# Quantitative Finance: Model-Free Implied Volatility Group Project

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# 1 Introduction

Volatility is an important statistical measure to show the level of uncertainty of how a specific item will change from the mean. In the financial market, volatility is one of the most significant concepts, which is related to decision-making, option pricing, and market forecasting, but could not be observed directly through the market data. In the past decade, different types of metrics were introduced to measure market volatility, for example, historical volatility (a measure of the realized volatility of an asset over a given period of time, based only on past data) and implied volatility (a measure of the market's expectation of the future volatility of an asset, implied by the prices of options on that asset).

According to the definition of quadratic variation, $\langle log(S)\rangle_T$ can implies the volatility as:

$$\langle log(S)\rangle_T = \lim_{|\Delta|\to 0}\sum_{k=1}^{n}\left(log(S_{t_k}) - log(S_{t_{k-1}})\right)^2 = \lim_{|\Delta|\to 0}\sum_{k=1}^{n}\left[log\left(\frac{S_{t_k}}{S_{t_{k-1}}}\right)\right]^2$$

where $|\Delta| = \sup_{1\le i\le n}|t_i - t_{i-1}|$ is for the partition $0 = t_0 < t_1 < ... < t_n = T$. In this report, the risk-neutral expectations of the quadratic variations of the market index over a fixed expiration period are regarded as the measure of market volatility, which is model-free and depends only on the corresponding index options. However, due to the limited finite number of observable options, which should be continuous and have an infinite range in theory, approximation errors always exist and here we introduce and compare two different approximation methods, the first of which is the VIX index issued by the Chicago Board Options Exchange [CBOE](2) and the second is an approximation method introduced by Fukasawa *et al.* (1)

In VIX index, the numerical approximation is applied as:

$$\begin{aligned}\frac{1}{T}\mathbb{E}\left[\langle log(S)\rangle_T\right] \approx 2\sum_{K=K_{min}}^{K_{-1}}\frac{P(K)}{K^2}\Delta K + \frac{P(K_0)+C(K_0)}{K_0^2}\Delta K_0 \\ + 2\sum_{K=K_1}^{K_{max}}\frac{C(K)}{K^2}\Delta K - \left(\frac{F}{K_0} - 1\right)^2\end{aligned} \tag{1}$$

In the new method, given a continuous semi-martingale $\{S_t\}_{t\ge 0}$ representing the price process of an underlying asset, the expectation of the quadratic variation is calculated through an integral as:

$$\frac{1}{T}\mathbb{E}\left[\langle log(S)\rangle_T\right] = \int_{-\infty}^{\infty}\sigma(g(z))^2\phi(z)dz \tag{2}$$

where $\sigma(g(z))^2$ is chosen to be a piecewise cubic polynomial with coefficients achieved with some settings, and details of (2) are further discussed in Section 2. Therefore, the combination of cubic polynomials and the standard normal density function as the integrand avoids the numerical integration with discretization, which is used in CBOE VIX, and thus reduces the approximation errors.

We generate two sets of artificial option price data under the Heston stochastic volatility model, with the restriction of a narrow range and a wide range of available strikes, respectively. The results show that the new method performs significantly better with a closer result to the true value when the strike range is narrow, and the advantage becomes less significant under a wide range of strikes. More comparisons are also done with different settings of parameters in the Heston model to simulate the performances under different market situations.

The structure of this report is organized as follows. In Section 2, we introduce the methodology of CBOE VIX and the new algorithm for computing the volatility index established by Fukasawa *et al.*(1), including the theory and step-by-step algorithm. We then, in Section 3, compare these two approximation algorithms of the volatility and further elaborate on the advantages and disadvantages along with the numerical results. Section 4 draws the conclusion of this report and provides some possible further improvements.

# 2 Methodology

Let $S_t$ be the price process of a risky asset, a continuous semi-martingale; $S_t^0$ be the price process of a risk-free asset, a deterministic process of locally bounded variation. By Itô's formula:

$$\log\left(\frac{S_T}{S_0}\right) = \int_0^T \frac{dS_t}{S_t} - \frac{1}{2}\int_0^T \frac{d\left\langle S\right\rangle_t}{S_t^2} = \int_0^T \frac{S_t^0}{S_t} d\left(\frac{S_t}{S_t^0}\right) + \log\left(\frac{S_T^0}{S_0^0}\right) - \frac{1}{2}\left\langle \log(S)\right\rangle_T$$

Then taking any risk-neutral expectation $\mathbb{E}$ on both sides gives:

$$\begin{aligned} \mathbb{E}[\langle \log(S)\rangle_T] &= -2\mathbb{E}\left[\log\left(\frac{S_T}{S_0}\right)\right] + 2\mathbb{E}\left[\log\left(\frac{S_T^0}{S_0^0}\right)\right] = -2\mathbb{E}\left[\log\left(\frac{S_T}{S_0\frac{S_T^0}{S_0^0}}\right)\right] \\ &= -2\mathbb{E}\left[\log\left(\frac{S_T}{\mathbb{E}[S_T]}\right)\right] =: -2\mathbb{E}\left[\log\left(\frac{S_T}{F}\right)\right] \end{aligned} \tag{3}$$

where $F = \mathbb{E}[S_T]$ is the $T$-expiry forward price of the asset $S$.

In the following parts of this section, we introduce two different approximations of the expectation $\frac{1}{T}\mathbb{E}[\langle \log(S)\rangle_T]$, the first of which is the numerical approximation used in VIX index and the second is based on a formula induced by the model-free link in pricing variance swaps as (2) mentioned in Section 1.

