Minimizing Condition Number via Convex Programming

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Abstract

In this paper we consider minimizing the spectral condition number of a positive semidefinite matrix over a nonempty closed convex set Ω. We show that it can be solved as a convex programming problem, and moreover the optimal value of the latter problem is achievable. As a consequence, when Ω is positive semidefinite representable, it can be cast into a semidefinite programming problem. We also study a closely related problem, that is, finding a *diagonal* preconditioner for a positive definite matrix. We show that it can be found by solving a convex programming problem.

Key words: condition number, diagonal preconditioner, convex programming, semidefinite programming

AMS 2000 subject classification: 90C22, 90C25, 15A12, 65F35

1 Introduction

Inspired by Maréchal and Ye [5], we consider the problem of the form

$$\kappa^* = \inf \left\{ \kappa(X) : X \in \mathcal{S}^n_+ \cap \Omega \right\}, \tag{1}$$

where $\Omega \subseteq \Re^{n\times n}$ is a nonempty closed convex set, $\mathcal{S}^n_+$ is the cone of symmetric positive semidefinite $n \times n$ matrices, and $\kappa(X)$ denotes the spectral condition number of X. We denote by $\lambda_{\max}(X)$ (resp. $\lambda_{\min}(X)$) the maximal (resp. minimal) eigenvalue of a real symmetric matrix X. As in [5], for any $X \in \mathcal{S}^n_+$, the function κ is defined as

$$\kappa(X) = \begin{cases} \lambda_{\max}(X)/\lambda_{\min}(X) & \text{if } \lambda_{\min}(X) > 0, \\ \infty & \text{if } \lambda_{\min}(X) = 0 \text{ and } \lambda_{\max}(X) > 0, \\ 0 & \text{if } X = 0. \end{cases}$$

It is clear that κ achieves the global minimum value of (1) at 0 if $0 \in \Omega$. To avoid some trivial cases, we make the following assumptions regarding (1) throughout the paper:

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A.1 Ω does not contain the zero matrix;

A.2 The optimal value κ^* of problem (1) is finite.

Problem (1) arises in several applications. For example, Guigues [2] recently applied (1) to estimate the covariance matrix for the Markowitz portfolio selection model (see also [5]). It is easy to show that $\kappa(\cdot)$ is a quasi-convex function. Moreover, an approximate solution of problem (1) can be found by solving a sequence of convex feasibility problems. Indeed, suppose that $\bar{\kappa}$ and $\underline{\kappa}$ are the known upper and lower bounds on the optimal value κ^* of (1). Let $\kappa_l = \underline{\kappa}$, $\kappa_u = \bar{\kappa}$, and $v = (\kappa_l + \kappa_u)/2$. Consider the convex feasibility problem:

$$\text{find } X \in \mathcal{F}_v := \left\{X \in \mathcal{S}^n_+ \cap \Omega,\ \lambda_{\max}(X) - v\lambda_{\min}(X) \leq 0\right\}. \tag{2}$$

If $F_v = \emptyset$, we know $\kappa^* \geq v$ and update κ_l by setting $\kappa_l \leftarrow v$. Otherwise, $\kappa^* \leq v$ and set $\kappa_u \leftarrow v$. By repeating this bisection scheme, one can find an ϵ-optimal solution of (1) in $\mathcal{O}(\log \frac{\bar{\kappa}-\underline{\kappa}}{\epsilon})$ number of accesses to the oracle (2) for any given $\epsilon > 0$. Though this scheme looks quite simple, it may not be easily implementable as checking whether F_v is empty or not can be highly numerically unstable.

Recently, Maréchal and Ye [5] studied problem (1) under the assumption that Ω is a compact convex set. They showed that an optimal solution of (1) can be approximated by an exact or an inexact solution of a nonsmooth convex programming problem

$$\min\{\kappa_p(X) : X \in \mathcal{S}^n_+ \cap \Omega\}, \tag{3}$$

for some sufficiently large $p > 0$, where $\kappa_p(X) := (\lambda_{\max}(X))^{(p+1)}/(\lambda_{\min}(X))^p$. In particular, it is proven in [5] that $\kappa_p(\cdot)$ is convex for any $p \geq 0$, and moreover every accumulation point of the sequence $\{X_{p_k}\}$ is an optimal solution of (1) for any $\{p_k\} \subseteq \Re_+ \to \infty$, where X_{p_k} is an optimal solution of (3) for $p = p_k$. It is not clear, however, how large p is enough to ensure an exact or inexact solution of (3) is an ϵ-optimal solution of (1) for a given $\epsilon > 0$.

