Assignment 1

29 January 2021 Due: 12 February, 2021

Part I : Theoretical Exercises

Question 1 (Markowtiz Efficient Frontier, (**15 marks**)). The minimum variance portfolio problem

$$\min_x \frac{1}{2} x^\top \Sigma x \text{ s.t. } e^\top x = 1,\ \mu^\top x = R$$

where the target return must be achieved exactly, short-selling is allowed, and the covariance matrix is assumed to be positive definite. Here, e is the n-dimensional vector of all ones. Assume that there are at least two assets whose mean returns are not equal to each other.

Denote three scalars $A = e^\top \Sigma^{-1} e$, $B = \mu^\top \Sigma^{-1} e$ and $C = \mu^\top \Sigma^{-1} \mu$.

1. Prove that $AC - B^2 = \left(\frac{B}{C}\mu - e\right)^\top \Sigma^{-1}(B\mu - Ce)$. (**4 marks**)

   Solution. It will be easier to simplify the right-hand side which is equal to $\frac{1}{C}(B\mu - Ce)^\top \Sigma^{-1}(B\mu - Ce)$. We have

$$\begin{aligned}(B\mu - Ce)^\top \Sigma^{-1}(B\mu - Ce) &= (B\mu - Ce)^\top (\Sigma^{-1} B\mu - \Sigma^{-1} Ce) \\ &= B\mu^\top \Sigma^{-1} B\mu - B\mu^\top \Sigma^{-1} Ce - Ce^\top \Sigma^{-1} B\mu + Ce^\top \Sigma^{-1} Ce \\ &= B^2 \mu^\top \Sigma^{-1} \mu - BC\mu^\top \Sigma^{-1} e - BCe^\top \Sigma^{-1} \mu + C^2 e^\top \Sigma^{-1} e \\ &= B^2 C - B^2 C - B^2 C + AC^2.\end{aligned}$$

   Dividing by C yields $AC - B^2$. □

   Marking. 3 marks for the chain of equations, 1 mark for simplifying right-hand side and dividing by C.

2. Prove that $AC - B^2 > 0$. (**4 marks**)

   Hint: You may need the fact that a positive definite matrix has an inverse which is also positive definite

   Solution. Since Σ is p.d., its inverse is also p.d. Definition of p.d. and $\mu \neq 0$ implies $C > 0$.

   Using the expression for $AC - B^2$ from above, we get that $AC - B^2 = \frac{1}{C}(B\mu - Ce)^\top \Sigma^{-1}(B\mu - Ce)$. Hence, $AC - B^2 > 0$ if and only if $(B\mu - Ce)^\top \Sigma^{-1}(B\mu - Ce) > 0$.

   The assumption that $\mu_i \neq \mu_j$ for some i, j implies that there does not exist any scalar λ such that $e = \lambda\mu$. Hence, $B\mu - Ce \neq 0$, because otherwise we would have the contradiction that $e = \frac{B}{C}\mu$.

   We again use Σ^{-1} is p.d. combined with $B\mu - Ce \neq 0$ to obtain that $AC - B^2 > 0$. □

   Marking. 1 mark for each of the following — derive $C > 0$, obtain if and only if for $AC - B^2 > 0$, derive $B\mu - Ce \neq 0$, final conclusion.

Let x_R^* be the minimum variance portfolio and let $\sigma_R^2 = {x_R^*}^\top \Sigma x_R^*$ be the minimum variance. The expected value of the portfolio is $\mu^\top x_R^*$, which is equal to R.

3. Prove that for every R, the point (σ_R, R) lies on the hyperbola $\frac{\sigma_R^2}{1/A} - \frac{(R-B/A)^2}{(C/A-B^2/A^2)} = 1$ in the (σ, R)-plane. (**4 marks**)

   Solution. From the lecture notes we have that
   $$x_R^* = \lambda_1 \Sigma^{-1} e + \lambda_2 \Sigma^{-1} \mu = \frac{C - RB}{AC - B^2} \Sigma^{-1} e + \frac{RA - B}{AC - B^2} \Sigma^{-1} \mu.$$
   Hence the minimum variance is
   $$\begin{aligned} \sigma_R^2 = x_R^\top \Sigma x_R &= x_R^\top \Sigma (\lambda_1 \Sigma^{-1} e + \lambda_2 \Sigma^{-1} \mu) \\ &= \lambda_1 x_R^\top e + \lambda_2 x_R^\top \mu \\ &= \lambda_1 + \lambda_2 \mu \end{aligned}$$
   where the last equality is due to $e^\top x_R = 1$ and $\mu^\top x_R = R$. Substituting back λ we get
   $$\begin{aligned} \sigma_R^2 &= \frac{AR^2 - 2BR + C}{AC - B^2} \\ &= \frac{1}{A} \frac{A^2 R^2 - 2ABR + AC}{AC - B^2} = \frac{1}{A} \left[1 + \frac{(AR - B)^2}{AC - B^2} \right]. \end{aligned}$$
   The hyperbola follows after rearranging terms. □

   Marking. 1 mark for substituting x_R into variance. 2 marks for computing variance with λ substituted back. 1 mark for scaling by A to obtain right-hand side of hyperbola.

