

Homework 05

Due: February 28, 9PM

Point total: 60

Instructions:

- Submit your PDF and/or .py file to Blackboard by the due date and time. Please do not zip your files together, as this interferes with Blackboard's preview functionality. Always show all your work, and for full credit, you must use the method that the problem instructs you to use (unless none is mentioned). Handwritten or typeset solutions are both acceptable, but unreadable submissions will be penalized. You may discuss problems with other students, but you may not write up solutions together, copy solutions from a common whiteboard, or otherwise share your written work or code. Do not use code or language that is copied from the Internet or other students; attribute the ideas *and* rephrase in your own words.

Problem 1 (9 points, 3 each)

Find the gradient of each function f at the specified point P , then indicate a vector of movement we should follow to maximally *decrease* f (for gradient descent, for example).

- $f(x, y) = x^2 + 2xy^2 + y^3$; $P = (2, 2)$
- $f(x, y) = 0$ if $x \leq 0$ or $y \leq 0$, else $f(x, y) = xy$; $P = (2, 3)$
- $f(w, x, y, z) = 2w + 3x + 4y + 5z$, $P = (6, 7, 8, 9)$ (The gradient works analogously for more dimensions than two.)

Problem 2 (12 points, 6 each)

For each function, identify $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, and then find the location(s) where its gradient is zero, and thus, we potentially have a local minimum or maximum. Then find one of those points that is definitely a minimum, and demonstrate that it is by showing f_{xx} and f_{yy} are both positive (indicating curvature upward along these axes) and $f_{xx}f_{yy} > f_{xy}^2$ (this happens to prevent the case of having non-positive curvature in some other, non-axis-aligned direction).

- $f(x, y) = 3x^2 + 10y^2 + 3$
- $f(x, y) = 6x^2 + xy + y^2$

Problem 3 (15 points [4, 3, 4, 4])

- An animator wants to show someone sledding down a hill. The hilly terrain is modeled with sine waves, $z = f(x, y) = 2\sin x + 2\sin y$. To model the base of the sled, find the equation for the tangent plane at $(\pi/6, 0, 1)$. Give the equation in the form $ax + by + cz = d$.

- ii. The force on the simulated snow is applied in a direction normal to the sled (tangent plane) and the surface itself. Using the equation and point from the previous part, find this normal vector. (It can point down in this problem, since we're using it for downward force.)
- iii. Lambert's cosine law says that the brightness of light reflecting off a surface is proportional to $\cos \theta$, where θ is the angle between the normal vector for the surface (pointing up) and a vector from the surface to the light source. Suppose all points on the surface act as if light is coming from a direction described by the vector $(1,1,1)$. (This isn't a "point source" of light, because the angle is the same for all points on the surface; it's more like the sun's rays, which all seem to be coming down at more or less the same angle.) Find an x, y coordinate of a hill position which maximizes the brightness of the light shining off it. (Give your coordinates in terms of π .)
- iv. Later in the animation, the hero has found a more extreme slope, modeled by $f(x, y) = 1/x + 2/y$ (with 0 undefined and off-camera). At $(2, 1/2, 9/2)$, find the tangent plane and what fraction of the light is shining off the sled (i.e. $\cos \theta$), using the same equations and light source direction as before. You can give the fraction of light rounded to the nearest percent (or equivalently, as a decimal smaller than one rounded to two decimal places).

Problem 4 (24 points)

This part can be found in the file `hw05_grad_descent.ipynb`. Complete the exercise in the notebook, and turn it in with your answer PDF.