

COMP9020 Week 5

Term 3, 2019

Algorithmic analysis

Summary of topics

- Motivation
- Standard approach
- Examples
- Simplifying with worst-case and big-O
- Recursive examples

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Algorithmic analysis: motivation

Want to compare algorithms – particularly ones that can solve *arbitrarily large* instances.

We would like to be able to talk about the resources (running time, memory, energy consumption) required by a program/algorithm as a function $f(n)$ of some parameter n (e.g. the size) of its input.

Example

How long does a given sorting algorithm take to run on a list of n elements?

Issues

Problems

- The exact resources required for an algorithm are difficult to pin down. Heavily dependent on:
 - Environment the program is run in (hardware, software, choice of language, external factors, etc)
 - Choice of inputs used

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Problems

- The exact resources required for an algorithm are difficult to pin down. Heavily dependent on:
 - Environment the program is run in (hardware, software, choice of language, external factors, etc)
 - Choice of inputs used
- Cost functions can be complex, e.g.

$$2n \log(n) + (n - 100) \log(n)^2 + \frac{1}{2^n} \log(\log(n))$$

Need to identify the “important” aspects of the function.

Order of growth

Example

Consider two time-cost functions:

- $f_1(n) = \frac{1}{10}n^2$ milliseconds, and
- $f_2(n) = 10n \log n$ milliseconds

Input size	$f_1(n)$	$f_2(n)$
100	0.01s	2s
1000	1s	30s
10000	1m40s	6m40s
100000	2h47m	1h23m
1000000	11d14h	16h40h
10000000	3y3m	8d2h

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Algorithmic analysis

Asymptotic analysis is about how costs **scale** as the input increases.

Standard (default) approach:

- Consider **asymptotic growth** of cost functions
- Consider **worst-case** (highest cost) inputs
- Consider **running time** cost: number of **elementary operations**

NB

Other common analyses include:

- *Average-case analysis*
- *Space (memory) cost*

Elementary operations

Informally: A single computational “step”; something that takes a constant number of computation cycles.

Examples:

- Arithmetic operations
- Comparison of two values
- Assignment of a value to a variable
- Accessing an element of an array
- Calling a function
- Returning a value
- Printing a single character

NB

Count operations up to a constant factor, $O(1)$, rather than an exact number.

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Examples

Example

Squaring a number (First version):

```
square( $n$ ) :  
    return  $n * n$ 
```

Examples

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Running time: $O(1)$

Running time vs Execution time

Previous example shows one difference between running time and execution time.

In general, **running time only approximates execution time**:

- Simplifying assumptions about elementary operations
- Hidden constants in big-O
- Big-O only looks at limiting performance as n gets large.

Examples

- Implementations of `square(n)` will take longer as n gets bigger
- A program that “solves chess” will run in $O(1)$ time.

Examples

Example

Squaring a number (Second version):

```
square( $n$ ) :  
   $r := 0$   
  for  $i = 1$  to  $n$  :  
     $r := r + n$   
  return  $r$ 
```


Examples

Example

Squaring a number (Second version):

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square( $n$ ) :
```

```
   $r := 0$ 
```

```
  for  $i = 1$  to  $n$  :
```

```
     $r := r + n$ 
```

```
  return  $r$ 
```

$O(1)$

Examples

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```
  for  $i = 1$  to  $n$  :
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     $r := r + n$ 
```

```
  return  $r$ 
```

$O(1)$

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Examples

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Squaring a number (Second version):

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square( $n$ ) :  
   $r := 0$   $O(1)$   
  for  $i = 1$  to  $n$  :  $O(1)$   
     $r := r + n$   $O(1)$   
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```

Examples

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Squaring a number (Second version):

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$O(1)$
 $O(1)$ | n times
 $O(1)$

Examples

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Squaring a number (Second version):

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n times $O(n)$

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n times $O(n)$

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Squaring a number (Second version):

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square( $n$ ) :  
   $r := 0$   $O(1)$   
  for  $i = 1$  to  $n$  :  $O(1)$   
     $r := r + n$   $O(1)$  |  $n$  times  $O(n)$   
  return  $r$   $O(1)$ 
```

Running time: $O(1) + O(n) + O(1) = O(n)$

Examples

Example

Cubing a number (using second squaring program):

