Reading of Advanced Financial Modelling – Part 2

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1. Lognormal Property of Stock Price

- We assume that the percentage changes in the stock price in a very short period of time are normally distributed. Define

μ: Expected return on stock per year
σ: Volatility of the stock price per year.

The mean and standard deviation of the return in time Δt are approximately $\mu\,\Delta t$ and $\sigma\sqrt{\Delta t}$, so that

$$\frac{\Delta S}{S} \sim \phi(\mu\,\Delta t,\ \sigma^2 \Delta t) \qquad \textbf{(15.1)}$$

  - The model implies that

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right]$$

so that

$$\ln \frac{S_T}{S_0} \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right] \qquad \textbf{(15.2)}$$

and

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right] \qquad \textbf{(15.3)}$$

- **Example**
  - Consider a stock with an initial price of \$40, an expected return of 16% per annum, and a volatility of 20% per annum. The probability distribution of the stock price in 6 months' time is given by

$$\ln S_T \sim \phi[\ln 40 + (0.16 - 0.2^2/2) \times 0.5,\ 0.2^2 \times 0.5]$$
$$\ln S_T \sim \phi(3.759,\ 0.02)$$

  - There is a 95% probability that a normally distributed variable has a value within 1.96 standard deviation of its mean. Hence

$$3.759 - 1.96 \times 0.141 < \ln S_T < 3.759 + 1.96 \times 0.141$$

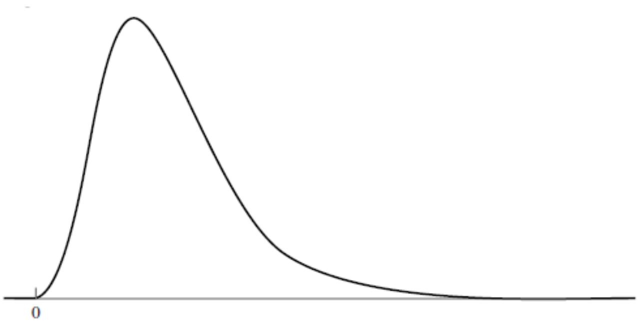
This can be written

$$e^{3.759-1.96\times0.141} < S_T < e^{3.759+1.96\times0.141}$$

or

$$32.55 < S_T < 56.56$$

- The following figure shows the shape of a lognormal distribution



- It can be shown that the expected value of S_T is

$$E(S_T) = S_0 e^{\mu T} \tag{15.4}$$

- The variance of S_T is

$$\mathrm{var}(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1) \tag{15.5}$$

- **Example**
  - Consider a stock where the current price is \$20, the expected return is 20% per annum, and the volatility is 40% per annum. The expected stock price and variance of the stock price in 1 year is given by

$$E(S_T) = 20e^{0.2\times 1} = 24.43 \quad \text{and} \quad \mathrm{var}(S_T) = 400e^{2\times 0.2\times 1}(e^{0.4^2\times 1} - 1) = 103.54$$

- Note that the property of lognormal distribution follows

$$\mathrm{E}[X] = \exp\left(\mu + \frac{1}{2}\sigma^2\right)$$

$$\mathrm{Var}[X] = \exp\left(2\mu + 2\sigma^2\right) - \exp\left(2\mu + \sigma^2\right)$$

1. The distribution of the rate of return

- If we define the continuously compounded rate of return per annum realized between times 0 and T as x, then

$$S_T = S_0 e^{xT}$$

so that

$$x = \frac{1}{T}\ln\frac{S_T}{S_0} \tag{15.6}$$

From equation (15.2), it follows that

$$x \sim \phi\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}\right) \tag{15.7}$$

  - As T increases, the standard deviation of x declines.
  - To understand the reason for this, consider two cases: T=1 and T=20. We are more certain about the average return per year over 20 years than we are about the return in any one year.