4. Explain why the efficient frontier is produced by all portfolios x_R^* having $R \in \left[\frac{B}{A}, \infty\right)$. (**3 marks**)

   Solution. The right part of the hyperbola has turning point at $R = B/A$ because of the term $(R - B/A)^2$ in the equation of the hyperbola. With σ_R on the x-axis, the highest expected return for a fixed value of standard deviation is attained beyond the turning point in this right part of the hyperbola. □

   Marking. 1 mark for using the hyperbola from previous question. 1 mark for identifying turning point of hyperbola. 1 mark for observing that the hyperbola forms the efficient frontier.

Question 2 (Coherent Risk Measures, (**15 marks**))**.** Let X be a random variable.

1. Prove that $\mathbf{MAD}[X]$ is positive homogenous and subadditive but not translation equivariant. (**6 marks**)

   Solution. The definition is $\mathbf{MAD}[X] := \mathbf{E}\left[\left|X - \mathbf{E}[X]\right|\right]$. For $c > 0$ we have
   $$\mathbf{MAD}[cX] = \mathbf{E}\left[\left|cX - \mathbf{E}[cX]\right|\right] = \mathbf{E}\left[\left|cX - c\mathbf{E}[X]\right|\right] = \mathbf{E}\left[c\left|X - \mathbf{E}[X]\right|\right] = c\mathbf{E}\left[\left|X - \mathbf{E}[X]\right|\right],$$
   where the third equality uses $c > 0$, and throughout we use linearity of expectation. Hence $\mathbf{MAD}[cX] = c\mathbf{MAD}[X]$.
   For arbitrary scalar c,
   $$\mathbf{MAD}[X+c] = \mathbf{E}\left[\left|X + c - \mathbf{E}[X + c]\right|\right] = \mathbf{E}\left[\left|X + c - \mathbf{E}[X] - c\right|\right] = \mathbf{E}\left[\left|X - \mathbf{E}[X]\right|\right] = \mathbf{MAD}[X],$$
   and so MAD is not translation equivariant, rather it is translation *invariant* because it does not change value under addition by a constant.

Now take $X, Y \in RV(\Omega)$.

$$\begin{aligned}\mathbf{MAD}[X+Y] &= \mathbf{E}\left[\left|X+Y-\mathbf{E}[X+Y]\right|\right] = \mathbf{E}\left[\left|X+Y-\mathbf{E}[X]-\mathbf{E}[Y]\right|\right] \\ &= \mathbf{E}\left[\left|X-\mathbf{E}[X]+Y-\mathbf{E}[Y]\right|\right] \\ &\leq \mathbf{E}\left[\left|X-\mathbf{E}[X]\right|+\left|Y-\mathbf{E}[Y]\right|\right] \\ &= \mathbf{E}\left[\left|X-\mathbf{E}[X]\right|\right]+\mathbf{E}\left[\left|Y-\mathbf{E}[Y]\right|\right] = \mathbf{MAD}[X]+\mathbf{MAD}[Y],\end{aligned}$$

where the $\leq$ is due to subadditivity of the absolute value function, i.e., for any scalars a, b we have $|a+b| \leq |a| + |b|$. □

Marking. 2 marks for each of the properties.

2. Prove that the following risk measure is positive homogenous, subadditive and translation equivariant for any $\delta > 0$ (**5 marks**)

$$\rho_\delta(X) \coloneqq \mathbf{E}[X] + \delta\,\mathbf{MAD}[X].$$

Solution. The first two properties are similar to previous question. They can also be established by showing that those properties are retained by adding two functions that satisfy them individually, and observing that $\mathbf{E}[X]$ is positive homogenous and subadditive.

For translation equivariance we need addition by the expectation operator.

$$\rho_\delta(X+c) = \mathbf{E}[X+c] + \delta\,\mathbf{MAD}[X+c] = \mathbf{E}[X] + c + \delta\,\mathbf{MAD}[X] = \rho_\delta(X) + c,$$

where we have used $\mathbf{MAD}[X+c] = \mathbf{MAD}[X]$ that was shown in the first part. □

Marking. 2 marks for each of the first two properties, 1 mark for translation equivariance.

