Pricing, Loss and Sensitivity Analysis of Barrier Options via Regression

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Abstract

Monte Carlo simulation methods for the valuation of financial instruments have become a staple of empirical economic analysis. A specific focus is dedicated to the feasibility of the stock price losses and barrier options as well as the sensitivity of partial time barrier options in dependence on their barrier values, which, to the best of our knowledge, hasn't been analysed in academic papers so far. This paper compares the performance of analytical results with empirical regression-based Monte Carlo outcomes, as proposed by Broadie et. al. (2015), where outer stage samples are used to generate financial risk factors and an inner stage simulation is applied to price the barrier options given the outer stage scenarios. Furthermore, potential correlation characteristics between portfolio pricing and loss regarding infinitesimal small changes in the underlying asset (Delta) are exhibited. Thus, a theoretical assumption approach to the explicit loss function is considered, as yet such an analytical loss function has not occurred in related literature.

Keywords: *Barrier Option, Sensitivity, Delta, Loss, Monte-Carlo, Regression*

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Contents

1 Introduction

Over the past decades, the possibilities of appropriate risk evaluation strategies have grown significantly. Being capable of assessing and therefore minimizing potential risk factors, e.g., exchange rates, stock prices, interest rates, etc., is essential for banks, financial institutions and private investors. Financial risk assessment is an important tool for monitoring and evaluating risks in a portfolio consisting of securities. Large financial companies, such as mutual or hedge funds, tend to re-evaluate the change of such portfolios under hundreds of thousands of potential realizations of risk factors to a future date, called risk horizon. Note, that the huge number of necessary simulations over many risk factors are extremely computationally challenging. There are many different techniques and risk measures used nowadays. Currently various distinct types of risk measures are in use, weighting numerous aspects of potential losses differently, Value at Risk and Expected shortfall being among the most prominent examples.

In order to optimize trading strategies, it is crucial to understand what kind of factors may have a significant influence on the resulting option price. The "Greeks" are the quantities representing the sensitivity of an option with respect to the change in the underlying parameters. They are often used for static as well as dynamic hedging of a portfolio and there exist many strategies to achieve good results.

In particular, this paper examines the potential stock price losses and investigates pricing functions for barrier options. The stochastic future paths of our underlying asset price are assumed to follow a geometric Brownian motion process. The final risk estimation is achieved through a specific regression method which is based on an extension of the well-known Monte Carlo simulation approach presented by Broadie et al (2015). Furthermore, we analyze the sensitivity (the Greeks) of partial-time barrier options under varying parameter restrictions. Numerical results are visualized and discussed at the end of this paper.

2 Related Literature

Broadie et al (2015) introduced a regression-based Monte Carlo simulation method in order to estimate potential risk. He compares the mean-squared-error of the normal standard estimator with the regression approach according to well chosen basis functions. His analysis relies on two important criteria: accuracy and computational effort. His results show that the regression based method significantly outperforms the standard nested simulation.
Westermark (2008) examines the performance of different pricing models with respect to the pricing of barrier options.The models include the Black-Scholes model and four stochastic volatility models ranging from the single-factor stochastic volatility model first proposed by Heston (1993) to a multi-factor stochastic volatility model with jumps in the spot price process. He shows that the choice of different loss functions have only little effect on the obtained barrier option prices.
Andersson, Nilsson (2007) presents a comparison of static and dynamic delta-hedging strategies. Two dynamic hedging methods are used as benchmarks; the delta hedging and delta-gamma

hedging strategy, respectively. He finds that the static hedging is more successful in reducing the risk and delivers a higher average return.
Faias, Santa-Clara (2011) take a new look at option asset allocation and try to determine optimal portfolio weights. They demonstrate that traditional allocation strategies, such as the mean-variance optimization, aren't adequate for considering options because the return distribution is non-normal. As a result, their new method proposed delivers a Sharpe ratio of 0.50 between 1996 and 2010.
Wang, Wang (2011) analyze the pricing of path-dependent options of four different Monte Carlo methods. They compare the results with the analytical pricing formulas and show that efficiency of a different variance reduction techniques can be improved significantly.
Wang, Fu, Marcus (2009) present a Monte Carlo simulation-based method of sensitivity analysis for barrier options for a general form of discontinuous sample payoff function.
Duffy (?) describes a class of finite-difference schemes which are commonly used to approximate analytical results. He extends theorie to explicit options pricing methods and focuses on the approximation of the Greeks. His results show that classical finite-difference methods perform well when parameters are neither too small nor too large.
Stoklosa (2007) presents numerical results for sensitivity estimation. She uses finite differences to estimate the values of two Greeks, the Delta and the Eta, that characterise the changes in the specified options prices in response to small changes in the initial asset price and barrier height. Furthermore, she presents a totally new pricing formula.

3 Barrier Options

Since this paper examines the potential loss of a portfolio consisting of barrier options, the following sections give specific preliminaries on the structure of this type of options.

3.1 Overview

Barrier options are path dependent options. In contrast to normal Call and Put options, their final payoff depends on whether the underlying asset price S_τ has reached a pre-defined barrier value H during the time to maturity T or not. We therefore distinguish between "knock-in" and "knock-out" options: Knock-in options start their lives worthless and are activated as soon as the pre-set barrier level H is breached. On the other hand, knock-out options are already active at time t_0 and get extinguished immediately when the asset price S_τ reaches the barrier H.
The payoff functions of the four main barrier options are as following:

$$\Psi_{UIC}(S_T, M_S) = \begin{cases} (S_T - K)^+ & M_S \geq H \\ 0 & M_S < H \end{cases} \tag{3.1}$$

$$\Psi_{UOC}(S_T, M_S) = \begin{cases} (S_T - K)^+ & M_S < H \\ 0 & M_S \geq H \end{cases} \tag{3.2}$$

$$\Psi_{DIP}(S_T, m_S) = \begin{cases} (K - S_T)^+ & m_S \leq H \\ 0 & m_S > H \end{cases} \tag{3.3}$$

$$\Psi_{DOP}(S_T, m_S) = \begin{cases} (K - S_T)^+ & m_S > H \\ 0 & m_S \leq H \end{cases} \tag{3.4}$$

where Ψ_{UIC}, Ψ_{UOC} is defined as the payoff of an up-and-in and up-and-out call option and Ψ_{DIP}, Ψ_{DOP} is the payoff of a down-and-in and down-and-out put option. M_S is defined as the maximum and m_S as the minimum of the stock price.

$$M_S = max\{S_\tau; 0 < \tau < T\} \tag{3.5}$$

$$m_S = min\{S_\tau; 0 < \tau < T\} \tag{3.6}$$

3.2 Put-Call Parity

One of the most important principles is the so-called Put-Call parity, which defines the relationship between European put and call options of the same category and parameter restrictions. As for normal European put and call options, one can also extend this relationship to any kind of path dependent options such as knock-in and knock-out options, as introduced in chapter 4.1.

A down-and-out put option expires worthless if the asset price falls below the barrier H and a down-and-in put option gets activated only if S_τ falls below H. Holding both a down-and-out (DO) and a down-and-in (DI) put option is equivalent to holding a plain vanilla put option in your portfolio.

This parity relationship can be described with the following equation.

$$P = P_{DI} + P_{DO} \tag{3.7}$$

where P denotes the price of the vanilla put and P_{DI} and P_{DO} is the price of the down-and-in and down-and-out put option, respectively. Nonetheless, this equation can only be applied if P_{DI} and P_{DO} are subject to the same parameter restrictions as strike price $K_{DI,DO}$, time to maturity $T_{DI,DO}$ and barrier $H_{DI,DO}$. Furthermore, we assume that there is no divergence between P_{DI} and P_{DO} (no arbitrage opportunity).

