1.

$\forall$ and $\exists$ are known as quantifiers (量词):

pitfall (陷阱, 诱惑)

$A \Rightarrow B$: Assume A and prove B

$A \Leftrightarrow B$: Prove "if A, then B" and "if B, then A".

$\forall x.A$: show A holds for every possible x

$\exists x A$: Find a value of x that makes A true.

Proposition	Its negation
A and B	not A or not B
A or B	not A and not B
$A \Rightarrow B$	A and not B
$A \Leftrightarrow B$	A and not B, or B and not A
$\forall x.A$	$\exists x$ not A
$\exists x.A$	$\forall x$ not A

2.

自然数集: $N = \{0, 1, 2, \cdots\}$

整数集: $Z = \{\cdots, -1, 0, 1, 2, \cdots\}$

正整数集: $N_{>0} = Z_{>0} = \{1, 2, \cdots\}$

有理数集: $Q = \{m/n : m, n \in Z, n \neq 0\}$ also called "fractions"

Real numbers: R

$\lfloor\ \rfloor : R \to Z$ — floor of x.

$\lceil\ \rceil : R \to Z$ — ceiling of x.

for $m, n \in Z$, m divides n if $n = km$ for some $k \in Z$

$\Rightarrow m|n \Rightarrow \frac{n}{m} = k, k \in Z$

$0|0 = 1$

$d|m$ and $d|n$

the greatest common divisor of m and n: $gcd(m, n)$ 最大公约数

the least common multiple of m and n: $lcm(m, n)$ 最小公倍数

$gcd(-4, 6) = gcd(4, -6) = gcd(-4, -6) = gcd(4, 6) = 2$
$lcm(-5, 5) = lcm(-5, 5) = lcm(5, -5) = lcm(5, 5) = 5$ } 都是正数结果

A number $n > 1$ is prime: only divisible by ± 1, and $\pm n$.

if $gcd(m, n) = 1 \Rightarrow$ m and n are relatively prime (相关质数)

$gcd(m, n) \cdot lcm(m, n) = |m| \cdot |n|$ ← 绝对值

$gcd(0, n) = n$

$m, p \in Z, n \in Z_{>0}$:

$m \text{ div } n = \lfloor \frac{m}{n} \rfloor$

$m \% n = m - (m \text{ div } n) \cdot n$

fact: $m \% n \in [0, n)$

$m = p \pmod n \Leftrightarrow (m \% n) = (p \% n)$

$m = p \pmod n$ if $n | (m - p)$ that is: $\frac{m-p}{n} = k$

~~m~~ if: $m = m' \pmod n$ and $p = p' \pmod n$
$\Downarrow$
$m + p = m' + p' \pmod n$ and
$m \cdot p = m' \cdot p' \pmod n$

Euclid (算法): $gcd(m, n) = \begin{cases} m & (m = n) \\ gcd(m-n, n) & (m > n) \\ gcd(m, n-m) & (m < n) \end{cases}$

faster Euclid (算法): $gcd(m, n) = \begin{cases} m & (m = n \text{ or } n = 0) \\ n & (m = 0) \\ gcd(m \% n, n) & (m > n > 0) \\ gcd(m, n \% m) & (0 < m < n) \end{cases}$

$f : S \to T$:

S domain of f (定义域) $Dom(f)$ inputs

T co-domain of f (值域) $Codom(f)$ possible output

f(S) image of f (相值) $Im(f)$ actual outputs

复合函数满足结合律: $h \circ (g \circ f) = (h \circ g) \circ f = h \circ g \circ f$ (associative)

Fuction is called surjective or onto (满射的) if $Im(f) = Codom(f)$

Fuction is called injective or 1-1 (1对1的) if $f(x) = f(y)$, then $x = y$

Function is called bijective if it is both surjective and injective.

A^c — complement (补集) (关于一个 universal set $\mathcal{U}$)

A, B are disjoint if $A \cap B = \phi$

$A \setminus B = A \cap B^c$

$A \oplus B$ — symmetric difference (异或)

$A \oplus B = (A \setminus B) \cup (B \setminus A)$

Cardinality: $|X| = \#(X) = card(X)$

$|Pow(X)| = 2^{|X|}$

To show sets equality:
1. Examine all elements
2. show $A \subseteq B$ and $B \subseteq A$
3. use the laws of Set operations

Σ (alphabet: 字母表): a finite, nonempty set.

