Quiz 3 — CSE 105, Winter 2006

Name (print): ____________________________

Student I.D.: __________________

- This quiz is closed book. You are only allowed to use one page of notes (double sided is fine)
- No form of collaboration is allowed during the quiz, including sharing notes, borrowing pencils, etc.
- Your solution will be evaluated both for correctness and clarity. A poorly written solution won't get full credit even if correct.
- Read all the problems first before you start working on any of them, so you can manage your time wisely.
- Good luck!

Problem	Points	Score
1	7	
2	7	
3	6	
4	10	
5	10	
Total	40	

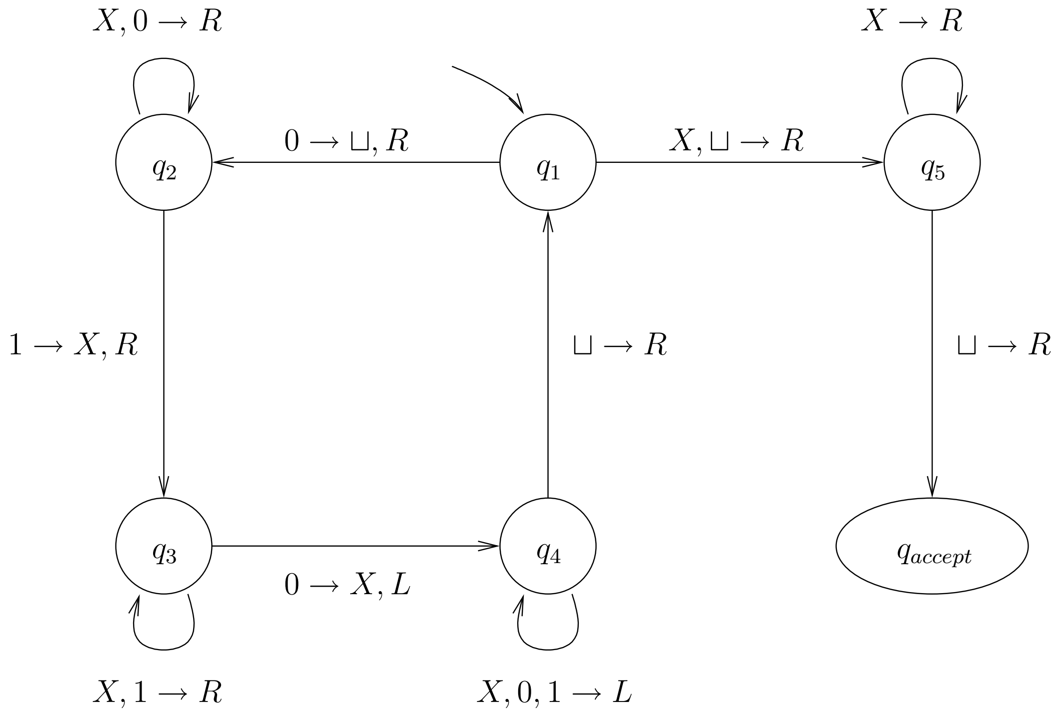

Figure 1: M_1. The input alphabet is $\Sigma = \{0, 1\}$ and the tape alphabet is $\Gamma = \{0, 1, X, \sqcup\}$. To save space, the reject state has been omitted; any transitions not explicitly shown go directly to the reject state q_{reject}.

Problem 1 [7 points]

Show the sequence of configurations that M_1 enters into on the following input strings (see figure 1). Note that any transitions not explicitly shown in the diagram go directly to the reject state, q_{reject} (which also isn't shown in the figure) and move the tape head to the right without changing the tape contents.

