Assignment 1

29 January 2021 Due: 12 February, 2021

Part I : Theoretical Exercises

Question 1 (Markowtiz Efficient Frontier, (**15 marks**)). The minimum variance portfolio problem

$$\min_x \frac{1}{2} x^\top \Sigma x \quad \text{s.t.} \quad e^\top x = 1, \ \mu^\top x = R$$

where the target return must be achieved exactly, short-selling is allowed, and the covariance matrix is assumed to be positive definite. Here, e is the n-dimensional vector of all ones. Assume that there are at least two assets whose mean returns are not equal to each other.

Denote three scalars $A = e^\top \Sigma^{-1} e$, $B = \mu^\top \Sigma^{-1} e$ and $C = \mu^\top \Sigma^{-1} \mu$.

1. Prove that $AC - B^2 = \left(\frac{B}{C}\mu - e\right)^\top \Sigma^{-1}(B\mu - Ce)$. (**4 marks**)

2. Prove that $AC - B^2 > 0$. (**4 marks**)

   Hint: You may need the fact that a positive definite matrix has an inverse which is also positive definite

Let x_R^* be the minimum variance portfolio and let $\sigma_R^2 = {x_R^*}^\top \Sigma x_R^*$ be the minimum variance. The expected value of the portfolio is $\mu^\top x_R^*$, which is equal to R.

3. Prove that for every R, the point (σ_R^2, R) lies on the hyperbola $\frac{\sigma_R^2}{1/A} - \frac{(R-B/A)^2}{(C/A-B^2/A^2)} = 1$ in the (σ, R)-plane. (**4 marks**)

4. Explain why the efficient frontier is produced by all portfolios x_R^* having $R \in \left[\frac{B}{A}, \infty\right)$. (**3 marks**)

Question 2 (Coherent Risk Measures, (**10 marks**)). Let X be a random variable.

1. Prove that $\mathbf{MAD}[X]$ is positive homogenous and subadditive but not translation equivariant. (**6 marks**)

2. Prove that the following risk measure is positive homogenous, subadditive and translation equivariant for any $\delta > 0$ (**5 marks**)

$$\rho_\delta(X) := \mathbf{E}[X] + \delta\,\mathbf{MAD}[X].$$

3. It can also be shown that $\rho_\delta(X)$ is monotone for $\delta < 1/2$, thereby making it coherent. Analyse whether ρ_1, which has $\delta = 1$, is monotone by considering two random variables: a constant $X = 1$ and the Bernoulli r.v. $Y \in \{0, 1\}$ with $\mathbf{Pr}[Y = 1] = p \in (0, 1)$. (**4 marks**)

Part II : Computational Exercises

Numerical answers to be submitted to the Online Assignment on Gradescope upto a precision of 4 decimal places. Mathematical models and code to be submitted along with Part I

Question 3 (Type-B Arbitrage, (**5 marks**)). Determine whether there is a type-B arbitrage opportunity given the following five risky securities and assuming the risk-free interest rate is 2.5%.

Security	Spot price	Value at maturity						
		ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
1	1.30	4	3	6	2	7	5.5	6.25
2	3.56	7	9	1	5	4	6	3
3	2.49	3.5	4	1	7.2	3.15	2	4.5
4	5.07	6	7	2	2.5	3	8	4
5	4.11	2.25	1	4	5	3	6.1	4.2

Question 4 (Portfolio Optimization, (**20 marks**)). Download the file `indices.csv` from Learn. It has mostly closing values for 7 leading stock market indices, Dow-Jones (USA), FTSE (UK), Dax (Germany), CAC (France), Nikkei, HSI (Hongkong), BOVESPA (Brazil) as well as monthly prices for Gold for the 8 years from January 2008 to January 2016.

1. Compute the covariance matrix and state the variance for the Nikkei index. (**4 marks**)

2. What are the smallest and largest values of the target returns which seem sensible, i.e., beyond which the minimum variance portfolio model will not change its answer ? (**2 marks**)

3. Plot the efficient frontier and composition of optimal portfolio for the above range of target return values. State the optimal portfolio for a 0.24% return rate. (**6 marks**)

4. Replace variance with mean absolute deviation as the risk measure in the objective and solve your model for a target return rate of 0.24%. State the optimal portfolio. (**5 marks**)

   You can use the `norm` or `abs` function inside Matlab to model MAD

5. For the optimal portfolio in part (4), state the values of its standard deviation, mean absolute deviation, and semi deviation. (**3 marks**)