- **Example**
  - Consider a stock with an expected return of 17% per annum and a volatility of 20% per annum. The probability distribution for the average rate of return (continuously compounded) realized over 3 years is normal, with mean

$$0.17 - \frac{0.2^2}{2} = 0.15$$

or 15% per annum, and standard deviation

$$\sqrt{\frac{0.2^2}{3}} = 0.1155$$

  - Because there is a 95% chance that a normally distributed variable will lie within 1.96 standard deviations of its mean, we can be 95% confident that the average return realized over 3 years will be between 15-1.96*11.55 = -7.6% and 15+1.96*11.55 = 37.6% per annum

2. The expected return

- The expected return, μ, required by investors from a stock depends on the riskiness of the stock.
  - The higher the risk, the higher the expected return
- It also depends on the level of interest rates in the economy.
  - The higher the level of interest rates, the higher the expected return required on any given stock

- Our model of stock price behavior implies that, in a very short period of time, the mean return is $\mu\Delta t$.
  - It is natural to assume from this that μ is the expected continuously compounded return on the stock.
  - However, this is not the case. The continuously compounded return, x, actually realized over a period of time of length T is given by

$$x = \frac{1}{T}\ln\frac{S_T}{S_0}$$

  - The expected value $E(x)$ is $\mu - \sigma^2/2$, not μ

- For another explanation, consider

$$E(S_T) = S_0 e^{\mu T}$$

Taking logarithms, we get

$$\ln[E(S_T)] = \ln(S_0) + \mu T$$

  - It is not tempting to set $\ln[E(S_T)] = E[\ln(S_T)]$, so that $E[\ln(S_T/S_0)] = \mu T$, which leads to $E(x) = \mu$
  - However, we cannot do this because ln is a nonlinear function. In fact, $\ln E(S_T) > E(\ln(S_T))$, so that $E[\ln(S_T/S_0)] < \mu T$, which leads to $E(x) < \mu$

3. Volatility

- The volatility, σ, of a stock is a measure of our uncertainty about the returns provided by the stock.
  - Stocks typically have a volatility between 15% and 60%
  - From equation (15.7), the volatility of a stock price can be defined as the standard deviation of the return provided by the stock in 1 year when the return is expressed using continuous compounding
  - When Δt is small, equation (15.1) shows that $\sigma^2 \Delta t$ is approximately equal to the variance of the percentage change in the stock price in time Δt
    - This means that $\sigma\sqrt{\Delta t}$ is approximately equal to the standard deviation of the percentage change in the stock price in time Δt
  - Suppose that $\sigma = 0.3$ per annum and the current stock price is \$50. The standard deviation of the percentage change in the stock price in 1 week is approximately

$$30 \times \sqrt{\frac{1}{52}} = 4.16\%$$

    - A 1-standard-deviation move in the stock price in 1 week is therefore 50*0.0416=2.08

- Uncertainty about a future stock price, as measured by its standard deviation, increases – at least approximately – with the square root of how far ahead we are looking.
  - For example, the standard deviation of the stock price in 4 weeks is approximately twice the standard deviation in 1 week

Estimating Volatility from Historical Data

- To estimate the volatility of a stock price empirically, the stock price is usually observed at fixed intervals of time. Define

$n+1$: Number of observations
S_i: Stock price at end of ith interval, with $i = 0, 1, \dots, n$
τ: Length of time interval in years

and let

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right) \quad \text{for } i = 1, 2, \dots, n$$

  - The usual estimate, s, of the standard deviation of the u_i is given by

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2}$$

or

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} u_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} u_i\right)^2}$$

  - From equation (15.2), the standard deviation of the u_i is $\sigma\sqrt{\tau}$. The variable s is therefore an estimate of $\sigma\sqrt{\tau}$. It follows that σ itself can be estimated as $\hat{\sigma}$, where

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}}$$

  - The standard error of this estimate can be shown to be approximately $\hat{\sigma}/\sqrt{2n}$

- Choosing an appropriate value for n is not easy
  - More data generally lead to more accuracy, but σ does change over time and data that are too old may not be relevant for predicting the future volatility.
  - A compromise that seems to work reasonably well is to using closing prices from daily data over the most recent 90 to 180 days.
  - Alternatively, as a rule of thumb, n can be set equal to the number of days to which the volatility is to be applied.
    - Thus, if the volatility estimate is to be used to value a 2-year option, daily data for the last 2 years are used.

