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SHORT COMMUNICATION

A computational study on robust portfolio selection based on a joint ellipsoidal uncertainty set

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Abstract The "separable" uncertainty sets have been widely used in robust portfolio selection models [e.g., see Erdoğan et al. (Robust portfolio management. manuscript, Department of Industrial Engineering and Operations Research, Columbia University, New York, 2004), Goldfarb and Iyengar (Math Oper Res 28:1–38, 2003), Tütüncü and Koenig (Ann Oper Res 132:157–187, 2004)]. For these uncertainty sets, each type of uncertain parameters (e.g., mean and covariance) has its own uncertainty set. As addressed in Lu (A new cone programming approach for robust portfolio selection, technical report, Department of Mathematics, Simon Fraser University, Burnaby, 2006; Robust portfolio selection based on a joint ellipsoidal uncertainty set, manuscript, Department of Mathematics, Simon Fraser University, Burnaby, 2008), these "separable" uncertainty sets typically share two common properties: (i) their actual confidence level, namely, the probability of uncertain parameters falling within the uncertainty set is unknown, and it can be much higher than the desired one; and (ii) they are fully or partially box-type. The associated consequences are that the resulting robust portfolios can be too conservative, and moreover, they are usually highly non-diversified as observed in the computational experiments conducted in this paper and Tütüncü and Koenig (Ann Oper Res 132:157–187, 2004). To combat these drawbacks, the author of this paper introduced a "joint" ellipsoidal uncertainty set (Lu in A new cone programming approach for robust portfolio selection, technical report, Department of Mathematics, Simon Fraser University, Burnaby, 2006; Robust portfolio selection based on a joint ellipsoidal uncertainty set, manuscript, Department of Mathematics, Simon Fraser University, Burnaby, 2008) and showed that it can be

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constructed as a confidence region associated with a statistical procedure applied to estimate the model parameters. For this uncertainty set, we showed in Lu (A new cone programming approach for robust portfolio selection, technical report, Department of Mathematics, Simon Fraser University, Burnaby, 2006; Robust portfolio selection based on a joint ellipsoidal uncertainty set, manuscript, Department of Mathematics, Simon Fraser University, Burnaby, 2008) that the corresponding robust maximum risk-adjusted return (RMRAR) model can be reformulated and solved as a cone programming problem. In this paper, we conduct computational experiments to compare the performance of the robust portfolios determined by the RMRAR models with our "joint" uncertainty set (Lu in A new cone programming approach for robust portfolio selection, technical report, Department of Mathematics, Simon Fraser University, Burnaby, 2006; Robust portfolio selection based on a joint ellipsoidal uncertainty set, manuscript, Department of Mathematics, Simon Fraser University, Burnaby, 2008) and Goldfarb and Iyengar's "separable" uncertainty set proposed in the seminal paper (Goldfarb and Iyengar in Math Oper Res 28:1–38, 2003). Our computational results demonstrate that our robust portfolio outperforms Goldfarb and Iyengar's in terms of wealth growth rate and transaction cost, and moreover, ours is fairly diversified, but Goldfarb and Iyengar's is surprisingly highly non-diversified.

Keywords Robust portfolio selection · Ellipsoidal uncertainty set · Cone programming

Mathematics Subject Classification (2000) 91B28 · 90C20 · 90C22

1 Introduction

It is well known that the optimal portfolios determined by the classical mean-variance model [7] are often sensitive to perturbations in the problem parameters (e.g., see [8]). Recently, robust portfolio selection models have been proposed to alleviate such a sensitivity [1–4,10,11]. The "separable" uncertainty sets have been widely used in the models (e.g., see [3,4,10]). For these uncertainty sets, each type of uncertain parameters (e.g., mean and covariance) has its own uncertainty set. As addressed in [5,6], these "separable" uncertainty sets typically share two common properties: (i) their actual confidence level, namely, the probability of uncertain parameters falling within the uncertainty set is unknown, and it can be much higher than the desired one; and (ii) they are fully or partially box-type. The associated consequences are that the resulting robust portfolios can be too conservative, and moreover, they are usually highly non-diversified as observed in the computational experiments conducted in this paper and [10].

