COMP9020 Week 7
Term 3, 2019
Graph Theory

- Textbook (R & W) - Ch. 3, Sec. 3.2; Ch. 6, Sec. 6.1–6.5
- A. Aho & J. Ullman. Foundations of Computer Science in C, p. 522–526 (Ch. 9, Sec. 9.10)

Summary of topics

- Motivation and applications
- Terminology and notation
- Graph traversals
- Properties of graphs

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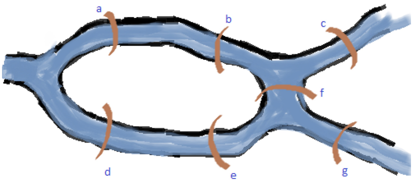
Graph theory: Historical Motivation

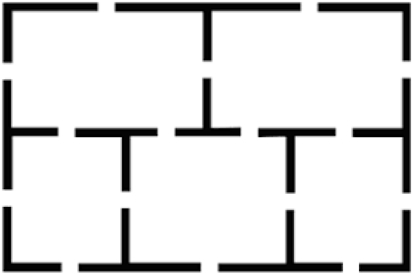
Bridges of Königsberg problem



Can you find a route which crosses each bridge exactly once?

Graph theory: Historical Motivation

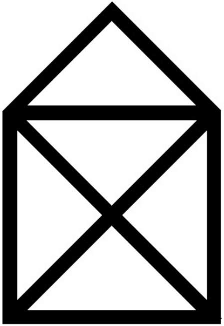
Five rooms problem



Can you find a route which passes through each door exactly once?

Graph theory: Historical Motivation

Crossed house problem



Can you draw this without taking your pen off the paper?

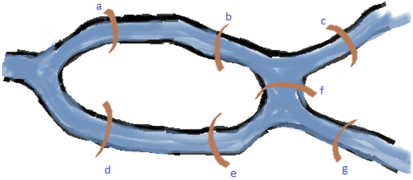
Graph theory: Historical Motivation

Three utilities problem

Can you connect all utilities to all houses without crossing connections?

Graph theory: Historical Motivation

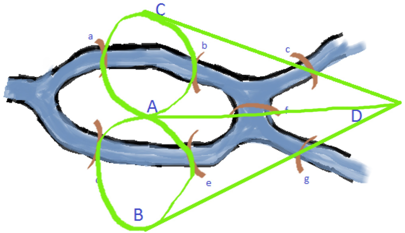
Bridges of Königsberg problem



Can you find a route which crosses each bridge exactly once?

Graph theory: Historical Motivation

Bridges of Königsberg problem



Can you find a route which crosses each bridge exactly once?

Graphs in Computer Science

Examples

1. The WWW can be considered a massive graph where the nodes are web pages and arcs are hyperlinks.
2. The possible states of a program form a directed graph.
3. Circuit components and their connections form a graph.
4. Social networks can be viewed as a graph where the nodes are users and the edges are connections.
5. The map of the earth can be represented as an undirected graph where edges delineate countries.

Graphs in Computer Science

Applications of graphs in Computer Science are abundant, e.g.

- route planning in navigation systems, robotics
- optimisation, e.g. timetables, utilisation of network structures, bandwidth allocation
- compilers using "graph colouring" to assign registers to program variables
- circuit layout (Untangle game)
- determining the significance of a web page (Google's pagerank algorithm)
- modelling the spread of a virus in a computer network or news in social network

Summary of topics

- Motivation and applications
- Terminology and notation
- Graph traversals
- Properties of graphs

Graphs

Terminology (the most common; there are many variants):

Graph — pair (V, E) where V – set of vertices (or nodes)
E – set of edges

Undirected graph: Every edge $e \in E$ is a two-element set of vertices, i.e. $e = \{x, y\} \subseteq V$ where $x \neq y$

Directed graph: Every edge (or arc) $e \in E$ is an ordered pair of vertices, i.e. $e = (x, y) \in V \times V$, note x may equal y.

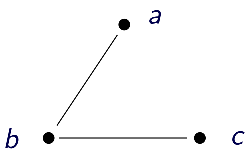
NB

Binary relations on finite sets correspond to directed graphs.
Symmetric, antireflexive relations correspond to undirected graphs.

