COMP9020 Week 8
Term 3, 2019
Combinatorics

- Textbook (R & W) - Ch. 5, Sec. 5.1–5.3; Ch. 9
- Supplementary Exercises Ch. 5, 9 (R & W)

Counting Techniques

General idea: find methods, algorithms or precise formulae to count the number of elements in various sets or collections derived, in a structured way, from some basic sets.

Examples

Single base set $S = \{s_1, \ldots, s_n\}$, $|S| = n$; find the number of

- all subsets of S
- ordered selections of r different elements of S
- unordered selections of r different elements of S
- selections of r elements from S such that ...
- functions $S \longrightarrow S$ (onto, 1-1)
- partitions of S into k equivalence classes
- graphs/trees with elements of S as labelled vertices/leaves

Applications of counting in CS

- Algorithmic analysis
- Data management
- Enumeration techniques
- Probability calculations

Overview

- Basic counting rules
- Combinations and Permutations
- Difficult counting problems

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Basic Counting Rules (1)

Union rule — S and T *disjoint*

$$|S \cup T| = |S| + |T|$$

$S_1, S_2, \ldots, S_n$ pairwise disjoint ($S_i \cap S_j = \emptyset$ for $i \neq j$)

$$|S_1 \cup \ldots \cup S_n| = \sum |S_i|$$

Example

How many numbers in $A = [1, 2, \ldots, 999]$ are divisible by 31 or 41?

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Example

How many numbers in $A = [1, 2, \ldots, 999]$ are divisible by 31 or 41?

$\lfloor 999/31 \rfloor = 32$ divisible by 31
$\lfloor 999/41 \rfloor = 24$ divisible by 41
No number in A divisible by both
Hence, $32 + 24 = 56$ divisible by 31 or 41

Basic Counting Rules (1)

Union rule: Inferences
For arbitrary sets $S, T, \ldots$

$$|S \cup T| = |S| + |T| - |S \cap T|$$
$$|T \setminus S| = |T| - |S \cap T|$$
$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3|$$
$$- |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3|$$
$$+ |S_1 \cap S_2 \cap S_3|$$

Basic Counting Rules (2)

Product rule

$$|S_1 \times \ldots \times S_k| = |S_1| \cdot |S_2| \cdots |S_k| = \prod_{i=1}^{k} |S_i|$$

If all $S_i = S$ (the same set) and $|S| = m$ then $|S^k| = m^k$

NB

This counts the number of sequences where the first item is from S_1, the second is from S_2, and so on.

Example

Let $\Sigma = \{a, b, c, d, e, f, g\}$.
How many 5-letter words?

$$|\Sigma^5| = |\Sigma|^5 = 7^5 = 16,807$$

How many with no letter repeated?

Basic Counting Rules (2)

Product rule: Sequences of selections

Question

How can we count sequences when the underlying set changes?

Answer

- *Define an order on the whole underlying set*
- *Select from $[1, n]$, where n is the size of the "remaining" set, and a selection of i represents choosing the i-th element in that set*

Basic Counting Rules (2)

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Example

Let $\Sigma = \{a, b, c, d, e, f, g\}$.
How many 5-letter words with no letter repeated?

$$\prod_{i=0}^{4}(|\Sigma| - i) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2,520$$

Basic Counting Rules (2)

Product rule: Sequences with restrictions/duplications

Question

How can we count sequences when we have equivalences in the underlying order?

Example

Let $\Sigma = \{a, b, c, d, e\}$.
How many 5-letter words with no letter repeated and *a* before *b* before *c*?

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Answer

- $S_1 = $ *sequences with equivalences,*
  $S_2 = $ *ways to distinguish within the equivalences,*
  $S = $ *sequences without restriction*
- $S_1 \times S_2 = S$, *so*
- $|S_1| = |S|/|S_2|$

Basic Counting Rules (2)

Example

Let $\Sigma = \{a, b, c, d, e\}$.
How many 5-letter words with no letter repeated and a before b before c?