## 2.1 CBOE Volatility Index (VIX)

### 2.1.1 Approximation Formula

The CBOE Volatility Index (VIX) is a measure of the stock market's expectation of volatility over the next 30 days calculated from the prices of S&P 500 index options. According to *CBOE Volatility Index Mathematics Methodology*(2), the generalized formula used in the volatility calculation for a fixed expiration $T$ is:

$$\sigma^2_{CBOE}(T) = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{rT} Q(K_i) - \frac{1}{T}\left(\frac{F}{K_0} - 1\right)^2 \tag{4}$$

and notations are explained as follows:

1. $T$ is the time to expiration (in years).
2. $r$ is the risk-free interest rate to expiration.
3. $F$ is the option-implied forward price level, determined through the put-call parity applied to the pair of put and call options with the same strike K that has the closest prices (choose the smallest K for multiple closest pairs), i.e.

$$\begin{aligned} K^* &= \min\left(\arg\min_K |C(K,T) - P(K,T)|\right) \\ F &= e^{rT}\left(C(K^*,T) - P(K^*,T)\right) + K^* \end{aligned} \tag{5}$$

4. $K_0$ is the largest strike price such that $K_0 \leq F$ and chosen as the ATM strike price. In practice, equivalently, take $K_0 = K^*$ in (5).
5. $K_i$ is the strike price of the $i^{th}$ OTM option. The calculation of VIX at time t includes all put options with $K_i \leq K_0$ and all call options with $K_i \geq K_0$.
6. $\Delta K_i$ is the interval between strike spreads. Considering the extreme cases, we have:

$$\Delta K_i = \begin{cases} K_i - K_{i-1}, & K_i = K_{max} \\ K_{i+1} - K_i, & K_i = K_{min} \\ \frac{K_{i+1} - K_{i-1}}{2}, & \text{otherwise} \end{cases}$$

7. $Q(K_i)$ is the option price of the OTM option with strike $K_i$ for $i \neq 0$, and $Q(K_0)$ is the average of the put option price and call option price with strike $K_0$.

Recall the following fact in analysis that for every $C^2$ function $f : (0, \infty) \mapsto \mathbb{R}$ and $M > 0$:

$$f(S) = f(M) + f'(M)(S - M) + \int_M^S f''(v)(S - v)^+ dv + \int_0^M f''(v)(v - S)^+ dv$$

Taking $f(S) = \log(S)$ and $M = K_0$ gives:

$$\begin{aligned} \log(S_T) &= \log(K_0) + \frac{S_T - K_0}{K_0} - \int_{K_0}^{S_T} \frac{1}{K^2}(S_T - K)^+ dK - \int_0^{K_0} \frac{1}{K^2}(K - S_T)^+ dK \\ &= \log(K_0) + \frac{S_T - K_0}{K_0} - \int_{K_0}^{\infty} \frac{(S_T - K)^+}{K^2} dK - \int_0^{K_0} \frac{(K - S_T)^+}{K^2} dK \end{aligned}$$

Taking expectation and then subtracting $\log(F) = \log(\mathbb{E}[S_T])$ on both sides, we have:

$$\begin{aligned} \mathbb{E}\left[\log\left(\frac{S_T}{F}\right)\right] &= \log\left(\frac{K_0}{F}\right) + \frac{F - K_0}{K_0} - \int_{K_0}^{\infty} \frac{\mathbb{E}[(S_T - K)^+]}{K^2} dK - \int_0^{K_0} \frac{\mathbb{E}[(K - S_T)^+]}{K^2} dK \\ &= -\int_0^{K_0} \frac{P(K)}{K^2} dK - \int_{K_0}^{\infty} \frac{C(K)}{K^2} dK - \int_{K_0}^{F} \frac{K - F}{K^2} dK \end{aligned}$$

where $C(K) = \mathbb{E}[(S_T - K)^+)]$ and $P(K) = \mathbb{E}[(K - S_T)^+)]$ are the undiscounted call and put option prices of the asset $S$ with strike $K$ and maturity $T$, respectively.
Finally, we have the model-free formula as: for any $K_0 > 0$

$$-2\mathbb{E}\left[\log\left(\frac{S_T}{F}\right)\right] = 2\int_0^{K_0} \frac{P(K)}{K^2} dK + 2\int_{K_0}^{\infty} \frac{C(K)}{K^2} dK + 2\int_{K_0}^{F} \frac{K - F}{K^2} dK \tag{6}$$

Hence, the general formula (4) can be regarded as a numerical approximation of the expectation of the quadratic variation, i.e. $\frac{1}{T}\mathbb{E}\left[\langle \log(S) \rangle_T\right] = -\frac{2}{T}\mathbb{E}\left[\log\left(\frac{S_T}{F}\right)\right] \approx \sigma^2_{CBOE}(T)$.

### 2.1.2 Algorithm

Here we summarise the COBE algorithm by referring to the official release in the following four steps: Choosing the expiration dates of near-term and next-term, selecting the options to be involved, calculating the volatility $\sigma^2_{CBOE,i}$, and determining VIX by linear interpolation.

Step 1: **Choosing the expiration dates of near-term and next-term**

In CBOE VIX computation, two volatilities are first calculated by the formula (4) corresponding to two constant expirations, i.e.near-term and next-term, and linear interpolation is used to get the index for exactly 30-day expiration $T$. Based on the list of available options in the market, the near-term expiration $T_1$ is chosen as the nearest to and before the 30-day expiration, whereas the next-term expiration $T_2$ is the first expiration on the list after the 30-day expiration. Note that all $T, T_1, T_2$ are in year fraction, i.e. $T = 30/365$. For each $T = T_i, i = 1, 2$, go through Steps 2 to 3.

Step 2: **Selecting the options to be involved**

Based on the formula (5), the option-implied forward price $F$ is determined, and then ATM strike $K_0$ is chosen to be the largest strike smaller than or equal to $F$. All put options at strike smaller than or equal to $K_0$, i.e $K_{min}, ..., K_{-1}, K_0$, and all call options at strike larger than or equal to $K_0$, i.e. $K_0, K_1, ..., K_{max}$, are selected and to be involved in the calculation of the volatility.