$\Sigma^0 = \{\lambda\}$ $\quad \Sigma^1 = \Sigma$ $\quad \Sigma^* =$ set of all words of all lengths

$\Sigma^+ =$ set of all nonempty words.

de Morgan's Laws: $(A \cap B)^c = A^c \cup B^c$
$(A \cup B)^c = A^c \cap B^c$

$\Leftarrow$

交换律 (Commutativity): $A \cup B = B \cup A$ $\quad A \cap B = B \cap A$

结合律 (Associativity): $(A \cup B) \cup C = A \cup (B \cup C)$ $\quad (A \cap B) \cap C = A \cap (B \cap C)$

分配律 (Distribution): $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $\quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

恒等: $A \cup \phi = A$ $\quad A \cap \mathcal{U} = A$ (Identity)

互补: $A \cup A^c = \mathcal{U}$ $\quad A \cap A^c = \phi$ (complementation)

(R) 自反性: reflexive: ~~Fail~~ For all $x \in S$, $(x, x) \in R$

(AR) 反自反性: antireflexive: For all $x \in S$, $(x, x) \notin R$

(S) 对称性: symmetric: For all $x, y \in S$, if $(x, y) \in R$, then $(y, x) \in R$

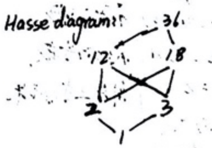
(AS) 反对称性: antisymmetric: For all $x, y \in S$, if (x, y) and $(y, x) \in R$, then $x = y$.

(T) 传递性: transitive: For all $x, y, z \in S$, if (x, y) and $(y, z) \in R$, the $(x, z) \in R$

等价关系: equivalence relations: is:
① Reflexive
② Symmetric
③ Transitive

poset: $(\{1, 2, 3, 12, 18, 36\}, |)$

Hasse diagram:



upper bound$(\{2, 3\}) = \{12, 18, 36\}$ ← no minimum

Maximum element: 36

Minimum element: 1

set of upper bounds
lower bounds
都是针对某个集合而言的

偏序关系: partial order $\preceq$ is satisfied:
① Reflexive
② Antisymmetric
③ Transitive

$A = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 3 & 2 & -1 & 2 \\ 4 & 0 & 1 & 3 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{bmatrix}$

$O(f(n))$: the class of all functions g that are asymptotically less than f.

$\Omega(f(n))$: the class of all functions g that are asymptotically greater than f.

$\Theta(f(n))$: the class of all functions g that have the same order of growth as f.

$1, \log(\log n), \log n, \sqrt{n}, \sqrt{n}\log n, n,$
$n\log(\log n), n\log n, n^{1.5}, n^2, n^3,$
$2^n, 2^n\log n, n \cdot 2^n, 3^n, n!, n^n, n^{n^2}, n^{2^n}$

Degree of Vertex:

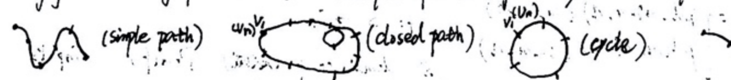
无向图: Undirected graphs: $deg(v) = |\{w \in V : \{v, w\} \in E\}|$, $\sum_{v \in V} deg(v) = 2 \cdot e(G)$

有向图: directed graphs: $\sum_{v \in V} outdeg(v) = \sum_{v \in V} indeg(v) = e(G)$

length of path: the number of edges: n. (Path of length 0: single vertex V_0)

Connected graph (undirected): — each pair of vertices joined by a path. 连通图: 任意两点之间都有通路相连

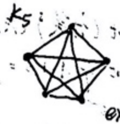
Strongly connected graph (directed): — each pair of vertices joined by a directed path in both directions. (强连通图)



acyclic graph (无环图) $\Rightarrow$ Tree: connected acyclic [undirected] graph.

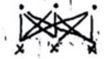
完全图: Complete graph K_n: n个点, 两两相连, 有 $\frac{n(n-1)}{2}$ 条边

K_5:



偶图: (bipartite)

完全偶图: $K_{m,n}$ (complete bipartite)



$(K_{3,3})$

同构的: (isomorphism) 自同构: (automorphism)

Hamiltonian path: visits every vertex of graph exactly once

Hamiltonian cycle: visits every vertex exactly once except ~~last~~ the last one, which duplicates the first.

