Delta hedging under Black-Scholes model
Derivative pricing: Group Project 2.1

Bingran Yu(bingran.yu22@imperial.ac.uk, 02271794)
Zhihao Xu(william.xu22@imperial.ac.uk, 02274977)
Zhaozhi Liu(zhaozhi.liu22@imperial.ac.uk, 01977539)
Zhengnan Dong(zhengnan.dong22@imperial.ac.uk, 02298872)

Contents

1 Introduction

Volatility is considered the most important characteristic describing the macro price movement of an asset. If there is a series of historical price data, the actual volatility, also called realized volatility, can be calculated by simply taking the standard deviation of the sample. On the other hand, there is another type of volatility known as implied volatility. It estimates the volatility implied by the options market price after making certain model assumptions (such as the Black-Scholes setting) for derivative pricing. Implied volatility is often described as a forward-looking measure, as it reflects the market's expectation of the future movement of an asset's price, rather than its historical movement. Implied volatility and realized volatility may converge under certain conditions, where market expectations about the future are consistent with historical occurrences.

The volatility smile phenomenon, nevertheless, implies that in the real world, given the Black-Scholes model, realized and implied volatility generally have different values. This situation creates some kind of profit opportunity. Suppose an investor believes that stock prices follow Black-Scholes dynamics and observes greater actual volatility than implied volatility. In that case, he or she will profit on average from a portfolio that involves the option and corresponding delta hedging strategy. Choice of volatility used in hedging is worth studying in PnL realization and strategy risk management.

There is much related literature addressing volatility hedging. Section 2 briefly goes through those, and one of the papers by *Ahmad and Wilmott*(1) is the main article for reference. This paper illustrates expected profit and variance of profit under different hedging strategies from both theoretical and simulation perspectives. The project basically sorts the theoretical illustration in a more general and notation-friendly way in Section 3, as well as correcting some mistakes in formulas. Furthermore, a dynamic delta-hedging simulator is built with generic hedging volatility input, in order to replicate the performance of different delta-hedging strategies. Section 5 presents the simulation results without constraint concerns. Values and statistical properties of simulated path-dependent PnLs are compared between different strategies. Intuitively, you will get different return and risk evaluations, whichever volatility is used for hedging. Instead of extending the study from a single option to a portfolio scope like *Ahmad and Wilmott* (1) did, this project further investigates delta hedging by adding more realistic settings, such as considering the possible exit midway in Section 6 and drawdown limits in Section 7.

2 Literature Review

Making money from volatility arbitrage has been extensively focused on and studied for a long time. There are many thousands of papers on forecasting volatility using either complex or simple statistical techniques, designed to help one exploit mispricings in derivatives and profit from volatility arbitrage. There is a similar order of magnitude of papers on various volatility models calibrated to vanilla option prices and used to price fancy exotic contracts. However, some of the papers may not apply the volatility forecasts to options and how to profit from that volatility arbitrage via delta hedging. Some need to check further the accuracy of the volatility models against data for the underlying.

Therefore, in the referred paper, the authors addressed how to make money from volatility arbitrage by returning to the basics and keeping the model and analysis simple. They focused on analyzing different delta-hedging strategies in a world of constant volatility. This paper examined the profit to be made from hedging options that are mispriced by the market, which is the subject of how to delta hedge when your estimate of future actual volatility differs from that of the market as measured by the implied volatility (*Natenberg*)(2). Furthermore, using different volatilities (implied volatility, actual volatility, and something in between) may also cause different impacts in the classical delta formula (*Black and Scholes*)(3).

The referred paper is mainly based on the excellent work of *Carr*(4) and *Henrard*(16). Carr used various volatilities to derive the expression for profit from hedging. Henrard arrived at these conclusions and ran simulations to examine the statistical characteristics of possible path-dependent profits. Additionally, he made crucial observations that asset growth rate contributes to the gain of hedging portfolios. The referred paper extended their analyses in several directions. It explored the joint effect of drift parameter and strike price on the expected profit, and found that higher growth rate relates to a higher strike at the maximum. Furthermore, it combined different call and put options into a portfolio, and applied results on portfolio optimization, which absolutely depends on risk/return profiles with particular hedging volatility.

The two papers are also widely used as a basis for conducting research. For example, *Wang and Chen*(7) used empirical evidence from the Chinese stock market to conclude volatility forecasting and options trading strategy. *Liu and Li*(8) finished forecasting realized volatility using different time scales. *Zhang and Sun*(9) further analyzed the impact of option implied volatility on stock market volatility. *Lee and Kim*(10) brought new insights from realized range-based volatility measures. *Lee and Chen*(11) compared different statistical techniques for forecasting volatility in financial markets. *Johnson*(12) also searched mispricings in the options market and implications for volatility arbitrage.

There is also other relevant literature in this area. For instance, the paper by *Carr and Verma*(4), which elaborates on the issue of hedging using implied volatility but with implied volatility varying stochastically, is another piece of relevant research in this field. The benefits of using volatility hedging are discussed by *Dupire*(13) based on the realized quadratic variation in the stock price. *Dupire*(13) also showed how to balance the pros and cons of hedging with actual and implied by hedging with another volatility altogether. *Forde*(14) presented related concepts in relation to Asians and the hedging of barrier options.

3 Theoretical Illustration

Theoretical derivation with different choices of volatility used in the hedging strategy (denoted as σ_h) is developed by deriving the formula of PnL. The following parameters (Table 1) are used throughout this report, and the derivation is based on the paper by *Henrard*(16).

Table 1: Declaration of parameters

Parameter	Definition
S_t	Stock price at time t
K	Pre-fixed strike price
r	(Continuously compounded) risk-free interest rate
q	(Continuously compounded) dividend yield rate
μ	Growth rate/drift
$\sigma_a, \sigma_i, \sigma_h$	Actual volatility, implied volatility, volatility used in hedging
T	Expiry time (equals to time to maturity if $t = 0$)
C^a, C^i, C^h	Option price at time t calculated by σ_a, σ_i, σ_h
V^h	Value of portfolio that constructed for hedging purpose
$\Delta^a, \Delta^i, \Delta^h$	Delta (derivative of C with respect to S_t) at time t calculated by σ_a, σ_i, σ_h
Θ^h	Theta at time t calculated by σ_h
Γ^h	Gamma at time t calculated by σ_h
W	Brownian motion
D	Discounted process

