\begin{aligned}0 &= 1 + y - x^2 - y^2, \\ 0 &= 1 - x - x^2 - y^2.\end{aligned}$$

That is,  $x^2 + y^2 - 1 = y = -x$ . So,  $y = -x$ , and  $2x^2 - 1 = -x$ . Thus, there are two equilibria,  $(1/2, -1/2)$  and  $(-1, 1)$ . To classify them, compute the Jacobian

$$J = \begin{bmatrix} -2x & 1 - 2y \\ -1 - 2x & -2y \end{bmatrix}.$$

At  $(1/2, -1/2)$ ,

$$J = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix},$$

so the eigenvalues are  $\lambda = \pm\sqrt{3}i$ , the origin of the linearized system is a center, but linearization cannot determine the type of the equilibrium for the original system. At  $(-1, 1)$ ,

$$J = \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix},$$

the eigenvalues are  $\lambda = \pm\sqrt{3}$ . So the equilibrium is a saddle. To show that there is a periodic solution, rewrite the system into polar coordinates. i.e.,  $x = r \cos \theta$  and  $y = r \sin \theta$ , and thus

$$\begin{aligned}r^2 &= x^2 + y^2, \\ \theta &= \tan^{-1} \frac{y}{x}.\end{aligned}$$

Note that

$$r \frac{dr}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt} = (x + y)(1 - x^2 + y^2),$$

and

$$\frac{d\theta}{dt} = \frac{d}{dt} \tan^{-1} \frac{y}{x} = \frac{\frac{dy}{dt}x - \frac{dx}{dt}y}{x^2 + y^2} = \frac{-(x^2 + y^2) + (x - y)(1 - x^2 - y^2)}{x^2 + y^2}.$$

So under the polar coordinate, the system is

$$\begin{aligned} \frac{dr}{dt} &= (\cos \theta + \sin \theta)(1 - r^2), \\ \frac{d\theta}{dt} &= -1 + \frac{1 - r^2}{r}(\cos \theta - \sin \theta) \end{aligned}$$

Note that  $r = 1$ ,  $\theta = -t$ , i.e.,  $\phi(t) = (\cos t, -\sin t)$ , is a periodic solution. Linearize about this periodic solution, denote the new variable (the deviations of  $x$  and  $y$  from the periodic solution) as  $u$  and  $v$ ,

$$\frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -2 \cos t & 1 + 2 \sin t \\ -1 - 2 \cos t & 2 \sin t \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Note that  $\phi'(t) = (-\sin t, -\cos t)$  is a solution. (In general, if  $\phi'(t) = f(\phi)$ , then  $\phi''(t) = Df(\phi(t))\phi'(t)$ , i.e.,  $\phi'(t)$  is a solution to the linearized system.) Thus, one of the characteristic exponents (denote as  $\lambda_1, \lambda_2$ ) must have zero real part. Note that

$$\lambda_1 + \lambda_2 = \frac{1}{2\pi} \int_0^{2\pi} -2 \cos t + 2 \sin t dt = 0,$$

Both must have zero real part. As  $\lambda = \frac{1}{2\pi} \ln \rho$  where  $\rho$  is the eigenvalue of the matrix  $e^{2\pi B}$ , which have eigenvalues  $e^{2\pi \kappa}$  where  $\kappa$  is an eigenvalue of  $B$ , clearly  $\lambda = \kappa$ . As  $B$  is real, all eigenvalues must be real or complex conjugates. So, eigen  $\lambda_1 = \lambda_2 = 0$  or they are pure imaginary.