NB

Same as asking how many 5-letter words from a, a, a, d, e

Basic Counting Rules (2)

Example

Let $\Sigma = \{a, b, c, d, e\}$.

How many 5-letter words with no letter repeated and a before b before c?

Let $\Sigma' = \{a, b, c\}$.

$$S = \prod_{i=0}^{4}(|\Sigma| - i) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

$$S_2 = \prod_{i=0}^{2}(|\Sigma'| - i) = 3 \cdot 2 \cdot 1 = 6$$

So $S_1 = 120/6 = 20$

NB

Same as asking how many 5-letter words from a, a, a, d, e

Exercises

Exercises

S, T finite. How many functions $S \longrightarrow T$ are there?

$\boxed{5.1.19}$ Consider a *complete* graph on n vertices.

(a) No. of paths of length 3

(b) paths of length 3 with all vertices distinct

(c) paths of length 3 with all edges distinct

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Exercises

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5.3.1 200 people. 150 swim or jog, 85 swim and 60 do both.
How many jog?

5.6.38 (Supp) There are 100 problems, 75 of which are 'easy' and
40 'important'.
What's the smallest number of easy *and* important problems?

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Exercise

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>
> $\boxed{5.3.2}$ $S = [100 \ldots 999]$, thus $|S| = 900$.
>
> (b) How many numbers have a 3 *and* a 7?
>
> (a) How many numbers have at least one digit that is a 3 or 7?
>
> ?

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Corollaries

- If $|S \cup T| = |S| + |T|$ then S and T are disjoint
- If $|\bigcup_{i=1}^{n} S_i| = \sum_{i=1}^{n} |S_i|$ then S_i are pairwise disjoint
- If $|T \setminus S| = |T| - |S|$ then $S \subseteq T$

These properties can serve to identify cases when sets are disjoint (resp. one is contained in the other).

Proof.

$|S| + |T| = |S \cup T|$ means $|S \cap T| = |S| + |T| - |S \cup T| = 0$

$|T \setminus S| = |T| - |S|$ means $|S \cap T| = |S|$ means $S \subseteq T$ $\qquad \square$

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Combinatorial Objects: How Many?

permutations

Ordering of all objects from a set S; equivalently: Selecting all objects while *recognising* the order of selection.

The number of permutations of n elements is

$$n! = n \cdot (n-1) \cdots 1, \quad 0! = 1! = 1$$

r-permutations (sequences without repetition)

Selecting any r objects from a set S of size n without repetition while *recognising* the order of selection.

Their number is

$$\Pi(n, r) = n \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$

Permutations with duplicates

Example

How many anagrams of ASSESS?

Permutations with duplicates

Example

How many anagrams of ASSESS?

Label S's: $AS_1S_2ES_3S_4$: 6!

In each anagram we can label the S's in 4! ways.

Suppose there are m anagrams. So $m.4! = 6!$, i.e. $m = \frac{6!}{4!}$

Permutations with duplicates

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Example

Number of anagrams of MISSISSIPPI?

Permutations with duplicates

Example

How many anagrams of ASSESS?
Label S's: $AS_1S_2ES_3S_4$: $6!$
In each anagram we can label the S's in $4!$ ways.
Suppose there are m anagrams. So $m.4! = 6!$, i.e. $m = \frac{6!}{4!}$

Example

Number of anagrams of MISSISSIPPI? $\frac{11!}{3!4!2!}$

r-selections (or: **r-combinations**)

Collecting any r distinct objects without repetition;
equivalently: selecting r objects from a set S of size n and *not* recognising the order of selection.