In this paper we will show that problem (1) can be solved as a convex programming problem, and moreover the optimal value of the latter problem is achievable. As a consequence, when Ω is positive semidefinite representable, it can be cast into a semidefinite programming problem. We also consider a closely related problem, that is, finding a *diagonal* preconditioner for a positive definite matrix. In particular, assume that $C \in \Re^{m\times n}$ has full column rank. Consider finding a diagonal preconditioner X for C^TC so that $\kappa(X^TC^TCX)$ is minimized, which can be formulated as

$$\begin{array}{ll} \inf & \kappa(X^TC^TCX) \\ \text{s.t.} & X \in \mathcal{D}^n_+ \cap \Omega, \end{array} \tag{4}$$

where $\mathcal{D}^n_+$ denotes the set of $n \times n$ nonnegative diagonal matrices and Ω is the same as defined above. We will show that problem (4) can be solved as a convex programming problem. Especially, when Ω is a box, it can be cast into a cone programming problem.

The rest of paper is organized as follows. In Section 2 we consider problem (1) and show that it can be solved as a convex programming problem. In Section 3 we study problem (4) and show that it can be solved as a convex programming problem. Finally we present some concluding remarks in Section 4.

1.1 Notation

The symbols $\Re^n$ and $\Re^n_+$ denote the n-dimensional Euclidean space and its nonnegative orthant, respectively. The space of symmetric $n \times n$ matrices will be denoted by $\mathcal{S}^n$. If $X \in \mathcal{S}^n$ is positive semidefinite, we write $X \succeq 0$. Also, we write $X \preceq Y$ (resp., $X \succeq Y$) to mean $Y - X \succeq 0$ (resp., $X - Y \succeq 0$). The cone of positive semidefinite (resp., definite) matrices is denoted by $\mathcal{S}^n_+$ (resp., $\mathcal{S}^n_{++}$). The cone of (resp., nonnegative) diagonal $n \times n$ matrices will be denoted by $\mathcal{D}^n$ (resp., $\mathcal{D}^n_+$). The n-dimensional second-order cone will be denoted by $\mathcal{L}^n$, that is,

$$\mathcal{L}^n := \left\{ x \in \Re^n : x_1 \geq \sqrt{x_2^2 + \cdots + x_n^2} \right\}.$$

We denote by I the identity matrix, whose dimension should be clear from the context. Given a convex set C, cl C and ri C denote the closure and relative interior of C, respectively. In addition, the recession cone of a closed convex set C is denoted by C_∞. Given any operator $\mathcal{A}$, $\Re(\mathcal{A})$ denotes the range of $\mathcal{A}$.

Finally, consider the problem of minimizing a real-valued function $f(x)$ over a certain nonempty feasible region $\mathcal{F}$ contained in the domain of f and let $\bar{f} := \inf\{f(x) : x \in \mathcal{F}\}$. For $\epsilon \geq 0$, we say that x_ϵ is an *ϵ-optimal solution* of this problem if $x_\epsilon \in \mathcal{F}$ and $f(x_\epsilon) \leq \bar{f} + \epsilon$.

2 Minimizing condition number

In this section we show that the condition number minimization problem, that is, problem (1), can be solved as a convex programming problem, and moreover the optimal value of the latter problem is achievable.

We first show that problem (1) can be reformulated as the following problem:

$$\lambda^* = \inf \left\{ \lambda_{\max}(X) : X \in t\Omega, t \geq 0, X \succeq I \right\}. \quad (5)$$

Theorem 2.1 *The following statements hold:*

i) problem (5) has the same optimal value as (1), that is, $\lambda^* = \kappa^*$;

ii) for any $\epsilon \geq 0$, if X_ϵ is an ϵ-optimal solution of (1), then $(1/\lambda_{\min}(X_\epsilon), X_\epsilon/\lambda_{\min}(X_\epsilon))$ is an ϵ-optimal solution of (5);

iii) for any $\epsilon \geq 0$, if (t_ϵ, X_ϵ) is an ϵ-optimal solution of (5), then X_ϵ/t_ϵ is an ϵ-optimal solution of (1).

Proof. By Assumptions A.1 and A.2, we know that $\mathcal{S}^n_{++} \cap \Omega \neq \emptyset$. It implies that problem (5) is feasible. Let (t, X) be a feasible solution of (5). It is easy to observe that $t > 0$ and $X/t \in \mathcal{S}^n_{++} \cap \Omega$. Hence, X/t is a feasible solution of (1). Moreover, we have

$$\kappa(X/t) = \kappa(X) = \lambda_{\max}(X)/\lambda_{\min}(X) \leq \lambda_{\max}(X), \quad (6)$$

where the last inequality is due to $X \succeq I$. It then immediately implies $\kappa^* \le \lambda^*$. Now suppose that X_ϵ is an ϵ-optimal solution of (1) for some $\epsilon \ge 0$. Then, $X_\epsilon \in \mathcal{S}^n_+ \cap \Omega$ and $\kappa(X_\epsilon) \le \kappa^* + \epsilon$, which together with Assumptions A.1 and A.2 implies $\lambda_{\min}(X_\epsilon) > 0$. It is then straightforward to verify that $(1/\lambda_{\min}(X_\epsilon), X_\epsilon/\lambda_{\min}(X_\epsilon))$ is a feasible solution of (5). Furthermore, we have

$$\lambda^* \le \lambda_{\max}\left(X_\epsilon/\lambda_{\min}(X_\epsilon)\right) = \kappa(X_\epsilon) \le \kappa^* + \epsilon. \tag{7}$$

