Week 02b: Graph Data Structures

Graph Definitions

Graphs

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Many applications require

- a collection of *items* (i.e. a set)
- *relationships*/connections between items

Examples:

- maps: items are cities, connections are roads
- web: items are pages, connections are hyperlinks

Collection types you're familiar with

- lists … linear sequence of items (last week; COMP9021)
- trees … branched hierarchy of items (COMP9021)

Graphs are more general … allow arbitrary connections

... Graphs

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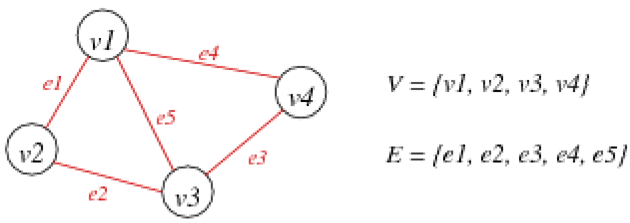
A *graph G = (V,E)*

- *V* is a set of *vertices*
- *E* is a set of *edges* (subset of $V \times V$)

Example:



... Graphs

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A real example: Australian road distances

Distance	Adelaide	Brisbane	Canberra	Darwin	Melbourne	Perth	Sydney
Adelaide	-	2055	1390	3051	732	2716	1605
Brisbane	2055	-	1291	3429	1671	4771	982
Canberra	1390	1291	-	4441	658	4106	309
Darwin	3051	3429	4441	-	3783	4049	4411
Melbourne	732	1671	658	3783	-	3448	873

Perth	2716	4771	4106	4049	3448	-	3972
Sydney	1605	982	309	4411	873	3972	-

Notes: vertices are cities, edges are distance between cities, symmetric

... Graphs

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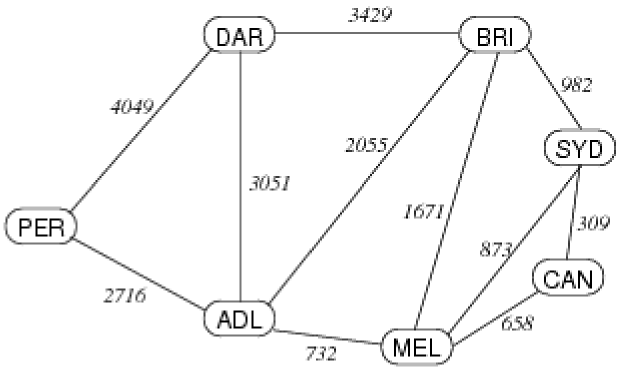
Alternative representation of above:



... Graphs

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Questions we might ask about a graph:

- is there a way to get from item A to item B?
- what is the best way to get from A to B?
- which items are connected?

Graph algorithms are generally more complex than tree/list ones:

- no implicit order of items
- graphs may contain cycles
- concrete representation is less obvious
- algorithm complexity depends on connection complexity

Properties of Graphs

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Terminology: $|V|$ and $|E|$ (cardinality) normally written just as V and E.

A graph with V vertices has at most $V(V-1)/2$ edges.

The ratio $E{:}V$ can vary considerably.

- if E is closer to V^2, the graph is *dense*
- if E is closer to V, the graph is *sparse*
  - Example: web pages and hyperlinks

Knowing whether a graph is sparse or dense is important

- may affect choice of data structures to represent graph
- may affect choice of algorithms to process graph

Exercise #1: Number of Edges

The edges in a graph represent pairs of connected vertices. A graph with V has V^2 such pairs.

Consider $V = \{1,2,3,4,5\}$ with all possible pairs:

$E = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), \ldots, (4,5), (5,5) \}$

Why do we say that the maximum #edges is $V(V-1)/2$?

… because

- (v,w) and (w,v) denote the same edge (in an undirected graph)
- we do not consider loops (v,v)

Graph Terminology

For an edge e that connects vertices v and w

- v and w are *adjacent* (neighbours)
- e is *incident* on both v and w

Degree of a vertex v

- number of edges incident on v

Synonyms:

- vertex = node, edge = arc = link (Note: some people use arc for *directed* edges)

... Graph Terminology

Path: a sequence of vertices where

- each vertex has an edge to its predecessor

Simple path: a path where

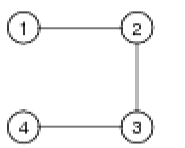
- all vertices and edges are different

Cycle: a path

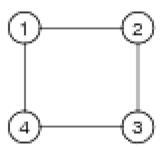
- that is simple except last vertex = first vertex

Length of path or cycle:

- #edges



Path: 1–2, 2–3, 3–4



Cycle: 1–2, 2–3, 3–4, 4–1

... Graph Terminology

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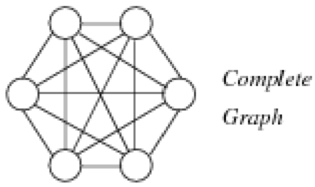
Connected graph

- there is a *path* from each vertex to every other vertex
- if a graph is not connected, it has ≥2 *connected components*

Complete graph K_V

- there is an *edge* from each vertex to every other vertex
- in a complete graph, $E = V(V-1)/2$



... Graph Terminology

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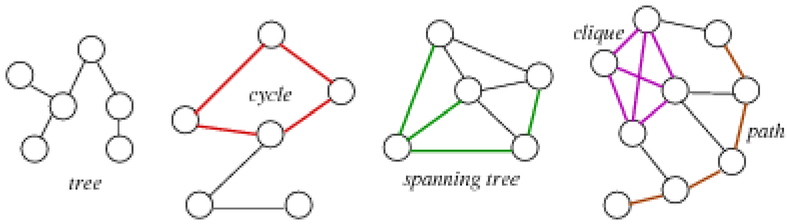
Tree: connected (sub)graph with no cycles

Spanning tree: tree containing all vertices

Clique: complete subgraph

Consider the following single graph:



This graph has 26 vertices, 33 edges, and 4 connected components

Note: The entire graph has no spanning tree; what is shown in green is a spanning tree of the third connected component

