MATH 442/551 Assignment #1

Due Thursday September 20, in class

1. Use the following Banach Fixed-Point Theorem to prove the Picard-Lindelof version fo teh Existence and Uniqueness Theorem (listed below).

   - Definition: Let X be a norned vector space, the map $T : X \to X$ is called a contraction mapping on X if there exists a constant $q \in [0, 1)$ such that $\|T(x) - T(y)\| \leq q\|x - y\|$ for all $x, y \in X$.
   - Banach Fixed-Point Theorem: Let X be a complete Banach space, and T be a contraction mapping on X, then T has a unique fixed point $x^* \in X$, i.e., $T(x^*) = x^*$.
   - Picard-Lindelof Theorem: Consider the initial value problem

     $$\frac{dx}{dt} = f(t, x),\ x(t_0) = x_0,$$

     , where $x \in \mathbb{R}^n$, $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous in t and Lipschitz continuous in X on a region $G \subset \mathbb{R}^{n+1}$ with non-empty interior, which interior contains (t_0, x_0). Then, there exists a positive constatn $h \leq a$ such that the initial value problem has a unique solution on $[t_0 - h, t_0 + h]$.

Proof: Integrate on both sides of the equation and rewrite it as an integral equation:

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \tag{1}$$

Consider the Banach space $C_{[t_0-h,t_0+h]}$ for some small positive constantg h, which value will be determined later. Then the right hand side of the integral equation defines an operator on $C_{[t_0-h,t_0+h]}$, namely,

$$F.(x) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

Nerxt, we will show that, for small enough h, this map is a contraction mapping. Note that, for any $x(t), y(t) \in C_{[t_0-h,t_0+h]}$,

$$
\begin{aligned}
\|F(x(t)) - F(y(t))\| = \| \int_{t_0}^{t} f(s, x(s)) - f(s, y(s)) ds\| \\
&\leq \int_{t_0}^{t} \|f(s, x(s)) - f(s, y(s))\| ds \\
&\leq | \int_{t_0}^{t} L\|x(t) - y(t)\| ds| \\
&= L|t - t_0| \|x(t) - y(t)\|.
\end{aligned}
$$

Thus, as long as $h < 1/L$, then, $L(t - t_0| \leq Lh < 1$, and thus F is a contraction map. By the Banach fixed point theorem, F has a unique fixed point, i.e., there is a unique $x(t)$ satisfying (1), which is thus a unique solution of the initial value problem.

2. Continuous Dependence on parameters: Consider the initial value problem

$$\frac{dx}{dt} = f(t, x; p), \ x(t_0) = x_0,$$

where $p \in \mathbb{R}^m$ is a parameter of the model (a constant vector), $x \in \mathbb{R}^n$, and $f : \mathbb{R}^{n+m+1} \to \mathbb{R}^n$ is a continuous function on a region $G \subset \mathbb{R}^{n+m+1}$ with non-empty interior, and Lipscitz continuous in (x, p) on G with a Lipschitz constant L. If $x_1(t; t_0, x_0, p_1)$ and $x_2(t; t_0, x_0, p_2)$ are two solutions to the IVP defined on an interval I, with parameter valuex $p = p_1$ and $p = p_2$, respectively.

(a) Show that, for all $t \in I$,

$$|x_1(t; t_0, x_0, p_1) - x_2(t; t_0, x_0, p_2)| \leq |p_1 - p_2| e^{L(t-t_0)}.$$

(b) Show that, for all fixed t, $x(t; t_0, x_0, p)$ is a continuous function of the parameter p.

