Due: Sunday, 3rd November, 23:59

Submission is through WebCMS/give and should be a single pdf file, maximum size 2Mb. Prose should be typed, not handwritten. Use of LaTeX is encouraged, but not required.

Discussion of assignment material with others is permitted, but the work submitted *must* be your own in line with the University's plagiarism policy.

Problem 1 (20 marks)

Recall the relation composition operator ; defined as:

$$R_1; R_2 = \{(a, c) : \text{there is a } b \text{ with } (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$$

For any set S, and any binary relations $R_1, R_2, R_3 \subseteq S \times S$, prove or give a counterexample to disprove the following:

(a) $(R_1; R_2); R_3 = R_1; (R_2; R_3)$ (4 marks)

(b) $I; R_1 = R_1; I = R_1$ where $I = \{(x, x) : x \in S\}$ (4 marks)

(c) $(R_1; R_2)^{\leftarrow} = R_1^{\leftarrow}; R_2^{\leftarrow}$ (4 marks)

(d) $(R_1 \cup R_2); R_3 = (R_1; R_3) \cup (R_2; R_3)$ (4 marks)

(e) $R_1; (R_2 \cap R_3) = (R_1; R_2) \cap (R_1; R_3)$ (4 marks)

Problem 2 (30 marks)

Let $R \subseteq S \times S$ be any binary relation on a set S. Consider the sequence of relations $R^0, R^1, R^2, \ldots$, defined as follows:

$$\begin{array}{rcl} R^0 & := & I = \{(x, x) : x \in S\}, \text{ and} \\ R^{i+1} & := & R^i \cup (R; R^i) \text{ for } i \geq 0 \end{array}$$

(a) Prove that if there is an i such that $R^i = R^{i+1}$, then $R^j = R^i$ for all $j \geq i$. (4 marks)

(b) Prove that if there is an i such that $R^i = R^{i+1}$, then $R^k \subseteq R^i$ for all $k \geq 0$. (4 marks)

(c) Let $P(n)$ be the proposition that for all $m \in \mathbb{N}$: $R^n; R^m = R^{n+m}$. Prove that $P(n)$ holds for all $n \in \mathbb{N}$. (8 marks)

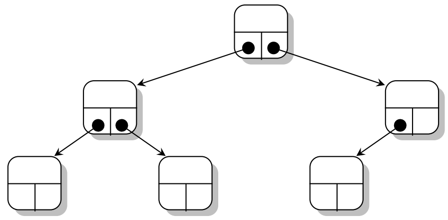
(d) If $|S| = k$, explain why $R^k = R^{k+1}$. (*Hint: Show that if* $(a, b) \in R^{k+1}$ *then* $(a, b) \in R^i$ *for some* $i < k + 1$.) (4 marks)

(e) If $|S| = k$, show that R^k is transitive. (4 marks)

(f) If $|S| = k$, show that $(R \cup R^{\leftarrow})^k$ is an equivalence relation. (6 marks)

Problem 3 (26 marks)

A *binary tree* is a data structure where each node is linked to at most two successor nodes:



If we allow empty binary trees (trees with no nodes), then we can simplify the description by saying a node has *exactly two children* which are binary trees.

(a) Give a recursive definition of the binary tree data structure. *Hint: review the recursive definition of a Linked List* (6 marks)

A *leaf* in a binary tree is a node that has no successors (i.e. it has two empty trees as children). A *fully-internal node* in a binary tree is a node that has two successors. The example above has 3 leaves and 2 fully-internal nodes.

(b) Based on your recursive definition above, define the function $\mathsf{count}(\mathsf{T})$ that counts the number of nodes in a binary tree T. (4 marks)

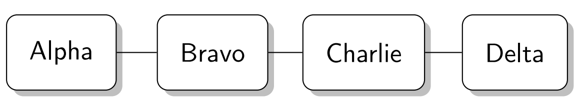
(c) Based on your recursive definition above, define the function $\mathsf{leaves}(\mathsf{T})$ that counts the number of leaves in a binary tree T. (4 marks)

(d) Based on your recursive definition above, define the function $\mathsf{internal}(\mathsf{T})$ that counts the number of fully-internal nodes in a binary tree T. *Hint: it is acceptable to define an empty tree as having* -1 *fully-internal nodes.* (4 marks)

(e) If T is a binary tree, let $P(\mathsf{T})$ be the proposition that $\mathsf{leaves}(\mathsf{T}) = 1 + \mathsf{internal}(\mathsf{T})$. Prove that $P(\mathsf{T})$ holds for all binary trees T. (8 marks)

Problem 4 (24 marks)

Four wifi networks, Alpha, Bravo, Charlie and Delta, all exist within close proximity to one another as shown below.



Networks connected with an edge in the diagram above can interfere with each other. To avoid interference networks can operate on one of two channels, hi and lo. Networks operating on different channels will not interfere; and neither will networks that are not connected with an edge.

Our goal is to determine (algorithmically) whether there is an **assignment of channels to networks** so that there is no interference. To do this we will transform the problem into a problem of determining if a propositional formula can be satisfied.

(a) Carefully defining the propositional variables you are using, (4 marks)
write **propositional formulas** for each of the following requirements:

(i) φ_1: Alpha uses channel hi or channel lo; and so does Bravo, Charlie and Delta. (4 marks)

(ii) φ_2: Alpha does not use both channel hi and lo; and the same for Bravo, Charlie and Delta. (4 marks)

(iii) φ_3: Alpha and Bravo do not use the same channel; and the same applies for all other pairs of networks connected with an edge. (4 marks)

(b) (i) Show that $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ is satisfiable; so the requirements can all be met. Note that it is sufficient to give a satisfying truth assignment, you do not have to list all possible combinations. (4 marks)

(ii) Based on your answer to the previous question, which channels should each network use in order to avoid interference? (4 marks)

Advice on how to do the assignment

All submitted work must be done individually without consulting someone else's solutions in accordance with the University's "Academic Dishonesty and Plagiarism" policies.

- Assignments are to be submitted via WebCMS (or give) as a single pdf file.
- Be careful with giving multiple or alternative answers. If you give multiple answers, then we will give you marks only for your worst answer, as this indicates how well you understood the question.
- Some of the questions are very easy (with the help of the lecture notes or book). You can use the material presented in the lecture or book (without proving it). You do not need to write more than necessary (see comment above).
- When giving answers to questions, we always would like you to prove/explain/motivate your answers.
- If you use further resources (books, scientific papers, the internet,...) to formulate your answers, then add references to your sources.