CS2810 Mathematics of Data Models	Gold
Spring 2019	February 14, 2020

Homework 04

Due: February 21, 9PM
Point total: 60

Instructions:

- Submit your PDF and/or .py file to Blackboard by the due date and time. Please do not zip your files together, as this interferes with Blackboard's preview functionality. Always show all your work, and for full credit, you must use the method that the problem instructs you to use (unless none is mentioned). Handwritten or typeset solutions are both acceptable, but unreadable submissions will be penalized. You may discuss problems with other students, but you may not write up solutions together, copy solutions from a common whiteboard, or otherwise share your written work or code. Do not use code or language that is copied from the Internet or other students; attribute the ideas *and* rephrase in your own words.

Problem 1 (12 points [3,2,5,2])

You are programming some logic for the behavior of a video game character, and want to solve for what some test cases should do by hand. Bound to the ground, the game character can effectively move in two dimensions.

i. Moving into a wall, the game character should only move at a speed equal to the magnitude of $\text{proj}_{\vec{s}}(\vec{v})$, where $\vec{v}$ is the 2D velocity and $\vec{s}$ points along the wall. Find the speed (velocity magnitude) if the character is moving with velocity $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$ on contacting the wall, and the wall runs from 2D coordinates (7,7) to (14, 3). Round your answer to one decimal place.

<u>*Solution:*</u> The wall vector is $\begin{bmatrix} 7 \\ -4 \end{bmatrix}$. The projection is then

$$\frac{\begin{bmatrix} 7 \\ -4 \end{bmatrix} \bullet \begin{bmatrix} -3 \\ 5 \end{bmatrix}}{\begin{bmatrix} 7 \\ -4 \end{bmatrix} \bullet \begin{bmatrix} 7 \\ -4 \end{bmatrix}} \begin{bmatrix} 7 \\ -4 \end{bmatrix} = \frac{-41}{65} \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$

which has a length of $41/65\ \sqrt{7^2 + 4^2}$ or about 5.1.

Applying the correct equation to the wrong inputs is worth 1/3. Doing the right things but having an arithmetic error is 2/3.

ii. The wall is spiky, and does damage to the character that is proportional to the character's velocity component headed directly into the wall (orthogonal to it). Using your work from the previous problem, find the magnitude of this velocity component, rounding to one decimal place.

Solution: If $\vec{v}$ is the component we need, we have that -41/65 $\begin{bmatrix} 7 \\ -4 \end{bmatrix} + \vec{v} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$. Solving for $\vec{v}$, we get $\vec{v} = \begin{bmatrix} 1.4 \\ 2.5 \end{bmatrix}$.

Some kind of more complex approach, like solving for what makes the dot product work, is -1 if it also does not "use work from the previous problem." But if it does, it can be full credit with the right answer. -1 for no work shown. An answer that is incorrect only because the work in the previous problem is incorrect is -1.

iii. The management has decided that this is actually going to be a 3D game. The two basis vectors for the plane describing the wall are $\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Find the projection onto the 2D surface of the wall of the velocity $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ of a jumping character. (You don't need to find the magnitude this time. Remember, show work sufficient to demonstrate that you could solve the problem by hand.)

Solution: For this 2D projection, $A = \begin{bmatrix} 2 & 0 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}$, so $A^T = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A^TA = \begin{bmatrix} 20 & 0 \\ 0 & 1 \end{bmatrix}$. Because this matrix just scales the first component, the inverse is easily found to be the one that scales the first component back: $(A^TA)^{-1} = \begin{bmatrix} 1/20 & 0 \\ 0 & 1 \end{bmatrix}$. The full projection matrix $A(A^TA)^{-1}A^T$ can be found through multiplication to be $\begin{bmatrix} 1/5 & 2/5 & 0 \\ 2/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which when applied to our velocity vector gives $\begin{bmatrix} 1.2 \\ 2.4 \\ 1 \end{bmatrix}$.

Up to -2 if there isn't enough work shown. Finding the inverse of A^TA is worth 1 point, finding the correct projection matrix is 3 points, and finding the final vector is worth 1 point.

iv. Finally, find the component of the velocity in the last question that is orthogonal to the wall, and show that the component is orthogonal to your previous answer with a dot product.

Solution: This vector is the difference between the original velocity and the projection, or

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1.2 \\ 2.4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.8 \\ -0.4 \\ 0 \end{bmatrix}$$

A quick dot product with the previous answer gives 24/25 - 24/25 = 0.

1 point for finding the vector, and 1 point for the dot.

Problem 2 (10 points [2, 4, 4])

In *dimensionality reduction*, we project a dataset down to a basis with fewer dimensions, perhaps for the purpose of visualizing the high dimensional data. But sometimes we can also use dimensionality reduction to extract some higher-level meaning from the data, squishing together vectors that appear to be about the same subject.

Suppose we have the 5 distinct sentence vectors from homework 1 for "good car works," "terrible vacuum," "good car," "car works," and "vacuum works." Recall that the dimensions from first to last represented "good," "works," "terrible," "car," and "vacuum," and a vector had a 1 if a word was present and 0 if a word was absent.

i. Consider the two vectors $\vec{a} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$. What words are these two vectors each pointing towards and away from? What contrast in meaning does the line along each vector seem to capture?

Solution: The first is pointed in the direction of positive words, "good" and "works," and away from the negative word, "terrible." The second is pointed in the direction of "car" and away from "vacuum." So an axis along the first seems to be measuring sentiment, and an axis along the second seems to distinguish what kind of object we're talking about.

1 point per statement. This is meant to be an easy warmup.

ii. Find a matrix that will give us the *coordinates* of a vector $\vec{v}$'s projection into the subspace described by those two basis vectors, where the coordinates are relative to that basis. ($\vec{a}$ should come first in the basis matrix.)

