COMP9414: Artificial Intelligence
Solutions Week 9: First-Order Logic

1. (i) All birds fly.
   (If an object x is a bird, then it flies.)

   (ii) Everyone has a mother.

   (iii) There is someone who is everyone's mother.

2. (i) $\forall x\, (cat(x) \rightarrow mammal(x))$

   (ii) $\neg\exists x\, (cat(x) \wedge reptile(x))$
   or, equivalently, $\forall x\, (cat(x) \rightarrow \neg reptile(x))$

   (iii) $\forall x\, (computer\_scientist(x) \rightarrow \exists y\, (operating\_system(y) \wedge likes(x, y)))$

3. (i) $\text{CNF}(\forall x\, (bird(x) \rightarrow flies(x)))$
   $\equiv \forall x\, (\neg bird(x) \vee flies(x))$ [Remove $\rightarrow$]
   $\equiv \neg bird(x) \vee flies(x)$ [Drop $\forall$)]

   (ii) $\text{CNF}(\exists x\, \forall y\, \forall z\, (person(x) \wedge ((likes(x,\ y) \wedge y \neq z) \rightarrow \neg likes(x,\ z))))$
   $\equiv \exists x\, \forall y\, \forall z\, (person(x) \wedge (\neg(likes(x,\ y) \wedge y \neq z) \vee \neg likes(x,\ z)))$ [Remove $\rightarrow$]
   $\equiv \exists x\, \forall y\, \forall z\, (person(x) \wedge (\neg likes(x,\ y) \vee y = z \vee \neg likes(x,\ z)))$ [De Morgan]
   $\equiv \forall y\, \forall z\, (person(c) \wedge (\neg likes(c,\ y) \vee y = z \vee \neg likes(c,\ z)))$ [Skolem constant c]
   $\equiv person(c) \wedge (\neg likes(c,\ y) \vee y = z \vee \neg likes(c,\ z))$ [Drop $\forall$]

4. (i) $\text{CNF}(\forall x\, (P(x) \rightarrow Q(x)))$
   $\equiv \forall x\, (\neg P(x) \vee Q(x))$ [Remove $\rightarrow$]
   $\equiv \neg P(x) \vee Q(x)$ [Drop $\forall$]

   $\text{CNF}(\neg\forall y\, (\neg Q(y) \rightarrow \neg P(y)))$
   $\equiv \neg\forall y\, (\neg\neg Q(y) \vee \neg P(y))$ [Remove $\rightarrow$]
   $\equiv \exists y\, \neg(\neg\neg Q(y) \vee \neg P(y))$ [De Morgan]
   $\equiv \exists y\, \neg(Q(y) \vee \neg P(y))$ [Double Negation]
   $\equiv \exists y\, (\neg Q(y) \wedge \neg\neg P(y))$ [De Morgan]
   $\equiv \exists y\, (\neg Q(y) \wedge P(y))$ [Double Negation]
   $\equiv \neg Q(c) \wedge P(c)$ [Skolemization]

   Proof:

1.	$\neg P(x) \vee Q(x)$	[Premise]
2.	$\neg Q(c)$	[Negated Query]
3.	$P(c)$	[Negated Query]
4.	$\neg P(c)$	[1, 2 Resolution $\{x/c\}$]
5.	$\Box$	[3, 4 Resolution]

   (ii) Works exactly as in (i)

   $\text{CNF}(\forall x\, (P(x) \rightarrow Q(x)))$
   $\equiv \forall x\, (\neg P(x) \vee Q(x))$ [Remove $\rightarrow$]
   $\equiv \neg P(x) \vee Q(x)$ [Drop $\forall$]

   $\text{CNF}(\neg\forall x\, (\neg Q(x) \rightarrow \neg P(x)))$
   $\equiv \neg\forall x\, (\neg\neg Q(x) \vee \neg P(x))$ [Remove $\rightarrow$]
   $\equiv \neg\forall x\, (Q(x) \vee \neg P(x))$ [Double Negation]
   $\equiv \exists x\, \neg(Q(x) \vee \neg P(x))$ [De Morgan]
   $\equiv \exists x\, (\neg Q(x) \wedge \neg\neg P(x))$ [De Morgan]

$\equiv \exists x\,(\neg Q(x) \land P(x))$ [Double Negation]
$\equiv \neg Q(c) \land P(c)$ [Skolemization]

Proof:

1.	$\neg P(x) \lor Q(x)$	[Premise 3]
2.	$\neg Q(c)$	[Negated Query]
3.	$P(c)$	[Negated Query]
4.	$\neg P(c)$	[1, 2 Resolution $\{x/c\}$]
5.	$\square$	[3, 4 Resolution]