Graph representations

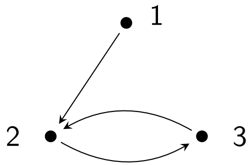
Graph:
$V = \{a, b, c\}$
$E = \{\{a, b\}, \{b, c\}\}$

Pictorially:



Directed graph:
$V = \{1, 2, 3\}$
$E = \{(1, 2), (2, 3), (3, 2)\}$

Pictorially:



Graph representations

Graph:
$V = \{a, b, c\}$
$E = \{\{a, b\}, \{b, c\}\}$

Adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Directed graph:
$V = \{1, 2, 3\}$
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Graph representations

Graph:
$V = \{a, b, c\}$
$E = \{\{a, b\}, \{b, c\}\}$

Adjacency list:

$$\begin{array}{ll} a: & b \\ b: & a, c \\ c: & b \end{array}$$

Directed graph:
$V = \{1, 2, 3\}$
$E = \{(1, 2), (2, 3), (3, 2)\}$

Adjacency list:

$$\begin{array}{ll} 1: & 2 \\ 2: & 3 \\ 3: & \end{array}$$

Graph representations

Graph:
$V = \{a, b, c\}$
$E = \{\{a, b\}, \{b, c\}\}$

Incidence matrix (vertices=rows, edges=columns):

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Directed graph:
$V = \{1, 2, 3\}$
$E = \{(1, 2), (2, 3), (3, 2)\}$

Incidence matrix (vertices=rows, edges=columns):

$$\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

Vertex Degrees (Undirected graphs)

- **Degree** of a vertex

$$\deg(v) = |\{ w \in V : \{v, w\} \in E \}|$$

  i.e., the number of edges attached to the vertex
- **Regular graph** — all degrees are equal
- *Degree sequence* $D_0, D_1, D_2, \ldots, D_k$ of graph $G = (V, E)$, where D_i = no. of vertices of degree i

Question

What is $D_0 + D_1 + \ldots + D_k$?

- $\sum_{v \in V} \deg(v) = 2 \cdot e(G)$; thus the sum of vertex degrees is always even.
- There is an even number of vertices of odd degree ($\boxed{6.1.8}$)

Vertex Degrees (Directed graphs)

- **Out-degree** of a vertex

$$outdeg(v) = |\{ w \in V : (v, w) \in E \}|$$

i.e., the number of edges going out of the vertex

- **In-degree** of a vertex

$$indeg(v) = |\{ w \in V : (w, v) \in E \}|$$

i.e., the number of edges going in to the vertex

- $\sum_{v \in V} outdeg(v) = \sum_{v \in V} indeg(v) = e(G)$.

Paths

- A **(directed) path** in a (directed) graph (V, E) is a sequence of edges that link up

$$v_0 \xrightarrow{\{v_0,v_1\}} v_1 \xrightarrow{\{v_1,v_2\}} \dots \xrightarrow{\{v_{n-1},v_n\}} v_n$$

  where $e_i = \{v_{i-1}, v_i\} \in E$ (or $e_i = (v_{i-1}, v_i) \in E$)
- **length** of the path is the number of edges: n
  neither the vertices nor the edges have to be all different
- Subpath of length r: $(e_m, e_{m+1}, \dots, e_{m+r-1})$
- Path of length 0: single vertex v_0
- **Connected graph (undirected)** — each pair of vertices joined by a path
- **Strongly connected graph (directed)** — each pair of vertices joined by a directed path in both directions

Exercises

Exercises

$\boxed{6.1.13(a)}$ Draw a connected, regular graph on four vertices, each of degree 2

$\boxed{6.1.13(b)}$ Draw a connected, regular graph on four vertices, each of degree 3

$\boxed{6.1.13(c)}$ Draw a connected, regular graph on five vertices, each of degree 3

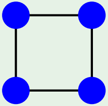
$\boxed{6.1.14(a)}$ Graph with 3 vertices and 3 edges

$\boxed{6.1.14(b)}$ Two graphs each with 4 vertices and 4 edges

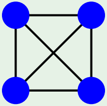
Exercises

Exercises

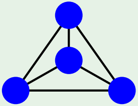
$\boxed{6.1.13}$ Connected, regular graphs on four vertices



(a)



(b)

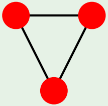


(b)

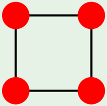
none

(c)

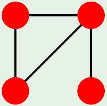
$\boxed{6.1.14}$ Graphs with 3 vertices and 3 edges must have a *cycle*



(a) the only one



(b)



(b)

Exercises

NB

We use the notation
$n = v(G) = |V|$ *for the no. of vertices of graph* $G = (V, E)$
$m = e(G) = |E|$ *for the no. of edges of graph* $G = (V, E)$

Exercises

$\boxed{\text{6.1.20(a)}}$ Graph with $e(G) = 21$ edges has a degree sequence
$D_0 = 0, D_1 = 7, D_2 = 3, D_3 = 7, D_4 = ?$
Find $v(G)$

$\boxed{\text{6.1.20(b)}}$ How would your answer change, if at all, when $D_0 = 6$?