Step 3: **Calculating the volatility $\sigma^2_{CBOE,i}$**

Substituting the set of selected options to the general formula (4), the volatility $\sigma_i^2(T_i)$ is computed as:

$$\begin{aligned}\sigma^2_{CBOE,i}(T_i) = \frac{2}{T_i}\sum_{K=K_{min}}^{K_{-1}} \frac{P(K)}{K^2}\Delta K + \frac{1}{T_i}\frac{P(K_0)+C(K_0)}{K_0^2}\Delta K_0 \\ + \frac{2}{T_i}\sum_{K=K_1}^{K_{max}} \frac{C(K)}{K^2}\Delta K - \frac{1}{T_i}\left(\frac{F}{K_0}-1\right)^2\end{aligned}$$

Note that here we can regard this calculation as the numerical approximation by CBOE VIX of (6) where the main approximation error comes[1]:

$$\begin{aligned}-2\mathbb{E}\left[\log\left(\frac{S_T}{F}\right)\right] \approx \sigma^2_{CBOE}T = 2\sum_{K=K_{min}}^{K_{-1}} \frac{P(K)}{K^2}\Delta K + \frac{P(K_0)+C(K_0)}{K_0^2}\Delta K_0 \\ + 2\sum_{K=K_1}^{K_{max}} \frac{C(K)}{K^2}\Delta K - \left(\frac{F}{K_0}-1\right)^2\end{aligned}$$

Step 4: **Determining VIX by interpolation**

$$VIX = 100 \times \sqrt{\frac{1}{T}\left[\frac{T_2-T}{T_2-T_1}\sigma^2_{CBOE,1}T_1 + \frac{T-T_1}{T_2-T_1}\sigma^2_{CBOE,2}T_2\right]} \tag{7}$$

[1]Note that $\Delta K$ is not explicitly defined in the formula given by Fukasawa *et al.*(1). Here in practice, we set $\Delta K_i = K_{i+1} - K_i$ for $i \leq -1$, $\Delta K_i = K_i - K_{i-1}$ for $i \geq 1$ and $\Delta K_0 = 2$.

## 2.2 New Method

We begin by presenting the theoretical basis of our new algorithm and then expand on the original paper by providing additional details in the following four steps.

### 2.2.1 Approximation Formula

Define $k := \log\left(\frac{K}{F}\right)$ be the log-moneyness, where $K$ is the strike price and $F = \mathbb{E}[S_T]$ is the forward price, then we can write $K = Fe^k$, and also define the mapping:

$$k \mapsto d_2(k) := d_2(k, \sigma(k)) = -\frac{k}{\sigma(k)\sqrt{T}} - \frac{\sigma(k)\sqrt{T}}{2} \tag{8}$$

Recall the Black-Scholes formula, we can define the undiscounted Black-Scholes price of a European put option as a function $P_{BS} : \mathbb{R} \times (0, \infty) \to (0, \infty)$ by:

$$P_{BS}(k, \sigma) := Fe^k\Phi(-d_2(k, \sigma)) - F\Phi(-d_1(k, \sigma))$$

where $\Phi$ is the standard normal distribution function, and $d_1 := d_2 + \sigma\sqrt{T}$.

Since $P_{BS}$ is increasing with respect to the volatility parameter, the inverse function $P_{BS}(k, \cdot)^{-1}$ is well-defined and gives the well-defined Black-Scholes implied volatility $\sigma : \mathbb{R} \to [0, \infty)$ by

$$\sigma(k) := P_{BS}(k, \cdot)^{-1}(P(Fe^k)) \tag{9}$$

where $P(Fe^k)$ is the undiscounted price of the put option with strike $K = Fe^k$.
Notice that $P(K) = \mathbb{E}[(K - S_T)^+]$ gives that

$$\begin{aligned}
\frac{dP}{dK}(K) &= \frac{d}{dK}\int_0^K (K - x) f_{S_T}(x) dx \\
&= \frac{d}{dK}\left(K\int_0^K f_{S_T}(x)dx\right) - \frac{d}{dK}\int_0^K x f_{S_T}(x)dx \\
&= \int_0^K f_{S_T}(x)dx + Kf_{S_T}(K) - Kf_{S_T}(K) \\
&= \int_0^K f_{S_T}(x)dx \\
\Rightarrow \frac{d^2P}{dK^2}(K) &= f_{S_T}(K)
\end{aligned} \tag{10}$$

which implies that

$$\begin{aligned}
-2\mathbb{E}\left[\log\left(\frac{S_T}{F}\right)\right] &= -2\int_0^\infty \log\left(\frac{K}{F}\right)\frac{d^2P}{dK^2}(K)dK \\
&= -2\int_{-\infty}^\infty k\frac{d^2P}{dK^2}(Fe^k)dk
\end{aligned} \tag{11}$$

For simplicity, we take $T = 1$ in the following part.

Define $D(K) := \mathbb{E}[K > S_T] = \frac{dP}{dK}(K)$ and $D_{BS}(K) := \frac{dP_{BS}}{dK}(K) = \Phi(-d_2(k, \sigma(k)))$. Then:

$$\begin{aligned}
D(K) = \frac{dP}{dK}(K) &= \frac{d}{dK}P_{BS}\left(\log\left(\frac{K}{F}\right), \sigma\left(\log\left(\frac{K}{F}\right)\right)\right) \\
&= D_{BS}(K) + \phi\left(-d_2\left(\log\left(\frac{K}{F}\right), \sigma\left(\log\left(\frac{K}{F}\right)\right)\right)\right)\frac{d\sigma}{dk}\left(\log\left(\frac{K}{F}\right)\right)
\end{aligned}$$

Define $g$ as the inverse function of $d_2$ as $g^{-1}(k) = d_2(k, \sigma(k))$. Then we can write:

$$D(Fe^k) = D_{BS}(Fe^k) + \phi(g^{-1}(k))\frac{d\sigma}{dk}(k)$$

where $\phi(x) = \phi(-x)$ is used.