```
cube( $n$ ) :  
   $r := 0$   
  for  $i = 1$  to  $n$  :  
     $r := r + \text{square}(n)$   
  return  $r$ 
```


Examples

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Cubing a number (using second squaring program):

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  return  $r$   $O(1)$ 
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Running time: $O(1) + O(n^2) + O(1) = O(n^2)$

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Worst-case and big-O

Worst-case input assumption and big-O combine to *simplify* the analysis:

Example

Sum of squares (Using second squaring program):

```
sumOfSquares( $n$ ) :  
   $r := 0$   
  for  $i = 1$  to  $n$  :  
     $r := r + \text{square}(i)$   
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     $r := r + \text{square}(i)$   $O(i)$   $O(?)$   
  return  $r$   $O(1)$ 
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```

Running time: $O(1) + O(n^2) + O(1) = O(n^2)$

Worst-case and big-O

Worst-case input assumption and big-O combine to *simplify* the analysis:

Example

Finding an element (x) in an array (L) of length n :

```
find( $x, L$ ):  
    for  $i = 0$  to  $n - 1$ :  
        if  $L[i] == x$ :  
            return  $i$   
    return  $-1$ 
```

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find( $x, L$ ):  
  for  $i = 0$  to  $n - 1$ :       $O(1)$   
    if  $L[i] == x$ :           $O(1)$   
      return  $i$             $O(1)$   
  return  $-1$                 $O(1)$ 
```

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Finding an element (x) in an array (L) of length n :

<code>find(x, L):</code>			
<code>for $i = 0$ to $n - 1$:</code>	$O(1)$		
<code>if $L[i] == x$:</code>	$O(1)$? times	$O(?)$
<code>return i</code>	$O(1)$		
<code>return -1</code>			$O(1)$

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Finding an element (x) in an array (L) of length n :

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find( $x, L$ ) :  
  for  $i = 0$  to  $n - 1$  :       $O(1)$   
    if  $L[i] == x$  :           $O(1)$     $O(n)$  times    $O(n)$   
      return  $i$               $O(1)$   
  return  $-1$                   $O(1)$ 
```

Running time: $O(n) + O(1) = O(n)$

Worst-case and big-O

Worst-case input assumption and big-O combine to *simplify* the analysis:

NB

Simplifications might lead to sub-optimal bounds, may have to do a better analysis to get best bounds:

- *Finer-grained upper bound analysis*
- *Analyse specific cases to find a matching lower bound (big- Ω)*

NB

*Big- Ω is a **lower bound** analysis of the worst-case; NOT a “best-case” analysis.*

Worst-case and big-O

Analyse specific cases to find a matching lower bound (big- Ω)

Example

Let L_n be an n -element array of 0's.

Finding an element (x) in an array (L) of length n :

```
find( $x, L$ ):  
  for  $i = 0$  to  $n - 1$ :  
    if  $L[i] == x$ :  
      return  $i$   
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```

Worst-case and big-O

Analyse specific cases to find a matching lower bound (big- Ω)

Example

Let L_n be an n -element array of 0's.

Finding an element (x) in an array (L) of length n :

```
find( $x, L$ ):  
  for  $i = 0$  to  $n - 1$ :       $\Omega(1)$   
    if  $L[i] == x$ :           $\Omega(1)$   
      return  $i$              $\Omega(1)$   
  return  $-1$                  $\Omega(1)$ 
```

Worst-case and big-O

Analyse specific cases to find a matching lower bound (big- Ω)

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find( $x, L$ ):  
  for  $i = 0$  to  $n - 1$ :       $\Omega(1)$   
    if  $L[i] == x$ :           $\Omega(1)$  |  $\Omega(n)$  times  
      return  $i$             $\Omega(1)$   
  return  $-1$                  $\Omega(1)$ 
```

Worst-case and big-O

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      return  $i$   
  return  $-1$ 
```

$\Omega(1)$		$\Omega(n)$ times	$\Omega(n)$
$\Omega(1)$			
$\Omega(1)$			
			$\Omega(1)$

Worst-case and big-O

Analyse specific cases to find a matching lower bound (big- Ω)

Example

Let L_n be an n -element array of 0's.

Finding an element (x) in an array (L) of length n :

<code>find(x, L):</code>			
for $i = 0$ to $n - 1$:	$\Omega(1)$		
if $L[i] == x$:	$\Omega(1)$	$\Omega(n)$ times	$\Omega(n)$
return i	$\Omega(1)$		
return -1			$\Omega(1)$

Running time of `find(1, L_n)`: $\Omega(n)$

Worst-case and big-O

Analyse specific cases to find a matching lower bound (big- Ω)

Example

Let L_n be an n -element array of 0's.

Finding an element (x) in an array (L) of length n :