Exercises

NB

We use the notation
$n = v(G) = |V|$ *for the no. of vertices of graph* $G = (V, E)$
$m = e(G) = |E|$ *for the no. of edges of graph* $G = (V, E)$

Exercises

$\boxed{\text{6.1.20(a)}}$ Graph with $e(G) = 21$ edges has a degree sequence
$D_0 = 0, D_1 = 7, D_2 = 3, D_3 = 7, D_4 = ?$
Find $v(G)$

$\sum_v \deg(v) = 2|E|$; here
$7 \cdot 1 + 3 \cdot 2 + 7 \cdot 3 + x \cdot 4 = 2 \cdot 21$ giving $x = 2$, thus
$v(G) = \sum D_i = 19$.

$\boxed{\text{6.1.20(b)}}$ How would your answer change, if at all, when $D_0 = 6$?
No change to D_4; $v(G) = 25$.

Cycles

Recall paths $v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \ \ldots \ \xrightarrow{e_n} v_n$

- *simple path* — $e_i \neq e_j$ for all edges of the path $(i \neq j)$
- *closed path* — $v_0 = v_n$
- **cycle** — closed path, all other v_i pairwise distinct and $\neq v_0$
- *acyclic path* — $v_i \neq v_j$ for *all* vertices in the path $(i \neq j)$

NB

1. *$C = (e_1, \ldots, e_n)$ is a cycle iff removing any single edge leaves an acyclic path. (Show that the 'any' condition is needed!)*
2. *C is a cycle if it has the same number of edges and vertices and no proper subpath has this property.*
   (Show that the 'subpath' condition is needed, i.e., there are graphs G that are **not** *cycles and $|E_G| = |V_G|$; every such G must contain a cycle!)*

Trees

- **Acyclic graph** — graph that doesn't contain any cycle
- **Tree** — connected acyclic [undirected]graph
- A graph is acyclic *iff* it is a *forest* (collection of disjoint trees)

NB

Graph G is a tree iff

$\leftrightarrow$ *it is acyclic and $|V_G| = |E_G| + 1$.*
(Show how this implies that the graph is connected!)

$\leftrightarrow$ *there is exactly one simple path between any two vertices.*

$\leftrightarrow$ *G is connected, but becomes disconnected if any single edge is removed.*

$\leftrightarrow$ *G is acyclic, but has a cycle if any single edge on already existing vertices is added.*

Trees

A tree with one vertex designated as its *root* is called a *rooted tree*. It imposes an ordering on the edges: 'away' from the root — from parent nodes to children. This defines a *level number* (or: *depth*) of a node as its distance from the root.

Another very common notion in Computer Science is that of a *DAG* — a *directed, acyclic graph*.

Exercise (Supplementary)

Exercises

$\boxed{6.7.3}$ (Supp) Tree with n vertices, $n \geq 3$.
Always true, false or could be either?
(a) $e(T) \stackrel{?}{=} n$
(b) at least one vertex of deg 2
(c) at least two v_1, v_2 s.t. $\deg(v_1)=\deg(v_2)$
(d) exactly one path from v_1 to v_2

Exercise (Supplementary)

Exercises

$\boxed{6.7.3}$ (Supp) Tree with n vertices, $n \geq 3$.
Always true, false or could be either?

(a) $e(T) \stackrel{?}{=} n$ — False
(b) at least one vertex of deg 2 — Could be either
(c) at least two v_1, v_2 s.t. $\deg(v_1)=\deg(v_2)$ — True
(d) exactly one path from v_1 to v_2 — True (characterises a tree)

Special Graphs

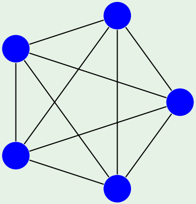
- **Complete graph K_n**
  n vertices, all pairwise connected, $\frac{n(n-1)}{2}$ edges.