Applying the fact that $\frac{d\phi}{dx}(x) = x\phi(x)$, we have

$$\begin{aligned}\frac{d^2P}{dK^2}(Fe^k) &= \frac{dD}{dK}(Fe^k) \\ &= \frac{1}{Fe^k}\phi(g^{-1}(k))\left\{-\frac{dg^{-1}}{dk}(k)\left(1 + g^{-1}(k)\frac{d\sigma}{dk}(k)\right) + \frac{d^2\sigma}{dk^2}(k)\right\}\end{aligned} \tag{12}$$

Substituting (12) to (11) under the condition that $\mathbb{E}[S_T^p] < \infty$ for some $p > 0$ that implies:

$$\lim_{k\to\pm\infty} \sigma(k)\phi(g^{-1}(k)) = 0 \quad \text{and} \quad \lim_{k\to\pm\infty} k\frac{d\sigma}{dk}(k)\phi(g^{-1}(k)) = 0$$

which is required for integration by part, and then we can get:

$$-2\mathbb{E}\left[\log\left(\frac{S_T}{F}\right)\right] = \int_{-\infty}^{\infty} \sigma(g(z))^2\phi(z)dz \tag{13}$$

Here $\sigma(\cdot)$ is the Black-Scholes implied volatility of put options.

Moreover, note that under the Black-Scholes model, the implied volatility given by the put and call options should be the same given the same strike and maturity since it assumes that the underlying asset follows a lognormal distribution. By similar process as (10), we can get:

$$\frac{d^2C}{dK^2}(K) = f_{S_T}(K) = \frac{d^2P}{dK^2}(K)$$

Hence the formula (13) also holds when $\sigma(\cdot)$ is the Black-Scholes implied volatility of call options.

### 2.2.2 Choice of Approximating Function of $\sigma(g(z))^2$

During the procedure of getting the approximation formula, it can be seen that it is enough for $\sigma(\cdot)$ to be continuously differentiable and its derivative to be absolutely continuous, which implies taking a $C^1-$interpolation/extrapolation by a piecewise polynomial as the approximating function of $\sigma(\cdot)$.

Moreover, the piecewise cubic polynomial is chosen to meet the following requirements implied by the features of $\sigma(g(z))^2$:

1. The extrapolation scheme should not induce a rapid decay of $\sigma(g(z))^2$ as $|z| \to \infty$
2. The interpolation scheme should not produce excessive oscillations, since the mapping $k \mapsto d_2(k, \sigma(k))^2 - d_1(k, \sigma(k))^2 = 2k$ is increasing, implies the following inequalities:
   (a) For $z > z_0 \geq 0$: $\hat{\sigma}(z) > \alpha_+(z; z_0) > (\hat{\sigma}(z_0) + z_0 - z)^+$
   (b) For $z_0 > z \geq 0$: $\hat{\sigma}(z) < \alpha_+(z; z_0) < \hat{\sigma}(z_0) + z_0 - z$
   (c) For $z < z_0 \leq 0$: $\hat{\sigma}(z) > \alpha_-(z; z_0)$
   (d) For $z < z_0 \leq z^*$ where $z^* < 0$ such that $\hat{\sigma}(z^*) = z^*$: $-z > \hat{\sigma}(z) > \alpha_-(z; z_0) > 0$

   where $\hat{\sigma}(z) := \sigma(g(z))$ and $\alpha_\pm(z; z_0) := -z \pm \sqrt{\hat{\sigma}(z_0)^2 + 2z_0\hat{\sigma}(z_0) + z^2}$ for $z, z_0 \in \mathbb{R}$.

### 2.2.3 Algorithm

Step 1: **Selecting the options to be involved**

Before calculation, we do a similar selection procedure of options used for computation as those in CBOE VIX. First, transactions data are used to determine the ATM strike price $K_0$, which is the strike price that minimizes the put-call differences and chooses the highest strike for equal differences case, i.e.

$$K_0 = \max\left(\arg\min_K |C(K,T) - P(K,T)|\right)$$

and then using the 1-Month Treasury Rate, the forward price $F$ is computed as:

$$F = K_0 + e^{rT}\left(C(K,T) - P(K,T)\right)$$

Then OTM puts are selected with strikes less than $K_0$, whereas OTM calls are selected with strikes greater than $K_0$ where all of the selected options have well-defined bid/ask prices. Moreover, the ratio of ask-to-bid prices is used to justify whether the mid-quote is reliable or not so that options with such a ratio greater than or equal to $c = 2$ are discarded.

Step 2: **Converting the data and further filtering the selected options**

For each option data included in Step 1, the corresponding implied volatility $\sigma(k)$ can be computed and gives data point $\left(d_2(k), \sigma(k)^2\right)$. Moreover, based on the decreasing monotonicity of $d_2(k)$, further filtration of options involved can be done by disregarding all put options with strike equal to and below the highest strike $K < K_0$ that breaks the monotonicity, and similarly disregarding all calls with strike equal to and above the lowest strike $K > K_0$ that breaks the monotonicity of $d_2$. Therefore, we achieve the longest interval for data points where the decreasing monotonicity of $d_2$ holds.

Step 3: **Constructing an approximation of $\sigma(g(z))^2$**

After the selection and further filtration of data points, assume that M data points are involved. Making $d_2(\cdot)$ data in ascending ordering and denoting by $x_1 < ... < x_M$, the corresponding $\sigma(\cdot)^2$ are denoted by $y_1, ..., y_M$, respectively. Then we are going to approximate a cubic polynomial $y(z) \approx \sigma(g(z))^2$ for $z \in \mathbb{R}$.

On $(-\infty, x_1]$ and $[x_M, \infty)$, constant extrapolation are used, i.e. $y = y_1$ and $y = y_M$[2].

On each interval $[x_i, x_{i+1}]$, $i = 1, ..., M-1$, defining a cubic polynomial to ensure the continuity and differentiability of the union and every point $(x_i, y_i)$ can be gone through, where the polynomial can be represented by:

$$y(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \quad x_i \le x \le x_{i+1}$$

To determine the coefficients, $y(x_i) = y_i$ and $y(x_{i+1}) = y_{i+1}$ are already settled. Moreover, we add restrictions on the slope of the cubic polynomial at each point to eliminate the two more degrees of freedom.

[2]Here, we fix a mistake in the original paper by changing "$y = x_0$ and $y = x_M$" to "$y = y_1$ and $y = y_M$" to get the correct extrapolation.

