MATH 442/551 Assignment #2 Solutions

October 12, 2018

1. Consider the following system

$$\frac{dx}{dt} = \begin{bmatrix} e^{-t} & \frac{t^2+1}{t^2} \\ (2-a)\frac{1-t}{t} & a\frac{1-t}{t} \end{bmatrix} x,$$

   where $x \in \mathbb{R}^3$. Determine the range of a that makes the origin unstable (ignore that case where eigenvalues has zero real parts).

Solution: Note that the system is an asymptotically autonomous system, i.e.,

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -(2-a) & -a \end{bmatrix} x + \begin{bmatrix} e^{-t} & \frac{1}{t^2} \\ \frac{2-a}{t} & \frac{a}{t} \end{bmatrix} x,$$

The origin is unstable if and only if the origin of the limit system

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -(2-a) & -a \end{bmatrix} x$$

is unstable, i.e., one of its eigenvalues λ_1 and λ_2 having a positive real part (ignoring the case with eigenvalues having zero real part). In the case where one eigenvalue is positive and the other is negative, $\lambda_1\lambda_2 = 2-a < 0$, i.e. $a > 2$; in the case where both are positive, or are complex conjugates with positive real part, $\lambda_1\lambda_2 = 2-a > 0$ and $\lambda_1 + \lambda_2 = -a > 0$, i.e., $a < 0$. So the origin is unstable is $a > 2$ or $a < 0$.

2. Consider the following linear system,

$$\frac{dx}{dt} = Ax + B(t)x,$$

where $x \in \mathbb{R}^n$, A is an $n \times n$ matrix which eigenvalues all have negative real parts. $B(t)$ is a continuous $n \times n$ matrix. Show that, there exists a constant $b > 0$, such that the origin is asymptotically stable as long as $\|B(t)\| < b$ for all $t \geq 0$.

Proof: Rewrite the system as an integral equation by treating $B(t)x$ as a forcing term and solving it.

$$\frac{d}{dt}(e^{-At}x) = e^{-At}\frac{dx}{dt} + e^{-At}Ax = e^{-At}B(t)x.$$

Integrate from 0 to t on both sides,

$$e^{-At}x - x(0) = \int_0^t e^{-As}B(s)x(s)ds,$$

i.e.,

$$x = e^{At}x(0) + \int_0^t e^{A(t-s)}B(s)x(s)ds.$$

Thus,

$$\|x(t)\| \leq \|e^{At}\|\|x(0)\| + \int_0^t \|e^{A(t-s)}\|\|B(s)\|\|x\|ds.$$

Since eigenvalues of A all have negative real part, there exists positive constants C and μ_0 such that $\|e^{At}\| \leq Ce^{-\mu_0 t}$ for all t. Together with the fact $\|B(t)\| \leq b$ uniformly in t,

$$\|x(t)\| \leq C\|x(0)\|e^{-\mu_0 t} + \int_0^t Cbe^{-\mu_0(t-s)}\|x(s)\|ds.$$

That is,

$$e^{\mu_0 t}\|x(t)\| \leq C\|x(0)\| + \int_0^t Cbe^{\mu_0 s}\|x(s)]\|ds.$$

Aoopky the Gronwall's inequality,

$$e^{\mu_0 t}\|x(t)\| \leq C\|x(0)\|e^{Cbt},$$

i.e.,

$$\|x(t)\| \leq \|x(0)\|Ce^{-(\mu_0 - Cb)t}.$$

Thus, pick $b < \mu_0/C$, then the origin is stable because for all $\varepsilon > 0$, pick $\delta = \varepsilon/C$, then for all $\|x(0)\| < \delta$, $\|x(t)\| \leq \varepsilon$. In addition, the origin is asymptotically stable, because $\|x(t)\| \to 0$ as $t \to \infty$.

3. Consider the system

$$\frac{dx}{dt} = 1 + y - x^2 - y^2,$$
$$\frac{dy}{dt} = 1 - x - x^2 - y^2.$$

   (a) Find the equilibria and classify them (a saddle, or an unstable spiral node, etc).
   (b) Show that there is a periodic solution (using polar coordinates).
   (c) Linearize about the periodic solution, and determine the characteristic exponents of the linearized system.

Solution: The equilibrium can be found by solving

$$0 = 1 + y - x^2 - y^2,$$
$$0 = 1 - x - x^2 - y^2.$$

That is, $x^2 + y^2 - 1 = y = -x$. So, $y = -x$, and $2x^2 - 1 = -x$. Thus, there are two equilibria, $(1/2, -1/2)$ and $(-1, 1)$. Top classify them, compute the Jacobian

$$J = \begin{bmatrix} -2x & 1-2y \\ -1-2x & -2y \end{bmatrix}.$$

At $(1/2, -1/2)$,

$$J = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix},$$

so the eigernvalues are $\lambda = \pm\sqrt{3}i$, the origin of the linearized system is a center, but linearization cannot determine the type of the equilibrium for the original system. At $(-1, 1)$,

$$J = \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix},$$

the eigenvalues are $\lambda = \pm\sqrt{3}$. So the equilibrium is a saddle. To show that there is a periodic solution, rewrite the system into polar coordinates. i.e., $x = r\cos\theta$ and $y = r\sin\theta$, and thus

$$r^2 = x^2 + y^2,$$
$$\theta = \tan^{-1}\frac{y}{x}.$$

Note that

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} = (x+y)(1-x^2+y^2),$$

and

$$\frac{d\theta}{dt} = \frac{d}{dt}\tan^{-1}\frac{y}{x} = \frac{\frac{dy}{dt}x - \frac{dx}{dt}y}{x^2+y^2} = \frac{-(x^2+y^2)+(x-y)(1-x^2-y^2)}{x^2+y^2}.$$

So under the polar coordinate, the system is

$$\frac{dr}{dt} = (\cos\theta + \sin\theta)(1-r^2),$$
$$\frac{d\theta}{dt} = -1 + \frac{1-r^2}{r}(\cos\theta - \sin\theta)$$

Those are coming from original assumption last page

Note that $r = 1$, $\theta = -t$, i.e., $\phi(t) = (\cos t, -\sin t)$, is a periodic solution. Linearize about this periodic solution, denote the new variable (the deviations of x and y from the periodic solution) as u and v,

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -2\cos t & 1+2\sin t \\ -1-2\cos & 2\sin t \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}.$$

Note that $\phi'(t) = (-\sin t, -\cos t)$ is a solution. (In general, if $\phi'(t) = f(\phi)$, then $\phi''(t) = Df(\phi(t))\phi'(t)$, i.e., $\phi'(t)$ is a solution to the linearized system.) Thus, one of the characteristic exponents (denote as λ_1, λ_2) must have zero real part. Note that

$$\lambda_1 + \lambda_2 = \frac{1}{2\pi}\int_0^{2\pi} -2\cos t + 2\sin t dt = 0,$$

Both must have zero real part. As $\lambda = \frac{1}{2\pi}\ln\rho$ where ρ is the eigenvalue of the matrix $e^{2\pi B}$, which have eigenvalues $e^{2\pi\kappa}$ where κ is an eigenvalue of B, clearly $\lambda = \kappa$. As B is real, all eigenvalues must be real or complex conjugates. So, eigen $\lambda_1 = \lambda_2 = 0$ or they are pure imaginary.