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Synthesis of Fast-Decaying Window Functions

A window function is a mathematical function that is zero valued outside some chosen interval [1], [2]. For applications like filtering, detection, and estimation, the window functions take the form of limited time functions, which are in general real and even functions [3], [4], while for applications like beamforming and image processing, they are limited spatial functions. A spatial window can be a complex function for optimizing the beams in magnitude as well as in phase, as in the case of certain antenna arrays, where the phasor currents in the array are complex numbers [5]. The windows for filtering applications are broadly categorized into two types based on their usage: the ones designed for suppression of cochannel interference and the ones for suppressing the far-off harmonic components [2], [3], [6]. Some of the recent window functions picked from the exhaustive literature are generalized normal windows, confined Gaussian windows, hyperbolic cosine windows, polynomial windows, cosine windows with coefficient optimization, and sine windows [7]–[12].

The signals coming out of a mixer stage in a communication receiver contain several undesired harmonic frequency components along with the desired frequency components. The undesired signals tend to deteriorate the signal-to-noise ratio (SNR) and introduce significant signal distortion at the next stage. Filters designed using fast-decaying windows can greatly suppress these undesired harmonic components and thereby facilitate an improved SNR and a reduction in signal distortion.

A survey of the literature on window functions with fast-decaying sidelobes reveals many windows, such as cosine-powered windows [3], odd- and even-series cosine windows [6], polynomial windows [10], and so on. The cosine-powered window [3] is designed to provide the maximum possible decay of sidelobes for a given number of terms in the window; for example, the three-term cosine powered window yields a main-lobe width of $\pm 3/T$ and a peak sidelobe level of –46.74 dB for a sidelobe decay of 30 dB/octave. However, with this window it is not possible to obtain a different set of main-lobe widths and peak sidelobe levels for the same sidelobe decay of 30 dB/octave.

Filters designed using fast-decaying windows can greatly suppress these undesired harmonic components and thereby facilitate an improved SNR and a reduction in signal distortion.

The aim of this lecture note is to propose a new three-term cosine window function that can yield several sets of main-lobe widths and peak sidelobe levels for a given high asymptotic decay. More specifically, this note proposes two types of window functions. In the first type, the window is designed for a maximum possible sidelobe decay (of 30 dB/octave) with many options for the main-lobe width and the peak sidelobe level. The second type is designed to have a null at the desired frequency location and a sidelobe decay of 24 dB/octave.

Relevance

Different applications of harmonic suppression require different sets of main-lobe widths and peak sidelobe levels for a given high sidelobe decay. The three-term cosine-powered window (also known as the *generalized cosine window*) does not have the flexibility of providing more options on main-lobe widths and peak sidelobes for a given sidelobe decay, while the window function proposed here has this kind of flexibility. This lecture note provides a detailed synthesis procedure for designing a simple, yet useful, window function with fast-decaying sidelobes (at a rate of 24 dB/octave as well as 30 dB/octave) with several options for main-lobe widths and peak sidelobe levels.

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Prerequisites

The prerequisites required are the basic knowledge of simple window functions, like the Hann window, the Hamming window, and the Blackman window, and the concepts involved in their synthesis. Those who are unfamiliar with these topics may read the book [1] and the two articles [2], [3], which provide sufficient background information to understand the contents of this note.

Problem statement and solution

Problem statement

The cosine window function mentioned in an article by Nuttall [3] has the form

$$w(t) = \frac{1}{T}\sum_{k=0}^{K} a_k \cos\left(\frac{2\pi kt}{T}\right),\quad \text{for } |t| \le \frac{T}{2},$$
$$= 0,\ \text{for } |t| > \frac{T}{2},$$

where T is the window duration, and there are $K+1$ unknowns (a_k). Normalizing the window at $t = 0$ results in K degrees of freedom. This window is also called the generalized cosine window or *cosine-sum window*.

Notice that when there is only one term ($K = 0$), we obtain a rectangular window, whose degree of freedom is zero; the window has a main-lobe width of $\pm 1/T$, a peak sidelobe level of –13 dB, and a sidelobe decay of 6 dB/octave. When a two-term cosine window ($K = 1$) is designed for the maximum sidelobe decay criterion, it results in the Hann window, which has a main-lobe width of $\pm 2/T$, a peak sidelobe level of –31.47 dB, and a decay of 18 dB/octave.