The slopes at two end points $x_1$ and $x_M$ are settled to be 0, i.e. $y'(x_1) = y'(x_M) = 0$. Then for $2 \leq j \leq M-1$, the slope of the cubic polynomial at point $(x_j, y_j)$ is chosen so that the tangent line at $(x_j, y_j)$ makes the same angle with the line linking $(x_{j-1}, y_{j-1})$, $(x_j, y_j)$ and the line linking $(x_j, y_j)$, $(x_{j+1}, y_{j+1})$, which gives the formula:

$$y'(x_j) = -\left(\frac{x_{j+1}-x_j}{l_{j+1}} - \frac{x_j - x_{j-1}}{l_j}\right) / \left(\frac{y_{j+1}-y_j}{l_{j+1}} - \frac{y_j - y_{j-1}}{l_j}\right)$$

Then on each interval $[x_i, x_{i+1}]$, $i = 1, ..., M-1$, substituting the four constraints of the polynomial, i.e. $y(x_i)$, $y(x_{i+1}$, $y'(x_i)$ and $y'(x_{i+1})$, we can get all the coefficients as follows:

$$\begin{aligned} a_i &= y_i \\ b_i &= y'(x_i) \\ c_i &= \frac{3\Delta y_i - \Delta x_i y'(x_{i+1}) - 2\Delta x_i y'(x_i)}{\Delta x_i^2} \\ d_i &= \frac{\Delta y_i - y'(x_i)\Delta x_i - c_i \Delta x_i^2}{\Delta x_i^3} \end{aligned} \tag{14}$$

where $\Delta x_i = x_{i+1} - x_i$ and $\Delta y_i = y_{i+1} - y_i$.
In summary, the cubic polynomial approximation of $\sigma(g(z))^2$ is:

$$\sigma(g(x))^2 \approx \begin{cases} y_1, & x < x_1 \\ a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3, & x_i \leq x \leq x_{i+1} \quad i = 1, ..., M-1 \\ y_M, & x > x_M \end{cases}$$

Step 4: **Integrating**

Here we compute the integral as follows:

$$\begin{aligned} \int_{-\infty}^{\infty} \sigma(g(z))^2 \phi(z) dz = & \int_{-\infty}^{x_1} \sigma(g(z))^2 \phi(z) dz + \sum_{i=1}^{M-1} \int_{x_i}^{x_{i+1}} \sigma(g(z))^2 \phi(z) dz \\ & + \int_{x_M}^{\infty} \sigma(g(z))^2 \phi(z) dz \end{aligned}$$

For each $1 \leq i \leq M-1$, we have:

$$\begin{aligned} \int_{x_i}^{x_{i+1}} \sigma(g(z))^2 \phi(z) dz \approx & \int_{x_i}^{x_{i+1}} \left(a_i + b_i(z-x_i) + c_i(z-x_i)^2 + d_i(z-x_i)^3\right) \phi(z) dz \\ & = a_i A_i + b_i B_i + c_i C_i + d_i D_i \end{aligned}$$

where $\Phi$ is the cdf of standard normal distribution and[3]

$$\begin{aligned} A_i &:= \Phi(x_{i+1}) - \Phi(x_i) \\ B_i &:= -\left(\phi(x_{i+1}) - \phi(x_i)\right) - x_i\left(\Phi(x_{i+1}) - \Phi(x_i)\right) \\ C_i &:= -\left(x_{i+1}\phi(x_{i+1}) - x_i\phi(x_i)\right) + 2x_i\left(\phi(x_{i+1}) - \phi(x_i)\right) + (1 + x_i^2)\left(\Phi(x_{i+1}) - \Phi(x_i)\right) \\ D_i &:= (1 - x_{i+1}^2)\phi(x_{i+1}) - (1 - x_i^2)\phi(x_i) + 3x_i\left(x_{i+1}\phi(x_{i+1}) - x_i\phi(x_i)\right) \\ & \quad - 3(1 + x_i^2)\left(\phi(x_{i+1}) - \phi(x_i)\right) - x_i(3 + x_i^2)(\Phi(x_{i+1}) - \Phi(x_i)) \end{aligned} \tag{15}$$

[3]For $C_i$, we fix a mistake in the original paper by eliminating the term "$x_i$" before $\phi(x_i)$ in the second term.

Hence, we have:

$$\int_{-\infty}^{\infty} \sigma(g(z))^2 \phi(z) dz \approx y_1 \Phi(x_1) + \sum_{i=1}^{M-1} (a_i A_i + b_i B_i + c_i C_i + d_i D_i) + y_M \Phi(-x_M)$$

where for $i = 1, ...M-1$, $a_i, b_i, c_i, d_i$ are defined as (14) , and $A_i, B_i, C_i, D_i$ are defined as (15).

# 3 Comparison and Empirical Evidence

In this section, we first theoretically analyzed the source of approximation errors, and then the new approach is contrasted with the CBOE process based on empirical tests using data generated by the Heston model, in order to verify the validation and improvement of the new method.

## 3.1 Source of approximation errors

There are multiple resources of approximation errors during this process. Firstly, the most obvious drawback of the CBOE approximation is that there may exist underestimation due to the diminished contributions of $P(K)$ when $K < K_{\min}$ and $C(K)$ on $K > K_{\max}$. When the available strike price range is not sufficiently broad, this is made even worse. Based on the thorough description of the new algorithm in the previous section, it can be seen that for CBOE approximation, there will be an influence when the ranges of strikes for the corresponding options are different. For instance, when the price of the put option at $K_{\min}$ becomes significantly high, the integrand $P(K)/K^2$ is not small enough around $K = K_{\min}$, the cutting-off of the integral over $[0, K_{\min}]$ will severely affect the approximation. Therefore, a non-negligible negative bias may appear for the CBOE procedure, especially during financial crises.

The new method improves this approximation through the process of extrapolation. Unlike other methods that may cause a positive bias(*Jiang and Tian* (3)), a constant extrapolation is chosen here to avoid it.

The second type of approximation error for the CBOE approximation is the discretization of the integral with respect to $K$. The new algorithm introduced a $C^1$ -interpolation by cubic polynomials in the Black-Scholes implied volatility scale since theoretically, $P(K) = \mathbb{E}\left[(K - S_T)_+\right]$ and $C(K) = \mathbb{E}\left[(S_T - K)_+\right]$ are continuous functions and they are $C^1$ if $S_T$ admits a density.

