Numerical methods
Computer project №4

Gleb Zhdanko

March 2025

General Information

The computations and visualizations were performed on a MacBook Pro (13-inch, 2020, Two Thunderbolt 3 Ports) equipped with a 1.4 GHz Quad-Core Intel Core i5 processor.

I used Python 3.12.4. The following Python libraries were utilized in the implementation:

- `numpy`: for numerical computations,
- `scipy`: for solving linear systems and other numerical methods,
- `matplotlib`: for creating high-quality visualizations.

I am using notation for last elements like in python: u_{-1} - means last element, u_{-2} - second element from the end and so on.

Problem 1

Consider the linear advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = f, \tag{1}$$

where a is a real number. Demonstrate that the following modified Lax-Friedrichs scheme for Eq. (1), given by

$$v_j^{n+1} = \frac{1}{2}(v_{j+1}^n + v_{j-1}^n) - \frac{a\lambda}{1+(a\lambda)^2}(v_{j+1}^n - v_{j-1}^n) + \Delta t f_j^n,$$

is stable for all values of $\lambda = \Delta t/\Delta x$. Discuss how this unconditionally stable explicit scheme relates to the following well-known theorem:

Theorem: There are no explicit, unconditionally stable, consistent finite difference schemes for hyperbolic systems of partial differential equations

Solution:

Let's perform a von Neumann stability analysis. To analyze the numerical scheme, we consider the homogeneous equation, meaning that $f = 0 \Leftrightarrow f_i = 0\ \forall i$. This is because the numerical behavior of the scheme is determined by how it propagates errors in the absence of external forcing. The source term f can only contribute a bounded effect; thus, if the scheme is unstable for the homogeneous equation, it will also be unstable for the full equation.

Let us assume a solution of the form $u_j^n = \sigma^n e^{ikx_j}$. Then, the numerical scheme can be written as:

$$
\begin{aligned}
\sigma^{n+1}e^{ikx_j} &= \frac{1}{2}(\sigma^n e^{ikx_{j+1}} + \sigma^n e^{ikx_{j-1}}) - \frac{a\lambda}{1+(a\lambda)^2}(\sigma^n e^{ikx_{j+1}} - \sigma^n e^{ikx_{j-1}}) \quad | : \sigma^n e^{ikx_j}
\end{aligned}
$$

$$
\Rightarrow \sigma = \frac{1}{2}(e^{ik\Delta x} + e^{-ik\Delta x}) - \frac{a\lambda}{1+(a\lambda)^2}(e^{ik\Delta x} - e^{-ik\Delta x})
$$

$$
\Leftrightarrow \sigma = \cos(k\Delta x) - \frac{a\lambda}{1+(a\lambda)^2} \cdot 2i\sin(k\Delta x)
$$

$$
|\sigma|^2 \leq 1 \Leftrightarrow \cos(k\Delta x)^2 + \frac{a^2\lambda^2}{(1+(a\lambda)^2)^2} 4\sin(k\Delta x)^2 \leq 1
$$

$$
\Leftrightarrow \cos(k\Delta x)^2 + 2(a\lambda)^2\cos(k\Delta x)^2 + \cos(k\Delta x)^2 \cdot (a\lambda)^4 + 4(a\lambda)^2\sin(k\Delta x)^2 \leq 1 + 2(a\lambda)^2 + (a\lambda)^4
$$

$$
\Leftrightarrow 1 - \cos(k\Delta x)^2 - 2(a\lambda)^2\sin(k\Delta x)^2 + (a\lambda)^4(1-\cos(k\Delta x)^2) \geq 0
$$

$$
\Leftrightarrow \sin(k\Delta x)^2 - 2(a\lambda)^2\sin(k\Delta x)^2 + ((a\lambda)^2\sin(k\Delta x))^2 \geq 0
$$

$$
\Leftrightarrow (\sin(k\Delta x) - (a\lambda)^2\sin(k\Delta x))^2 \geq 0
$$

Since this inequality always holds, the method is stable for all values of $\lambda = \Delta t/\Delta x$.

Next, we check the consistency of the method using an operator representation:

$$
u_{j+r}^{n+k} = e^{D_x r\Delta x}e^{D_t k\Delta t}u_j^n.
$$

Substituting this into the numerical scheme gives:

$$
e^{D_t\Delta t}u_j^n = \left[\frac{1}{2}(e^{D_x\Delta x} + e^{-D_x\Delta x}) - \frac{a\lambda}{1+(a\lambda)^2}(e^{D_x\Delta x} - e^{-D_x\Delta x})\right]u_j^n + \Delta t f_j^n
$$

$$
(1 + D_t\Delta t + \frac{1}{2}(D_t\Delta t)^2 + O(\Delta t^3))u_j^n =
$$

$$
\left[\frac{1}{2}(2 + (D_x\Delta x)^2 + O(\Delta x^4)) - \frac{a\lambda}{1+(a\lambda)^2}(2D_x\Delta x + \frac{1}{3}(D_x\Delta x)^3 + O(\Delta x^5))\right]u_j^n + \Delta t f_j^n
$$

$$
(D_t\Delta t + \frac{2a\lambda}{1+(a\lambda)^2}D_x\Delta x)u_j^n - \Delta t f_j^n =
$$

$$
\left[\frac{1}{2}(D_x\Delta x)^2 - \frac{a\lambda}{3+3(a\lambda)^3}(D_x\Delta x)^3 - \frac{1}{2}(D_t\Delta t)^2 + O(\Delta x^4, \Delta t^3)\right]u_j^n.
$$

To recover the initial equation, we set $\frac{2a\lambda}{1+(a\lambda)^2} = 1$, which leads to $a\lambda = 1 \Rightarrow \Delta t = a\Delta x$. In this case, the leading error term is:

$$
\frac{1}{2}(D_x\Delta x)^2 - \frac{a\lambda}{3+3(a\lambda)^3}(D_x\Delta x)^3 - \frac{1}{2}(D_t\Delta t)^2 + O(\Delta x^4, \Delta t^3) =
$$

$$
\frac{1}{2}(D_x\Delta x)^2 - \frac{1}{6}(D_x\Delta x)^3 - \frac{1}{2}(D_t\Delta t)^2 + O(\Delta x^4, \Delta t^3) =
$$

$$
\frac{1}{2}(D_x\Delta x)^2 - \frac{1}{2}(D_t a\Delta x)^2 + O(\Delta x^3) = \frac{1}{2}\Delta x^2(D_x^2 - a^2D_t^2) + O(\Delta x^3).
$$

Thus, the leading error term is $\frac{1}{2}\Delta x^2(D_x^2 - a^2D_t^2)$. In cases where $\frac{2a\lambda}{1+(a\lambda)^2} \neq 1$, the scheme is not consistent with the initial equation.