3. It can also be shown that $\rho_\delta(X)$ is monotone for $\delta < 1/2$, thereby making it coherent. Analyse whether ρ_1, which has $\delta = 1$, is monotone by considering two random variables: a constant $X = 1$ and the Bernoulli r.v. $Y \in \{0, 1\}$ with $\mathbf{Pr}[Y = 1] = p \in (0, 1)$. (**4 marks**)

Solution. Since $Y(\omega) \in \{0, 1\}$ for all $\omega \in \Omega$, we have $Y(\omega) \leq X(\omega)$ for all $\omega \in \Omega$. Obviously $\mathbf{MAD}[X] = 0$. We have $\mathbf{MAD}[Y] = \mathbf{E}\left[|Y - p|\right]$ since $\mathbf{E}[Y] = p$, and so $\mathbf{MAD}[Y] = (1-p)|0-p| + p|1-p| = (1-p)p + p(1-p) = 2p(1-p)$. Thus $\rho_1(X) = 1$ and $\rho_1(Y) = p + 2p(1-p) = 3p - 2p^2$. The quadratic $p \mapsto 2p^2 - 3p + 1$ is ≤ 0 over $[0.5, 1]$ and ≥ 0 over $[0, 0.5]$. Hence, monotonicity does not hold when $p \in [0.5, 1]$ but holds when $p \in [0, 0.5]$. □

Marking. 2 marks for deriving $\rho_1(X)$ and $\rho_1(Y)$. 1 mark for analysing the quadratic in $\rho_1(Y)$. 1 mark for obtaining the correct ranges for p.

Part II : Computational Exercises

Numerical answers to be submitted to the Online Assignment on Gradescope upto a precision of 4 decimal places. Mathematical models and code to be submitted along with Part I

Question 3 (Type-B Arbitrage, (**5 marks**))**.** Determine whether there is a type-B arbitrage opportunity given the following five risky securities and assuming the risk-free interest rate is 2.5%.

Security	Spot price	Value at maturity						
		ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
1	1.30	4	3	6	2	7	5.5	6.25
2	3.56	7	9	1	5	4	6	3
3	2.49	3.5	4	1	7.2	3.15	2	4.5
4	5.07	6	7	2	2.5	3	8	4
5	4.11	2.25	1	4	5	3	6.1	4.2

Solution. Let M be 6×7 matrix (rows = assets, columns = scenarios) whose first row is $(1.025, \ldots, 1.025)$ for risk-free asset and the remaining rows are the values from the table. Let its columns be indexed by M_j. Take $m = 7$ scenarios, $n = 5$ (so that we have 6 assets). The spot prices are $S_0 = (1, 1.30, 3.56, 2.49, 5.07, 4.11)$. Solve the following linear program with variables $y_0, \ldots, y_5$

$$
\begin{aligned}
OPT \;=\; \min_{y} \;\; & -\frac{1}{m}\sum_{j=1}^{m} M_j^\top y \\
& M^\top y \geq 0 \\
& S_0^\top y \leq 0
\end{aligned}
$$

Type-B arbitrage exists if and only if $OPT = -\infty$. Solving the LP gives $OPT = -\infty$. □

Marking. 2 marks for the formulation. 1 mark for noting that type-B arbitrage exists if and only if optimal value is $-\infty$. 2 marks for solving and getting the right answer on Gradescope.

Question 4 (Portfolio Optimization, (**20 marks**))**.** Download the file `indices.csv` from Learn. It has mostly closing values for 7 leading stock market indices, Dow-Jones (USA), FTSE (UK), Dax (Germany), CAC (France), Nikkei, HSI (Hongkong), BOVESPA (Brazil) as well as monthly prices for Gold for the 8 years from January 2008 to January 2016.

1. Compute the covariance matrix and state the variance for the Nikkei index. (**4 marks**)
2. What are the smallest and largest values of the target returns which seem sensible, i.e., beyond which the minimum variance portfolio model will not change its answer ? (**2 marks**)
3. Plot the efficient frontier and composition of optimal portfolio for the above range of target return values. State the optimal portfolio for a 0.24% return rate. (**6 marks**)
4. Replace variance with mean absolute deviation as the risk measure in the objective and solve your model for a target return rate of 0.24%. State the optimal portfolio. (**5 marks**)

   You can use the `norm` or `abs` function inside Matlab to model MAD
5. For the optimal portfolio in part (4), state the values of its standard deviation, mean absolute deviation, and semi deviation. (**3 marks**)