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... Graph Terminology

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A *spanning tree* of connected graph $G = (V,E)$

- is a subgraph of G containing all of V
- and is a single tree (connected, no cycles)

A *spanning forest* of non-connected graph $G = (V,E)$

- is a subgraph of G containing all of V
- and is a set of trees (not connected, no cycles),
  - with one tree for each *connected component*

Exercise #2: Graph Terminology

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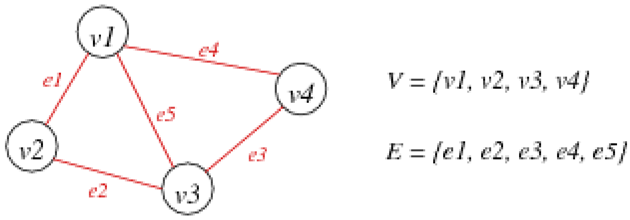

1. How many edges to remove to obtain a spanning tree?
2. How many different spanning trees?

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1. 2
2. $\frac{5 \cdot 4}{2} - 2 = 8$ spanning trees (no spanning tree if we remove *{e1,e2}* or *{e3,e4}*)

... Graph Terminology

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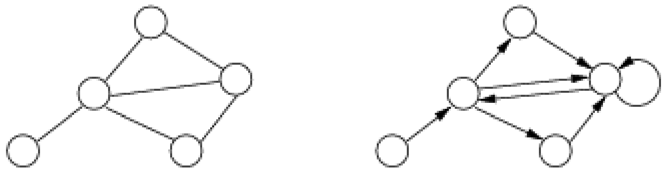
Undirected graph

- *edge(u,v)* = *edge(v,u)*, no self-loops (i.e. no *edge(v,v)*)

Directed graph

- *edge(u,v)* ≠ *edge(v,u)*, can have self-loops (i.e. *edge(v,v)*)

Examples:



Undirected graph *Directed graph*

... Graph Terminology

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Other types of graphs …

Weighted graph

- each edge has an associated value (weight)
- e.g. road map (weights on edges are distances between cities)

Multi-graph

- allow multiple edges between two vertices
- e.g. function call graph (`f()` calls `g()` in several places)

Graph Data Structures

Graph Representations

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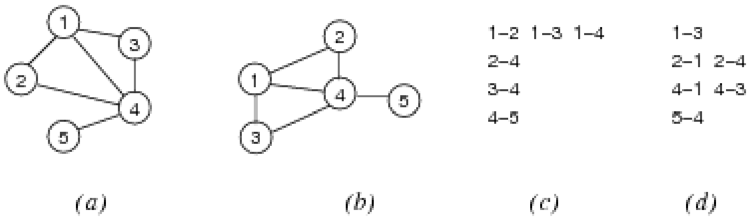
Defining graphs:

- need some way of identifying vertices
- could give diagram showing edges and vertices
- could give a list of edges

E.g. four representations of the same graph:



... Graph Representations

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We will discuss three different graph data structures:

1. Array of edges
2. Adjacency matrix
3. Adjacency list

Array-of-edges Representation

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Edges are represented as an array of `Edge` values (= pairs of vertices)

- space efficient representation
- adding and deleting edges is slightly complex
- undirected: order of vertices in an `Edge` doesn't matter
- directed: order of vertices in an `Edge` encodes direction

[(0,1), (1,2), (1,3), (2,3)]   [(1,0), (1,1), (0,2), (0,3), (2,3)]

For simplicity, we always assume vertices to be numbered `0..V-1`

... Array-of-edges Representation

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Graph initialisation

```
newGraph(V):
|  Input  number of nodes V
|  Output new empty graph
|
|  g.nV = V   // #vertices (numbered 0..V-1)
|  g.nE = 0   // #edges
|  allocate enough memory for g.edges[]
|  return g
```

How much is enough? … No more than *V(V-1)/2* … Much less in practice (sparse graph)

... Array-of-edges Representation

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Edge insertion

```
insertEdge(g,(v,w)):
|  Input graph g, edge (v,w)
|
|  g.edges[g.nE]=(v,w)
|  g.nE=g.nE+1
```

... Array-of-edges Representation

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Edge removal

```
removeEdge(g,(v,w)):
|  Input graph g, edge (v,w)
|
|  i=0
|  while (v,w)≠g.edges[i] do
|     i=i+1
|  end while
|  g.edges[i]=g.edges[g.nE-1] // replace (v,w) by last edge in array
|  g.nE=g.nE-1
```

Cost Analysis

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Storage cost: *O(E)*

Cost of operations:

- initialisation: *O(1)*
- insert edge: *O(1)* (assuming edge array has space)
- find/delete edge: *O(E)* (need to find edge in edge array)

If array is full on insert

- allocate space for a bigger array, copy edges across ⇒ *O(E)*

If we maintain edges in order

- use binary search to insert/find edge ⇒ *O(log E)*

Exercise #3: Array-of-edges Representation

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Assuming an array-of-edges representation …

Write an algorithm to output all edges of the graph

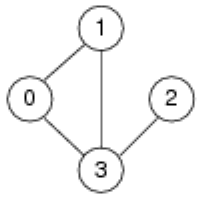
```
show(g):
|  Input graph g
|
```

```
|   for all i=0 to g.nE-1 do
|      print g.edges[i]
|   end for
```

Time complexity: *O(E)*

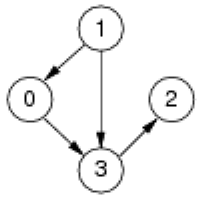
Adjacency Matrix Representation

Edges represented by a $V \times V$ matrix



Undirected graph

A	0	1	2	3
0	0	1	0	1
1	1	0	0	1
2	0	0	0	1
3	1	1	1	0



Directed graph

A	0	1	2	3
0	0	0	0	1
1	1	0	0	1
2	0	0	0	0
3	0	0	1	0

... Adjacency Matrix Representation

Advantages

- easily implemented as 2-dimensional array
- can represent graphs, digraphs and weighted graphs
  - graphs: symmetric boolean matrix
  - digraphs: non-symmetric boolean matrix
  - weighted: non-symmetric matrix of weight values

