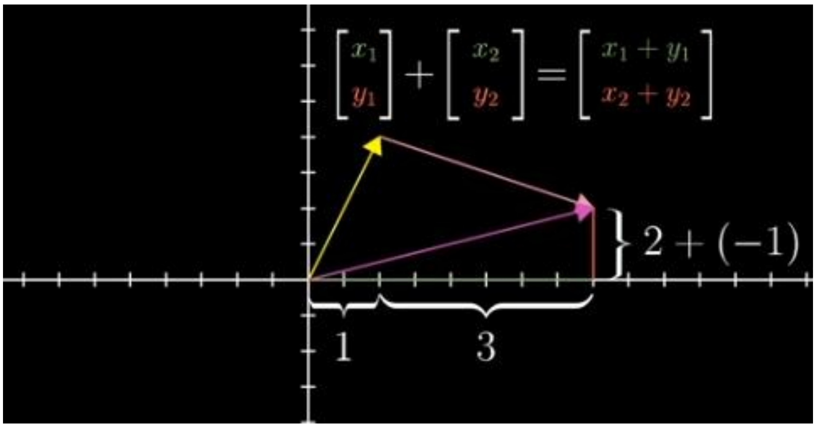
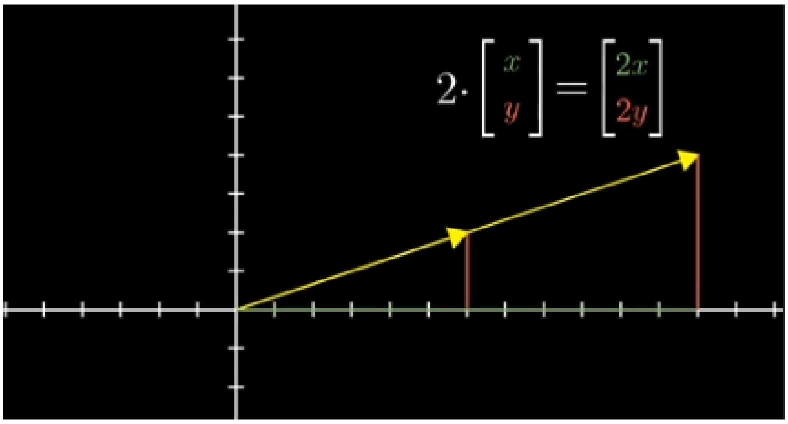
Credit: https://www.3blue1brown.com/topics/linear-algebra

Vectors, what are they

- Vector addition

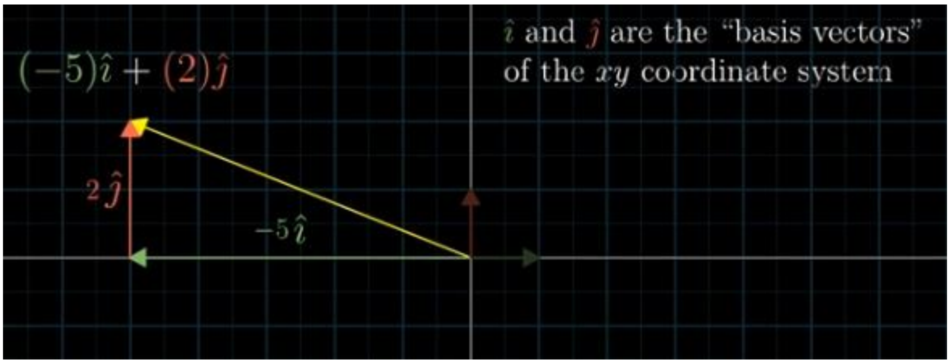


- Scaler multiplication



Linear combinations, span, and basis vectors

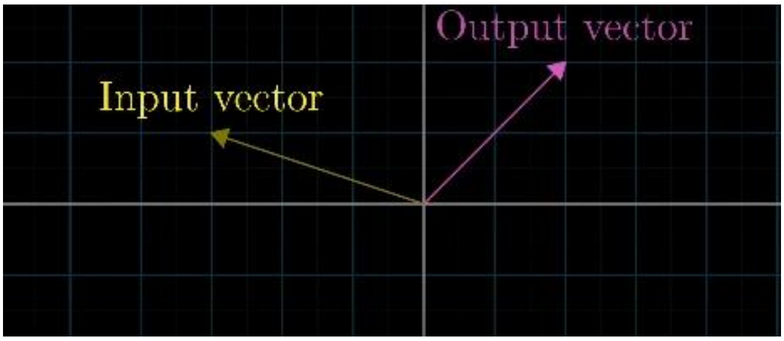
- Basis



- Span of the two vectors are all the possible linear combinations of the vector
    - In 2D, it represents all the points in the 2D plane
    - In 3D, it represents all the points on a plane which went through the origin in the 3D space

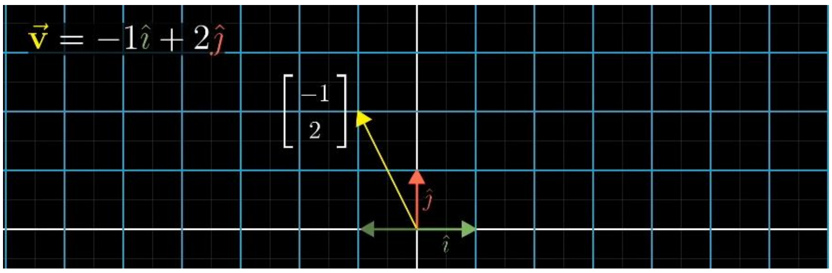
Matrices and transforming space

- If a linear transformation takes a vector and spit out another vector, consider this transformation as moving the original vector to the output vector

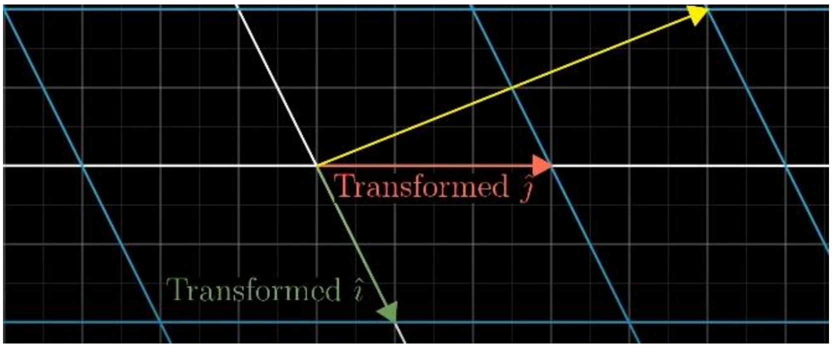


- Conceptualize each vector as a point in space as opposed to arrow
    - So we move from point to point
- Linear transformation satisfies
    - 1. Lines remain lines
    - 2. Origin remains fixed
    - It keeps grid lines remain parallel and evenly spaced

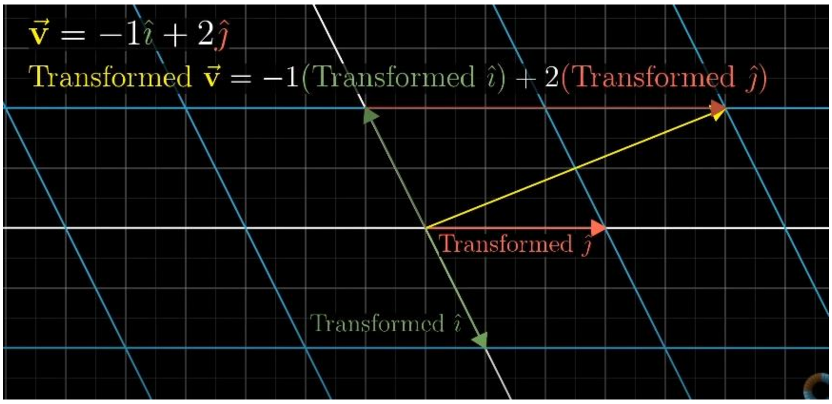
- Question: how to describe transformations numerically?
    - Just record where the [1, 0] and [0, 1] land, and everything else follows
- Consider



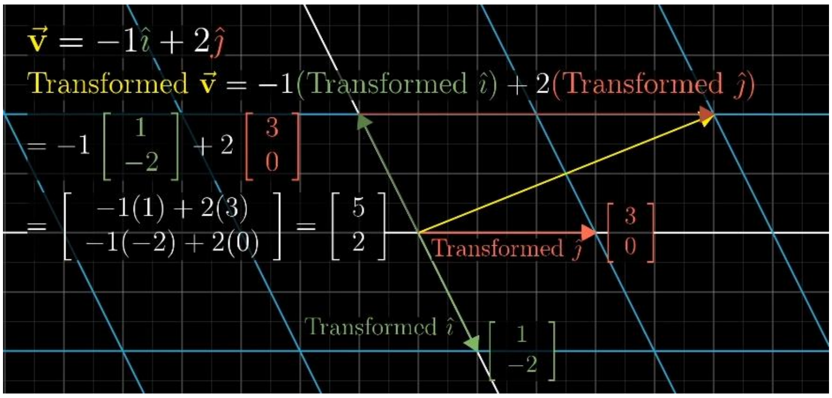
    - After transformation



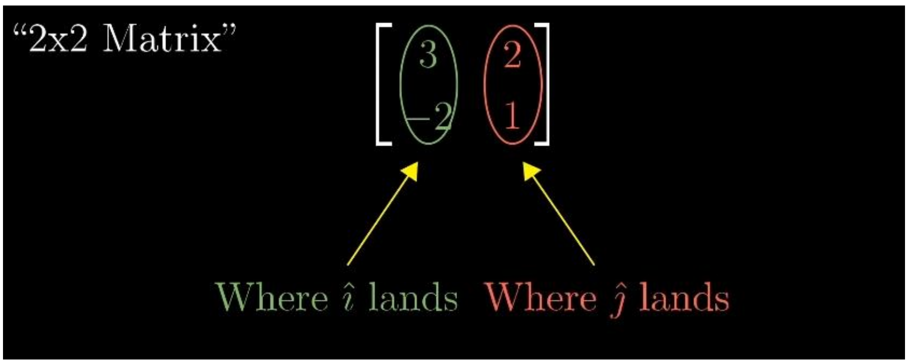
➢ Then, v remains the same linear combination of the two basis



