Derivative pricing: Project 2.1
Delta hedging under Black-Scholes model

April 20, 2023

Overview

This presentation is divided into following parts.

1. Theoretical formula for PnL at maturity
2. PnL discussion of different strategies without constraints
3. PnL discussion if exiting the position midway
4. PnL discussion with drawdown limits
5. Conclusion

Theoretical formula for PnL at maturity

Consider an option with payoff $C(S_t, T; \sigma_i)$ having the expiry T and strike price K. Denote C_t to be the value of this option at time t. Under the Black-Scholes model, this option is priced by its implied volatility σ_i: $C_t := C_t^i$.
Suppose that S_t follows a geometric brownian motion in real market,

$$dS_t = (\mu - q)S_t dt + \sigma_a dW_t$$

and thus, we can define $C_t = C(t, S_t)$, i.e. a function of t and S_t by Markov property of S_t.
At the meantime, we hedge this option using a delta based on σ_h. The time t value of the portfolio created for hedging purposes has the notation V_t^h. Therefore, the PnL at time $t \in [0, T]$ of the hedged option using the Δ^h portfolio is

$$-\mathrm{PnL}_t = V_t^h - C^i(t, S_t)$$

Theoretical formula for PnL at maturity

Now we use discounted value to simplify the computation, where an uppercase D represents the discounted value:

$$-D\text{PnL}_t = DV_t^h - DC^i(t, S_t)$$

The discounted value of portfolio is

$$DV_t^h = \varphi_t + \phi_t DS_t$$

where (φ_t, ϕ_t) is a self-financing strategy. More specifically, define $\phi_t := D\Delta^h$. Thus, with the consideration of dividend payment rate q, the self-financing condition with dividend is:

$$DV_t^h = DV_0^h + \int_0^t q\phi_z DS_z dz + \int_0^t \phi_z dDS_z \tag{1}$$

By Feynman-Kac theorem, we know that $DC^h(t, S_t)$ satisfies the PDE

$$D\Theta^h - qDS_t D\Delta^h + \frac{1}{2} DS_t^2 \sigma_h^2 D\Gamma^h = 0$$

Theoretical formula for PnL at maturity

Applying Ito's formula to $DC^h(t, S_t)$, and assume $\sigma = \sigma_h$, we have

$$\begin{aligned} d(DC^h(t, S_t)) &= D\Theta^h dt + D\Delta^h dDS_t + \frac{1}{2} D\Gamma^h \sigma_a^2 DS_t^2 dt \\ &= D\Delta^h (qDS_t dt + dDS_t) + \frac{1}{2} DS_t^2 (\sigma_a^2 - \sigma_h^2) D\Gamma^h dt \end{aligned}$$

Integrating at both sides, we have

$$\begin{aligned} \int_0^t \phi_z dDS_z = DC^h(t, S_t) - DC^h(0, S_0) - \int_0^t \phi_z qDS_z dz \\ - \int_0^t \frac{1}{2} DS_z^2 (\sigma_a^2 - \sigma_h^2) D\Gamma^h dz \end{aligned} \tag{2}$$

Theoretical formula for PnL at maturity

Combining 2 with self-financing condition 1:

$$
\begin{aligned}
-D\mathrm{PnL}_t &= DV_0^h - DC^i(t,S_t) + DC^h(t,S_t) - DC^h(0,S_0) \\
&\quad + \int_0^t \frac{1}{2} DS_z^2(\sigma_h^2 - \sigma_a^2) D\Gamma^h dz \\
\implies -\mathrm{PnL}_t &= e^{rt}(V_0^h - C^h(0,S_0)) + C^h(t,S_t) - C^i(t,S_t) \\
&\quad + \int_0^t \frac{1}{2} S_z^2(\sigma_h^2 - \sigma_a^2)\Gamma^h dz
\end{aligned}
$$

Notice that, V_0^h is the option price at $t = 0$, which means that it equals to the value of option priced by actual volatility σ_a at $t = 0$, though it has Superscript h on it since we want to create a strategy to hedge this kind of option.