Euler path: path containing every edge exactly once (有欧拉path ⟨ either: 有Euler circuit; or: exactly 2 vertices of odd degree)

Euler circuit: closed Euler path. (有欧拉环 $\Leftrightarrow$ $deg(v)$ is even for all $v \in V$)

chromatic number: 最小着色数 $\chi(G)$

$\chi(G) \geq K(G)$

A clique is a complete subgraph (完全子图 $\Rightarrow$ 团)

最大完全子图: clique number $\Rightarrow$ $K(G)$

If a graph is nonplanar then it must contain a subdivision of K_5 ~~or~~ or $K_{3,3}$.

K_n for $n \geq 5$ is nonplanar; $K_{m,n}$ is nonplanar when $m \geq 3$ and $n \geq 3$

~~finding a subdivision H of graph G~~
To show G contains subdivision of H

I: ① start with H
② Subdivide an edge
③ add a vertex
④ add an edge
⑤ finish with G

II: ① start with G
② Delete an edge
③ Delete a vertex
④ replace $a - b - c \Rightarrow a - c$
⑤ finish with H

Boolean algebra:
is a structure $(T, \vee, \wedge, ', 0, 1)$ where
$0, 1 \in T$
$\vee$: $T \times T \to T$ (called join)
$\wedge$: $T \times T \to T$ (called meet)
$'$: $T \to T$ (called complementation)
and following laws hold for all $x, y, z \in T$:
commutative: $x \vee y = y \vee x$
$x \wedge y = y \wedge x$
associative: $(x \vee y) \vee z = x \vee (y \vee z)$
$(x \wedge y) \wedge z = x \wedge (y \wedge z)$
distributive: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
identity: $x \vee 0 = x$, $x \wedge 1 = x$
complementation: $x \vee x' = 1$, $x \wedge x' = 0$
Dual: dual(E) is the expression obtained by replacing $\wedge$ with $\vee$

conditional probability:
$$P(E|S) = \frac{P(E \cap S)}{P(S)} \quad (P(S) \neq 0)$$

r-permutations:
$$\Pi(n, r) \triangleq n \cdot (n-1)(n-2) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$
r-selections (r-combinations):
$$\binom{n}{r} = \frac{\Pi(n,r)}{r!} = \frac{n!}{(n-r)!\,r!} = \frac{n \cdot (n-1)(n-2) \cdots (n-r+1)}{r \cdots 2 \cdot 1}$$

Selecting k items from a set of n items:

with replacement	Order matters	Examples	Formula
Yes	Yes	words of length k	n^k
No	Yes	k-permutations	$\Pi(n,k)$
No	NO	subset of size k	$\binom{n}{k}$
Yes	NO	multisets of size k	$\binom{n+k-1}{k}$

Case	Balls	Balls per box	Number	
1A	Indist	At most 1	$\binom{n}{k}$	~~[illegible]~~
1B	Indist	Any number	$\binom{n+k-1}{k}$	$k < n$
2A	Dis	At most 1	$\Pi(n,k)$	
2B	Dis	Any number	n^k	

have n "distinguishable" boxes, k balls with 1. Indist 2. Dist

A and B are independent $(A \perp B)$ if $P(A \cap B) = P(A) \cdot P(B)$
and $A \perp B \Leftrightarrow A^c \perp B \Leftrightarrow A \perp B^c \Leftrightarrow A^c \perp B^c$

The expected value of a random variable X is: $E(X) = \sum_{k \in \Omega} P(X=k) \cdot k$
For any random variables X, Y and integer k: $E(X+Y) = E(X) + E(Y)$ $E(k \cdot X) = k \cdot E(X)$
For random variable X with expected value $\mu = E(X)$
the standard deviation of X is: $\sigma = \sqrt{E(X-\mu)^2}$ and variance of X is σ^2
the variance can be calculated as $E((X-\mu)^2) = E(X^2) - \mu^2$

A ~~well-defined~~ well-formed formula (wff)
over a set of propositional variables PROP is:
(B) $\top$ is a wff
(B) $\perp$ is a wff
(B) p is a wff for all $p \in PROP$
(R) if φ is a wff then $\neg\varphi$ is a wff
(R) if φ and ψ are wff, then:
$(\varphi \wedge \psi)$
$(\varphi \vee \psi)$
$(\varphi \to \psi)$ and
$(\varphi \leftrightarrow \psi)$ are wff