- **Complete bipartite graph $K_{m,n}$**
  Has $m + n$ vertices, partitioned into two (disjoint) sets, one of n, the other of m vertices.
  All vertices from different parts are connected; vertices from the same part are disconnected. No. of edges is $m \cdot n$.

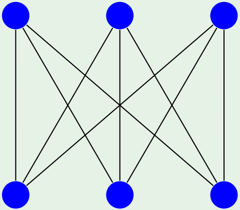
- **Complete k-partite graph $K_{m_1,\ldots,m_k}$**
  Has $m_1 + \ldots + m_k$ vertices, partitioned into k disjoint sets, respectively of $m_1, m_2, \ldots$ vertices.
  No. of edges is $\sum_{i<j} m_i m_j = \frac{1}{2} \sum_{i \neq j} m_i m_j$
  - These graphs generalise the complete graphs $K_n = K_{\underbrace{1,\ldots,1}_{n}}$

Example

K_5 :



$K_{3,3}$:



Graph Isomorphisms

$\phi : G \longrightarrow H$ is a *graph isomorphism* if

(i) $\phi : V_G \longrightarrow V_H$ is a bijection

(ii) $(x, y) \in E_G$ iff $(\phi(x), \phi(y)) \in E_H$

Two graphs are called *isomorphic* if there exists (at least one) isomorphism between them.

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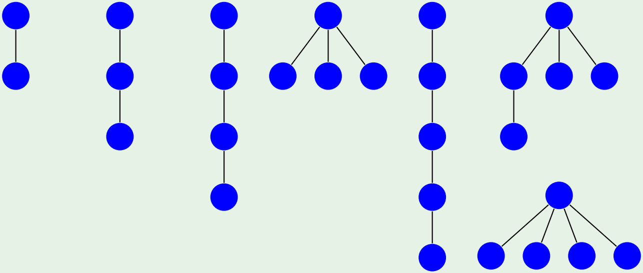
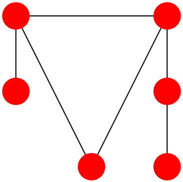
Example

All nonisomorphic trees on $2, 3, 4$ and 5 vertices.



Automorphisms and Asymmetric Graphs

An isomorphism from a graph to itself is called *automorphism*.
Every graph has at least the trivial automorphism;
(trivial meaning $\phi(v) = v$ for all $v \in V_G$)
Graphs with no non-trivial automorphisms are called *asymmetric*.
The smallest non-trivial asymmetric graphs have 6 vertices.



(Can you find another one with 6 nodes? There are seven more.)

Summary of topics

- Motivation and applications
- Terminology and notation
- Graph traversals
- Properties of graphs

Edge Traversal

Definition

- **Euler path** — path containing every edge exactly once
- **Euler circuit** — closed Euler path

Characterisations

- G (connected) has an Euler circuit iff $\deg(v)$ is even for all $v \in V$.
- G (connected) has an Euler path iff either it has an Euler circuit (above) or it has exactly two vertices of odd degree.

NB

- *These characterisations apply to graphs with loops as well*
- *For directed graphs the condition for existence of an Euler circuit is $indeg(v) = outdeg(v)$ for all $v \in V$*

Exercises

Exercises

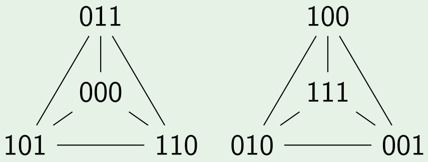
$\boxed{6.2.11}$ Construct a graph with vertex set $\{0,1\} \times \{0,1\} \times \{0,1\}$ and with an edge between vertices if they differ in exactly two coordinates.
(a) How many components does this graph have?
(b) How many vertices of each degree?
(c) Euler circuit?

$\boxed{6.2.12}$ As Ex. 6.2.11 but with an edge between vertices if they differ in two or three coordinates.

Exercises

Exercises

$\boxed{6.2.11}$ This graph consists of all the *face diagonals* of a cube. It has two disjoint components.

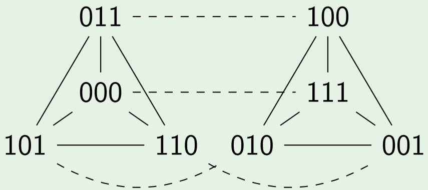


No Euler circuit

Exercises

Exercises

$\boxed{6.2.12}$ (Refer to Ex. 6.2.11 and connect the vertices from different components in pairs)



Must have an Euler circuit (why?)