Theoretical formula for PnL at maturity

Hence we can have a final form for PnL:

$$\begin{aligned}\mathrm{PnL}_t = & - e^{rt}(C^i(0,S_0) - C^h(0,S_0)) - C^h(t,S_t) + C^i(t,S_t) \\ & + \int_0^t \frac{1}{2} S_z^2(\sigma_a^2 - \sigma_h^2)\Gamma^h dz\end{aligned} \tag{3}$$

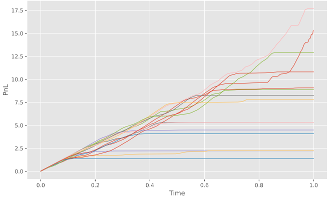
Corresponding differential form:

$$d\mathrm{PnL} = \frac{1}{2}(\sigma_a^2 - \sigma_i^2)S^2\Gamma^i dt + (\Delta^i - \Delta^h)((\mu - r + D)Sdt + \sigma_a S dW_t)$$

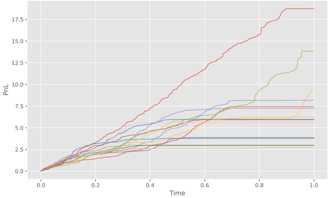
- When $\sigma_h = \sigma_a$, $\mathrm{PV}(\mathrm{PnL}_T) = C(S_t, T; \sigma_a) - C(S_t, T; \sigma_i)$.
- When $\sigma_h = \sigma_i$, $\mathrm{PV}(\mathrm{PnL}_T) = \frac{1}{2}(\sigma_a^2 - \sigma_i^2)\int_t^T e^{-r(s-t)} S^2 \Gamma^i ds$.
- Overlapping part: $\frac{1}{2}(\sigma_a^2 - \sigma_i^2)S^2\Gamma^h dt$

Theoretical formula for PnL at maturity

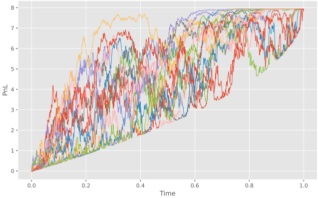
*Comparison between theoretical and simulated results



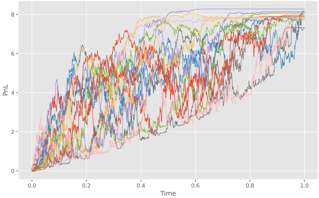
(a) $\sigma_h = 10\%$ hedging continuously



(b) $\sigma_h = 10\%$ hedging discretely



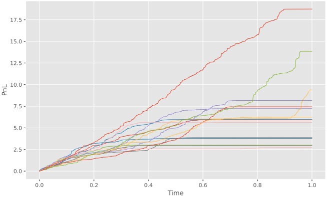
(c) $\sigma_h = 30\%$ hedging continuously



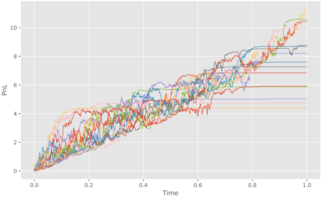
(d) $\sigma_h = 30\%$ hedging discretely

Discussion 1: PnL Without Hedging Constraints

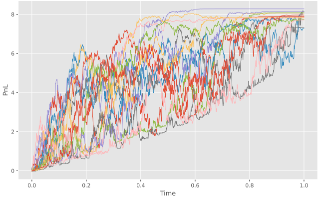
(i) Results and discussion between strategies:
Simulated paths under four hedging strategies with 15 paths each



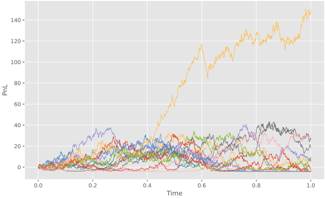
(a) $\sigma_h = \sigma_i = 10\%$ hedging



(b) $\sigma_h = 20\%$ hedging



(c) $\sigma_h = \sigma_a = 30\%$ hedging



(d) strategy without hedging

Discussion 1: PnL Without Hedging Constraints

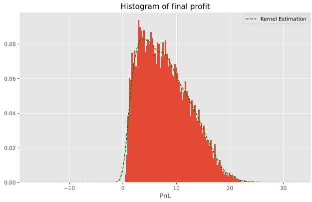
(i) Results and discussion between strategies

