Quantitative Finance
Model-Free Implied Volatility

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Overview

This presentation is divided into the following parts.

1. Introduction
2. Methodology
    1. CBOE Volatility Index (VIX)
    2. New Method by Fukasawa *et al.*
3. Comparison and Empirical Evidence
4. Conclusion and Further Improvement

1 Introduction

Definition

The quadratic variation of log price $\langle log(S) \rangle_T$ is defined as as:

$$\begin{aligned}
\langle log(S) \rangle_T &= \lim_{|\Delta| \to 0} \sum_{k=1}^{n} \left(log(S_{t_k}) - log(S_{t_{k-1}}) \right)^2 \\
&= \lim_{|\Delta| \to 0} \sum_{k=1}^{n} \left[log \left(\frac{S_{t_k}}{S_{t_{k-1}}} \right) \right]^2
\end{aligned}$$

where $|\Delta| = \sup_{1 \le i \le n} |t_i - t_{i-1}|$ is for the partition $0 = t_0 < t_1 < ... < t_n = T$

- Sum of the squared log-return as time interval tends to 0
- Use $\frac{1}{T}\mathbb{E}[\langle log(S) \rangle_T]$ as the measure of volatility

2 Methodology

Two different approximations of the expectation

$$\mathbb{E}[\langle log(S)\rangle_T] = -2\mathbb{E}\left[log\left(\frac{S_T}{\mathbb{E}[S_T]}\right)\right] =: -2\mathbb{E}\left[log\left(\frac{S_T}{F}\right)\right]$$

1. CBOE Volatility Index (VIX): numerical approximation
2. New Method: the model-free link in pricing variance swaps

2.1 CBOE Volatility Index (VIX)

Proposition

Let $f : (0, \infty) \mapsto \mathbb{R}$ *be a* C^2 *function and* $M > 0$*:*

$$\begin{aligned} f(S) =& f(M) + f'(M)(S - M) \\ &+ \int_M^S f''(v)(S - v)^+ dv + \int_0^M f''(v)(v - S)^+ dv \end{aligned}$$

Taking $f(S) = log(S)$ and $M = K_0$ and taking expectation gives:

$$-2\mathbb{E}\left[log\left(\frac{S_T}{F}\right)\right] = 2\int_0^{K_0} \frac{P(K)}{K^2} dK + 2\int_{K_0}^{\infty} \frac{C(K)}{K^2} dK + 2\int_{K_0}^{F} \frac{K - F}{K^2} dK$$

where $C(K) = \mathbb{E}[(S_T - K)^+)]$ and $P(K) = \mathbb{E}[(K - S_T)^+)]$ are the undiscounted call and put option prices of the asset S with strike K and maturity T, respectively.

2.1 CBOE Volatility Index (VIX)

Definition

In CBOE VIX, the volatility for a fixed expiration T is defined as:

$$\sigma^2(T) = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{rT} Q(K_i) - \frac{1}{T}\left(\frac{F}{K_0} - 1\right)^2$$

- F is the option-implied forward price level
- K_0 is the chosen ATM strike price
- K_i is the strike price of the i^{th} OTM option. The calculation of VIX at time t includes all put options with $K_i \leq K_0$ and all call options with $K_i \geq K_0$.
- ΔK_i is the interval between strike spreads.
- $Q(K_i)$ is the option price of the OTM option with strike K_i, and $Q(K_0)$ is the average of the put and call option price with strike K_0.

2.1 CBOE Volatility Index (VIX)-Algorithm

- Step 1: **Choosing near-term T_1 and next-term T_2**
- Step 2: **Selecting the options to be involved**
- Step 3: **Calculating the volatility σ_i^2**
- Step 4: **Determining VIX by linear interpolation**

$$VIX = 100 \times \sqrt{\frac{1}{T}\left[\frac{T_2 - T}{T_2 - T_1}\sigma_1^2 T_1 + \frac{T - T_1}{T_2 - T_1}\sigma_2^2 T_2\right]}$$

2.2 New Method

Definition

Let K be the strike price and $F = \mathbb{E}[S_T]$ is the forward price. Define $k := log\left(\frac{K}{F}\right)$ be the log-moneyness, i.e. $K = Fe^k$, and define the mapping:

$$k \mapsto d_2(k) := d_2(k, \sigma(k)) = -\frac{k}{\sigma(k)\sqrt{T}} - \frac{\sigma(k)\sqrt{T}}{2}$$

Definition

Define g as the inverse function of d_2 as:

$$g^{-1}(k) = d_2(k, \sigma(k))$$

2.2 New Method

Definition

The undiscounted Black-Scholes price of a European put option is defined as a function $P_{BS} : \mathbb{R} \times (0, \infty) \to (0, \infty)$ by:

$$P_{BS}(k, \sigma) := Fe^k \Phi(-d_2(k, \sigma)) - F\Phi(-d_1(k, \sigma))$$

where Φ is the cdf of $N(0, 1)$, and $d_1 := d_2 + \sigma\sqrt{T}$.