Due to the arbitrariness of ϵ, we immediately conclude that $\lambda^* \le \kappa^*$. Thus, we have $\lambda^* = \kappa^*$, and so statement (i) holds. Moreover, it follows from (7) and statement (i) that $(1/\lambda_{\min}(X_\epsilon), X_\epsilon/\lambda_{\min}(X_\epsilon))$ is an ϵ-optimal solution of (5), and hence statement (ii) holds. Next we show that statement (iii) holds. Indeed, suppose (t_ϵ, X_ϵ) is an ϵ-optimal solution of (5) for some $\epsilon \ge 0$. We clearly see that $t_\epsilon > 0$ and X_ϵ/t_ϵ is a feasible solution of (1). Replacing t and X by t_ϵ and X_ϵ, respectively in (6), and using statement (i), we obtain that

$$\kappa(X_\epsilon/t_\epsilon) \le \lambda_{\max}(X_\epsilon) \le \lambda^* + \epsilon = \kappa^* + \epsilon,$$

which implies that X_ϵ/t_ϵ is an ϵ-optimal solution of (1). ∎

We see from Theorem 2.1 that problem (1) can be solved as (5). Clearly, the objective function and the feasible region of (5), denoted by $\mathcal{F}$, are convex. Nevertheless, $\mathcal{F}$ generally is not closed. For example, let $\Omega = \{X \in \mathcal{S}^n : X \succeq I\}$. It is easy to observe that $\{(t_k, X_k)\} = \{(1/k, I)\} \subseteq \mathcal{F}$ and $(t_k, X_k) \to (0, I) \notin \mathcal{F}$, which implies that $\mathcal{F}$ is not closed. We next provide a necessary and sufficient condition for the closedness of $\mathcal{F}$.

Theorem 2.2 *The feasible region $\mathcal{F}$ of (5) is a closed convex set if and only if $\mathcal{S}^n_{++} \cap \Omega_\infty = \emptyset$.*

Proof. It is straightforward to show that $\mathcal{F}$ is convex. Then it remains to show that $\mathcal{F}$ is closed if and only if $\mathcal{S}^n_{++} \cap \Omega_\infty = \emptyset$. By Assumption A.2 and statement (i) of Theorem 2.1, we know that the optimal value of problem (5) is finite, which immediately implies that $\mathcal{S}^n_{++} \cap \Omega \neq \emptyset$. Let $X \in \text{ri}\ \Omega$ and $Y \in \mathcal{S}^n_{++} \cap \Omega$. It follows that $\{\alpha X + (1-\alpha)Y : \alpha \in (0,1]\} \subseteq \text{ri}\ \Omega$. Clearly, $\alpha X + (1-\alpha)Y \in \mathcal{S}^n_{++} \cap \text{ri}\ \Omega$ when $0 < \alpha \ll 1$. Thus $\mathcal{S}^n_{++} \cap \text{ri}\ \Omega \neq \emptyset$. We now define

$$\mathcal{K} = \{t(1, X) : t \ge 0, X \in \Omega\}, \quad \tilde{\mathcal{K}} = \{(t, X) : t \in \Re, X \succeq I\}. \tag{8}$$

It follows from Corollary 6.8.1 of [6] and Proposition 2.1.11 (pp. 107 of [3]) that

$$\text{ri}\ \mathcal{K} = \{t(1, X) : t > 0, X \in \text{ri}\ \Omega\}, \quad \text{ri}\ \tilde{\mathcal{K}} = \{(t, X) : t \in \Re, X \succ I\}.$$