Remark: From the lecture notes: **Consistency:** A finite difference equation is consistent with a differential equation if the difference between the finite difference equation and the differential equation (i.e., the truncation error) goes to zero as the grid spacings independently tend to zero. However, it is convenient to say that a scheme is *consistent with the initial equation* if the error goes to zero as the grid size vanishes, possibly with some dependencies.

To summarize, the scheme is always stable and consistent with the initial equation only when $\Delta t = a\Delta x$. This does not contradict the theorem, as the theorem requires consistency in the general case.

Problem 2

The difference approximation

$$
\frac{u_j^{n+1} - (1-\theta)u_j^n - \theta u_j^{n-1}}{(1+\theta)\Delta t} = \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2}
$$

is used for approximating a solution of the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

in the domain $0 \le x \le 1, t \ge 0$. Show that the difference scheme is consistent. Investigate the stability properties for different values of θ, $(0 \le \theta \le 1)$ and $r = \Delta t/\Delta x^2$.

Solution: Let's use operator representation for analysing consistency:

$$u_{j+r}^{n+k} = e^{D_x r\Delta x} e^{D_t k\Delta t} u_j^n.$$

$$\frac{[e^{D_t\Delta t} - (1-\theta) - \theta e^{-D_t\Delta t}]u_j^n}{(1+\theta)\Delta t} = \frac{[e^{-D_x\Delta x} - 2 + e^{D_x\Delta x}]u_j^n}{\Delta x^2}$$

Let's consider only operators for simplicity of notation, without u_j^n:

$$\frac{1 + D_t\Delta t + \frac{(D_t\Delta t)^2}{2} + \frac{(D_t\Delta t)^3}{6} + O(\Delta t^4) - 1 + \theta - \theta + \theta D_t\Delta t - \theta\frac{(D_t\Delta t)^2}{2} + \theta\frac{(D_t\Delta t)^3}{6} + O(\Delta t^4)}{(1+\theta)\Delta t} =$$

$$= \frac{1 - D_x\Delta x + \frac{(D_x\Delta x)^2}{2} - \frac{(D_x\Delta x)^3}{6} + \frac{(D_x\Delta x)^4}{24} - 2 + 1 + D_x\Delta x + \frac{(D_x\Delta x)^2}{2} + \frac{(D_x\Delta x)^3}{6} + \frac{(D_x\Delta x)^4}{24} + O(\Delta x^6)}{\Delta x^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{(1+\theta)D_t\Delta t + (1-\theta)\frac{(D_t\Delta t)^2}{2} + (1+\theta)\frac{(D_t\Delta t)^3}{6} + O(\Delta t^4)}{(1+\theta)\Delta t} =$$

$$= \frac{(D_x\Delta x)^2 + \frac{1}{12}(D_x\Delta x)^4 + O(\Delta x^6)}{\Delta x^2} \Leftrightarrow$$

$$\Leftrightarrow D_t - D_x^2 + \frac{1-\theta}{1+\theta}\frac{D_t^2\Delta t}{2} + \frac{D_t^3\Delta t^2}{6} - \frac{1}{12}D_x^4\Delta x^2 + O(\Delta t^3, \Delta x^4) = 0$$

If $\theta = 1$ then leading error term is: $\frac{D_t^3\Delta t^2}{6} - \frac{1}{12}D_x^4\Delta x^2$.

Else, leading error term is: $\frac{1-\theta}{1+\theta}\frac{D_t^2\Delta t}{2} - \frac{1}{12}D_x^4\Delta x^2$. However, for any $\theta \in [0, 1]$ the method is consistent.

Let's investigate stability. For that we will use von Neumann method:

$$u_j^n = \sigma^n e^{ikx_j}.$$

Substitute into the difference scheme:

$$\frac{\sigma^{n+1}e^{ikx_j} - (1-\theta)\sigma^n e^{ikx_j} - \theta\sigma^{n-1}e^{ikx_j}}{(1+\theta)\Delta t} = \frac{\sigma^n e^{ikx_{j-1}} - 2\sigma^n e^{ikx_j} + \sigma^n e^{ikx_{j+1}}}{\Delta x^2} \quad \Big| : \sigma^{n-1}e^{ikx_j}$$

$$\frac{\sigma^2 - (1-\theta)\sigma - \theta}{(1+\theta)\Delta t} = \frac{\sigma e^{-ik\Delta x} - 2\sigma + \sigma e^{ik\Delta x}}{\Delta x^2}$$

Let's simplify the right hand side:

$$\frac{\sigma e^{-ik\Delta x} - 2\sigma + \sigma e^{ik\Delta x}}{\Delta x^2} = \frac{2\sigma}{\Delta x^2}(cos(k\Delta x) - 1) = \frac{-4\sigma sin^2(k\Delta x/2)}{\Delta x^2}$$

Let's make substitution: $r = \frac{\Delta t}{\Delta x^2}$. Then multiplying both sides by $(1+\theta)\Delta t$ we get:

$$\sigma^2 - (1-\theta)\sigma - \theta + 4r\sigma sin^2(k\Delta x/2)(1+\theta) = 0$$

$$\begin{cases} \sigma_1 = \frac{1-\theta-4r(1+\theta)sin^2(k\Delta x/2) + \sqrt{(1-\theta-4r(1+\theta)sin^2(k\Delta x/2))^2 + 4\theta}}{2} \\ \sigma_2 = \frac{1-\theta-4r(1+\theta)sin^2(k\Delta x/2) - \sqrt{(1-\theta-4r(1+\theta)sin^2(k\Delta x/2))^2 + 4\theta}}{2} \end{cases}$$

Since we are interested when $|\sigma| \le 1$, we can find maximum absolute value of σ_1, σ_2. From method of staring we notice that $\sigma_1 \ge 0, \sigma_2 \le 0$, $\sin^2(k\Delta x/2) \ge 0$.