For a three-term cosine window, when its two degrees of freedom ($K = 2$) are used for obtaining the maximum sidelobe decay, it results in a window that has a main-lobe width of $\pm 3/T$, a peak sidelobe of –46.74 dB, and a decay of 30 dB/octave. Similarly, when a four-term cosine window ($K = 3$) is designed to get maximum sidelobe decay, it results in a window that has a main-lobe width of $\pm 4/T$, a peak sidelobe level of –60.95 dB, and a sidelobe decay of 42 dB/octave.

These are some examples of fast-decaying window functions with only one set of main-lobe widths and peak sidelobe levels for a given sidelobe decay or, equivalently, for a given number of terms in the window. Now, the problem is that of obtaining more than one set of features for a given number of terms in the window. We attempt to answer this question in the next section.

This led to searching for a metric that will indicate whether a window is on par with the other known efficient windows.

Solution

Notice that the coefficients $(2\pi k/T)$ of t in the cosine functions of the previous window function are known numbers; and if these numbers are turned into unknown quantities, they will increase the degrees of freedom in the window, which, in turn, leads to more options for the main-lobe width and the peak sidelobe level for a given decay of sidelobes. With this concept in mind, a new three-term cosine window function is proposed in this article as explained next.

Synthesis of fast-decaying window function with more options

We propose here a simple three-term window function, existing in the duration $-T/2$ to $+T/2$, as

$$w(t) = \frac{1}{1+a+c}\left[1 + a\cos\left(\frac{bt}{T}\right) + c\cos\left(\frac{dt}{T}\right)\right],\quad \text{for } |t| \le \frac{T}{2},$$
$$= 0,\ \text{for } |t| > \frac{T}{2}, \tag{1}$$

where a, b, c, and d are the window parameters unknown at present ($abcd \neq 0$ and $1 + a + c \neq 0$), and they will be determined from the specifications imposed on the window function. The corresponding frequency function of the window is obtained by taking the Fourier transform of (1) as follows:

$$W(f) = \int_{-\infty}^{+\infty} w(t)\, e^{-j2\pi ft}\, dt. \tag{2}$$

Evaluation of the preceding integral yields the expression for the frequency function of the window (1) as

$$W(f) = \frac{T}{1+a+c} \times \left\{ \frac{\sin(x)}{x} + \frac{a}{2}\left[\frac{\sin\left(x+\frac{b}{2}\right)}{x+\frac{b}{2}} + \frac{\sin\left(x-\frac{b}{2}\right)}{x-\frac{b}{2}}\right] + \frac{c}{2}\left[\frac{\sin\left(x+\frac{d}{2}\right)}{x+\frac{d}{2}} + \frac{\sin\left(x-\frac{d}{2}\right)}{x-\frac{d}{2}}\right]\right\}, \tag{3}$$

where $x = \pi fT$. Observe that the expressions (1) and (3), for the window function in the time and frequency domain, will remain the same for both positive as well as negative values of b and d, which means that b and d can be considered to be positive valued ($b > 0$ and $d > 0$).

Since, in general, any two frequency functions will have two different values at $f = 0$, it is essential to normalize the frequency functions with respect to their values at $f = 0$ to compare their salient features. So, we define a normalized frequency function, $W_n(f) = W(f)/W(0)$, where $W(0)$ is the value of $W(f)$ at $f = 0$. Accordingly, we make use of $W_n(f)$ for plotting and comparing the frequency responses of various window functions. The quantity $W(0)$ is evaluated from the relation (see [1])

$$W(0) = \int_{-\infty}^{+\infty} w(t)\, dt.$$

For the window function (1), $W(0)$ is determined as

$$W(0) = \frac{T}{1+a+c}\left[1 + \left(\frac{2a}{b}\right)\sin\left(\frac{b}{2}\right) + \left(\frac{2c}{d}\right)\sin\left(\frac{d}{2}\right)\right]. \tag{4}$$

So, the normalized frequency function $[W_n(f)]$ is obtained from (3) and (4) as

$$W_n(f) = \frac{1}{A} \times \left\{ \frac{\sin(x)}{x} + \frac{a}{2}\left[\frac{\sin\left(x+\frac{b}{2}\right)}{x+\frac{b}{2}} + \frac{\sin\left(x-\frac{b}{2}\right)}{x-\frac{b}{2}}\right] + \frac{c}{2}\left[\frac{\sin\left(x+\frac{d}{2}\right)}{x+\frac{d}{2}} + \frac{\sin\left(x-\frac{d}{2}\right)}{x-\frac{d}{2}}\right]\right\}, \tag{5}$$

where A is given by

$$A = 1 + \left(\frac{2a}{b}\right)\sin\left(\frac{b}{2}\right) + \left(\frac{2c}{d}\right)\sin\left(\frac{d}{2}\right).$$

Notice that we have not yet determined the unknown window parameters a, b, c, and d. They are determined based on whether the window function has to have maximum decay of sidelobes, maximum suppression of close-in sidelobes, or some decay of sidelobes in combination with some null points at the desired locations.

