Assignment 2

25 February 2021 Due: 10 March, 2021

Part I : Theoretical Exercises

Question 1 (Recourse function, (**15 marks**)). Consider $\min_{x\in X} c^\top x + \mathbf{E}\left[Q(x,\omega)\right]$ where the recourse function is

$$Q(x,\omega) = \min_y f_0(y,\omega) \text{ s.t. } f_i(y,\omega) + g_i(x,\omega) \le h_i(\omega),\ i = 1,\ldots,m, \quad y \in Y,$$

where Y is a convex set and for each $i = 1,\ldots,m$, $g_i(\cdot,\omega)$ is a convex function of x for every $\omega \in \Omega$.

1. Suppose that for each $i = 0,\ldots,m$, $f_i(\cdot,\omega)$ is a concave function of y for every $\omega \in \Omega$. Suppose Y is a polytope, which is a convex set with finitely many extreme points[1]. Let the extreme points of Y be the vectors $\{\bar{y}^1,\ldots,\bar{y}^K\}$ for some finite integer K.

   Use Lagrange weak duality[2] to argue that for every $\lambda \ge \mathbf{0}$ the recourse function can be lower bounded as (**8 marks**)

   $$Q(x,\omega) \ge -h(\omega)^\top\lambda + \sum_{i=1}^m \lambda_i g_i(x,\omega) + \min_{k=1,\ldots,K} f_0(\bar{y}^k,\omega) + \sum_{i=1}^m \lambda_i f_i(\bar{y}^k,\omega)$$

   Observe that this lower bound is a convex function of x for every ω. To establish this lower bound, you will first have to prove the following lemma:

   > *When minimising a concave function over a compact convex set, there exists at least one extreme point of the set that is an optimal solution.* ***(4 marks)***

2. Suppose that for each $i = 0,\ldots,m$, $f_i(\cdot,\omega)$ is a convex function of y for every $\omega \in \Omega$. Prove that $Q(\cdot,\omega)$ is a convex function of x for every $\omega \in \Omega$. (**5 marks**)

   Hint: This is using basic definition of convexity and independent of the first part.

3. Explain how the theorem from lectures week 5 about convexity of the recourse function $Q(x,\omega) = \min_y\{q(\omega)^\top y : W(\omega)y + T(\omega)x = h(\omega),\ y \ge \mathbf{0}\}$ follows as a simple corollary of above. (**2 marks**)

Question 2 (Cones of Matrices, (**20 marks**)). A set $\mathcal{C}$ is a cone if for every $x \in \mathcal{C}$ we have $\lambda x \in \mathcal{C}$ for every $\lambda > 0$, and it is a convex cone if it is a cone and for every $x, y \in \mathcal{C}$ we have $x + y \in \mathcal{C}$. In this question we will analyse some cones in the space of symmetric real $n \times n$ matrices $\mathcal{S}_n := \{A \in \mathbb{R}^{n\times n} \colon A = A^\top\}$. In the lectures we saw the convex cone formed by positive semidefinite matrices

$$\mathcal{PSD}_n := \{A \in \mathcal{S}_n \colon v^\top A v \ge 0,\ \forall v\}.$$

Throughout, let $\mathbf{0}$ denote either a vector or matrix of all zeros, as appropriate by context. Let $\mathcal{S}_n^+ := \{A \in \mathcal{S}_n \colon A \ge \mathbf{0}\}$ be the set of nonnegative symmetric matrices, meaning that every such matrix has each entry in it ≥ 0. Now consider the following sets of matrices,

$$\begin{aligned} \mathcal{DNN}_n &:= \mathcal{PSD}_n \cap \mathcal{S}_n^+ && \text{(Doubly nonnegative matrices)} \\ \mathcal{COP}_n &:= \{A \in \mathcal{S}_n \colon v^\top A v \ge 0,\ \forall v \ge \mathbf{0}\} && \text{(Copositive matrices)} \\ \mathcal{CP}_n &:= \{A \in \mathcal{S}_n \colon A = BB^\top \text{ for some } B \ge \mathbf{0}\} && \text{(Completely positive matrices)} \end{aligned}$$

Note that in the definition of $\mathcal{CP}_n$, the matrix B is of dimension $n \times k$ for some integer k.