Definition

The Black-Scholes implied volatility $\sigma : \mathbb{R} \to [0, \infty)$ is defined by

$$\sigma(k) := P_{BS}(k, \cdot)^{-1}(P(Fe^k))$$

where $P(Fe^k)$ is the undiscounted price of put with strike $K = Fe^k$.

2.2 New Method

Proposition

Let $P(K) = \mathbb{E}[(K - S_T)^+]$ *and* $f_{S_T}(K)$ *be the density function of* S_T*. Then*

$$\frac{d^2 P}{dK^2}(K) = f_{S_T}(K)$$

Hence

$$\begin{aligned} -2\mathbb{E}\left[log\left(\frac{S_T}{F}\right)\right] &= -2\int_0^\infty log\left(\frac{K}{F}\right)\frac{d^2P}{dK^2}(K)dK \\ &= -2\int_{-\infty}^\infty k\frac{d^2P}{dK^2}(Fe^k)dk \end{aligned} \tag{1}$$

2.2 New Method

Proposition

If $\mathbb{E}[S_T^p] < \infty$ *for some* $p > 0$*, then:*

$$\lim_{k\to\pm\infty} \sigma(k)\phi(g^{-1}(k)) = 0 \quad \textit{and} \quad \lim_{k\to\pm\infty} k\frac{d\sigma}{dk}(k)\phi(g^{-1}(k)) = 0$$

Under condition $\mathbb{E}[S_T^p] < \infty$, using integration by part, we can get:

$$-2\mathbb{E}\left[log\left(\frac{S_T}{F}\right)\right] = \int_{-\infty}^{\infty} \sigma(g(z))^2\phi(z)dz \tag{2}$$

2.2 New Method

Theorem

Let $\hat{\sigma}(z) := \sigma(g(z))$, $\alpha_{\pm}(z; z_0) := -z \pm \sqrt{\hat{\sigma}(z_0)^2 + 2z_0\hat{\sigma}(z_0) + z^2}$.
For $z, z_0 \in \mathbb{R}$:

1. *For* $z > z_0 \geq 0$: $\hat{\sigma}(z) > \alpha_+(z; z_0) > (\hat{\sigma}(z_0) + z_0 - z)^+$
2. *For* $z_0 > z \geq 0$: $\hat{\sigma}(z) < \alpha_+(z; z_0) < \hat{\sigma}(z_0) + z_0 - z$
3. *For* $z < z_0 \leq 0$: $\hat{\sigma}(z) > \alpha_-(z; z_0)$
4. *For* $z < z_0 \leq z^*$ *where* $z^* < 0$ *such that* $\hat{\sigma}(z^*) = z^*$:
   $-z > \hat{\sigma}(z) > \alpha_-(z; z_0) > 0$

To satisfy the above characteristics of $\sigma(g(z))$, the piecewise cubic polynomial is chosen to approximate $\sigma(g(z))$ so that:

- The extrapolation scheme does not induce a rapid decay of $\sigma(g(z))^2$ as $|z| \to \infty$.
- The interpolation scheme does not produce excessive oscillations.

2.2 New Method-Algorithm

- Step 1: **Selecting the options to be involved**
  In addition, discard the option with the ratio of ask-to-bid prices greater than or equal to $c = 2$ for which the mid-quote is not reliable.
- Step 2: **Converting data & further filtering options**
  Disregard all options breaks the monotonicity of d_2
- Step 3: **Constructing an approximation of $\sigma(g(z))^2$**
  - Constant extrapolation on $(-\infty, x_1]$ and $[x_M, \infty)$
  - Piecewise cubic polynomial on $[x_i, x_{i+1}]$, $i = 1, ..., M - 1$