Let's find how $\sigma_{1,2}$ depends on $A = \sin^2(k\Delta x/2)$:

$$\frac{d\sigma_{1,2}}{dA} = \frac{-4r(1+\theta) \pm \frac{(1-\theta-4r(1+\theta)sin^2(k\Delta x/2))\cdot(-4r(1+\theta))}{\sqrt{(1-\theta-4r(1+\theta)sin^2(k\Delta x/2))^2+4\theta}}}{2} =$$

$$= -2r(1+\theta)\cdot\left(1 \pm \frac{(1-\theta-4r(1+\theta)sin^2(k\Delta x/2))}{\sqrt{(1-\theta-4r(1+\theta)sin^2(k\Delta x/2))^2+4\theta}}\right)$$

$$-2r(1+\theta) \leq 0 \text{ (since } r \geq 0,\ \theta \in [0,1]),$$

$$1 \pm \frac{(1-\theta-4r(1+\theta)sin^2(k\Delta x/2))}{\sqrt{(1-\theta-4r(1+\theta)sin^2(k\Delta x/2))^2+4\theta}} \geq 0 \text{ (since absolute value of fraction is always less than 1)} \Rightarrow$$

$$\Rightarrow \sigma_{1,2} \downarrow \text{ as } A \uparrow \Rightarrow$$

$$\Rightarrow |\sigma_1|_{max} = |\sigma_1(A=0)|_{max},\ |\sigma_2|_{max} = |\sigma_2(A=1)|_{max}$$

Max of σ_1 is when $sin^2(k\Delta x/2) = 0 \Rightarrow$

$$\sigma_1 \leq \frac{1-\theta+2\theta}{2} = \frac{1+\theta}{2} \leq 1 \text{ always.}$$

Minimum of σ_2 when $\sin^2(k\Delta x/2) = 1$:

$$\sigma_2 \geq \frac{1-\theta-4r(1+\theta)-\sqrt{(1-\theta-4r(1+\theta))^2+4\theta}}{2} \geq -1 \Rightarrow$$

$$\Rightarrow 1-\theta-4r(1+\theta)-\sqrt{(1-\theta-4r(1+\theta))^2+4\theta} \geq -2.$$

Let's solve equation (when lhs is equal to -2) as $r(\theta)$ with substitution $y = 1-\theta-4r(1+\theta)$:

$$y - \sqrt{y^2+4\theta} = -2 \Rightarrow y = \theta - 1 \Rightarrow r = \frac{1-\theta}{2(1+\theta)}.$$

We are interested when lhs is ≥ -2, then we need to take $r \leq \frac{1-\theta}{2(1+\theta)}$.

Maximum value of r is: $r_{max} = r(0) = \frac{1}{2}$. (this aligns with our theory when we use FCTS scheme)

To sum up: The scheme is consistent and is stable for $r \leq \frac{1-\theta}{2(1+\theta)}$.

Problem 3

Consider the linear advection-diffusion-dispersion equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3},$$

where c, α, and β are real non-negative constants. The general solution of the above equation can be written in the form

$$u(x,t) = Aexp(\lambda t + ikx),$$

where A is the amplitude, $\lambda = \sigma + i\omega$ is the complex frequency, and k is the wavenumber. Using each of the following initial conditions:

1. $$u(x,0) = \begin{cases} sin(\pi x) & \text{for } 0 \leq x \leq 1, \\ 0 & \text{for } x > 1 \end{cases}$$

2. $$u(x,0) = \begin{cases} 1-|2x-1| & \text{for } 0 \leq x \leq 1, \\ 0 & \text{for } x > 1 \end{cases}$$

consider the following problems:

- Convection: $\alpha = \beta = 0, c = 0.1, u(0,t) = 0, x \in [0,\infty]$, and the Lax-Wendroff scheme (LW);
- Diffusion: $c = \beta = 0, \alpha = 0.1/\pi^2, u(0,t) = u(1,t) = 0, x \in [0,1]$, and the scheme consisting of forward Euler in time and central difference in space(FTCS);

- Dispersion: $c = \alpha = 0, \beta = 0.001/\pi^2, u(0,t) = 0, u(x,t) \to 0$ as $x \to \infty, x \in [0, \infty]$, and the Crank-Nicolson scheme(CN).

In each case, discuss the quantitive and qualitative behavior of the numeral solution at times $t = 0(2)20$, by making the use of corresponding modified equations and analytical solutions.