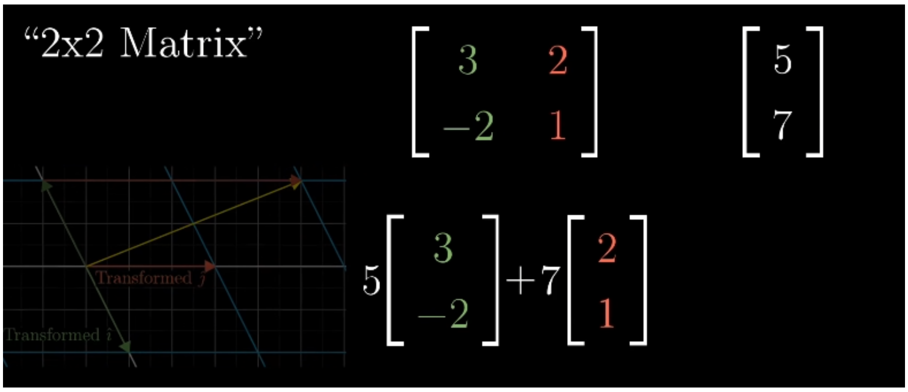
➢ Now, I can deduce where v is based on where i, and j is



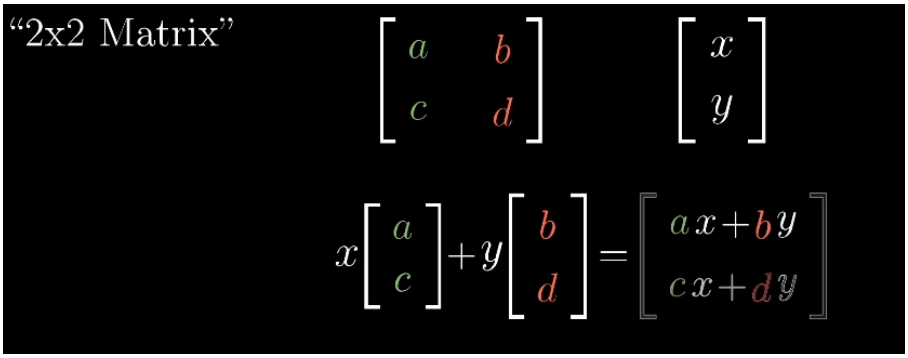
- We can package these coordinates into a 2*2 matrix



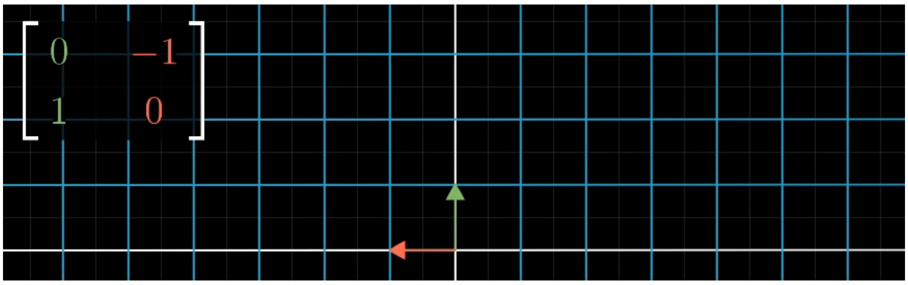
➢ If the matrix describes a linear transformation, and some specific vector, and you want to know where that linear transformation takes the vector



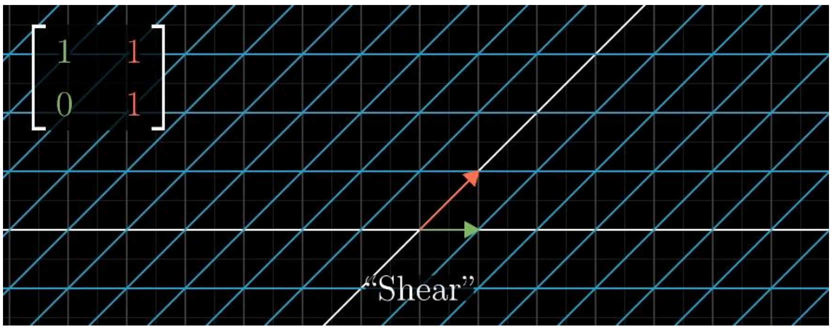
- More generally



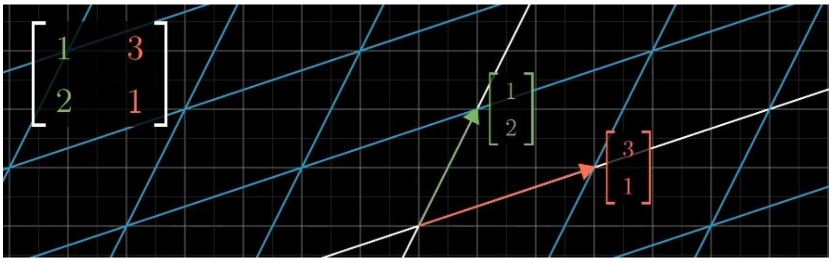
- If we take the transformation of 90 degree counter-clockwise, the matrix that describe this transformation is



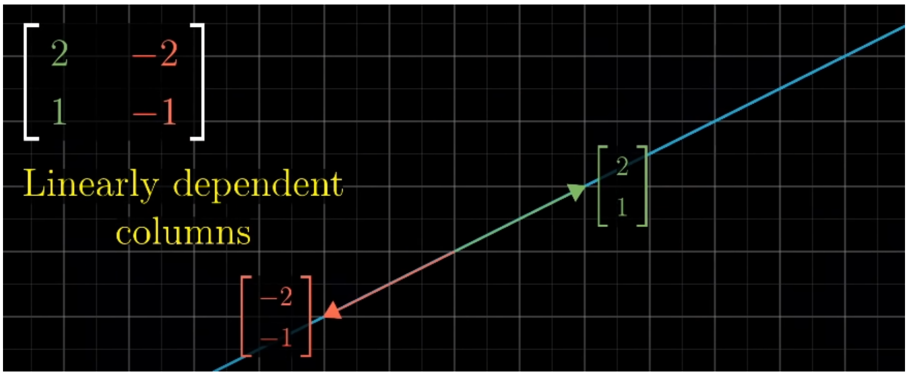
- "shear" transformation



- Now, go the other way around, known the matrix, and deduce what transformation it is