```
find( $x, L$ ):  
  for  $i = 0$  to  $n - 1$ :       $\Omega(1)$   
    if  $L[i] == x$ :           $\Omega(1)$  |  $\Omega(n)$  times       $\Omega(n)$   
      return  $i$              $\Omega(1)$   
  return  $-1$                  $\Omega(1)$ 
```

Running time of $\text{find}(1, L_n)$: $\Omega(n)$

Therefore, running time of $\text{find}(x, L)$: $\Theta(n)$

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Recursive examples

Example

Factorial:

```
fact( $n$ ) :  
    if  $n == 0$  :  
        return 1  
    else :  
        return  $n * \text{fact}(n - 1)$ 
```


Recursive examples

Example

Factorial:

```
fact( $n$ ) :  
  if  $n == 0$  :  $O(1)$   
    return 1  $O(1)$   
  else :  
    return  $n * \text{fact}(n - 1)$   $O(1) + ?$ 
```

Recursive examples

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Factorial:

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Running time for $\text{fact}(n)$: $T(n)$

Recursive examples

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Factorial:

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  if  $n == 0$  :  $O(1)$   
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```

Running time for $\text{fact}(n)$: $T(n)$

Recursive examples

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Factorial:

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fact(n) :  
  if n == 0 :            $O(1)$   
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```

Running time for $\text{fact}(n)$: $T(n)$, where:

$$T(0) \in O(1) + O(1) = O(1)$$

$$T(n) = T(n - 1) + O(1)$$

Recursive examples

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Factorial:

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fact(n) :  
  if n == 0 : O(1)  
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Running time for $\text{fact}(n)$: $T(n)$, where:

$$\begin{aligned} T(0) &\in O(1) + O(1) = O(1) \\ T(n) &= T(n - 1) + O(1) \\ &\in O(n) \end{aligned}$$

Recursive examples

Example

Factorial:

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fact(n) :  
  if n == 0 :           O(1)  
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  else :  
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Running time for $\text{fact}(n)$: $T(n)$, where:

$$\begin{aligned}T(0) &\in O(1) + O(1) = O(1) \\T(n) &= T(n - 1) + O(1) \\&\in O(n)\end{aligned}$$

Running time: $T(n) \in O(n)$

Recursive examples

Example

Summing elements of a linked list (length n):

```
sum(L) :  
  if L.isEmpty() :  
    return 0  
  else :  
    return L.data + sum(L.next)
```

Recursive examples

Example

Summing elements of a linked list (length n):

```
sum(L) :  
  if L.isEmpty() : O(1)  
    return 0 O(1)  
  else :  
    return L.data + sum(L.next) O(1) +
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Recursive examples

Example

Summing elements of a linked list (length n):

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sum(L) :  
  if L.isEmpty() : O(1)  
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Running time for `sum(L)`: $T(n)$

Recursive examples

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Summing elements of a linked list (length n):

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sum(L) :  
  if L.isEmpty() :  $O(1)$   
    return 0  $O(1)$   
  else :  
    return L.data + sum(L.next)  $O(1) + T(n - 1)$ 
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Running time for `sum(L)`: $T(n)$

Recursive examples

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Summing elements of a linked list (length n):

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Running time for $\text{sum}(L)$: $T(n)$, where:

$$T(0) \in O(1) + O(1) = O(1)$$

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Recursive examples

Example

Insertion sort (L has n elements):

```
sort(L) :  
  if L.isEmpty() :  
    return L  
  else :  
    L2 := sort(L.next)  
    insert L.data into L2  
    return L2
```

Recursive examples

Example

Insertion sort (L has n elements):

```
sort(L) :  
  if L.isEmpty() :  $O(1)$   
    return L  $O(1)$   
  else :  
    L2 := sort(L.next)  
    insert L.data into L2  
    return L2  $O(1)$ 
```

