CS2810 Mathematics of Data Models	Gold
Spring 2019	January 30, 2019

Homework 03

Due: February 7, 9PM
Point total: 75

Instructions:

- Submit your PDF and/or .py file to Blackboard by the due date and time. Please do not zip your files together, as this interferes with Blackboard's preview functionality. Always show all your work, and for full credit, you must use the method that the problem instructs you to use (unless none is mentioned). Handwritten or typeset solutions are both acceptable, but unreadable submissions will be penalized. You may discuss problems with other students, but you may not write up solutions together, copy solutions from a common whiteboard, or otherwise share your written work or code. Do not use code or language that is copied from the Internet or other students; attribute the ideas *and* rephrase in your own words.

Problem 1 (16 points, 4 each)

For each matrix, try to find the (two-sided) inverse using Gauss-Jordan elimination. If this process shows that no inverse exists, finish the Gauss-Jordan elimination process and report the rank and nullity of the matrix. If an inverse does exist, use it to find a solution to the system $A\vec{c} = \vec{1}$, where A is the given matrix and $\vec{1}$ is a ones vector with as many elements as A has rows. (Remember that it's easy to check your work for your inverse and system solution.)

i. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution: This is halfway there already in Gauss-Jordan elimination, since it's already in echelon form. With a leading one in each row, we know the matrix must have an inverse. Performing Gauss-Jordan elimination on the following matrix,

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

we first zero the final column to get the augmented matrix

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

It would be easy at this point to assume a pattern holds where we just fill the columns with -1 on the right hand side, but remember that the second and third rows now contain additional nonzero elements. So the next steps, zeroing the 3rd and 2nd columns, result in

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

and finally

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Multiplying by the ones vector gives $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, which actually makes sense because the ones vector is the last column of this matrix, and those would be the coordinates of the all-ones vector if we were using the columns as a basis.

ii. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 5 & 6 & 7 & 8 \\ 6 & 8 & 10 & 12 \end{bmatrix}$

Solution: Gauss-Jordan elimination immediately zeroes the second row, so there is no inverse. Continuing without looking for an inverse produces the echelon form $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which results in reduced row echelon form $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

With two rows that aren't zeroed, the rank is 2. Since this matrix was 4×4, the nullity is 4-2 = 2.

iii. $\begin{bmatrix} 1 & 2 & 3 \\ 0.5 & 0.25 & 0.5 \\ 2 & 1 & 0 \end{bmatrix}$

Solution: Performing Gauss-Jordan elimination on the matrix

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0.5 & 0.25 & 0.5 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

produces the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 2 & 1/6 \\ 0 & 1 & 0 & 2/3 & -4 & 2/3 \\ 0 & 0 & 1 & 0 & 2 & -1/2 \end{array}\right]$$

So the inverse is $\begin{bmatrix} -1/3 & 2 & 1/6 \\ 2/3 & -4 & 2/3 \\ 0 & 2 & -1/2 \end{bmatrix}$

Applying this inverse to the vector $\vec{1}$ using matrix multiplication gives the solution $\vec{c} = \begin{bmatrix} 11/6 \\ -8/3 \\ 3/2 \end{bmatrix}$. (We can check the inverse by multiplying by the original matrix and verifying that it produces the identity, and we can check our solution by multiplying $A\vec{c}$ and checking that it produces the ones vector.)

iv. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix}$

Solution: Not being square, this matrix can't have an inverse to begin with. Gauss-Jordan creates a matrix with all ones in the first row, and a zeroed out second row, so the matrix has rank 1. Since it has 4 columns and rank 1, the nullity is 4-1=3. It produces output only in a line, losing 3 dimensions from the input.

Problem 2 (9 points)

Determine whether each statement is true or false, and explain how we know.

i. An $m \times n$ matrix can never have a larger rank than the smaller of m and n.

Solution: True, because the column rank is equal to the row rank, so it both must have at least that many columns, and at least that many rows.

ii. If a set of vectors is linearly dependent, this can always be demonstrated with a subset of no more than three of the vectors, where one is a linear combination of the other two.

Solution: False, because it may be the case that $n-1$ of the vectors are necessary to be linearly combined to produce the nth vector. The solution to last week's problem about a set of four vectors that are dependent, but any three are independent, is an example.

iii. There exists a basis where the origin's coordinates contain at least one nonzero element. (Hint: consider whether the zero vector always works as the origin's coordinates.)

Solution: False – using all zeros for coordinates must produce the origin, and we know that points can't have multiple representations in the same basis because it was proved in class.

Problem 3 (13 points [2,3,4,4])

For each general type of matrix described, determine whether the matrix is always guaranteed to be invertible. If it is, explain how we know, using only facts covered in this course and/or reasonable arguments. If it isn't, give a counterexample.

i. Rotation matrices - that is, any matrix that represents a rotation in three-dimensional space. (Recall that any such rotation can be achieved by multiplying matrices that represent rotations around the three individual axes.)