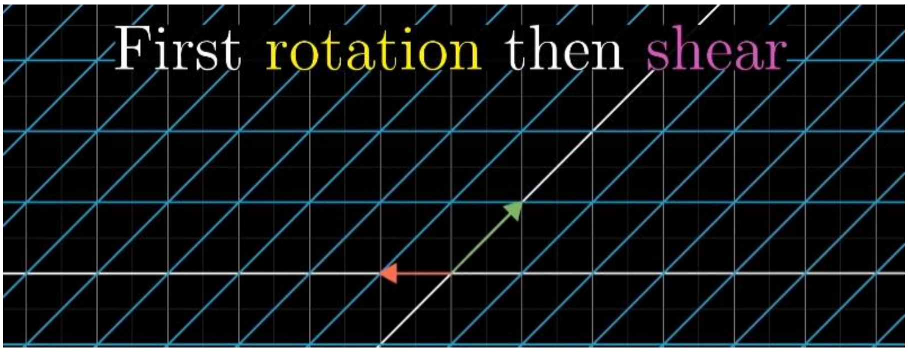


- If the linear transformation matrix is dependent

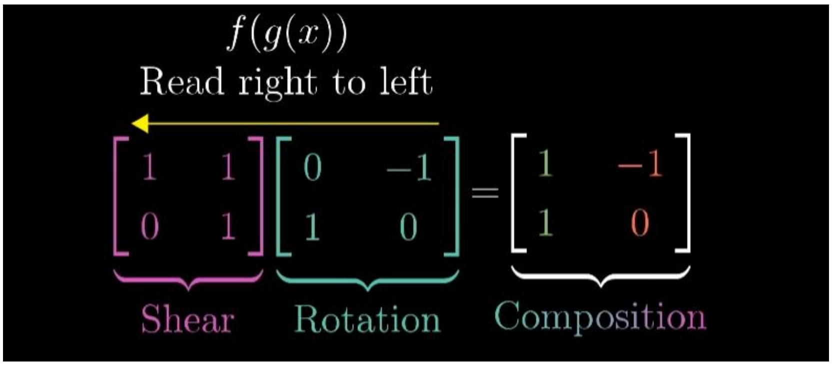


Matrix multiplication as composition

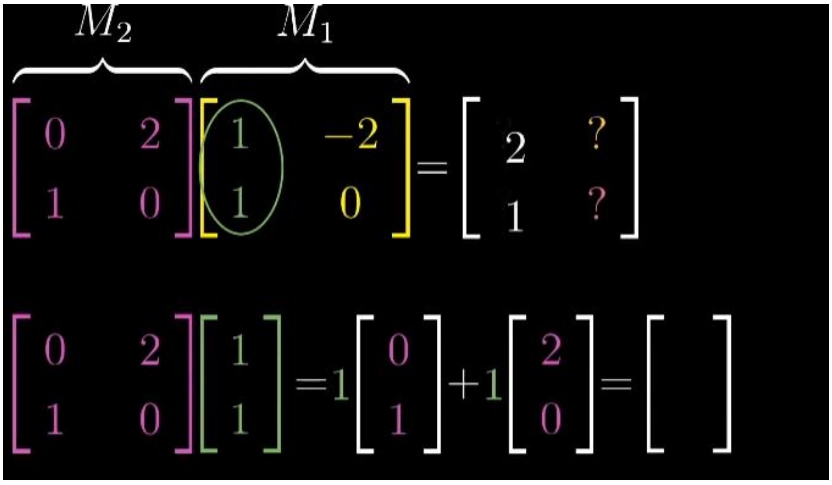
- To describe multiple transformation

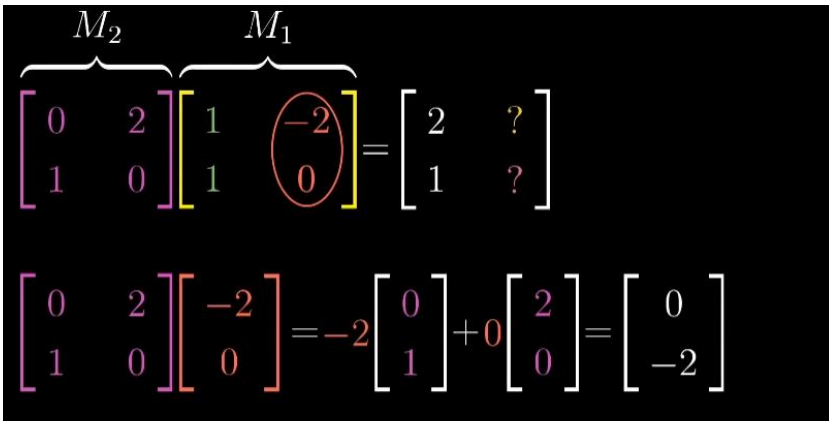


➢ This is multiplication of two matrices representing each transformation
➢ Multiplying the resulting (composition) matrix, it represents one transformation that perform both transformations in one step

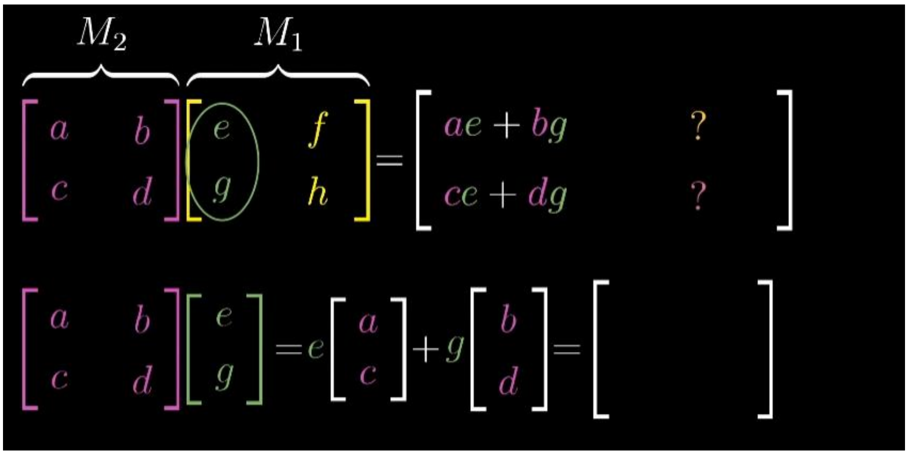


- Tracking where the basis go

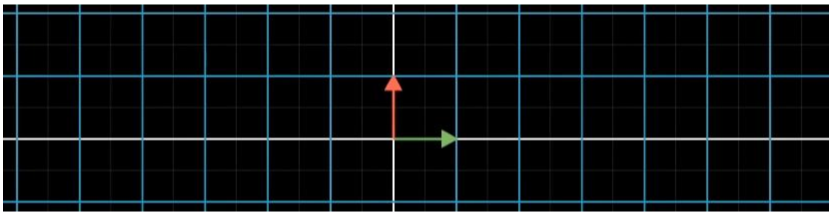


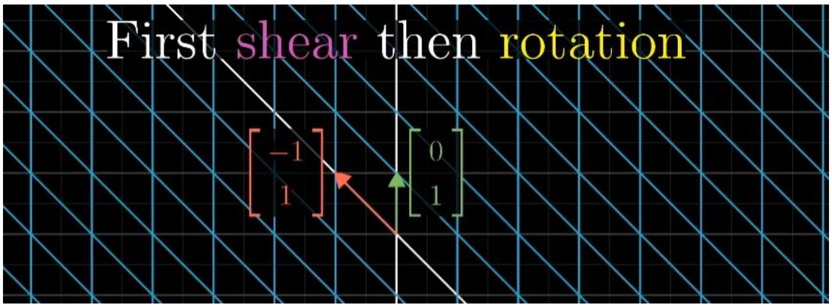


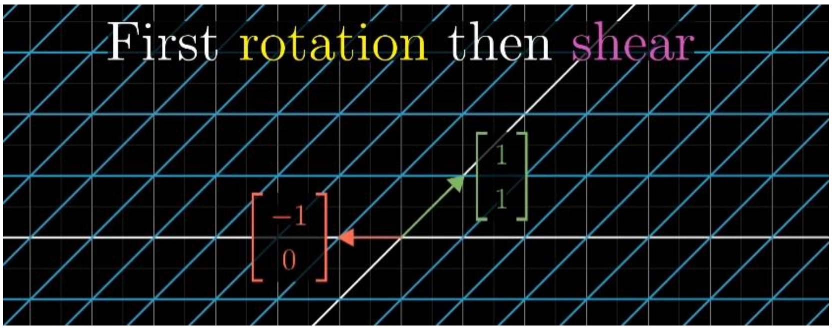
➢ In general



- Does the order matter?