Example

- The following table shows a possible sequence of stock prices during 21 consecutive trading days. In this case, n=20, so that

$$\sum_{i=1}^{n} u_i = 0.09531 \quad \text{and} \quad \sum_{i=1}^{n} u_i^2 = 0.00326$$

and the estimate of the standard deviation of the daily return is

$$\sqrt{\frac{0.00326}{19} - \frac{0.09531^2}{20 \times 19}} = 0.01216$$

Table 15.1 Computation of volatility.

Day i	*Closing stock price (dollars),* S_i	*Price relative* S_i/S_{i-1}	*Daily return* $u_i = \ln(S_i/S_{i-1})$
0	20.00		
1	20.10	1.00500	0.00499
2	19.90	0.99005	−0.01000
3	20.00	1.00503	0.00501
4	20.50	1.02500	0.02469
5	20.25	0.98780	−0.01227
6	20.90	1.03210	0.03159
7	20.90	1.00000	0.00000
8	20.90	1.00000	0.00000
9	20.75	0.99282	−0.00720
10	20.75	1.00000	0.00000
11	21.00	1.01205	0.01198
12	21.10	1.00476	0.00475
13	20.90	0.99052	−0.00952
14	20.90	1.00000	0.00000
15	21.25	1.01675	0.01661
16	21.40	1.00706	0.00703
17	21.40	1.00000	0.00000
18	21.25	0.99299	−0.00703
19	21.75	1.02353	0.02326
20	22.00	1.01149	0.01143

  - Assuming that there are 252 trading days per year, $\tau = 1/252$ and the data give an estimate for the volatility per annum of $0.01216\sqrt{252} = 0.193$
  - The standard error of this estimate is

$$\frac{0.193}{\sqrt{2 \times 20}} = 0.031$$

- The forging analysis assumes that the stock pays no dividends, but it can be adapted to accommodate dividend-paying stocks.
  - The return, u_i, during a time interval that includes an ex-dividend day is given by

$$u_i = \ln \frac{S_i + D}{S_{i-1}}$$

where D is the amount of the dividend. The return in other time intervals is still

$$u_i = \ln \frac{S_i}{S_{i-1}}$$

Trading days vs. calendar days

- An important issue is whether time should be measured in calendar days or trading days when volatility parameters are being estimated and used.
  - Research shows that volatility is much higher when the exchange is open for trading than when it is closed.
  - As a result, practitioner tend to ignore days when the exchange is closed when estimating volatility from historical data and when calculating the life of an option.
  - The volatility per annum is calculated from the volatility per trading day using the formula

$$\begin{array}{c}\text{Volatility}\\ \text{per annum}\end{array} = \begin{array}{c}\text{Volatility}\\ \text{per trading day}\end{array} \times \sqrt{\begin{array}{c}\text{Number of trading days}\\ \text{per annum}\end{array}}$$

  - The number of trading days in a year is usually assumed to be 252 for stocks
  - The life of an option is also usually measured using trading days rather than calendar days. It is calculated as T years, where

$$T = \frac{\text{Number of trading days until option maturity}}{252}$$

What Causes Volatility?

- It's natural to assume that the volatility of a stock is caused by new information reaching the market.
  - The new information causes people to revise their opinions about the value of the stock.
  - The price of the stock changes and volatility results.
  - This view of what causes volatility is not supported by research.
    - They showed that volatility between close of training on Friday and that of Mondy is far less than 3 times than the volatility between two adjacent training days
  - The only reasonable conclusion from all this is that volatility is to a large extent cased by trading itself.