Figure 1 visualizes an example of the parity relationship for a DI and a DO put option, both having maturities of one year ($T = 1$) and strike prices $K_{DI,DO} = 100$. We presume the underlying asset being a stock with initial price $S_0 = 100$, annualized volatility of $\sigma = 0.2$ and risk-free interest rate of $r = 0.03$. For simplicity, we denote the stocks dividend yield to be zero ($d = 0.0$).

As we can see in figure 1, vertically adding up the prices of the DI and DO at any H_i ($i = 50, 51, ..., 100$) always results in the price P of the plain vanilla put option, as described by the parity formula (3.7).

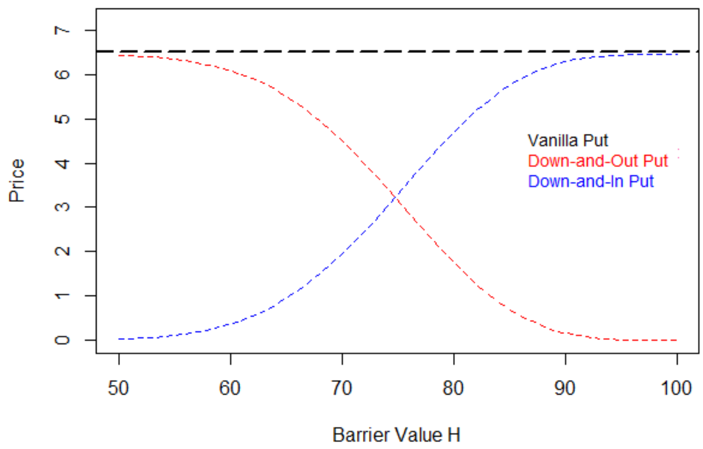

Figure 1: Put-Call parity under different barriers H

4 Preliminaries

This section provides a general overview of the model set-up, the essential parameter restrictions needed for the regression algorithm and a brief summary of the standard nested simulation. For more detailed information I refer to Broadie et al (2015).

4.1 Model Presentation

The regression algorithm is based on a nested Monte Carlo simulation method to estimate financial risk. The main idea is assembled out of two different computational steps. First, an outer simulation is used to produce an a quantity of uncertain financial risk factors and an inner simulation is then applied to price the derivatives and finally compute potential portfolio losses based on the risk factor outcomes. The regression approach combines information from different risk factor realizations in order to provide significantly better approximations of our portfolio losses.

A huge advantage of this regression method is that it can be assigned to a wide range of risk measures, such as e.g. value at risk, expected shortfall and many others used. Best results can thus be achieved, when an optimal number of inner as well as outer stage samples are taken into consideration, or by applying best possible regression weights to our calculations. To purposefully capture the entire information captured in the simulations, it is essential to choose the right basis function and interpolation approaches. Unfortunately, the ideal set of basis functions can not be established by mathematical guidance and is therefore practically challenging. We will have a closer look at this later on.

4.2 Parameter Restrictions

As previously indicated, we are interested in the potential loss of our underlying asset. In our case, we will focus on a portfolio consisting of exotic (path-dependent) barrier options. Losses are evaluated at a future time τ with respect to the initial value at time $t = 0$. The portfolio

loss at time τ is dependent on the risk factors ω. All the possible realizations of the risk factors at time τ are stated as $\Omega \subset \mathbb{R}^N$. We define each $\omega \in \Omega$ to be one scenario, which determines the value of the derivatives at time τ. The portfolio loss can indeed be interpreted as a function of ω: $L(\omega)$.
For the simulation processes, we assume that the underlying stock price follows a geometric Brownian Motion with the following stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{4.1}$$

where W_j denotes the Brownian Motion component, $S_{j,0}$ is the price of the stock at time $t = 0$, $\mu_j \in \mathbb{R}$ is the annualized drift and σ_j the annualized volatility of this specific stock. Itô's Lemma can be used to solve this equation and we therefore get the solution:

$$S_{j,\tau}(\omega) = S_{j,0} \cdot e^{\left(\mu_j - \frac{\sigma_j^2}{2}\right)\tau + \sigma_j \sqrt{\tau} W_j} \tag{4.2}$$

where $S_{j,\tau}(\omega)$ is the simulated stochastic future price of the jth stock at time τ. Some examples are shown in figure 2.

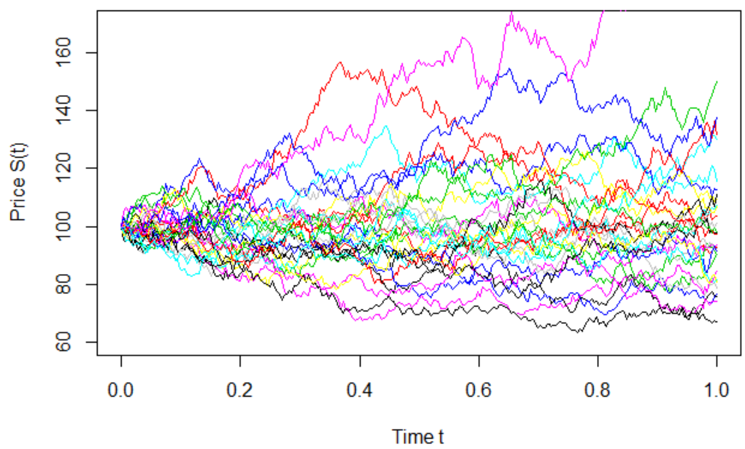

Figure 2: Stochastic future paths of a stock

4.2.1 Risk Measure

For the model we assume that each scenario ω is distributed towards the real-world distribution of the risk factors. The portfolio loss $L(\omega)$ can be expressed by a single scalar α:

$$\alpha \triangleq E[f(L(\omega))] \tag{4.3}$$

There are many different risk measures ($f(\cdot)$) used to approximately simulate portfolio losses, such as the probability of a large loss, squared tracking error, expected excess loss, or solely the absolute loss. We will focus on the *Absolute Loss* which can be determined on a pre-defined loss threshold c. The numerical approximations rely on the Monte Carlo estimates. The risk measures merely serve as analytical constructs. For instance, in case of concentrating on the expected excess loss, α is the expected value of losses in excess of the loss threshold c. In this case, we can define:

$$f(L) \triangleq (L-c)^+ \tag{4.4}$$

and determine our expected excess loss α according to the expression (4.1).
Note, that the loss threshold c may play an important role to any kind of investor or financial institution since this value specifies how much risk or loss one can tolerate. For instance, an entirely risk averse investor therefore will not use a large parameter value for c during their predictions.