Since $\mathcal{S}^n_{++} \cap \text{ri}\ \Omega \neq \emptyset$, we easily see that $\text{ri}\ \mathcal{K} \cap \text{ri}\ \tilde{\mathcal{K}} \neq \emptyset$. Using this result and Theorems 6.5 and 8.2 of [6], we obtain that

$$\begin{aligned} \text{cl}\ (\mathcal{K} \cap \tilde{\mathcal{K}}) &= \text{cl}\ \mathcal{K} \cap \text{cl}\ \tilde{\mathcal{K}} = (\mathcal{K} \cup \{(0, X) : X \in \Omega_\infty\}) \cap \tilde{\mathcal{K}}, \\ &= (\mathcal{K} \cap \tilde{\mathcal{K}}) \cup \{(0, X) : X \in \Omega_\infty, X \succeq I\}, \end{aligned}$$

which together with the definitions of $\mathcal{K}$ and $\tilde{\mathcal{K}}$ implies that $\mathcal{F} = \mathcal{K} \cap \tilde{\mathcal{K}}$ is closed if and only if $\{(0, X) : X \in \Omega_\infty, X \succeq I\} \subseteq \mathcal{K}$. Further, by the definition of $\mathcal{K}$, the latter condition holds if and only if $\{(0,X) : X \in \Omega_\infty, X \succeq I\} = \emptyset$, or equivalently, $\mathcal{S}^n_{++} \cap \Omega_\infty = \emptyset$. Thus the conclusion holds. ∎

We observe from Theorems 2.1 and 2.2 that when $\mathcal{S}^n_{++} \cap \Omega_\infty = \emptyset$, problem (1) can be solved as convex programming problem (5). In particular, for the case where Ω is a compact convex set that is assumed in [5], $\mathcal{S}^n_{++} \cap \Omega_\infty = \emptyset$ evidently holds since $\Omega_\infty = \{0\}$. Given that $\mathcal{S}^n_{++} \cap \Omega_\infty = \emptyset$ generally does not hold, we will further consider a relaxation of (5):

$$\mu^* = \inf \{\lambda_{\max}(X) : (t,X) \in \Xi, \ X \succeq I\}, \tag{9}$$

where

$$\Xi := \{t(1,X) : t \geq 0, X \in \Omega\} \cup \{(0,X) : X \in \Omega_\infty\}. \tag{10}$$

In view of (8) and Theorem 8.2 of [6], we see that $\Xi = \mathrm{cl}\, \mathcal{K}$, and hence Ξ is closed and convex. We next show that problem (1) can be solved as convex programming problem (9).

Theorem 2.3 *The following statements hold:*

i) problem (9) has the same optimal value as (1), that is, $\mu^* = \kappa^*$*;*

ii) for any $\epsilon \geq 0$*, if* X_ϵ *is an* ϵ*-optimal solution of (1), then* $(1/\lambda_{\min}(X_\epsilon), X_\epsilon/\lambda_{\min}(X_\epsilon))$ *is an* ϵ*-optimal solution of (9);*

iii) for any $\epsilon \geq 0$*, if* (t_ϵ, X_ϵ) *is an* ϵ*-optimal solution of (9) for some* $t_\epsilon > 0$*, then* X_ϵ/t_ϵ *is an* ϵ*-optimal solution of (1);*

iv) for any $\epsilon > 0$*, if* $(0, X_\epsilon)$ *is an* $\epsilon/2$*-optimal solution of (9), then* $\bar{X} + \alpha X_\epsilon$ *is an* ϵ*-optimal solution of (1), provided that* $\alpha \geq \underline{\alpha} := \max\{2[\lambda_{\max}(\bar{X}) - (\mu^*+\epsilon)\lambda_{\min}(\bar{X})]/\epsilon, 1 - \lambda_{\min}(\bar{X}), 0\}$*, where* $\bar{X}$ *is an arbitrary point in* Ω*.*

Proof. In view of (5) and (9), we clearly see that $\mu^* \leq \lambda^*$, which together with Theorem 2.1 (i) implies that $\mu^* \leq \kappa^*$. For any $\epsilon \geq 0$, suppose (t_ϵ, X_ϵ) is an ϵ-optimal solution of (9) for some $t_\epsilon > 0$. Then $\lambda_{\max}(X_\epsilon) \leq \mu^* + \epsilon \leq \lambda^* + \epsilon$. Thus (t_ϵ, X_ϵ) is an ϵ-optimal solution of (5). It immediately follows from Theorem 2.1 (iii) that statement (iii) holds. Now, for any $\epsilon > 0$, suppose $(0, X_\epsilon)$ is an $\epsilon/2$-optimal solution of (9). Then, we clearly have

$$I \preceq X_\epsilon \in \Omega_\infty, \quad \lambda_{\max}(X_\epsilon) \leq \mu^* + \epsilon/2. \tag{11}$$

Let $\bar{X}$ be an arbitrary point in Ω and $\underline{\alpha}$ be defined above. In view of (11) and the definition of $\underline{\alpha}$, it is not hard to show that when $\alpha \geq \underline{\alpha}$, we have $\bar{X} + \alpha X_\epsilon \in \mathcal{S}^n_{++} \cap \Omega$ and

$$\frac{\lambda_{\max}(\bar{X}) + \alpha\lambda_{\max}(X_\epsilon)}{\lambda_{\min}(\bar{X}) + \alpha} \leq \frac{\lambda_{\max}(\bar{X}) + \alpha(\mu^* + \epsilon/2)}{\lambda_{\min}(\bar{X}) + \alpha} \leq \mu^* + \epsilon. \tag{12}$$

