# Homework 05

**Due:** February 28, 9PM
**Point total:** 60

**Instructions:**

- Submit your PDF and/or .py file to Blackboard by the due date and time. Please do not zip your files together, as this interferes with Blackboard's preview functionality. Always show all your work, and for full credit, you must use the method that the problem instructs you to use (unless none is mentioned). Handwritten or typeset solutions are both acceptable, but unreadable submissions will be penalized. You may discuss problems with other students, but you may not write up solutions together, copy solutions from a common whiteboard, or otherwise share your written work or code. Do not use code or language that is copied from the Internet or other students; attribute the ideas *and* rephrase in your own words.

## Problem 1 (9 points, 3 each)

Find the gradient of each function $f$ at the specified point $P$, then indicate a vector of movement we should follow to maximally *decrease* $f$ (for gradient descent, for example).

In each case, 1 point for finding correct partial derivatives, 1 point for finding the gradient at the point, 1 point for the flip.

**i.** $f(x, y) = x^2 + 2xy^2 + y^3$; $P = (2, 2)$

*Solution:* $\frac{\partial f}{\partial x} = 2x + 2y^2$
$\frac{\partial f}{\partial y} = 4xy + 3y^2$
At (2,2), these evaluate to 12 and 28, respectively. So the gradient is $\begin{bmatrix} 12 \\ 28 \end{bmatrix}$

and the direction that will reduce $f$ the most at this point is
$\begin{bmatrix} -12 \\ -28 \end{bmatrix}$

**ii.** $f(x, y) = 0$ if $x \leq 0$ or $y \leq 0$, else $f(x, y) = xy$; $P = (2, 3)$

*Solution:* Since we're in a part of the function with both values greater than zero, the other stipulation doesn't apply. $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = x$, so the gradient at (2,3) is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and the direction to minimize it is $\begin{bmatrix} -3 \\ -2 \end{bmatrix}$

**iii.** $f(w, x, y, z) = 2w + 3x + 4y + 5z$, $P = (6, 7, 8, 9)$ (The gradient works analogously for more dimensions than two.)

*Solution:* The gradient is $\begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$, regardless of where it is evaluated, and the direction that best minimizes it is $\begin{bmatrix} -2 \\ -3 \\ -4 \\ -5 \end{bmatrix}$.

## Problem 2 (12 points, 6 each)

For each function, identify $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, and then find the location(s) where its gradient is zero, and thus, we potentially have a local minimum or maximum. Then find one of those points that is definitely a minimum, and demonstrate that it is by showing $f_{xx}$ and $f_{yy}$ are both positive (indicating curvature upward along these axes) and $f_{xx}f_{yy} > f_{xy}^2$ (this happens to prevent the case of having non-positive curvature in some other, non-axis-aligned direction).

For each problem, 1 point each for each first derivative, 1 point for identifying the extrema where the gradient is 0, all), 1 point each for the main second derivatives ($f_{xx}$ and $f_{yy}$, and 1 point for doing the final check that there's no hidden curvature through $f_{xy}$ (or just showing $f_{xy}$ is 0).

**i.** $f(x, y) = 3x^2 + 10y^2 + 3$

*Solution:* $\frac{\partial f}{\partial x} = 6x$, $\frac{\partial f}{\partial y} = 20y$

These are both 0 at $(0, 0)$. Both $f_{xx}$ and $f_{yy}$ are positive (6 and 20), and $f_{xy} = 0$, so this is a minimum.

**ii.** $f(x, y) = 6x^2 + xy + y^2$

*Solution:*

The gradient is $\begin{bmatrix} 12x + y \\ x + 2y \end{bmatrix}$, which is zero for the solution of the system where both equations are equal to 0. We can see through Gauss-Jordan elimination that the only solution is (0,0). The second derivatives there are $f_{xx} = 12, f_{yy} = 2, f_{xy} = 1$, so both $f_{xx}$ and $f_{yy}$ are positive, and $24 > 1^2$, so this is a real local minimum.

