Problem 1

(a) : All possible functions:

F1 : { (a,0), (b,0), (c,0) }

F2 : { (a,0), (b,0), (c,1) }

F3 : { (a,0), (b,1), (c,0) }

F4 : { (a,0), (b,1), (c,1) }

F5 : { (a,1), (b,0), (c,0) }

F6 : { (a,1), (b,0), (c,1) }

F7 : { (a,1), (b,1), (c,0) }

F8 : { (a,1), (b,1), (c,1) }

(b) : Pow(X) = { A : A $\subseteq$ X } and |Pow(X)| = $2^{|X|}$

➔ Pow({a, b, c}) = $2^{|\{a, b, c\}|}=2^3=8$

We can consider like this : "0" and "1" represent "exist" and "not exist", so if the element in { a, b, c } all point to 0, they represent an empty set. Therefore, the subset of { a, b, c } can represent the existence and non-existence of a, b, c. So the number of functions is equal to |Pow({a, b, c})|.

➔when |co-domain| = 2, Pow({a, b, c}) = the answer for (a).

(c) : if card(A) = m and card(B) = n:

i. Number of fuctions from A to B : n^m

ii. For the symmetric relation, you can analyze half triangle in the relation matrix. Every elements in the matrix can exist and not exist. So the number of relations from A to B : |Pow(m x n)| = $2^{|m \times n|}$

iii. For the symmetric relation, you can analyze half triangle in the relation matrix, and it can be concluded that half triangle has (1+m) /2 elements, So the number of symmetric relations from A to B is $2^{(1+m)/2}$.

Problem 2

(a) $S_{2,-3}$ = { 2m - 3n : m, n ∈ Z}

1 (where m = 2, n = 1); 3 (where m = 3, n = 1);

5 (where m = 4, n = 1); 7 (where m = 5, n = 1);

4 (where m = 5, n = 2);

(b) $S_{12,16}$ = { 12m + 16n : m, n ∈ Z}

0 (where m = 0, n = 0); 12 (where m = 1, n = 0);

16 (where m = 0, n = 1); 28 (where m = 1, n = 1);

24 (where m = 2, n = 0);

(c) ∵ d = gcd(x , y) and x, y ∈ Z ∴ x = k1*d, for some k1 ∈ Z;

Same reason for y = k2*d, for some k2 ∈ Z;

∴ $S_{x,y}$ = { (k1 ⋆ d) m + (k2 ⋆ d) n : m, n ∈ Z }, for some k1, k2 ∈ Z;

= { (k1 ⋆ m+ k2 ⋆ n) d : m, n ∈ Z };

And then we can see that (k1 ⋆ m+ k2 ⋆ n) ∈ Z because of k1, k2, m and n ∈ Z;

Then, $\{n : n \in Z \text{ and } d \mid n\} = \{n : n = k*d \text{ and } k \in Z \text{ and } n \in Z\}$;

∵ $\{(k1 * m + k2 * n)\} \subseteq Z$

∴ $\{(k1 * m + k2 * n)\, d : m, n \in Z\} \subseteq \{n : n = k*d \text{ and } k \in Z \text{ and } n \in Z\}$ and that is : $S_{x,y} \subseteq \{n : n \in Z \text{ and } d|n\}$.

(d) ∵ $\{n : n \in Z \text{ and } z \mid n\}$;

∴ $\{n : n \in Z \text{ and } n = p*z, \text{ for some } p \in Z\}$;

∵ z is the smallest positive number in Sx,y, so z can be standed for $(p1 * m + p2 * n)\, d$, for some p1, p2, m and n, making z be the smallest positive number in Sx,y, that is let $(p1 * m + p2 * n)$ be the smallest positive number since $d \geqslant 0$ and $z > 0$.

∴ $\{n : n = p*(p1 * m + p2 * n)\, d \text{ and } m, n, p, k1, k2 \in Z\}$

∵ p1, p2 are specific two numbers to satisfy z to be the smallest positive number in Sx,y.