- 0

  Solution: $\{q_1 0\} \to \{\sqcup q_2\} \to \{\sqcup \sqcup q_{reject}\}$

- 01

  Solution: $\{q_1 01\} \to \{\sqcup q_2 1\} \to \{\sqcup X q_3\} \to \{\sqcup X \sqcup q_{reject}\}$

- 010

  Solution: $\{q_1 010\} \to \{\sqcup q_2 10\} \to \{\sqcup X q_3 0\} \to \{\sqcup XX q_4\} \to \{\sqcup X q_4 X\} \to \{\sqcup q_4 XX\} \to \{q_4 \sqcup XX\} \to \{\sqcup q_1 XX\} \to \{\sqcup X q_5 X\} \to \{\sqcup XX q_5\} \to \{\sqcup XX \sqcup q_{accept}\}$

What language does M_1 decide?

Solution: $L(M_1) = \{0^n 1^n 0^n, n \geq 0\}$

Problem 2 [7 points]

Give a high-level description of a Turing machine that decides the following language,

$$L = \{1^n\#w \mid w \text{ is the binary encoding of n }\}.$$

For example, the strings 111#11, 1111#0100, and #0 are in L, but 111#1 and 1111#10 aren't.

Solution:

$M_L =$ "On input x

- Parse x into $c = 1^n$ and w.
- Set $w' \leftarrow w$ and $c' \leftarrow c$.
- Loop until c' contains all zeros
  - Set one of the 1s in c' to a 0.
  - If w' contains all zeros, then stop looping and reject.
  - If $w' == s1$ then set $w' \leftarrow s0$
  - If $w' == s10^m$ $(m > 0)$, then set $w' \leftarrow s01^m$
- If w' contains all zeros then accept. Otherwise reject.

Problem 3 [6 points]

Define languages $\{L_A, ..., L_F\}$ in terms of languages L_1 and L_2 as follows,

- $L_A = L_1 \cup L_2$
- $L_B = L_1 \cap L_2$
- $L_C = L_1 \setminus L_2$ (the set difference between L_1 and L_2)
- $L_D = \overline{L_1}$
- $L_E = L_1^*$
- $L_F = L_1 \circ L_2$

If L_1 and L_2 are decidable languages, which of the following are decidable (circle your answers)?

Solution: $\boxed{L_A}$ $\boxed{L_B}$ $\boxed{L_C}$ $\boxed{L_D}$ $\boxed{L_E}$ $\boxed{L_F}$

If L_1 and L_2 are Turing-recognizable, which of the following are Turing-recognizable?

Solution: $\boxed{L_A}$ $\boxed{L_B}$ L_C L_D $\boxed{L_E}$ $\boxed{L_F}$

Problem 4 [10 points]

For each of the following, circle true or false and give a brief explanation.

$\boxed{\text{T}}$ / F Define a *static input Turing Machine* as a TM that is identical to a normal TM, except that it can't write to the input portion of its tape. If M is a static input TM, $L(M)$ can always be recognized by some DFA.
Explanation:

Solution: True. Static input TMs are equivalent to DFAs that can read the input in both directions which are equivalent to standard DFAs. (For the curious, the equivalence of the two types of DFA was shown by Rabin and Scott some 50 years ago in their paper *"Finite automata and their decision problems", IBM Journal of Research and Development, 3 (1959), 114-125.*)

T / $\boxed{\text{F}}$ The union of a countable set of regular languages is always regular.
Explanation:

Solution: False. In fact, every language is the union of a countable set of regular languages, each containing a single string. Clusure under pairwise union does not imply closure under infinite union.

$\boxed{\text{T}}$ / F The set of decidable languages is closed under symmetric difference. Recall, the symmetric difference operator is defined as,

$$L_1 \oplus L_2 = \left(L_1 \cap \overline{L_2}\right) \cup \left(\overline{L_1} \cap L_2\right)$$

Explanation:

Solution: True. Since decidable languages are closed under complement, union, and intersection it directly follows that decidable languages are closed under symmetric difference.