In this lecture note, two cases are presented; in the first case (see next section), the window will have the maximum decay of sidelobes using all of the four unknowns for this purpose. In the second case, one null point is placed at a desired location using one unknown, and the remaining three unknowns are used to get the maximum possible decay of sidelobes.

Window with maximum decay of sidelobes

If a window function has to have maximum asymptotic decay of sidelobes, then its time function, $w(t)$, and all of its possible derivatives, $w'(t), w''(t), w'''(t), \ldots$ must vanish at $t = \pm T/2$; see [2], [3]. In our case, since there are four unknowns (a, b, c, and d), the time function $w(t)$ and its three derivatives, $w'(t), w''(t)$, and $w'''(t)$, have to become zero at $t = \pm T/2$. These conditions result in the following four expressions, after some algebraic manipulations:

$$c = -\frac{1 + a\cos\left(\frac{b}{2}\right)}{\cos\left(\frac{d}{2}\right)} \tag{6}$$

$$cd\sin\left(\frac{d}{2}\right) = -ab\sin\left(\frac{b}{2}\right) \tag{7}$$

$$cd^2\cos\left(\frac{d}{2}\right) = -ab^2\cos\left(\frac{b}{2}\right) \tag{8}$$

$$cd^3\sin\left(\frac{d}{2}\right) = -ab^3\sin\left(\frac{b}{2}\right). \tag{9}$$

Dividing (7) by (8) yields

$$\tan\left(\frac{d}{2}\right) = \left(\frac{d}{b}\right)\tan\left(\frac{b}{2}\right). \tag{10}$$

Eliminating c from (8) using (6) results in an expression for a,

$$a = \frac{d^2}{\left[(b^2 - d^2)\cos\left(\frac{b}{2}\right)\right]}. \tag{11}$$

The use of (7) in (9) results in

$$ab(b^2 - d^2)\sin\left(\frac{b}{2}\right) = 0. \tag{12}$$

Let us examine (12) more closely. We know that $ab \neq 0$, and since $b^2 - d^2$ is in the denominator of (11), $b^2 - d^2 \neq 0$. Therefore, the only option is $\sin(b/2) = 0$, implying $(b/2) = n\pi$, where $n = 1, 2, 3, \ldots$. Using $(b/2) = n\pi$ in (10) yields $\tan(d/2) = 0$, which implies $(d/2) = p\pi$, where $p = 1, 2, 3, \ldots$. Further, the use of $(b/2) = n\pi$ in (11) yields two expressions for a as given next,

$$a = \frac{(-1)^n p^2}{(n^2 - p^2)}, \tag{13}$$

depending on whether n is even or odd. Let n be even, so, $n = 2m$ where $m = 1, 2, 3, \ldots$; using it in (13) leads to

$$a = \frac{p^2}{(4m^2 - p^2)}, \quad p \neq 2m. \tag{14}$$

We use (14) and $(b/2) = 2m\pi$ in (6) to determine c as shown next:

$$c = \frac{(-1)^{p+1} 4m^2}{(4m^2 - p^2)}. \tag{15}$$

Notice that for the case of n even, all of the four unknowns in the window function (1) are determined using the criterion for the maximum decay of sidelobes, and the window decays as $1/f^5$ at a rate of 30 dB/octave.

For n odd, let $n = 2m - 1$, where $m = 1, 2, 3, \ldots$; we obtain $(b/2) = (2m - 1)\pi$. Using this in (10), (11), and (6), the unknowns d, a, and c are determined in a similar manner as before:

$$d = 2\pi p, \quad a = \frac{p^2}{p^2 - (2m-1)^2},$$
$$c = \frac{(-1)^{p+1}(2m-1)^2}{(2m-1)^2 - p^2}, \tag{16}$$

for $p \neq 2m - 1$. Thus, for n odd, the previous values of a, b, c, and d result in a window that decays at a rate of 30 dB/octave.

Observe that, with the various combinations of m and p (for n even and odd), the window function (1) will now yield several sets of main-lobe widths and peak sidelobe levels for a sidelobe decay of 30 dB/octave (see Table 1).

For example, when $m = 1$, $p = 1$, and n is even, the window (denoted as $W_{e,1,1}$ in Table 1) has a main-lobe width of $\pm 3/T$ and a peak sidelobe level of –46.74 dB with a decay of 30 dB/octave. Notice that this window is the same as the three-term cosine-powered window mentioned in [3].

