MATH 442/551 Assignment #3 Solutions

December 3, 2018

1. Show that, if an orbit Γ is not a closed orbit, and its ω limit set $\omega(\Gamma)$ is a closed orbit, then, there exists a positively invariant set bounded by the orbit Γ and $\omega(\Gamma)$, such that $\omega(\Gamma)$ is the ω limit set of all orbits starting in this positively invariant set. (Hint, firstly, show that there exists a neighborhood of a closed orbit that contains no equilibrium; secondly, consider a cross section ℓ of $\omega(\Gamma)$).

Proof: Here, to stress, wee are considering an autonomous planer system $\frac{dx}{dt} = f(x)$ where $x \in \mathbb{R}^2$ and $f(x)$ satisfy the Existence and Uniqueness Theorem so that the orbits exist and is unique from any initial condition. Without loss of generality, suppose that the orbit $\Gamma = \{\phi^t(x_0) : t \geq 0\}$ (where $\phi^t(x_0)$ is the solution with initial condition x_0) is on the outside of the closed orbit $\omega(\Gamma)$. We will cary out the proof in the following steps:

a) There is a neighborhood U of the closed orbit $\omega(\Gamma)$ that contains no equilibrium. This can be shown because for any $x \in \omega(\Gamma)$, the vector field $f(x) \neq 0$. Thus, by continuity, the exists a neighborhood $\mathcal{N}_x$ such that $f(p) \neq 0$ for all $p \in \mathcal{N}_x$. Let $\Omega = \cup_{x\in\omega(\Gamma)}\mathcal{N}_x$, then there is no equilibrium in Ω.

b) There is a point on the close orbit, $x \in \omega(\Gamma)$, such that a transverse ℓ at $x$also intersects with Γ multiple times. To show this, pick any $x \in \omega(\Gamma)$, then there exists a time sequence $t_n \to \infty$ as $n \to \infty$, such that $x_n = \phi^{t_n}(x_0) \to x$. Take any transveral ℓ at x. Note there exists an $\varepsilon > 0$ such that any orbit starting in the ε ball $B_\varepsilon = \{y : \|y - x\| \leq \varepsilon\}$ intersects with ℓ. Since $x_n \in B_\varepsilon$ for all $n$large enough, the orbit starting with x_n (i.e., Γ itself) must intersect with ℓ, infinite many times.

c) Let x_1 be the first intersection of Γ and the transversal ℓ in U, then the Poincare map $P(x_1)$ is defined, and $P(x_1)$ is between x_1 and x. This is because the intersections with ℓ must be ordered. Thus, if $P(x_1)$ is outside of the segment between x_1 and x on l, then all future intersections are bounded by x_1, and thus x_n cannot approach x.

d) Then the region $\Omega \subset U$ bounded by Γ (the segment between x_1 and $P(x_1)$), $\omega(\Gamma)$, and the transversal ℓ form a ring like region that is positively invariant. The region Ω is positively invariant because any orbit starting in Ω, say $\Gamma_y = \{\phi^t(y) : t \geq 0, y \in \Omega\}$, it cannot leave the region Ω through the orbits $\omega(\Gamma)$ and Γ, because orbits cannot intersect.

e) By the Poincare-Bendixon theorem, for any orbit Γ_y starting in Ω (as defined above), its ω limit set $\omega(\Gamma_y)$ must be either an equilibrium or a closed orbit. But there is no equilibrium in U, so it has to be a closed orbit.

f) Note that $\omega(\Gamma_y) = \omega(\Gamma)$, otherwise, Γ is bounded inside by $\omega(\Gamma_y)$, and thus cannot approach $\omega(\Gamma)$. This completes the proof.

2. Consider the following system

$$\frac{dx}{dt} = x(3 - 2x - y),$$
$$\frac{dy}{dt} = y(3 - x - 2y).$$

(a) Show that all orbit starting in the first quadrant are positively bounded. (Hint, show that the square $[0, M] \times [0, M]$ is positively invariant for large enough M).

(b) Find the equilibria and classify them.

(c) Show that all orbits with a positive initial condition must approach the positive equilibrium. (Hint, we have shown in class that there is no closed orbit in the first quadrant).