$\boxed{\text{T}}$ / F If L_1 is undecidable and L_2 is decidable, then $L_1 \oplus L_2$ must be undecidable, where $\oplus$ denotes the symmetric difference operator,

$$L_1 \oplus L_2 = \left(L_1 \cap \overline{L_2}\right) \cup \left(\overline{L_1} \cap L_2\right)$$

Explanation:

Solution: True. Assume for the purpose of contradiction that $L_1 \oplus L_2$ is decidable. Then $(L_1 \oplus L_2)\, L_2$ is decidable, because the set of decidable languages is closed under symmetric difference. However $(L_1 \oplus L_2)\, L_2 = L_1$, which is undecidable by assumption. This gives a contradiction.

T / $\boxed{\text{F}}$ The intersection of a recognizable language and an unrecognizable language is always unrecognizable.
Explanation:

Solution: False. If L_1 is any unrecognizable language and L_2 is the recognizable language $\emptyset$, then $L_1 \cap L_2 = \emptyset$ is recognizable.

Problem 5 [10 points]

Let language L be defined as,

$$L = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are Turing machines and } L(M_1) \subseteq L(M_2)\}.$$

Prove that L is undecidable by showing a reduction from one of the following languages: A_{TM}, E_{TM}, EQ_{TM}, or $HALT_{TM}$.

Solution:

Note there are many correct solutions to this problem. Here we show two proofs, a Turing reduction and a mapping reduction.

Method 1: Turing Reduction

Claim. *L is undecidable.*

Proof. By reduction from E_{TM}.

Assume L is decidable. Then there exists at TM M_L that decides L. We now construct a TM S to decide E_{TM}, which operates as follows,

$S =$ "On input $\langle M\rangle$, an encoding of a TM M:

1. Construct a new TM M' that rejects on all inputs (ie, $L(M') = \emptyset$).
2. Run M_L on input $\langle M, M'\rangle$
3. If M_L accepts, then accept
4. Otherwise reject.

To see that S decides E_{TM}, note that $M_L(\langle M, M'\rangle)$ accepts if $L(M) = \emptyset$ and rejects if $L(M) \neq \emptyset$ (due to the fact that $L(M') = \emptyset$). Therefore $S(\langle M\rangle)$ accepts if $L(M) = \emptyset$ and rejects otherwise. Since each step of S is guaranteed not to loop, S is a decider for E_{TM} implying E_{TM} is decidable. However, this is a contradiction because we know that E_{TM} is undecidable. Therefore our assumption that L is decidable must be false. $\square$

Method 2: Mapping Reduction

Claim. *L is undecidable.*

Proof. $E_{TM} \leq_m L$.

Consider a mapping function f that takes input of the form $\langle M\rangle$ and returns output of the form $\langle M_1, M_2\rangle$, where f is computed by the following machine,

$F =$ "On input $\langle M\rangle$:

1. Set $M_1 = M$.

2. Construct a new TM M_2 that rejects on all inputs.

3. Output $\langle M_1, M_2 \rangle$.

To show that f is a mapping function from E_{TM} to L, we need to show that f is computable and that $\langle M \rangle \in E_{TM}$ if and only if $\langle M_1, M_2 \rangle \in L$. f is trivially computable because each step in F halts, implying F always halts. It is also simple to verify that $\langle M \rangle \in E_{TM}$ if and only if $\langle M_1, M_2 \rangle \in L$,

- $\langle M \rangle \in E_{TM} \implies \langle M_1, M_2 \rangle \in L$: If $\langle M \rangle \in E_{TM}$ then $L(M) = L(M_1) = \emptyset$. Since $\emptyset \subseteq L(M_2)$ this implies $\langle M_1, M_2 \rangle \in L$.

- $\langle M_1, M_2 \rangle \in L \implies \langle M \rangle \in E_{TM}$: If $\langle M_1, M_2 \rangle \in L$ then $L(M_1) = L(M) \subseteq L(M_2)$. Since $L(M_2) = \emptyset$, this implies $L(M) = \emptyset$ which means $\langle M \rangle \in E_{TM}$.

The existence of a computable mapping function from E_{TM} to L and the fact that E_{TM} is undecidable means that L is also undecidable.

$\square$