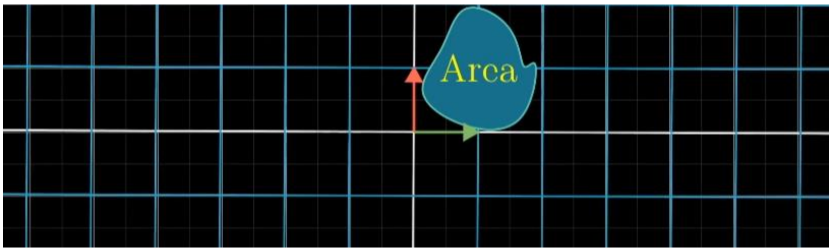




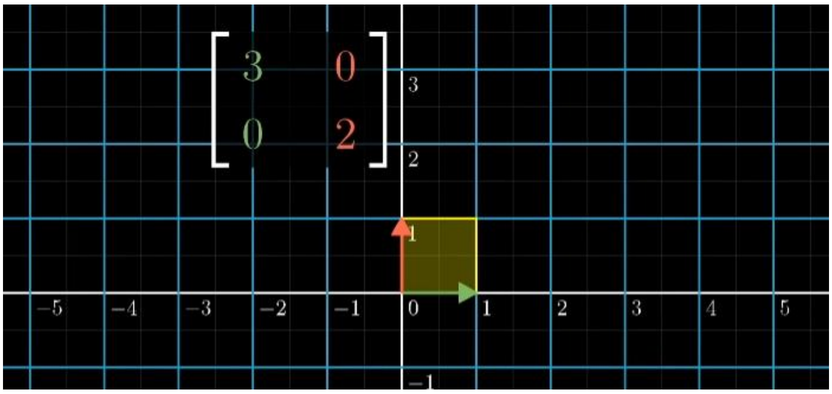


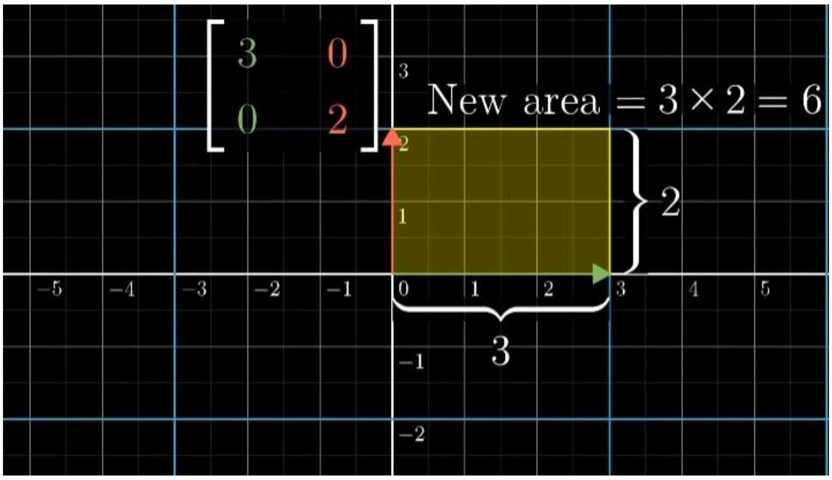
The determinant

- We want to see how a given area changes after transformation

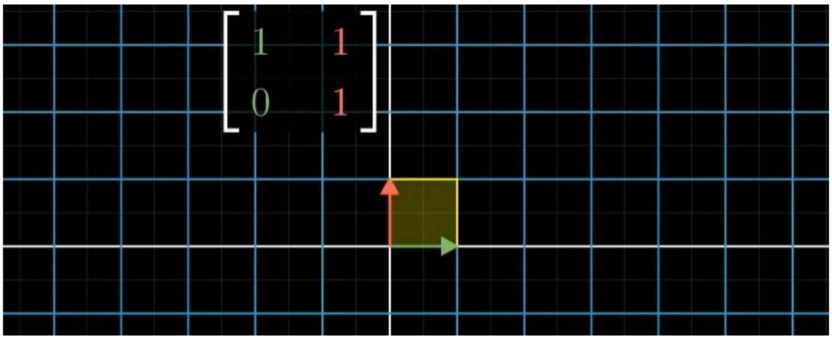


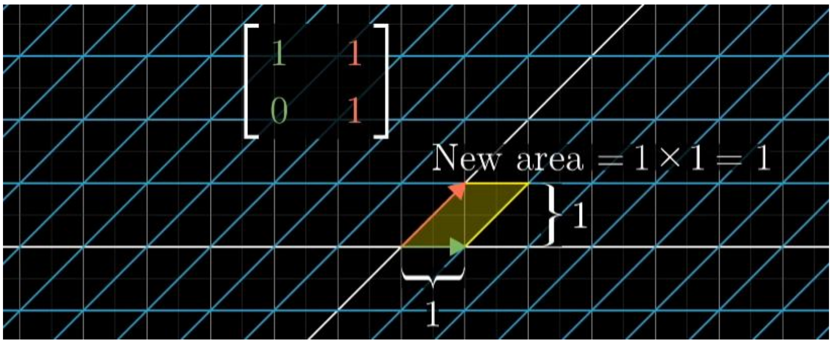
  - Consider



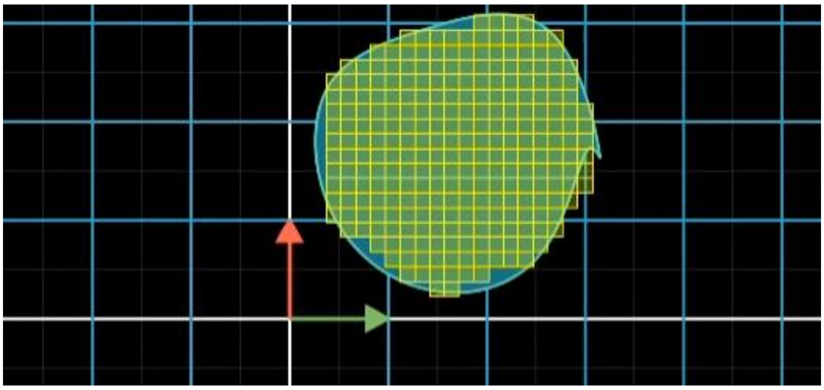


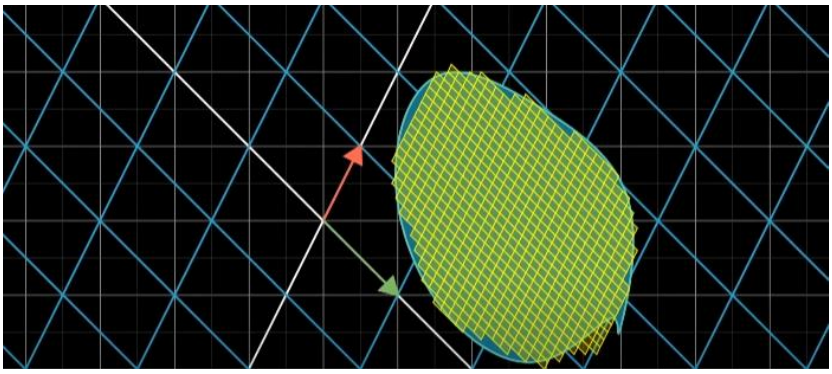
➢ Consider



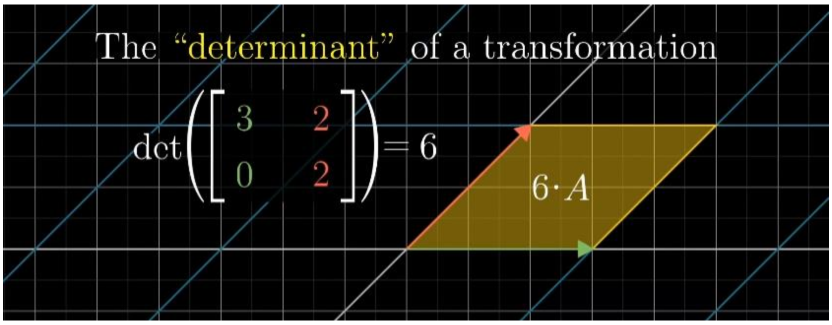


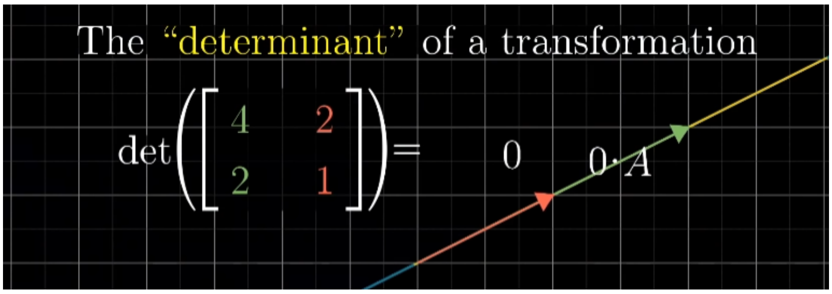
➢ Than we can approximate the uneven shape with squares



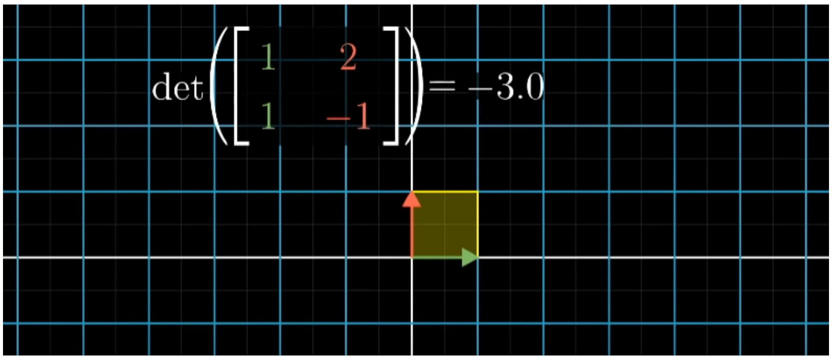


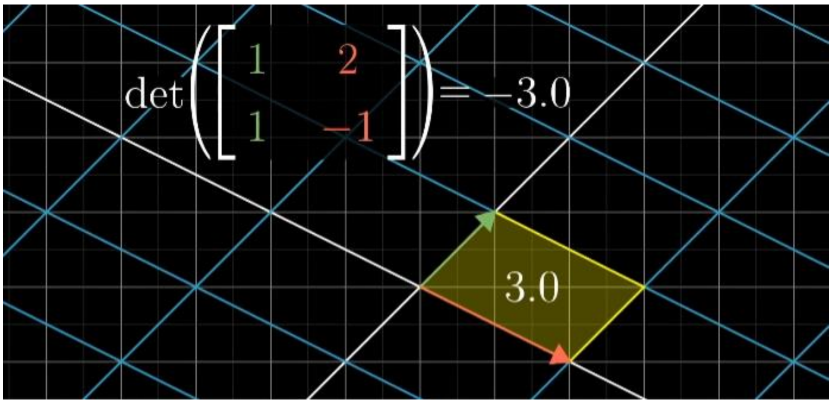
- Determinant



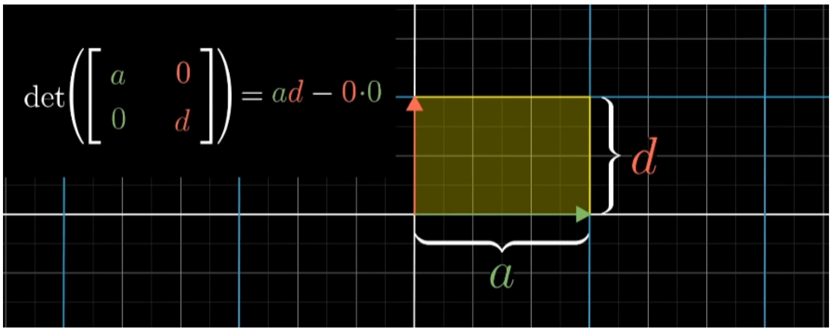


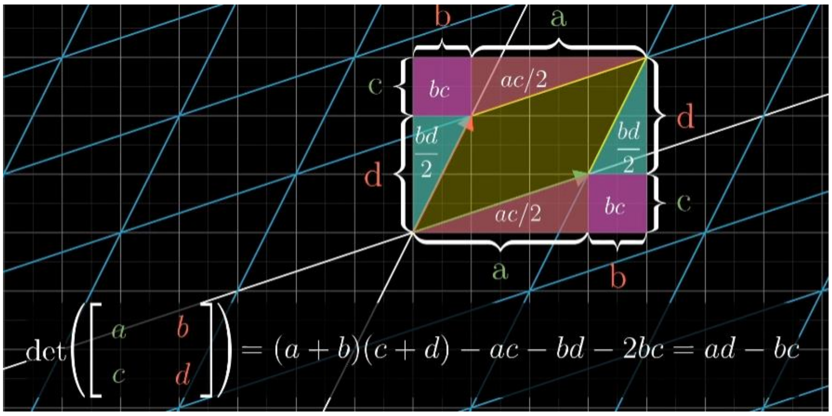
➢ When orientation of space is reversed, the determinant is negative