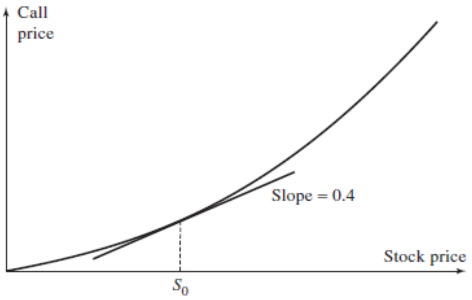
4. The idea underlying BSM DE

- The Black-Scholes-Merton differential equation is an equation that must be satisfied by the price of any derivative dependent on a non-dividend-paying stock
  - The arguments involve setting up a riskless portfolio consisting of a position in the derivative and a position in the stock.
  - In the absence of arbitrage opportunities, the return from the portfolio must be the risk-free interest rate, r. This leads to the Black-Scholes-Merton differential equation

- The reason a riskless can be set up is that in any short period time, the price of the derivative is perfectly correlated with the price of the underlying stock
  - Suppose that a particular point in time the relationship between a small change in stock price resultant small change in price of a European call option is given by

$$\Delta c = 0.4\,\Delta S$$

Figure 15.2 Relationship between call price and stock price. Current stock price is S_0.



  - A riskless portfolio would consist of
    - A long position in 40 shares
    - A short position in 100 call options

- Note that in BSM, the position in the stock and the derivative is riskless for only a very short period of time
  - Theoretically, it remains riskless only for an instantaneously short period of time
  - To remain riskless, it must be adjusted, or ***rebalanced***, frequently

Assumptions

- The assumptions we use to derive the BSM DE are
  - The stock price follows the geometric Brownian motion with μ, σ constant
  - The short selling of securities with full use of proceeds is permitted
  - There are no transaction costs or taxes. All securities are perfectly divisible
  - There are no dividends during the life of the derivative

- There are no riskless arbitrage opportunities
- Security trading is continuous
- The risk-free rate of interest, r, is constant and the same for all maturities

5. Derivation of the BSM DE

- The stock price process we are assuming is

$$dS = \mu S\,dt + \sigma S\,dz \tag{15.8}$$

  - Suppose that f is the price of a derivative contingent on S. the variable f must be some function of S and t, hence, the Ito's Lemma states

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S\,dz \tag{15.9}$$

The discrete versions of equations (15.8) and (15.9) are

$$\Delta S = \mu S\,\Delta t + \sigma S\,\Delta z \tag{15.10}$$

and

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S\,\Delta z \tag{15.11}$$

  - Note that the Δz in the equations are the same
  - It follows that a portfolio of the stock and the derivative can be constructed so that the Wiener process is eliminated by

-1: derivative

$+\partial f/\partial S$: shares.

  - Define π as the value of the portfolio

$$\Pi = -f + \frac{\partial f}{\partial S}S \tag{15.12}$$

The change $\Delta\Pi$ in the value of the portfolio in the time interval Δt is given by

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S \tag{15.13}$$

Substituting equations (15.10) and (15.11) into equation (15.13) yields

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t \tag{15.14}$$

  - Since the equation does not involve Δz, the portfolio must be riskless during time Δt
  - The assumptions listed in the preceding section imply that the portfolio must instantaneously earn the same rate of return as other short-term risk-free securities. It follows that

$$\Delta\Pi = r\Pi\,\Delta t \tag{15.15}$$

where r is the risk-free interest rate. Substituting from equations (15.12) and (15.14) into (15.15), we obtain

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t = r\left(f - \frac{\partial f}{\partial S}S\right)\Delta t$$

so that

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf \tag{15.16}$$

  - This is the Black-Sholes-Merton differential equation

- It has many solutions, corresponding to all the different derivatives that can be defined with S as the underlying variable
- The particular derivative that is obtained when the equation is solved depends on the ***boundary conditions*** that are used.
- In the case of a European call option, the key boundary condition is

$$f = \max(S - K, 0) \quad \text{when } t = T$$

In the case of a European put option, it is

$$f = \max(K - S, 0) \quad \text{when } t = T$$

Example

- A forward contract on a non-dividend-paying stock is a derivative dependent on the stock.
  - As such, it should satisfy the BSM DE, we know that the value of the forward contract, f, at a general time t, is given in terms of the stock price S at this time by

$$f = S - Ke^{-r(T-t)}$$

where K is the delivery price. This means that

$$\frac{\partial f}{\partial t} = -rKe^{-r(T-t)}, \qquad \frac{\partial f}{\partial S} = 1, \qquad \frac{\partial^2 f}{\partial S^2} = 0$$