To combat the aforementioned drawbacks, we considered the following factor model for asset returns in [5,6], which was first studied in [4]. Suppose that a discrete-time market has n traded assets. The vector of asset returns over a single market period is denoted by $r \in \Re^n$. The returns on the assets in different market periods are assumed to be independent. The single period return r is assumed to be a random vector given by

$$r = \mu + V^T f + \epsilon, \tag{1}$$

where $\mu \in \Re^n$ is the vector of mean returns, $f \sim \mathcal{N}(0, F) \in \Re^m$ denotes the returns of the m factors driving the market, $V \in \Re^{m \times n}$ denotes the factor loading matrix of the n assets, and $\epsilon \sim \mathcal{N}(0, D) \in \Re^n$ is the vector of residual returns. Further, it is assumed that D is a positive semidefinite diagonal matrix, and the residual return vector ϵ is independent of the factor return vector f. Suppose the market data consists of asset returns $\{r^t : t = 1, \ldots, p\}$ and factor returns $\{f^t : t = 1, \ldots, p\}$ for p trading periods. Let $B = (f^1, f^2, \ldots, f^p) \in \Re^{m \times p}$ denote the matrix of factor returns, and let $e \in \Re^p$ denote an all-one vector. Further, let $A = (eB^T)$, $y_i = (r_i^1, r_i^2, \ldots, r_i^p)^T$, $\bar{x}_i = (A^T A)^{-1} A^T y_i$, $s_i^2 = \|y_i - A\bar{x}_i\|^2/(p-m-1)$ for $i = 1, \ldots, n$. In [5,6], we proposed a "joint" ellipsoidal uncertainty set of (μ, V) with ω-confidence level in the form of

$$\begin{aligned} \mathcal{S}_{\mu,v} &\equiv \mathcal{S}_{\mu,v}(\omega) \\ &= \left\{ (\tilde{\mu}, \tilde{V}) \in \Re^n \times \Re^{m \times n} : \sum_{i=1}^n \frac{(\tilde{x}_i - \bar{x}_i)^T (A^T A)(\tilde{x}_i - \bar{x}_i)}{s_i^2} \leq (m+1)\tilde{c}(\omega) \right\} \end{aligned} \tag{2}$$

for some $\tilde{c}(\omega)$, where $\tilde{x}_i = (\tilde{\mu}_i, \tilde{V}_{1i}, \tilde{V}_{2i}, \ldots, \tilde{V}_{mi})^T$ for $i = 1, \ldots, n$. We showed that it can be constructed as an ω-confidence region associated with a statistical procedure applied to estimate the model parameters (μ, V). For the details, see [5,6].

Based on the "joint" ellipsoidal uncertainty set $\mathcal{S}_{\mu,v}$, we studied the following robust maximum risk-adjusted return (RMRAR) problem in [5,6]:

$$\max_{\phi \in \Phi} \min_{(\mu, V) \in \mathcal{S}_{\mu,v}} \mathrm{E}[r_\phi] - \theta \mathrm{Var}[r_\phi], \tag{3}$$

where $\theta \geq 0$ represents a risk-aversion parameter, and

$$\mathrm{E}[r_\phi] = \phi^T \mu, \quad \mathrm{Var}[r_\phi] = \phi^T (V^T F V + D)\phi, \quad \Phi = \{\phi : e^T \phi = 1, \phi \geq 0\}.$$

We showed that the RMRAR problem (3) can be reformulated and solved as a cone programming problem (see Theorems 4.4 and 4.5 of [5,6]).

In this paper, we conduct computational experiments to compare the performance of the robust portfolios determined by the RMRAR models with our "joint" uncertainty set and Goldfarb and Iyengar's "separable" uncertainty set proposed in the seminal paper [4]. Our computational results demonstrate that our robust portfolio outperforms Goldfarb and Iyengar's in terms of wealth growth rate and transaction cost, and moreover, ours is fairly diversified, but Goldfarb and Iyengar's is highly non-diversified.

