1. Consider the system (this is a SIRS disease model with imperfect vaccination, where $V = 1 - S - I - R$ is the fraction of vaccinated population)

$$\frac{dS}{dt} = -rSI - \phi S + \theta(1 - S - I - R) + \rho R$$
$$\frac{dI}{dt} = rSI + \sigma r(1 - S - I - R)I - I,$$
$$\frac{dR}{dt} = I - \rho R$$

   (a) Show that
$$(\frac{\theta}{\phi + \theta}, 0, 0)$$
   is always an equilibrium.

   (b) Discuss the bifurcartion of this equilibrium with the parameter r, and draw bifurcation diagram (equilibrium as a function of r) about the bifurcation point for the two cases:

      i. θ is very small,

      ii. θ is very large.

Solution:

a) Simply plugging the equilibrium into the right hand side of the equations to verify. Details omitted.

b) The Jacobian matrix is

$$J = \begin{bmatrix} -rI - \phi - \theta & -rS - \theta & \rho - \theta \\ rI - \sigma rI & rS + \sigma r(1 - S - 2I - R) - 1 & -\sigma rI \\ 0 & 1 & -\rho \end{bmatrix}$$

At the equilibrium,

$$J = \begin{bmatrix} -\phi - \theta & -r\frac{\theta}{\phi+\theta} - \theta & \rho - \theta \\ 0 & r\frac{\theta+\sigma\phi}{\phi+\theta} - 1 & 0 \\ 0 & 1 & -\rho \end{bmatrix}$$

Check (SN1): When
$$r\frac{\theta + \sigma\phi}{\phi + \theta} = 1,$$
i.e.,
$$r = r^* = \frac{\theta + \phi}{\theta + \sigma\phi},$$

$$J = \begin{bmatrix} -\phi - \theta & -r^*\frac{\theta}{\phi+\theta} - \theta & \rho - \theta \\ 0 & 0 & 0 \\ 0 & 1 & -\rho \end{bmatrix}$$

has a simple zero eigenvalue, with a left eigenvector $w = (0, \frac{1}{\rho}, 0)$. and a right eigenvector

$$v = (\frac{\rho - \theta - \theta\rho - \frac{\theta}{\theta+\phi} r^* \rho}{\theta + \phi}, \rho, 1)$$

Thus, (SN1) is satisfied.

Check (SN2):

$$\begin{aligned}
\alpha &= w \frac{\partial f}{\partial r}(\frac{\theta}{\theta + \phi}, 0, 0, r^*) \\
&= w \left[\begin{array}{c} -SI \\ SI + \sigma(1 - S - I - R)I \\ 0 \end{array} \right]_{I=0} \\
&= w \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right] = 0
\end{aligned}$$

Thus, the condition (SN2) is not satisfied.

Check (TC2):

$$\begin{aligned}
\alpha &= w \frac{\partial J}{\partial r}(\frac{\theta}{\theta + \phi}, 0, 0, r^*) v \\
&= w \left[\begin{array}{ccc} 0 & -\frac{\theta}{\phi+\theta} & 0 \\ 0 & \frac{\theta+\sigma\phi}{\phi+\theta} & 0 \\ 0 & 0 & 0 \end{array} \right] v \\
&= \frac{\theta + \sigma\phi}{\phi + \theta} > 0,
\end{aligned}$$

Thus, (TC2) is satisfied.

Check (TC3):

$$\begin{aligned}
\beta &= w \frac{1}{2} \left[\begin{array}{c} v^T D^2 f_1(\frac{\theta}{\theta+\phi}, 0, 0, r^*) v \\ v^T D^2 f_2(\frac{\theta}{\theta+\phi}, 0, 0, r^*) v \\ v^T D^2 f_3(\frac{\theta}{\theta+\phi}, 0, 0, r^*) v \end{array} \right] \\
&= w \left[\begin{array}{c} \partial_{SI} f_1 v_1 v_2 \\ \partial_{SI} f_2 v_1 v_2 + \frac{1}{2} \partial_{II} f_2 v_2^2 + \partial_{IR} f_2 v_2 v_3 \\ 0 \end{array} \right] \\
&= \frac{1}{\rho} \left(\partial_{SI} f_2 v_1 v_2 + \frac{1}{2} \partial_{II} f_2 v_2^2 + \partial_{IR} f_2 v_2 v_3 \right) \\
&= \frac{1}{\rho} \left(r^*(1 - \sigma) v_1 v_2 - \frac{r^* \sigma}{2} v_2^2 - r^* \sigma v_2 v_3 \right) \\
&= r^* \left((1 - \sigma) v_1 - \frac{\sigma}{2} v_2 - \sigma v_3 \right) \\
&= r^* \left((1 - \sigma) \frac{\rho - \theta - \theta\rho - \frac{\theta}{\theta+\phi} r^* \rho}{\theta + \phi} - \frac{\sigma\rho}{2} - \sigma \right).
\end{aligned}$$

i) When $\theta \ll 1$, $r^* \approx 1/\sigma$,

$$\beta \approx \frac{(1-\sigma)\frac{\rho}{\phi} - \frac{\sigma\rho}{2} - \sigma}{\sigma}.$$

If $\beta \neq 0$, i.e., $(1-\sigma)\frac{\rho}{\phi} \neq \sigma(\frac{\rho}{2} - 1)$, (TC3) is satisfied, and thus on the center manifold the system can be simplified to

$$\frac{dx}{dt} = \alpha(r - r^*) + \beta x^2 + o(x^2),$$

and there is transcritical bifurcation at $r = r^*$.

Since $\alpha > 0$, the origin is asymptotically stable if $r < r^*$, and unstable if $r > r^*$.

If $\beta > 0$, i.e., $(1-\sigma)\frac{\rho}{\phi} > \sigma(\frac{\rho}{2} - 1)$, then there is a positive equilibrium that is unstable when $r < r^*$, and a negative equilibrioum that is asymptotically stable when $r > r^*$. If $\beta < 0$, i.e., $(1-\sigma)\frac{\rho}{\phi} < \sigma(\frac{\rho}{2} - 1)$, then there is a negative equilibrium that is unstable when $r < r^*$, and a positive equilibrioum that is asymptotically stable when $r > r^*$.

ii) When $\theta \gg 1$, $r^* \approx 1$,

$$\beta \approx (1-\sigma)(1-\rho) - \frac{\sigma\rho}{2}.$$

If $\beta > 0$, i.e., $(1-\sigma)(1-\rho) > \frac{\sigma\rho}{2}$, then there is a positive equilibrium that is unstable when $r < r^*$, and a negative equilibrioum that is asymptotically stable when $r > r^*$. If $\beta < 0$, i.e., $(1-\sigma)(1-\rho) < \frac{\sigma\rho}{2}$, then there is a negative equilibrium that is unstable when $r < r^*$, and a positive equilibrioum that is asymptotically stable when $r > r^*$.

The bifurcation diagrams are omitted.