4.2.2 Analytical Loss

The exact expression for the loss of the portfolio can be determined using analytical formulas to price the option. Since the analytical loss hasn't been presented in related literature so far, we make the assumption, that it could be derived as following:

$$L = Payoff - OptionPrice \tag{4.5}$$

where the payout has to be calculated path dependently. That means, that for one stochastic future path of our stock we need to simulate a Geometric Brownian Motion and then examine whether the pre-defined barrier value has been breached in $[\tau, T]$. If this is the case, the down-and-out put option expires worthless even though we have paid the options price. This is exactly where the loss appears.
Since our further investigations are focused on down-and-out put options, we limit ourselves to this kind of derivative. Further details can be found in Haug (2006) and Qi-Min Fei (2011). The analytical formula for the price of a DO-put option (Qi-Min Fei (2011)) is:

$$\begin{aligned} P_{DO,put} &= K \cdot e^{-r(T-\tau)} \cdot N(-d2) - S_0 \cdot N(-d1) + S_0 \cdot N(-x1) \\ &- K \cdot e^{-r(T-\tau)} \cdot N(-x1 + \sigma\sqrt{T-\tau}) - S_0 \cdot (\frac{H}{S_0})^{2\lambda} \cdot [N(y) - N(y1)] \\ &+ K \cdot e^{-r(T-\tau)} \cdot (\frac{H}{S_0})^{2\lambda-2} \cdot [N(y - \sigma\sqrt{T-\tau}) - N(y1 - \sigma\sqrt{T-\tau})] \end{aligned} \tag{4.6}$$

with $d1 = \sigma \cdot \sqrt{T-\tau} + \dfrac{ln(\frac{S_0}{K}) + (r-0.5\sigma^2)(T-\tau)}{\sigma \cdot \sqrt{T-\tau}}$, $d2 = d1 - \sigma\sqrt{T-\tau}$, $\lambda = \dfrac{r+0.5\sigma^2}{\sigma^2}$,
$y = \dfrac{ln(\frac{H^2}{S_0 K})}{\sigma\sqrt{T-\tau}} + \lambda \cdot \sigma\sqrt{T-\tau}$, $x1 = \dfrac{ln(\frac{S_0}{H})}{\sigma\sqrt{T-\tau}} + \lambda \cdot \sigma\sqrt{T-\tau}$ and $y1 = \dfrac{ln(\frac{H}{S_0})}{\sigma\sqrt{T-\tau}} + \lambda \cdot \sigma\sqrt{T-\tau}$.

For the detailed derivation of the mathematical components we refer to Hull (2007) and Qi-Min Fei (2011).

4.3 Standard Nested Simulation

The standard nested simulation is the procedure used to estimate the risk measure α as defined in (4.1) through Monte Carlo simulations. We consider a set of n scenarios $\omega^{(1)}, ..., \omega^{(n)}$ which we define to be the outer stage samples. Secondly, for a given scenario $\omega^{(i)}$, the portfolio loss $L(\omega^{(i)})$ can be estimated with a Monte Carlo simulation.
This is what we call the inner stage sample. In other words, for each scenario $\omega^{(i)}$, there will be produced m inner stage simulations, indicated with $\zeta^{(i)}$. $\zeta^{(1)}$,...,$\zeta^{(i)}$ are independent and identical distributed variables that secure the stochastic randomness of the inner simulation. Finally, the portfolio asset loss can be estimated by $\hat{L}(\omega^{(i)}, \zeta^{(i)})$.

$$L(\omega) = E[\hat{L}(\omega, \zeta)|\omega] \tag{4.7}$$

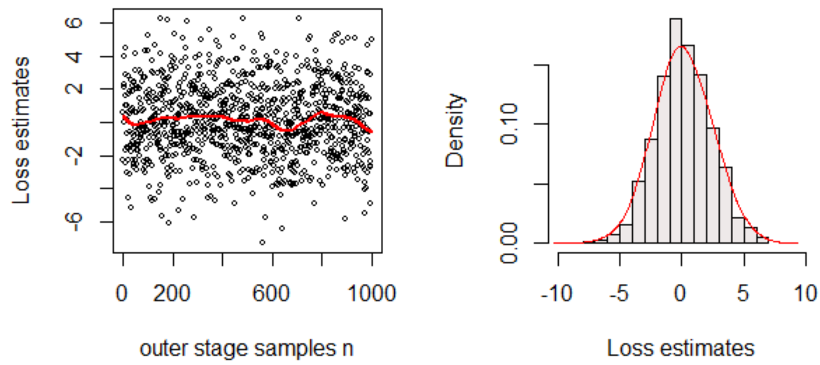

Figure 3: Smooth Monte Carlo loss estimates and histogram

Figure 3 shows an example of the stock loss estimates $\hat{L}(\omega^{(1)}, \zeta^{(1)})$,...,$\hat{L}(\omega^{(n)}, \zeta^{(n)})$ for an outer sample of $n = 1000$ and inner sample size $m = 500$. The histogram shows that the Monte Carlo loss estimates are approximately normally distributed with no appearance of heavy tails.
With given loss estimates $\hat{L}(\omega^{(1)}, \zeta^{(1)})$,...,$\hat{L}(\omega^{(n)}, \zeta^{(n)})$, the risk measure α is then approximated via the standard nested estimator, which is defined as:

$$\hat{\alpha}_{(m,n)} \approxeq \frac{1}{n} \sum_{i=1}^{n} f(\hat{L}(\omega^{(i)}, \zeta^{(i)})) \tag{4.8}$$

This is where the computational difficulty appears: Given n outer stage scenarios and m inner stage samples, the total number of simulations k required is given by $k = nm$. Unfortunately, k must be chosen sufficiently large in order to achieve reliable approximations. For further information about accuracy and optimal choice of m and n, I refer to chapter 2.1 (Broadie et. al. (2015)).

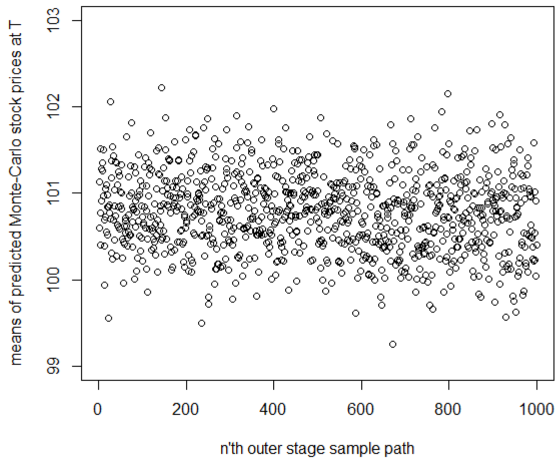

Figure 4: Means of predicted Monte-Carlo stock prices at T

In figure 4 one can see the n means of the simulated Monte-Carlo stock prices at the final time to maturity T. Every dot in the above figure corresponds to the arithmetic mean of all m inner stage stock samples for each n outer stage scenario. One may observe that the data points vertically lie within a very narrow interval ($\approx [99, 102]$). As for a mathematical consequence, this indeed makes sense since for each n outer scenario we have computed the mean of m values.

As we will see later on, a main idea is to visualize the estimated Monte-Carlo losses according to these simulated mean stock prices and finally compare the results with the regression-based loss estimates.

5 Empirical Portfolio Pricing

The value of our portfolio consisting of barrier options can empirically be priced on basis of n simulated stock prices $S_\tau(\omega)^i$ using Monte Carlo simulations of the inner stage sample paths.

5.1 Algorithm

The following steps will clarify this pricing algorithm based on a step-by-step guidance.

5.1.1 Step 1

As mentioned above, the empirical pricing of our options portfolio relies on simulated stock prices between time τ and time to maturity T, which are generated under the outer stage risk factors ω^i (chapter 4.2) and inner stage risk factors ζ^i. Specifically, the (risk-neutral) stock price on the pth path for the jth stock is given by:

$$S_{j,t}^{(p)}(\omega, \zeta) = S_{j,\tau}(\omega) \cdot e^{\left(r_f - \frac{\sigma_j^2}{2}\right)(t-\tau) + \sigma_j \sqrt{\tau} W_{j,t}^{(p)}} \tag{5.1}$$

where $S_{j,\tau}(\omega)$ is the predicted stock price at τ (chapter 4.2), r_f represents the continuously compounded riskless rate of interest and σ_j is the annual volatility of the jth stock. $W_{j,t}^{(p)}$ is defined to be a Brownian Motion for the jth security and for each inner stage sample path $p = 1, ..., m$. For simplicity, we specify that the m inner stage sample paths are independent and therefore there isn't any correlation between $W^{(p')}_{j',t}$ and $W^{(p'')}_{j'',t}$ if $p' \neq p''$ (Broadie et. al. (2015)).