Disadvantages:

- if few edges (sparse) ⇒ memory-inefficient

... Adjacency Matrix Representation

Graph initialisation

```
newGraph(V):
|   Input  number of nodes V
|   Output new empty graph
|
|   g.nV = V   // #vertices (numbered 0..V-1)
|   g.nE = 0   // #edges
|   allocate memory for g.edges[][]
|   for all i,j=0..V-1 do
|      g.edges[i][j]=0   // false
|   end for
|   return g
```

... Adjacency Matrix Representation

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Edge insertion

```
insertEdge(g,(v,w)):
|  Input graph g, edge (v,w)
|
|  if g.edges[v][w]=0 then  // (v,w) not in graph
|     g.edges[v][w]=1       // set to true
|     g.edges[w][v]=1
|     g.nE=g.nE+1
|  end if
```

... Adjacency Matrix Representation

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Edge removal

```
removeEdge(g,(v,w)):
|  Input graph g, edge (v,w)
|
|  if g.edges[v][w]≠0 then  // (v,w) in graph
|     g.edges[v][w]=0       // set to false
|     g.edges[w][v]=0
|     g.nE=g.nE-1
|  end if
```

Exercise #4: Show Graph

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Assuming an adjacency matrix representation …

Write an algorithm to output all edges of the graph (no duplicates!)

... Adjacency Matrix Representation

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```
show(g):
|  Input graph g
|
|  for all i=0 to g.nV-2 do
|  |  for all j=i+1 to g.nV-1 do
|  |     if g.edges[i][j] then
|  |        print i"—"j
|  |     end if
|  |  end for
|  end for
```

Time complexity: $O(V^2)$

Exercise #5:

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Analyse storage cost and time complexity of adjacency matrix representation

Storage cost: $O(V^2)$

If the graph is sparse, most storage is wasted.

Cost of operations:

- initialisation: $O(V^2)$ (initialise $V \times V$ matrix)
- insert edge: $O(1)$ (set two cells in matrix)
- delete edge: $O(1)$ (unset two cells in matrix)

... Adjacency Matrix Representation

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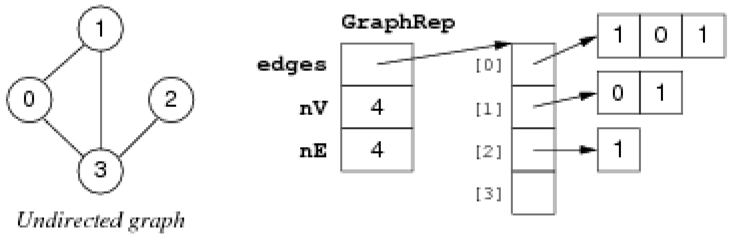
A storage optimisation: store only top-right part of matrix.



Undirected graph

New storage cost: V-1 int ptrs + $V(V+1)/2$ ints (but still $O(V^2)$)

Requires us to always use edges (v,w) such that $v < w$.

Adjacency List Representation

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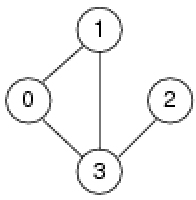
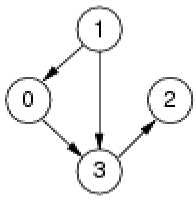
For each vertex, store linked list of adjacent vertices:



Undirected graph

A[0] = <1, 3>
A[1] = <0, 3>
A[2] = <3>
A[3] = <0, 1, 2>



Directed graph

A[0] = <3>
A[1] = <0, 3>
A[2] = < >
A[3] = <2>

... Adjacency List Representation

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Advantages

- relatively easy to implement in languages like C
- can represent graphs and digraphs
- memory efficient if E:V relatively small

Disadvantages:

- one graph has many possible representations
  (unless lists are ordered by same criterion e.g. ascending)

... Adjacency List Representation

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Graph initialisation

```
newGraph(V):
|  Input  number of nodes V
|  Output new empty graph
|
|  g.nV = V   // #vertices (numbered 0..V-1)
|  g.nE = 0   // #edges
|  allocate memory for g.edges[]
|  for all i=0..V-1 do
|     g.edges[i]=NULL   // empty list
|  end for
|  return g
```

... Adjacency List Representation

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Edge insertion:

```
insertEdge(g,(v,w)):
|  Input graph g, edge (v,w)
|
|  insertLL(g.edges[v],w)
|  insertLL(g.edges[w],v)
|  g.nE=g.nE+1
```

... Adjacency List Representation

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Edge removal:

```
removeEdge(g,(v,w)):
|  Input graph g, edge (v,w)
|
|  deleteLL(g.edges[v],w)
|  deleteLL(g.edges[w],v)
|  g.nE=g.nE-1
```

Exercise #6:

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Analyse storage cost and time complexity of adjacency list representation

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Storage cost: *O(V+E)* (*V* list pointers, total of 2·*E* list elements)

Cost of operations:

- initialisation: *O(V)* (initialise *V* lists)
- insert edge: *O(1)* (insert one vertex into list)
  - if you don't check for duplicates
- find/delete edge: *O(V)* (need to find vertex in list)

If vertex lists are sorted

- insert requires search of list ⇒ *O(V)*
- delete always requires a search, regardless of list order

Comparison of Graph Representations

	array of edges	adjacency matrix	adjacency list
space usage	*E*	V^2	*V+E*
initialise	*1*	V^2	*V*
insert edge	*1*	*1*	*1*
find/delete edge	*E*	*1*	*V*

Other operations:

	array of edges	adjacency matrix	adjacency list
disconnected(v)?	*E*	*V*	*1*
isPath(x,y)?	*E·log V*	V^2	*V+E*
copy graph	*E*	V^2	*E*
destroy graph	*1*	*V*	*E*

Graph Abstract Data Type

Graph ADT

Data:

- set of edges, set of vertices

Operations:

- building: create graph, add edge
- deleting: remove edge, drop whole graph
- scanning: check if graph contains a given edge

Things to note:

- set of vertices is fixed when graph initialised
- we treat vertices as `int`s, but could be arbitrary `Item`s

... Graph ADT

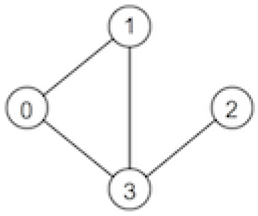
Graph ADT interface **`graph.h`**