Solution: Always invertible, because the rotation in the opposite direction must also have a matrix. This is true even when we compose rotations, because we could apply the same composition but in reverse order, and with the negative of the angle.

ii. A square matrix that is filled left-to-right with the Fibonacci numbers, $0, 1, 1, 2, 3, 5, 8 \ldots$, where $F_i = F_{i-2} + F_{i-1}$. If the last element of a row is F_k, the first element of the next row is F_{k+1}. For example, the 2×2 matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ while the first two rows of the 5×5 matrix are $\begin{bmatrix} 0 & 1 & 1 & 2 & 3 \\ 5 & 8 & 13 & 21 & 34 \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}$

Solution: This is not invertible for $n \geq 3$, because it's not of full rank; for $n \geq 3$, the first two columns will sum to the third.

iii. Any matrix that represents a connected undirected graph's adjacency matrix, where $A_{ij} = 1$ if vertices i and j have an edge and 0 otherwise. (Recall that a connected graph is one where a path exists between any two vertices; we mention this just to rule out degenerate graphs that don't have any edges, for example.)

Solution: Not necessarily invertible. We could easily have two rows that are identical, where two vertices have the same neighbors (like in a ring ABCD). Then the rank isn't full.

iv. All square matrices of the form $\begin{bmatrix} 1 & 0 & 0 & \ldots \\ 1 & 2 & 0 & \ldots \\ 1 & 2 & 3 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix}$ with whole numbers counting from 1 to i on row i (and zeros afterward on the row).

Solution: Invertible. The transpose has the same rank, and is already in echelon form for Gauss-Jordan elimination. So the matrix must always be of full rank. (In fact, any "lower triangular" matrix with nonzero values along the diagonal would similarly be invertible.)

Problem 4 (12 points)

You are working for *SimplyTheTest.com*[1], purveyor of pop culture tests where users online answer some questions, and then the site tells the user how much Jon Snow or Little Mermaid they have in them. You have come to the realization that you could add a nice feature where the user only needs to take one test, and then can get the results phrased in whatever universe they like with just a change of basis. Vectors representing famous characters from that universe can serve as the basis, and the results can be reported in terms of basis coefficients for those characters.

In a surprising nod toward scientific rigor, the original test results that your site obtains from users actually do measure the five OCEAN traits of legitimate personality tests: openness to experience, conscientiousness, extraversion, agreeableness, and neuroticism. So your reported results are actually just a change of basis from what a psychologist might say.

[1] "Better than other tests!"

Given these 5-dimensional results for Alice and Bob, report their coordinates in *Star Wars* space, where each coordinate corresponds to a different Star Wars character. (The basis vectors are listed in the order you should list their coordinates.) Instead of solving these by hand, you may call numpy.linalg.inv and perform a matrix multiplication to get your answer, but you must still explain why you went through the steps that you did.

Alice's OCEAN results: $\begin{bmatrix} 5 \\ 4 \\ -3 \\ 0 \\ 4 \end{bmatrix}$

Bob's OCEAN results: $\begin{bmatrix} 0 \\ -2 \\ 4 \\ -1 \\ -1 \end{bmatrix}$

Star Wars basis:

Rei: $\begin{bmatrix} 4 \\ 3 \\ -1 \\ -2 \\ 2 \end{bmatrix}$, Leia: $\begin{bmatrix} 2 \\ 5 \\ 0 \\ -1 \\ 3 \end{bmatrix}$, Darth Vader: $\begin{bmatrix} -1 \\ 4 \\ -3 \\ -5 \\ 2 \end{bmatrix}$, BB-8: $\begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \\ -5 \end{bmatrix}$, The Mandalorian: $\begin{bmatrix} 2 \\ 4 \\ -2 \\ 0 \\ -4 \end{bmatrix}$

<u>*Solution:*</u> We are trying to solve for a vector of coefficients $\vec{c}$ such that $A\vec{c} = \vec{v}$, where A is a matrix with column vectors equal to the basis vectors, and $\vec{v}$ is the original vector we are trying to characterize with those basis vectors. If we solve for the inverse of A, then according to the equation, $\vec{c} = A^{-1}\vec{v}$, and we can find each set of coordinates by applying the inverse to the vector $\vec{v}$.

The *Star Wars* basis matrix is:

$$\begin{bmatrix} 4 & 2 & -1 & 5 & 2 \\ 3 & 5 & 4 & 5 & 4 \\ -1 & 0 & -3 & 5 & -2 \\ -2 & -1 & -5 & 5 & 0 \\ 2 & 3 & 2 & -5 & -4 \end{bmatrix}$$

The inverse of that matrix (found with numpy.linalg.inverse) is:

```
>>> Ainv = np.linalg.inv(A)
>>> Ainv
array([[ 0.2745098 , -0.1372549 ,  0.03921569, -0.19607843, -0.01960784],
       [-0.1127451 ,  0.18137255, -0.23039216,  0.40196078,  0.24019608],
       [-0.07843137,  0.03921569,  0.2745098 , -0.37254902, -0.1372549 ],
       [ 0.00882353,  0.02058824,  0.24411765, -0.17058824, -0.09705882],
       [ 0.00245098,  0.06127451, -0.32107843,  0.23039216, -0.02696078]])
```

Multiplying that by Alice and Bob's vectors give the two coordinate vectors,

`array([ 0.62745098,  1.81372549, -1.60784314, -0.99411765,  1.1127451 ])`

and

`array([ 0.64705882, -1.92647059,  1.52941176,  1.20294118, -1.61029412])`

So we might tell the user that Alice is mostly Leia with a heavy streak of Mandalorian and a little Rei, while Bob is mostly Darth with a little BB-8 to lighten him up.

Problem 5 (25 Points)

This problem is described in `hw3_programming.ipynb`. As usual, we will want a .py file turned in with this content; recall that you can convert a notebook to .py with `file->download as...->.py`.