When these are substituted into the left-hand side of equation (15.16), we obtain

$$-rKe^{-r(T-t)} + rS$$

This equals rf, showing that equation (15.16) is indeed satisfied.

A Perpetual Derivative

- Consider a perpetual derivative that pays off a fixed amount Q when the stock price equals H for the first time.
  - Note that, the value of the derivative for a particular S has no dependence on t.

- Suppose first that S < H
  - The boundary conditions for the derivative are f=0 when S=0 and f=Q when S=H.
  - The simple solution f = QS / H satisfies both the boundary condition and the differential equation.
  - It must therefore be the value of the derivative

- Suppose next that S>H.
  - The boundary conditions are now f=0 as S tends to infinity and f=Q when S=H.
  - The derivative price

$$f = Q\left(\frac{S}{H}\right)^{-\alpha}$$

where α is positive, satisfies the boundary conditions. It also satisfies the differential equation when

$$-r\alpha + \tfrac{1}{2}\sigma^2\alpha(\alpha+1) - r = 0$$

or $\alpha = 2r/\sigma^2$. The value of the derivative is therefore

$$f = Q\left(\frac{S}{H}\right)^{-2r/\sigma^2} \tag{15.17}$$

The Prices of Tradeable Derivatives

- Any function f(S, t) that is a solution of the differential equation is the theoretical price of a derivative that could be traded.
  - ➢ If a derivative with the price existed, it would not create any arbitrage opportunities.
  - ➢ To illustrate this point, consider first the function e^S.
    - ✧ This does not satisfy the differential equation. It is therefore not a candidate for being the price of a derivative dependent on the stock price.

$$f = e^S : \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 f + rSf \neq rf$$

$\frac{\partial f}{\partial t} \downarrow 0$ $\qquad \frac{\partial^2 f}{\partial S^2}$ $\qquad \frac{\partial f}{\partial S}$

$$\frac{1}{2}\sigma^2 S^2 + rS = r$$

6. *Risk-Neutral Valuation*

- A key property of BSM DE is that the equation does not involve any variables that are affected by the risk preferences of investors.
  - ➢ The variables that do appear in the equation are the current stock price, time, stock price volatility, and the risk-free rate of interest. All are independent of risk preferences
  - ➢ The BSM DE would not be independent of risk preference if it involved the expected return, μ, on the stock.
    - ✧ This is because the value of μ does depend on risk preferences
  - ➢ If risk preferences do not enter the equation, they cannot affect its solution.
    - ✧ Any set of risk preferences can, therefore, be used when evaluating f.
    - ✧ In particular, the very simple assumption that all investors are risk neutral can be made.

- In a world where investors are risk neutral, the expected return on all investment assets is the risk-free rate of interest, r.
    - It is also true that the **present value of any cash flow** in a risk-neutral world can be obtained by discounting its expected value at the risk-free rate.
    - Consider a derivative that provides a payoff at one particular time. It can be valued using risk-neutral valuation by using the following procedure:
        - 1. Assume that the expected return from the underlying asset is the risk-free interest rate, r
        - 2. Calculate the expected payoff from the derivative
        - 3. Discount the expected payoff at the risk-free interest rate

- It is important to appreciate that risk-neutral valuation is merely an artificial device for obtaining solutions to be BSM DE.
    - The solutions that are obtained are valid in all worlds, not just those where investors are risk neutral.
    - When we move from a risk-neutral world to a risk-averse world, two things happen
        - The expected growth rate in the stock price changes and the discount rate that must be used for any payoffs from the derivative changes
        - It happens that there two changes always offset each other exactly

