MATH 442/551 Assignment #1

Due Thursday September 20, in class

1. Use the following Banach Fixed-Point Theorem to prove the Picard-Lindelof version fo teh Existence and Uniqueness Theorem (listed below).

   - Definition: Let X be a norned vector space, the map $T : X \to X$ is called a contraction mapping on X if there exists a constant $q \in [0, 1)$ such that $\|T(x) - T(y)\| \leq q\|x - y\|$ for all $x, y \in X$.
   - Banach Fixed-Point Theorem: Let X be a complete Banach space, and T be a contraction mapping on X, then T has a unique fixed point $x^* \in X$, i.e., $T(x^*) = x^*$.
   - Picard-Lindelof Theorem: Consider the initial value problem

     $$\frac{dx}{dt} = f(t, x),\ x(t_0) = x_0,$$

     , where $x \in \mathbb{R}^n$, $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous in t and Lipschitz continuous in X on a region $G \subset \mathbb{R}^{n+1}$ with non-empty interior, which interior contains (t_0, x_0). Then, there exists a positive constatn $h \leq a$ such that the initial value problem has a unique solution on $[t_0 - h, t_0 + h]$.
   - Hint: consider the vector space $C_{[t_0-h,t_0+h]}$, and show that for small enough h, the map

     $$F(x) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

     is a contraction mapping.

2. Continuous Dependence on parameters: Consider the initial value problem

   $$\frac{dx}{dt} = f(t, x; p),\ x(t_0) = x_0,$$

   where $p \in \mathbb{R}^m$ is a parameter of the model (a constant vector), $x \in \mathbb{R}^n$, and $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a continuous function on a region $G \subset \mathbb{R}^{n+1}$ with non-empty interior, and Lipscitz continuous in x on G with a Lipschitz constant L. If $x_1(t; t_0, x_0, p_1)$ and $x_2(t; t_0, x_0, p_2)$ are two solutions to the IVP defined on an interval I, with parameter valuex $p = p_1$ and $p = p_2$, respectively.

(a) Show that, for all $t \in I$,

$$|x_1(t; t_0, x_0, p_1) - x_2(t; t_0, x_0, p_2)| \leq |p_1 - p_2| e^{L(t-t_0)} .$$

(b) Show that, for all fixed t, $x(t; t_0, x_0, p)$ is a continuous function of the parameter p.

- Hint: Consider the system $\frac{dz}{dt} = f(t, z)$ where $z = (x^T, p^T)^T$, and $\frac{dp}{dt} = 0$. That is, extend the system to include p as constant state variables.

3. Consider the initial value problem

$$\frac{dx}{dt} = Ax + B(t)x,\ x(t_0) = x_0,$$

where $x \in \mathbb{R}^n$, A is a constant $n \times n$ matrix which eigenvalues all have negative real part; $B(t)$ is an $n \times n$ continuous matrix function, with $\int_{t_0}^{\infty} \|B(s)\| ds < M$ for some constant $M > 0$ (note that this condition guarantees that $B(t) \to 0$ as $t \to \infty$). Here, for each t, the matrix norm is defined as

$$\|B(t)\| = \max_{\|x\| \neq 0} \frac{\|B(t)x\|}{\|x\|} ,$$

and thus $\|B(t)x\| \leq \|B(t)\| \|x\|$ for all x.

(a) Show that its solution satisfies

$$x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-s)} B(s) x(s) ds .$$

(b) Note that all eigenvalues of A has negative real parts, thus $\exists C > 0$ such that $\|e^{At}\| \leq C$ for all t. Use this fact to show that the origin is stable, i.e., $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\|x(t)\| \leq \varepsilon$ as long as $\|x_0\| \leq \delta$.

- Hint: Apply Gronwall's inequality to $\|x(t)]\|$.