Consider an option with payoff $C(S_t, T; \sigma_i)$ having the expiry T and strike price K. Denote C_t to be the value of this option at time t. Under the Black-Scholes model, this option is priced by its implied volatility σ_i: $C_t := C_t^i$. Suppose that S_t follows a geometric brownian motion in real market,

$$dS_t = (\mu - q)S_t dt + \sigma_a dW_t$$

and thus, we can define $C_t = C(t, S_t)$, i.e. a function of t and S_t by Markov property of S_t. At the meantime, we hedge this option using a delta based on σ_h. The time t value of the portfolio created for hedging purposes has the notation V_t^h. Denote the profit and loss by PnL at time $t \in [0, T]$ of the hedged option using the Δ^h portfolio, we have

$$-\mathrm{PnL}_t = V_t^h - C^i(t, S_t)$$

Now we use discounted value to simplify the computation, where an uppercase D represents the discounted value:

$$-D\mathrm{PnL}_t = DV_t^h - DC^i(t, S_t)$$

The discounted value of portfolio is

$$DV_t^h = \varphi_t + \phi_t DS_t$$

where (φ_t, ϕ_t) is a self-financing strategy. More specifically, define $\phi_t := D\Delta^h$. Thus, with the consideration of dividend payment rate q, the self-financing condition with dividend is:

$$DV_t^h = DV_0^h + \int_0^t q\phi_z DS_z dz + \int_0^t \phi_z dDS_z \tag{1}$$

By Feynman-Kac theorem, we know that $DC^h(t, S_t)$ satisfies the PDE

$$D\Theta^h - qDS_t D\Delta^h + \frac{1}{2}DS_t^2\sigma_h^2 D\Gamma^h = 0$$

Applying Ito's formula to $DC^h(t, S_t)$, and assume $\sigma = \sigma_h$, we have

$$\begin{aligned} d(DC^h(t, S_t)) &= D\Theta^h dt + D\Delta^h dDS_t + \frac{1}{2}D\Gamma^h\sigma_a^2 DS_t^2 dt \\ &= D\Delta^h(qDS_t dt + dDS_t) + \frac{1}{2}DS_t^2(\sigma_a^2 - \sigma_h^2)D\Gamma^h dt \end{aligned}$$

Integrating at both sides, we have

$$\int_0^t \phi_z dDS_z = DC^h(t,S_t) - DC^h(0,S_0) - \int_0^t \phi_z qDS_z dz - \int_0^t \frac{1}{2} DS_z^2(\sigma_a^2 - \sigma_h^2) D\Gamma^h dz \quad (2)$$

Combining 2 with self-financing condition 1:

$$-D\mathrm{PnL}_t = DV_0^h - DC^i(t,S_t) + DC^h(t,S_t) - DC^h(0,S_0) + \int_0^t \frac{1}{2} DS_z^2(\sigma_h^2 - \sigma_a^2) D\Gamma^h dz$$

$$\implies -\mathrm{PnL}_t = e^{rt}(V_0^h - C^h(0,S_0)) + C^h(t,S_t) - C^i(t,S_t) + \int_0^t \frac{1}{2} S_z^2(\sigma_h^2 - \sigma_a^2) \Gamma^h dz$$

Notice that, V_0^h is the option price at $t = 0$, which means that it equals to the value of option priced by actual volatility σ_a at $t = 0$, though it has Superscript h on it since we want to create a strategy to hedge this kind of option. Hence we can have a final form for PnL:

$$\mathrm{PnL}_t = -e^{rt}(C^i(0,S_0) - C^h(0,S_0)) - C^h(t,S_t) + C^i(t,S_t) + \int_0^t \frac{1}{2} S_z^2(\sigma_a^2 - \sigma_h^2) \Gamma^h dz \quad (3)$$

We have found that the equation (2) in original paper is wrong: it forgets the part $e^{rt}(C^i(0,S_0) - C^h(0,S_0))$ and equation(2) is -PnL not PnL. Notice that equation 3 can also expressed in differential form:

$$d\mathrm{PnL} = \frac{1}{2}(\sigma_a^2 - \sigma_i^2) S^2 \Gamma^i dt + (\Delta^i - \Delta^h)((\mu - r + D) S dt + \sigma_a S dW_t)$$

- When $\sigma_h = \sigma_a$, $\mathrm{PV}(\mathrm{PnL_T}) = C(S_t, T; \sigma_a) - C(S_t, T; \sigma_i)$.
- When $\sigma_h = \sigma_i$, $\mathrm{PV}(\mathrm{PnL_T}) = \frac{1}{2}(\sigma_a^2 - \sigma_i^2) \int_t^T e^{-r(s-t)} S^2 \Gamma^i ds$.

4 Simulator Construction

This project uses the idea of Monte Carlo simulation and builds a simulator in Python to see how profits are realized under different settings. Our simulator allows generic volatility inputs that the user can change the realized, implied and hedging volatilities to get different strategy performances under the corresponding market situation. Note that the hedging volatility is constrained between realized and implied volatilities. In the later discussion of results, all simulations with hedging are carried out with 30% actual volatility and 10% implied volatility.

Other simulator inputs include parameters that influence the underlying asset price dynamics, such as initial price S_0 and drift μ. Therefore, it is capable of testing PnL realizations with a changing drift by this simulator. Information of the vanilla option, like time to maturity T and strike price K, is also part of the simulator's input. Default values are set to be $T = 1$, $K = 100$.

There is a separate function to generate the performance that does not hedge at all(which corresponds to the no hedging strategy). This function has the same inputs as the simulator but excludes the hedging volatility.

When adding hedging constraints, the generic simulator remains unmodified, and extra functions are written on this basis. For both midway exit and limited drawdown cases, portfolio paths are generated by the simulator at first and then crossed out if they do not satisfy particular constraints when calculating the PnL performance.

5 Discussion 1: PnL Without Hedging Constraints

5.1 Results and discussion between strategies

This section presents the simulated results under different hedging strategies. To be specific, there are four strategies of interest in total: (A) hedging using $\sigma_i = 10\%$; (B) hedging using something in between σ_h; (C) hedging using $\sigma_a = 30\%$; (D) do not hedge. In strategy (B), hedging volatility is chosen to have equidistant values $\sigma_h = 15, 20, 25\%$ for comparison. Other parameters are fixed in simulations: interest rate $r = 0$, initial price $S_0 = 100$, and drift $\mu = 0.1$. Under each situation, the simulator generates $N_{path} = 10000$ paths and takes average value for expected PnL. In each path, time to maturity is divided into $N_{interval} = 1000$ sub-intervals, where at each grid hedging position is recalculated. Table 2 summarizes all results of expected PnL and variance of PnL at maturity $T = 1$ for all strategies.