Solution: Just like the line of best fit application, we can ignore the first mention of the basis matrix in our projection equation, because we're just interested in coordinates.

$$A^T A = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Since that matrix just scales its input, the inverse is just the matrix that scales it back, $\begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$

So the matrix that gives us our coordinates is

$$(A^T A)^{-1} A^T = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & -1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & -1/2 \end{bmatrix}$$

1 point for finding the inverse of $A^T A$, 2 points for realizing that the needed equation doesn't have a leading A, and 1 point for the final matrix numbers.

iii. Find the coordinates in that subspace of "good car works" and "terrible vacuum."

Solution: This is just a straight multiplication by the previous matrix, producing $\begin{bmatrix} 2/3 \\ 1/2 \end{bmatrix}$ for "good car works" and $\begin{bmatrix} -1/3 \\ -1/2 \end{bmatrix}$ for "terrible vacuum."

2 points for each statement, half credit if the projection matrix was wrong, half credit if the approach is right but the numbers are wrong.

Problem 3 (12 points [7, 2, 3])

i. Diagonalize the matrix $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$. (Choose D so that the largest eigenvalue is on the left.) Remember to show all steps necessary to compute the answer by hand.

Solution: The characteristic polynomial, derived from the formula $\det(A - I\lambda) = 0$, is $\lambda^2 - 5\lambda + 6 = 0$, which factors to eigenvalues $\lambda = 3, \lambda = 2$. Substituting each of these back into $A - I\lambda = \vec{0}$ gives the corresponding eigenvectors of $c \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for $\lambda = 3$ and $c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $\lambda = 2$. That means P is $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and D is $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. We can find using Gauss-Jordan elimination that the inverse of P is $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. So the final answer is:

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Each eigenvalue worth 1, each eigenvector worth 1, finding the inverse worth 2, putting it all together worth 1. Failing to show work for a part (eigenstuff or inverse) is worth half.

ii. Describe the hundredth power of A in terms of P,D, and P^{-1}. If it involves exponentiating a matrix, show the elements of the exponentiated matrix. You can leave powers of numbers in the form a^b.

Solution:

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^{100} & 0 \\ 0 & 2^{100} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Basic idea of exponentiating the diagonal matrix worth 1, window dressing of the other matrices worth 1.

iii. Find the rough growth rate as a function of N for the magnitude of $A^N v_1$, $A^N v_2$, and $A^N v_3$, where $v_1 = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$, $v_2 = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 30 \\ 20 \end{bmatrix}$. (In each case, we are looking for an exponential function of the form λ^N, and you can think about behavior "in the limit" as N gets very large.)

Solution: In the case of v_1, it's an eigenvector with eigenvalue 3, so the growth rate is 3^N. In the case of v_2, it's an eigenvector with eigenvalue 2, so the growth rate is 2^N. In the case of

v_3, it's a mix of the two eigenvectors, so the larger one will dominate in the end, for a growth rate of 3^N as well.

1 point per growth rate.

Problem 4 (12 points [8,2,2])

In a particular board game, players have three kinds of resources: research, military, and cities. Players have discovered a strategy that seems very effective, where by taking the same actions every turn, they can cause the resources to obey the following dynamical system:

$$\begin{aligned} r_{t+1} &= 2r_t + m_t + c_t \\ m_{t+1} &= r_t + 2m_t + c_t \\ c_{t+1} &= r_t + m_t + 2c_t \end{aligned}$$

Players have claimed that this generally causes all the resources to increase exponentially.

i. Use

```
w, v = numpy.linalg.eig()
```

to find the eigenvectors and eigenvalues of the associated matrix to verify this claim of exponential growth. If so, what exactly is the exponential rate?

Solution: The eigenvalues are $\lambda = 1$, which occurs twice, and $\lambda = 4$, which is associated with the all ones vector. (This corresponds to the column where all the values are the same, a scaling of the ones vector.) np.linalg.eig() returns two other eigenvectors orthogonal to the ones vector, but any two vectors in those vectors' span would do as well.

```
>>> w
array([1., 4., 1.])
>>> v
array([[-0.81649658,  0.57735027,  0.381008  ],
       [ 0.40824829,  0.57735027, -0.81590361],
       [ 0.40824829,  0.57735027,  0.43489561]])
```

So the growth rate is on the order of 4^N.

2 points for showing eig() work, 2 points for correctly identifying eigenvectors, 1 point for correctly identifying eigenvalues, 3 points for correctly identifying the exponential growth rate.

ii. If all the resources are always strictly positive, is there any way to achieve a steady state here?

Solution: No, because such a vector would need to be totally orthogonal to the all ones vector, and that's not possible if all the resources are positive (the dot product can't be zero). Also, a just as good answer is that we can inspect the matrix and determine that clearly every value is going to increase every turn when the values are positive.

An answer of "yes because there's an eigenvalue of 1" is worth 0 here. They don't need both justifications for full credit – just noting that everything will always grow without bound if it's positive is fine.

iii. If it were possible for the values to be negative, then would it be possible to have a steady state? If so, exactly which vectors achieve this steady state?

Solution: Yes, any vector orthogonal to the ones vector would have an eigenvalue of 1, since both its nonzero eigenvector components would have that eigenvalue. An example would be $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$.

Full credit here if they just now give the eigenvectors from their matrix; they don't have to realize that any basis vectors for that plane would do.

Problem 5 (14 points)

Download `fish_regression.ipynb` and `fish_data.csv` from Blackboard, and complete parts A-D. For this assignment, you can simply turn in the Jupyter notebook directly instead of saving it as a .py file.