2 Computational results

In this section, we present computational experiments on the RMRAR models. We conduct two types of computational tests. The first type of tests are based on simulated data, and the second type of tests use real market data. The main objective of these computational tests is to compare the performance of the RMRAR models with our "joint" uncertainty set (2) and Goldfarb and Iyengar's "separable" uncertainty set described in (3) and (4) of [4]. All computations are performed using SeDuMi V1.1R2 [9]. Throughout this section, the symbols "LROB" and "GIROB" are used to label the robust portfolios determined by the RMRAR models with our "joint" and Goldfarb and Iyengar's "separable" uncertainty sets, respectively. The following terminology will also be used in this section.

Definition 1 The diversification number of a portfolio is defined as the number of its components that are above 1%.

2.1 Computational results for simulated data

In this subsection, we conduct computational tests for simulated data. The data is generated in the same manner as described in Section 7 of [4]. Indeed, we fix the number of assets $n = 50$ and the number of factors $m = 5$. A symmetric positive definite factor covariance matrix F is randomly generated, and it is assumed to be certain. The nominal factor loading matrix V is also randomly generated. The covariance matrix D of the residual returns ϵ is assumed to be certain and set to $D = 0.1\ \mathrm{diag}(V^T F V)$, that is, the linear model explains 90% of the asset variance. The nominal asset returns $\mu \in \Re^n$ are chosen independently according to a uniform distribution on [0.5, 1.5%]. Finally, we generate a sequence of asset and factor return vectors r and f according to the normal distributions $\mathcal{N}(\mu, V^T F V + D)$ and $\mathcal{N}(0, F)$ for an investment period of length $p = 90$, respectively. We also randomly generate the asset returns, denoted by $R \in \Re^{n \times p}$, for next period of length p according to $\mathcal{N}(\mu, V^T F V + D)$. In addition, given a desired confidence level $\omega > 0$, our "joint" uncertainty set $\mathcal{S}_{\mu,v}$ is built as in Section 3 of [5,6], and Goldfarb and Iyengar's "separable" uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$ is built as in Section 2 of [5,6] with $\tilde{\omega} = \omega^{1/n}$. As discussed in Section 2 of [5,6], $\mathcal{S}_m \times \mathcal{S}_v$ has at least ω-confidence level, but its actual confidence level is unknown.

Let ϕ_r be a robust portfolio computed from the data of the current period. Suppose that ϕ_r is held constant for the investment over next period. The *wealth growth rate* of ϕ_r over next period is defined as

$$(\Pi_{1 \le k \le p}(e + R_k))^T \phi_r - 1, \tag{4}$$

where $e \in \Re^n$ denotes the all-one vector and $R_k \in \Re^n$ denotes the kth column of R for $k = 1, \ldots, p$.

We next report the performance of the RMRAR models with our "joint" uncertainty set (2) and Goldfarb and Iyengar's "separable" uncertainty set described in (3) and (4) of [4] as the risk aversion parameter θ ranges from 0 to 10. The computational results averaged over 10 randomly generated instances are shown in Fig. 1 that consists of three groups of plots for $\omega = 0.05, 0.50, 0.95$, respectively. In each of these three groups, the left plot is about the diversification number of the robust portfolios, and the right plot is about the wealth growth rate of the robust portfolios over next period. We first observe that our robust portfolio is fairly diversified, but Goldfarb and Iyengar's