5.1.2 Step 2

Having simulated the future stock paths in $[\tau, T]$, we proceed with checking whether or not one $S^{(p)}_{j,t}$ has touched or even fallen below the defined barrier value between times τ and T (down-and-out put). In the event of one specific path breaching the barrier, the corresponding derivative expires worthless and the resulting payout is equal to zero- otherwise you get a payoff. Mathematically, this payout structure can be written as follows:

$$\Psi_{DOP}(S_T, m_S) = \begin{cases} e^{(-r_f(T-\tau))} \cdot (K - S_T)^+ & m_S > H \\ 0 & m_S \leq H \end{cases} \tag{5.2}$$

with $m_S = min\{S_t; \tau < t < T\}$ being the minimum stock price between τ and T. The mathematical term $(e^{-r_f \cdot (T-\tau)})$ is essential due to the fact that the payout needs to be discounted. Notice, that this formula is only valid for partial time barrier options that can be knocked in or out only between τ and T (see, e.g., section 4.17.4, Haug (2006)).

5.1.3 Step 3

Having identified the k cases for which a payout has been realized, the value for the option can then be computed via the arithmetic mean of all $S^{(i)}_T$:

$$S_{p,i} = \frac{1}{k} \sum_{i=1}^{k} e^{-r_f \cdot (T-\tau)} \cdot (K - S^{(i)}_T)^+ \tag{5.3}$$

In the case of the portfolio consisting of $n = 1, ..., h$ different down-and-out put options with $H_{1,...,h}$ and $K_{1,...,h}$, the presented procedure can be carried out for each nth barrier option without any deviation. The resulting portfolio value for every $S_{j,\tau}(\omega)$ is then obtained by summing up the n option values.

This pricing algorithm is illustratively presented in our results (chapter 8), as well as an impressive comparison to the analytical portfolio prices (formula (4.6)). Furthermore, we will present the empirical portfolio price evaluations after applying the regression method to the previously obtained Monte Carlo estimates. Thereafter, we will compare and discuss the performance of the analytical, (empirical) Monte Carlo and (empirical) regression-based portfolio price outcomes.

5.2 Probability

As a further extension to related literature, we now dedicate our interest to the analytical probability of one sampled jth stock $S_{j,t}(\omega)$ *exceeding* the barrier level H. The derivation of this probability is exhibited extensively. The probability Γ is derived as follows:

$$\begin{aligned}
P[S_{j,t}(\omega) > H] &= P[\underbrace{S_{j,\tau}(\omega) \cdot e^{((r-0.5\sigma^2)(t-\tau))}}_{=c} \cdot e^{(\sigma_j W_{j,t})} > H] \quad \forall t \in [\tau, T] \\
&= P[c \cdot e^{(\sigma_j W_{j,t})} > H] \\
&= P[W_{j,t} > (\frac{1}{\sigma_j}) \cdot log(\frac{H}{c})] \qquad\qquad \forall t \in [\tau, T] \\
&= P[min(W_{j,t}) > \underbrace{(\frac{1}{\sigma_j}) \cdot log(\frac{H}{c})}_{=L}] \qquad \forall t \in [\tau, T] \\
&= 1 - P[min(W_{j,t}) < L] \\
&= 1 - 2 \cdot P[W_{j,T-\tau} < L] \\
&= 1 - 2 \cdot \Phi\left(\frac{L}{\sqrt{T-\tau}}\right)
\end{aligned}$$

where

$$L = (\frac{1}{\sigma_j}) \cdot log(\frac{H}{c}) = (\frac{1}{\sigma_j}) \cdot log\left(\frac{H}{S_{j,\tau}(\omega) \cdot e^{((r-0.5\sigma^2)(t-\tau))}}\right) \tag{5.4}$$

with $W_{j,t}$ being a Brownian Motion for the jth stock and $t \in [\tau, T]$. With the above derived formula for the probability of exceeding the barrier, one can imagine that we actually slowly approach to the barrier level from above, since the Brownian Motion process is considered to be time discrete. As a consequence, the discrete Brownian Motion tends to approximatively underestimate the probability of failure. Nevertheless, this is negligible when the time steps in $[\tau, T]$ are set to a considerably big number.

6 Regression Algorithm

The regression algorithm represents a new method to approximate our portfolio stock loss $L(\cdot)$. The main idea is that we consider d so-called basis functions $\phi_1(\cdot)$,...$\phi_d(\cdot)$ for each scenario ω, which can be defined as a row vector:

$$\vec{\Phi}(\omega) \triangleq \begin{pmatrix} \phi_1(\omega) \\ \cdots \\ \phi_d(\omega) \end{pmatrix}^T ; \qquad d \in \mathbb{R} \tag{6.1}$$

In a second step, we try to determine a column vector $\vec{r} \in \mathbb{R}$ and estimate our portfolio loss $L(\omega)$ with the following approximation:

$$L(\omega) \approx \vec{\Phi}(\omega) \cdot \vec{r} \tag{6.2}$$

If we manage to find a good set of basis functions, we are then able to proceed by approximating an optimal solution for $\vec{r}$ by solving a mean squared error problem. In order to achieve this, we have to consider the Monte Carlo asset loss estimates $(\hat{L}(\omega^{(i)}, \zeta^{(i)})$ as described in chapter 4.3:

$$\vec{r}_{opt} \approx min\langle \frac{1}{n} \sum_{i=1}^{n} (\hat{L}(\omega^{(i)}, \zeta^{(i)}) - \Phi(\omega^{(i)})r)^2 \rangle \tag{6.3}$$

Finally, this optimization delivers our optimal coefficient vector $\vec{r}_{opt}$ and we are now capable of estimating our risk measure α by:

$$\hat{\alpha}_{reg} \triangleq E[f(\Phi(\omega)\vec{r}_{opt})] \tag{6.4}$$

where $\hat{\alpha}_{reg}$ is called the regression estimator. Thus, this regression algorithm indeed can be applied for pricing our options portfolio (see, e.g. chapter 5).

6.1 Practical Example

However, the practical biggest challenge is choosing a clever set of basis functions. An adequate choice of basis functions can always be set up, when you rely on low order polynomials. Nevertheless, having a complex portfolio consisting of many different exotic derivatives, one could establish more appropriate basis functions via the analytical pricing of the corresponding vanilla options. Because this would blow up the frame, we proceed with low order polynomials:

For example, if we hold three different down-and-out put options with strikes K_1, K_2, K_3 and barrier levels H_1, H_2, H_3 and with identical times to maturity, then the first and most simple set of basis functions would include the quadratic functions of the stock price $S_\tau(\omega)$: $\vec{\Phi}^{(1)} = \langle 1, S_\tau(\omega), (S_\tau(\omega))^2 \rangle$. The second basis set includes all elements in $\Phi^{(1)}$ as well as the quadratic functions inclusive of the corresponding barrier prices:

$$\vec{\Phi}^{(2)} = \begin{pmatrix} 1 \\ S_\tau(\omega) \\ (S_\tau(\omega))^2 \\ S_\tau(\omega) - H_1 \\ S_\tau(\omega) - H_2 \\ S_\tau(\omega) - H_3 \\ (S_\tau(\omega) - H_1)^2 \\ (S_\tau(\omega) - H_2)^2 \\ (S_\tau(\omega) - H_3)^2 \end{pmatrix}^T \tag{6.5}$$

where $S_\tau(\omega)$ denotes the underlying asset price at time τ. It is very important to understand that the strikes K_i do not have any influence on the basis functions set up since we are only interested in the potential portfolio loss - and that is only dependent on whether the barriers H_i get touched between times τ and T or not. Strikes K_i would only be considered when potential payoffs of the portfolio are analysed (see e.g., chapter 5).