2.2 New Method-Algorithm

- Step 4: **Integrating**

$$\int_{-\infty}^{\infty} \sigma(g(z))^2 \phi(z) dz \approx y_1 \Phi(x_1) + \sum_{i=1}^{M-1} (a_i A_i + b_i B_i + c_i C_i + d_i D_i) + y_M \Phi(-x_M)$$

where for $i = 1, \dots M-1$, a_i, b_i, c_i, d_i are solved as the coefficients of piecewise cubic polynomials on $[x_i, x_{i+1}]$, and A_i, B_i, C_i, D_i can be computed as:

$$\begin{aligned}
A_i &:= \Phi(x_{i+1}) - \Phi(x_i) \\
B_i &:= -\left(\phi(x_{i+1}) - \phi(x_i)\right) - x_i \left(\Phi(x_{i+1}) - \Phi(x_i)\right) \\
C_i &:= -\left(x_{i+1}\phi(x_{i+1}) - x_i\phi(x_i)\right) + 2x_i \left(\phi(x_{i+1}) - \phi(x_i)\right) \\
&\quad + \left(1 + x_i^2\right)\left(\Phi(x_{i+1}) - \Phi(x_i)\right) \\
D_i &:= (1 - x_{i+1}^2)\phi(x_{i+1}) - (1 - x_i^2)\phi(x_i) + 3x_i \left(x_{i+1}\phi(x_{i+1}) - x_i\phi(x_i)\right) \\
&\quad - 3(1 + x_i^2)\left(\phi(x_{i+1}) - \phi(x_i)\right) - x_i(3 + x_i^2)(\Phi(x_{i+1}) - \Phi(x_i))
\end{aligned}$$

2.2 New Method-Algorithm

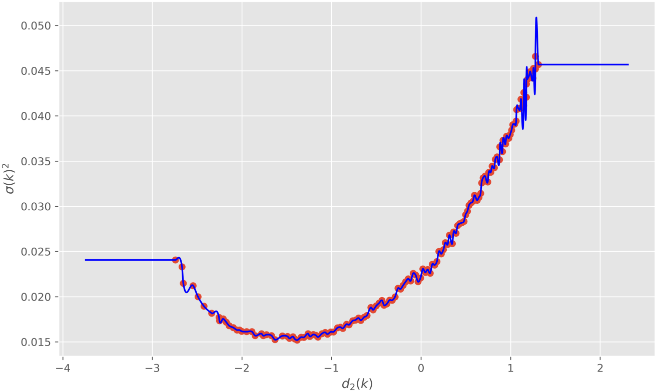

Figure: Piecewise cubic approximation of $\sigma(k)^2$

3.1 Source of approximation errors

- **Error 1:** The most obvious drawback of the CBOE approximation is that there may exist **underestimation** due to the diminished contributions of $P(K)$ when $K < K_{\text{min}}$, and $C(K)$ on $K > K_{\text{max}}$.
- **Error 2:** The second type of approximation error for the CBOE approximation is the discretization of the integral with respect to K.
- **Error 3:** CBOE procedure may not run properly when the market data is not ideal. This is because if quotes of the K_0 put option or the K_0 call option are null or the bid price is higher than the ask price

3.2 The Heston Model

$$
\begin{aligned}
\mathrm{d}S_t &= S_t\sqrt{\nabla_t}\left[\rho \mathrm{d}W_t^1 + \sqrt{1-\rho^2}\mathrm{d}W_t^2\right] \\
\mathrm{d}V_t &= \lambda\left(v - V_t\right)\mathrm{d}t + \eta\sqrt{\nabla_t}\mathrm{d}W_t^1
\end{aligned}
$$

The parameters are:

- λ mean reversion coefficient of the variance process
- v long term mean of the variance process
- η volatility coefficient of the variance process
- ρ correlation between W^1 and W^2

$$
\frac{1}{T}\mathbb{E}\left[\langle \log(S)\rangle_T\right] = \frac{1}{T}\mathbb{E}\left[\int_0^T V_t \,\mathrm{d}t\right] = v + \frac{1-e^{-\lambda T}}{\lambda T}\left(V_0 - v\right)
$$

