MATH 442/551 Assignment #4

Due Monday November 19, in class

1. Consider a Hamiltonian system

$$\frac{dp}{dt} = -\frac{\partial H(p,q)}{\partial q},$$
$$\frac{dq}{dt} = \frac{\partial H(p,q)}{\partial p}.$$

   (a) Show that, if the point (p^*, q^*) is a local minimum of the Hamiltonian $H(p,q)$, then it is a stable equilibrium of the system.

   (b) Show that the pendulum equation

$$\dot{x} = y$$
$$\dot{y} = -\omega^2 \sin x$$

   is a Hamiltonian system, (i.e., identify the Hamiltonian $H(y,x)$), and show that the origin is locally stable. Is it locally asymptotically stable?

Solution:

a) If the point (p^*, q^*) is a local minimum of the Hamiltonian $H(p,q)$, then there exists a neighborhood $\mathcal{N}$ of (p^*, q^*), such that $H(p,q) \geq H(p^*, q^*)$ for all $(p,q) \in \mathcal{N}$. Thus, the function

$$V(p,q) = H(p,q) - H(p^*, q^*)$$

is positively definite in $\mathcal{N}$. We use $V(p,q)$ in $\mathcal{N}$ as a Lyaponov function. The orbital derivative

$$
\begin{aligned}
L_t V(p,q) &= \nabla H \cdot f \\
&= \begin{bmatrix} \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial q} \end{bmatrix} \cdot \begin{bmatrix} -\frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} \\
&= 0,
\end{aligned}
$$

which is negatively semi-definite. Thus, the equilibrium (p^*, q^*) is stable.

b) If the Hamiltonian $H(y,x)$ exists, then $\frac{\partial H}{\partial y} = y$, so

$$H = \frac{1}{2}y^2 + g(x)$$

where g is an unknown function. Also, since $\frac{\partial H}{\partial x} = -(-\omega^2 \sin x)$,

$$g'(x) = \omega^2 \sin x,$$

and thus $g(x) = -\omega^2 \cos x + C$. Without loss of generality, we can pick $C = 0$. That is,

$$H = \frac{1}{2}y^2 - \omega^2 \cos x,$$

which has a local minimum at $(0,0)$ because

$$\frac{\partial H}{\partial x}(0,0) = \frac{\partial H}{\partial y}(0,0) = 0,$$

and the Hessian matrix

$$D^2 H(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & \omega^2 \end{bmatrix}$$

has two positive eigenvalues. Thus, from part a),

$$V(y,x) = \frac{1}{2}y^2 + \omega^2(1 - \cos x)$$

is a Lyapunov function, and its minimum $(0,0)$ is stable.

But it is no asymptotically stable, because otherwise, all orbits starting close to theorigin must approach $(0,0)$, thus, by continuity, along any orbit, $V(y(t), x(t))$ must also approach 0. But, starting with any nonzero initial condition (x_0, y_0),

$$V(x(t), y(t)) = V(x_0, y_0) > 0,$$

and thus cannot approach 0. Thus, by contradiction, the origin $(0,0)$ is not asymptotically stable.

2. Consider the system

$$\frac{dx}{dt} = -ax + y,$$
$$\frac{dy}{dt} = 1 + x^2 - y,$$

where the parameter $a > 0$.

(a) Find the bifurcation point (equilibrium and the corresponding parameter value).

(b) Approximate the extended center manifold (including a) with a polynomial up to the second order. (First shift the equilibrium at the bifurcation point to the origin, and shift the parameter value to zero).

(c) Show that a saddle node bifurcation occurs at the bifurcation point.

Solution:

a) We first find the equilibria, which must satisfy

$$0 = -ax + y,$$
$$0 = 1 + x^2 - y,$$

that is, $y = ax$ and

$$1 + x^2 - ax = 0. \tag{1}$$

So,

$$x = \frac{a \pm \sqrt{a^2 - 4}}{2}, y = ax.$$

If an equilibrium is a bifurcation point, then its Jacobian J must has a zero eigenvalue, i.e., $|J| = 0$,

$$J = \begin{bmatrix} -a & 1 \\ 2x & -1 \end{bmatrix}$$

So,

$$|J| = a - 2x = 0 \tag{2}$$

That is, a bifurcation point must satisfy both (1) and (2). Solving them together yields $a = 2$, $x = 1$ and $a = -2$, $x = -1$. Since we assume $a > 0$, we pick

$$a = 2,\ x = 1,\ y = ax = 2.$$

b) We first extend the system as

$$\frac{dx}{dt} = -ax + y,$$
$$\frac{dy}{dt} = 1 + x^2 - y,$$
$$\frac{da}{dt} = 0.$$

Then we shift the bifurcation point $(1, 2, 2)$ to the origin. Let $x = X + 1$, $y = Y + 2$ and $a = A + 2$. So,

$$\begin{aligned}
\frac{dX}{dt} &= -(A+2)(X+1) + Y + 2 \\
&= -AX - 2X - A + Y \\
\frac{dY}{dt} &= 1 + (1+X)^2 - Y - 2 \\
&= 2X - Y - X^2 \\
\frac{dA}{dt} &= 0
\end{aligned}$$