It is worth noting that in the process of generating the model, it is found that the CBOE procedure may not run properly when the market data is not ideal. This is because if quotes of the $K_0$ put option or the $K_0$ call option are null or the bid price is higher than the ask price, then the close volatility index cannot be calculated, While the CBOE procedure is no longer viable, the new method remains effective.

### 3.2 Heston Method

To examine the above two types of approximation errors and test the extent to which our method indeed improves the approximation, the Heston model is introduced here($r = 0$, no drift term in the dynamic of $S_t$):

$$\begin{aligned}
\mathrm{d}S_t &= S_t\sqrt{\nabla_t}\left[\rho \mathrm{d}W_t^1 + \sqrt{1-\rho^2}\ \mathrm{d}W_t^2\right] \\
\mathrm{d}V_t &= \lambda\left(v - V_t\right)\mathrm{d}t + \eta\sqrt{\nabla_t}\ \mathrm{d}W_t^1
\end{aligned}$$

Where the stock price $S_t$ follows a geometric Brownian motion with stochastic volatility. The square of the volatility $V_t$ (the variance) follows a CIR process. And $\left(W^1, W^2\right)$ is a two-dimensional standard Brownian motion. The parameters are:
- $\lambda$ mean reversion coefficient of the variance process
- $v$ long term mean of the variance process
- $\eta$ volatility coefficient of the variance process
- $\rho$ correlation between $W^1$ and $W^2$ We will also require that $2\kappa\theta > \sigma^2$ (Feller condition) for this model. And the pseudocode for the Heston model is shown below.

Heston model can not only capture more realistic market dynamics compared to other models, such as the Black-Scholes model, which assumes constant volatility, but it also has a closed-form solution for option prices, which makes it computationally efficient for pricing these options, so that here we can determine the theoretical call and put option prices and derive the explicit formula for the expected annualized quadratic variation:

$$\frac{1}{T}\mathbb{E}\left[\langle\log(S)\rangle_T\right] = \frac{1}{T}\mathbb{E}\left[\int_0^T V_t\ \mathrm{d}t\right] = v + \frac{1-e^{-\lambda T}}{\lambda T}\left(V_0 - v\right)$$

### 3.3 Comparison

Here we started the comparison between the CBOE process and the new method based on the artificial options data generated by the Heston model. Firstly, We generate call and put option prices with the same set of strikes on the basis of the option data. Unlike the original article, which chose two different maturities to compare the differences in the data, here we generate two different datasets, both the $T$ is set to be $30/365 \approx 0.0822$. The initial price $S_0$ of the underlying asset is set to 4100 and the interest rate is set to be 0. one with a relatively narrow strike range and the other with a wider range, so as to measure the difference between the two algorithms. See Table 1 and Table 2 for the generated data narrow maturity range and wide maturity range, respectively.

To further approach normal market conditions, the author randomizes the bid-ask spread using geometric random variables with success probability $p = 0.8$. From our point of view, the reason for using geometric random variables is as follows:

- **Skewness towards lower values** The geometric distribution is positively skewed, meaning that it has a longer tail on the right side of the distribution. In the context of the bid-ask spread, this implies that prices are more likely to be closer to the true value, with decreasing probability for prices further away. This property aligns with

| **Strike** | **Call** | | | **Put** | | |
|---|---|---|---|---|---|---|
| | **Ask** | **Theoretical** | **Bid** | **Ask** | **Theoretical** | **Bid** |
| 2000 | 2099.3 | 2100.3 | 2101.3 | 0.285 | 0.3 | 0.315 |
| 2100 | 1999.58 | 2000.58 | 2001.58 | 0.5510 | 0.58 | 0.609 |
| 2200 | 1900.05 | 1901.05 | 1902.05 | 0.9975 | 1.05 | 1.1025 |
| 2300 | 1800.81 | 1801.81 | 1802.81 | 1.7195 | 1.81 | 1.9005 |
| 2400 | 1702.01 | 1703.01 | 1704.01 | 2.8595 | 3.01 | 3.1605 |
| 2500 | 1603.81 | 1604.81 | 1605.81 | 4.5695 | 4.81 | 5.0505 |
| 2600 | 1506.44 | 1507.44 | 1508.44 | 6.44 | 7.44 | 9.4400 |
| 2700 | 1410.14 | 1411.14 | 1412.14 | 10.14 | 11.14 | 12.14 |
| 2800 | 1315.2 | 1316.2 | 1318.2 | 15.2 | 16.2 | 17.2 |
| 2900 | 1221.97 | 1222.97 | 1223.97 | 21.97 | 22.97 | 23.97 |
| ... | ... | ... | ... | ... | ... | ... |
| 6500 | 3.5435 | 3.73 | 3.9165 | 2401.73 | 2403.73 | 2404.73 |
| 6600 | 2.7835 | 2.93 | 3.0765 | 2501.93 | 2502.93 | 2503.93 |
| 6700 | 2.061 | 2.29 | 2.4045 | 2601.29 | 2602.29 | 2603.29 |
| 6800 | 1.691 | 1.78 | 1.958 | 2700.78 | 2701.78 | 2703.78 |
| 6900 | 1.242 | 1.38 | 1.4490 | 2800.38 | 2801.38 | 2802.38 |
| 7000 | 1.0070 | 1.06 | 1.113 | 2900.06 | 2901.06 | 2902.06 |
| 7100 | 0.7790 | 0.82 | 0.943 | 2999.82 | 3000.82 | 3001.82 |
| 7200 | 0.5985 | 0.63 | 0.6615 | 3099.63 | 3100.63 | 3101.63 |
| 7300 | 0.4560 | 0.48 | 0.528 | 3199.48 | 3200.48 | 3201.48 |
| 7400 | 0.3515 | 0.37 | 0.407 | 3299.37 | 3300.37 | 3302.37 |

Table 1: Artificial data with narrow strike range and parameter set A

the observation that spreads are generally tighter in liquid markets, while they may occasionally widen during periods of volatility or illiquidity.