```
// graph representation is hidden
typedef struct GraphRep *Graph;

// vertices denoted by integers 0..N-1
typedef int Vertex;

// edges are pairs of vertices (end-points)
typedef struct Edge { Vertex v; Vertex w; } Edge;

// operations on graphs
Graph newGraph(int V);                      // new graph with V vertices
void  insertEdge(Graph, Edge);
void  removeEdge(Graph, Edge);
bool  adjacent(Graph, Vertex, Vertex); /* is there an edge
                                          between two vertices */
void  freeGraph(Graph);
```

Exercise #7: Graph ADT Client

Write a program that uses the graph ADT to

- build the graph depicted below
- print all the nodes that are incident to vertex `1` in ascending order



```
#include <stdio.h>
#include "Graph.h"

#define NODES 4
#define NODE_OF_INTEREST 1

int main(void) {
   Graph g = newGraph(NODES);

   Edge e;
   e.v = 0; e.w = 1; insertEdge(g,e);
   e.v = 0; e.w = 3; insertEdge(g,e);
   e.v = 1; e.w = 3; insertEdge(g,e);
   e.v = 3; e.w = 2; insertEdge(g,e);

   int v;
   for (v = 0; v < NODES; v++) {
      if (adjacent(g, v, NODE_OF_INTEREST))
         printf("%d\n", v);
   }

   freeGraph(g);
   return 0;
}
```

Graph ADT (Array of Edges)

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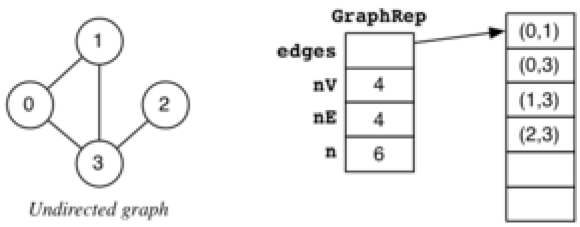
Implementation of GraphRep (array-of-edges representation)

```
typedef struct GraphRep {
   Edge *edges; // array of edges
   int   nV;    // #vertices (numbered 0..nV-1)
   int   nE;    // #edges
   int   n;     // size of edge array
} GraphRep;
```



Graph ADT (Adjacency Matrix)

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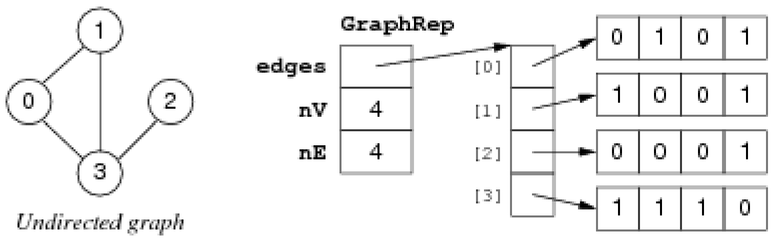
Implementation of GraphRep (adjacency-matrix representation)

```
typedef struct GraphRep {
   int  **edges; // adjacency matrix
   int    nV;    // #vertices
   int    nE;    // #edges
} GraphRep;
```



... Graph ADT (Adjacency Matrix)

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Implementation of graph initialisation (adjacency-matrix representation)

```
Graph newGraph(int V) {
   assert(V >= 0);
   int i;

   Graph g = malloc(sizeof(GraphRep));      assert(g != NULL);
   g->nV = V;  g->nE = 0;

   // allocate memory for each row
   g->edges = malloc(V * sizeof(int *));    assert(g->edges != NULL);
   // allocate memory for each column and initialise with 0
   for (i = 0; i < V; i++) {
      g->edges[i] = calloc(V, sizeof(int)); assert(g->edges[i] != NULL);
   }
```

```
   return g;
}
```

standard library function `calloc(size_t nelems, size_t nbytes)`

- allocates a memory block of size `nelems*nbytes`
- and sets all bytes in that block to *zero*

... Graph ADT (Adjacency Matrix)

Implementation of edge insertion/removal (adjacency-matrix representation)

```
// check if vertex is valid in a graph
bool validV(Graph g, Vertex v) {
   return (g != NULL && v >= 0 && v < g->nV);
}

void insertEdge(Graph g, Edge e) {
   assert(g != NULL && validV(g,e.v) && validV(g,e.w));

   if (!g->edges[e.v][e.w]) {  // edge e not in graph
      g->edges[e.v][e.w] = 1;
      g->edges[e.w][e.v] = 1;
      g->nE++;
   }
}

void removeEdge(Graph g, Edge e) {
   assert(g != NULL && validV(g,e.v) && validV(g,e.w));

   if (g->edges[e.v][e.w]) {   // edge e in graph
      g->edges[e.v][e.w] = 0;
      g->edges[e.w][e.v] = 0;
      g->nE--;
   }
}
```

Exercise #8: Checking Neighbours (i)

Assuming an adjacency-matrix representation …

Implement a function to check whether two vertices are directly connected by an edge

```
bool adjacent(Graph g, Vertex x, Vertex y) { … }
```

```
bool adjacent(Graph g, Vertex x, Vertex y) {
   assert(g != NULL && validV(g,x) && validV(g,y));

   return (g->edges[x][y] != 0);
}
```

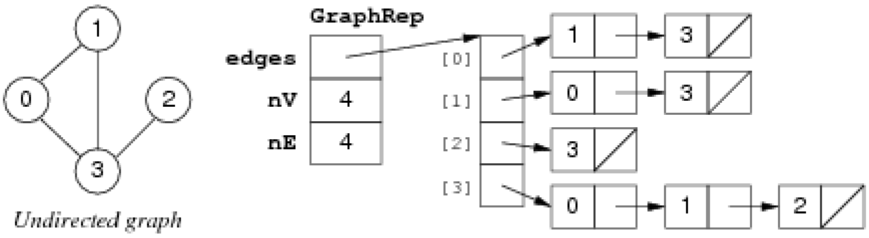
Graph ADT (Adjacency List)

Implementation of `GraphRep` (adjacency-list representation)