When $m = 2$, $p = 1$, and n is even, the window, $W_{e,2,1}$, has a main-lobe width of $\pm 2/T$ and a peak sidelobe level of –27.6 dB; however, this window has its second sidelobe level at –28.4 dB, i.e., almost the same as that of the first sidelobe level, which is a not-so-desirable feature for a fast-decaying window function. Thereafter, the window decays at a rate of 30 dB/octave.

When $m = 1$, $p = 3$, and n is odd, the window, $W_{o,1,3}$, has a main-lobe width of $\pm 2/T$ but a higher peak sidelobe level of –21.8 dB. However, the window decays monotonously at a rate of 30 dB/octave thereafter.

So, clearly one has many options to choose, like a lower peak sidelobe with a wider main-lobe width, or a higher peak sidelobe with a narrower main-lobe width, all of these with the same sidelobe decay of 30 dB/octave (see Figure 1). Figure 2 shows the time functions of these windows.

Table 1. Window parameters, main-lobe widths, and peak sidelobe levels of the windows with a decay of 30 dB/octave.

n	m	p	Window	a	b	c	d	Main-Lobe Width	Peak Sidelobe Level
Even	1	1	$W_{e,1,1}$	1/3	4π	4/3	2π	$\pm 3/T$	–46.74 dB
Even	2	1	$W_{e,2,1}$	1/15	8π	16/15	2π	$\pm 2/T$	–27.6 dB
Odd	1	3	$W_{o,1,3}$	9/8	2π	–1/8	6π	$\pm 2/T$	–21.8 dB

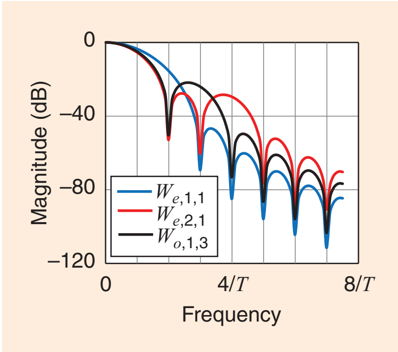

FIGURE 1. The frequency responses of windows with 30-dB/octave decay.

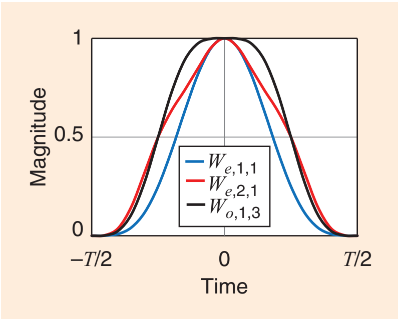

FIGURE 2. The time functions of windows with 30-dB/octave decay.

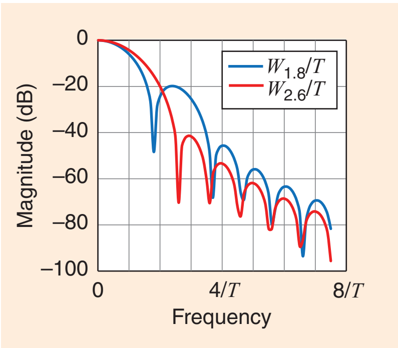

FIGURE 3. The frequency responses of windows with 24-dB/octave decay.

Table 2. The salient features of the windows with a decay of 24 dB/octave.

Window	Main-Lobe Width	Peak Sidelobe Level
$W_{1.8/T}$	$\pm 1.8/T$	–19.8 dB
$W_{2.6/T}$	$\pm 2.6/T$	–41.4 dB

Placing a first null in the frequency function

If it is required to place the first null of the frequency function at some desired location that is different than that provided by the previously mentioned windows, then we need to use the criterion $W(f_0) = 0$, where f_0 is the desired first null point, in place of the fourth criterion, $w'''(T/2) = 0$, used to design the windows with a 30-dB/octave decay. As a result the window will now decay at a rate of 24 dB/octave instead of 30 dB/octave. In this section, we present the synthesis of such a window function.