Table: Simulated performances of hedging strategies without constraints

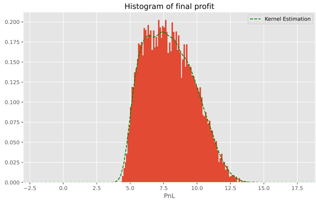
Hedging Strategy	**Value of σ_h**	$\mathbf{E[PnL_T]}$	$\mathbf{Var[PnL_T]}$
Hedging using σ_i	10%	7.949240	20.661090
Hedging using something else in between	15%	7.948893	9.192208
	20%	7.944231	3.279438
	25%	7.948760	0.761217
Hedging with σ_a	30%	7.939477	0.113254
Do not hedge	N.A.	14.546687	701.807327

Discussion 1: PnL Without Hedging Constraints

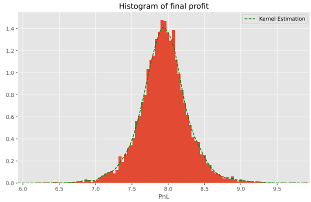
(i) Results and discussion between strategies:
Distributions of realized PnLs under four hedging strategies with 10000 paths each



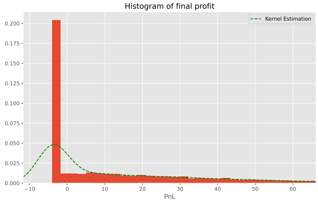
(a) $\sigma_h = \sigma_i = 10\%$ hedging



(b) $\sigma_h = 20\%$ hedging



(c) $\sigma_h = \sigma_a = 30\%$ hedging



(d) strategy without hedging

Discussion 1: PnL Without Hedging Constraints

(ii) Strategy performance with changing drift

Parameters are $S_0 = 100$, $\sigma_h = \sigma_i = 10\%$, $\sigma_a = 30\%$, $r = 0$, $q = 0$, $K = 100$, $T = 1$.

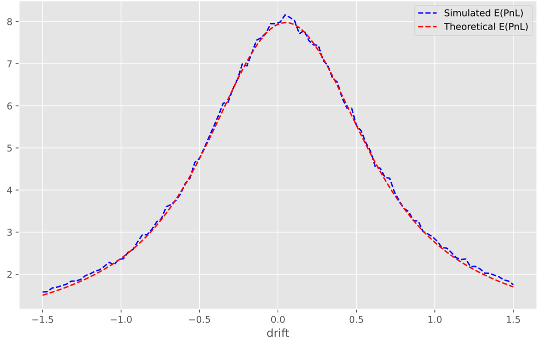
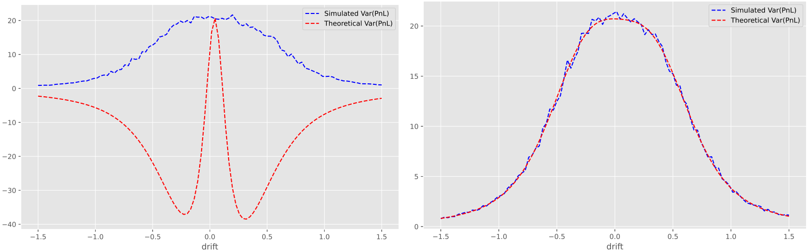


Figure: Theoretical and simulated expected PnL under implied volatility hedging versus drift

Discussion 1: PnL Without Hedging Constraints

(ii) Strategy performance with changing drift

Parameters are $S_0 = 100$, $\sigma_h = \sigma_i = 10\%$, $\sigma_a = 30\%$, $r = 0$, $q = 0$, $K = 100$, $T = 1$.