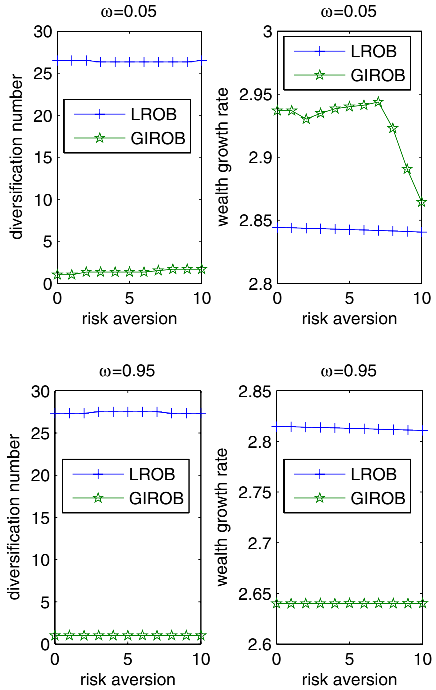

Fig. 1 Performance of portfolios for $\omega = 0.05$, 0.50 and 0.95

is highly non-diversified. Indeed, for $\omega = 0.05, 0.50, 0.95$, the diversification number of our robust portfolio is around 26, and that of Goldfarb and Iyengar's is around one or two. One possible interpretation of this phenomenon is that our uncertainty set $\mathcal{S}_{\mu,v}$ is ellipsoidal, but Goldfarb and Iyengar's uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$ is partially box-type. It seems that the ellipsoidal uncertainty structure tends to produce more diversified robust portfolio than does the fully or partially box-type one. In addition, we observe that for $\omega = 0.05$ or $\omega = 0.50$ with a relatively small θ, the wealth growth rate of our robust portfolio is lower than that of Goldfarb and Iyengar's. But for $\omega = 0.95$ or $\omega = 0.50$ with a relatively large θ, our wealth growth rate is higher than Goldfarb and Iyengar's. This phenomenon is actually not surprising. Indeed, we know that $\mathcal{S}_{\mu,v}$ has confidence ω while $\mathcal{S}_m \times \mathcal{S}_v$ has at least ω confidence, but its actual confidence level can be much higher than ω. Hence for a small ω, the associated model with $\mathcal{S}_m \times \mathcal{S}_v$ can be more robust than that with $\mathcal{S}_{\mu,v}$. However, for a relatively large ω, the uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$ can be over confident, and its corresponding robust model can be conservative. This phenomenon becomes more prominent as the risk aversion parameter θ gets larger.

2.2 Computational results for real market data

In this subsection, we perform experiments on real market data for the RMRAR models with our "joint" and Goldfarb and Iyengar's "separable" uncertainty sets. The

Table 1 Assets

Aerospace and Defense		Telecommunications	
BA	Boeing Corp.	VZ	Verizon Communications
UTX	United Technologies	T	AT&T
LMT	Lockheed Martin	S	Sprint Nextel
NOC	Northrop Grumman	CMCSK	Comcast
HON	Honeywell Intl.	BLS	BellSouth
Semiconductors and Other Electronic Components		Computer Software	
INTC	Intel Corp.	MSFT	Microsoft
TXN	Texas Instruments	ORCL	Oracle
SANM	Sanmina-SCI	CA	CA
SLR	Solectron	ERTS	Electronic Arts
JBL	Jabil Circuit	SYMC	Symantec
Computers and Office Equipment		Pharmaceuticals	
IBM	Intl. Business Machines	PFE	Pfizer
HPQ	Hewlett-Packard	JNJ	Johnson & Johnson
DELL	Dell	ABT	Abbott Laboratories
XRX	Xerox	MRK	Merck
AAPL	Apple Computer	BMY	Bristol-Myers Squibb
Network and Other Communications Equipment		Chemicals	
MOT	Motorola	DOW	Dow Chemical
CSCO	Cisco Systems	DD	DuPon
LU	Lucent Technologies	LYO	Lyondell Chemical
QCOM	Qualcomm	PPG	PPG Industries
Electronics and Electrical Equipment		Utilities (Gas & Electric)	
EMR	Emerson Electric	DUK	Duke Energy
WHR	Whirlpool	D	Dominion Resources
ROK	Rockwell Automation	EXC	Exelon
SPW	SPX	SO	Southern
		PEG	Public Service Enterprise Group

universe of assets that are chosen for investment are those ranked at the top of each of 10 industry categories by Fortune 500 in 2006. In total there are $n = 47$ assets in this set (see Table 1). The set of factors are 10 major market indices (see Table 2). The data sequence consists of daily asset returns from 25July 2002 through 10 May 2006. It shall be mentioned that the data used in this experiment was collected on 11 May 2006. The most recent data available at that time was the one on 10 May 2006.