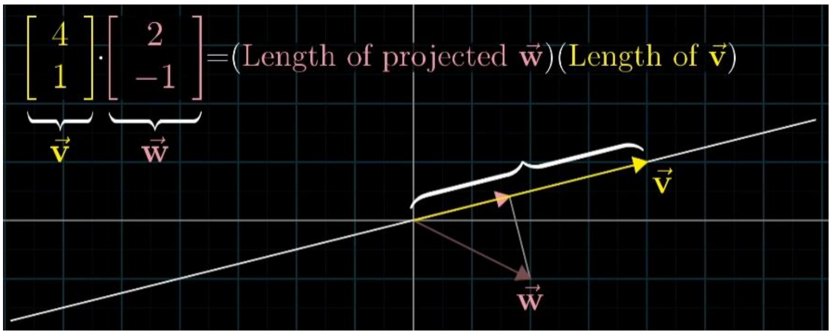


- Intuition of the formula

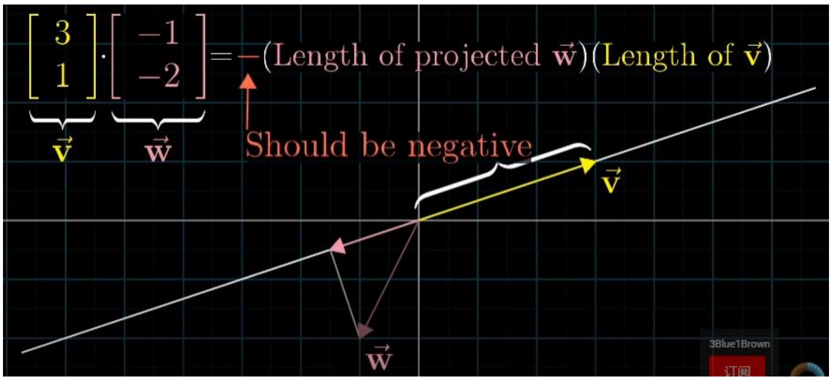




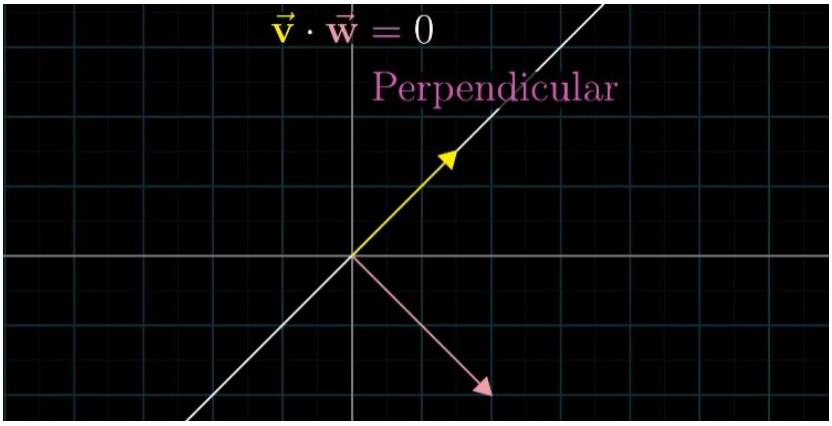
Dot product

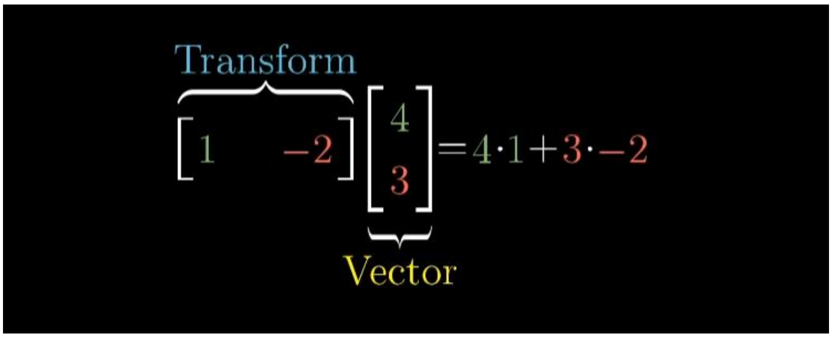


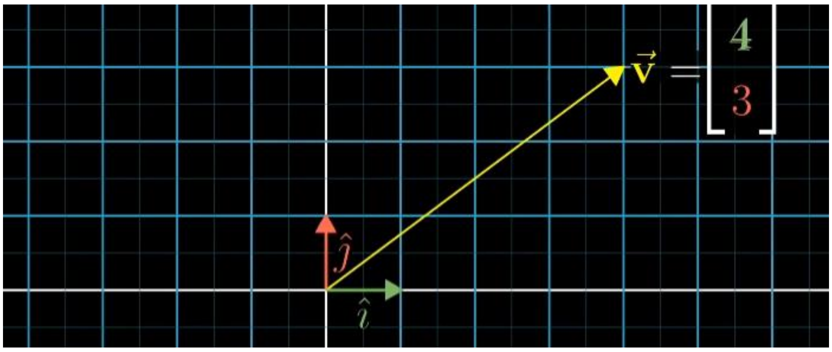
- When w is pointing at a direction opposite from v