$\wedge$ and {but, ; , although, nevertheless}
$\vee$ or either ... or ..
$\neg$ not not the case
$\to$ if...then... {implies, whenever, is sufficient}
$\leftrightarrow$ if and only if {bi-implies, exactly when, just in case}

真值指派函数: truth assignment: v: Prop $\to \mathbb{B}$
$v(\top) = $ True $v(\varphi \wedge \psi) = v(\varphi) \,\&\&\, v(\psi)$
$v(\perp) = $ False $v(\varphi \vee \psi) = v(\varphi) \,||\, v(\psi)$
$v(\neg\varphi) = !v(\varphi)$ $v(\varphi \to \psi) = v(\varphi) \to v(\psi)$
$v(\varphi \leftrightarrow \psi) = v(\varphi) \leftrightarrow v(\psi)$

A formula φ is:
satisfiable if $v(\varphi) = $ True for some (某个) truth assignment v (v satisfies φ)
a tautology if $v(\varphi) = $ True for all (所有) truth assignments v.
unsatisfiable or a contradiction if $v(\varphi) = $ false for all truth assignments v.
Two formulas, φ and ψ, are logically equivalent, $\varphi \equiv \psi$, if $v(\varphi) = v(\psi)$ for all truth assignments v.

$\boxed{\equiv \text{ is an equivalence relation}}$

R (自反)
S (对称)
T (传递)

for all propositions P, Q, R:
$P \vee Q \equiv Q \vee P$ $P \to q \equiv \neg P \vee q$
$P \wedge Q \equiv Q \wedge P$ $\neg\neg P \equiv P$
$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$ $P \to q \equiv \neg q \to \neg P$
$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$ $\neg(P \vee q) = \neg P \wedge \neg q$
$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
$P \vee \perp \equiv P$
$P \wedge \top \equiv P$
$P \vee \neg P \equiv \top$
$P \wedge \neg P \equiv \perp$

a set of formulas is a Theory.
A truth assignment v satisfies a Theory T if $v(\varphi) = $ true for all $\varphi \in T$
A theory T entails a formula φ, $T \models \varphi$, if $v(\varphi) = $ true for all assignments v which satisfy T.

Conjunctive normal form: (CNF): (eg. $p \vee q \vee \bar{r}$)
$\bigwedge_i C_i$, where C_i is a ~~disj~~ disjunction of literals.

Disjunctive normal form: (DNF): eg. $(p \wedge q \wedge \bar{r})$
$\bigvee_i C_i$, where C_i is a conjunction of literals.

karnaugh maps:

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	+	+		+
$\bar{x}$	+		+	+

解递归 Solving Recurrences:
Approach 1: Unwinding the recurrence
eg. $f(0) = 1$ $f(n) = 2f(n-1)$:
$f(n) = 2f(n-1) = 2(2f(n-2)) = 2 \cdot (2 \cdot (2 f(n-3))) \ldots$
$= 2^n \cdot f(0) = 2^n$
Approach 2: Approximating with big-O:
eg: $f(0) = 1$ $f(1) = 1$ $f(n) = f(n-1) + f(n-2)$
~~f(n-2)~~ Assuming $f(n)$ is increasing:
$f(n-2) \le f(n-1) \Rightarrow f(n) \le 2f(n-1) \Rightarrow f(n) \le 2^n$
$\Rightarrow f(n) \in O(2^n)$
Approach 3: Master Theorem:
$T(n) = a \cdot T(\frac{n}{b}) + f(n)$, where $f(n) \in O(n^c (\log n)^k)$
Let $d = \log_b a$ then:
case 1: If $c < d$ then $T(n) = O(n^d)$
case 2: If $c = d$ then $T(n) = O(n^d (\log n)^{k+1})$
case 3: If $c > d$ then $T(n) = O(f(n))$
eg. $T(n) = T(\frac{n}{2}) + n^2$, $T(1) = 1$
$\Rightarrow$ case 3: $T(n) = O(n^c) = O(n^2)$
推论: Master Theorem 的推论
$T(n) = a \cdot T(n-1) + b \cdot n^k$
then $T(n) = \begin{cases} O(n^{k+1}) & \text{if } a = 1 \\ O(a^n) & \text{if } a > 1 \end{cases}$

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