Now we use $W(f_0) = 0$ in place of $w'''(T/2) = 0$, which means discarding the expressions (9) and (12). This yields [see (3)]

$$\frac{\sin(x_0)}{x_0} + \frac{a}{2}\left[\frac{\sin\left(x_0 + \frac{b}{2}\right)}{x_0 + \frac{b}{2}} + \frac{\sin\left(x_0 - \frac{b}{2}\right)}{x_0 - \frac{b}{2}}\right] + \frac{c}{2}\left[\frac{\sin\left(x_0 + \frac{d}{2}\right)}{x_0 + \frac{d}{2}} + \frac{\sin\left(x_0 - \frac{d}{2}\right)}{x_0 - \frac{d}{2}}\right] = 0, \quad (17)$$

where $x_0 = \pi f_0 T$. The parameters a, c, and d are eliminated from (17) using (11), (6), and (10), and, after some algebraic manipulations, we obtain the following compact expression relating b and x_0:

$$\left(\frac{b}{2}\right)\cot\left(\frac{b}{2}\right) = x_0 \cot(x_0). \quad (18)$$

For a given x_0, (18) is a transcendental equation in b, which can be solved numerically to determine b. For example, if the desired first null point (f_0) is $2.6/T$, b is obtained from (18) as $b = 10.37088825$. The remaining window parameters d, a, and c are determined from (10), (11), and (6), respectively, in that order. Similarly, if the desired first null point (f_0) is $1.8/T$, b is determined numerically from (18) as $b = 17.1802790008$. How narrow can a main-lobe width be? We answer this question in the next section.

From Figure 3, we notice that when the desired null point is chosen as $1.8/T$, the window ($W_{1.8/T}$) has a peak sidelobe level of –19.8 dB, and when the desired null point is $2.6/T$, the window ($W_{2.6/T}$) has a peak sidelobe level of –41.4 dB. Both windows decay at a rate of 24 dB/octave. Table 2 lists the main-lobe widths and first sidelobe levels of these windows, and Table 3 gives the list of the desired null frequencies and the corresponding window parameters. Figure 4 shows these windows in the time domain.

Minimum limit on the first null point

To determine the minimum value of the first null point (f_0), consider the expression (18); differentiating it with respect to b and equating $d(x_0)/db$ to zero yields

$$\sin^2(x_0)[\sin(b) - b] = 0. \quad (19)$$

Equating the factor $[\sin(b) - b]$ in (19) to zero implies that a minimum value of x_0 (and hence f_0) may be obtained when $b = 0$, so using this value in (18) yields $x_0 = \pi/2$. The remaining parameters a, c, and d are required to be determined for $b = 0$. For this purpose, (10) is rearranged as

$$\frac{\tan\left(\frac{d}{2}\right)}{d} = \frac{\tan\left(\frac{b}{2}\right)}{b}, \quad (20)$$

which becomes indeterminate when $b = 0$. We apply L'Hospital's rule for (20) as $b \to 0$, which results in

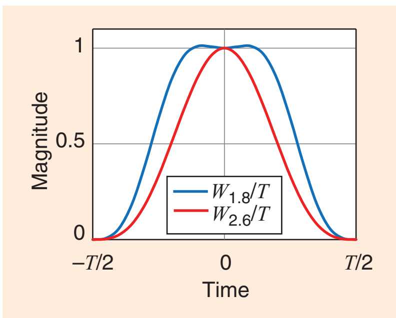

FIGURE 4. Time functions of windows with 24-dB/octave decay.

Table 3. The parameters of the windows with a decay of 24 dB/octave.

Window	*a*	*b*	*c*	*d*
$W_{1.8/T}$	–0.1765718786	17.1802790008	1.1885742524	5.5932866481
$W_{2.6/T}$	0.601726718	10.3708882464	1.7195749353	4.8106166523

$$\tan\left(\frac{d}{2}\right) = \left(\frac{d}{2}\right),$$

and its one solution, $d = 0$, is discarded since $d \neq b$ [see (11)]. Searching for another solution, we obtain $d = 8.9868189158\ldots$; using this d and $b = 0$ in (11) yields $a = -1$, and consequently from (6) we obtain $c = 0$. Substituting these values of a, b, c, and d in (3), we notice that $W(f)$ vanishes for all f. Hence, we conclude that $b = 0$ does not yield the minimum null point.

Next, the other factor in (19), $\sin^2(x_0)$, is equated to zero, which yields $x_0 = \pi$ (or $f_0 = 1/T$). Even though $x_0 = 0$ is also a solution of $\sin^2(x_0) = 0$, we discard it since it implies zero main-lobe width. If we use $x_0 = \pi$ in (18) directly, we encounter a division by zero; so (18) is rearranged as

$$\frac{\tan\left(\frac{b}{2}\right)}{\frac{b}{2}} = \frac{\tan(x_0)}{x_0}, \qquad (21)$$

and the substitution of $x_0 = \pi$ in (21) results in $\tan(b/2) = 0$, which, in turn, yields $(b/2) = j\pi$, where $j = 1, 2, 3, \ldots$. We need to determine the remaining parameters; so, using $(b/2) = j\pi$ in (20) determines $(d/2) = k\pi$, where $k = 1, 2, 3, \ldots$. Note that $k \neq j$ since $d \neq b$. From (11) and (6), the parameters a and c are determined as

$$a = \frac{(-1)^j k^2}{j^2 - k^2}, \quad c = \frac{(-1)^{k+1} j^2}{j^2 - k^2}. \qquad (22)$$

