Financial Risk Theory - 2020/1 Project

1 Section A - Theoretical Part

Let $(\Omega, \mathcal{F}, P)$ be a probability space. We denote by $\mathbb{E}[X]$ the expectation of a random variable X. For $1 \leq p < \infty$, L^p is used to denote the usual space of p-integrable real-valued random variables.

Fix an integer $d \geq 1$. For an $\mathbb{R}^d$-valued random variable X, its law on $\mathcal{B}(\mathbb{R}^d)$ (the Borel sigma-algebra of $\mathbb{R}^d$) is denoted by $\mathcal{L}(X)$. Scalar product is denoted by $\langle \cdot, \cdot \rangle$, with $|\cdot|$ standing for the corresponding norm (where the dimension of the space may vary depending on the context).

For any integer $q \geq 1$, let $\mathcal{P}(\mathbb{R}^q)$ denote the set of probability measures on $\mathcal{B}(\mathbb{R}^q)$.

For $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, let $\mathcal{C}(\mu, \nu)$ denote the set of probability measures ζ on $\mathcal{B}(\mathbb{R}^{2d})$ such that its respective marginals are μ, ν. For two probability measures μ and ν, the Wasserstein distance of order $p \geq 1$ is defined as

$$W_p(\mu, \nu) := \inf_{\zeta \in \mathcal{C}(\mu,\nu)} \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\theta - \theta'|^p \zeta(\mathrm{d}\theta \mathrm{d}\theta') \right)^{1/p}, \ \mu, \nu \in \mathcal{P}(\mathbb{R}^d). \tag{1}$$

Consider a stochastic differential equation (SDE), which is given by,

$$\mathrm{d}X_t = -h(X_t)dt + \sqrt{2\beta^{-1}}dB_t, \qquad \forall\, t > 0, \tag{2}$$

with a deterministic initial condition $X_0 \in \mathbb{R}^d$, where β is a positive constant, $h : \mathbb{R}^d \to \mathbb{R}^d$ and $(B_t)_{t \geq 0}$ is a d-dimensional Brownian motion. Moreover, assume that:

Assumption 1 *There exists a positive constant L such that, for all $\theta, \theta' \in \mathbb{R}^d$,*

$$|h(\theta) - h(\theta')| \leq L|\theta - \theta'|\,. \tag{3}$$

Assumption 2 *There exists a constant $m > 0$ such that, for all $\theta, \theta' \in \mathbb{R}^d$,*

$$-\left\langle \theta - \theta', h(\theta) - h(\theta') \right\rangle \leq -m|\theta - \theta'|^2.$$

Suppose also that there exists a unique $\theta^* \in \mathbb{R}^d$ such that $h(\theta^*) = 0$.

Question 1 *Let $T > 0$ and $d = 1$. Prove that*

$$\sup_{t \le T} \mathbb{E}[|X_t|^2] < \infty.$$

Recall that you need to justify why the expectation of a stochastic integral is zero, every time you use this property (for stochastic integrals) since it is not always true. Moreover, if you decide to use Gronwall's lemma, you would need to justify first why the RHS (of the inequality under consideration) is finite prior to applying Gronwall's lemma.

Question 2 *Let $d = 1$. Prove that*

$$\sup_{t \ge 0} \mathbb{E}[|X_t|^2] < \infty.$$

Question 3 *Let $d = 1$. Consider also the SDE which is given by*

$$\mathrm{d}Z_t = -h(Z_t)dt + \sqrt{2\beta^{-1}}dB_t \tag{4}$$

with a deterministic initial condition $Z_0 \in \mathbb{R}^d$, which is different that X_0. Prove that

$$\lim_{t \to \infty} W_2(\mu_t, \nu_t) = 0$$

where $\mu_t = \mathrm{Law}(X_t)$ and $\nu_t = \mathrm{Law}(Z_t)$.