Recalling that $X_\epsilon \succeq I$, and $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ are convex and concave functions, respectively, we obtain that for any $\alpha \geq \underline{\alpha}$,

$$\kappa(\bar{X} + \alpha X_\epsilon) = \frac{\lambda_{\max}\left((\bar{X} + \alpha X_\epsilon)/(1+\alpha)\right)}{\lambda_{\min}\left((\bar{X} + \alpha X_\epsilon)/(1+\alpha)\right)} \leq \frac{\lambda_{\max}(\bar{X}) + \alpha\lambda_{\max}(X_\epsilon)}{\lambda_{\min}(\bar{X}) + \alpha\lambda_{\min}(X_\epsilon)} \leq \frac{\lambda_{\max}(\bar{X}) + \alpha\lambda_{\max}(X_\epsilon)}{\lambda_{\min}(\bar{X}) + \alpha},$$

which together with (12) and the fact that $\bar{X} + \alpha X_\epsilon \in \mathcal{S}^n_{++} \cap \Omega \ \forall \alpha \geq \underline{\alpha}$, implies that

$$\kappa^* \leq \kappa(\bar{X} + \alpha X_\epsilon) \leq \mu^* + \epsilon \quad \forall \alpha \geq \underline{\alpha}. \tag{13}$$

Due to the arbitrariness of ϵ, we see from (13) that $\kappa^* \leq \mu^*$. It then follows that $\kappa^* = \mu^*$ and hence statement (i) holds. In view of (13) and the relation $\kappa^* = \mu^*$, we immediately see that statement (iv) holds. Finally, recall from Theorem 2.1 (i) that $\kappa^* = \lambda^*$. Hence, $\lambda^* = \mu^*$. Using this relation and Theorem 2.1 (ii), we easily conclude that statement (ii) holds. ■

We next show that problem (9) is solvable, that is, its optimal value is achievable. Before proceeding, we provide some upper bounds on any ϵ-optimal solution of (9) for some $\epsilon \geq 0$.

Lemma 2.4 *Suppose that (t_ϵ, X_ϵ) is an ϵ-optimal solution of (9) for some $\epsilon \geq 0$. Define*

$$\underline{\lambda}^* = \inf \left\{ \lambda_{\max}(X) : X \in \mathcal{S}^n_+ \cap \Omega \right\}. \tag{14}$$

Then

$$0 \leq t_\epsilon \leq (\mu^* + \epsilon)/\underline{\lambda}^*, \qquad I \preceq X_\epsilon \preceq (\mu^* + \epsilon) I, \tag{15}$$

where μ^* is the optimal value of (9).

Proof. By Assumptions A.1 and A.2, we easily observe that $\underline{\lambda}^* \in (0, \infty)$. Since (t_ϵ, X_ϵ) is an ϵ-optimal solution of (9), we know that $\lambda_{\max}(X_\epsilon) \leq \mu^* + \epsilon$ and $X_\epsilon \succeq I$, which implies that the second relation of (15) holds. If $t_\epsilon = 0$, the first relation of (15) evidently holds. We now assume that $t_\epsilon > 0$. It follows from Theorem 2.3 (iii) that $X_\epsilon / t_\epsilon \in \mathcal{S}^n_+ \cap \Omega$. This relation together with the definition of $\underline{\lambda}^*$ implies that $\lambda_{\max}(X_\epsilon)/t_\epsilon \geq \underline{\lambda}^*$. Using this inequality and the relation $\lambda_{\max}(X_\epsilon) \leq \mu^* + \epsilon$, we see that the first relation of (15) holds. ■

We are now ready to show that problem (9) is solvable.

Theorem 2.5 *Problem (9) is solvable, that is, its optimal value can be achieved at some feasible point.*

Proof. Given an arbitrary $\epsilon > 0$, define

$$\Pi := \{(t, X) : 0 \leq t \leq (\mu^* + \epsilon)/\underline{\lambda}^*, I \preceq X \preceq (\mu^* + \epsilon) I\},$$

where μ^* is the optimal value of (9) and $\underline{\lambda}^*$ is defined in (14). It follows from Lemma 2.4 that problem (9) is equivalent to

$$\inf \{\lambda_{\max}(X) : (t, X) \in \Xi \cap \Pi\}, \tag{16}$$

where Ξ is defined in (10). We know that Ξ is closed. Hence, $\Xi \cap \Pi$ is compact. In addition, $\lambda_{\max}(\cdot)$ is continuous. It follows that problem (16) is solvable, which implies that problem (9) is solvable. ■

In view of Theorems 2.3 and 2.5, we see that problem (1) can be reformulated as a solvable convex programming problem (9). Moreover, an optimal solution of (9) can provide either an optimal or approximate solution of (1).