```
typedef struct GraphRep {
   Node **edges;  // array of lists
```

```
   int    nV;     // #vertices
   int    nE;     // #edges
} GraphRep;

typedef struct Node {
   Vertex      v;
   struct Node *next;
} Node;
```



Exercise #9: Checking Neighbours (ii)

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Assuming an adjacency list representation …

Implement a function to check whether two vertices are directly connected by an edge

```
bool adjacent(Graph g, Vertex x, Vertex y) { … }
```

```
bool adjacent(Graph g, Vertex x, Vertex y) {
   assert(g != NULL && validV(g,x));

   return inLL(g->edges[x], y);
}
```

`inLL()` checks if linked list contains an element

Problems on Graphs

Problems on Graphs

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What kind of problems do we want to solve on/via graphs?

- is the graph fully-connected?
- can we remove an edge and keep it fully-connected?
- is one vertex reachable starting from some other vertex?
- which vertices are reachable from *v*? (transitive closure)
- is there a cycle that passes through all vertices? (circuit)
- what is the cheapest cost path from *v* to *w*?
- is there a tree that links all vertices? (spanning tree)
- what is the minimum spanning tree?
- what is the maximal flow through a graph?
- …
- can a graph be drawn in a plane with no crossing edges? (planar graphs)
- are two graphs "equivalent"? (isomorphism)

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Graph Algorithms

In this course we examine algorithms for

- graph traversal (simple graphs)
- reachability (directed graphs)
- minimum spanning trees (weighted graphs)
- shortest path (weighted graphs)
- maximum flow (weighted graphs)

Graph Traversal

Finding a Path

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Questions on paths:

- is there a path between two given vertices (*src*,*dest*)?
- what is the sequence of vertices from *src* to *dest*?

Approach to solving problem:

- examine vertices adjacent to *src*
- if any of them is *dest*, then done
- otherwise try vertices two edges from *src*
- repeat looking further and further from *src*

Two strategies for graph traversal/search: *depth-first*, *breadth-first*

- DFS follows one path to completion before considering others
- BFS "fans-out" from the starting vertex ("spreading" subgraph)

... Finding a Path

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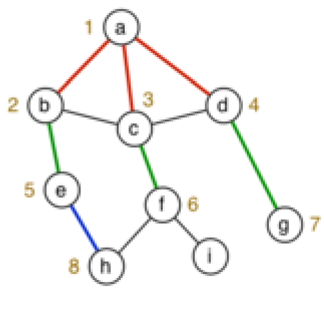
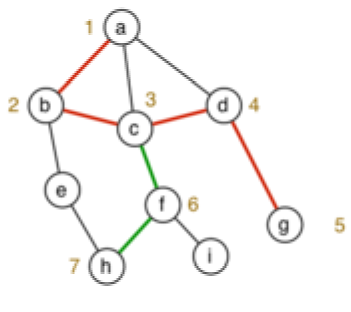
Comparison of BFS/DFS search for checking if there is a path from *a* to *h* …



Breadth-first Search



Depth-first Search

Both approaches ignore some edges by remembering previously visited vertices.

Depth-first Search

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Depth-first search can be described recursively as

```
depthFirst(G,v):
```

1. mark v as visited
2. for each (v,w)∈*edges*(G) do
   if w has not been visited then
     depthFirst(w)

The recursion induces *backtracking*

... Depth-first Search

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Recursive DFS path checking

```
hasPath(G,src,dest):
|  Input  graph G, vertices src,dest
|  Output true if there is a path from src to dest in G,
|         false otherwise
|
|  mark all vertices in G as unvisited
|  return dfsPathCheck(G,src,dest)

dfsPathCheck(G,v,dest):
|  mark v as visited
|  if v=dest then       // found dest
|     return true
|  else
|  |  for all (v,w)∈edges(G) do
|  |  |  if w has not been visited then
|  |  |     return dfsPathCheck(G,w,dest) // found path via w to dest
|  |  |  end if
|  |  end for
|  end if
|  return false         // no path from v to dest
```

Exercise #10: Depth-first Traversal (i)

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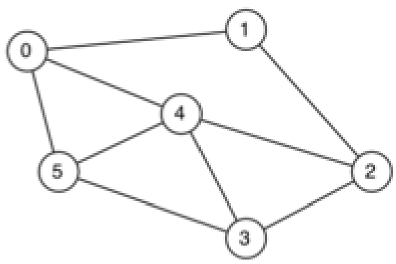
Trace the execution of `dfsPathCheck(G,0,5)` on:



Consider neighbours in ascending order

Answer:

0 - 1 - 2 - 3 - 4 - 5

... Depth-first Search

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Cost analysis:

- all vertices marked as unvisited, each vertex visited at most once ⇒ cost = $O(V)$
- visit all edges incident on visited vertices ⇒ cost = $O(E)$
  - assuming an adjacency list representation

Time complexity of DFS: $O(V+E)$ (adjacency list representation)

- the larger of V,E determines the complexity

... Depth-first Search

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Note how different graph data structures affect cost:

- array-of-edges representation
  - visit all edges incident on visited vertices ⇒ cost = $O(V \cdot E)$
  - cost of DFS: $O(V \cdot E)$
- adjacency-matrix representation
  - visit all edges incident on visited vertices ⇒ cost = $O(V^2)$
  - cost of DFS: $O(V^2)$

For *dense graphs* … $E \cong V^2 \Rightarrow O(V+E) = O(V^2)$
For *sparse graphs* … $E \cong V \Rightarrow O(V+E) = O(E)$

... Depth-first Search

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Knowing whether a path exists can be useful

Knowing what the path is even more useful
⇒ record the previously visited node as we search through the graph (so that we can then trace path through graph)

Make use of global variable:

- `visited[]` … array to store previously visited node, for each node being visited

---

... Depth-first Search

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```
visited[] // store previously visited node, for each vertex 0..nV-1

findPath(G,src,dest):
|  Input graph G, vertices src,dest
|
|  for all vertices v∈G do
|     visited[v]=-1
|  end for
|  visited[src]=src                  // starting node of the path
|  if dfsPathCheck(G,src,dest) then  // show path in dest..src order
|  |  v=dest
|  |  while v≠src do
|  |     print v"-"
|  |     v=visited[v]
|  |  end while
```

```
|   |  print src
|   end if

dfsPathCheck(G,v,dest):
|   if v=dest then                    // found edge from v to dest
|      return true
|   else
|   |  for all (v,w)∈edges(G) do
|   |  |  if visited[w]=-1 then
|   |  |  |  visited[w]=v
|   |  |  |  if dfsPathCheck(G,w,dest) then
|   |  |  |     return true           // found path via w to dest
|   |  |  |  end if
|   |  |  end if
|   |  end for
|   end if
|   return false                      // no path from v to dest
```