It is now required to investigate which values of j and k provide us the minimum first null point. After several trials, it is observed that many sets of $[j, k]$ yield the first null point (f_0) at $1/T$, such as [2, 3], [3, 4], [2, 5], and so on. Let us denote these windows as $W_{j,k}$. Interestingly, the windows ($W_{j,k}$) satisfy the criterion $w'''(\pm T/2) = 0$ or the expression (9); hence, they decay at a rate of 30 dB/octave.

A representative frequency plot is shown in Figure 5, which reveals that the peak sidelobe of the window ($W_{2,5}$) is much higher (–4 dB) than that provided by the rectangular window (–13 dB). This result was expected since the sidelobes of $W_{2,5}$ decay at a much faster rate (30 dB/octave) than that of the rectangular window (6 dB/octave). Hence, even though the narrow main-lobe width and the fast rolloff of sidelobes are attractive features of this window, its peak sidelobe of –4 dB is grossly insufficient to suppress the nearby tones to a reasonable extent.

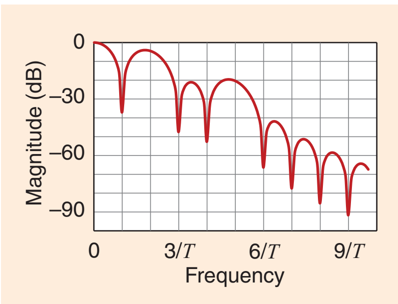

FIGURE 5. The frequency response of the window, $W_{2,5}$, with first null at $1/T$.

Comparison with the traditional windows

To see how the proposed windows compare with the traditional cosine window functions mentioned in [3], we construct two plots. In the first one (Figure 6), the sidelobe decay of the windows is plotted versus the first null frequency (which is nothing but half of the main-lobe width), and in the second one (Figure 7), the peak sidelobe level of the windows is plotted versus the first null frequency. The traditional windows chosen for comparison are all fast-decaying window functions, such as the Hann window, Blackman window, three-term cosine window (Figure 8 in [3]),

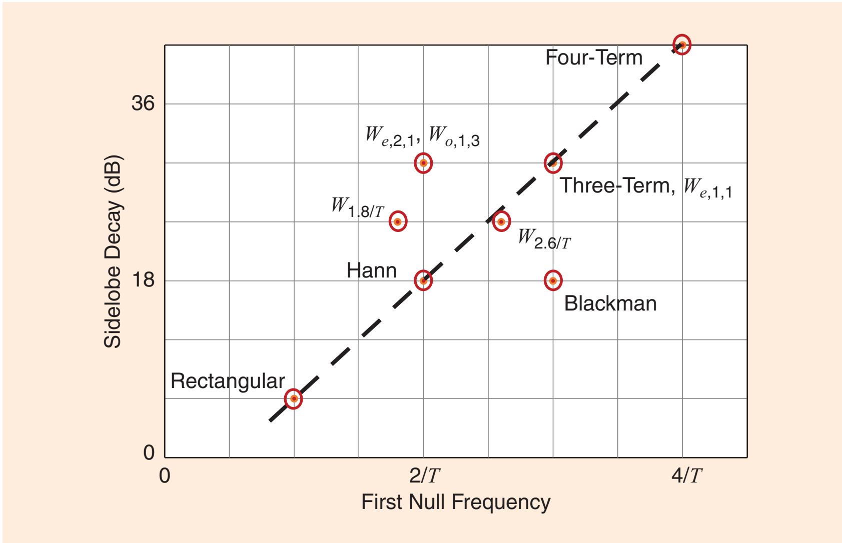

FIGURE 6. A comparison of the sidelobe decays of various windows.

