## Problem1

**(a)**

For $R_1$ , $R_2$ and $R_3$, we let $R_1 \subseteq A\times B$, $R_2 \subseteq B\times C$ and $R_3 \subseteq C\times D$;

And we let $a \in A$, $b \in B$, $c \in C$, $d \in D$

Suppose $(a, d) \in (R_1 ; R_2) ; R_3$ , we have :

there is a c with $(a, c) \in R_1 ; R_2$ and $(c, d) \in R_3$

$\leftrightarrow \exists c \{ \exists b [ (a, b) \in R_1 \text{ and } (b, c) \in R_2 ] \text{ and } (c, d) \in R_3 \}$

$\leftrightarrow \exists c \exists b \{ (a, b) \in R_1 \text{ and } (b, c) \in R_2 \text{ and } (c, d) \in R_3 \}$

$\leftrightarrow \exists b \exists c \{ (a, b) \in R_1 \text{ and } (b, c) \in R_2 \text{ and } (c, d) \in R_3 \}$

$\leftrightarrow \exists b \{ [ (a, b) \in R_1 ] \text{ and } \exists c [ (b, c) \in R_2 \text{ and } (c, d) \in R_3 ] \}$

$\leftrightarrow \exists b \{ (a, b) \in R_1 \text{ and } (b, d) \in R_3 \}$

$\leftrightarrow (a, d)\ (a, d) \in R_1 ; (R_2 ; R_3)$

**(b)**

For $R_1$ and I, we let $R_1 \subseteq A\times B$, $I \subseteq S\times S$;

Suppose $(a, b) \in I ; R_1$ , we have :

there is an a with $(a, a) \in I$ and $(a, b) \in R_1$

$\leftrightarrow \exists a \{ [ (a, a) \in I ] \text{ and } (a, b) \in R_1 \}$

$\leftrightarrow \exists a \{ (a, a) \in I \text{ and } (a, b) \in R_1 \}$

$\leftrightarrow \exists a \exists b \{ (a, a) \in I \text{ and } (a, b) \in R_1 \text{ and } (b, b) \in I \}$

$\leftrightarrow \exists b \{ (a, b) \in R_1 \text{ and } (b, b) \in I \}$

$\leftrightarrow$ $(a, b) \in R_1 ; I$

$\leftrightarrow$ $\exists a \exists b \{ (a, b) \in R_1 \} \leftrightarrow (a, b) \in R_1$

**(c)**

Counterexample: let $R_1 = \{(1, 3)\}$, $R_2 = \{(3, 7)\}$

So $R_1 ; R_2 = \{ (1, 7) \}$ : there is a 3 with $(1, 3) \in R_1$ and $(3, 7) \in R_2$

And $(R_1 ; R_2)^{\leftarrow} = \{ (7, 1) \}$

$R_1^{\leftarrow} ; R_2^{\leftarrow} = \Phi$

because there doesn't exist an S with $(3, S) \in R_1^{\leftarrow}$ and $(S, 3) \in R_2^{\leftarrow}$

so it is not True.

**(d)**

Suppose $(a, c) \in (R_1 \cup R_2) ; R_3$, we have :

there is a b with $(a, b) \in R_1 \cup R_2$ and $(b, c) \in R_3$

$\leftrightarrow$ $\exists b \{ [ (a, b) \in R_1 \lor (a, b) \in R_2 ] \land (b, c) \in R_3 \}$

$\leftrightarrow$ $\exists b \{ [ (a, b) \in R_1 \land (b, c) \in R_3 ] \lor [ (a, b) \in R_2 \land (b, c) \in R_3 ] \}$ (using distribution)

$\leftrightarrow$ $\exists b \{ [ (a, b) \in R_1 \land (b, c) \in R_3 ] \} \lor \exists b [ (a, b) \in R_2 \land (b, c) \in R_3 ] \}$

$\leftrightarrow$ $(a, c) \in (R_1 ; R_3) \lor (a, c) \in (R_1 ; R_3)$

$\leftrightarrow$ $(a, c) \in (R_1 ; R_3) \cup (R_2 ; R_3)$

$\leftrightarrow$ $(R_1 \cup R_2) ; R_3 = (R_1 ; R_3) \cup (R_2 ; R_3)$

**(e)**

Suppose $(a, c) \in R_1 ; (R_2 \cap R_3)$, we have :

there is a b with $(a, b) \in R_1$ and $(b, c) \in (R_2 \cap R_3)$

$\leftrightarrow \exists b \{ (a, b) \in R_1 \wedge [ (b, c) \in R_2 \wedge (b, c) \in R_3 ] \}$

