Homework 03

Due: February 7, 9PM
Point total: 75

Instructions:

- Submit your PDF and/or .py file to Blackboard by the due date and time. Please do not zip your files together, as this interferes with Blackboard's preview functionality. Always show all your work, and for full credit, you must use the method that the problem instructs you to use (unless none is mentioned). Handwritten or typeset solutions are both acceptable, but unreadable submissions will be penalized. You may discuss problems with other students, but you may not write up solutions together, copy solutions from a common whiteboard, or otherwise share your written work or code. Do not use code or language that is copied from the Internet or other students; attribute the ideas *and* rephrase in your own words.

Problem 1 (16 points, 4 each)

For each matrix, try to find the (two-sided) inverse using Gauss-Jordan elimination. If this process shows that no inverse exists, finish the Gauss-Jordan elimination process and report the rank and nullity of the matrix. If an inverse does exist, use it to find a solution to the system $A\vec{c} = \vec{1}$, where A is the given matrix and $\vec{1}$ is a ones vector with as many elements as A has rows. (Remember that it's easy to check your work for your inverse and system solution.)

i. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

ii. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 5 & 6 & 7 & 8 \\ 6 & 8 & 10 & 12 \end{bmatrix}$

iii. $\begin{bmatrix} 1 & 2 & 3 \\ 0.5 & 0.25 & 0.5 \\ 2 & 1 & 0 \end{bmatrix}$

iv. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{bmatrix}$

Problem 2 (9 points)

Determine whether each statement is true or false, and explain how we know.

i. An $m \times n$ matrix can never have a larger rank than the smaller of m and n.

ii. If a set of vectors is linearly dependent, this can always be demonstrated with a subset of no more than three of the vectors, where one is a linear combination of the other two.

iii. There exists a basis where the origin's coordinates contain at least one nonzero element. (Hint: consider whether the zero vector always works as the origin's coordinates.)

Problem 3 (13 points [2,3,4,4])

For each general type of matrix described, determine whether the matrix is always guaranteed to be invertible. If it is, explain how we know, using only facts covered in this course and/or reasonable arguments. If it isn't, give a counterexample.

i. Rotation matrices - that is, any matrix that represents a rotation in three-dimensional space. (Recall that any such rotation can be achieved by multiplying matrices that represent rotations around the three individual axes.)

ii. A square matrix that is filled left-to-right with the Fibonacci numbers, $0, 1, 1, 2, 3, 5, 8 \ldots$, where $F_i = F_{i-2} + F_{i-1}$. If the last element of a row is F_k, the first element of the next row is F_{k+1}. For example, the 2×2 matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ while the first two rows of the 5×5 matrix are $\begin{bmatrix} 0 & 1 & 1 & 2 & 3 \\ 5 & 8 & 13 & 21 & 34 \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}$

iii. Any matrix that represents a connected undirected graph's adjacency matrix, where $A_{ij} = 1$ if vertices i and j have an edge and 0 otherwise. (Recall that a connected graph is one where a path exists between any two vertices; we mention this just to rule out degenerate graphs that don't have any edges, for example.)

iv. All square matrices of the form $\begin{bmatrix} 1 & 0 & 0 & \ldots \\ 1 & 2 & 0 & \ldots \\ 1 & 2 & 3 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix}$ with whole numbers counting from 1 to i on row i (and zeros afterward on the row).

Problem 4 (12 points)

You are working for *SimplyTheTest.com*[1], purveyor of pop culture tests where users online answer some questions, and then the site tells the user how much Jon Snow or Little Mermaid they have in them. You have come to the realization that you could add a nice feature where the user only needs to take one test, and then can get the results phrased in whatever universe they like with just a change of basis. Vectors representing famous characters from that universe can serve as the basis, and the results can be reported in terms of basis coefficients for those characters.

In a surprising nod toward scientific rigor, the original test results that your site obtains from users actually do measure the five OCEAN traits of legitimate personality tests: openness to experience, conscientiousness, extraversion, agreeableness, and neuroticism. So your reported results are actually just a change of basis from what a psychologist might say.

[1] "Better than other tests!"

Given these 5-dimensional results for Alice and Bob, report their coordinates in *Star Wars* space, where each coordinate corresponds to a different Star Wars character. (The basis vectors are listed in the order you should list their coordinates.) Instead of solving these by hand, you may call numpy.linalg.inv and perform a matrix multiplication to get your answer, but you must still explain why you went through the steps that you did.

Alice's OCEAN results: $\begin{bmatrix} 5 \\ 4 \\ -3 \\ 0 \\ 4 \end{bmatrix}$

Bob's OCEAN results: $\begin{bmatrix} 0 \\ -2 \\ 4 \\ -1 \\ -1 \end{bmatrix}$

Star Wars basis:

Rei: $\begin{bmatrix} 4 \\ 3 \\ -1 \\ -2 \\ 2 \end{bmatrix}$, Leia: $\begin{bmatrix} 2 \\ 5 \\ 0 \\ -1 \\ 3 \end{bmatrix}$, Darth Vader: $\begin{bmatrix} -1 \\ 4 \\ -3 \\ -5 \\ 2 \end{bmatrix}$, BB-8: $\begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \\ -5 \end{bmatrix}$, The Mandalorian: $\begin{bmatrix} 2 \\ 4 \\ -2 \\ 0 \\ -4 \end{bmatrix}$

Problem 5 (25 Points)

This problem is described in `hw3_programming.ipynb`. As usual, we will want a .py file turned in with this content; recall that you can convert a notebook to .py with `file->download as...->.py`.