Solution:

1. Convection equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

Lax-Wendroff scheme:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}c\left[u_{i+1}^n - u_{i-1}^n\right] + \frac{\Delta t^2}{2\Delta x^2}c^2\left[u_{i+1}^n - 2u_i^n + u_{i-1}^n\right]$$

I have chosen time step with accordance to stability region $\frac{c\Delta t}{\Delta x} < 1$:

$$\Delta t = 0.5\frac{\Delta x}{c}.$$

2. Diffusion equation:

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$$

FTCS scheme:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha\frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2}$$

$$u_i^{n+1} = u_i^n + \alpha\frac{\Delta t}{\Delta x^2}(u_{i-1}^n - 2u_i^n + u_{i+1}^n)$$

I have chosen time step with accordance to stability region $\frac{2\alpha\Delta t}{\Delta x^2} < 1$:

$$\Delta t = 0.5\frac{\Delta x^2}{2\alpha}.$$

3. Dispersion equation:

$$\frac{\partial u}{\partial t} = \beta\frac{\partial^3 u}{\partial x^3}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\beta}{2}\left(\frac{u_{i-1}^{n+1} - 3u_i^{n+1} - 3u_{i+1}^{n+1} - u_{i+2}^{n+1}}{2\Delta x^3} + \frac{u_{i-1}^n - 3u_i^n - 3u_{i+1}^n - u_{i+2}^n}{2\Delta x^3}\right)$$

$$u_{i-1}^{n+1}\left(\frac{-\beta\Delta t}{4\Delta x^3}\right) + u_i^{n+1}\left(1 + \frac{3\beta\Delta t}{4\Delta x^3}\right) + u_{i+1}^{n+1}\left(\frac{3\beta\Delta t}{4\Delta x^3}\right) + u_{i+2}^{n+1}\frac{\beta\Delta t}{4\Delta x^3} =$$
$$= u_{i-1}^n\frac{\beta\Delta t}{4\Delta x^3} + u_i^n\left(1 - \frac{3\beta\Delta t}{4\Delta x^3}\right) + u_{i+1}^n\frac{-3\beta\Delta t}{4\Delta x^3} + u_{i+2}^n\frac{-\beta\Delta t}{4\Delta x^3}$$

Here we need to clarify how to set boundary conditions. We have that $u(x,t) \to 0$ as $x \to \infty$. I have approximated infinity with $x = 10$. Since our scheme uses u_{i+1} and u_{i+2} we need to set some boundary condition for $x = 10$. I have set $u_{-1} = u_{-2} = 0$ - as approximation to condition that function decays to zero as x goes to infinity.

Crank-Nicolson scheme is stable everywhere, however on the internet I have found that it's better to satisfy the following condition:

$$\frac{\beta\Delta t}{\Delta x^4} << 1.$$

Therefore I have set:

$$\Delta t = 0.01\frac{\Delta x^4}{\beta}.$$

For Crank-Nicolson scheme I used first order approximation of the third derivative, because it contains less terms and easier to implement. This accuracy is enough to get the idea of the solution and it's behaviour and problems. However, I understand that first order accuracy is not enough(it can be seen on the plot for sin function) and if I need to use the Crank-Nicolson in the future, I will improve it's accuracy.

Discussion:

Figure 1: Evolution in time of Convection equation

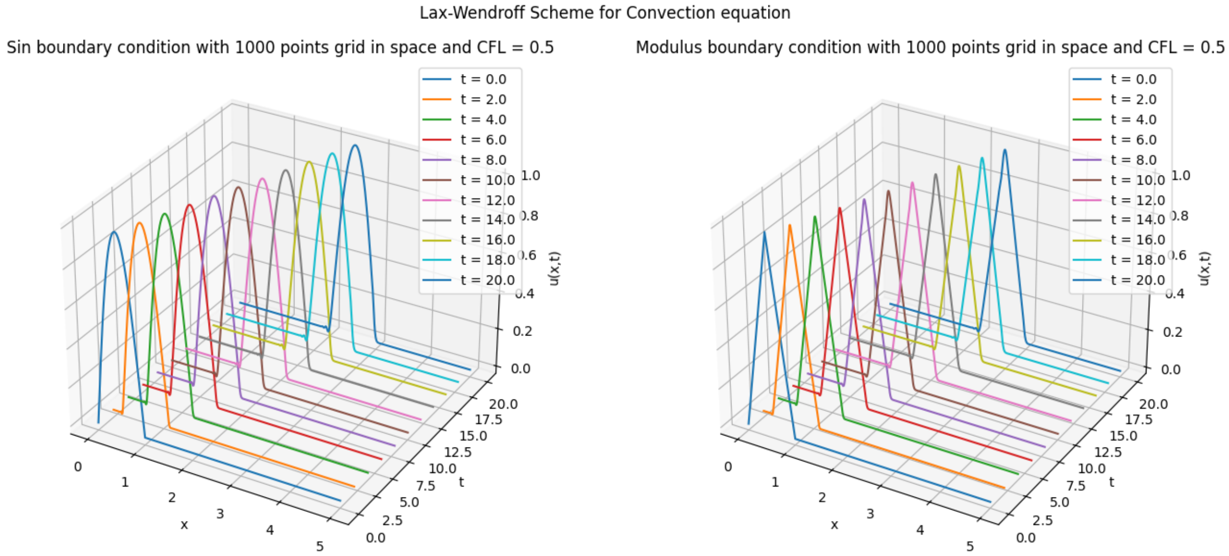
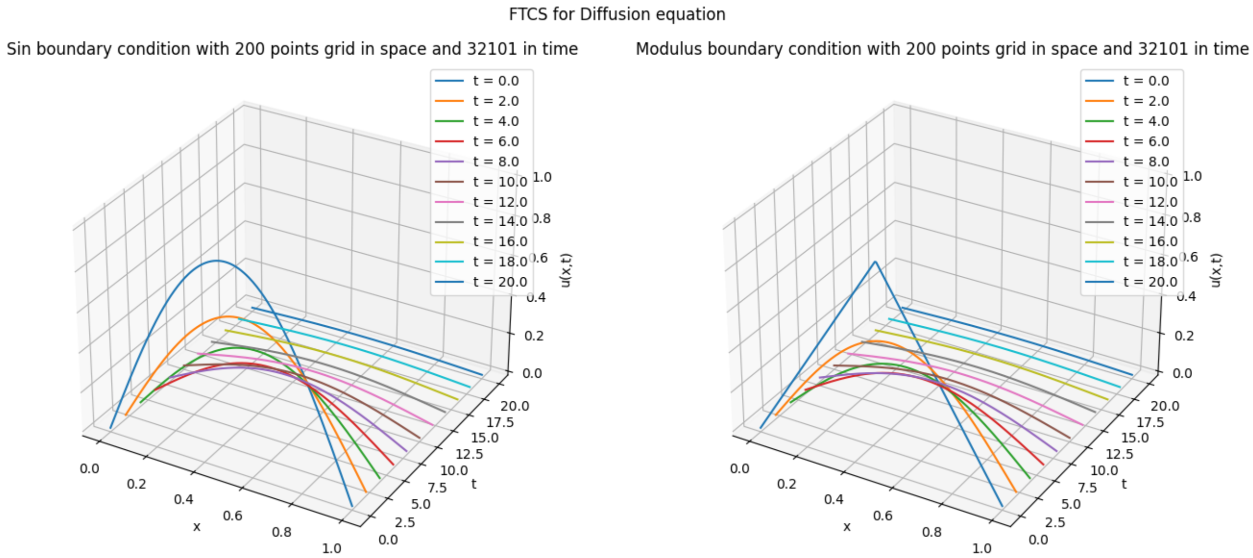
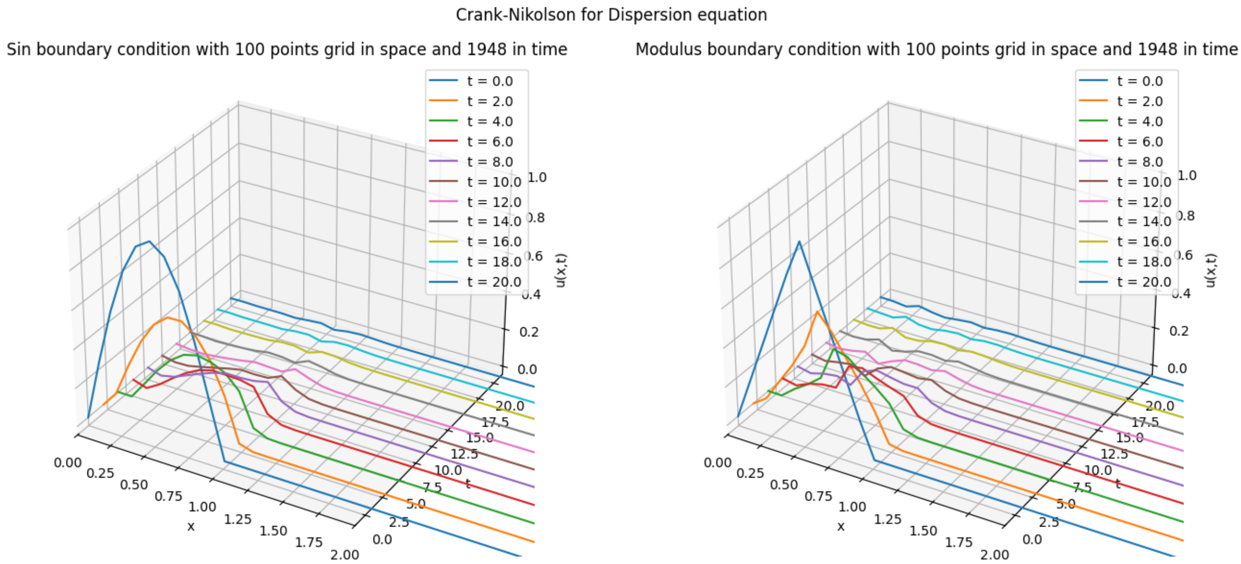