$\leftrightarrow \exists b \{ [ (a, b) \in R_1 \wedge (b, c) \in R_2 ] \wedge [ (a, b) \in R_1 \wedge (b, c) \in R_3 ] \}$ ①

Here in ①, we have the same b with above condition, that means ① can be able to imply :

$\rightarrow \exists b \{ [ (a, b) \in R_1 \wedge (b, c) \in R_2 ]$ and $\exists b \{ [ (a, b) \in R_1 \wedge (b, c) \in R_3 ] \}$ ②

And here in ②, this two b can be different value, eg: like the first b have value 1 and the second b with the value of 3;

we can get ① → ②, but we cannot get ② → ① due to ∃ b difference;

②$\leftrightarrow (a, c) \in (R_1 ; R_2) \wedge (a, c) \in (R_1 ; R_3)$

$\leftrightarrow (a, c) \in (R_1 ; R_2) \cap (R_1 ; R_3)$

So, finally, we have $R_1 ; (R_2 \cap R_3) \rightarrow (R_1 ; R_2) \cap (R_1 ; R_3)$;

but $(R_1 ; R_2) \cap (R_1 ; R_3)$ -/-> $R_1 ; (R_2 \cap R_3)$

## Problem2

**(a)**

if there is an i such that $R^i = R^{i+1}$, that is :

① **[Basic Case]** when j = i, $R^j = R^i$ ;

② **[Inductive Step]** when j = i, we let j = k, $R^k = R^j = R^i$ ;

And then $j = k+1 > i$: $R^j = R^{k+1} = R^k \cup (R ; R^k) = R^i \cup (R ; R^i) = R^{i+1} = R^i$;

$\therefore$ if there is an i such that $R^i = R^{i+1}$, then $R^j = R^i$ for all $j \geq i$

**(b)**

① As I proved in (a) : if $k \geq i$, then we have $R^k = R^i \rightarrow R^k \subseteq R^i$ ;

② now if $0 \leq k < i$, we have $R^{k+1} = R^k \cup (R ; R^k) \rightarrow R^k \subseteq R^{k+1}$

Because $k<i$, so we have $R^k \subseteq R^i$;

That is : for $k \geq 0$, we have $R^k \subseteq R^i$

**(c)**

P(n) be the proposition that for all $m \in$ N: $R^n ; R^m = R^{n+m}$ ;

① **[Basic Case]** P(0) is that for all $m \in$ N: $R^0 ; R^m = R^m$;

We have proved that $I ; R_1 = R_1 ; I = R_1$ (in problem1), so $R^0 ; R^m = I ; R^m = R^m$ ;

So P(0) holds;

② **[Inductive Step]** Assume P(n) holds : and that is : for all $m \in$ N: $R^n ; R^m = R^{n+m}$;

P(n+1) means for all $m \in$ N: $R^{n+1} ; R^m = R^{n+1+m} \rightarrow (R^n \cup (R ; R^n)) ; R^m =$

We have $R^{n+1} ; R^m = (R^n \cup (R ; R^n)) ; R^m = (R^n ; R^m) \cup ((R ; R^n) ; R^m)$

$= \quad (R^n ; R^m) \cup (R ; (R^n ; R^m)) = R^{n+m} \cup (R ; R^{m+n}) = R^{n+1+m}$;

So P(n+1) holds;

Finally, that is P(n) holds for all $n \in$ N;

**(d)**

For R and $R^0$, $R \subseteq S \times S$ can be any binary relation on set S, $R^0 = \{(x,x) : x \in S\}$;

To prove $R^k = R^{k+1}$, we can prove $R^k \subseteq R^{k+1}$ and $R^{k+1} \subseteq R^k$ instead;

By definition, $R^{k+1} = R^k \cup (R ; R^k)$;

→ $R^k \subseteq R^{k+1}$ because of the definition;

Now we explain why $R^{k+1} \subseteq R^k$ :

If $(a, b) \in R^{k+1}$ then $(a, b) \in R^k$ or $(a, b) \in (R ; R^k)$ (by definition)

First case ① : if $(a, b) \in R^k$ :

That is for any tuple $(a, b) \in R^{k+1}$, $(a, b) \in R^k$;

In this case, $R^{k+1} \subseteq R^k$ holds.

Second case ② : if $(a, b) \in (R ; R^k)$ :

Assuming that for any $k >= 0$ : That means $\exists\, m_k \{ (a, m_k) \in R$ and $(m_k, b) \in R^k \}$

→$\exists\, m_{k-1} \{ (m_k, m_{k-1}) \in R$ and $(m_{k-1}, b) \in R^{k-1} \}$

→$\exists\, m_{k-2} \{ (m_{k-1}, m_{k-2}) \in R$ and $(m_{k-2}, b) \in R^{k-1} \}$

→ ... ...