[1] An extreme point is a point that cannot be written as a convex combination of two other points.
[2] This was covered in week 2 in `Quadratic.pdf`.

1. What is the difference between a psd matrix and a copositive matrix ? (**1 mark**)

2. Prove that each of $\mathcal{DNN}_n, \mathcal{COP}_n, \mathcal{CP}_n$ is a convex cone. (**5 marks**)

   You can skip showing convexity of $\mathcal{CP}_n$ since that requires using an alternate definition.

3. Prove the following relationships, (**9 marks**)

$$\mathcal{CP}_n \subseteq \mathcal{DNN}_n \subseteq \mathcal{PSD}_n \subseteq \mathcal{PSD}_n + \mathcal{S}_n^+ \subseteq \underbrace{\mathcal{COP}_n \subseteq \mathcal{CP}_n^*}_{\textbf{(4 marks)}}.$$

   Here, $\mathcal{C}^*$ denotes the dual cone of a cone $\mathcal{C}$ as defined in lectures, and note that while forming the dual cone in the matrix space we take the inner product to be the Frobenius inner product $A \bullet B = \sum_{i,j} A_{ij} B_{ij}$. Also, + between two sets denotes Minkowski addition, i.e., for two sets X and Y, we have $X + Y := \{x + y \colon x \in X, y \in Y\}$.

   In fact, $\mathcal{COP}_n$ is equal to $\mathcal{CP}_n^*$, but you are asked only to show the $\subseteq$-inclusion.

4. Consider a portfolio optimization problem that is a quadratically constrained quadratic program (QCQP) which maximises the expected return while imposing restrictions on multiple risk measures $\mathbf{Risk}_1, \ldots, \mathbf{Risk}_m$ that are quadratic risk measures (but not necessarily convex)

$$z^* = \max_x \ \mu^\top x \quad \text{s.t.} \quad \mathbf{Risk}_i(x) \le b_i, \ i = 1, \ldots, m, \ x \in F.$$

   Here, $\mathbf{Risk}_i(x) = x^\top Q_i x + f_i^\top x$ is the i^{th} quadratic risk measure and F is the set of feasible portfolios given by some linear constraints, including no short-selling constraint. Define the following set where $\bullet$ denotes the Frobenius inner product between matrices

$$\mathcal{F} := \left\{ (x, X) \in \mathbb{R}^n \times \mathbb{R}^{n \times n} \colon Q_i \bullet X + f_i^\top x \le b_i \ \forall i, \ x \in F \right\}$$

   and consider the following problem for some convex cone $\mathcal{C}$.

$$z_{\mathcal{C}} = \max_x \ \mu^\top x \quad \text{s.t.} \quad (x, X) \in \mathcal{F}, \ \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix} \in \mathcal{C}.$$

   Depending on which $\mathcal{C}$ we choose, either $\mathcal{PSD}_{n+1}, \mathcal{DNN}_{n+1}, \mathcal{COP}_{n+1}, \mathcal{CP}_{n+1}$, we get the values $z_{psd}, z_{dnn}, z_{cop}, z_{cp}$, respectively. Compare these values to each other and also to z^*, as in are any of them $=, \le, \ge$ than another ? (**5 marks**)

   Please highlight your final answer comparing all the values.

Part II : Computational Exercises

Numerical answers to be submitted to the Online Assignment on Gradescope upto a precision of 4 decimal places. Mathematical models and code to be submitted along with Part I.

For this part, we will consider the risk-neutral stochastic portfolio problem. The general formulation is

$$\min_x \ z(x) := c^\top x + \mathbf{E}\left[Q(x, \omega)\right] \quad \text{s.t.} \quad x \in X$$

where c is the expense ratio (investment cost), x is the portfolio, $X = \{x \ge \mathbf{0} \colon \sum_{i=1}^n x_i = 1\}$ is the set of feasible portfolios, and the recourse function $Q(x, \omega)$ is given by

$$Q(x, \omega) = \min_y \sum_{t=1}^T b_t y_t(\omega) \quad \text{s.t.} \quad \sum_{i=1}^n r_{i,t}(\omega) x_i + y_t(\omega) \ge R_t, \ t = 1, \ldots, T, \ \ y_t(\omega) \ge 0 \ \forall t.$$

Here, b is the penalty cost on shortfall, y is the shortfall variable, $r_{i,t}$ is the random return of asset i in time t and R_t is the target return in time t.