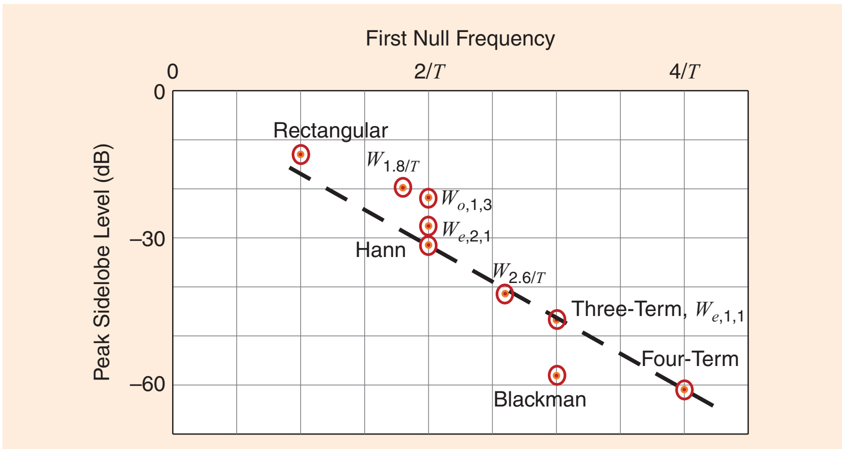

FIGURE 7. A comparison of the peak sidelobe levels of various windows.

and four-term cosine window (Figure 10 in [3]). The basic rectangular window is also included in the comparison. We have avoided comparison with the Hamming window and the "minimum" three-term cosine window (Figure 4 in [3]) as these are designed for suppressing close-in sidelobes, while the proposed window functions are designed for obtaining rapidly decaying sidelobes.

Notice that in Figure 6 the sidelobe decays of the rectangular, Hann, three-term cosine, and four-term cosine windows fall in a line. This is due to the fact that the windows belong to the same cosine family and their available degrees of freedom are utilized only for obtaining the maximum possible sidelobe decay. The windows above this line are considered as more efficient and those below this line are considered as less efficient than the windows on the line (as far as sidelobe decay is considered).

Similarly, in Figure 7 the peak sidelobe levels of the rectangular, Hann, three-term cosine, and four-term cosine windows fall in a line, The windows below this line are considered as more efficient and those above this line are considered as less efficient than the windows located on the line, with respect to the feature peak sidelobe level.

The number of samples in the window must be sufficiently large to minimize the aliasing effects and obtain a frequency response close to the response for the continuous case.

Based on these yardsticks, the proposed windows $W_{e,2,1}$, $W_{o,1,3}$, and $W_{1.8/T}$ appear to be more efficient as far as the sidelobe decay is concerned, but less efficient in peak sidelobe level, when compared with the windows placed on the line; see Figures 6 and 7. Similarly, the $W_{2.6/T}$ and Blackman windows appear to be more efficient when looking at the peak sidelobe level, but less efficient in the sidelobe decay, in comparison to the windows on the line (Figures 6 and 7). Thus, one can note that, from these figures alone, it is difficult to conclude whether a window is efficient or not. This led to searching for a metric that will indicate whether a window is on par with the other known efficient windows.

The landmark article by Harris [2] listed many figures of merit, like peak sidelobe, sidelobe fall-off, coherent gain, equivalent noise bandwidth, scallop loss, and so on. However, it was found that individually these figures of merit do not serve our purpose of determining whether a window is more efficient or not compared to the traditional cosine windows designed for maximum sidelobe rolloff. Therefore, it became imperative to define a new metric to judge the windows on their relative ability to suppress the harmonics, which led to proposing a new metric, the efficiency index (η):

$$\eta = \frac{(\alpha\,|SD| + \beta\,|PSL|)}{f_N},$$

where $|SD|$ and $|PSL|$ are the magnitudes of the sidelobe decay and peak sidelobe level, and f_N is the first null frequency of the window. The quantities α and β are real numbers, which depend on whether a window is a fast-decaying window, or it is designed to suppress close-in sidelobes. For the fast-decaying windows (like the ones discussed here), $\alpha = 1$ and $\beta = 0.5$. If the degrees of freedom in the window are equally utilized for sidelobe decay and sidelobe suppression, then the values of α and β could be 1 each. If the window is designed with all of its degrees of freedom utilized only for sidelobe suppression, then one can assign $\alpha = 0.5$ or less, and $\beta = 1$ or more.

The efficiency indexes of various windows discussed in this lecture note are listed in Table 4. The proposed windows, $W_{e,1,1}$, $W_{e,2,1}$, and $W_{o,1,3}$, are compared with the traditional three-term cosine window (Figure 8 in [3]) due to the fact that they all have a sidelobe decay of 30 dB/octave. In the same way, the proposed windows, $W_{1.8/T}$ and $W_{2.6/T}$, are compared with the Blackman window since, in all of these windows, one degree of freedom is not utilized for sidelobe rolloff. From this comparison, we note that the proposed windows are equally or more efficient than the traditional ones.

Table 4. A comparative list of salient features of the windows.