Table 2: Simulated performances of hedging strategies without constraints

Hedging Strategy	**Value of σ_h**	$\mathbf{E[PnL_T]}$	$\mathbf{Var[PnL_T]}$
Hedging using σ_i	10%	7.949240	20.661090
Hedging using something else in between	15%	7.948893	9.192208
	20%	7.944231	3.279438
	25%	7.948760	0.761217
Hedging with σ_a	30%	7.939477	0.113254
Do not hedge	N.A.	14.546687	701.807327

From the above table, there is a significant decrease in the variance of profit at maturity with the increased value of hedging volatility. By utilizing a realized volatility hedge, we are essentially replicating a correctly priced short option position. The returns of the long option and the replicated short option will offset each other, resulting in a profit that is equal to the difference between the Black-Scholes option prices derived from the realized and implied volatilities. In theory, by hedging with realized volatility, we are able to lock in profits

$$C(\sigma_a, T) - C(\sigma_i, T) \approx 7.9358$$

Indeed, as stock prices fluctuate, option prices will also change. A portfolio hedged using realized volatility cannot fully eliminate this randomness. Consequently, there is a possibility of incurring losses while holding this portfolio, which is not ideal from a risk management perspective. It is crucial for investors to be aware of the potential risks and uncertainties associated with such a strategy and to consider other risk management techniques in conjunction with this hedging approach to minimize potential losses.

Now, consider the corresponding alternative situation: if we hedge using implied volatility, we adjust the marked-to-market price fluctuations of the held options based on the stock price volatility. In conjunction with the previous theoretical results, this value change will be non-decreasing, meaning that, under ideal conditions without considering transaction costs and price impacts etc., the value of our hedged portfolio will almost never incur losses. However, the variance of our expected returns will be significantly large (see 2), this is due to its highly path-dependency. Based on prior theoretical derivations, the value dynamics of a portfolio hedged with implied volatility does not contain diffusion terms, making it a better approach from a risk management perspective. The trade-off, however, is that we can no longer lock in profits.

Additionally, compared to hedging with realized volatility, an advantage of using implied volatility is that we do not need to know the actual volatility. We only need to know the relationship between the magnitudes and adopt corresponding actions accordingly.

When hedging with the volatilities between realized and implied, the variance PnL tends to increase as the volatility decreases. Furthermore, the drawdown of the hedged portfolio's value becomes progressively smaller. This implies that as the chosen volatility decreases, the risk associated with the portfolio may rise, but the fluctuations in the portfolio's value will be more contained. It is crucial for investors to carefully assess their risk tolerance and choose an appropriate hedging strategy accordingly to maintain a balance between risk and return. Therefore, choosing the hedging strategy indeed depends on investors' risk preference.

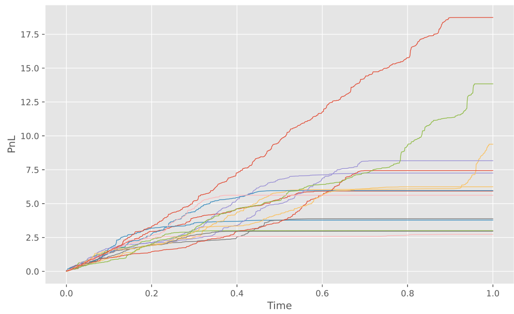
Figure 1 shows how the values of portfolios in four strategies move with respect to time to maturity. Under each set, 15 paths are simulated with 1000 sub-intervals, i.e., $N_{path} = 15$, $N_{interval} = 1000$. These graphs give an intuitive display that the higher volatility is used in delta hedging, the larger fluctuations occur in the value dynamics of the portfolio.

After describing the distribution of realized PnLs by statistics, including their expectation and variance in Table 2, another way is to plot the histograms (see Figure 2). Each histogram collects 10000 paths. When increasing the hedging volatility from implied one ($\sigma_i = 10\%$) to actual volatility ($\sigma_a = 30\%$), there is a decrease in the skewness of the PnL distribution, which implies a more centered distribution around its expectation.

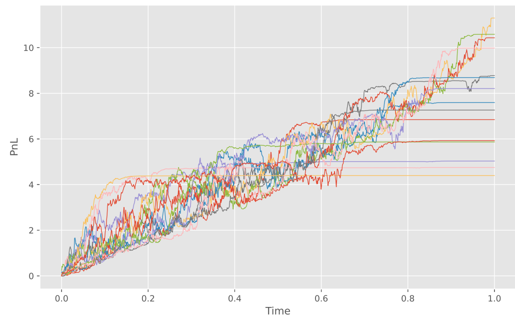
5.2 Comparison between theoretical and simulated results

As what we derived in Section 3, when $\sigma_h = \sigma_i$, $\mathrm{PV}(\mathrm{PnL_T}) = \frac{1}{2}(\sigma_a^2 - \sigma_i^2)\int_t^T e^{-r(s-t)} S^2 \Gamma^i ds$. Set $t = 0$, and theoretically, it is practical to compute the integral after stock price simulations at each intermediate point. Therefore, Figure 3 is drawn with 15 generated paths. Comparing with (a) in Figure 1, these two graphs have similar trends, but the one from the theoretical integral has smoother curves. This discrepancy is due to the implementation of discrete hedging, while the theoretical results are based on continuous hedging formulas. Our simulator operates on daily marked-to-market valuations and carries out discrete hedging accordingly.

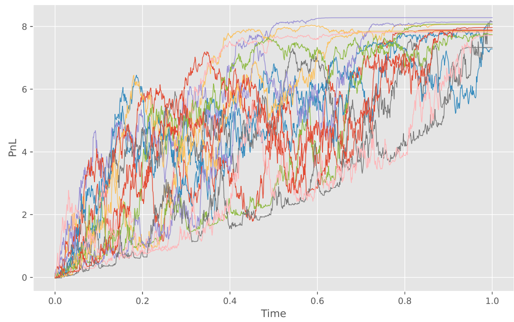
When $\sigma_h = \sigma_a$, as we can see from Figure 4, the theoretical result give us a locking profit when hedging using actual volatility σ_a at $t = T$. However, we can only see that the PnL_T oscillates around 7.9358 when we hedge discretely, which may influence whether we choose