Solution:

a) We consider the square $\Omega = \{(x, y) : 0 \leq x, y \leq M\}$. Note that when $x = 0$, $\frac{dy}{dt} = y(3 - 2y)$, so the y axis consists of orbits (i.e., is invariant). Similarly, the x axis is also invariant. Thus, any orbit starting in the interior Ω cannot intersect with the two axes, otherwise, the whole orbit must be contained in the axis it intersects. So orbits starting in Ω cannot leave Ω through the two axes. We will then show that they cannot leave throughthe other two boundaries of Ω either. On the boundary $\{M\} \times [0, M]$, i.e. $x = M$ and $0 \leq y \leq M$, the normal vector is $\vec{n} = (1, 0)$, and the vector field f on the boundary satisfies

$$f \cdot \vec{n} = \left[\begin{array}{c} x(3-2x-y) \\ y(3-x-2y) \end{array} \right]_{x=M} \cdot \vec{n} = M(3-2M-y) \leq 0$$

for large $M > 3/2$. That is, for large M, orbits can only enter Ω through this boundary. Similarly we can show that orbits can only enter Ω through $[0, M] \times \{M\}$. Thus, Ω is positively invariant. any orbit starting in Ω is positively bounded by M in both x and y. Since M is arbitrary, any orbit, suppose its initial condition is (x_0, y_0) is bounded by such an M. So any orbit is positively bounded.

b) To find equilibrium, we solve

$$\begin{aligned} 0 &= x(3-2x-y), \\ 0 &= y(3-x-2y). \end{aligned}$$

Here, if $x = 0$, $y = 0$ or $y = 3/2$. Similarly, if $y = 0$, $x = 0$ or $x = 3/2$. If both x and y are not zero, then

$$\begin{aligned} 0 &= 3-2x-y, \\ 0 &= 3-x-2y, \end{aligned}$$

which has a unique solution $(1, 1)$. Thus, there are four equilibria, $(0, 0)$, $(3/2, 0)$, $(0, 3/2)$, and $(1, 1)$. To classify them, we need to compute their Jacobian and compute the eigenvalues. The Jacobian is

$$J(x, y) = \left[\begin{array}{cc} 3-4x-y & -x \\ -y & 3-x-4y \end{array} \right].$$

At $(0, 0)$,

$$J(0, 0) = \left[\begin{array}{cc} 3 & 0 \\ 0 & 3 \end{array} \right],$$

which eigenvalues are 3 and 3, thus it is an unstable node.

At $(3/2, 0)$,

$$J(3/2, 0) = \left[\begin{array}{cc} -3 & -3/2 \\ 0 & 3/2 \end{array} \right],$$

which eigenvalues are $3/2$ and $-3/2$, thus it is a saddle.

Similarly, $(0, 3/2)$ is also a saddle.

At $(1,1)$,

$$J(x,y) = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix},$$

which eigenvaluies are -1 and -3. This, this is a stable node.

c) As shown in class, there is no closed orbit. Recall that this is shown by taking $\phi = \frac{1}{xy}$, then

$$\nabla \cdot (\phi f) = \nabla \cdot \begin{bmatrix} 3 - 2x - y \\ 3 - x - 2y \end{bmatrix} = -4 < 0.$$

Thus, by the Bendixson Theorem, there is no closed orbit. From a), any orbit starting in the interior of the first quadrant, say $\Gamma = \{\phi^t(x_0, y_0) : x_0, y_0 > 0\}$ is positively bounded. Thus, from the Poincare-Bendixson theorem, $\omega(\Gamma)$ must then be a fixed point.

The fixed point cannot be $(0,0)$ because it is repelling, i.e., there is no $t_n \to \infty$ such that $\phi^{t_n}(x_0, y_0) \to 0$.

The fixed point in $\omega(\Gamma)$ cannot be a saddle, because otherwise it has to include its stable manifold W_s, and thus also $\omega(W_s)$. Since $\omega(W_s)$ must also be a fixed point, and it cannot be $(0,0)$ with the reasoning above. If it is $(0, 3/2)$ or $(3/2, 0)$, then we have a hetero- or hmoclinic orbit, which does not exist in this system (excluded by the Bendixson Theorem). This $\omega(W_s) = (1,1)$. But $(1,1)$ is attractive, thus, there is an $\varepsilon > 0$ such that for all solution starting in the ε neighborhood $(1,1)$, namely $\mathcal{N}_\varepsilon$, the solution must appraoch $(1,1)$, and since W_s eventually falls into $\mathcal{N}_\varepsilon$, and so does Γ, Γ must approach $(1,1)$, which conflicts with the assumption that a saddle is in $\omega(\Gamma)$.

This leaves with only one possibility, $\omega(\Gamma) = (1,1)$, that is, any orbit Γ must approach the equilibrium $(1,1)$.