## Problem 3 (15 points [4, 3, 4, 4])

**i.** An animator wants to show someone sledding down a hill. The hilly terrain is modeled with sine waves, $z = f(x, y) = 2\sin x + 2\sin y$. To model the base of the sled, find the equation for the tangent plane at $(\pi/6, 0, 1)$. Give the equation in the form $ax + by + cz = d$.

*Solution:* The directional derivatives are $\frac{\partial f}{\partial x} = 2\cos x$ and $\frac{\partial f}{\partial y} = 2\cos y$. So the equation for the plane is $z - 1 = (2\cos\pi/6)(x - \pi/6) + (2\cos 0)(y - 0) = (\sqrt{3})x - \pi\sqrt{3}/6 + 2y$, which is $-\sqrt{3}x - 2y + z = 1 - \pi\sqrt{3}/6$.

1 point each for the correct directional derivatives, 1 point for trying to apply the correct tangent plane equation, and 1 point for succeeding.

**ii.** The force on the simulated snow is applied in a direction normal to the sled (tangent plane) and the surface itself. Using the equation and point from the previous part, find this normal vector. (It can point down in this problem, since we're using it for downward force.)

*Solution:* This is just the vector $\begin{bmatrix} (\frac{\partial f}{\partial x})_0 \\ (\frac{\partial f}{\partial y})_0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2\cos\pi/6 \\ 2\cos 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ 2 \\ -1 \end{bmatrix}$

1 point for each correct element of the final normal vector.

**iii.** Lambert's cosine law says that the brightness of light reflecting off a surface is proportional to $\cos\theta$, where $\theta$ is the angle between the normal vector for the surface (pointing up) and a vector from the surface to the light source. Suppose all points on the surface act as if light is coming from a direction described by the vector (1,1,1). (This isn't a "point source" of light, because the angle is the same for all points on the surface; it's more like the sun's rays, which all seem to be coming down at more or less the same angle.) Find an $x, y$ coordinate of a hill position which maximizes the brightness of the light shining off it. (Give your coordinates in terms of $\pi$.)

*Solution:* The sled's normal vector must be in the same direction as the light to maximize this quantity. The normal vector takes the form $\begin{bmatrix} -2\cos x \\ -2\cos y \\ 1 \end{bmatrix}$ if we flip it to point upward toward the light. We can solve to set the first two coordinates to 1, and we have $\cos x = \cos y = -1/2$, so coordinates of $(2\pi/3, 2\pi/3)$ would do that.

Half credit for knowing roughly how to do this but messing up arithmetic or signs. 1/4 for a good effort.

**iv.** Later in the animation, the hero has found a more extreme slope, modeled by $f(x, y) = 1/x + 2/y$ (with 0 undefined and off-camera). At $(2, 1/2, 9/2)$, find the tangent plane and what fraction of the light is shining off the sled (i.e. $\cos\theta$), using the same equations and light source direction as before. You can give the fraction of light rounded to the nearest percent (or equivalently, as a decimal smaller than one rounded to two decimal places).

*Solution:* Our partial derivatives are $\frac{\partial f}{\partial x} = -1/x^2$ and $\frac{\partial f}{\partial y} = -2/y^2$. So our equation for a plane is $z - 9/2 = (-1/(2^2)(x-2) + -2/(1/2)^2(y - 1/2)$, which simplifies to $x/4 + 8y + z = 9$. This makes the normal vector in the direction of the light $\vec{n} = \begin{bmatrix} 1/4 \\ 8 \\ 1 \end{bmatrix}$, so the proportion of light is $cos\theta = \frac{\vec{l} \bullet \vec{n}}{|\vec{l}||\vec{n}|} = \frac{(1/4+8+1)}{(\sqrt{3})(\sqrt{(1/4)^2+8^2+1})} = 66\%$.

2 points for the tangent plane, 2 points for the light fraction. Half credit for right approach but wrong arithmetic. In this and similar problems, you don't need to take off any points if they round to an incorrect number of decimal places or round the wrong direction, so long as their answer is recognizably correct.

## Problem 4 (24 points)

This part can be found in the file `hw05_grad_descent.ipynb`. Complete the exercise in the notebook, and turn it in with your answer PDF.