Figure 2: Evolution in time of Diffusion equation



1. Convection: we can see a wave travelling to the right, matching the expected behavior of the real solution. Differences with the real solution include:
   - The sharp edge at $t = 0$ becomes rounded in subsequent plots due to numerical dispersion inherent in the Lax-Wendroff scheme. Analysis of the Lax-Wendroff scheme shows that the leading error term contains a third derivative, which produces dispersive behavior. (more details in NMCL classes).
   - Perturbations appear behind the wave (left side) due to numerical dispersion, where different frequency components travel at slightly different speeds in the numerical solution, causing trailing oscillations particularly near steep gradients. (I may be wrong with terms here, we discuss similar effect in NMCL classes.)
2. Diffusion: we can see how initial wave is diffused and become almost flat - this is correct. As in convection case solution for modulus boundary condition become round - this is also correct, since our initial wave dispersed and lost it's sharpness.
3. Dispersion: we can see how the initial wave spreads out as it propagates. This is the correct physical behavior since in a dispersive medium, different frequency components travel at different speeds.

Figure 3: Evolution in time of Dispersion equation



Problem 4

Consider unsteady viscous flow in an inclined open rectangular channel. Initially, the fluid is at rest. One may assume that streamwise velocity distribution at any cross-sectional area is the same. Due to viscous effects the fluid adheres to the walls, resulting in zero velocity at the walls, while the open top boundary remains stress-free at all times. The equation describing the evolution of the flow in the open channel is given by:

$$\rho \frac{\partial U}{\partial t} = \rho g_z + \mu \left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \right),$$

where $U(X, Y, t)$ is the streamwise velocity, g_z is the streamwise component of gravitational acceleration, and X, Y , and Z are the horizontal, vertical, and streamwise coordinates, respectively. Here, ρ is the fluid density, and μ is the viscosity. This problem can be rewritten in non-dimensional form as:

$$\frac{\partial u}{\partial \tau} = Fr + \frac{1}{Re} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} \right),$$

where u is the nondimensional streamwise velocity, x and y are nondimensional coordinates, τ is nondimensional time, and Fr and Re are the Froude and Reynolds numbers, respectively. The initial conditions are given by:

$$u(x, y, 0) = 0,$$

while the boundary conditions are:

$$u(x, 0, \tau) = u(0, y, \tau) = u(l, y, \tau) = 0, \; \frac{\partial u}{\partial y}(x, h, \tau) = 0.$$

Discretize the problem using explicit Euler time discretization and central difference in space.

- What is the accuracy of the solution in terms of $\Delta\tau$, Δx, and Δy?
- Is the scheme consistent?
- Is the numerical scheme stable?
- What is the maximum time step Δ_τ^{max} you can take in terms of Δx and Δy?