Application to Forward Contracts on a Stock

- In this section we derive the pricing formula of forward contracts from risk-neutral valuation
    - We make the assumption that interest rates are constant and equal to r.
    - Consider a long forward contract that matures at time T with delivery price, K. the value of the contract at maturity is

$$S_T - K$$

    - From the risk-neutral valuation argument, the value of the forward contract at time 0 is its expected value at time T in a risk-neutral world discounted at the risk-free rate of interest.
    - Denoting the value of the forward contract at time zero by f, this means that

$$f = e^{-rT}\hat{E}(S_T - K)$$

where $\hat{E}$ denotes the expected value in a risk-neutral world. Since K is a constant, this equation becomes

$$f = e^{-rT}\hat{E}(S_T) - Ke^{-rT} \tag{15.18}$$

The expected return μ on the stock becomes r in a risk-neutral world. Hence, from equation (15.4), we have

$$\hat{E}(S_T) = S_0 e^{rT} \tag{15.19}$$

Substituting equation (15.19) into equation (15.18) gives

$$f = S_0 - Ke^{-rT}$$

This is in agreement with equation (5.5).

7. BSM pricing formulas

- The most famous solution to the differential equation are the BSM formulas for the prices of European call and put options. These formulas are

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{15.20}$$

and

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \tag{15.21}$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

  - The function N(x) is the probability that a variable with a standard normal distribution will be less than x

Figure 15.3 Shaded area represents $N(x)$.

Risk-Neutral approach to BSM option pricing formula

- Another approach is to use risk-neutral valuation. Consider a European call option. The expected value of the option at maturity in a risk-neutral world is

$$\hat{E}[\max(S_T - K, 0)]$$

  - Where, $\hat{E}$ denotes the expected value in a risk-neutral world.
  - From the risk-neutral valuation argument, the European call option price c is this expected value discounted at the risk-free rate of interest, that is

$$c = e^{-rT} \hat{E}[\max(S_T - K, 0)] \tag{15.22}$$

- **A key result** before proving the BS result

Key Result

If V is lognormally distributed and the standard deviation of $\ln V$ is w, then

$$E[\max(V-K,\,0)] = E(V)N(d_1) - KN(d_2) \tag{15A.1}$$

where

$$d_1 = \frac{\ln[E(V)/K] + w^2/2}{w}$$

$$d_2 = \frac{\ln[E(V)/K] - w^2/2}{w}$$

and E denotes the expected value.

Proof of Key Result

Define $g(V)$ as the probability density function of V. It follows that

$$E[\max(V-K,0)] = \int_K^{\infty}(V-K)g(V)\,dV \tag{15A.2}$$

The variable $\ln V$ is normally distributed with standard deviation w. From the properties of the lognormal distribution, the mean of $\ln V$ is m, where[16]

$$m = \ln[E(V)] - w^2/2 \tag{15A.3}$$

Define a new variable

$$Q = \frac{\ln V - m}{w} \tag{15A.4}$$

This variable is normally distributed with a mean of zero and a standard deviation of 1.0. Denote the density function for Q by $h(Q)$ so that

$$h(Q) = \frac{1}{\sqrt{2\pi}}e^{-Q^2/2}$$

Using equation (15A.4) to convert the expression on the right-hand side of equation (15A.2) from an integral over V to an integral over Q, we get

$$E[\max(V-K,\,0)] = \int_{(\ln K-m)/w}^{\infty}(e^{Qw+m}-K)\,h(Q)\,dQ$$

or

$$E[\max(V-K,\,0)] = \int_{(\ln K-m)/w}^{\infty} e^{Qw+m}h(Q)dQ - K\int_{(\ln K-m)/w}^{\infty} h(Q)dQ \tag{15A.5}$$