We now present three examples to illustrate how problem (1) can be solved as convex programming problem. In the first two examples we choose Ω to be the same sets as used in [2] for estimating the covariance matrix for the Markowitz portfolio selection model (see also [5]). In the third example we consider a positive semidefinite representable set Ω. We show that for all these sets Ω, problem (1) can be cast into a semidefinite programming (SDP) problem, which can be suitably solved by interior point solvers (e.g., SeDuMi [8] and SDPT3 [9]) and first-order methods (see, for example, [1, 4]).

Corollary 2.6 *Let $Q_1, \ldots, Q_m \in \mathcal{S}^n$ be given. Assume that $\Omega = \mathrm{co}\,\{Q_1, \ldots, Q_m\}$, where* co C *denotes the convex hull of the set C. Then, problem (1) can be solved as the following SDP problem:*

$$\begin{array}{ll} \min\limits_{s,y,X} & s \\ s.t. & \sum\limits_{i=1}^{m} Q_i y_i - X = 0, \\ & y \in \Re^m_+, \ I \preceq X \preceq sI. \end{array} \tag{17}$$

Proof. Since $\Omega = \mathrm{co}\,\{Q_1, \ldots, Q_m\}$, it is clear that Ω is a compact convex set and so $\Omega_\infty = \{0\}$. Using this result, the definition of Ω and Theorem 2.3, we immediately see that problem (1) can be solved as the following SDP problem:

$$\begin{array}{ll} \min\limits_{s,t,y,X} & s \\ \text{s.t.} & \sum\limits_{i=1}^{m} Q_i y_i - X = 0, \\ & \sum\limits_{i=1}^{m} y_i - t = 0, \\ & t \geq 0, \ y \in \Re^m_+, \ I \preceq X \preceq sI, \end{array}$$

which is clearly equivalent to problem (17). Thus the conclusion holds. ∎

Corollary 2.7 *Let $Q \in \mathcal{S}^n$ be given. Assume that $\Omega = \{X \in \mathcal{S}^n : |X_{ij} - Q_{ij}| \leq \eta \ \forall ij\}$ for some $\eta > 0$. Then, problem (1) can be solved as the following SDP problem:*

$$\begin{array}{ll} \min\limits_{s,t,X} & s \\ s.t. & (Q_{ij} - \eta)t \ \leq \ X_{ij} \ \leq \ (Q_{ij} + \eta)t \quad \forall ij \\ & t \geq 0, \ I \preceq X \preceq sI. \end{array}$$

Proof. The conclusion immediately follows from the definition of Ω and Theorem 2.3. ∎

Corollary 2.8 *Assume that $\emptyset \neq \Omega$ is positive semidefinite representable, i.e., there exists $C \in \mathcal{S}^m$ and linear operators $\mathcal{A} : \mathcal{S}^n \to \mathcal{S}^m$ and $\mathcal{B} : \Re^k \to \mathcal{S}^m$ such that*

$$\Omega = \left\{ X \in \mathcal{S}^n : \mathcal{A}(X) + \mathcal{B}(u) + C \succeq 0 \text{ for some } u \in \Re^k \right\}. \tag{18}$$

Suppose that $\Re(\mathcal{B})+\mathcal{S}_+^m$ is closed. Then, problem (1) can be solved as the following SDP problem:

$$\begin{array}{cl} \min\limits_{s,t,u,X} & s \\ s.t. & \mathcal{A}(X)+\mathcal{B}(u)+tC \succeq 0, \\ & t \geq 0,\ I \preceq X \preceq sI. \end{array} \tag{19}$$

Proof. It is easy to observe that $\Omega = \mathcal{A}^{-1}(\Re(\mathcal{B})+\mathcal{S}_+^m - C)$. Since $\Re(\mathcal{B})+\mathcal{S}_+^m$ is a closed convex cone, we clearly see that Ω is closed and convex, and $(\Re(\mathcal{B})+\mathcal{S}_+^m - C)_\infty = \Re(\mathcal{B})+\mathcal{S}_+^m$. It then follows from Proposition 2.2.5 (pp. 110 of [3]) that

$$\Omega_\infty = \mathcal{A}^{-1}((\Re(\mathcal{B})+\mathcal{S}_+^m - C)_\infty) = \mathcal{A}^{-1}(\Re(\mathcal{B})+\mathcal{S}_+^m). \tag{20}$$