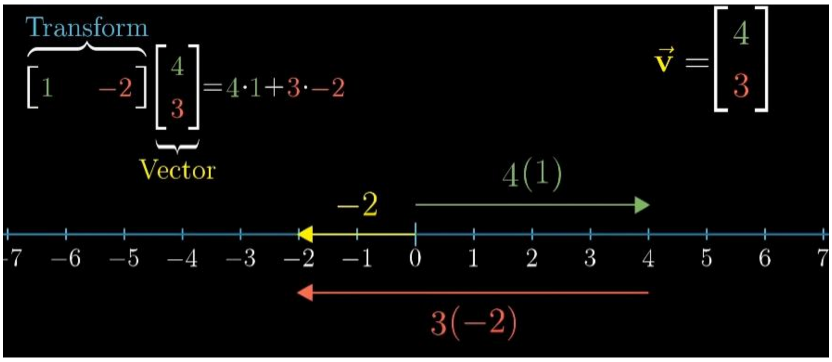


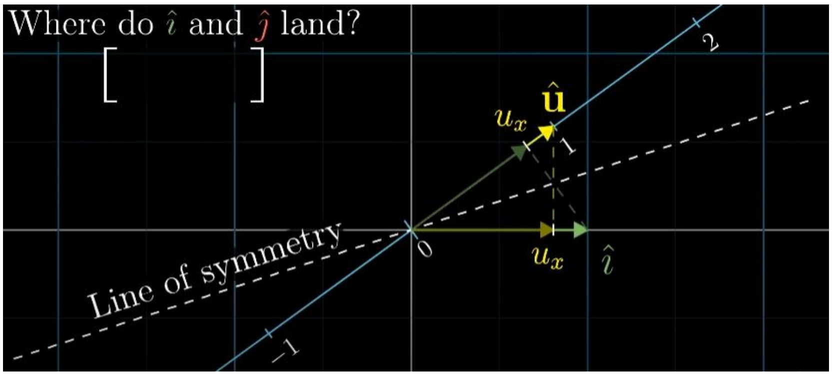
- When one is perpendicular to another

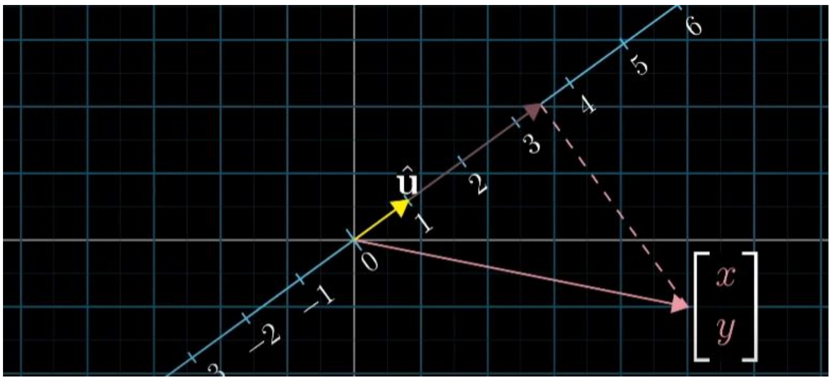


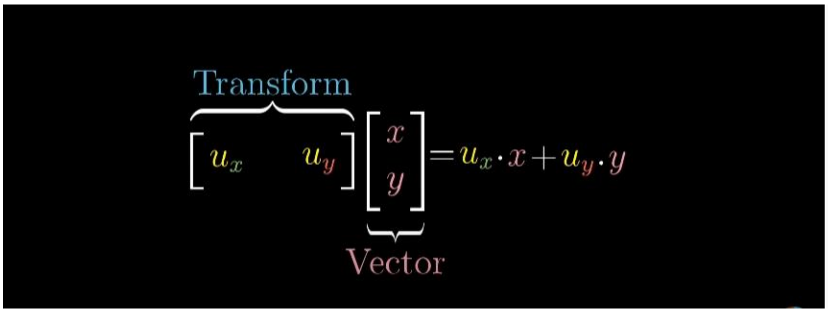




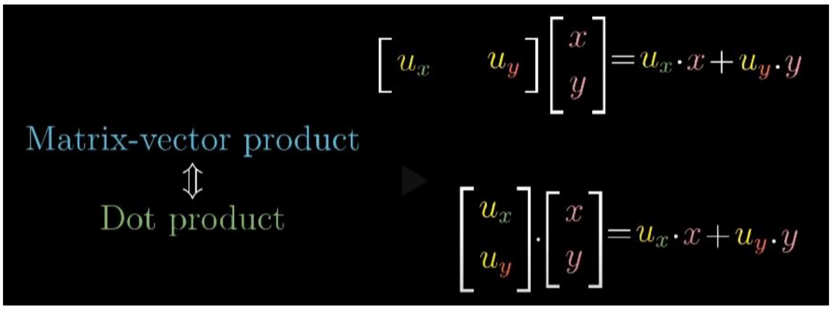


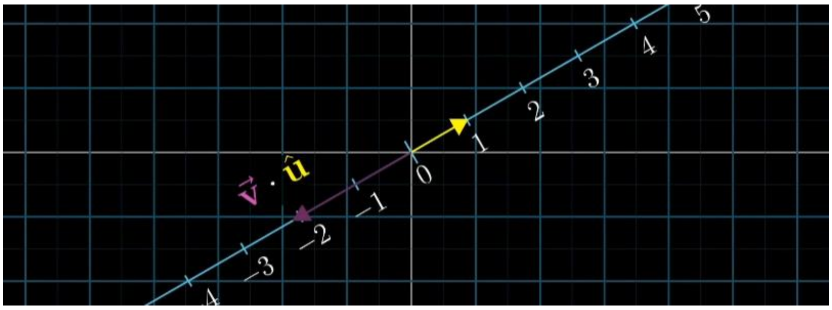




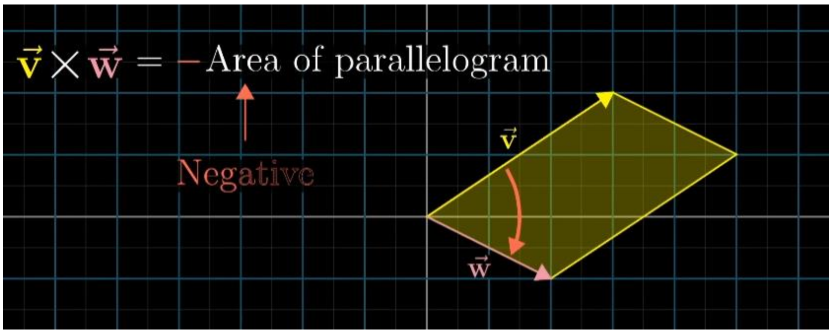


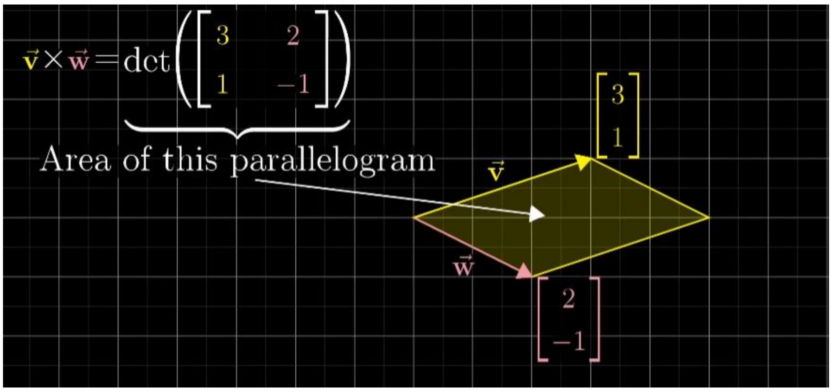
- This is why taking the dot product of a unit vector can be interpreted as projecting a vector onto the span of the unit vector, and taking the length