(a) Theoretical Var(PnL) is derived from the equation in paper

(b) Theoretical Var(PnL) is derived from the modified equation

Figure: Theoretical and simulated PnL variance under implied volatility hedging versus drift

Discussion 1: PnL Without Hedging Constraints

(iii) Strategy performance with changing strike

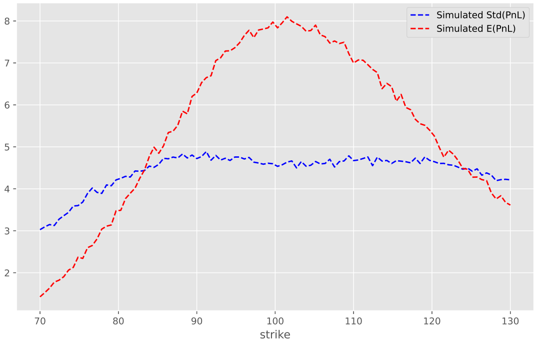
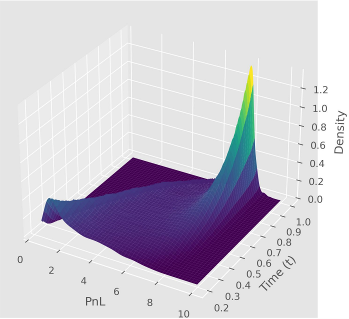


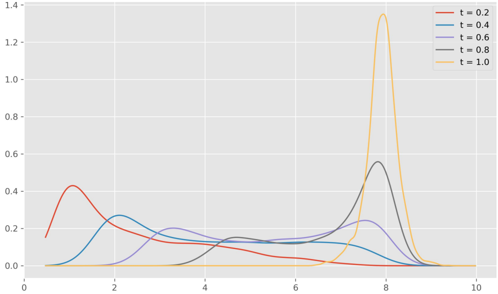
Figure: Simulated expected PnL and its standard deviation under implied volatility hedging versus strike

Discussion 2: PnL if Exiting the Position Midway

(i) Hedging using actual volatility



(a) KDE density for PnL

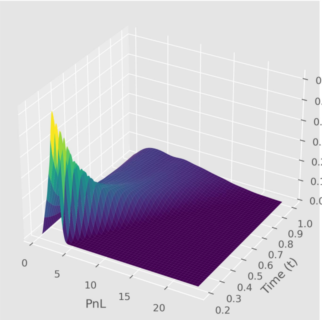


(b) Selected density for PnL, t=0.2,0.4,0.6,0.8,1

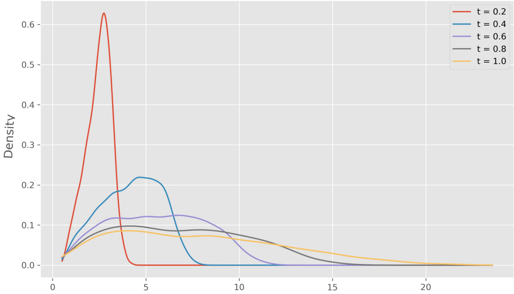
Figure: Distribution of PnL at different exiting time with hedging volatility = 30%

Discussion 2: PnL if Exiting the Position Midway

(ii) Hedging using implied volatility



(a) KDE density for PnL

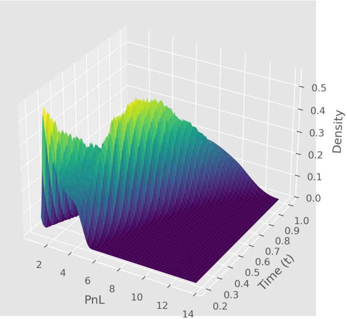


(b) Selected density for PnL, t=0.2,0.4,0.6,0.8,1

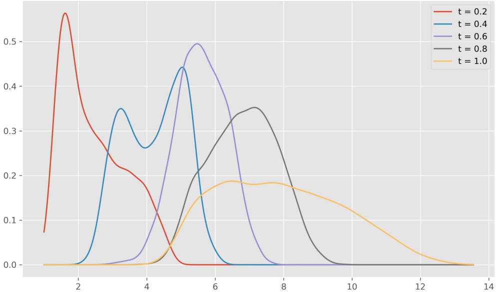
Figure: Distribution of PnL at different exiting time with hedging volatility = 10%

