Homework 04

Due: February 21, 9PM
Point total: 60

Instructions:

- Submit your PDF and/or .py file to Blackboard by the due date and time. Please do not zip your files together, as this interferes with Blackboard's preview functionality. Always show all your work, and for full credit, you must use the method that the problem instructs you to use (unless none is mentioned). Handwritten or typeset solutions are both acceptable, but unreadable submissions will be penalized. You may discuss problems with other students, but you may not write up solutions together, copy solutions from a common whiteboard, or otherwise share your written work or code. Do not use code or language that is copied from the Internet or other students; attribute the ideas *and* rephrase in your own words.

Problem 1 (12 points [3,2,5,2])

You are programming some logic for the behavior of a video game character, and want to solve for what some test cases should do by hand. Bound to the ground, the game character can effectively move in two dimensions.

i. Moving into a wall, the game character should only move at a speed equal to the magnitude of $\text{proj}_{\vec{v}}(\vec{s})$, where $\vec{v}$ is the 2D velocity and $\vec{s}$ points along the wall. Find the speed (velocity magnitude) if the character is moving with velocity $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$ on contacting the wall, and the wall runs from 2D coordinates (7,7) to (14, 3). Round your answer to one decimal place.

ii. The wall is spiky, and does damage to the character that is proportional to the character's velocity component headed directly into the wall (orthogonal to it). Using your work from the previous problem, find the magnitude of this velocity component, rounding to one decimal place.

iii. The management has decided that this is actually going to be a 3D game. The two basis vectors for the plane describing the wall are $\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Find the projection onto the 2D surface of the wall of the velocity $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ of a jumping character. (You don't need to find the magnitude this time. Remember, show work sufficient to demonstrate that you could solve the problem by hand.)

iv. Finally, find the component of the velocity in the last question that is orthogonal to the wall, and show that the component is orthogonal to your previous answer with a dot product.

Problem 2 (10 points [2, 4, 4])

In *dimensionality reduction*, we project a dataset down to a basis with fewer dimensions, perhaps for the purpose of visualizing the high dimensional data. But sometimes we can also use dimensionality reduction to extract some higher-level meaning from the data, squishing together vectors that appear to be about the same subject.

Suppose we have the 5 distinct sentence vectors from homework 1 for "good car works," "terrible vacuum," "good car," "car works," and "vacuum works." Recall that the dimensions from first to last represented "good," "works," "terrible," "car," and "vacuum," and a vector had a 1 if a word was present and 0 if a word was absent.

i. Consider the two vectors $\vec{a} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$. What words are these two vectors each pointing towards and away from? What contrast in meaning does the line along each vector seem to capture?

ii. Find a matrix that will give us the *coordinates* of a vector $\vec{v}$'s projection into the subspace described by those two basis vectors, where the coordinates are relative to that basis. ($\vec{a}$ should come first in the basis matrix.)

iii. Find the coordinates in that subspace of "good car works" and "terrible vacuum."

Problem 3 (12 points [7, 2, 3])

i. Diagonalize the matrix $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$. (Choose D so that the largest eigenvalue is on the left.) Remember to show all steps necessary to compute the answer by hand.

ii. Describe the hundredth power of A in terms of P,D, and P^{-1}. If it involves exponentiating a matrix, show the elements of the exponentiated matrix. You can leave powers of numbers in the form a^b.

iii. Find the rough growth rate as a function of N for the magnitude of $A^N v_1$, $A^N v_2$, and $A^N v_3$, where $v_1 = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$, $v_2 = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 30 \\ 20 \end{bmatrix}$. (In each case, we are looking for an exponential function of the form λ^N, and you can think about behavior "in the limit" as N gets very large.)

Problem 4 (12 points [8,2,2])

In a particular board game, players have three kinds of resources: research, military, and cities. Players have discovered a strategy that seems very effective, where by taking the same actions every turn, they can cause the resources to obey the following dynamical system:

$$\begin{aligned} r_{t+1} &= 2r_t + m_t + c_t \\ m_{t+1} &= r_t + 2m_t + c_t \\ c_{t+1} &= r_t + m_t + 2c_t \end{aligned}$$

Players have claimed that this generally causes all the resources to increase exponentially.

i. Use

```
w, v = numpy.linalg.eig()
```

to find the eigenvectors and eigenvalues of the associated matrix to verify this claim of exponential growth. If so, what exactly is the exponential rate?

ii. If all the resources are always strictly positive, is there any way to achieve a steady state here?

iii. If it were possible for the values to be negative, then would it be possible to have a steady state? If so, exactly which vectors achieve this steady state?

Problem 5 (14 points)

Download `fish_regression.ipynb` and `fish_data.csv` from Blackboard, and complete parts A-D. For this assignment, you can simply turn in the Jupyter notebook directly instead of saving it as a .py file.