For spatial discretization, use a uniform grid with $\Delta x = \Delta y = 1/25$. Solve the problem numerically up to a time of $\tau = 10$ for the following parameter sets:

a) $l = 1, \; h = 1, \; Fr = 1, \; Re = 1,$

b) $l = 1, \; h = 1, \; Fr = 1, \; Re = 10.$

Show the solutions for both cases for the final time. For each case find the net flow rate through the channel

$$Q(\tau) = \int_0^l \int_0^h u(x, y, \tau) dx dy$$

by numerically integrating the solution at each time step and plot it as a function of time. Compare the solutions for both cases and explain the differences.

Soltuion:

Let's introduce notation: $u(x_i, y_j, t_n) = u_{i,j}^n$. Discretisation:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = Fr + \frac{1}{Re}\left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2}\right)$$

To analyse accuracy and consistency of solution let's use Taylor expansion and decompose functions: $u_{i,j}^{n+1}, u_{i\pm1,j}^n, u_{i,j\pm1}^n$:

$$u_{i,j}^{n+1} = u_{i,j}^n + \frac{\partial u}{\partial t}|_{i,j,n}\Delta t + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}|_{i,j,n}\Delta t^2 + O(\Delta t^3)$$

$$u_{i\pm1,j}^n = u_{i,j}^n \pm \frac{\partial u}{\partial x}|_{i,j,n}\Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}|_{i,j,n}\Delta x^2 \pm \frac{1}{6}\frac{\partial^3 u}{\partial x^3}|_{i,j,n}\Delta x^3 + \frac{1}{24}\frac{\partial^4 u}{\partial x^4}_{i,j,n}\Delta x^4 \pm \frac{1}{120}\frac{\partial^5 u}{\partial x^5}|_{i,j,n}\Delta x^5 + O(\Delta x^6)$$

$$u_{i,j\pm1]}^n = u_{i,j}^n \pm \frac{\partial u}{\partial y}|_{i,j,n}\Delta y + \frac{1}{2}\frac{\partial^2 u}{\partial y^2}|_{i,j,n}\Delta x^2 \pm \frac{1}{6}\frac{\partial^3 u}{\partial y^3}|_{i,j,n}\Delta y^3 + \frac{1}{24}\frac{\partial^4 u}{\partial y^4}_{i,j,n}\Delta y^4 \pm \frac{1}{120}\frac{\partial^5 u}{\partial y^5}|_{i,j,n}\Delta x^5 + O(\Delta x^6)$$

Then substituing into finite difference scheme we get:

$$\frac{\partial u}{\partial t}|_{i,j,n} + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}|_{i,j,n}\Delta t + O(\Delta t^2) =$$

$$= Fr + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}|_{i,j,n} + \frac{1}{12}\frac{\partial^4 u}{\partial x^4}|_{i,j,n}\Delta x^2 + O(\Delta x^4) + \frac{\partial^2 u}{\partial y^2}|_{i,j,n} + \frac{1}{12}\frac{\partial^4 u}{\partial y^4}|_{i,j,n}\Delta y^2 + O(\Delta y^4)\right) \Rightarrow$$

$\Rightarrow$ accuracy of solution: first order in time, second order in space both in x and y direction:

$$\frac{\partial u}{\partial t}|_{i,j,n} = Fr + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}|_{i,j,n} + \frac{\partial^2 u}{\partial y^2}|_{i,j,n}\right) + O(\Delta t, \Delta x^2, \Delta y^2).$$

From here it's clear that scheme is consistent.

To analyse stability let's use von Neumann method:

$$u_{i,j}^n = \sigma^n e^{ik_1 x_i} e^{ik_2 y_j}$$

Let's substitute that into equation (for stability we can ommit Fr, since it's constant and doesn't effect the stability of solution):

$$\frac{\sigma^{n+1} e^{ik_1 x_i} e^{ik_2 y_j} - \sigma^n e^{ik_1 x_i} e^{ik_2 y_j}}{\Delta t} =$$

$$= \frac{1}{Re}\left(\frac{\sigma^n e^{ik_1 x_{i+1}} e^{ik_2 y_j} - 2\sigma^n e^{ik_1 x_i} e^{ik_2 y_j} + \sigma^n e^{ik_1 x_{i-1}} e^{ik_2 y_j}}{\Delta x^2} + \frac{\sigma^n e^{ik_1 x_i} e^{ik_2 y_{j+1}} - 2\sigma^n e^{ik_1 x_i} e^{ik_2 y_j} + \sigma^n e^{ik_1 x_i} e^{ik_2 y_{j-1}}}{\Delta y^2}\right) \Rightarrow$$

$$\Big| : \sigma^n e^{ik_1 x_i} e^{ik_2 y_j} \Rightarrow \frac{\sigma - 1}{\Delta} = \frac{1}{Re}\left(\frac{-2 + 2cos(k_1 \Delta x)}{\Delta x^2} + \frac{-2 + 2cos(k_2 \Delta y)}{\Delta y^2}\right) \Leftrightarrow$$

$$\Leftrightarrow \sigma = \frac{-4\Delta t}{Re\Delta x^2}\sin^2(\frac{k_1 \Delta x}{2}) + \frac{-4\Delta t}{Re\Delta y^2}\sin^2(\frac{k_2 \Delta y^2}{2}) + 1 \Rightarrow$$

$$\Rightarrow \max|\sigma| = \left|\frac{-4\Delta t}{Re\Delta x^2} + \frac{-4\Delta t}{Re\Delta y^2} + 1\right| \leq 1 \Rightarrow$$

$$\Rightarrow -1 \leq \frac{-4\Delta t}{Re\Delta x^2} + \frac{-4\Delta t}{Re\Delta y^2} + 1 \leq 1 \Leftrightarrow$$

$$\Leftrightarrow -2 \leq \frac{-4\Delta t}{Re\Delta x^2} + \frac{-4\Delta t}{Re\Delta y^2} \leq 0 \Rightarrow \Delta t \leq \frac{Re}{2}\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)^{-1}.$$