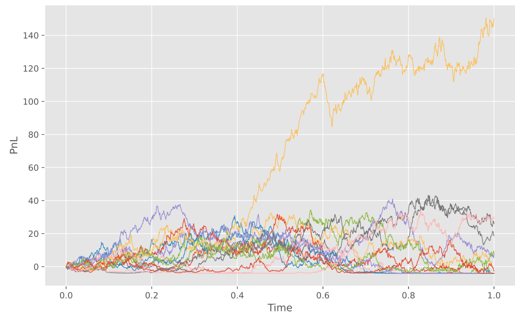
(a) $\sigma_h = \sigma_i = 10\%$ volatility hedging



(b) $\sigma_h = 20\%$ volatility hedging

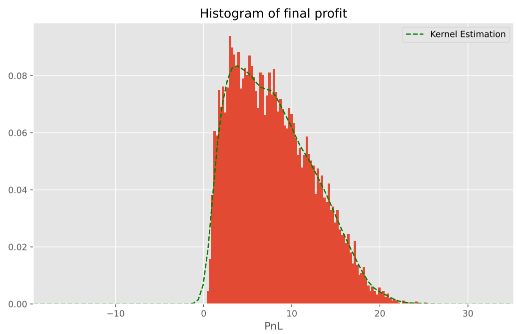


(c) $\sigma_h = \sigma_a = 30\%$ volatility hedging

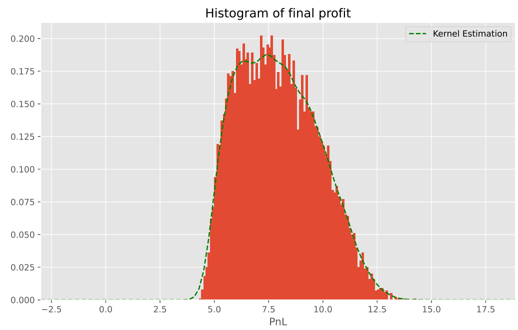


(d) strategy without hedging

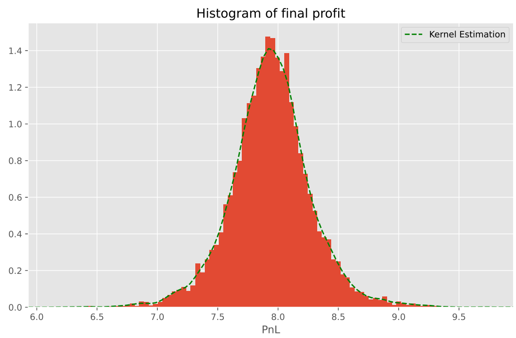
Figure 1: Simulated paths under four hedging strategies with 15 paths each



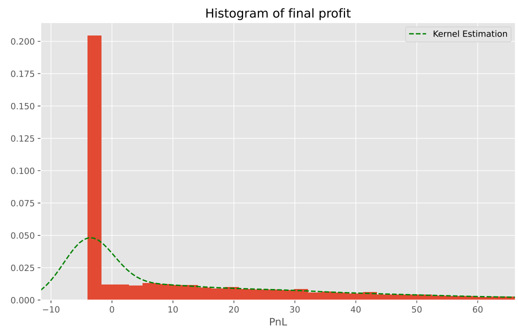
(a) $\sigma_h = \sigma_i = 10\%$ volatility hedging



(b) $\sigma_h = 20\%$ volatility hedging



(c) $\sigma_h = \sigma_a = 30\%$ volatility hedging



(d) strategy without hedging

Figure 2: Distributions of realized PnLs under four hedging strategies with 10000 paths each

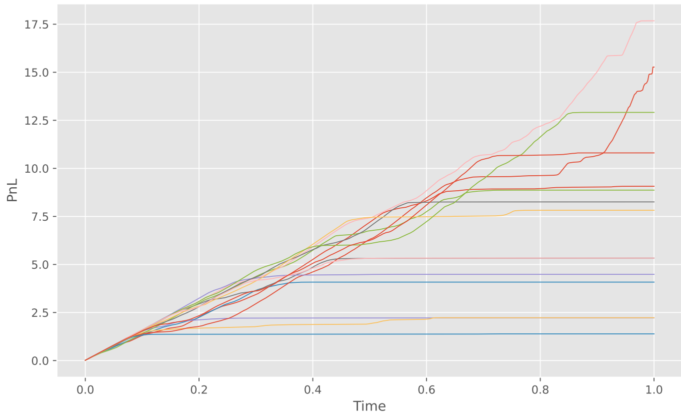


Figure 3: Theoretical paths ($N_{path} = 15$) under implied volatility hedging

this hedge portfolio. Hedging continuously in the real market is impossible. Also, it will cause many transaction costs. However, a proper understanding of the idealized case is critical to understanding the subtleties of real market applications.

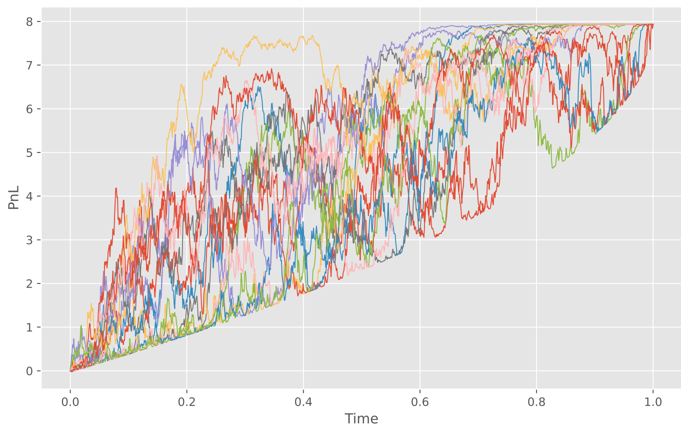


Figure 4: Theoretical paths ($N_{path} = 15$) under actual volatility hedging

Furthermore, there is a very important part $\frac{1}{2}(\sigma_a^2 - \sigma_i^2)\int_t^T e^{-r(s-t)}S^2\Gamma^i ds$ in all PnL for hedge using different σ_h, which tells us that "one part of the PnL of a hedged portfolio is its cash gamma multiplied by the square difference between realized volatility and implied volatility. To make sure we have positive PnL, we should necessarily buy the option when actual volatility is greater than implied volatility and hedge since S^2 and Γ^i are greater than 0 a.s.

