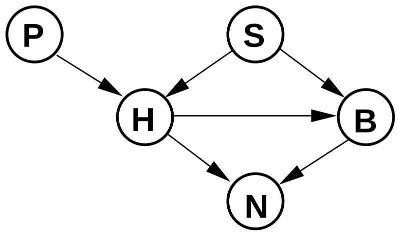
COMP9414: Artificial Intelligence
Solutions Week 6: Reasoning with Uncertainty

1. $P(A \wedge B) = P(A|B).P(B)$
   $P(B \wedge A) = P(B|A).P(A)$
   Now $P(A \wedge B) = P(B \wedge A)$ [why exactly?]
   Therefore $P(A|B).P(B) = P(B|A).P(A)$
   Rearranging gives Bayes' Rule $P(A|B) = \frac{P(B|A).P(A)}{P(B)}$ if $P(B) > 0$

2. $P(Mumps|\neg Fever) = \frac{P(\neg Fever|Mumps).P(Mumps)}{P(\neg Fever)}$
   $= \frac{(1-P(Fever|Mumps)).P(Mumps)}{1-P(Fever)}$
   $= \frac{(1-\frac{3}{4}).\frac{1}{15000}}{1-\frac{1}{1000}}$
   $= 0.0000167$

3. (i)



$P(H \wedge B \wedge S \wedge P \wedge N) = P(H|P \wedge S).P(B|S \wedge H).P(S).P(P).P(N|H \wedge B)$

(ii) $P(H \wedge B \wedge S \wedge P \wedge \neg N) = P(H|S \wedge P).P(B|H \wedge S).P(S).P(P).P(\neg N|H \wedge B)$
$= 0.8 \times 0.4 \times 0.1 \times 0.2 \times 0.1$
$= 0.00064$

$P(H \wedge \neg B \wedge S \wedge P \wedge \neg N) = P(H|S \wedge P).P(\neg B|H \wedge S).P(S).P(P).P(\neg N|H \wedge \neg B)$
$= 0.8 \times 0.6 \times 0.1 \times 0.2 \times 0.5$
$= 0.00480$

$P(H \wedge B \wedge \neg S \wedge P \wedge \neg N) = P(H|\neg S \wedge P).P(B|H \wedge \neg S).P(\neg S).P(P).P(\neg N|H \wedge B)$
$= 0.4 \times 0.3 \times 0.9 \times 0.2 \times 0.1$
$= 0.00216$

$P(H \wedge \neg B \wedge \neg S \wedge P \wedge \neg N) = P(H|\neg S \wedge P).P(\neg B|H \wedge \neg S).P(\neg S).P(P).P(\neg N|H \wedge \neg B)$
$= 0.4 \times 0.7 \times 0.9 \times 0.2 \times 0.5$
$= 0.02520$

$P(H \wedge B \wedge S \wedge \neg P \wedge \neg N) = P(H|S \wedge \neg P).P(B|H \wedge S).P(S).P(\neg P).P(\neg N|H \wedge B)$
$= 0.6 \times 0.4 \times 0.1 \times 0.8 \times 0.1$
$= 0.00192$

$P(H \wedge \neg B \wedge S \wedge \neg P \wedge \neg N) = P(H|S \wedge \neg P).P(\neg B|H \wedge S).P(S).P(\neg P).P(\neg N|H \wedge \neg B)$
$= 0.6 \times 0.6 \times 0.1 \times 0.8 \times 0.5$
$= 0.0144$

$P(H \wedge B \wedge \neg S \wedge \neg P \wedge \neg N) = P(H|\neg S \wedge \neg P).P(B|H \wedge \neg S).P(\neg S).P(\neg P).P(\neg N|H \wedge B)$
$= 0.02 \times 0.3 \times 0.9 \times 0.8 \times 0.1$
$= 0.000432$

$P(H \wedge \neg B \wedge \neg S \wedge \neg P \wedge \neg N) = P(H|\neg S \wedge \neg P).P(\neg B|H \wedge \neg S).P(\neg S).P(\neg P).P(\neg N|H \wedge \neg B)$
$= 0.02 \times 0.7 \times 0.9 \times 0.8 \times 0.5$
$= 0.00504$

(iii) $P(P|H \wedge \neg N) = \frac{P(P \wedge H \wedge \neg N)}{P(H \wedge \neg N)} = \frac{0.0328}{0.054592} = 0.60082$

Note:
$P(P \wedge H \wedge \neg N) = \Sigma_{b,s} P(H \wedge b \wedge s \wedge P \wedge \neg N) = 0.00064 + 0.00480 + 0.00216 + 0.02520$
$P(H \wedge \neg N) = \Sigma_{b,s,p} P(H \wedge b \wedge s \wedge p \wedge \neg N) = 0.00064 + 0.00480 + 0.00216 + 0.02520 + 0.00192 + 0.0144 + 0.000432 + 0.00504 = 0.05452$