3.3 Comparison

Strike	Call			Put		
	Ask	Theoretical	Bid	Ask	Theoretical	Bid
2000	2099.3	2100.3	2101.3	0.285	0.3	0.315
2100	1999.58	2000.58	2001.58	0.5510	0.58	0.609
2200	1900.05	1901.05	1902.05	0.9975	1.05	1.1025
2300	1800.81	1801.81	1802.81	1.7195	1.81	1.9005
2400	1702.01	1703.01	1704.01	2.8595	3.01	3.1605
2500	1603.81	1604.81	1605.81	4.5695	4.81	5.0505
...	...	...	...	...	...	...
6900	1.242	1.38	1.4490	2800.38	2801.38	2802.38
7000	1.0070	1.06	1.113	2900.06	2901.06	2902.06
7100	0.7790	0.82	0.943	2999.82	3000.82	3001.82
7200	0.5985	0.63	0.6615	3099.63	3100.63	3101.63
7300	0.4560	0.48	0.528	3199.48	3200.48	3201.48
7400	0.3515	0.37	0.407	3299.37	3300.37	3302.37

Table: Artificial data with narrow strike range and parameter set A

3.3 Comparison

Strike	Call			Put		
	Ask	Theoretical	Bid	Ask	Theoretical	Bid
200.0	3899.0	3900.0	3903.0	-0.0	0.0	-0.0
400.0	3698.0	3700.0	3701.0	-0.0	0.0	-0.0
600.0	3499.0	3500.0	3501.0	0.0	0.0	0.0
800.0	3299.0	3300.0	3301.0	0.0	0.0	0.0
1000.0	3099.0	3100.0	3101.0	0.0	0.0	0.0
1200.0	2899.0	2900.0	2901.0	0.0	0.0	0.0
...	...	...	...	...	...	...
6400.0	4.5125	4.75	4.9875	2304.75	2299.75	2306.75
6600.0	2.637	2.93	3.0765	2502.93	2501.93	2503.93
6800.0	1.691	1.78	1.869	2701.78	2699.78	2702.78
7000.0	1.007	1.06	1.113	2901.06	2900.06	2902.06
7200.0	0.5985	0.63	0.6615	3100.63	3099.63	3101.63
7400.0	0.333	0.37	0.3885	3300.37	3299.37	3301.37

Table: Artificial data with wide strike range and parameter set A

3.3 Comparison

Randomizes the bid-ask spread using **geometric random variables** with success probability $p = 0.8$.

- Skewness towards lower values
- Discrete nature
- Others

Parameter Set A: $\lambda = 1, v = 0.2, \eta = 0.5, \rho = 0.8$ and $V_0 = 0.6$;
Parameter Set B: $\lambda = 1, v = 0.2, \eta = 1.0, \rho = 0.4$ and $V_0 = 0.6$;
Parameter Set C: $\lambda = 5, v = 0.04, \eta = 1.0, \rho = 0.4$ and $V_0 = 0.6$;
Parameter Set D: $\lambda = 1.5, v = 0.04, \eta = 0.3, \rho = 0.7$ and $V_0 = 0.04$;

- A, B, and C represent the market conditions during financial crises, with instantaneous volatility $\sqrt{V_0} \approx 77\%$.
- D is typical values obtained by calibrating on a regular day with instantaneous volatility $\sqrt{V_0} = 20\%$

3.3 Comparison

Table: Expected annualized quadratic variations.

Parameter	Narrow range			Wide range		
	True	New	CBOE	True	New	CBOE
A	0.5840	0.5842	0.5576	0.5840	0.5838	0.5735
B	0.5840	0.5844	0.5577	0.5840	0.5760	0.5736
C	0.4992	0.4990	0.4755	0.4992	0.4990	0.4906
D	0.0400	0.0402	0.0364	0.0400	0.0393	0.0392

The new method is stable in both settings, while the CBOE algorithm suffers more serious **underestimation** problems in the narrow strike range.

4 Conclusion and Further Improvement

Conclusion

- The referenced study presents a novel model-free method for estimating the expected quadratic variations in asset prices.
- The new approximation method circumvents numerical integration by leveraging the integral structure concerning the standard normal density and applying polynomial interpolation to the integrated.
- The new algorithm demonstrates superior numerical efficiency compared to the CBOE procedure. The improvement is more obvious during economic and financial shocks.

4 Conclusion and Further Improvement

Application

- the dynamics of model-free implied volatility surfaces,
- the statistical properties of volatility indices,
- a robust replication theory of volatility derivatives.

Improvement

- The final result is calculated by interpolating the data from the two T's before and after the 30 days when there is option data available in the market.
- However, in our numerical tests, we found that there may be dates with few available data of options (a small range of K), reducing the accuracy of the result.
- Therefore, it may be appropriate to consider choosing another close date, which may further improve the accuracy of the algorithm.

References:

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