- **Discrete nature** The bid-ask spread in financial markets is inherently discrete, with prices occurring at distinct levels such as in increments of a tick size.
- **Others** The geometric distribution exhibits a memoryless property, which allows the simulation to better capture the unpredictable nature of market fluctuations. And it is relatively simple and easy to work with, as it has only one parameter $p$ that needs to be estimated.

More specifically, the series of ask prices are sampled in such a way that, with probability $p$, it becomes the lowest price above the true value and, with probability $(1-p)p$, it becomes the second-lowest price and so on and so forth. It becomes the subsequent prices in the same geometric manner, and the series of bid prices are chosen similarly. It is worth noting that to avoid negative bid prices for options, we empirically use 5% of the stock prices as the tick size when the option prices are less than 5.
The expected annualized quadratic variations and the approximated values by the CBOE procedure and the new method are obtained based on the following parameter sets:

| Strike | Call | | | Put | | |
|---|---|---|---|---|---|---|
| | Ask | Theoretical | Bid | Ask | Theoretical | Bid |
| 200.0 | 3899.0 | 3900.0 | 3903.0 | -0.0 | 0.0 | -0.0 |
| 400.0 | 3698.0 | 3700.0 | 3701.0 | -0.0 | 0.0 | -0.0 |
| 600.0 | 3499.0 | 3500.0 | 3501.0 | 0.0 | 0.0 | 0.0 |
| 800.0 | 3299.0 | 3300.0 | 3301.0 | 0.0 | 0.0 | 0.0 |
| 1000.0 | 3099.0 | 3100.0 | 3101.0 | 0.0 | 0.0 | 0.0 |
| 1200.0 | 2899.0 | 2900.0 | 2901.0 | 0.0 | 0.0 | 0.0 |
| 1400.0 | 2699.0 | 2700.0 | 2701.0 | 0.0 | 0.0 | 0.0 |
| 1500.0 | 2599.0 | 2600.0 | 2601.0 | 0.0 | 0.0 | 0.0 |
| 1600.0 | 2498.01 | 2500.01 | 2502.01 | 0.01 | 0.0090 | 0.0105 |
| 1700.0 | 2398.03 | 2400.03 | 2401.03 | 0.03 | 0.0285 | 0.0345 |
| ... | ... | ... | ... | ... | ... | ... |
| 5600.0 | 26.97 | 27.97 | 28.97 | 1527.97 | 1525.97 | 1528.97 |
| 5800.0 | 17.41 | 18.41 | 20.41 | 1718.41 | 1717.41 | 1719.41 |
| 6000.0 | 10.91 | 11.91 | 12.91 | 1911.91 | 1910.91 | 1914.91 |
| 6200.0 | 6.58 | 7.58 | 8.58 | 2107.58 | 2106.58 | 2108.58 |
| 6400.0 | 4.5125 | 4.75 | 4.9875 | 2304.75 | 2299.75 | 2306.75 |
| 6600.0 | 2.637 | 2.93 | 3.0765 | 2502.93 | 2501.93 | 2503.93 |
| 6800.0 | 1.691 | 1.78 | 1.869 | 2701.78 | 2699.78 | 2702.78 |
| 7000.0 | 1.007 | 1.06 | 1.113 | 2901.06 | 2900.06 | 2902.06 |
| 7200.0 | 0.5985 | 0.63 | 0.6615 | 3100.63 | 3099.63 | 3101.63 |
| 7400.0 | 0.333 | 0.37 | 0.3885 | 3300.37 | 3299.37 | 3301.37 |

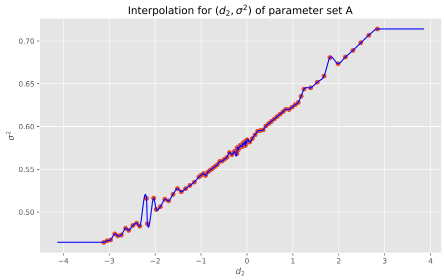
Table 2: Artificial data with wide strike range and parameter set A

**Parameter Set A**: $\lambda = 1, v = 0.2, \eta = 0.5, \rho = 0.8$ and $V_0 = 0.6$;
**Parameter Set B**: $\lambda = 1, v = 0.2, \eta = 1.0, \rho = 0.4$ and $V_0 = 0.6$;
**Parameter Set C**: $\lambda = 5, v = 0.04, \eta = 1.0, \rho = 0.4$ and $V_0 = 0.6$;
**Parameter Set D**: $\lambda = 1.5, v = 0.04, \eta = 0.3, \rho = 0.7$ and $V_0 = 0.04$;

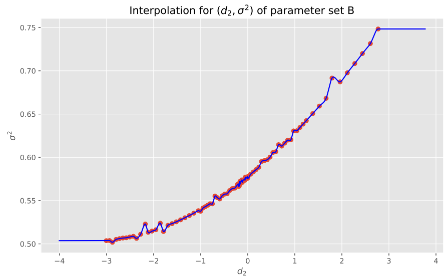
The authors chose these parameters based on the consideration: A, B and C represent the market conditions during financial crises, with instantaneous volatility $\sqrt{V_0} \approx 77\%$; The parameters in D are typical values obtained by calibrating on a regular day with instantaneous volatility $\sqrt{V_0} = 20\%$. Interpolation for $(d_2, \sigma^2)$ of parameter set A, B, C, and D for narrow strike range can be seen in Figure 1 and Figure 2, respectively.

Under this average volatility level, the tails of the distribution of $S_T$ are light, and hence the approximation errors due to the cutting-off of tails are negligible, which can be more visual to show the errors due to the numerical integration of the CBOE procedure. The result is shown in Table 3.