A complete description of our experimental procedure is as follows. The entire data sequence is divided into investment periods of length $p = 90$ days. For each investment period t, the factor covariance matrix F is computed based on the factor returns of the previous p trading days, and the variance d_i of the residual return is set to $d_i = s_i^2$, where s_i^2 is given in Sect. 1. In addition, given a desired confidence level $\omega > 0$, our "joint" uncertainty set $\mathcal{S}_{\mu,v}$ is built as in Section 3 of [5,6], and

Table 2 Factors

DJCMP65	Dow Jones Composite 65 Stock Average
DJINDUS	Dow Jones Industrials
DJUTILS	Dow Jones Utilities
DJTRSPT	Dow Jones Transportation
FRUSSL2	Russell 2000
NASA100	Nasdaq 100
NASCOMP	Nasdaq Composite
NYSEALL	NYSE Composite
S&PCOMP	S&P 500 Composite
WILEQTY	Dow Jones Wilshire 5000 Composite

Goldfarb and Iyengar's "separable" uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$ is built as in Sect. 2 of [5,6] with $\tilde{\omega} = \omega^{1/n}$. The robust portfolios are then obtained by solving the RMRAR models with these uncertainty sets, and they are held constant for the investment at each period t.

Since a block of data of length $p = 90$ is required to construct uncertainty sets or estimate the parameters, the first investment period indexed by $t = 1$ starts from $(p + 1)$th day. The time period 25 July 2002–10 May 2006 contains 11 periods of length $p = 90$, and hence in all there are 10 investment periods. Given a sequence of portfolios $\{\phi^t\}_{t=1}^{10}$, the corresponding *overall wealth growth rate* is defined as

$$\Pi_{1\le t\le 10}\left[\Pi_{tp\le k\le (t+1)p}(e + r_k)\right]^T \phi_r^t - 1,$$

and the *average diversification number* is defined as $\sum_{t=1}^{10} I(\phi^t)/10$, where $I(\phi^t)$ denotes the diversification number of the portfolio ϕ^t.

We now report the performance of the RMRAR models with our "joint" uncertainty set $\mathcal{S}_{\mu,v}$, and Goldfarb and Iyengar's "separable" uncertainty set $\mathcal{S}_m \times \mathcal{S}_v$ as the risk aversion parameter θ ranges from 0 to 10^4. The computational results for the confidence level $\omega = 0.05, 0.50, 0.95$ are shown in Fig. 2 that consists of three group of plots. In each of these groups, the left plot is about the average diversification number of robust portfolios, and the second plot is about the overall wealth growth rate over next 10 periods of the investment using robust portfolios. We observe that our robust portfolio is fairly diversified, but Goldfarb and Iyengar's is highly non-diversified. Also, the overall wealth growth rate of the investment based on our robust portfolio is higher than that using Goldfarb and Iyengar's robust portfolio.

The realization cost is another natural concern for any investment strategy. We next compare the cost of implementing the above investment strategies. For a sequence of portfolios $\{\phi^t\}_{t=1}^{10}$, its *average transaction cost* is defined as $\sum_{t=2}^{10} \|\phi^t - \phi^{t-1}\|_1/9$ (see also the discussion in [4]). In Fig. 3, we report the average transaction costs of the investments using the robust portfolios for the confidence levels $\omega = 0.05, 0.50, 0.95$, respectively. We observe that the investment based on our robust portfolio incurs lower average transaction cost than that using Goldfarb and Iyengar's robust portfolio.