To find the center manifold, we look at the Jacobian matrix at the origin of (X, Y, A),

$$J = \begin{bmatrix} -2 & 1 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$J = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 & \frac{2}{3} \\ 1 & 1 & \frac{1}{3} \\ 0 & 0 & -1 \end{bmatrix}$$

Let

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & \frac{2}{3} \\ 1 & 1 & \frac{1}{3} \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ A \end{bmatrix}$$

and thus

$$\begin{bmatrix} X \\ Y \\ A \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u + v + w \\ -u + 2v \\ -3w \end{bmatrix}$$

$$
\begin{aligned}
\frac{d}{dt}\begin{bmatrix} u \\ v \\ w \end{bmatrix} &= \frac{1}{3}\begin{bmatrix} 2 & -1 & \frac{2}{3} \\ 1 & 1 & \frac{1}{3} \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} -AX-2X-A+Y \\ 2X-Y-X^2 \\ 0 \end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix} -2AX-6X-2A+3Y+X^2 \\ -A-AX-X^2 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 2w(u+v+w)-3u+(u+v+w)^2 \\ w+w(u+v+w)-\frac{1}{3}(u+v+w)^2 \\ 0 \end{bmatrix}
\end{aligned}
$$

Let the center manifold be

$$
u = h(v,w) = \alpha v^2 + \beta vw + \gamma w^2 + \cdots ,
$$

and substitute into the equation,

$$
\begin{aligned}
\frac{du}{dt} &= 2\alpha v\dot{v} + \beta\dot{v}w + \beta v\dot{w} + 2\gamma w\dot{w} \\
&= (2\alpha v + \beta w)\dot{v}
\end{aligned}
$$

because $\dot{w} = 0$, and thus

$$
\begin{aligned}
&(2\alpha v + \beta w)[w + w(u+v+w) - \frac{1}{3}(u+v+w)^2] \\
=&2w(u+v+w) - 3u + (u+v+w)^2 \\
=&2w(\alpha v^2 + \beta vw + \gamma w^2 + \cdots + v + w) \\
&- 3(\alpha v^2 + \beta vw + \gamma w^2 + \cdots) \\
&+ (\alpha v^2 + \beta vw + \gamma w^2 + \cdots + v + w)^2
\end{aligned}
$$

Comparing the coefficients of v^2,vw and w^2 terms:

$$
\begin{aligned}
0 &= -3\alpha + 1, \\
2\alpha &= 2 - 3\beta + 2, \\
\beta &= 2 - 3\gamma + 1,
\end{aligned}
$$

so, $\alpha = 1/3$, $\beta = 10/3$, and $\gamma = -1/3$, i.e.,

$$
u = \frac{1}{3}(v^2 + 10vw - w^2) + \ldots
$$

and on the center manifold,

$$\begin{aligned}\frac{dv}{dt} =& w + w[\frac{1}{3}(v^2 + 10vw - w^2) + \cdots + v + w] \\ &- \frac{1}{3}[\frac{1}{3}(v^2 + 10vw - w^2) + v + w]^2 \\ =& w + \frac{8}{9}wv + \frac{8}{9}w^2 + \dots\end{aligned}$$

c) Show show that there is a saddle nbode bifurcation, one can either apply the Saddle-Node Bifurcation Theorem, or find the normal form of the center manifold, or show that for $a < 2$, there is no equilibrium, when $a = 2$, there is an unique equilibrium, and when $a > 2$ there is a saddle and a node. Here we use the first approach. We have already computed the Jacobian J at the bifurcation point $(1, 2)$ with $a = 2$,

$$J = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix},$$

which has a simple zero eigenvalue, with a left eigenvector $w = \frac{1}{3}(1, 1)$ and a right eigenvector $u = (1, 2)^T$.

$$\alpha = w\frac{\partial f}{\partial a}(1, 2, 2) = \frac{1}{3}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} -x \\ 0 \end{bmatrix}_{x=1} = -\frac{1}{3} \neq 0.$$

and

$$\begin{aligned}\beta &= w\frac{1}{2}\begin{bmatrix} \partial_{xx}f_1u_1^2 + 2\partial_{xy}f_1u_1u_2 + \partial_{yy}f_1u_2^2 \\ \partial_{xx}f_1u_1^2 + 2\partial_{xy}f_1u_1u_2 + \partial_{yy}f_1u_2^2 \end{bmatrix}_{x=1,y=2,a=2} \\ &= w\frac{1}{2}\begin{bmatrix} 0 \\ 2u_1^2 \end{bmatrix} = v_1^2 = \frac{1}{3} \neq 0\end{aligned}$$

Thus, the system satisfies the Saddle-Node Bifurcation Theorem.