Discussion 2: PnL if Exiting the Position Midway

(iii) Hedging using volatility between implied and actual



(a) KDE density for PnL

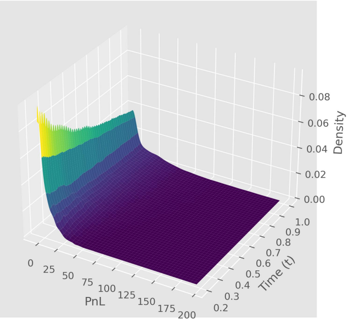


(b) Selected density for PnL, t=0.2,0.4,0.6,0.8,1

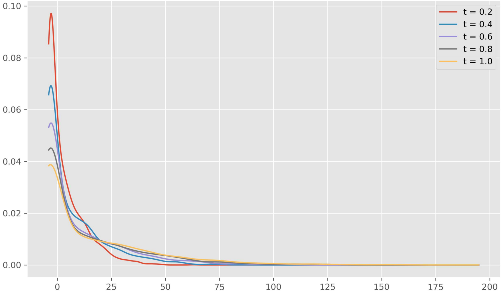
Figure: Distribution of PnL at different exiting time with hedging volatility = 20%

Discussion 2: PnL if Exiting the Position Midway

(iv) Do not hedge



(a) KDE density for PnL



(b) Selected density for PnL, t=0.2,0.4,0.6,0.8,1

Figure: Distribution of PnL at different exiting time without hedging

Discussion 3: PnL with Drawdown Limits

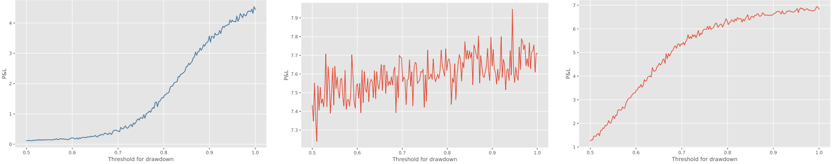
By letting $b = 0$ in our previous construction, comparing the term of $F_{0a}(1)$ defined previously, we have we first talk about the high water mark, which is the highest value the fund has achieved in the past:

$$\mathrm{HWM}_t = \max_{s \leq t} P_s.$$

Since investors often withdraw their money when the fund drops too far below the high-water mark, an important risk measure in this context is the (relative) drawdown

$$\mathrm{DD}_t = \frac{\mathrm{HWM}_t - P_t}{\mathrm{HWM}_t},$$

Discussion 3: PnL with Drawdown Limits



(a) Actual volatility (b) Implied volatility (c) Other volatility

Figure: PnL under different volatilities considering the drawdown

Discussion 3: PnL with Drawdown Limits

- We set *DD* to 0.9 and observe the change in PnL for volatility in the range 0.1 to 0.3.
- The closer volatility is to true volatility, the more PnL is affected, and the closer it is to implied volatility, the less PnL is affected.

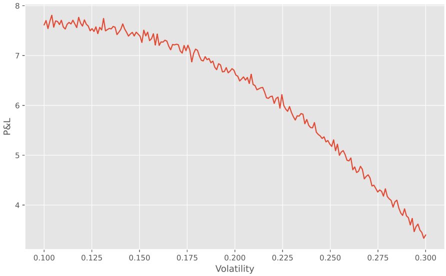


Figure: PnL for volatility between actual and implied volatilities

Conclusion

Table: Pros and Cons using different volatility to hedge

Choices of volatilities	**Pros**	**Cons**
Actual volatility	Know the exact value of the profit at expiration	The daunting fluctuations of PnL during the life of the option
Implied volatility	- No local fluctuations in PnL - Only need to be on the right side of the trade to profit. - Implied volatility is easy to observe	Unknown value of the profit at expiration
Volatilities in between	Quantifying the 'local' risk (the daily fluctuations in PnL), optimal volatility may be derived	

Conclusion

- **Exiting the midway:** the mean of the PnL gradually increases and stabilizes as the time approaches the expiration date in actual volatility hedging, while the PnL displays stronger path dependency and greater uncertainty in implied volatility hedging.
- **Drawdown:** implied volatility hedging strategies offer a distinct advantage in mitigating the risks associated with excessive drawdowns that may result in stop strategies.
- The implementation of any hedging strategy in actual trading involves multiple considerations, and each case warrants a meticulous analysis based on its unique merits.

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