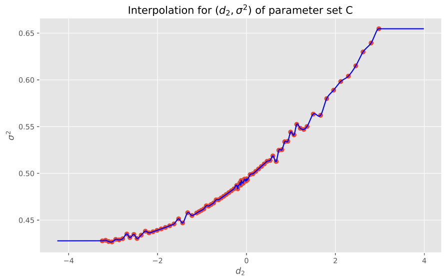
The column "New" stands for the values resulting from the application of the new method. it can be concluded from Table 3 that the CBOE method underestimates the expected quadratic variation where our method is found to improve the estimation accuracy both in two scenarios. To be specific, the new method is stable in both settings, while the CBOE algorithm suffers more serious underestimation problems in the narrow strike range.



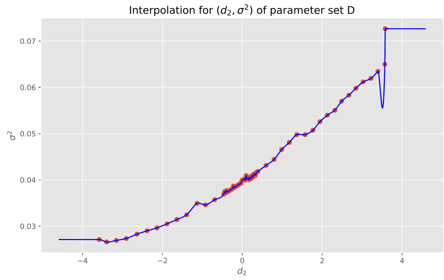

(a) Parameter Set A




(b) Parameter Set B

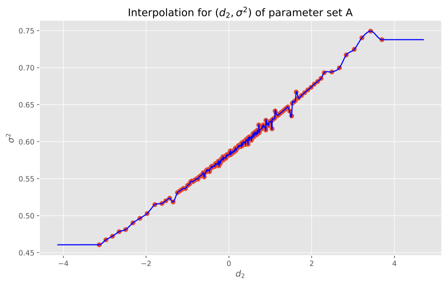



(c) Parameter Set C

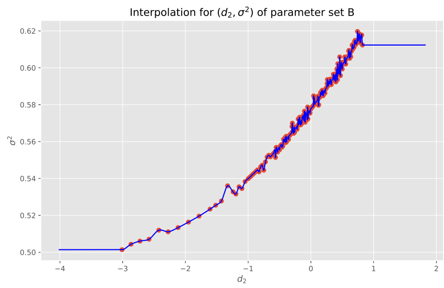



(d) Parameter Set D

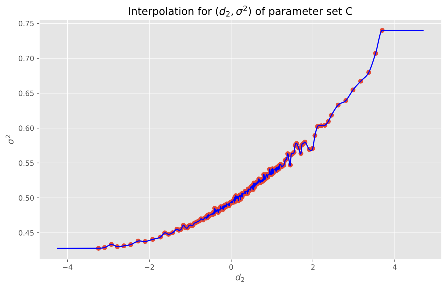
Figure 1: Interpolation for $(d_2, \sigma^2)$ of parameter set A, B, C, and D for narrow strike range



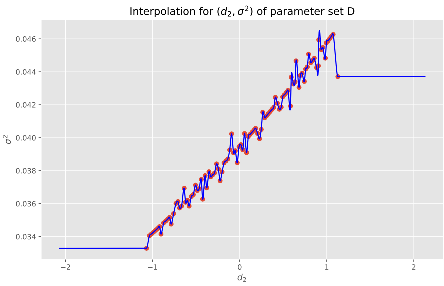

(a) Parameter Set A




(b) Parameter Set B




(c) Parameter Set C




(d) Parameter Set D

Figure 2: Interpolation for $(d_2, \sigma^2)$ of parameter set A, B, C, and D for wider strike range

Table 3: Expected annualized quadratic variations.

| Parameter | Narrow range | | | Wide range | | |
|---|---|---|---|---|---|---|
| | True | New | CBOE | True | New | CBOE |
| A | 0.5840 | 0.5842 | 0.5576 | 0.5840 | 0.5838 | 0.5735 |
| B | 0.5840 | 0.5844 | 0.5577 | 0.5840 | 0.5760 | 0.5736 |
| C | 0.4992 | 0.4990 | 0.4755 | 0.4992 | 0.4990 | 0.4906 |
| D | 0.0400 | 0.0402 | 0.0364 | 0.0400 | 0.0393 | 0.0392 |

Therefore, the new algorithm is especially theoretically meaningful in the context of this financial crisis setting.

# 4 Conclusion and Further Improvement

In this report, we thoroughly introduce the methodology of CBOE VIX and the new algorithm for computing the volatility index including the theory and step-by-step algorithm. Then the comparison between the two approximation algorithms is conducted based on data generated by the Heston model, and the numerical results are analyzed under different parameter settings and strike ranges.

The referenced study presents a novel model-free method for estimating the expected quadratic variations in asset prices. This connection between quadratic variations and option prices is model-free and demonstrated within the well-known Black-Scholes implied volatility framework. The new approximation method circumvents numerical integration by leveraging the integral structure concerning the standard normal density and applying polynomial interpolation to the integrated. Therefore, the new algorithm demonstrates superior numerical efficiency compared to the CBOE procedure, which depends on discretization for integral evaluations. It can also be seen that the improvement is more obvious during economic and financial shocks.

This model is an improvement built upon the foundation of many previous research papers, which leads to extremely accurate and robust implied volatility estimates, even for companies with no at-the-money option or those with non-normally distributed returns. Furthermore, this new method is cited and extended in other research papers by investigating the dynamics of model-free implied volatility surfaces, examining the statistical properties of volatility indices, or building a robust replication theory of volatility derivatives.

As the algorithm is already well established, we only propose a minor enhancement direction. The final result is calculated by interpolating the data from the two T's before and after the 30 days when there is option data available in the market. However, in our numerical tests, we found that there may be dates with few available data of options (a small range of $K$), reducing the accuracy of the result. Therefore, it may be appropriate to consider choosing another close date, which may further improve the accuracy of the algorithm.

# 5 Bibliography


[1] M. Fukasawa, I. Ishida, N. Maghrebi, K. Oya, M. Ubukatai, and K. Yamaza, Model-free implied volatility: From surface to index, *International Journal of Theoretical and Applied Finance* **14**(4) (2011) 433–463.

[2] Chicago Board Options Exchange, Cboe Volatility Index Mathematics Methodology (2022). URL: https://cdn.cboe.com/api/global/us_indices/governance/Cboe_Volatility_Index_Mathematics_Methodology.pdf

[3] G.T. Jiang, and Y.S. Tian, Extracting model-free volatility from option prices: An examination of the VIX index, *Journal of Derivatives* (2007) 35–60.