5.3 Strategy performance with changing drift

Thanks to the generic simulator, after fixing the hedging volatility and other parameters, we can study the performance of path-dependent PnLs with respect to different growth rates μ. Implied volatility hedging is more interesting since *Ahmad and Wilmott* (1) have shown in their paper (Figure 4) that theoretically expected profit versus the drift μ should have a maximum. Under this subsection, verification of how expected profits perform with changing μ is carried out by simulations. μ is tested every 0.03, and each point comes from 2000 simulated paths. The theoretical $\mathrm{E}[\mathrm{PnL_T}]$, calculated from the integration, are plotted together for comparison. Parameters are $S_0 = 100$, $\sigma_h = \sigma_i = 10\%$, $\sigma_a = 30\%$, $r = 0$, $q = 0$, $K = 100$, $T = 1$.

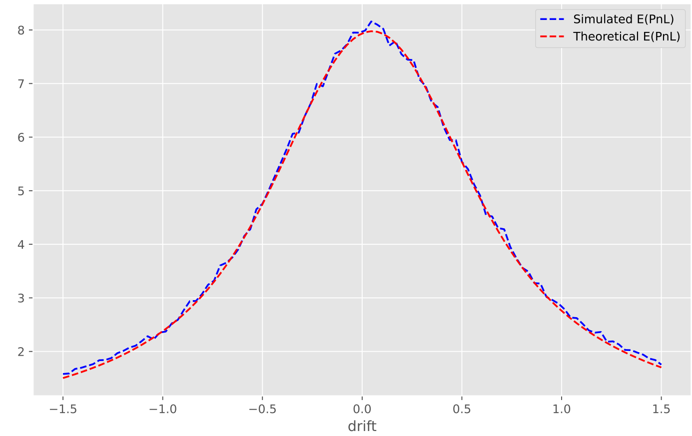


Figure 5: Theoretical and simulated expected PnL under implied volatility hedging versus drift

Figure 5 shows two nearly coincident curves, which is a pretty good validation. Also, with simple parameters, the peak is obtained around $\mu = 0$. This observation can be interpreted as the maximum expected PnL reached when the growth rate eventually ends up at a stock price close to the strike price.

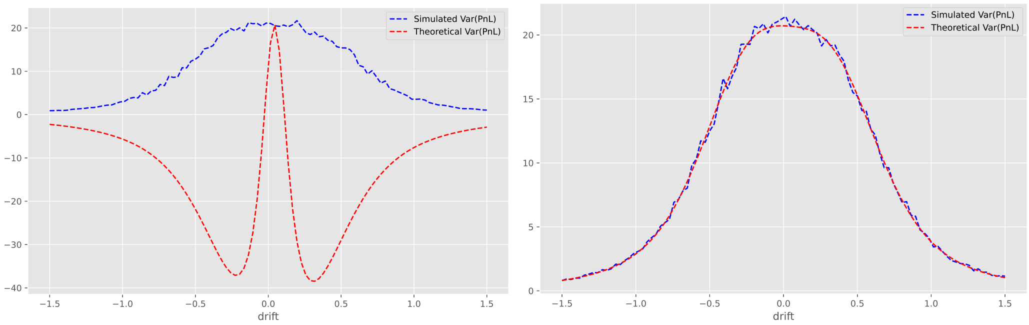
The same steps are taken to study the variance of PnL for further verification. However, to our surprise, with the variance equation given in the reference paper, the variance curve is plotted as the red line in Figure 6 (a), which has a huge discrepancy compared with the simulated results. This difference triggered us to derive the equation to see whether there is any mistake. After careful derivation, we conclude that there are sign typos in $p(u, s; S_0, t_0)$ expression of the variance equation. The correct $p(u, s; S_0, t_0)$ used in the variance of PnL with implied volatility hedging should be

$$
\begin{aligned}
&p(u, s; S_0, t_0) \\
&= -\frac{1}{2}\frac{(x-\alpha(T-s))^2}{\sigma_i^2(T-s)} - \frac{1}{2}\frac{(x-\alpha(T-u))^2}{\sigma_a^2(u-s)+\sigma_i^2(T-u)} + \frac{1}{2}\frac{\left(\frac{(x-\alpha(T-s))}{\sigma_i^2(T-s)} + \frac{(x-\alpha(T-u))}{\sigma_a^2(u-s)+\sigma_i^2(T-u)}\right)^2}{\frac{1}{\sigma_a^2(s-t_0)} + \frac{1}{\sigma_i^2(T-s)} + \frac{1}{\sigma_a^2(u-s)+\sigma_i^2(T-u)}}
\end{aligned} \tag{4}
$$

Results with the modified equation are shown in Figure 6 (b). Again, two nearly coincident curves suggest that this equation is more possible to be correct, which gives similar values as simulations.

5.4 Strategy performance with changing strike

Borrowing the same methodology in the above Section, simulations graph out how expected profits and standard deviation perform with changing strikes under the implied volatility hedging (see Figure 7). Parameters are set to be $S_0 = 100$, $\sigma_h = \sigma_i = 10\%$, $\sigma_a = 30\%$,



(a) Theoretical Var(PnL) is derived from the equation in paper

(b) Theoretical Var(PnL) is derived from the modified equation

Figure 6: Theoretical and simulated PnL variance under implied volatility hedging versus drift

$r = 0$, $q = 0$, $\mu = 0$, $T = 1$. These two metrics in Figure 7 have similar trends as in the referred paper Figure 10. Expected PnL is approximately realized at where the spot price is close to the strike.