∴ $(p1 * m + p2 * n) \in \{(k1 * m + k2 * n) : m, n, k1, k2 \in Z\}$

∵ $p * (p1 * m + p2 * n)$ is a multiple of $(p1 * m + p2 * n)$

∴ $\{p * (p1 * m + p2 * n)\} \subseteq \{(k1 * m + k2 * n)\}$ for some $p \in Z$

∴ $\{p*(p1 * m + p2 * n)\, d\} \subseteq \{(k1 * m + k2 * n)\, d\}$

And that is $\{n : n \in Z \text{ and } z|n\} \subseteq Sx,y$

(e) ∵ z is the smallest positive number in Sx,y, so we can express z as

$$z = (k1 * m + k2 * n) * d > 0$$

$\because$ $d = gcd(x, y) > 0$

$\therefore$ $(k1 * m + k2 * n) > 0$ and $(k1 * m + k2 * n) \in Z$

$\because$ z is the smallest positive number in Sx,y

$\therefore$ $(k1 * m + k2 * n) = 1$ that is $z / d = 1$

$\therefore$ $z = d$

$\therefore$ $z \geqslant d$

(f) According to the conclusion of (e):

$\because$ $z = d$

$\therefore$ $z \leqslant d$

Problem 3

(a) $(A * B) * (A * B)$

$= (A^c \cup B^c) * (A^c \cup B^c)$ (definition)

$= (A^c \cup B^c)^c \cup (A^c \cup B^c)^c$ (definition)

$= ((A^c)^c \cap (B^c)^c) \cup ((A^c)^c \cap (B^c)^c)$ (de Morgan's Laws)

$= (A \cap B) \cup (A \cap B)$ (double complementation)

$= A \cap (B \cup B)$ (distribution)

$= A \cap B$ (idempotence)

(b) $A^c = A^c \cup A^c$ (idempotence)

$= A * A$ (definition)

(c) $\Phi = A^c \cap A$ (Complementation)

$= (A^c * A) * (A^c * A)$ (by conclusion (a))

$= ((A * A) * A) * ((A * A) * A)$ (by conclusion (b))

(d) $A \setminus B = A \cap B^c$ (by definition)

$= A \cap (B * B)$ (by conclusion (b))

$= (A * (B * B)) * (A * (B * B))$ (by conclusion (a))

Problem 4

(a) w = a, v = ba;

$\because$ ba $\neq$ az, for $z \in \Sigma^*$ and a $\neq$ baz, for $z \in \Sigma^*$

(b) By definition of $R^{\leftarrow}(B)$:

$R^{\leftarrow}(\{aba\})$ is a set satisfying a condition that:

$R^{\leftarrow}(\{aba\}) = \{ w : aba = wz, z \in \Sigma^* \}$;

➔ $R^{\leftarrow}(\{aba\}) = \{ \lambda, a, ab, aba \}$

(c) let z be λ:

Reflexivity: For all $w \in \Sigma^*$, $(w, w) \in R$ for all w (due to $w = w\lambda$).

Antisymmetry: For all w, $v \in \Sigma^*$, if (w, v) $\in$ R and (v, w) $\in$ R:

That is : $v = w\lambda$ and $w = v\lambda$ ➔ $v = w$

Transitivity: For all w, $v \in \Sigma^*$, if (w, v) $\in$ R and (v, t) $\in$ R:

That is : $v = w\lambda$ and $t = v\lambda$ ➔ $t = w\lambda\lambda = w$

So (t, w) $\in$ R

Problem 5

Case x = 0:

If x = 0: $\because$ x|yz $\therefore$ yz = 0

$\because$ gcd(x, y) = gcd(0, y) = 1 $\therefore$ y = 1

$\therefore$ z = x = 0

$\therefore$ x|z

Case $x \neq 0$:

As we can see in problem 2, so d = gcd(x, y) can be represented in the form of (mx + ny) for some m, n $\in$ Z.

So we can write gcd(x , y) = 1 = $(m_0x + n_0y)$ for some m_0, $n_0 \in$ Z.

And x|yz ➔ yz = kx, for some k $\in$ Z

Multiply both sides of this equation by z, and we can get :

$z = m_0xz + n_0yz$, and we substitude yz by kx: $z = m_0xz + n_0kx$

$\therefore$ $z = (m_0z + n_0k)x$ $\because$ m_0, z, n_0, k $\in$ Z

$\therefore$ $(m_0z + n_0k)$ is an integer $\therefore$ x|z