→$\exists\, m_1 \{ (m_2, m_1) \in R$ and $(m_1, b) \in R^1 \}$

→$\exists\, m_0 \{ (m_1, m_0) \in R$ and $(m_0, b) \in R^0 \}$

$\exists\, m_0$ { $(m_1, m_0) \in R$ and $(m_0, b) \in R^0$ },

➔ $m_0 = b$ , we have $(b, b) \in R^0$ by definition of $R^0$.

$\therefore m_0 \cdot m_1 \cdot \ldots \ldots m_{k-1} \cdot m_k \in S$, from $m_0$ to $m_k$ : there are at least (k+1) elements.

And we $|S| = k$, at least two of them are equal :

→there must exists $m_i$ and $m_j$ such that $m_i = m_j$ for $0 \le i < j \le k$;

So $(a, b) \in R^i$ with $i < k + 1$;

that is for any $(a, b) \in R^{k+1}$ , then $(a, b) \in (R ; R^k)$

from the above, $R^{k+1} \subseteq R^k$ and $R^k \subseteq R^{k+1}$, so we have $R^k = R^{k+1}$

**(e)**

Assume that for all $a, b, c \in S$,

If $(a, b) \in R^k$, $(b, c) \in R^k$

It is clear that $R^k ; R^k = \{(a, c) :$ there is a b with $(a, b) \in R^k, (b, c) \in R^k \}$ (by the definition of " ; ")

In the proof of (c) : we know $R^n ; R^m = R^{n+m}$, so $R^k ; R^k = R^{2k}$;

$\therefore (a, c) \in R^k ; R^k = R^{2k}$ ;

In the proof of (d) : If $|S| = k$, then we get $R^k = R^{k+1}$;

And in the proof of (a) : we know that there is a k such that $R^k = R^{k+1}$, then $R^j = R^k$ for all $j \ge k$

$\because 2k > k$

$\therefore R^{2k} = R^k \rightarrow (a, c) \in R^k ; R^k = R^{2k} = R^k \rightarrow$ that is $(a, c) \in R^k$;

$\because (a, b) \in R^k, (b, c) \in R^k$ and $(a, c) \in R^k$, so $R^k$ transitive when $|S| = k$

**(f)**

Firstly, we have the common premise of if |S| = k: so we can use the proof of (d) and (e) as a part of our proof in (f).

① **Reflexive :**

$(R\cup R^{\leftarrow})^0 = I = \{(x,x) : x \in S\}$, so $(R\cup R^{\leftarrow})^0$ is reflexive;

We let L represents relation $R\cup R^{\leftarrow}$, and $L^0$ is sysmetric because $L^0 = I$;

By definition, we have $L^1 = L^0 \cup ( L ; L^0)$ ,

And in the proof of question (c), we have : $R^n ; R^m = R^{n+m}$ ;

So $(R\cup R^{\leftarrow})^0 \subseteq (R\cup R^{\leftarrow})^1 \subseteq (R\cup R^{\leftarrow})^2 \ldots \ldots (R\cup R^{\leftarrow})^{k-1} \subseteq (R\cup R^{\leftarrow})^k$;

That is : for all (x, x), $(x, x) \in (R\cup R^{\leftarrow})^k$ is reflexive;

② Sysmetric :

**[Basic Case]** we know $(R\cup R^{\leftarrow})^0 = I = \{(x,x) : x \in S\}$;

We let L represents relation $R\cup R^{\leftarrow}$, which we know that relation $R\cup R^{\leftarrow}$ is sysmetric by itself, and $L^0$ is sysmetric because $L^0 = I$;

By definition, we have $L^1 = L^0 \cup ( L ; L^0)$

$\because$ $L^0$ is sysmetric, and $( L ; L^0)$ is sysmetric due to $( L ; L^0) = L$;

$\therefore$ $L^1$ is sysmetric ;

**[Inductive Step]** we assume $L^k$ is sysmetric, and then based on the proof in question (c) : $R^n;R^m = R^{n+m}$; we can get $L^1; L^k = L^{1+k}$, we need to prove if P(k) : $L^k$ is sysmetric holds, then P(k+1) : $L^{k+1}$ is sysmetric holds.

$L^{k+1} = L^{1+k} = L^1; L^k$;

Assume $(a, c) \in L^{1+k} = L^1; L^k$, then there is a c such that $(a, b) \in L^1$ and $(b, c) \in L^k$,

$\because$ both $L^1$ and $L^k$ are sysmetric, so there are (b, a) and (c, b) such that $(b, a) \in L^1$ and $(c, b) \in L^k$;

$\therefore$ $(c, a) \in L^{k+1} = L^{1+k} = L^1; L^k \rightarrow P(k+1)$ holds

So $L^k = (R \cup R^{\leftarrow})^k$ is sysmetric.