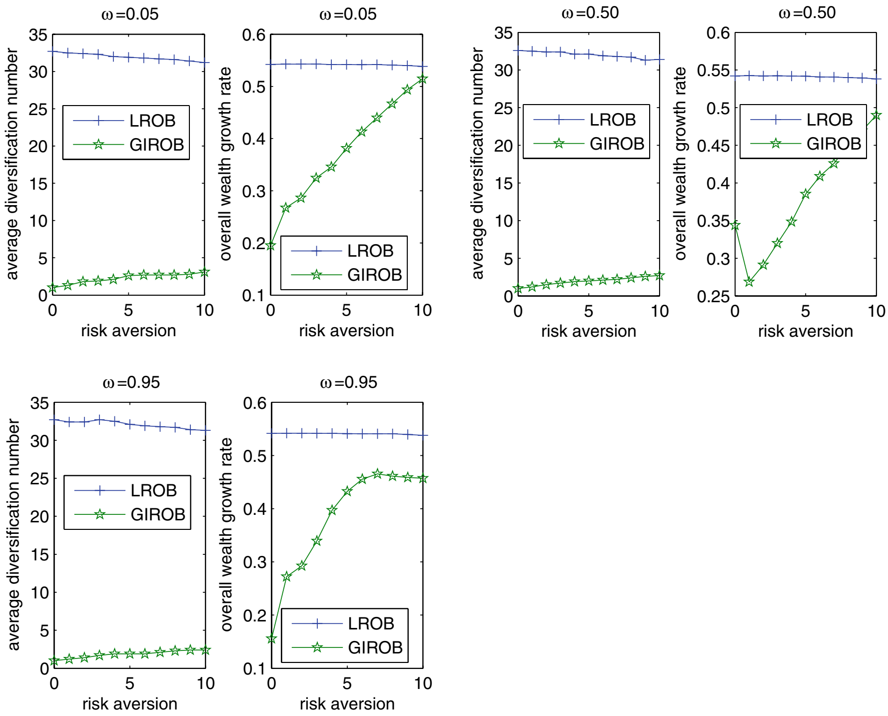

Fig. 2 Performance of portfolios for $\omega = 0.05$, 0.50 and 0.95

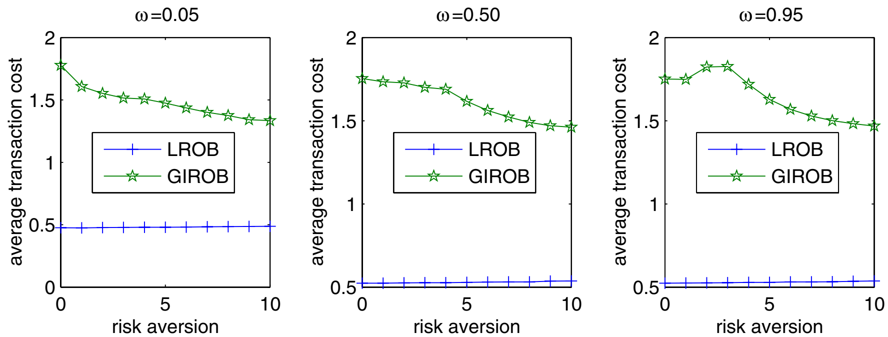

Fig. 3 Average cost of portfolios for $\omega = 0.05$, 0.50 and 0.95

3 Concluding remarks

In this paper, we conducted computational experiments to compare the performance of the RMRAR model with our "joint" uncertainty set and Goldfarb and Iyengar's "separable" uncertainty set. We observed that the RMRAR model with our uncertainty set is usually less conservative than that based on Goldfarb and Iyengar's. In particular, our robust portfolio outperforms Goldfarb and Iyengar's in terms of wealth growth rate and transaction cost. In addition, our robust portfolio is fairly diversified, but Goldfarb and Iyengar's is highly non-diversified. Though we only considered the RMRAR model in this paper, we expect that the similar phenomenon can also be observed in other robust portfolio selection models, e.g., robust maximum Sharpe ratio and robust value-at-risk models (see [4]).

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