Solution. The values in the given data file first need to be sorted by increasing values of date, and then return rates must be computed by taking ratios of consecutive rows. Otherwise, the return rates will be wrong, which will lead to wrong answers for rest of question. The μ vector is to be calculated using geometric means. The covariance matrix is

$$
\begin{pmatrix}
0.0020 & 0.0016 & 0.0020 & 0.0019 & 0.0018 & 0.0021 & 0.0019 & -0.0003 \\
0.0016 & 0.0018 & 0.0020 & 0.0020 & 0.0017 & 0.0021 & 0.0019 & -0.0003 \\
0.0020 & 0.0020 & 0.0034 & 0.0028 & 0.0025 & 0.0026 & 0.0022 & -0.0005 \\
0.0019 & 0.0020 & 0.0028 & 0.0028 & 0.0022 & 0.0023 & 0.0022 & -0.0005 \\
0.0018 & 0.0017 & 0.0025 & 0.0022 & 0.0039 & 0.0027 & 0.0021 & -0.0006 \\
0.0021 & 0.0021 & 0.0026 & 0.0023 & 0.0027 & 0.0044 & 0.0033 & 0.0001 \\
0.0019 & 0.0019 & 0.0022 & 0.0022 & 0.0021 & 0.0033 & 0.0043 & 0.0001 \\
-0.0003 & -0.0003 & -0.0005 & -0.0005 & -0.0006 & 0.0001 & 0.0001 & 0.0016
\end{pmatrix}
$$

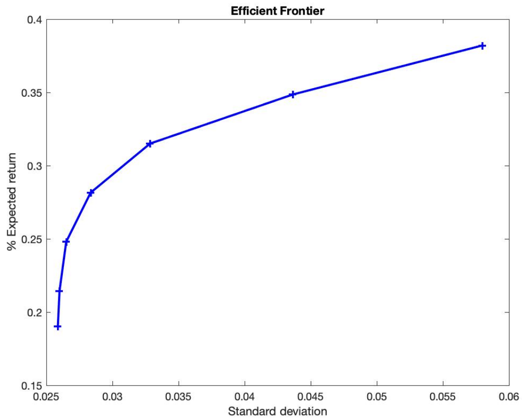

Figure 1: Efficient frontier for min var

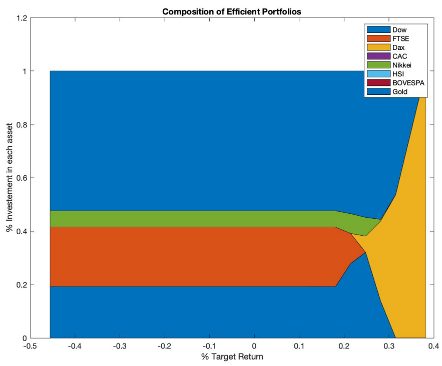

Figure 2: Composition for min var vs. Target return

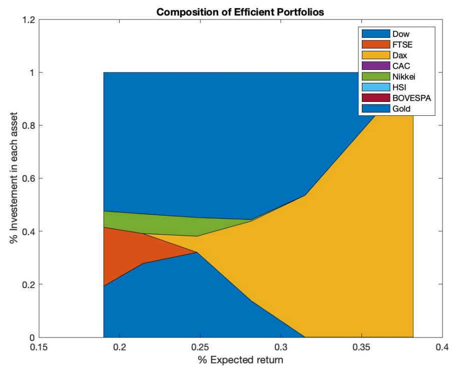

Figure 3: Composition for min var vs. Expected return

The composition plot vs. target return must start at lower limit of -0.46% and end at upper limit 0.38%. The MAD for portfolio return can be derived as

$$\mathbf{MAD}[R(x)] = \mathbf{MAD}[r(\omega)^\top x] = \mathbf{E}\left[\left|r(\omega)^\top x - \mathbf{E}\left[r(\omega)^\top x\right]\right|\right] = \mathbf{E}\left[\left|r(\omega)^\top x - \mu^\top x\right|\right]$$

$$= \mathbf{E}\left[\left|\sum_{i=1}^{n}(r_i(\omega) - \mu_i)x_i\right|\right] = \sum_{j=1}^{m} p_j \left|\sum_{i=1}^{n}(r_i(\omega) - \mu_i)x_i\right|$$

which is used as the objective function in the formulation. Use $m = 96$ and $p_j = 1/m$ for all j since we use the past historical values as equally likely scenarios for the future. □

Marking. 2 marks for computing the covariance matrix. 3 marks for plotting the efficient frontier and portfolio composition. 2 marks for formulating the model with MAD as risk measure. If wrong return rates or covariance matrix are used, then 2 marks deducted but you can still get the 3 marks for the plots if they look correct.

Remaining 13 marks are reserved for getting the right numerical answers on Gradescope.