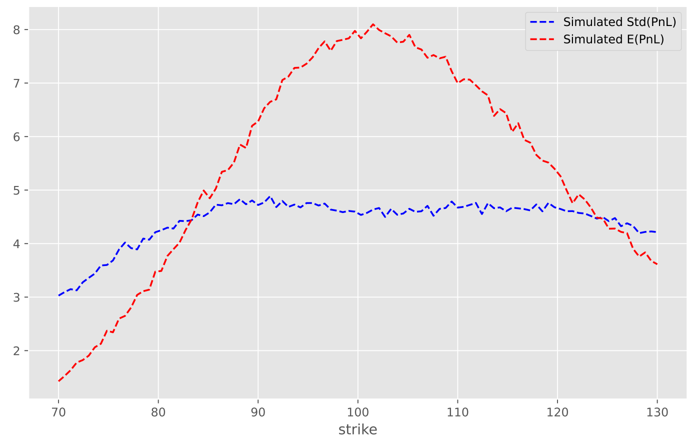


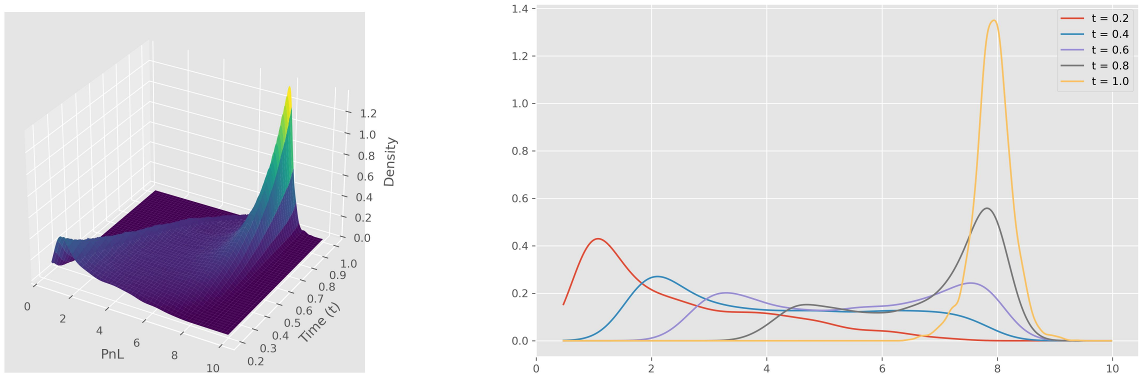
Figure 7: Simulated expected PnL and its standard deviation under implied volatility hedging versus strike

6 Discussion 2: PnL if Exiting the Position Midway

In this section, we focus on the scenarios of exiting the position midway and try to conclude the distribution of the PnL at that time under different hedging strategies: hedge using $\sigma_i = 10\%$, hedge using the $\sigma_a = 30\%$, something else in between, and do not hedge. The common parameters in the below simulations are set as $K = 100, r = 0, \mu = 0.1$.

6.1 Hedge using σ_a=30%

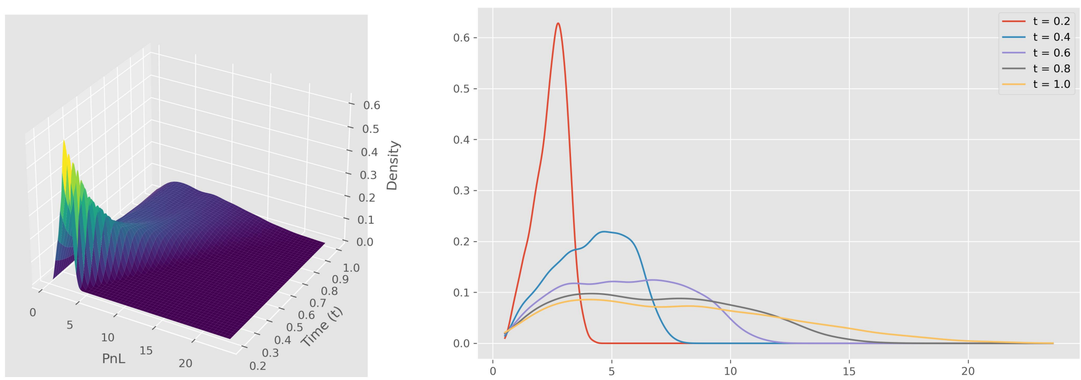
When employing actual volatility hedging, if we exit the hedging process midway, the distribution changes of our PnL can be illustrated as Figure 8a. We can observe that as the time approaches the expiration date, the mean of PnL gradually increases and stabilizes near the difference $C(\sigma_a, T) - C(\sigma_i, T) \approx 7.9358$ at the expiry.



(a) KDE density for PnL (b) Selected density for PnL of t=0.2,0.4,0.6,0.8,1

Figure 8: Distribution of PnL at different exiting time with hedging volatility = 30%

6.2 Hedge using σ_i=10%



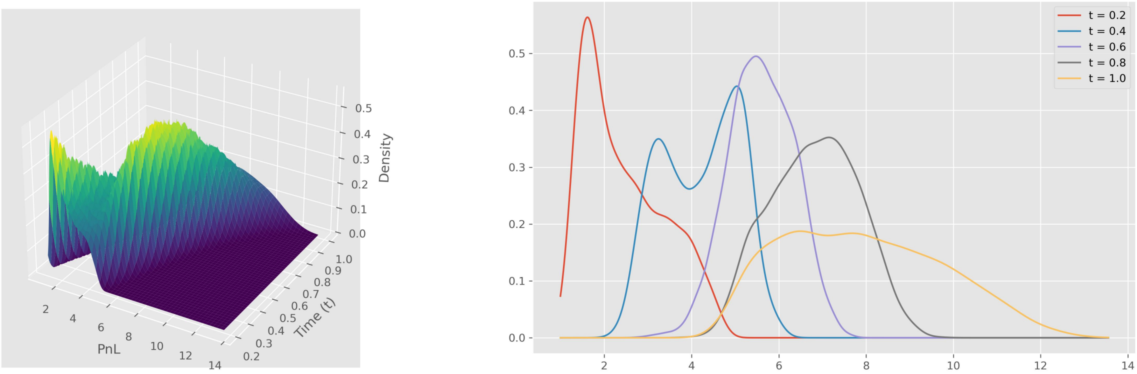
(a) KDE density for PnL (b) Selected density for PnL of t=0.2,0.4,0.6,0.8,1

Figure 9: Distribution of PnL at different exiting time with hedging volatility = 10%

When employing implied volatility hedging, the simulated PnL is initially quite concentrated. As time gradually approaches the expiration date, the variance of the PnL distribution progressively increases, exhibiting strong path dependency. The PnL distribution displays greater uncertainty compared to that of hedging with actual volatility.

6.3 Hedging with something else in between

For hedging with a volatility value lying between the implied volatility and the actual volatility, here we use $\sigma_h = 20\%$ as an example presented in Figure 10. The profit distribution under this circumstance appears to be a combination of the PnLs from hedging with implied and actual volatilities. Although the final distribution tends to stabilize, and the variance increases over time, this rate of increase is not as rapid as the rate for the implied volatility PnL.