Proof: For Part (a), consider the system

$$
\begin{aligned}
\frac{dx}{dt} &= f(t, x, p), \\
\frac{dp}{dt} &= 0,
\end{aligned}
$$

Note that, since $f(t, x, p)$ is Lipschitz continuous in (x, p), so is the right hand side $(f(t, x, p)^T, 0^T)^T$. Thus,b y the theorem of continuous dependence on initial conditions,

$$
\begin{aligned}
|x_1(t; t_0, x_0, p_1) - x_2(t; t_0, x_0, p_2)| &\leq \left| \begin{pmatrix} x_1(t; t_0, x_0, p_1) \\ p_1 \end{pmatrix} - \begin{pmatrix} x_2(t; t_0, x_0, p_2) \\ p_2 \end{pmatrix} \right| \\
&\leq \left| \begin{pmatrix} x_1(t_0) \\ p_1 \end{pmatrix} - \begin{pmatrix} x_2(t_0) \\ p_2 \end{pmatrix} \right| e^{L(t-t_0)} \\
&= |p_1 - p_2| e^{L(t-t_0)}
\end{aligned}
$$

For Part (b), note that for any fixed time t and all $\varepsilon > 0$, pick $|p_1 - p_2| \leq \varepsilon e^{-L(t-t_0)}$, then $|x(p_1) - x(p_1)| \leq \varepsilon$. Thus, for any fixed time t, the solution to the initial value problem is continuous as a funcrtion of the parameter p.

3. Consider the initial value problem

$$\frac{dx}{dt} = Ax + B(t)x,\ x(t_0) = x_0,$$

where $x \in \mathbb{R}^n$, A is a constant $n \times n$ matrix which eigenvalues all have negative real part; $B(t)$ is an $n \times n$ continuous matrix function, with $\int_{t_0}^{\infty} \|B(s)\| ds < M$ for some constant $M > 0$ (note that this condition guarantees that $B(t) \to 0$ as $t \to \infty$). Here, for each t, the matrix norm is defined as

$$\|B(t)\| = \max_{\|x\| \neq 0} \frac{\|B(t)x\|}{\|x\|},$$

and thus $\|B(t)x\| \leq \|B(t)\| \|x\|$ for all x.

(a) Show that its solution satisfies

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}B(s)x(s)ds\,.$$

(b) Note that all eigenvalues of A has negative real parts, thus $\exists C > 0$ such that $\|e^{At}\| \leq C$ for all t. Use this fact to show that the origin is stable, i.e., $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\|x(t)\| \leq \varepsilon$ as long as $\|x_0\| \leq \delta$.

Proof: For Part (a), we will treat $B(t)x$ as a forcing term (i.e., a known function), then the original system can be solved as a forced (non-homogeneous) linear system. With the integrating factor e^{-At},

$$\frac{d}{dt}(e^{-At}x) = e^{-At}\frac{dx}{dt} - e^{-At}Ax = e^{-At}B(t)x.$$

Integrate on both sides from t_0 to t,

$$e^{-At}x(t) - e^{-At_0}x_0 = \int_{t_0}^{t} e^{-As}B(s)x(s)ds,$$

and thus,

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}B(s)x(s)ds.$$

For Part (b), note that

$$\begin{aligned}
\|x(t)\| &\leq \|e^{A(t-t_0)}x_0\| + \|\int_{t_0}^{t} e^{A(t-s)}B(s)x(s)\|ds. \\
&\leq \|e^{A(t-t_0)}\|\|x_0\| + \int_{t_0}^{t} \|e^{A(t-s)}\|\|B(s)\|\|x(s)\|ds \\
&\leq C\|x_0\| + \int_{t_0}^{t} C\|B(s)\|\|x(s)\|ds.
\end{aligned}$$

Apply the Gronwall's inequality,

$$\|x(t)\| \leq C\|x_0\|e^{C\int_{t_0}^{t}\|B(s)\|ds}.$$

Let

$$M = e^{\int_{t_0}^{\infty}\|B(s)\|ds} < +\infty,$$

then, $\forall \varepsilon > 0$, take $\delta = \varepsilon/(M^C C)$, so that for all $\|x_0\| \leq \delta$,

$$\|x\| \leq C\delta M^C \leq \varepsilon.$$