Exercise #11: Depth-first Traversal (ii)

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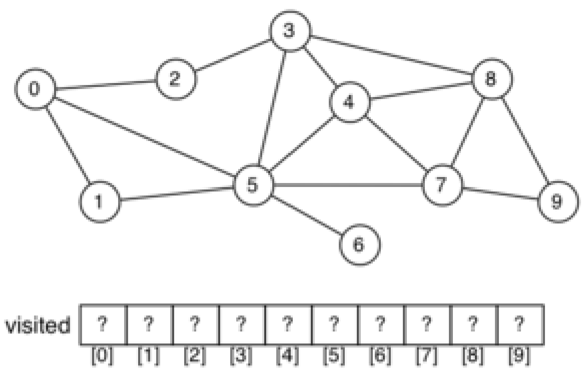
Show the DFS order in which we visit vertices in this graph when searching for a path from 0 to 6:



Consider neighbours in ascending order

0	**0**	**3**	**5**	**3**	**1**	**5**	**4**	**7**	**8**
[0]	**[1]**	**[2]**	**[3]**	**[4]**	**[5]**	**[6]**	**[7]**	**[8]**	**[9]**

Path: 6-5-1-0

... Depth-first Search

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DFS can also be described non-recursively (via a *stack*):

```
hasPath(G,src,dest):
|   Input  graph G, vertices src,dest
|   Output true if there is a path from src to dest in G,
|          false otherwise
|
|   mark all vertices in G as unvisited
|   push src onto new stack s
|   found=false
|   while not found and s is not empty do
```

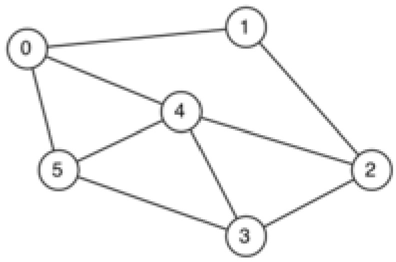
```
|   |   pop v from s
|   |   mark v as visited
|   |   if v=dest then
|   |      found=true
|   |   else
|   |   |  for each (v,w)∈edges(G) such that w has not been visited
|   |   |     push w onto s
|   |   |  end for
|   |   end if
|   end while
|   return found
```

Uses standard stack operations (push, pop, check if empty)

Time complexity is the same: *O(V+E)* (each vertex added to stack once, each element in vertex's adjacency list visited once)

Exercise #12: Depth-first Traversal (iii)

Show how the stack evolves when executing `findPathDFS(g,0,5)` on:



Push neighbours in *descending* order … so they get popped in ascending order

---

										4		5		
								3		5		5		5
				1		2		4		4		4		4
				4		4		4		4		4		4
(empty)	→	0	→	5	→	5	→	5	→	5	→	5	→	5

Breadth-first Search

Basic approach to breadth-first search (BFS):

- visit and mark current vertex
- visit all neighbours of current vertex
- then consider neighbours of neighbours

Notes:

- tricky to describe recursively
- a minor variation on non-recursive DFS search works
  ⇒ switch the *stack* for a *queue*

... Breadth-first Search

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BFS algorithm (records visiting order, marks vertices as visited when put *on* queue):

```
visited[] // array of visiting orders, indexed by vertex 0..nV-1

findPathBFS(G,src,dest):
|  Input  graph G, vertices src,dest
|
|  for all vertices v∈G do
|     visited[v]=-1
|  end for
|  enqueue src into new queue q
|  visited[src]=src
|  found=false
|  while not found and q is not empty do
|  |  dequeue v from q
|  |  if v=dest then
|  |     found=true
|  |  else
|  |  |  for each (v,w)∈edges(G) such that visited[w]=-1 do
|  |  |     enqueue w into q
|  |  |     visited[w]=v
|  |  |  end for
|  |  end if
|  end while
|  if found then
|     display path in dest..src order
|  end if
```

Uses standard queue operations (enqueue, dequeue, check if empty)

Exercise #13: Breadth-first Traversal

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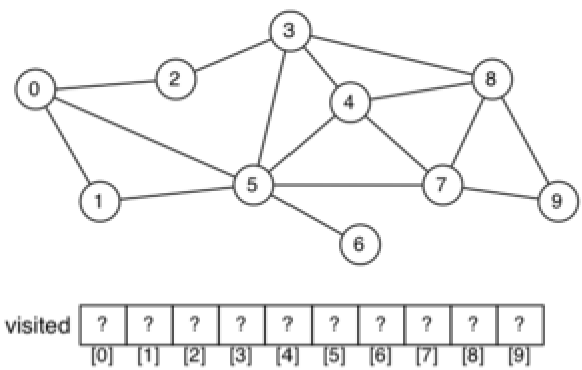
Show the BFS order in which we visit vertices in this graph when searching for a path from 0 to 6:



Consider neighbours in ascending order

0	0	0	2	5	0	5	5	3	-1
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]

Path: 6-5-0

... Breadth-first Search

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Time complexity of BFS: O(V+E) (adjacency list representation, same as DFS)

BFS finds a "shortest" path

- based on minimum # edges between *src* and *dest*.
- stops with first-found path, if there are multiple ones

In many applications, edges are weighted and we want path

- based on minimum sum-of-weights along path *src* .. *dest*

We discuss weighted/directed graphs later.