In case of a portfolio consisting of three DO-put options, the basis set (6.5) delivers very good loss approximations (see, e.g., section 5.2, Broadie et. al. (2015)). Notice, that the true portfolio loss can only be determined if the basis set $\Phi^{(2)}$ would additionally include the exact expression for the analytical loss, which can be derived using analytical formulas to value derivatives. Since (to the best of my knowledge) an explicit loss function hasn't been presented in related literature so far, this makes reliable *analytical* loss investigations extremely challenging, which could be examined in a further study.

7 Sensitivity

The "Greeks" play an important role in the financial industry because they declare how sensitive a derivative is with respect to small changes in the underlying parameters such as the asset price S_τ, volatility σ, risk-free interest rate r and time to maturity T. According to the Taylor series expansion, the change in an options price S_p can be approximated via:

$$dS_p \approx \frac{\partial S_p}{\partial S_\tau} dS_\tau + \frac{\partial S_p}{\partial r} dr + \frac{\partial S_p}{\partial \sigma} d\sigma + \frac{\partial S_p}{\partial T} dT + \frac{1}{2} \frac{\partial^2 S_p}{\partial S_\tau^2} dS_\tau^2 \tag{7.1}$$

where higher order terms are neglected. The partial derivatives in this expansion are known as the "Greeks". In this work we will limit ourselves to the first derivative, which is called Delta Δ and it measures the sensitivity of the options price with regard to infinitesimal small changes in the stock price. In other words, having a resulting $\Delta = 0.1$, that means that the price of your option increases by 0.1 when the underlying stock goes up by 1. It has the following equation:

$$\Delta = \frac{\partial S_p}{\partial S_\tau} dS_\tau \tag{7.2}$$

For our numerical calculations we approximate Δ with the finite difference method (FDM):

$$\Delta = \frac{\partial S_p}{\partial S_\tau} \approx \frac{S_p(S_\tau + \triangle S_\tau, T, \sigma, r) - S_p(S_\tau - \triangle S_\tau, T, \sigma, r)}{2 \cdot \triangle S_\tau} \tag{7.3}$$

where $\triangle S_\tau$ is a infinitesimal small number.
The delta Δ of plain vanilla put options can only fluctuate in $[-1, 0]$ and vanilla call options in $[0, 1]$ respectively.

8 Results

In this section we present the results for the empirical as well as the analytical portfolio pricing. Furthermore, we discuss portfolio losses and its sensitivity behaviour for delta. For our numerical and analytical calculations we presumed a stock with initial start price $S_0 = 100$, annualized volatility $\sigma = 0.2$, real-world drift $\mu = 0.08$, risk-free interest rate $r = 0.03$ and dividend yield $d = 0$.
For the simulation paths we set the number for the outer stage Monte Carlo samples to $n = 1000$ and the corresponding number of inner samples to $n = 500$. The sample size used to fit the base functions to the estimated loss function was set to $n_{reg} = 1000$ and the number of time steps for the Brownian Motion processes were set to $n_{BM} = 100$. The loss threshold was set to $c = 0.3608$ in accordance with chapter 5.2, Broadie et. al. (2015).

8.1 Portfolio Pricing

Our parameter setting is the same as in the single asset case, as in Broadie et. al. (2015). The portfolio consists of three down-and-out put options with a risk horizon of $\tau = \frac{1}{52}$ and time to maturity of one month ($T = \frac{1}{12}$). The corresponding strike and barrier prices of each derivative are given as:

1. Down-and-out put option with strike $K_1 = 101$ and barrier $H_1 = 91$.
2. Down-and-out put option with strike $K_2 = 110$ and barrier $H_2 = 100$.
3. Down-and-out put option with strike $K_3 = 114.5$ and barrier $H_3 = 104.5$.

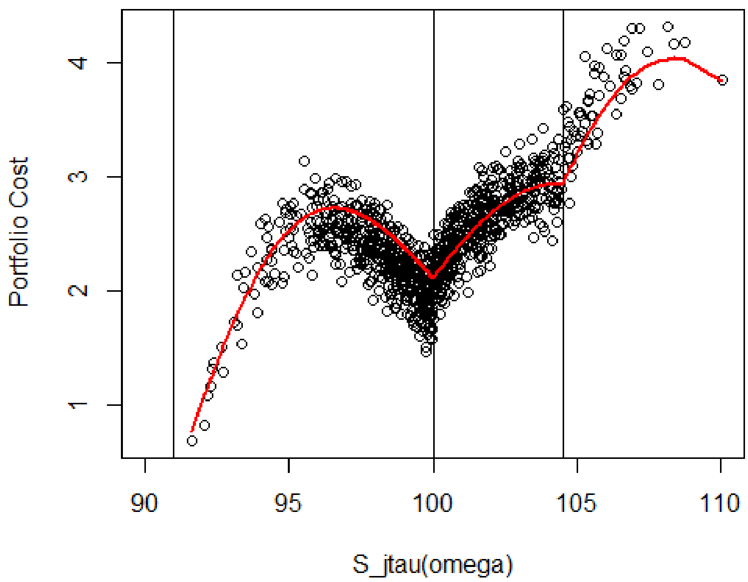

Figure 5: Comparison of empirical Monte Carlo estimates with analytical results

Figure 5 visualizes the empirical portfolio prices which were obtained as presented in the pricing algorithm (chapter 5). For each $S_{j,\tau}(\omega)$ on the x-axis, the corresponding dot in the scatter plot represents the final portfolio value, which in our case is the sum of the three down-and-out put options values, according to formula (5.3). The red line corresponds to the analytical price of the portfolio, which also was determined for every $S_{j,\tau}(\omega)$ (see formula 4.6).

One may observe, that the empirical portfolio price outcomes have a very nice overall distribution and trend, compared to the true analytical portfolio prices. Furthermore, one can establish remarkable discontinuities (as indicated with vertical lines) in both empirical and analytical trends. Notice, that these discontinuities appear at $S_{j,\tau}(\omega) = 100, 104.5$, coincident with the barrier values $H_2 = 100$ and $H_3 = 104.5$. If the outer and inner stage sample paths would be set to a bigger value, then the interval of $S_{j,\tau}(\omega)$ would increase and a discontinuity would also become visible at $S_{j,\tau}(\omega) = H_1 = 91$.