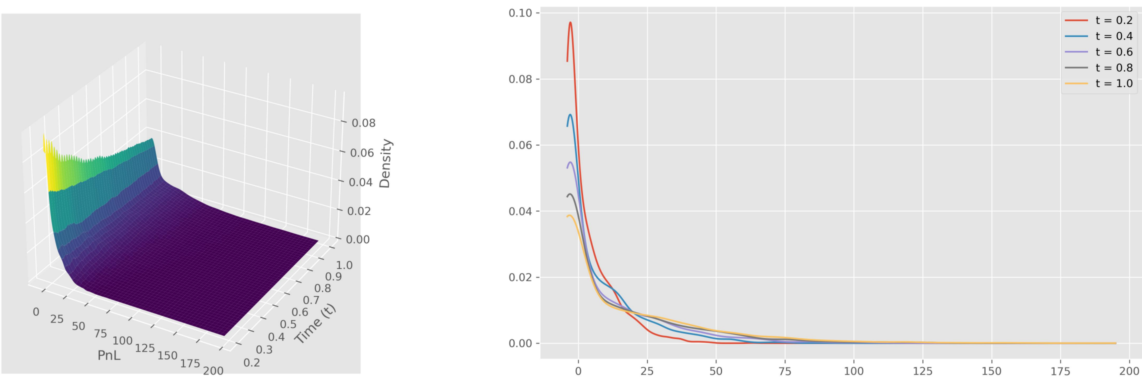


(a) KDE density for PnL

(b) Selected density for PnL of t=0.2,0.4,0.6,0.8,1

Figure 10: Distribution of PnL at different exiting time with hedging volatility = 20%

6.4 Strategy without hedging



(a) KDE density for PnL

(b) Selected density for PnL of t=0.2,0.4,0.6,0.8,1

Figure 11: Distribution of PnL at different exiting time without hedging

If choosing not to hedge, investors may obtain higher returns compared to hedging. Notice that the possible PnL can be relatively high compared with previous strategies since the value of the x-axis reaches 100. In contrast, strategies with hedging have a limited x-axis of 20 at most. Nevertheless, density also appears for negative PnLs, which means that a loss may occur if no hedging is involved. Investors would need to bear the risk of losing the principal investment though they can enjoy the probability of earning much more than other strategies.

7 Discussion 3: PnL with Drawdown Limits

In this part, extra limits on "drawdown" (i.e. on losing money at some point during the life of the position) are considered to determine which strategy is preferable. In order to introduce the concept of drawdown, we first talk about the high water mark, which is the highest value the fund has achieved in the past:

$$\mathrm{HWM}_t = \max_{s \leq t} P_s.$$

Since investors often withdraw their money when the fund drops too far below the high-water mark, an important risk measure in this context is the (relative) drawdown

$$\mathrm{DD}_t = \frac{\mathrm{HWM}_t - P_t}{\mathrm{HWM}_t},$$

where P_t is the cumulative return. Several methods are commonly used to set the threshold for drawdowns. Here we consider setting different thresholds DD by a fixed percentage. Without experience with real market manipulation, we observe the variation in PnL over $DD \in (0.5, 1)$. There are also several common strategies that market participants may adopt when a drawdown reaches a certain threshold. Here the strategy of stopping loss orders is considered for simplification. A stop-loss order is a predetermined order placed by an investor to sell a security when its price reaches a specified level. If the drawdown exceeds a predetermined threshold, the investor may use a stop-loss order to sell the investment in order to limit further losses automatically.

Then we analyze different PnL under three kinds of volatilities considering the drawdown. From Figure 12a, under the scenario of hedging with the actual volatility, the change of DD primarily influences the PnL. It is less than 1 when $DD < 1$ and sharply increases with the growth of DD. This observation is consistent with the situation we discussed earlier, knowing accurately what profit you will get at expiration, but with a wide range of variation in between. On a mark-to-market basis, you could lose before you gain.

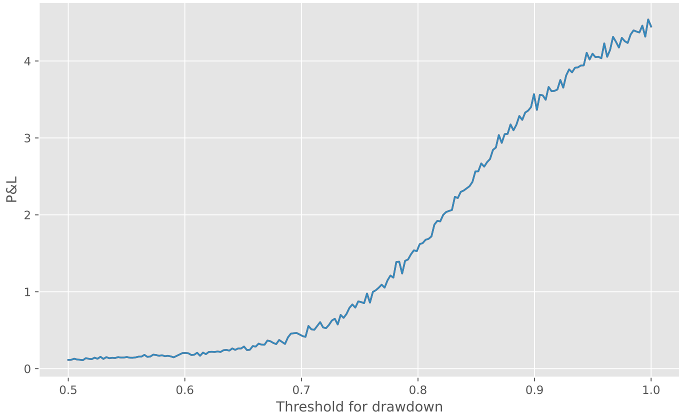
However, the situation is entirely different when hedging with implied volatility. Based on our previous theoretical analysis, by hedging with implied volatility, we are balancing the random fluctuations in the mark-to-market option value with the fluctuations in the stock price so that prices almost always trend upwards, and the drawdown limit basically do not affect the overall strategy. This conclusion is consistent with Figure 12b, where the PnL is always floating around 7, almost in line with the result of not setting a retraction threshold, which is contrary to the actual volatility.

It can be concluded from Figure 12c that The PnL will still be affected by the threshold for drawdown, but the degree of change here will be much smaller than the case of actual volatility. So for further discussion, we set DD to 0.9 and observe the change in PnL for volatility in the range 0.1 to 0.3 from Figure 13. Furthermore, the closer volatility is to actual volatility, the more PnL is affected, and the closer it is to implied volatility, the less PnL is affected.

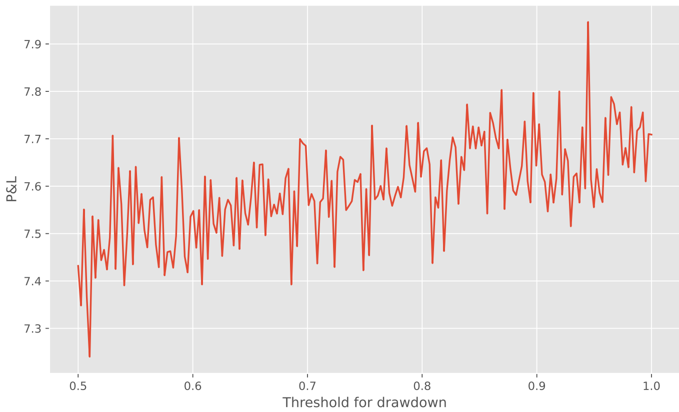
In summary, a drawdown threshold can be effectively set for hedging purposes, and employing implied volatility in such a strategy can lead to a more stable and substantial PnL. Furthermore, closer implied volatility between implied and actual volatility can also be considered, resulting in an acceptable PnL with a smaller drawdown.