Recursive examples

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Recursive examples

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sort(L) :  
  if L.isEmpty() :  $O(1)$   
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  else :  
    L2 := sort(L.next)  $T(n - 1)$   
    insert L.data into L2  $O(n)$   
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```

Running time for `sort(L)`: $T(n)$

Recursive examples

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Insertion sort (L has n elements):

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Running time for `sort(L)`: $T(n)$, where:

$$T(0) \in O(1) + O(1) = O(1)$$

$$T(n) = T(n - 1) + O(n) + O(1)$$

Recursive examples

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Running time for `sort(L)`: $T(n)$, where:

$$\begin{aligned}T(0) &\in O(1) + O(1) = O(1) \\T(n) &= T(n-1) + O(n) + O(1) \\&\in O(n^2)\end{aligned}$$

Recursive examples

Example

Euclidean algorithm for $\text{gcd}(m, n)$ ($N = m + n$):

```
gcd( $m, n$ ) :  
  if  $m > n$  :  
    return gcd( $m - n, n$ )  
  else if  $n > m$  :  
    return gcd( $m, n - m$ )  
  else :    return  $m$ 
```

Recursive examples

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Euclidean algorithm for $\text{gcd}(m, n)$ ($N = m + n$):

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gcd( $m, n$ ) :  
  if  $m > n$  :  $O(1)$   
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  else if  $n > m$  :  $O(1)$   
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Recursive examples

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    return gcd( $m, n - m$ )  
  else :  $O(1)$  return  $m$ 
```

Running time for $\text{gcd}(m, n)$: $T(N)$

Recursive examples

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Euclidean algorithm for $\text{gcd}(m, n)$ ($N = m + n$):

```
gcd( $m, n$ ) :  
  if  $m > n$  :  
    return gcd( $m - n, n$ )  
  else if  $n > m$  :  
    return gcd( $m, n - m$ )  
  else :  
    return  $m$ 
```

$O(1)$
 $\leq T(N - 1)$
 $O(1)$
 $\leq T(N - 1)$
 $O(1)$

Running time for $\text{gcd}(m, n)$: $T(N)$

Recursive examples

Example

Euclidean algorithm for $\text{gcd}(m, n)$ ($N = m + n$):

```
gcd( $m, n$ ) :  
  if  $m > n$  :  
    return gcd( $m - n, n$ )  $O(1)$   
 $\leq T(N - 1)$   
  else if  $n > m$  :  $O(1)$   
    return gcd( $m, n - m$ )  $\leq T(N - 1)$   
  else : return  $m$   $O(1)$ 
```

Running time for $\text{gcd}(m, n)$: $T(N)$, where:

$$\begin{aligned} T(1) &\in O(1) \\ T(N) &\leq T(N - 1) + O(1) \end{aligned}$$

Recursive examples

Example

Euclidean algorithm for $\text{gcd}(m, n)$ ($N = m + n$):

```
gcd( $m, n$ ) :  
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  else if  $n > m$  :  $O(1)$   
    return gcd( $m, n - m$ )  $\leq T(N - 1)$   
  else :  $O(1)$   
    return  $m$ 
```

Running time for $\text{gcd}(m, n)$: $T(N)$, where:

$$\begin{aligned} T(1) &\in O(1) \\ T(N) &\leq T(N - 1) + O(1) \\ &\in O(N) \end{aligned}$$

Recursive examples

Example

Euclidean algorithm for $\text{gcd}(m, n)$ ($N = m + n$):

Running time: $O(N)$

NB

N is not the input **size**. Input size is $\log(m) + \log(n)$

Recursive examples

Example

Faster Euclidean algorithm for $\text{gcd}(m, n)$ ($N = m + n$):

```
gcd( $m, n$ ) :  
  if  $m > n > 0$  :  
    return gcd( $m \% n, n$ )  
  else if  $n > m > 0$  :  
    return gcd( $m, n \% m$ )  
  else :      return max( $m, n$ )
```

Recursive examples

Example

Faster Euclidean algorithm for $\text{gcd}(m, n)$ ($N = m + n$):