Other DFS Examples

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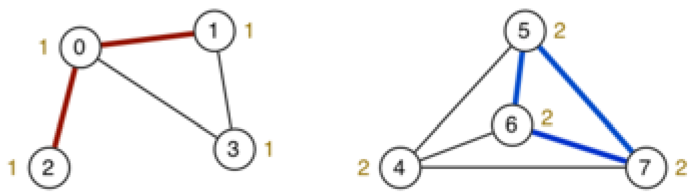
Other problems to solve via DFS graph search

- checking for the existence of a cycle
- determining which connected component each vertex is in



Graph with two connected components, a path and a cycle

Exercise #14: Buggy Cycle Check

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A graph has a *cycle* if

- it has a path of length > 1
- with start vertex *src* = end vertex *dest*
- and without using any edge more than once

We are not required to give the path, just indicate its presence.

The following DFS cycle check has two bugs. Find them.

```
hasCycle(G):
|  Input   graph G
|  Output  true if G has a cycle, false otherwise
|
|  choose any vertex v∈G
|  return dfsCycleCheck(G,v)

dfsCycleCheck(G,v):
|  mark v as visited
|  for each (v,w)∈edges(G) do
|  |  if w has been visited then    // found cycle
|  |     return true
|  |  else if dfsCycleCheck(G,w) then
```

```
|   |      return true
|   end for
|   return false                    // no cycle at v
```

1. Only one connected component is checked.
2. The loop

   ```
   for each (v,w)∈edges(G) do
   ```

   should exclude the neighbour of v from which you just came, so as to prevent a single edge *w-v* from being classified as a cycle.

Computing Connected Components

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Problems:

- how many connected subgraphs are there?
- are two vertices in the same connected subgraph?

Both of the above can be solved if we can

- build an array, one element for each vertex V
- indicating which connected component V is in

- `componentOf[]` … array [0..nV-1] of component IDs

... Computing Connected Components

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Algorithm to assign vertices to connected components:

```
components(G):
|   Input graph G
|
|   for all vertices v∈G do
|      componentOf[v]=-1
|   end for
|   compID=0
|   for all vertices v∈G do
|   |  if componentOf[v]=-1 then
|   |     dfsComponents(G,v,compID)
|   |     compID=compID+1
|   |  end if
|   end for

dfsComponents(G,v,id):
|   componentOf[v]=id
|   for all vertices w adjacent to v do
|      if componentOf[w]=-1 then
|         dfsComponents(G,w,id)
|      end if
|   end for
```

Exercise #15: Connected components

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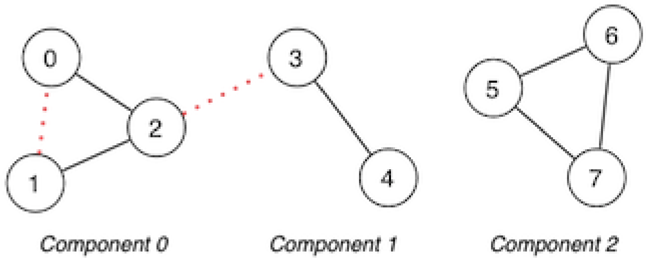
Trace the execution of the algorithm

1. on the graph shown below
2. on the same graph but with the dotted edges added



Consider neighbours in ascending order

1.

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
-1	-1	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1	-1
0	-1	0	-1	-1	-1	-1	-1
0	0	0	-1	-1	-1	-1	-1
0	0	0	1	-1	-1	-1	-1
...							
0	0	0	1	1	2	2	2

2.

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
-1	-1	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1	-1
0	0	-1	-1	-1	-1	-1	-1
0	0	0	-1	-1	-1	-1	-1
...							
0	0	0	0	0	1	1	1

Hamiltonian and Euler Paths

Hamiltonian Path and Circuit

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Hamiltonian path problem:

- find a path connecting two vertices *v*,*w* in graph *G*
- such that the path includes each *vertex* exactly once

If *v* = *w*, then we have a *Hamiltonian circuit*

Simple to state, but difficult to solve (*NP*-complete)

Many real-world applications require you to visit all vertices of a graph:

- Travelling salesman
- Bus routes
- …

Problem named after Irish mathematician, physicist and astronomer Sir William Rowan Hamilton (1805 — 1865)

... Hamiltonian Path and Circuit

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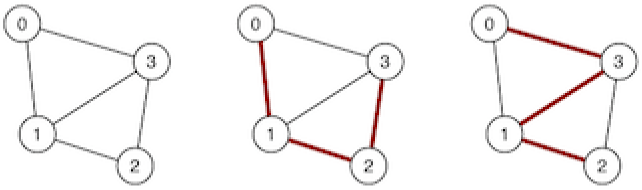
Graph and two possible Hamiltonian paths:



... Hamiltonian Path and Circuit

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Approach:

- generate all possible simple paths (using e.g. DFS)
- keep a counter of vertices visited in current path
- stop when find a path containing *V* vertices

Can be expressed via a recursive DFS algorithm

- similar to simple path finding approach, except
  - keeps track of path length; succeeds if length = *v*
  - resets "visited" marker after unsuccessful path

... Hamiltonian Path and Circuit

96/106

Algorithm for finding Hamiltonian path:

```
visited[] // array [0..nV-1] to keep track of visited vertices

hasHamiltonianPath(G,src,dest):
|  for all vertices v∈G do
|     visited[v]=false
|  end for
|  return hamiltonR(G,src,dest,#vertices(G)-1)

hamiltonR(G,v,dest,d):
|  Input G    graph
|        v    current vertex considered
|        dest destination vertex
|        d    distance "remaining" until path found
|
|  if v=dest then
|     if d=0 then return true else return false
|  else
|  |  mark v as visited
|  |  for each unvisited neighbour w of v in Gdo
|  |     if hamiltonR(G,w,dest,d-1) then
```

```
|   |         return true
|   |      end if
|   |   end for
|   end if
|   mark v as unvisited             // reset visited mark
|   return false
```