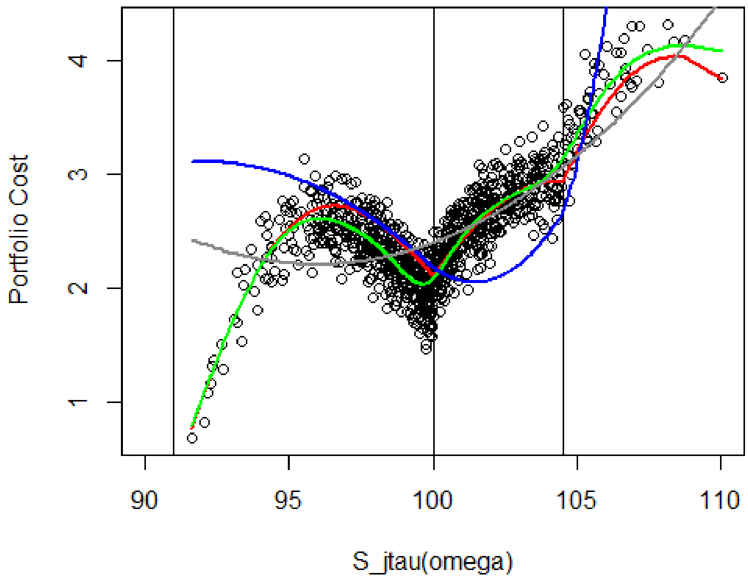

Figure 6: Comparison of regression method with analytical results

Figure 6 is an extension of the previously illustrated plot, where the analytical portfolio prices (red) are now broadened by three additional curves from which two are obtained via the regression-based method (grey and blue curve) and the green one was achieved via *cubic smoothing splines*. The grey curve is the result of a regression with basis set $\Phi^{(1)}$ and the blue curve corresponds to a regression with basis set $\Phi^{(2)}$ (for the detailed basis set compositions, see chapter 6). The green curve results from fitting cubic smoothing splines (Green et. al. (1994)), (which were not exemplified in Broadie et. al. (2015), neither in related literature), to the given data, which in our case are the estimated Monte Carlo portfolio prices (black dots). In this specific figure, the degrees of freedom (for fitting the splines) was set to 12 ($df = 12$).
First of all, one may recognize that compared to the true analytical portfolio prices, the regression method with basis set $\Phi^{(1)}$ comes off worst. Second, the basis set $\Phi^{(2)}$ tends to moderately approach the analytical prices. Evidently, the results obtained via the cubic smoothing splines significantly outperforms the two regression methods with basis sets $\Phi^{(1)}$ and $\Phi^{(2)}$. In addition, the discontinuities become visible at approximately $S_{j,\tau}(\omega) = H_{2,3} = 100, 104.5$.

8.2 Loss Investigation

Since the true analytical loss function has not been presented in the previous sections, the following results are based on the loss considering the empirical stock prices $(S_{j,\tau}(\omega,\zeta) - S_{(t=0)})$. The listed results do not correspond to the option portfolio loss.

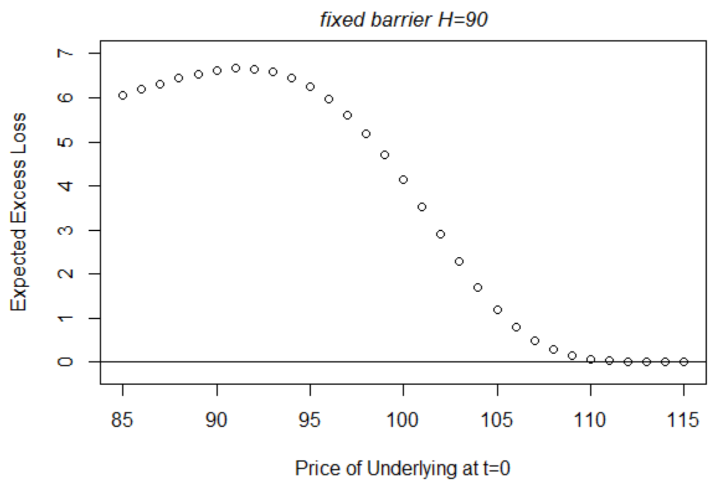

Figure 7: Expected excess loss under different stock start values $S_{0(i)}$

Figure 4 shows the risk measure $\alpha =$ expected excess loss under different underlying start prices $S_{0(i)}$. In this example, we have presumed the derivative to be a down-and-out put option with a fixed barrier $H = 90$ and strike $K = 100$. We can see that the simulation provided very nice results as the points in figure 4 show a very consistent trend. Since a down-and-out put option is considerer with $H = 90$, the losses for $S_{0(i)} = 85, ..., 90$ can be interpreted as for a up-and-out put option.

As presumed, the expected excess loss indicates a logical trend: the potential portfolio loss seems to be biggest when the initial stock price is set near the barrier. As the start price increases from 90 to 115, the expected excess loss decreases and is nearly zero at $S_0 = 109$. Between $S_0 \in [110, 115]$ the loss can be neglected since its value is approximately zero. This trend absolutely makes sense since the probability of getting knocked out vanishes as we increase the price of the underlying at time $t = 0$.

Table 1: Numerical results for a down-and-out put option

Asset price S_0	Barrier H	Option price S_p	Expected excess loss α
100	50	6.43653	0.0
100	55	6.35351	0.00021
100	60	6.09726	0.00112
100	65	5.51870	0.08514
100	70	4.51096	0.22382
100	75	3.16384	0.44241
100	80	1.76724	0.62117
100	85	0.70472	1.37295
100	90	0.15989	4.02089
100	95	0.01890	5.99150
100	100	0.00003	6.49231

Having a look at table 1, we can determine that increasing the absolute difference between the initial asset price S_0 and the barrier value H has a positive effect on the resulting option price. As the numerical difference gets larger, resultant the probability of hitting the barrier approaches to zero. As a logical result, the price S_p of the down-and-out put option converges to a normal plain vanilla put option, as verified analytically ($P_{vanilla,put} = 6.45231$).
A very similar relationship can be perceived when having a closer look at the change in the loss α, which here was computed as the expected excess loss (chapter 4.2.1). With a fixed initial stock price $S_0 = 100$, the potential expected excess loss seems to converge (nearly) exponentially fast towards zero when the barrier value H declines.
These positive correlations between start price S_0, barrier H, option price S_p and expected excess loss α are indeed sensible since in general a risk averse investor has to pay more for a down-and-out put option if he is not economically able to tolerate any big losses.

8.3 Delta

Table 2: Numerical results compared to delta Δ

Asset price S_0	Barrier H	Option price S_p	Expected excess loss α	Delta Δ
100	50	6.43653	0.0	-0.39739
100	55	6.35351	0.00021	-0.38393
100	60	6.09726	0.00112	-0.34871
100	65	5.51870	0.08514	-0.28149
100	70	4.51096	0.22382	-0.18543
100	75	3.16384	0.44241	-0.08486
100	80	1.76724	0.62117	-0.01250
100	85	0.70472	1.37295	0.01475
100	90	0.15989	4.02089	0.01072
100	95	0.01890	5.99150	0.00181
100	100	0.00003	6.49231	0.0

In table 2 we compare the delta Δ for a stock with same parameter compositions as indicated above. We can see that Δ has a very negative value when holding a down-and-out put option with a barrier level $H = 50$. As we decrease the difference between the initial spot price S_0 and barrier H, the delta Δ tends to converge to near zero. As described previously in chapter 6, the delta of a plain vanilla put option can only fluctuate within the interval $[-1, 0]$.
When analyzing the change of Δ, one can observe an interesting feature: Δ turns positive between the barrier values $H = 80$ and $H = 85$ and thus levels off at zero as the barrier converges to $S_0 = 100$. This sensitivity behaviour of Δ is quite impressive and could be examined elaborately in further studies.

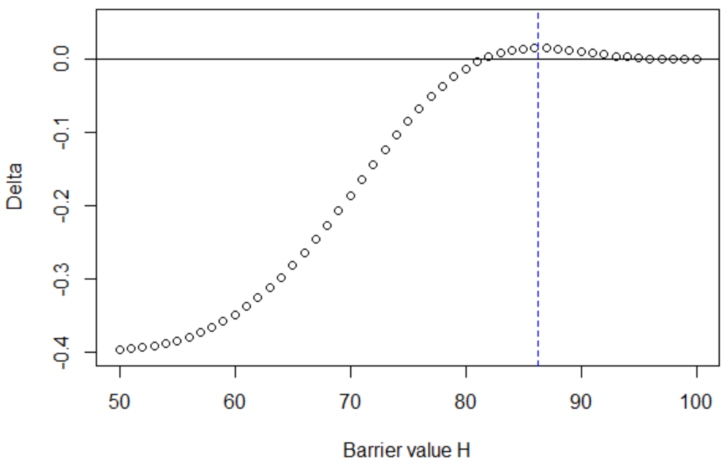

Figure 8: Delta Δ_i under different barriers H_i

The trade-off between different barrier levels H_i and corresponding delta Δ_i is visualized in figure 5. As already presented in table 2, we can recognize that Δ_i converges to zero as the barrier H_i gets bigger. At around $H = 80$, $\Delta^{(80)}$ changes it's sign to positive, and having a maximum at approximately $H = 87$ (blue vertical line).