Exercises

Exercises

$\boxed{6.2.14}$ Which complete graphs K_n have an Euler circuit?
When do bipartite, 3-partite complete graphs have an Euler circuit?

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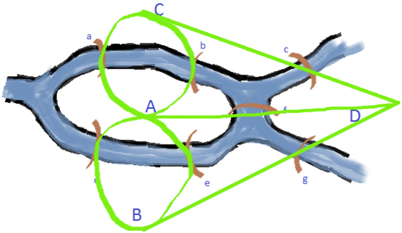
$\boxed{6.2.14}$ Which complete graphs K_n have an Euler circuit?
When do bipartite, 3-partite complete graphs have an Euler circuit?

K_n has an Euler circuit for n odd
$K_{m,n}$ — when both m and n are even
$K_{p,q,r}$ — when $p+q, p+r, q+r$ are all even, ie. when p, q, r are all even or all odd

Bridges of Köngisberg

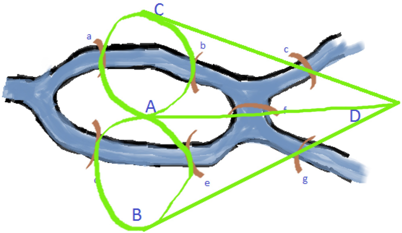
Bridges of Königsberg problem



Can you find a route which crosses each bridge exactly once?

Bridges of Köngisberg

Bridges of Königsberg problem



Can you find a route which crosses each bridge exactly once? No!

Vertex Traversal

Definition

- **Hamiltonian path** visits every vertex of graph exactly once
- **Hamiltonian cycle** visits every vertex exactly once except the last one, which duplicates the first

NB

Finding such a cycle, or proving it does not exist, is a difficult problem — the worst case is NP-complete.

Examples (when the cycle exists)

- All five regular polyhedra (verify!)
- n-cube; Hamiltonian circuit = *Gray code*
- K_m for all m; $K_{m,n}$ iff $m = n$; $K_{a,b,c}$ iff a, b, c satisfy the triangle inequalities: $a + b \geq c$, $a + c \geq b$, $b + c \geq a$
- Knight's tour on a chessboard (incl. rectangular boards)

Examples when a Hamiltonian cycle does not exist are much harder to construct.
Also, given such a graph it is nontrivial to verify that indeed there is no such a cycle: there is nothing obvious to specify that could assure us about this property.
In contrast, if a cycle is given, it is immediate to verify that it is a Hamiltonian cycle.
These situations demonstrate the often enormous discrepancy in difficulty of 'proving' versus (simply) 'checking'.

Exercise

Exercise

$\boxed{6.5.5(a)}$ How many Hamiltonian cycles does $K_{n,n}$ have?

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$\boxed{6.5.5(a)}$ How many Hamiltonian cycles does $K_{n,n}$ have?

Let $V = V_1 \cup V_2$

- start at any vertex in V_1
- go to any vertex in V_2
- go to any *new* vertex in V_1
- . . . . . .

There are $n!$ ways to order each part and two ways to choose the 'first' part, implying $c = 2(n!)^2$ circuits.

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Colouring

Informally: assigning a "colour" to each vertex (e.g. a node in an electric or transportation network) so that the vertices connected by an edge have different colours.
Formally: A mapping $c : V \longrightarrow [1 \,..\, n]$ such that for every $e = (v, w) \in E$

$$c(v) \neq c(w)$$

The minimum n sufficient to effect such a mapping is called the **chromatic number** of a graph $G = (E, V)$ and is denoted $\chi(G)$.

NB

This notion is extremely important in operations research, esp. in scheduling.
There is a dual notion of 'edge colouring' — two edges that share a vertex need to have different colours. Curiously enough, it is much less useful in practice.

Properties of the Chromatic Number

- $\chi(K_n) = n$
- If G has n vertices and $\chi(G) = n$ then $G = K_n$

Proof.

Suppose that G is 'missing' the edge (v, w), as compared with K_n. Colour all vertices, except w, using $n - 1$ colours. Then assign to w the same colour as that of v. $\square$

- If $\chi(G) = 1$ then G is totally disconnected: it has 0 edges.
- If $\chi(G) = 2$ then G is bipartite.
- For any tree $\chi(T) = 2$.
- For any cycle C_n its chromatic number depends on the parity of n — for n even $\chi(C_n) = 2$, while for n odd $\chi(C_n) = 3$.