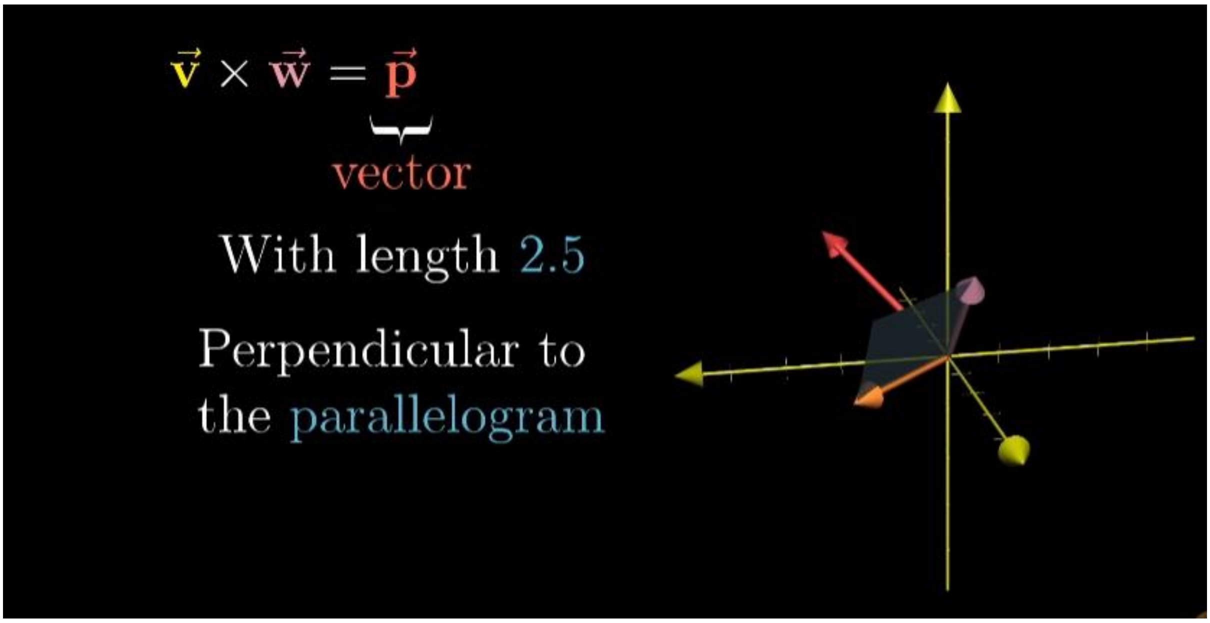


Cross Product

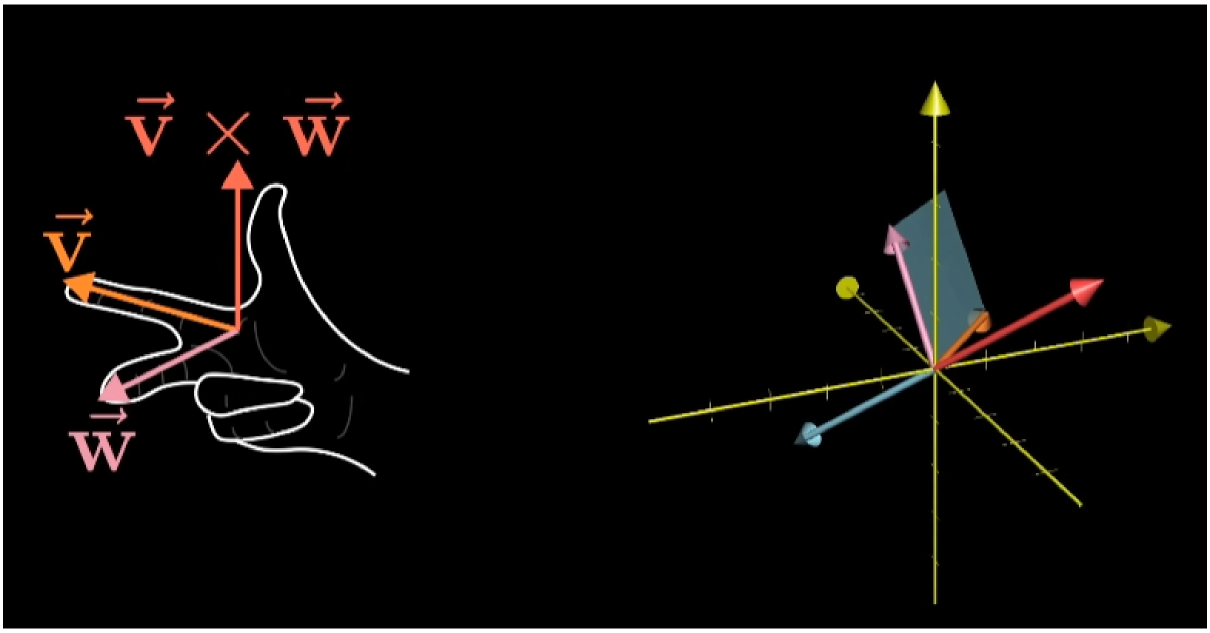




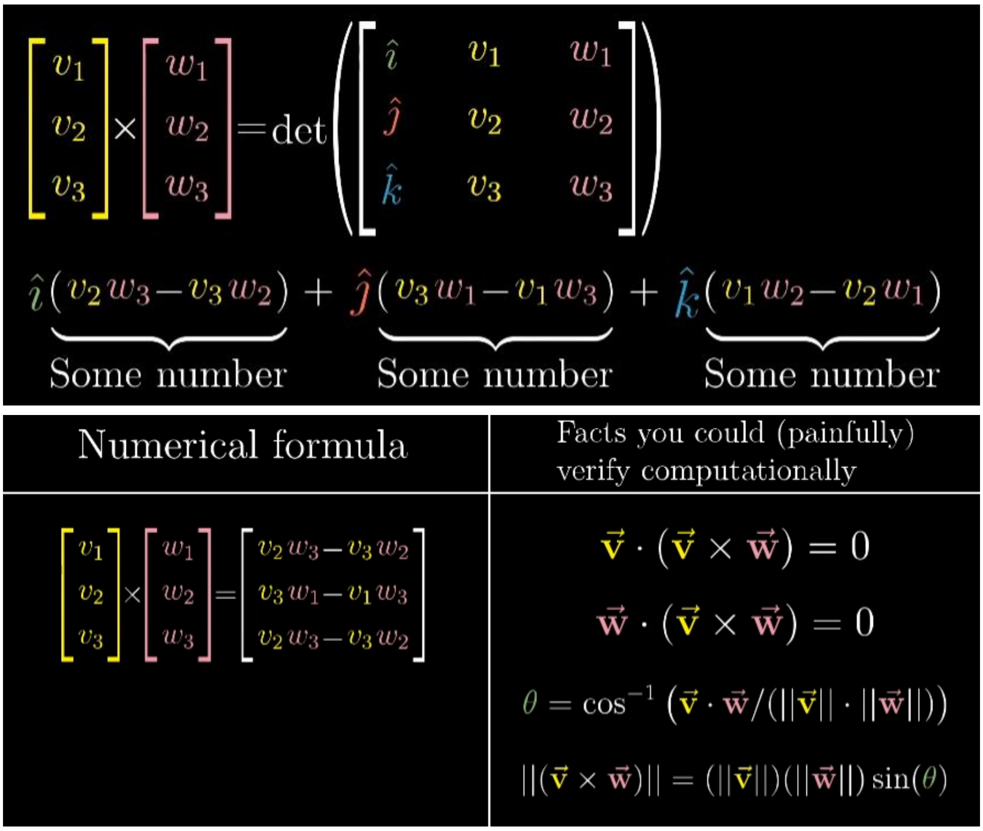
- More accurately



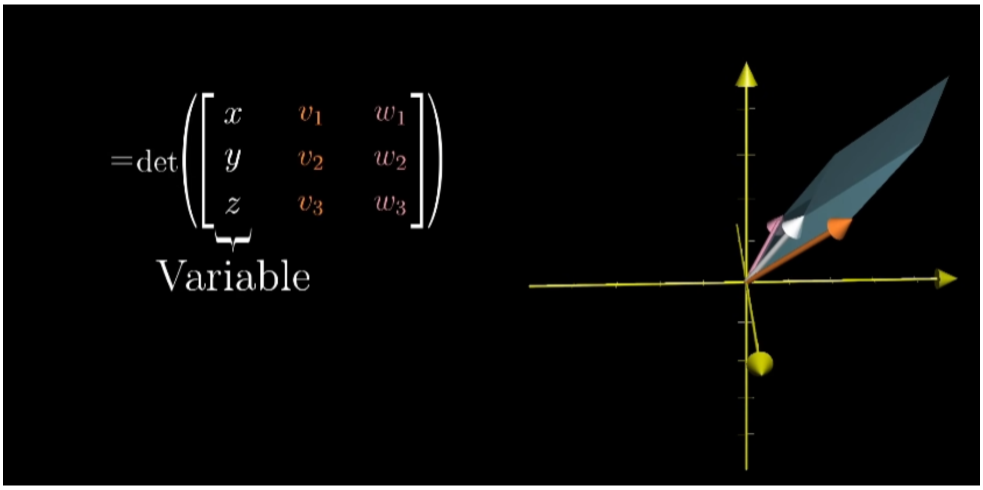
- In which direction? There are two vectors perpendicular to the plane