Recall from Theorem 2.3 that problem (1) can be solved as (9), which together with (20) immediately implies that the conclusion holds. ∎

Remark. If $\Re(\mathcal{B})\cap\mathcal{S}_+^n = \emptyset$ or $\Re(\mathcal{B})\cap\mathcal{S}_{++}^n \neq \emptyset$ holds, $\Re(\mathcal{B})+\mathcal{S}_+^n$ is closed (see [7]). Nevertheless, it generally may not be closed. For example, let $\mathcal{B}:\Re\to\mathcal{S}^2$ be defined as:

$$\mathcal{B}(u) = \begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} \quad \forall u \in \Re.$$

Consider the sequence $\{X_k\}$ defined as follows:

$$X_k = \begin{bmatrix} 0 & 1 \\ 1 & 1/k \end{bmatrix} = \begin{bmatrix} -k & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} k & 1 \\ 1 & 1/k \end{bmatrix} \quad \forall k \geq 1.$$

Clearly, $\{X_k\} \subseteq \Re(\mathcal{B})+\mathcal{S}_+^2$, but

$$\lim_{k\to\infty} X_k = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \notin \Re(\mathcal{B})+\mathcal{S}_+^2$$

since

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -u & 1 \\ 1 & 0 \end{bmatrix} \notin \mathcal{S}_+^2 \quad \forall u \in \Re.$$

Additionally, using similar arguments as in the proof of Theorem 2.3 (iii) and (iv), we can show that an approximate solution of (1) can be obtained by approximately solving (19) even when $\Re(\mathcal{B})+\mathcal{S}_+^n$ is not closed (in which case Ω may not be closed).

3 Finding a diagonal preconditioner

In this section we show that the diagonal preconditioner finding problem, that is, problem (4), can be solved as a convex programming problem.

We first show that problem (4) can be solved as the following problem:

$$\inf\left\{\lambda_{\max}(X^TC^TCX) : (t,X)\in\Xi,\ X^TC^TCX \succeq I,\ X\in\mathcal{D}_+^n\right\}, \tag{21}$$

where Ξ is defined in (10).

Proposition 3.1 *The following statements hold:*

i) problem (21) has the same optimal value as (4);

ii) for any $\epsilon \geq 0$, if X_ϵ is an ϵ-optimal solution of (4), then $(1/\sqrt{\lambda_{\min}(X_\epsilon^T C^T C X_\epsilon)}$, $X_\epsilon/\sqrt{\lambda_{\min}(X_\epsilon^T C^T C X_\epsilon)})$ is an ϵ-optimal solution of (21);

iii) for any $\epsilon \geq 0$, if (t_ϵ, X_ϵ) is an ϵ-optimal solution of (21) for some $t_\epsilon > 0$, then X_ϵ/t_ϵ is an ϵ-optimal solution of (4);

iv) for any $\epsilon > 0$, if $(0, X_\epsilon)$ is an $\epsilon/2$-optimal solution of (21), then $\bar{X} + \alpha X_\epsilon$ is an ϵ-optimal solution of (4), provided that α is sufficiently large, where $\bar{X}$ is an arbitrary diagonal matrix in $\mathcal{D}^n \cap \Omega$.

Proof. Given any X, $Y \in \Re^{n\times n}$, we can see that

$$\begin{aligned}
\lim_{\alpha\to\infty} \kappa((X+\alpha Y)^T C^T C(X+\alpha Y)) &= \lim_{\alpha\to\infty} \frac{\lambda_{\max}((X+\alpha Y)^T C^T C(X+\alpha Y))}{\lambda_{\min}((X+\alpha Y)^T C^T C(X+\alpha Y))} \\
&= \lim_{\alpha\to\infty} \frac{\lambda_{\max}((X/\alpha+Y)^T C^T C(X/\alpha+Y))}{\lambda_{\min}((X/\alpha+Y)^T C^T C(X/\alpha+Y))} \\
&= \lim_{\alpha\to\infty} \frac{\lambda_{\max}(Y^T C^T C Y)}{\lambda_{\min}(Y^T C^T C Y)} = \kappa(Y^T C^T C Y), \qquad (22)
\end{aligned}$$

where the third inequality is due to the continuity of eigenvalues. The conclusion of this proposition then follows from (22) and similar arguments as used in the proof of Theorem 2.3. ∎

It is not hard to observe that the set defined by the constraint $X^T C^T C X \succeq I$ is typically nonconvex. Thus problem (21) seems to be nonconvex. We will, however, show that problem (21) can be reformulated into a convex programming problem.