We could also use direct inference, though this provides no advantage in this example. To do this, we need a conditional version of Bayes' Rule: $P(B|A,C) = \frac{P(A|B,C)P(B|C)}{P(A|C)}$

Then $P(P|H, \neg N) = \frac{P(\neg N|H,P).P(P|H)}{P(\neg N|H)}$

Now $P(\neg N|H,P) = P(\neg N|H,B,P).P(B|H,P) + P(\neg N|H,\neg B,P).P(\neg B|H,P)$
$= P(\neg N|H,B).P(B|H,P) + P(\neg N|H,\neg B).P(\neg B|H,P)$
$= P(\neg N|H,B).(P(B|H,S,P).P(S|H,P) + P(B|H,\neg S,P).P(\neg S|H,P)) +$
$P(\neg N|H,\neg B).(P(\neg B|H,S,P).P(S|H,P) + P(\neg B|H,\neg S,P).P(\neg S|H,P))$
$= [P(\neg N|H,B).(P(B|H,S).P(H|S,P).P(S|P)+P(B|H,\neg S).P(H|\neg S,P).P(\neg S|P))+$
$P(\neg N|H,\neg B).(P(\neg B|H,S).P(H|S,P).P(S|P)+P(\neg B|H,\neg S).P(H|\neg S,P).P(\neg S|P))]/P(H|P)$
$= [P(\neg N|H,B).(P(B|H,S).P(H|S,P).P(S) + P(B|H,\neg S).P(H|\neg S,P).P(\neg S)) +$
$P(\neg N|H,\neg B).(P(\neg B|H,S).P(H|S,P).P(S)+P(\neg B|H,\neg S).P(H|\neg S,P).P(\neg S))]/P(H|P)$

Also $P(P|H) = P(H|P).P(P)/P(H)$, so cancelling $P(H|P)$
$P(\neg N|H,P).P(P|H)$
$= [P(\neg N|H,B).(P(B|H,S).P(H|S,P).P(S) + P(B|H,\neg S).P(H|\neg S,P).P(\neg S)) +$
$P(\neg N|H,\neg B).(P(\neg B|H,S).P(H|S,P).P(S)+P(\neg B|H,\neg S).P(H|\neg S,P).P(\neg S))].P(P)/P(H)$

This generates four terms exactly as above, and similarly $P(\neg N|H)$ generates eight terms as above. The extra $P(H)$ cancel each other out.

4. Let A stand for "Alarm", B for "Burglary" and E for "Earthquake".

   Then by Bayes' Rule
   $P(B|A) = P(A|B).P(B)/P(A) = (P(A|B \wedge E).P(E).P(B)+P(A|B \wedge \neg E).P(\neg E).P(B))/P(A)$,
   and as in lectures
   $P(A) = P(A|B \wedge E).P(E).P(B)+P(A|B \wedge \neg E).P(\neg E).P(B)+P(A|\neg B \wedge E).P(E).P(\neg B)+$
   $P(A|\neg B \wedge \neg E).P(\neg E).P(\neg B)$

   So $P(B|A) = (0.95 \times 0.002 \times 0.001 + 0.94 \times 0.998 \times 0.001)/P(A)$ and
   $P(A) = 0.95 \times 0.002 \times 0.001 + 0.94 \times 0.998 \times 0.001 + 0.29 \times 0.002 \times 0.999 +$
   $0.001 \times 0.998 \times 0.999$

   Thus $P(B|A) = 0.00094002/0.002516442 = 0.3735512$

   Intuitively, the "true positives" (when there really is a burglary) account for roughly only 10/26 of the cases when the alarm is ringing (around 0.001 of the time), while the "false positives" account for 16/26 cases (6/26 when the alarm is ringing because of an earthquake, due to a false positive rate around 0.3 and prior of 0.002, so around 0.0006 of the time, and 10/26 when there is neither a burglary nor an earthquake, due to a false positive rate of 0.001 and a prior close to 1, so around 0.001 of the time). The rough calculation is $10/26 = 0.001/(0.001 + 0.0006 + 0.001)$. That is, the false positives significantly outweigh the true positives in this scenario.

5. By the Chain Rule, $P(A \wedge B \wedge C) = P(B|A,C).P(A|C).P(C) = P(A|B,C).P(B|C).P(C)$. So, provided $P(C) \neq 0$ and $P(A|C) \neq 0$, the conditional version of Bayes' Rule follows.