(iii) CNF($\forall x\,(P(x) \to Q(x))$]
$\equiv \forall x\,(\neg P(x) \lor Q(x))$ [Remove $\to$]
$\equiv \neg P(x) \lor Q(x)$ [Drop $\forall$]

CNF($P(a)$]
$\equiv P(a)$

CNF($\neg Q(a)$]
$\equiv \neg Q(a)$

Proof:

1.	$\neg P(x) \lor Q(x)$	[Premise]
2.	$P(a)$	[Premise]
3.	$\neg Q(a)$	[Negated Query]
4.	$Q(a)$	[1, 2 Resolution $\{x/a\}$]
6.	$\square$	[3, 4 Resolution]

(iv) CNF($\forall x\,(P(x) \to Q(x))$)
$\equiv \forall x\,(\neg P(x) \lor Q(x))$ [Remove $\to$]
$\equiv \neg P(x) \lor Q(x)$ [Drop $\forall$]

CNF($\exists x\, P(x)$)
$\equiv P(a)$ [Skolemization]

CNF($\neg \exists x\, Q(x)$)
$\equiv \forall x\, \neg Q(x)$ [De Morgan]
$\equiv \neg Q(x)$ [Drop $\forall$]

Proof:

1.	$\neg P(x) \lor Q(x)$	[Premise]
2.	$P(a)$	[Premise]
3.	$\neg Q(y)$	[Copy of Negated Query]
4.	$Q(a)$	[1, 2 Resolution $\{x/a\}$]
5.	$\square$	[3, 4 Resolution $\{y/a\}$)

(v) CNF($\forall x\,(P(x) \to Q(x))$)
$\equiv \forall x\,(\neg P(x) \lor Q(x))$ [Remove $\to$]
$\equiv \neg P(x) \lor Q(x)$ [Drop $\forall$]

CNF($\forall x\,(Q(x) \to R(x))$)
$\equiv \forall x\,(\neg Q(x) \lor R(x))$ [Remove $\to$]
$\equiv \neg Q(x) \lor R(x)$ [Drop $\forall$]

CNF($\neg \forall x\,(P(x) \to R(x))$)
$\equiv \neg \forall x\,(\neg P(x) \lor R(x))$ [Remove $\to$]
$\equiv \exists x\,(\neg(\neg P(x) \lor R(x))$ [De Morgan]
$\equiv \exists x\,(\neg\neg P(x) \land \neg R(x))$ [De Morgan]

$\equiv \exists x\,(P(x) \land \neg R(x))$ [Double Negation]
$\equiv P(c) \land \neg R(c)$ [Skolemization]

Proof:

1.	$\neg P(x) \lor Q(x)$	[Premise]
2.	$\neg Q(y) \lor R(y)$	[Copy of Premise]
3.	$P(c)$	[Negated Query]
4.	$\neg R(c)$	[Negated Query]
5.	$\neg P(y) \lor R(y)$	[1, 2 Resolution $\{x/y\}$]
6.	$R(c)$	[3, 5 Resolution $\{y/c\}$]
9.	$\square$	[4, 6 Resolution]

5. (i) (A) $\exists x\,(cs(x) \land \forall y\,(os(y) \to likes(x,\ y)))$
   (B) $os(Linux)$
   (C) $\exists z\, likes(z,\ Linux)$

(ii) (A) $cs(c) \land (\neg os(y) \lor likes(c,\ y))$ [Skolemization and Drop $\forall$]
   (B) $os(Linux)$
   (C) $\neg likes(z,\ Linux)$ [De Morgan and Drop $\forall$]

(iii)

1.	$cs(c)$	[Premise A]
2.	$\neg os(y) \lor likes(c,\ y)$	[Premise A]
3.	$os(Linux)$	[Premise B]
4.	$\neg likes(z,\ Linux)$	[Negated Query]
5.	$likes(c,\ Linux)$	[2, 3 Resolution $\{y/Linux\}$]
6.	$\square$	[4, 5 Resolution $\{z/c\}$]

(iv) Yes. A, B, $\neg C$ in (ii) are Horn clauses so there must be an SLD resolution of the empty clause if there is a resolution of the empty clause, and there is as we have seen in (iii). The following is an SLD resolution of the empty clause starting with the negated query (line 4).

1.	$cs(c)$	[Premise A]
2.	$\neg os(y) \lor likes(c,\ y)$	[Premise A]
3.	$os(Linux)$	[Premise B]
4.	$\neg likes(z,\ Linux)$	[Negated Query]
5.	$\neg os(Linux)$	[4, 2 Resolution $\{z/c,\ y/Linux\}$]
6.	$\square$	[5, 3 Resolution]

(v) $A, B \models C$