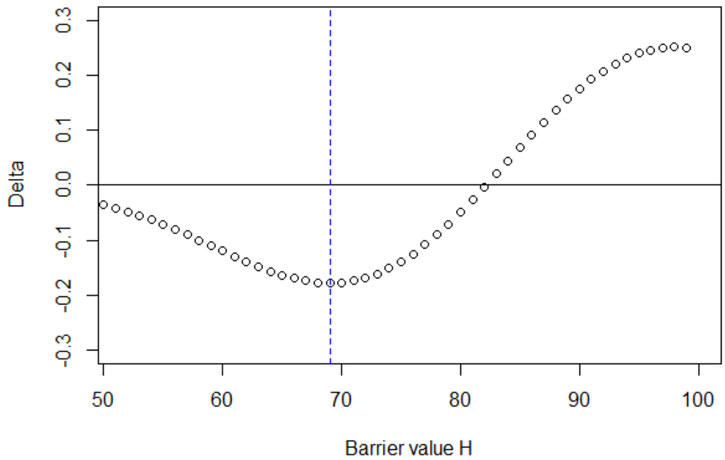

Figure 9: Delta Δ_i under constant $K - H$ difference

In figure 6 again the delta Δ is visualized but under constant barrier-strike difference $(K_i - H_i = 25)$. It now shows that Δ even comes up with more scattered negative as well as positive values. At around $H = 69$ (and corresponding strike $K = H + 25 = 94$, Δ indicates a minimum of about $\Delta \approx -0.19$ (blue vertical line). In this case, holding a down-and-out put option with a barrier $H \approx 81$ in your portfolio would represent the best possible solution since the option price S_p would not change if the stock increases its value by 1 ($\Delta = 0.0$).

9 Conclusion

Monte Carlo simulation methods for the valuation of financial instruments such as derivatives play an important and challenging role. Our results show that specific risk measures can empirically be obtained among realistic parameter restrictions. Absolute portfolio asset losses were visualized and their trends compared at different barrier prices. It was found that the portfolio consisting of down-and-out barrier options indeed exhibit positive correlations between asset price, barrier level and loss as well as the sensitivity to infinitesimal small changes in its underlying. Unlike plain vanilla put and call options, we have seen that Δ possesses heavy fluctuations in $[-1, 1]$, dependent on the corresponding barrier price.
Compared to analytical results, we have shown that the option portfolio pricing works well with the regression method, although a deliberate choice of basis set is essential. Nevertheless, the regression with basis set $\Phi^{(2)}$ tends to approach to the true analytical curve. However, fitting cubic smoothing splines (with a good choice of degrees of freedom) to the Monte Carlo estimates delivers best outcomes located very near to the analytical solution and, in addition, the discontinuities at barrier values become visible too.

Apart from this, we have proposed an assumption to the true analytical loss function. We hereby do not issue a guarantee for the arithmetic correctness but it surely represents a solid fundament for further loss investigations with regard to option pricing and other exotic derivatives.

10 References

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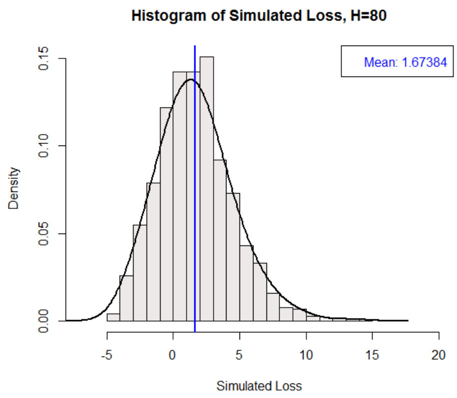
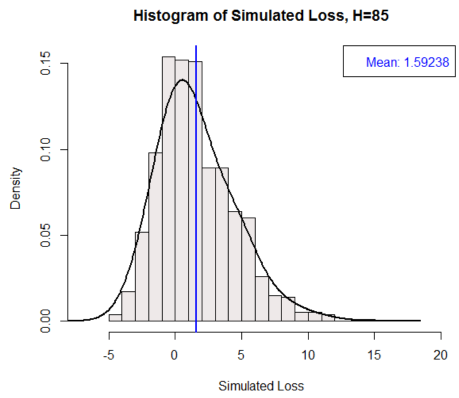
11 Appendix

11.1 A: Loss Distributions

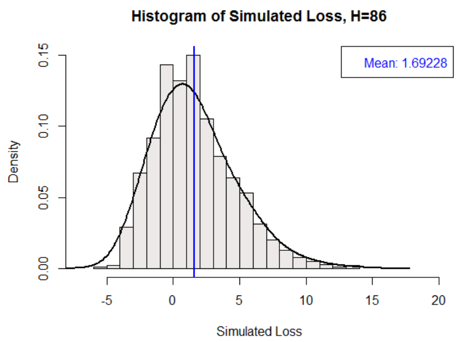
The following histograms are provided to have a look at the loss distributions under different barrier values of a down-and-out put option. Dependant on the corresponding barrier level, one may observe different distribution characteristics such as heavy tailings and asymmetry, compared to a normal Gaussian distribution.



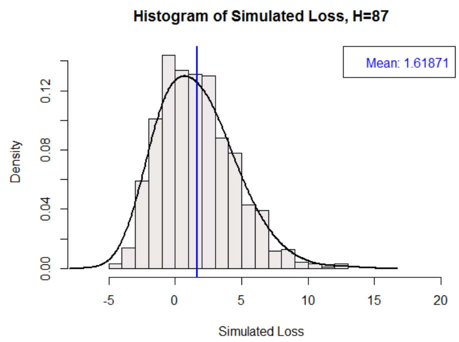
(a)



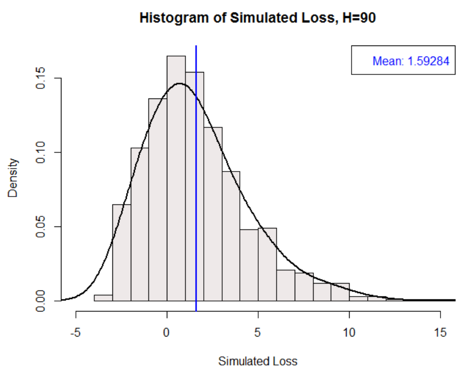
(b)



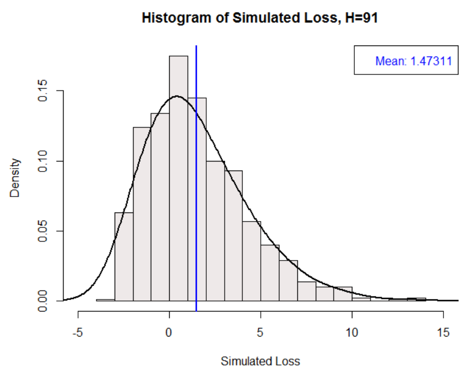
(c)