Theorem 3.2 *Problem (4) can be solved as the following convex programming problem:*

$$\begin{aligned}
\min_{s,t,Y,X} \quad & s \\
s.t. \quad & \begin{bmatrix} I & CX \\ X^T C^T & sI \end{bmatrix} \succeq 0, \\
& \begin{bmatrix} I & Y \\ Y & C^T C \end{bmatrix} \succeq 0, \\
& \begin{pmatrix} (Y_{ii}+X_{ii})/2 \\ (Y_{ii}-X_{ii})/2 \\ 1 \end{pmatrix} \in \mathcal{L}^3, \quad i=1,\ldots,n, \\
& (t,X) \in \Xi, \; X \in \mathcal{D}^n_+, \; Y \in \mathcal{D}^n_+.
\end{aligned} \qquad (23)$$

Proof. In view of Proposition 3.1, it suffices to show that problem (21) can be reformulated as (23). Indeed, we first observe that

$$s \geq \lambda_{\max}(X^T C^T C X) \;\Leftrightarrow\; sI - X^T C^T C X \succeq 0 \;\Leftrightarrow\; \begin{bmatrix} I & CX \\ X^T C^T & sI \end{bmatrix} \succeq 0.$$

In addition, for any $X \in \mathcal{D}^n_+$,

$$\begin{array}{rcl} X^T C^T C X \succeq I & \Leftrightarrow & C^T C \succeq X^{-2} \;\Leftrightarrow\; C^T C \succeq Y^2, YX \succeq I \text{ for some } Y \in \mathcal{D}^n_+, \\ & \Leftrightarrow & C^T C \succeq Y^2, Y_{ii}X_{ii} \geq 1 \; (i = 1, \ldots, n) \text{ for some } Y \in \mathcal{D}^n_+, \\ & \Leftrightarrow & \left[\begin{array}{cc} I & Y \\ Y & C^T C \end{array}\right] \succeq 0, \; \left(\begin{array}{c} (Y_{ii} + X_{ii})/2 \\ (Y_{ii} - X_{ii})/2 \\ 1 \end{array}\right) \in \mathcal{L}^3, \quad i = 1, \ldots, n. \end{array}$$

Using the above observations, we easily see that problem (21) is equivalent to (23). Therefore, the conclusion of this theorem holds. ∎

We next present an example to illustrate how problem (4) can be solved as a convex programming problem. In particular, we choose Ω to be a box. One can easily see from Theorem 3.2 that for such Ω, problem (4) can be cast into a cone programming problem, which can be suitably solved by interior point solvers (e.g., SeDuMi [8] and SDPT3 [9]) and first-order methods (see, for example, [4]).

Corollary 3.3 *Let $d \in \Re^n_{++}$ be given. Assume that $\Omega = \{X \in \mathcal{D}^n : |X_{ii} - d_i| \leq \eta \ \forall i\}$ for some $\eta > 0$. Then, problem (4) can be solved as the following cone programming problem:*

$$\begin{array}{ll} \min\limits_{s,t,Y,X} & s \\ s.t. & \left[\begin{array}{cc} I & CX \\ X^T C^T & sI \end{array}\right] \succeq 0, \\ & \left[\begin{array}{cc} I & Y \\ Y & C^T C \end{array}\right] \succeq 0, \\ & \left(\begin{array}{c} (Y_{ii} + X_{ii})/2 \\ (Y_{ii} - X_{ii})/2 \\ 1 \end{array}\right) \in \mathcal{L}^3, \quad i = 1, \ldots, n, \\ & (d_i - \eta)t \;\leq\; X_{ii} \;\leq\; (d_i + \eta)t, \quad i = 1, \ldots, n, \\ & t \geq 0, \; X \in \mathcal{D}^n_+, \; Y \in \mathcal{D}^n_+. \end{array}$$

4 Concluding remarks

In this paper we considered minimizing the spectral condition number of a positive semidefinite matrix over a nonempty closed convex set Ω. We showed that it can be solved as a convex programming problem, and moreover the optimal value of the latter problem is achievable. As a consequence, when Ω is positive semidefinite representable, it can be cast into an SDP problem. We also considered a closely related problem, that is, finding a diagonal preconditioner for a positive definite matrix. We showed that it can be found by solving a convex programming problem.

The results of this paper can be straightforwardly extended to the problem:

$$\inf\left\{ \frac{\sum_{i=1}^k \lambda_i(X)}{\sum_{j=0}^l \lambda_{n-j}(X)} : \; X \in \mathcal{S}^n_+ \cap \Omega \right\},$$

where $1 \leq k \leq n$, $0 \leq l \leq n-1$, and $\lambda_i(X)$ denotes the ith largest eigenvalue of X for $i = 1, \ldots, n$.

In Section 3 we showed that a *diagonal* preconditioner for a positive definite matrix can be found by solving a convex programming problem. It is not clear whether or not this result can be extended to find a general preconditioner.

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