Math 442/551 Lecture 2

September 6, 2018

Gronwall's inequality

We have seen estimates on functions given in differential or integral forms. For example, given $\phi'(t) \leq \psi'(t)$ for $t \in [t_0, t_0 + a]$, $\phi(t_0) = \psi(t_0)$, integrating on both sides gives

$$\phi(t) - \phi(t_0) \leq \psi(t) - \psi(t_0),$$

which is the same as

$$\phi(t) \leq \psi(t),$$

which holds for all $t \in [t_0, t_0 + a]$. Similarly, if $\phi'(t) \leq \psi(t)$ for $t \in [t_0, t_0 + a]$, then

$$\phi(t) \leq \phi(t_0) + \int_{t_0}^{t} \psi(s) ds\,,$$

which holds for all $t \in [t_0, t_0 + a]$.

Can we give an estimate on $\phi(t)$ that satisfies $\phi' \leq \psi(t)\phi$? Alternatively, in a more general form given by integrating on both sides, if

$$\phi(t) \leq \phi_0 + \int_0^t \phi(s)\psi(s) ds, \tag{1}$$

can we give an estimate on $\phi(t)$?

- Theorem (Gronwall): if $\phi(t) \leq \phi(t_0) + \int_{t_0}^{t} \psi(s)\phi(s) ds$, for $t \in [t_0, t_0 + a]$, where a is a positive constant, and ϕ and ψ are nonnegative continuous functions, then

$$\phi(t) \leq \phi(t_0) e^{\int_{t_0}^{t} \psi(s) ds}.$$

Note that this estimate is called the Gronwall's inequality. Why do we need such an estimate? We will see this estimate again and again in this course. Here we introduct a useful theorem that can be proved by this Gronwall's inequality.

- Theorem (Continuous dependency on initial conditions): Consider the following system $x' = f(t, x)$, where $x \in \mathbb{R}^n$, and $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is Lipschitz in x (with Lipschitz constant L) and continuous on an interval $I = [t_0, t_0 + a]$. Then, for two solutions $x_1(t)$ and $x_2(t)$ on the interval I,

$$\|x_1(t) - x_2(t)\| \leq \|x_1(t_0) - x_2(t_0)\| e^{L(t-t_0)}.$$

Note that a function $f(x)$ is called Lipschitz continuous on an interval I if

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

for any $x, y \in I$. Also, Lipscitch continuous implies continuous.

- Proof: We integrate the DE on I on both sides,

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))ds\,.$$

Since x_1 and x_2 are both solutions, then

$$x_1(t) = x_1(t_0) + \int_{t_0}^{t} f(s, x_1(s))ds\,,$$
$$x_2(t) = x_2(t_0) + \int_{t_0}^{t} f(s, x_1(s))ds\,.$$

take difference,

$$x_1(t) - x_2(t) = x_1(t_0) - x_2(t_0) + \int_{t_0}^{t} f(s, x_1(s)) - f(s, x_2(s))ds$$

thus

$$\begin{aligned} \|x_1(t) - x_2(t)\| &\leq \|x_1(t_0) - x_2(t_0)\| + \int_{t_0}^{t} \|f(s, x_1(s)) - f(s, x_2(s))\|ds \\ &\leq \|x_1(t_0) - x_2(t_0)\| + \int_{t_0}^{t} L\|x_1(s) - x_2(s)\|ds \end{aligned}$$

Apply the Gronwall's inequality to $\phi(t) = \|x_1(t) - x_2(t)\|$ and $\psi(t) = L$, and get

$$\|x_1(t) - x_2(t)\| \leq \|x_1(t_0) - x_2(t_0)\|e^{L(t-t_0)}.$$

This theorem shows that, on any finite time interval I, two solutions can be arbitrarily close if their initial conditions are close enough.

Now lets prove the Gronwall's inequality.

- Proof of Gronwall's inequality: Let $f(t) = \phi(t_0) + \int_{t_0}^{t} \psi(s)\phi(s)ds$. Note that $f(t) > 0$ and $f'(t) = \psi(s)\phi(s)$. Then the condition (1) becomes

$$\frac{\phi(t)}{f(t)} \leq 1\,.$$

Since $\psi(s)$ is a nonnegative function,

$$\frac{\phi(t)\psi(t)}{f(t)} = \frac{f'(x)}{f(x)} \leq \psi(t).$$

Integrating no both sides of this inequality gives

$$\int_{t_0}^{t} \frac{f'(s)}{f(s)} ds = \ln f(t) - \ln f(t_0) \leq \int_{t_0}^{t} \psi(s) ds\,,$$

which is the same as

$$f(t) \leq f(t_0) e^{\int_{t_0}^{t} \psi(s) ds} = \phi_0 e^{\int_{t_0}^{t} \psi(s) ds}$$

and from the condition (1), $\phi(t) \leq f(t)$. Thus

$$\phi(t) \leq \phi_0 e^{\int_{t_0}^{t} \psi(s) ds}.$$

A more general form: If $\phi(t) \leq \alpha(t) + \int_{t_0}^{t} \psi(s)\phi(s) ds$ on an interval I, where $\alpha$is a real valued function with negative part integrable on I, ϕ is a continuous function, and $\psi(t)$ is a nonnegative continuous function, then

$$\phi(t) \leq \alpha(t) + \int_{t_0}^{t} \alpha(s)\psi(s) e^{\int_{s}^{t} \psi(u) du} ds$$

and if, in addition, $\alpha(t)$ is nondecreasing, then

$$\begin{aligned}
\phi(t) &\leq \alpha(t) + \alpha(t) \int_{t_0}^{t} \psi(s) e^{\int_{s}^{t} \psi(u) du} ds \\
&= \alpha(t) + \alpha(t) \left[-e^{\int_{s}^{t} \psi(u) du} \right]_{t_0}^{t} \\
&= \alpha(t) e^{\int_{s}^{t} \psi(u) du}.
\end{aligned}$$