```
gcd( $m, n$ ) :  
  if  $m > n > 0$  :  $O(1)$   
    return gcd( $m \% n, n$ )  
  else if  $n > m > 0$  :  $O(1)$   
    return gcd( $m, n \% m$ )  
  else :  $O(1)$  return max( $m, n$ )
```

Recursive examples

Example

Faster Euclidean algorithm for $\text{gcd}(m, n)$ ($N = m + n$):

```
gcd( $m, n$ ) :  
  if  $m > n > 0$  : O(1)  
    return gcd( $m \% n, n$ )  
  else if  $n > m > 0$  : O(1)  
    return gcd( $m, n \% m$ )  
  else : O(1) return max( $m, n$ )
```

Running time for $\text{gcd}(m, n)$: $T(N)$

Recursive examples

Example

Faster Euclidean algorithm for $\text{gcd}(m, n)$ ($N = m + n$):

```
gcd(m, n) :  
  if m > n > 0 :  
    return gcd(m % n, n) O(1)  
  else if n > m > 0 : ≤ T(N/1.5)  
    return gcd(m, n % m) O(1)  
  else : ≤ T(N/1.5)  
    return max(m, n) O(1)
```

Running time for $\text{gcd}(m, n)$: $T(N)$

Recursive examples

Example

Faster Euclidean algorithm for $\text{gcd}(m, n)$ ($N = m + n$):

```
gcd(m, n) :  
  if m > n > 0 :  
    return gcd(m % n, n) O(1)  
     $\leq T(N/1.5)$   
  else if n > m > 0 : O(1)  
    return gcd(m, n % m)  $\leq T(N/1.5)$   
  else : return max(m, n) O(1)
```

Running time for $\text{gcd}(m, n)$: $T(N)$, where:

Recursive examples

Example

Faster Euclidean algorithm for $\text{gcd}(m, n)$ ($N = m + n$):

```
gcd(m, n) :  
  if m > n > 0 :  
    return gcd(m % n, n) O(1)  
  else if n > m > 0 : O(1)  
    return gcd(m, n % m) ≤ T(N/1.5)  
  else : return max(m, n) ≤ T(N/1.5)  
                                O(1)
```

Running time for $\text{gcd}(m, n)$: $T(N)$, where:

$$\begin{aligned} T(1) &\in O(1) \\ T(N) &\leq T(N/1.5) + O(1) \end{aligned}$$

Recursive examples

Example

Faster Euclidean algorithm for $\text{gcd}(m, n)$ ($N = m + n$):

```
gcd(m, n) :  
  if m > n > 0 :  
    return gcd(m % n, n) O(1)  
  else if n > m > 0 : O(1)  
    return gcd(m, n % m) ≤ T(N/1.5)  
  else : return max(m, n) ≤ T(N/1.5)  
                                O(1)
```

Running time for $\text{gcd}(m, n)$: $T(N)$, where:

$$\begin{aligned} T(1) &\in O(1) \\ T(N) &\leq T(N/1.5) + O(1) \\ &\in O(\log N) \end{aligned}$$

Recursive examples

Example

Faster Euclidean algorithm for $\gcd(m, n)$ ($N = m + n$):

What about lower bounds?

Recursive examples

Example

Faster Euclidean algorithm for $\gcd(m, n)$ ($N = m + n$):

What about lower bounds?

- Can show algorithm takes k steps to compute $\gcd(F_k, F_{k-1})$ where F_k is the k -th Fibonacci number
- Can show $1.5^k \leq F_k \leq 2^k$, so $k \in \Theta(\log F_k)$
- Therefore $\gcd(F_k, F_{k-1}) \in \Omega(\log(F_k + F_{k-1}))$

Exercise

Exercise

4.3.22 The following algorithm raises a number a to a power n .

```
exp( $a, n$ ) :  
   $p = 1$   
   $i = n$   
  while  $i > 0$  :  
     $p = p * a$   
     $i = i - 1$   
  return  $p$ 
```

Determine the running time of this algorithm.

Exercise

Exercise

4.3.21 The following algorithm gives a fast method for raising a number a to a power n .

```
fast-exp( $a, n$ ) :  
   $p = 1$   
   $q = a$   
   $i = n$   
  while  $i > 0$  :  
    if  $i$  is odd :  
       $p = p * q$   
     $q = q * q$   
     $i = \lfloor \frac{i}{2} \rfloor$   
  return  $p$ 
```

Determine the running time of this algorithm.