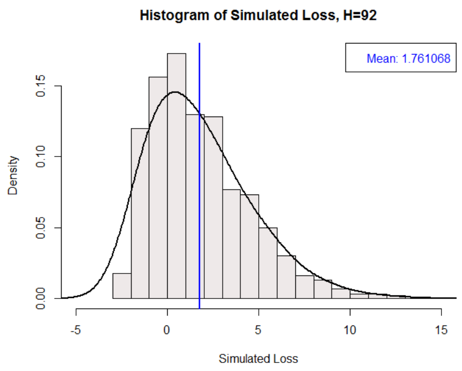
(d)



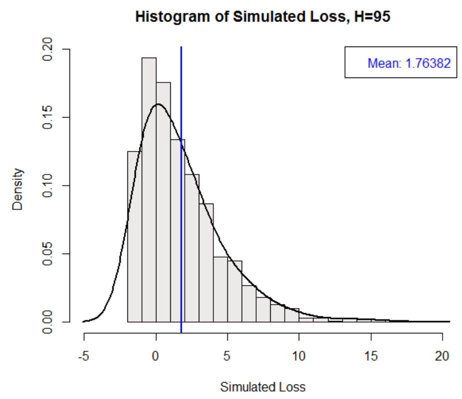
(e)



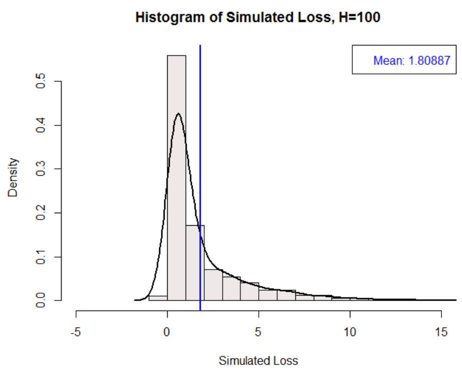
(f)


Histogram of Simulated Loss, H=92
Mean: 1.761068
Density
0.00 0.05 0.10 0.15
-5 0 5 10 15
Simulated Loss

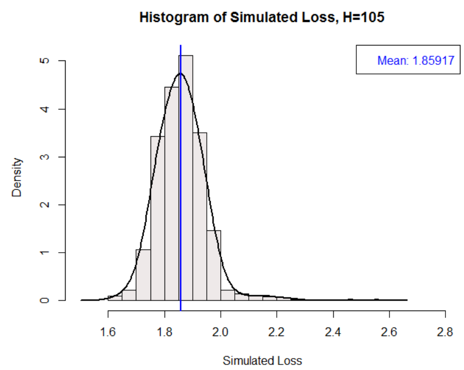
(a)


Histogram of Simulated Loss, H=95
Mean: 1.76382
Density
0.00 0.05 0.10 0.15 0.20
-5 0 5 10 15 20
Simulated Loss

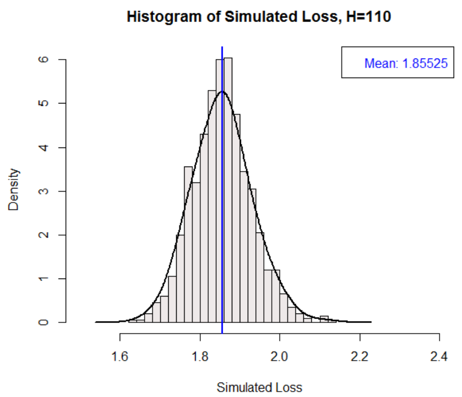
(b)


Histogram of Simulated Loss, H=100
Mean: 1.80887
Density
0.0 0.1 0.2 0.3 0.4 0.5
-5 0 5 10 15
Simulated Loss

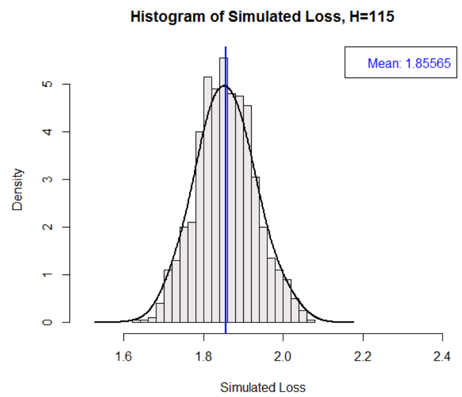
(c)


Histogram of Simulated Loss, H=105
Mean: 1.85917
Density
0 1 2 3 4 5
1.6 1.8 2.0 2.2 2.4 2.6 2.8
Simulated Loss

(d)


Histogram of Simulated Loss, H=110
Mean: 1.85525
Density
0 1 2 3 4 5 6
1.6 1.8 2.0 2.2 2.4
Simulated Loss

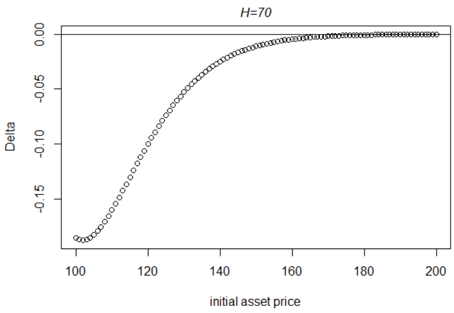
(e)


Histogram of Simulated Loss, H=115
Mean: 1.85565
Density
0 1 2 3 4 5
1.6 1.8 2.0 2.2 2.4
Simulated Loss

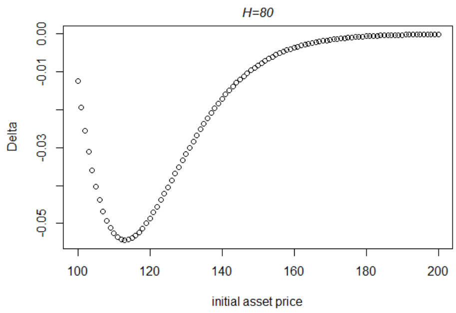
(f)

11.2 B: Delta under different barriers

Subsequent, different sensitivity behaviours (Δ) under different stock start prices ($S_0 = (100, ..., 200)$) are visualized for five different barrier levels. Note, that the trends are remarkably different with regards to the indicated barrier, as each denoted above the figure.



(a)



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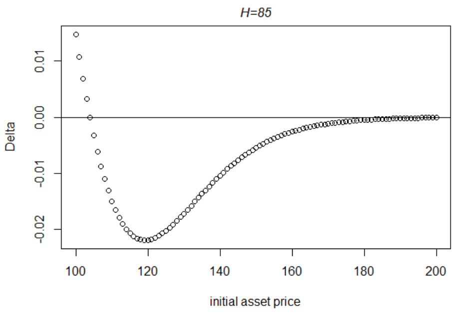

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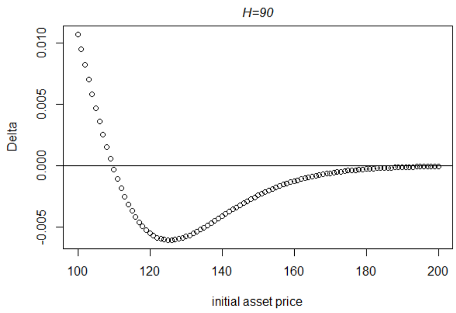

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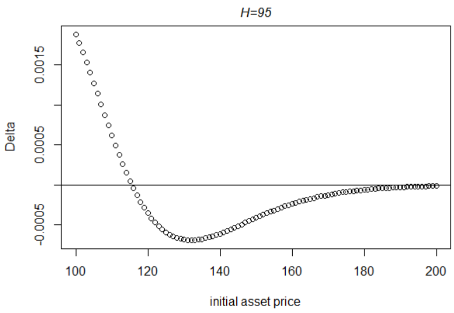

(e)