Their number is

$$\binom{n}{r} = \frac{\Pi(n,r)}{r!} = \frac{n!}{(n-r)!\,r!} = \frac{n \cdot (n-1) \cdots (n-r+1)}{1 \cdot 2 \cdots r}$$

NB

These numbers are usually called binomial coefficients due to

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + b^n = \sum_{i=0}^{n} \binom{n}{i}a^{n-i}b^i$$

Also defined for any $\alpha \in \mathbb{R}$ as $\dbinom{\alpha}{r} = \dfrac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!}$

Simple Counting Problems

Example

| 5.1.2 | Give an example of a counting problem whose answer is

(a) $\Pi(26, 10)$

(b) $\binom{26}{10}$

Simple Counting Problems

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Draw 10 cards from a half deck (eg. black cards only)
(a) the cards are recorded in the order of appearance
(b) only the complete draw is recorded

Examples

- Number of edges in a complete graph K_n
- Number of diagonals in a convex polygon
- Number of poker hands
- Decisions in games, lotteries etc.

Exercises

Exercises

5.1.6 From a group of 12 men and 16 women, how many committees can be chosen consisting of

(a) 7 members?

(b) 3 men and 4 women?

(c) 7 women or 7 men?

5.1.7 As above, but any 4 people (male or female) out of 9 and two, Alice and Bob, unwilling to serve on the same committee.

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Counting Poker Hands

Exercises

5.1.15 A poker hand consists of 5 cards drawn without replacement from a standard deck of 52 cards

$$\{A, 2\text{-}10, J, Q, K\} \times \{\text{club, spade, heart, diamond}\}$$

(a) Number of "4 of a kind" hands (e.g. 4 Jacks)

(b) Number of non-straight flushes, i.e. all cards of same suit but *not* consecutive (e.g. 8,9,10,J,K)

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Selecting items summary

Selecting k items from a set of n items:

With replacement	Order matters	Examples	Formula
Yes	Yes	Words of length k (sequences of length k)	n^k
No	Yes	k-permutations	$\Pi(n, k)$
No	No	Subsets of size k	$\binom{n}{k}$
Yes	No		

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Yes	No	Multisets of size k	$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$

"Balls in boxes"

Have n "distinguishable" boxes.
Have k balls which are either:

1. Indistinguishable
2. Distinguishable

How many ways to place balls in boxes with

A. At most one
B. Any number of

balls per box?

NB

Suppose K is a set with $|K| = k$ and N is a set with $|N| = n$:

- *2A counts the number of injective functions from K to N*
- *2B counts the number of functions from K to N*

"Balls in boxes"

Case	Balls	Balls per box	Number
1A	Indist.	At most 1	
1B	Indist.	Any number	
2A	Dist.	At most 1	
2B	Dist.	Any number	

"Balls in boxes"

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2B	Dist.	Any number	n^k

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Difficult Counting Problems

Example (Ramsay numbers)

An example of a *Ramsay number* is $R(3,3) = 6$, meaning that

> "K_6 is the smallest complete graph such that if all edges are painted using two colours, then there must be at least one monochromatic triangle"

This serves as the basis of a game called S-I-M (invented by Simmons), where two adversaries connect six dots, respectively using blue and red lines. The objective is to *avoid* closing a triangle of one's own colour. The second player has a winning strategy, but the full analysis requires a computer program.

Using Programs to Count

Two dice, a red die and a black die, are rolled.
(Note: one *die*, two or more *dice*)

Write a program to list all the pairs $\{(R, B) : R > B\}$

Similarly, for three dice, list all triples $R > B > G$

Generally, for *n* dice, all of which are *m*-sided ($n \leq m$), list all *decreasing n*-tuples

NB

In order to just find the number of such n-tuples, it is not necessary to list them all. One can write a recurrence relation for these numbers and compute (or try to solve) it.

Approximate Counting

NB

*A Count may be a precise value or an **estimate**.*

The latter should be asymptotically correct or at least give a good asymptotic bound, whether upper or lower. If S is the base set, $|S| = n$ its size, and we denote by $c(S)$ some collection of objects from S we are interested in, then we seek constants a, b such that

$$a \leq \lim_{n \to \infty} \frac{est(|c(S)|)}{|c(S)|} \leq b$$

In other words $est(|c(S)|) \in \Theta(|c(S)|)$.