8 Conclusion

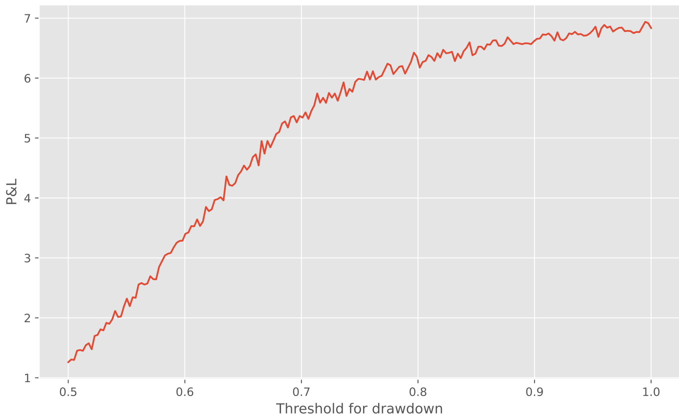
Based on the referred paper, the performance of different delta-hedging strategies is discussed and measured in this project considering three constant volatilities of interest in specific: implied, actual, and hedging volatility. A detailed theoretical illustration supporting volatility arbitrage is first given. A dynamic delta-hedging simulator is then built



(a) Actual volatility hedging



(b) Implied volatility hedging



(c) Other volatility

Figure 12: PnL under different volatilities considering the drawdown

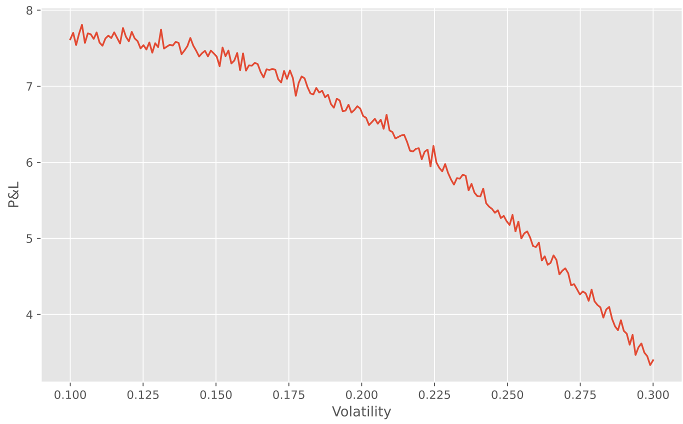


Figure 13: PnL for volatility changing between actual and implied volatilities with $DD = 0.9$

with generic hedging volatility input. Furthermore, simulation results without constraint concerns, including values and statistical properties of simulated path-dependent PnLs, are compared between different strategies. Finally, this project extends the study by adding more realistic settings, such as considering the possible exit midway and drawdown limits.

The pros and cons of hedging with each volatility can be concluded on the basis of the above research, which is shown in Table 3. In practice, the choice of which volatility to use often depends on whether one must mark to market or model. If mark-to-model is allowed ideally, then the day-to-day fluctuations in mark-to-market profit and loss may not be a concern, and actual volatility can be used for hedging to gain the exact value of profit. This approach is typically close to optimal in terms of expected total profit and does not have a standard deviation of final profit. However, it is common to have to report profit and loss based on market values, and it is more common to hedge based on implied volatility to mitigate the daily fluctuations in PnL.

To extend the study by adding more realistic settings and trying to find the optimal volatility in real-world scenarios, such as considering the possible exit midway and drawdown limits. When exiting the midway, the mean of the PnL gradually increases and stabilizes as the time approaches the expiration date when employing actual volatility hedging. At the same time, the PnL distribution displays stronger path dependency and more significant uncertainty when utilizing implied volatility hedging. Furthermore, with drawdown limits, using implied volatility in hedging strategies offers a distinct advantage in mitigating the risks associated with excessive drawdowns that may result in stop strategies. The absence of excessive volatility reduces the exposure to such drawdowns, thus enhancing the PnL of the hedge. Nevertheless, it is crucial to note that implementing any hedging strategy in actual trading involves multiple considerations, and each case warrants a meticulous analysis based on its unique merits.

Table 3: Pros and Cons using different volatility to hedge

Choices of volatilities	**Pros**	**Cons**
Actual volatility	Know the exact value of the profit at expiration	The daunting fluctuations of PnL during the life of the option
Implied volatility	- No local fluctuations in PnL - Only need to be on the right side of the trade to profit. - Implied volatility is easy to observe	Unknown value of the profit at expiration
Volatilities between implied and actual	Quantifying the 'local' risk (the daily fluctuations in PnL), optimal volatility may be derived	

9 Bibliography

[1] Riaz, Ahmad., & Paul Wilmott, 2005 Which Free Lunch Would You Like Today, Sir?: Delta Hedging, Volatility Arbitrage and Optimal Portfolios.

[2] Natenberg, S., 1994, Option Volatility and Pricing, McGraw–Hill.

[3] Black, F., & Scholes, M., 1973, The pricing of options and corporate liabilities, Journal of Political Economy 81 637–59.

[4] Carr, P., 2005, FAQs in option pricing theory, J. Derivatives.

[5] Carr, P., & Verma, A., 2005, Potential Problems in Market Models of Implied Volatility, Working Paper.

[6] Henrard, M., 2003, Parameter risk in the Black and Scholes model, Risk and Insurance 0310002, Economics Working Paper Archive at WUSTL.

[7] Wang, L., & Chen, S., 2017, Volatility forecasting and option trading strategy: Evidence from Chinese stock market, International Review of Economics & Finance, 48, 122-135.

[8] Liu, M., & Li, J., 2018, Forecasting realized volatility using different time scales: Evidence from Chinese stock market, Journal of Forecasting, 37(3), 282-299.

[9] Zhang, Y., & Sun, W., 2019, The impact of option implied volatility on stock market volatility: Evidence from China, Applied Economics, 51(26), 2822-2835.

[10] Lee, H., & Kim, Y., 2020, Forecasting stock market volatility: New insights from realized range-based volatility measures, International Journal of Forecasting, 36(4), 1590-1605.

[11] Lee, K. & Chen, T., 2009, Comparison of statistical techniques for forecasting volatility in financial markets, Journal of Financial Research, 32(4), 55-70.

[12] Johnson, L., 2008, Mispricings in options market and their implications for volatility arbitrage, International Journal of Economics, 40(1), 120-135.

[13] Dupire, B., 2005, Exploring Volatility Derivatives: New Advances in Modelling, Presented at Global Derivatives 2005 Paris.

[14] Forde, M., 2005, Static hedging of barrier options under local volatility with r 6 = q, and model-independent hedging of Asian options. PhD thesis Bristol University.

[15] Schoutens, W., Simons, E., & Tistaert, J., 2004, A perfect calibration! Now what? Wilmott magazine, March 2004, 66–78.

[16] Henrard, M., 2003, Parameter risk in the Black and Scholes model, Working Paper.