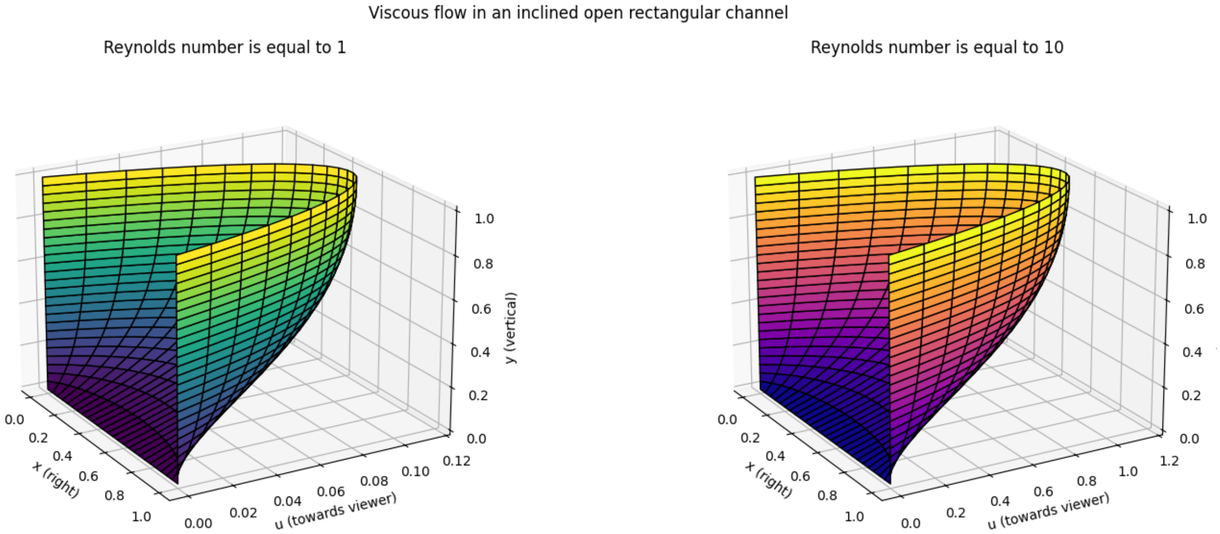
For numerical solution I have chosen $\Delta t = \frac{Re}{8}\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)^{-1}$. Main equation:

$$u_{i,j}^{n+1} = u_{i,j}^n + Fr \cdot \Delta t + \frac{\Delta t}{Re}\left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2}\right)$$

For boundary condition we also need to descritise the derivative. I chose backward difference of the second order to keep the order of error the same as main equation:

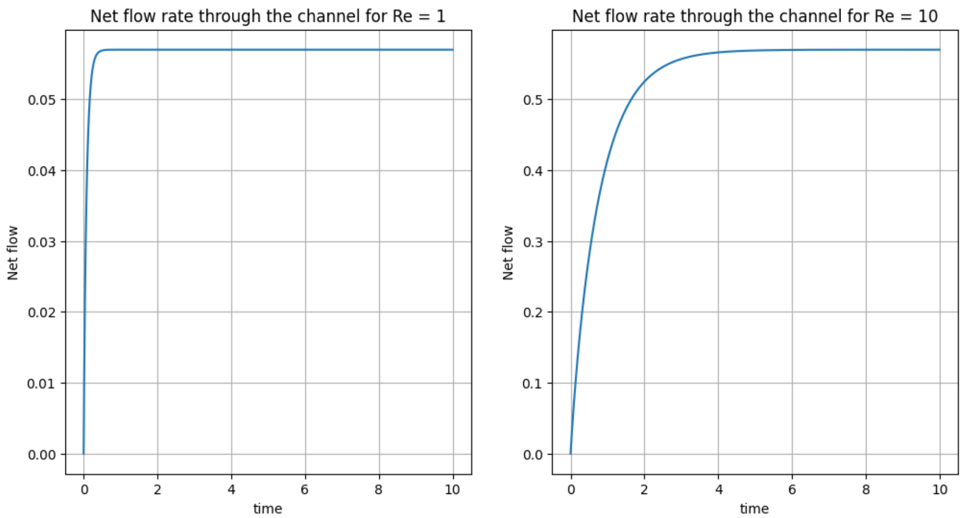
$$\frac{\partial u^n_{i,-1}}{\partial y} = \frac{-4u^n_{i,-2} + u^n_{i,-3} + 3u^n_{i,j}}{2\Delta y} = 0 \Rightarrow u^n_{i,j} = \frac{1}{3}\left(4u^n_{i,-2} - u^n_{i,-3}\right)$$

Figure 4: Velocity distribution for viscous unsteady flow on inclined surface



For net flow rate I have used two ways: trapezoidal integration from scipy library. It allows to integrate over a function which defined as values at points and it's second order accurate, so that's quite enough for our problem. And just summation: since our function is constant on the square with sides $\Delta x, \Delta y$ we can sum over all values of u for each time and multiply by area of unit cell: $\Delta x \cdot \Delta y$. Max difference between two cases for case 1 is: 0.00152, for case 2 is: 0.0152.Trapezoidal rule is more accurate, I will leave it.

Figure 5: Net flow rate through the channel



Discussion:

Two solutions are the similliar, however all values are differ by a factor of ten. For example, maxim values for velocity differ by factor of 10: with Re = 1 it's 0.12 and with Re = 10 it's 1.2.

Similiar picture with net flow: in first case we have 0.057, in second case we have: 0.57.

Also, time steps differ by factor of 10 (follows from the formula).