Now

$$e^{Qw+m}h(Q) = \frac{1}{\sqrt{2\pi}}e^{(-Q^2+2Qw+2m)/2} = \frac{1}{\sqrt{2\pi}}e^{[-(Q-w)^2+2m+w^2]/2}$$

$$= \frac{e^{m+w^2/2}}{\sqrt{2\pi}}e^{[-(Q-w)^2]/2} = e^{m+w^2/2}h(Q-w)$$

This means that equation (15A.5) becomes

$$E[\max(V-K,\,0)] = e^{m+w^2/2}\int_{(\ln K-m)/w}^{\infty} h(Q-w)dQ - K\int_{(\ln K-m)/w}^{\infty} h(Q)dQ \tag{15A.6}$$

If we define $N(x)$ as the probability that a variable with a mean of zero and a standard deviation of 1.0 is less than x, the first integral in equation (15A.6) is

$$1 - N[(\ln K - m)/w - w] = N[(-\ln K + m)/w + w]$$

Substituting for m from equation (15A.3) leads to

$$N\left(\frac{\ln[E(V)/K] + w^2/2}{w}\right) = N(d_1)$$

Similarly the second integral in equation (15A.6) is $N(d_2)$. Equation (15A.6), therefore, becomes

$$E[\max(V-K,\,0)] = e^{m+w^2/2}N(d_1) - KN(d_2)$$

Substituting for m from equation (15A.3) gives the key result.

- **BSM result**

We now consider a call option on a non-dividend-paying stock maturing at time T. The strike price is K, the risk-free rate is r, the current stock price is S_0, and the volatility is σ. As shown in equation (15.22), the call price c is given by

$$c = e^{-rT}\hat{E}[\max(S_T - K, 0)] \tag{15A.7}$$

where S_T is the stock price at time T and $\hat{E}$ denotes the expectation in a risk-neutral world. Under the stochastic process assumed by Black–Scholes–Merton, S_T is log-normal. Also, from equations (15.3) and (15.4), $\hat{E}(S_T) = S_0e^{rT}$ and the standard deviation of $\ln S_T$ is $\sigma\sqrt{T}$.

From the key result just proved, equation (15A.7) implies

$$c = e^{-rT}[S_0e^{rT}N(d_1) - KN(d_2)] = S_0N(d_1) - Ke^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln[\hat{E}(S_T)/K] + \sigma^2T/2}{\sigma\sqrt{T}} = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln[\hat{E}(S_T)/K] - \sigma^2T/2}{\sigma\sqrt{T}} = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

This is the Black–Scholes–Merton result.

Understanding N(d1) and N(d2)

- N(d2) is fairly simple, it is the probability that a call option will be exercised in a risk-neutral world.
- N(d1) is not quite so easy to interpret
  - The expression $S_0N(d_1)e^{rT}$ is the expected stock price at time T in a risk-neutral world when stock prices less than the strike price are counted as zero.
  - The strike price is only paid if the stock price is greater than K and as just mentioned this has a probability of N(d2).
  - The expected payoff in a risk-neutral world is therefore

$$S_0N(d_1)e^{rT} - KN(d_2)$$

Present-valuing this from time T to time zero gives the Black–Scholes–Merton equation for a European call option:

$$c = S_0N(d_1) - Ke^{-rT}N(d_2)$$

For another interpretation, note that the Black–Scholes–Merton equation for the value of a European call option can be written as

$$c = e^{-rT}N(d_2)[S_0e^{rT}N(d_1)/N(d_2) - K]$$

The terms here have the following interpretation:

e^{-rT}: Present value factor
$N(d_2)$: Probability of exercise
$e^{rT}N(d_1)/N(d_2)$: Expected percentage increase in stock price in a risk-neutral world if option is exercised
K: Strike price paid if option is exercised.