Download the file `indices2.csv`. This has index price data of eight leading stock market indices in US from 2013 — 2019. We use quarters as time periods, so $T = 4$ in the portfolio model.

Question 3 (Data Analysis, (**10 marks**)). 1. Batch this data to create year-to-year quarterly data, i.e. return rates for each index for each quarter for the seven years. Take geometric means while batching in each quarter. (**4 marks**)

After batching for each index and each quarter, you should get seven data points for return rates.

2. Using this batched data as a sample, compute the geometric mean $\mu_{i,t}$ for return of each index in each quarter and also the covariance matrix Σ_t among these returns for each quarter. (**4 marks**)

3. Generate a range of target return rates for each quarter to be the range between the smallest and largest mean returns across all assets for that quarter, i.e., take the interval

$$R_t(\delta) = \min_{i=1,\ldots,n} \mu_{i,t} + \delta \left(\max_{i=1,\ldots,n} \mu_{i,t} - \min_{i=1,\ldots,n} \mu_{i,t} \right),$$

for $\delta \in \{0,\ 0.05,\ 0.1,\ 0.15,\ \ldots,\ 1.0\}$. (**2 marks**)

Use the following investment costs for each index is

Index	S&P100	S&P500	S&P600	Dow	NASDAQ	Russell 2000	Barron's	Wilshire
Expense ratio c_i	0.45	1.15	0.65	0.8	1.25	1.1	0.9	0.7

Take the shortfall penalty costs for the four quarters to be $b_1 = 1.3, b_2 = 2.5, b_3 = 1.75, b_4 = 3.25$.

Question 4 (Stochastic gradient method, (**35 marks**)). Consider the batched data to represent future scenarios $\omega_1, \ldots, \omega_7$ for each quarter with equal likelihood. For e.g., the first scenario ω_1 has returns as the first of the seven data points for each index return.[3]

Use target $R_t(0.5)$. Start with the initial portfolio that invests equally among all the indices. You should terminate at iteration k if $k > 10,000$ or if[4]

$$\max\{\|x^k - x^{k-1}\|, \|x^k - x^{k-2}\|, \|x^k - x^{k-3}\|\} < \epsilon$$

for $\epsilon = 10^{-6}$. Take the solution x^* obtained upon termination of the algorithm as the "optimal" portfolio from that algorithm.

1. Implement the algorithm from slide 78 of lectures. In step 2, to choose a scenario, choose for each quarter the scenario ω_j with $j = k \mod 7$ where k is the iteration number of your algorithm[5]. For step 3, use `projunitsimplex.m` on Learn. (**10 marks**)

2. Implement the algorithm from slide 75 of lectures. In step 2, to choose scenarios, choose for each quarter all the 7 scenarios so that $N_k = 7$ for each iteration k. For step 3, use `projunitsimplex.m` on Learn. (**10 marks**)

3. For each of the above two algorithms, compute the *in-sample*[6] and *out-of-sample*[7] values of the total cost $z(x^*)$ where x^* was the solution at termination of the algorithm. Use equal likelihoods for scenarios while computing expected recourse cost. (**6 marks**)

4. Plot efficient frontiers for the two algorithm implementations for target returns $R_t(\delta)$ as defined earlier. Plot the frontiers separately and also in the same figure (using "hold on;" in Matlab) to compare them. (**4 marks**)

5. Plot composition of optimal portfolios (v/s. target return and v/s. expected return) for the two algorithm implementations for target returns $R_t(\delta)$ as defined earlier. (**5 marks**)

[3] We could create $56 = 7 \times 8$ scenarios by mixing scenarios across the 8 indices, but we avoid that for now.

[4] This means that the point has not moved too far away for 3 iterations. Note that SGD only has asymptotic convergence guarantee.

[5] Thus, after every 7 iterations, you choose the same scenario.

[6] This means the sample that was used as input to the model, batched data in this case.

[7] This means the data coming from the same population distribution as the in-sample data but was not input to the model. In this case, take this to be the data before any batching was done.