Cliques

Graph (V', E') *subgraph* of (V, E) — $V' \subseteq V$ and $E' \subseteq E$.

Definition

A **clique** in G is a *complete* subgraph of G. A clique of k nodes is called k-*clique*.
The size of the largest clique is called the *clique number* of the graph and denoted $\kappa(G)$.

Theorem

$\chi(G) \geq \kappa(G)$.

Proof.

Every vertex of a clique requires a different colour, hence there must be at least $\kappa(G)$ colours. $\square$

However, this is the only restriction. For any given k there are graphs with $\kappa(G) = k$, while $\chi(G)$ can be arbitrarily large.

NB

This fact (and such graphs) are important in the analysis of parallel computation algorithms.

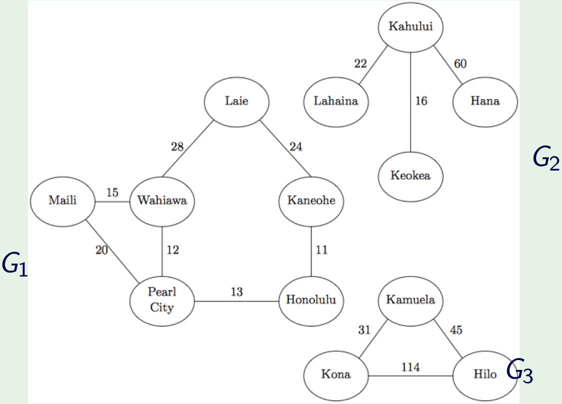
- $\kappa(K_n) = n$, $\kappa(K_{m,n}) = 2$, $\kappa(K_{m_1,\ldots,m_r}) = r$.
- If $\kappa(G) = 1$ then G is totally disconnected.
- For a tree $\kappa(T) = 2$.
- For a cycle C_n
  $\kappa(C_3) = 3, \quad \kappa(C_4) = \kappa(C_5) = \ldots = 2$

The difference between $\kappa(G)$ and $\chi(G)$ is apparent with just $\kappa(G) = 2$ — this does not imply that G is bipartite. For example, the cycle C_n for any odd n has $\chi(C_n) = 3$.

Exercise

Exercise

$\boxed{9.10.1}$ (Ullmann)

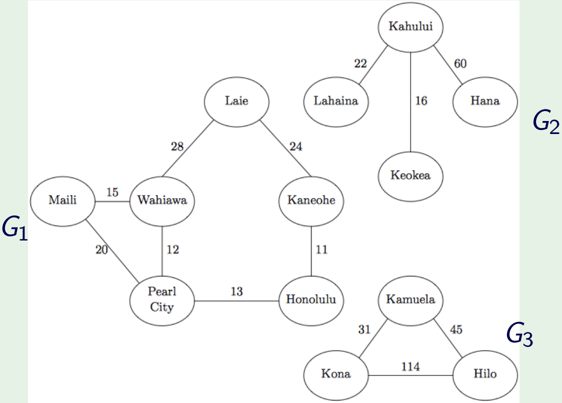


$\chi(G_i)$? $\kappa(G_i)$?

Exercise

Exercise

$\boxed{9.10.1}$ (Ullmann)



$$\chi(G_1) = \kappa(G_1) = 3; \quad \chi(G_2) = \kappa(G_2) = 2; \quad \chi(G_3) = \kappa(G_3) = 3$$

Exercise

Exercise

$\boxed{9.10.3}$ (Ullmann) Let $G = (V, E)$ be an undirected graph. What inequalities must hold between

- the maximal $deg(v)$ for $v \in V$
- $\chi(G)$
- $\kappa(G)$

Exercise

Exercise

$\boxed{9.10.3}$ (Ullmann) Let $G = (V, E)$ be an undirected graph. What inequalities must hold between

- the maximal $deg(v)$ for $v \in V$
- $\chi(G)$
- $\kappa(G)$

$max_{v \in V} deg(v) + 1 \geq \chi(G) \geq \kappa(G)$

Planar Graphs

Definition

A graph is **planar** if it can be embedded in a plane without its edges intersecting.

Theorem

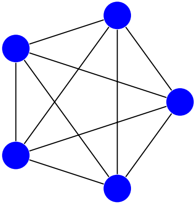
If the graph is planar it can be embedded (without self-intersections) in a plane so that all its edges are straight lines.