So for all k >= 0 :

$(R \cup R^{\leftarrow})^k$ is sysmetric.

③ Transitive :

As we have proved in (e), If $|S| = k$, show that $R^k$ is transitive, and R can be any binary relation on set S.

So we substitude R with $(R \cup R^{\leftarrow})^k$, so $(R \cup R^{\leftarrow})^k$ is transitive.

After we have proved ①②③ : $(R \cup R^{\leftarrow})^k$ is reflexive, sysmetric and transitive,so $(R \cup R^{\leftarrow})^k$ is an equivalence relation.

**Problem3**

**(a)**

recursive definition of the binary tree data structure is :

A binary tree is either :

- (Basic definition) an empty tree( with no successors), or
- (Recursive definition) a point pointing to two binary trees, one is left successor and the other is right successor.

**(b)** Counting the number of nodes in a binary tree T:

```
count(T):
    if(T.isEmpty()):                                        (B)
        return 0
    else:                                                   (R)
        return count(T.left_child) + count(T.right_child) + 1
```

**(c)** Counting the number of leaves in a binary tree T:

```
leaves(T):
    if(T.isEmpty()):                                        (B)
        return 0
    elif(T.left_child.isEmpty() && T.right_child.isEmpty()):   (B)
        return 1
```

```
    else:                                                       (R)
        return leaves(T.left) + leaves(T.right)
```

**(d)** Counting the number of fully-internal nodes in a binary tree T:

```
internal(T):
    if(T.isEmpty()):                                            (B)
        return 0
    elif(!T.left_child.isEmpty() && !T.right_child.isEmpty()):  (R)
        return internal(T.left) + internal(T.right) + 1
    else:                                                       (R)
        return internal(T.left) + internal(T.right)
```

**(e)**

we assign num_T(viewed as total number of nodes), num_$T_0$(viewed as leaves), num_$T_1$(viewed as a tree which has one child) and num_$T_2$(viewed as full-internal nodes)to represent three different kinds of Tree structures respectively:

and we create an equation according to the relation between the number of these three kinds of trees and the number of lines (using num_line to represent) connecting each node:

the fact is that a line comes from the head part of every node apart from the root node, so we get equations as below:

- num_T = num_$T_0$ + num_$T_1$ + num_$T_{2;}$

- num_line = num_T – 1
- num_line = $num\_T_1$ + 2 * $num\_T_{2;}$

→ $num\_T_0$ + $num\_T_1$ + $num\_T_2$ – 1 = $num\_T_1$ + 2 * $num\_T_{2;}$

→ $num\_T_2$ = $num\_T_0$ – 1

That is : leaves(T) = 1+internal(T)

∴ P(T) holds.

## Problem4

### (a)

Defining proposition “Alpha uses channel hi” as $A_H$, “Alpha uses channel lo” as $A_L$;

So does $B_H$, $B_L$, $C_H$, $C_L$, $D_H$, $D_L$;

**i.** $\phi 1 = ((((A_H \vee A_L) \wedge (B_H \vee B_L)) \wedge (C_H \vee C_L)) \wedge (D_H \vee D_L))$

**ii.** $\phi 2 = (((\neg(A_H \wedge A_L) \wedge \neg(B_H \wedge B_L)) \wedge \neg(C_H \wedge C_L)) \wedge \neg(D_H \wedge D_L))$

**iii.** $\phi 3 = ((\neg((A_H \wedge B_H) \vee (A_L \wedge B_L)) \wedge \neg((B_H \wedge C_H) \vee (B_L \wedge C_L))) \wedge \neg((C_H \wedge D_H) \vee (C_L \wedge D_L)))$

### (b)

**i.** in the situation of below:

| $A_H$ | $A_L$ | $B_H$ | $B_L$ | $C_H$ | $C_L$ | $D_H$ | $D_L$ |
|---|---|---|---|---|---|---|---|
| T | F | F | T | T | F | F | T |

$\phi 1$ = T, $\phi 2$ = T, $\phi 3$ = T; → so $\phi 1 \wedge \phi 2 \wedge \phi 3$ is satisfiable;

**ii.** Based on answer to the previous question, Alpha uses channel hi, Bravo uses channel lo, Charlie uses channel hi and Delta uses channel lo, they can avoid interfere with each other under this assignment, or

In another case that Alpha uses channel lo, Bravo uses channel hi, Charlie uses channel lo and Delta uses channel hi, they can also avoid interfere with each other.