Properties of the BSM Formulas

- We now show that the BSM formulas have the right general properties by considering what happens when some of the parameters take extreme values
  - When the stock price at time 0 becomes very large, a call option is almost certain to be exercised.
    - It then becomes very similar to a forward contract with delivery price K

- The call price then should be

$$S_0 - Ke^{-rT}$$

- This is accordance with BSM formula
- When S_0 becomes very large, both d1, d2 becomes very large, and N(d1), N(d2) become close to 1

➢ Consider next when the volatility approaches 0.

- Since the stock is virtually riskless, the price should be

$$\max(S_0e^{rT} - K,\ 0)$$

Discounting at rate r, the value of the call today is

$$e^{-rT}\max(S_0e^{rT} - K,\ 0) = \max(S_0 - Ke^{-rT},\ 0)$$

- This is again consistent with the BSM formulas, since if $S_0 > Ke^{-rT}$, then $\ln(S_0/K) + rT > 0$. As σ goes to infinity, d1, d2 → positive infinity, N(d1), N(d2) → 1, the equation becomes

$$c = S_0 - Ke^{-rT}$$

- In case of $S_0 < Ke^{-rT}$, $\ln(S_0/K) + rT < 0$. As σ goes to infinity, d1, d2 → negative infinity, N(d1), N(d2) → 0, the equation becomes 0
- Therefore, the call price is $\max(S_0 - Ke^{-rT}, 0)$

8. Dividend

European options

- European options can be analyzed by assuming that the stock price is the sum of two components: a riskless component that corresponds to the known dividends during the life of the option and a risky component.
  - ➢ The BSM formula is correct if S_0 is equal to the risky component of the stock price is equal to the risky component of the stock price and σ is the volatility of the process followed by the risky component
  - ➢ Operationally, this means that the BSM formulas can be used provided that the stock price is reduced by the present value of all the dividends during the life of the option, the discounting being done from the ex-dividend dates at the risk-free rate.

Example

- Consider a European call option on a stock when there are ex-dividend dates in two months and five months. The dividend on each ex-dividend date is expected to be $0.5. The current share price is $40, the exercise price is $40, the stock price volatility is 30% per annum, the risk-free rate of interest is 9% per annum, and the time to maturity is six months.
  - The present value of the dividends is

$$0.5e^{-0.09\times 2/12} + 0.5e^{-0.09\times 5/12} = 0.9742$$

  - The option price can therefore be calculated from the BSM formula, with initial stock price = 40-0.9742=39.0256, K=40, r=0.09, volatility = 0.3, and T=0.5

$$d_1 = \frac{\ln(39.0258/40) + (0.09 + 0.3^2/2)\times 0.5}{0.3\sqrt{0.5}} = 0.2020$$

$$d_2 = \frac{\ln(39.0258/40) + (0.09 - 0.3^2/2)\times 0.5}{0.3\sqrt{0.5}} = -0.0102$$

Using the NORMSDIST function in Excel gives

$$N(d_1) = 0.5800, \quad N(d_2) = 0.4959$$

and, from equation (15.20), the call price is

$$39.0258 \times 0.5800 - 40e^{-0.09\times 0.5} \times 0.4959 = 3.67$$

or $3.67.

American Call options

- Consider next American call options.
  - In the absence of dividends American options should never be exercised early.
  - An extension to the argument shows that, when there are dividends, it can only be optimal to exercise at a time immediately before the stock goes ex-dividend
  - If the option is exercised at time t_n, the investor receives

$$S(t_n) - K$$

  - Where S(t) denotes the stock price at time t.
  - If the option is not exercised, the stock price drops to $S(t_n) - D_n$.
    - The value of the options is then greater than

$$S(t_n) - D_n - Ke^{-r(T-t_n)}$$

It follows that, if

$$S(t_n) - D_n - Ke^{-r(T-t_n)} \geqslant S(t_n) - K$$

that is,

$$D_n \leqslant K\left[1 - e^{-r(T-t_n)}\right] \tag{15.24}$$

it cannot be optimal to exercise at time t_n. On the other hand, if

$$D_n > K\left[1 - e^{-r(T-t_n)}\right] \tag{15.25}$$

    - It is then optimal to exercise