NB

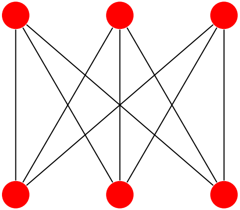
This notion and its related algorithms are extremely important to VLSI and visualizing data.

Two minimal nonplanar graphs

K_5 :



$K_{3,3}$:

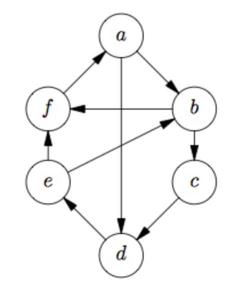


Try out K_5
Try out $K_{3,3}$

Exercise

Exercise

$\boxed{9.10.2}$ (Ullmann)



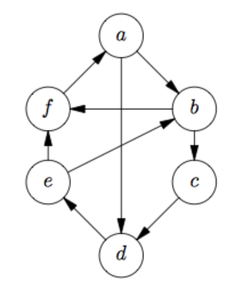
Is (the undirected version of) this graph planar?

Try it out

Exercise

Exercise

$\boxed{9.10.2}$ (Ullmann)



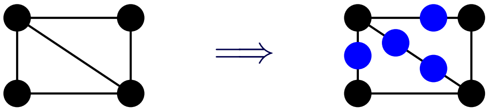
Is (the undirected version of) this graph planar? Yes

Try it out

Theorem

If graph G contains, as a subgraph, a nonplanar graph, then G itself is nonplanar.

For a graph, *edge subdivision* means to introduce some new vertices, all of degree 2, by placing them on existing edges.



We call such a derived graph a *subdivision* of the original one.

Theorem

If a graph is nonplanar then it must contain a subdivision of K_5 or $K_{3,3}$.

Theorem

K_n for $n \geq 5$ is nonplanar.

Proof.

It contains K_5: choose any five vertices in K_n and consider the subgraph they define. □

Theorem

$K_{m,n}$ is nonplanar when $m \geq 3$ and $n \geq 3$.

Proof.

They contain $K_{3,3}$ — choose any three vertices in each of two vertex parts and consider the subgraph they define. □

Question

Are all $K_{m,1}$ *planar?*

Question

Are all $K_{m,1}$ planar?

Answer

Yes, they are trees of two levels — the root and m leaves.

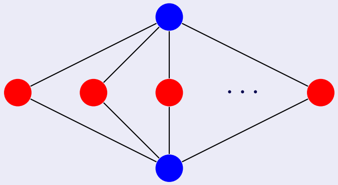
Question

Are all $K_{m,2}$ *planar?*

Answer

Yes; they can be represented by "glueing" together two such trees at the leaves.

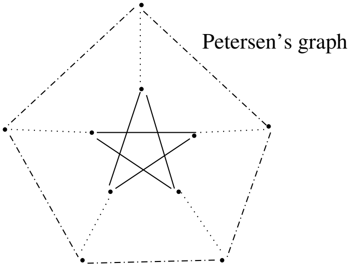
Sketching $K_{m,2}$



Also, among the k-partite graphs, planar are $K_{2,2,2}$ and $K_{1,1,m}$.
The latter can be depicted by drawing one extra edge in $K_{2,m}$, connecting the top and bottom vertices.

NB

Finding a 'basic' nonplanar obstruction is not always simple



It contains a subdivision $K_{3,3}$, but not K_5.

Strategy for finding a subdivision

To show G contains a subdivision of H:

Strategy I:

- Start at H
- Perform the following operations as many times as you need:
  1. (i) Subdivide an edge
  2. (ii) Add a vertex
  3. (iii) Add an edge
- Finish with G

NB

- *Each operation increases* $|V| + |E|$
- *Can do all (i) first, then all (ii), then all (iii)*

Strategy for finding a subdivision

To show G contains a subdivision of H:

Strategy II:

- Start at G
- Perform the following operations as many times as you need:
  1. (i) Delete an edge
  2. (ii) Delete a vertex (and all adjacent edges)
  3. (iii) Replace a vertex of degree 2 with an edge connecting its neighbours (contracting a vertex)
- Finish with H

NB

- *Each operation decreases* $|V| + |E|$
- *Can do all (i) first, then all (ii), then all (iii)*

Showing a graph does not contain a subdivision

Question

What does not change when performing the operations?