Problem 5

Consider the steady heat conduction problem in a square region with constant heat conductivity. The non-dimensional equation describing the temperature distribution in the region is given by:

$$\frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2} = 0,$$

where $\theta(X, Y)$ is the non-dimensional temperature, and X and Y are the non-dimensional coordinates. The boundary conditions are:

$$\frac{\partial \theta}{\partial Y}(X, 0) = Y, \qquad \theta(X, 1) = 1,$$
$$\theta(0, Y) = 1, \quad \frac{\partial \theta}{\partial X}(1, Y) = 1 - Y.$$

Discretize the problem using a central difference scheme. For spatial discretization, use a uniform grid with $\Delta x = \Delta y = 1/20$. Solve the steady problem numerically using the following iterative schemes:

a) Jacoby,

b) Gauss Seidel.

Stop the iterations for both methods when both of the following two convergence criteria are satisfied:

$$\max_{i,j} |R_{i,j}| < \delta_R$$

and

$$\max_{i,j} |\theta_{i,j}^{n+1} - \theta_{i,j}^{n}| < \delta_\theta$$

where $R_{i,j}$ is the residual of the discretized equation. Use $\delta R = 10^{-5}$ and $\delta_\theta = 10^{-7}$. Plot both norms as a function of iteration as well as the converged solution. Compare the convergence of both methods. Which method converges faster?

Solution: Let's discritise our equation: $u(x_i, y_j) = u_{i,j}$:

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = 0.$$

Let's discritise our boundary condtions(only Neumann conditons): For $\frac{\partial \theta}{\partial Y}(X, 0) = Y$ I will use second order forward scheme (since I want to keep second order accuracy of my scheme):

$$\frac{-3\theta_{i,0} + 4\theta_{i,1} - \theta_{i,2}}{2\Delta y} = x_i \Rightarrow \theta_{i,0} = \frac{1}{3}\left(4\theta_{i,1} - \theta_{i,2} - 2\Delta y x_i\right)$$

Similiarly for $\frac{\partial \theta}{\partial X}(1, Y) = 1 - Y$ I will use second order backward scheme:

$$\frac{3\theta_{-1,j} - 4\theta_{-2,j} + \theta_{-3,j}}{2\Delta x} = (1 - y_i) \Rightarrow \theta_{-1,j} = \frac{1}{3}\left(4\theta_{-2,j} - \theta_{-3,j} + 2\Delta x(1 - y_j)\right).$$

Using Jacoby scheme we get:

$$\frac{u_{i+1,j}^n - 2u_{i,j}^{n+1} + u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^{n+1} + u_{i,j-1}^n}{\Delta y^2} = 0,$$

where n corresponds to iteration number. We can find $u_{i,j}^{n+1}$ in the following form:

$$u_{i,j}^{n+1} = \left(\frac{u_{i+1,j}^n + u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i,j+1}^n + u_{i,j-1}^n}{\Delta y^2}\right) / \left(\frac{2}{\Delta x^2} + \frac{2}{\Delta y^2}\right).$$

Since we use Jacobi method everywhere on the rhs we use unupdated values.

Boundary conditions we will update at the end of each iteration, since in Jacobi method we don't use new values.

For Gauss Seidel scheme approach is the same, however we can keep only one massive of $u_{i,j}$ and update it in live. Since we are using 5 points scheme, we can update boundary values at the end of the iteration.

Residual in that case will be our laplas operator:

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = R_{i,j}.$$

For checking the condition on difference between previous value and new, I have a copy of values. It's not efficient, but it works.

Gauss-Seidel converged faster (1520 iteration with comparisson to Jacobi method: 4401 iteraions).

Figure 6: Solution

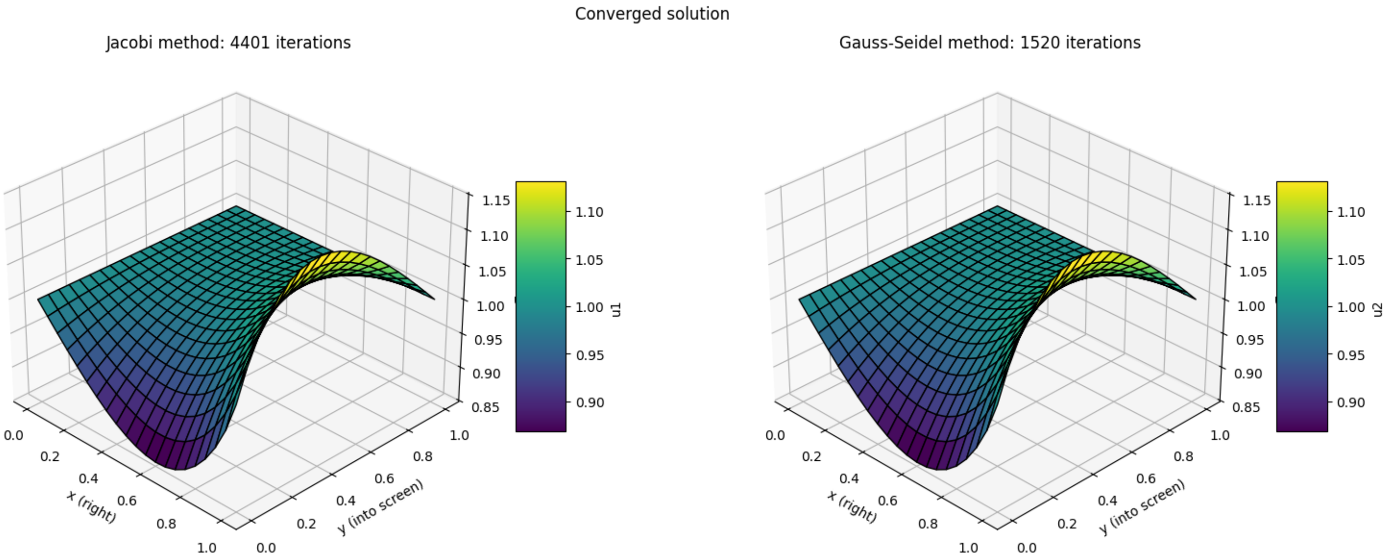


Figure 7: Residual and Difference convergence