Exercise #16: Hamiltonian Path

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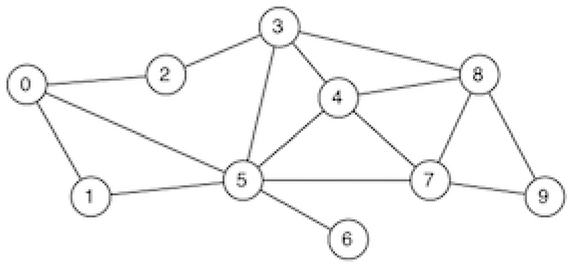
Trace the execution of the algorithm when searching for a Hamiltonian path from 1 to 6:



Consider neighbours in ascending order

1-0-2-3-4-5-6	d≠0
1-0-2-3-4-5-7-8-9	no unvisited neighbour
1-0-2-3-4-5-7-9-8	no unvisited neighbour
1-0-2-3-4-7-5-6	d≠0
1-0-2-3-4-7-8-9	no unvisited neighbour
1-0-2-3-4-7-9-8	no unvisited neighbour
1-0-2-3-4-8-7-5-6	d≠0
1-0-2-3-4-8-7-9	no unvisited neighbour
1-0-2-3-4-8-9-7-5-6	✓

Repeat on your own with `src=0` and `dest=6`

... Hamiltonian Path and Circuit

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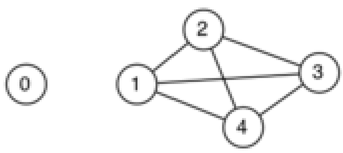
Analysis: worst case requires *(V-1)!* paths to be examined

Consider a graph with isolated vertex and the rest fully-connected



Checking `hasHamiltonianPath(g,`*x*`,0)` for any *x*

- requires us to consider every possible path
- e.g 1-2-3-4, 1-2-4-3, 1-3-2-4, 1-3-4-2, 1-4-2-3, …
- starting from any *x*, there are 3! paths ⇒ 4! total paths
- there is no path of length 5 in these *(V-1)!* possibilities

There is no known simpler algorithm for this task ⇒ *NP*-hard.

Note, however, that the above case could be solved in constant time if
we had a fast check for 0 and x being in the same connected component

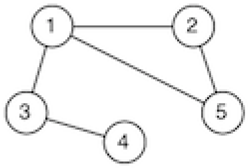
Euler Path and Circuit

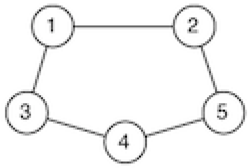
100/106

Euler path problem:

- find a path connecting two vertices *v,w* in graph *G*
- such that the path includes each *edge* exactly once
  (note: the path does not have to be simple ⇒ can visit vertices more than once)

If $v = w$, the we have an *Euler circuit*





Many real-world applications require you to visit all edges of a graph:

- Postman
- Garbage pickup
- …

Problem named after Swiss mathematician, physicist, astronomer, logician and engineer Leonhard Euler (1707 — 1783)

... Euler Path and Circuit

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One possible "brute-force" approach:

- check for each path if it's an Euler path
- would result in factorial time performance

Can develop a better algorithm by exploiting:

Theorem. *A graph has an Euler circuit* if and only if
it is connected and all vertices have even degree

Theorem. *A graph has a non-circuitous Euler path* if and only if
it is connected and exactly two vertices have odd degree

Exercise #17: Euler Paths and Circuits

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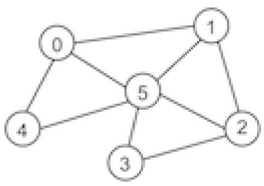
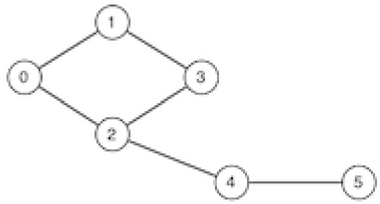
Which of these two graphs have an Euler path? an Euler circuit?





No Euler circuit

Only the second graph has an Euler path, e.g. 2-0-1-3-2-4-5

... Euler Path and Circuit

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Assume the existence of `degree(g,v)` (degree of a vertex, cf. week 5 problem set)

Algorithm to check whether a graph has an Euler path:

```
hasEulerPath(G,src,dest):
|  Input  graph G, vertices src,dest
|  Output true if G has Euler path from src to dest
|         false otherwise
|
|  if src≠dest then        // non-circuitous path
|     if degree(G,src) or degree(G,dest) is even then
|        return false
|     end if
|  else if degree(G,src) is odd then  // circuit
|     return false
|  end if
|  for all vertices v∈G do
|     if v≠src and v≠dest and degree(G,v) is odd then
|        return false
|     end if
|  end for
|  return true
```

... Euler Path and Circuit

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Analysis of `hasEulerPath` algorithm:

- assume that connectivity is already checked
- assume that degree is available via *O(1)* lookup
- single loop over all vertices ⇒ *O(V)*

If degree requires iteration over vertices

- cost to compute degree of a single vertex is *O(V)*
- overall cost is $O(V^2)$

⇒ problem tractable, even for large graphs (unlike Hamiltonian path problem)

For the keen, a linear-time (in the number of edges, *E*) algorithm to compute an Euler path is described in [Sedgewick] Ch.17.7.

Summary

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- Graph terminology
    - vertices, edges, vertex degree, connected graph, tree
    - path, cycle, clique, spanning tree, spanning forest
- Graph representations
    - array of edges
    - adjacency matrix
    - adjacency lists
- Graph traversal
    - depth-first search (DFS)
    - breadth-first search (BFS)
    - cycle check, connected components
    - Hamiltonian paths/circuits, Euler paths/circuits

- Suggested reading (Sedgewick):
    - graph representations … Ch. 17.1-17.5
    - Hamiltonian/Euler paths … Ch. 17.7
    - graph search … Ch. 18.1-18.3, 18.7

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