Question 4 *Let $d \ge 1$. Consider the Euler-Maruyama scheme for the SDE (2), which is given by*

$$Y_{(n+1)\lambda} = Y_{n\lambda} - \lambda h(Y_{n\lambda}) + \sqrt{\frac{2\lambda}{\beta}}\xi_{n+1}, \qquad \forall\, n \ge 0, \tag{5}$$

with initial condition $Y_0 \in \mathbb{R}^d$, which is the same as the initial value for SDE (2), i.e. $Y_0 = X_0$, $\lambda > 0$ and $\{\xi_n\}_{n \ge 1}$ is a sequence of independent and identically distributed (i.i.d.) standard Gaussian random variables in $\mathbb{R}^d$. Prove that

$$\sup_{n \ge 0} \mathbb{E}[|Y_{n\lambda}|^2] < \infty, \tag{6}$$

by first proving that

$$\sup_{n\geq 0} \mathbb{E}[|Y_{n\lambda} - \theta^*|^2] < \infty,$$

provided that λ *is less or equal than some constant* c*. Write down* c *in terms of* L *and* m*.*

Remark 1.1 *Note that the above representation of the Euler-Maruyama scheme* (5) *is due to the following iterative scheme*

$$Y_{(n+1)\lambda} = Y_{n\lambda} - h(Y_{n\lambda})\left((n+1)\lambda - n\lambda\right) + \sqrt{\frac{2}{\beta}}\left(B_{(n+1)\lambda} - B_{n\lambda}\right), \tag{7}$$

where the Brownian increment $B_{(n+1)\lambda} - B_{n\lambda}$ *is replaced by* $\sqrt{\lambda}\xi_{n+1}$*.*

Question 5 *Let* $d = 1$*. Consider now the continuous-time interpolation of the Euler-Maruyama scheme* (5)*,*

$$dY_t^\lambda = -h(Y^\lambda_{\kappa_\lambda(t)})dt + \sqrt{\frac{2}{\beta}}dB_t, \tag{8}$$

for any $t > 0$ *with the same initial value as before, i.e.* $Y_0^\lambda = X_0$*. Moreover,* $\kappa_\lambda(t) = \lambda\lfloor t/\lambda \rfloor$*, where* $\lfloor x \rfloor$ *returns the integer part of the real number* x*. Hence, the solutions of* (8) *and* (7) *coincide at grid points, i.e. when* $t \in \{\lambda, 2\lambda, 3\lambda, \ldots\}$ *since the integral form of* (8) *yields*

$$Y^\lambda_{(n+1)\lambda} = Y^\lambda_{n\lambda} - \int_{n\lambda}^{(n+1)\lambda} h(Y^\lambda_{\kappa_\lambda(u)})du + \sqrt{\frac{2}{\beta}}\left(B_{(n+1)\lambda} - B_{n\lambda}\right).$$

Prove that, for any $t \geq 0$*,*

$$\lim_{\lambda\to 0} W_2(\mu_t, \nu_t^\lambda) = 0$$

where $\mu_t = \mathrm{Law}(X_t)$ *and* $\nu_t^\lambda = \mathrm{Law}(Y_t^\lambda)$*. Hint: Consider the squared difference* $|X_t - Y_t^\lambda|^2$ *and apply Itô's formula.*

2 Section B - Programming & Simulation Part

Task 1 *Implement the* VaR-CVaR *algorithm from Section 5.2. of the article*

https://arxiv.org/pdf/2007.01672.pdf

with the same assumptions and parameter setting (single asset case) as per Table 1 & 2 of the aforementioned article. Your task is to reproduce Table 1 & 2 (approximately).

You need to include a (working) copy of your code for this task in your answer.

Task 2 *Implement the* VaR-CVaR *algorithm for the case of three assets, i.e. for a portfolio which consists of three risky assets, as described in subsection 5.2.2., which is entitled "Minimizing* CVaR *of portfolios of assets" of the article*

https://arxiv.org/pdf/2007.01672.pdf

by examining the following 5 different cases:

Asset 1	*Asset 2*	*Asset 3*
N(500,1)	N(0, 10^6)	N(0,10^{-4})
N(500,1)	N(0, 10^6)	N(0,1)
N(0, 10^3)	N(0, 1)	N(0, 4)
N(0, 1)	N(1, 4)	N(0,10^{-4})
N(0, 1)	N(1, 4)	N(2, 1)

at a confidence level 95% following the style of presentation of Table 3 of the aforementioned article.

You need to include a (working) copy of your code for this task in your answer.

3 Section C - Peer Evaluation Part

You would need to (peer) evaluate two (anonymous) submissions (Parts A & B only) without having access to a copy of the solutions.