Window	Number of Terms	Main-Lobe Width	Sidelobe Decay	Peak Sidelobe Level	Efficiency Index
Traditional					
Rectangular	1	$\pm 1/T$	6 dB/octave	–13 dB	12.5
Hann	2	$\pm 2/T$	18 dB/octave	–31.47 dB	16.87
Blackman	3	$\pm 3/T$	18 dB/octave	–58.11 dB	15.68
Cosine (3)	3	$\pm 3/T$	30 dB/octave	–46.74 dB	17.79
Cosine (4)	4	$\pm 4/T$	42 dB/octave	–60.95 dB	18.12
Proposed					
$W_{e,1,1}$	3	$\pm 3/T$	30 dB/octave	–46.74 dB	17.79
$W_{e,2,1}$	3	$\pm 2/T$	30 dB/octave	–27.6 dB	21.9
$W_{o,1,3}$	3	$\pm 2/T$	30 dB/octave	–21.8 dB	20.45
$W_{1.8/T}$	3	$\pm 1.8/T$	24 dB/octave	–19.8 dB	18.83
$W_{2.6/T}$	3	$\pm 2.6/T$	24 dB/octave	–41.4 dB	17.19

Low-pass filter design using the proposed windows

Two cases of low-pass filters designed using the proposed windows are given here. In both the cases, the cutoff frequency of the ideal low-pass filter is 0.25 MHz and the window duration is 16 μS. In the first case, the window $W_{o,1,3}$ is used; in the second case, the window $W_{e,1,1}$ is used. The corresponding frequency responses of the two filters are shown in Figures 8 and 9. One can notice from these plots that the filter designed with $W_{o,1,3}$ has its first null at 0.33 MHz with a peak sidelobe at

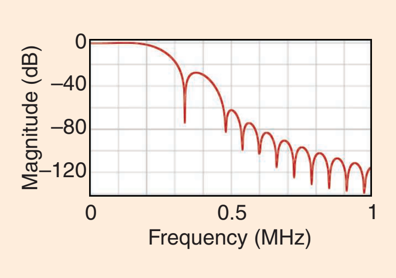

FIGURE 8. The frequency response of the low-pass filter using $W_{o,1,3}$.

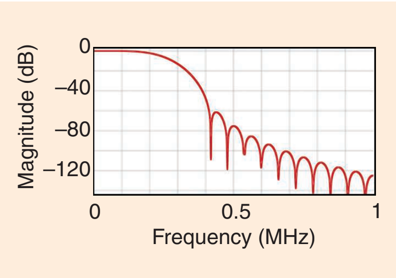

FIGURE 9. Frequency response of low-pass filter using $W_{e,1,1}$.

–27 dB, while the filter designed with $W_{e,1,1}$ has its first null at 0.42 MHz with a peak sidelobe at –61 dB.

Assuming that the desired beat frequency components are at 0.2 MHz, the third and the fifth harmonics are suppressed by more than 80 dB and 110 dB, respectively, in the filter designed with $W_{o,1,3}$. In the filter designed with $W_{e,1,1}$, these harmonics are suppressed by more than 90 dB and 120 dB (see Figures 8 and 9). By employing discrete Fourier transform techniques, finite-impulse response filters can be realized using the proposed windows. The number of samples in the window must be sufficiently large to minimize the aliasing effects and obtain a frequency response close to the response for the continuous case. For more details, see the articles [2] and [4].

What we have learned

This lecture note has described how to obtain several sets of window features (the main-lobe width, the peak sidelobe level, and the sidelobe decay) without increasing the number of terms in a cosine window function. It has also demonstrated that the proposed window is capable of placing a first null at any desired frequency ($> 1/T$); the sidelobe decay in this case will be 24 dB/octave. When the first null point is at one of the discrete frequencies, $1/T, 2/T$, or $3/T$, the sidelobe decay will be 30 dB/octave. The windows synthesized using various options are compared with the traditional cosine windows (Hann, three-term and four-term cosine, and Blackman) using a new metric termed the *efficiency index*, and they are found to be equally or more efficient. The two examples given in the previous section illustrate the usage of the proposed windows in the filter design.

Notice that the design concept introduced in this note for a three-term cosine window can be extended to a cosine window with more than three terms. For example, a four-term cosine window will contain six unknowns, and a five-term cosine window will contain eight unknowns. In general, an N-term cosine window will have $2(N-1)$ unknowns, and it can be designed for a maximum decay of $6(2N-1)$ dB/octave, with many options for main-lobe widths and sidelobe peaks. However, one encounters transcendental equations in the unknowns, which require the use of numerical methods to determine the unknowns.

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