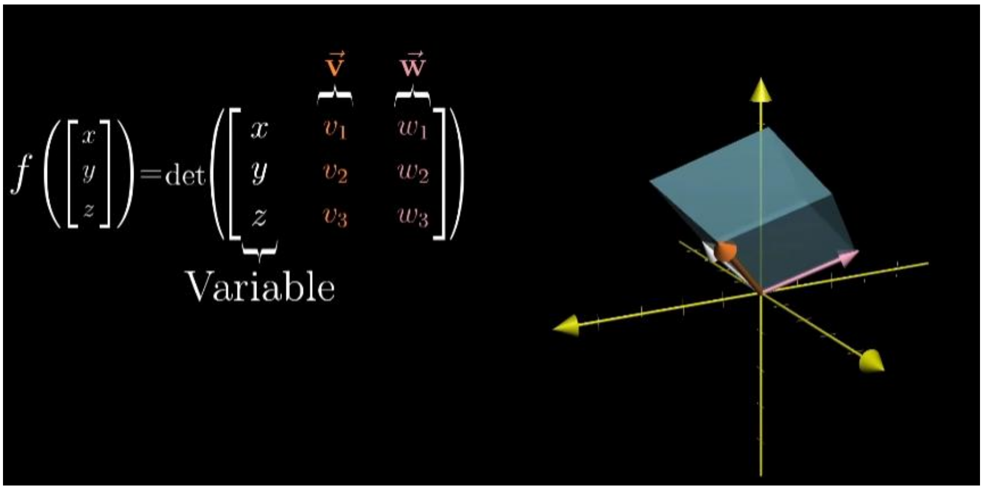


- Calculation

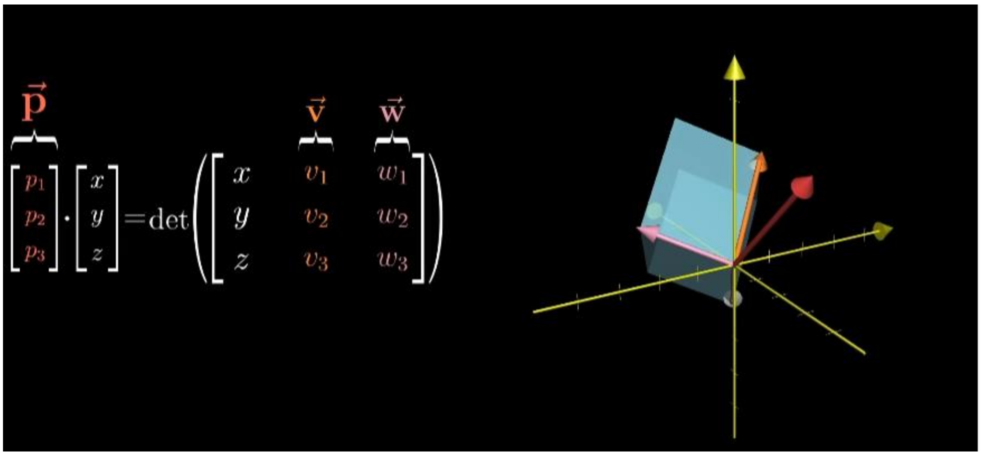


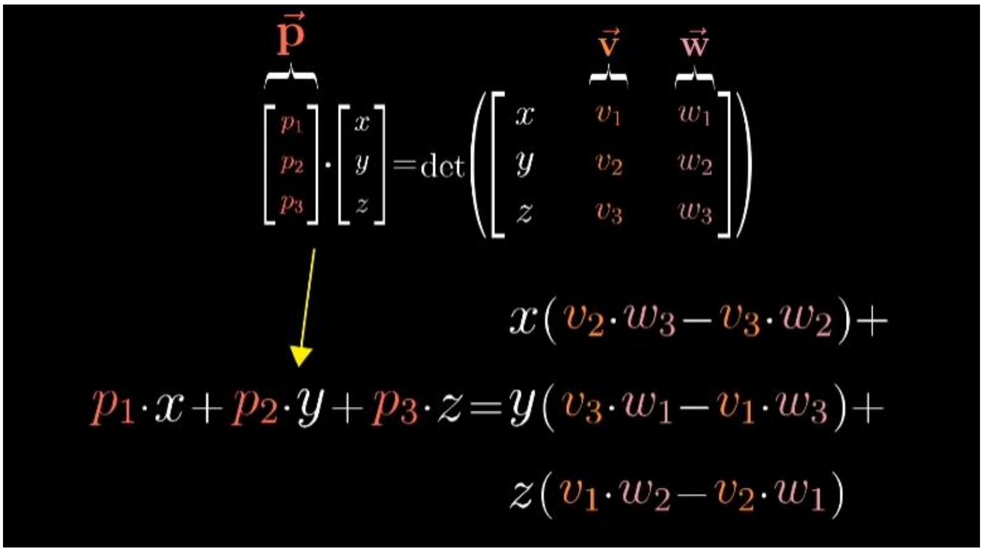
- Representation

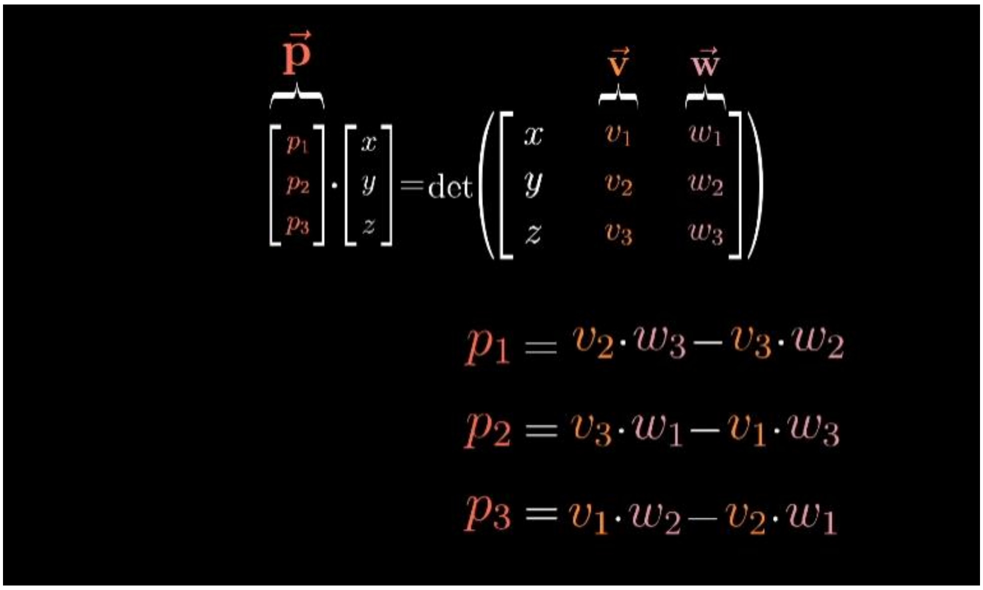




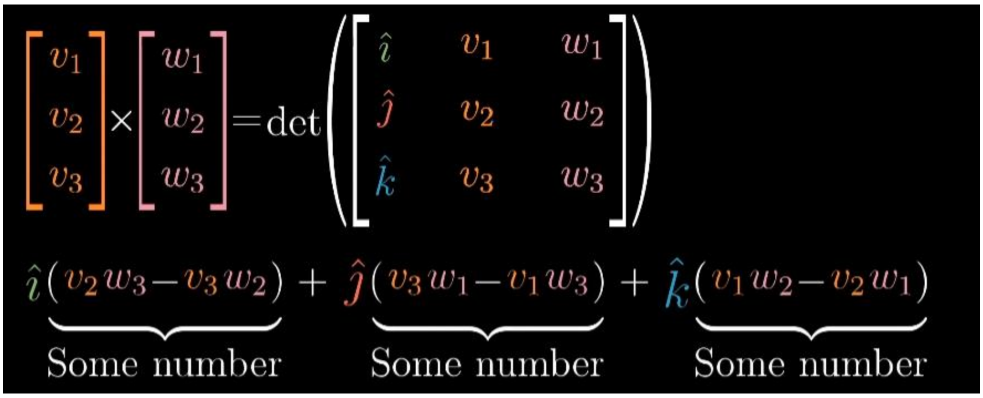
- We are looking for vector p where



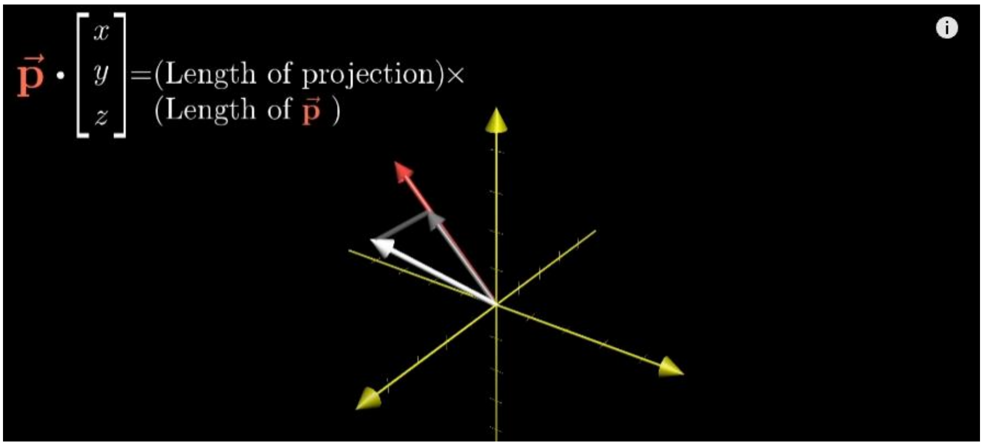




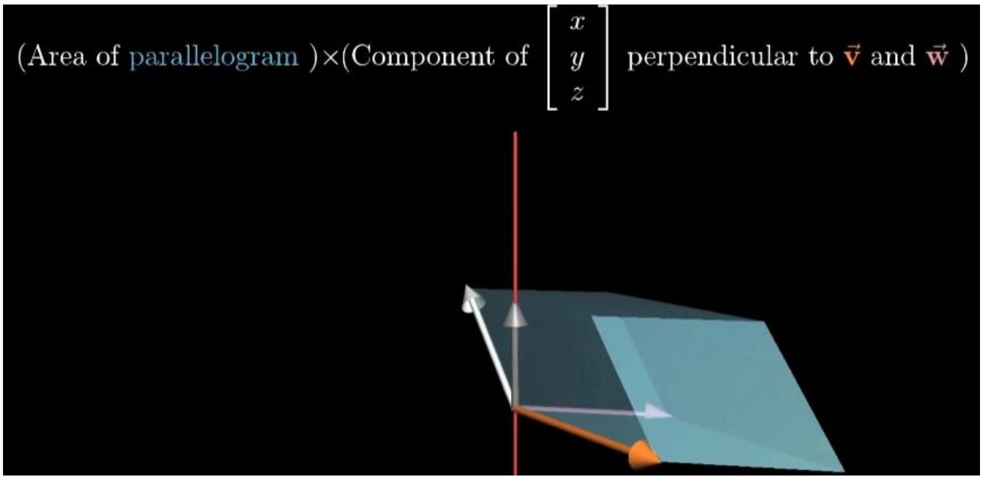
- ➢ Recall



- Geometric understanding
  - ➢ Recall that

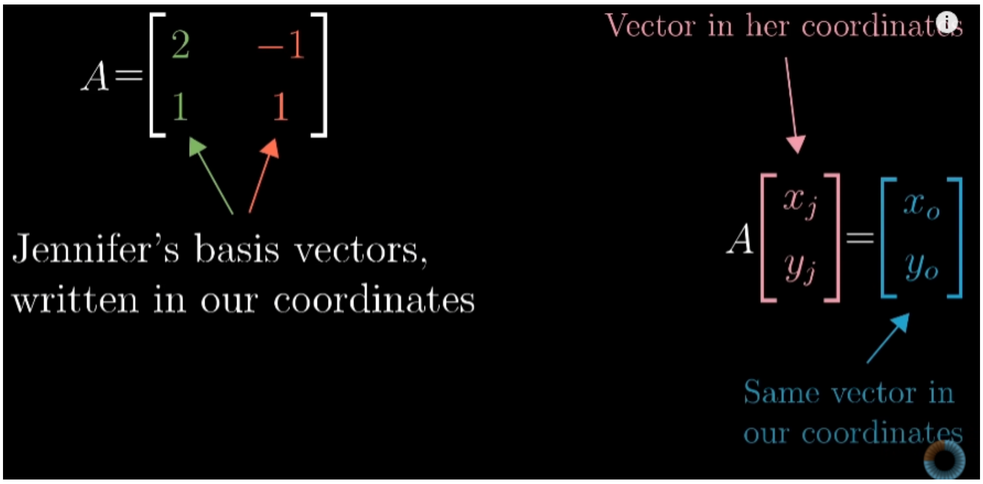


  - ➢ Consider multiplying the area of the parallelogram by the component of vector x, y, z that is perpendicular to vector v and w

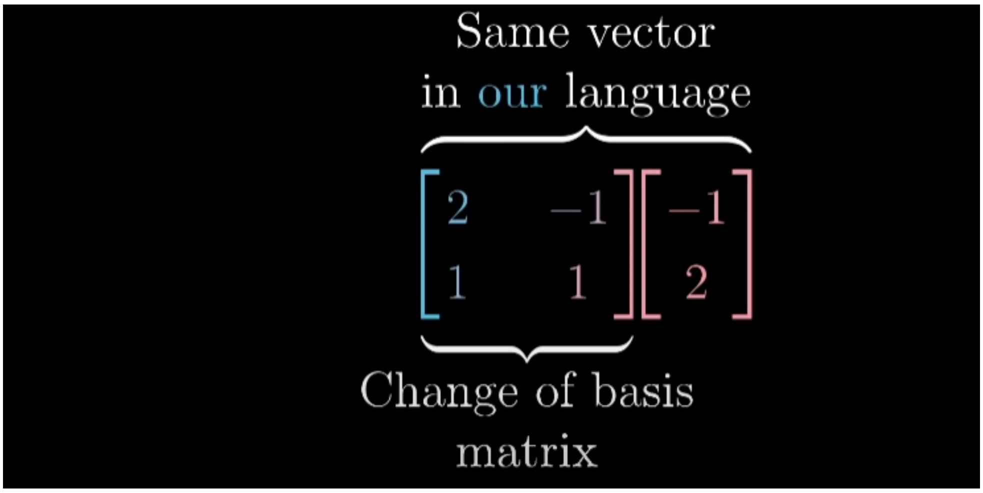


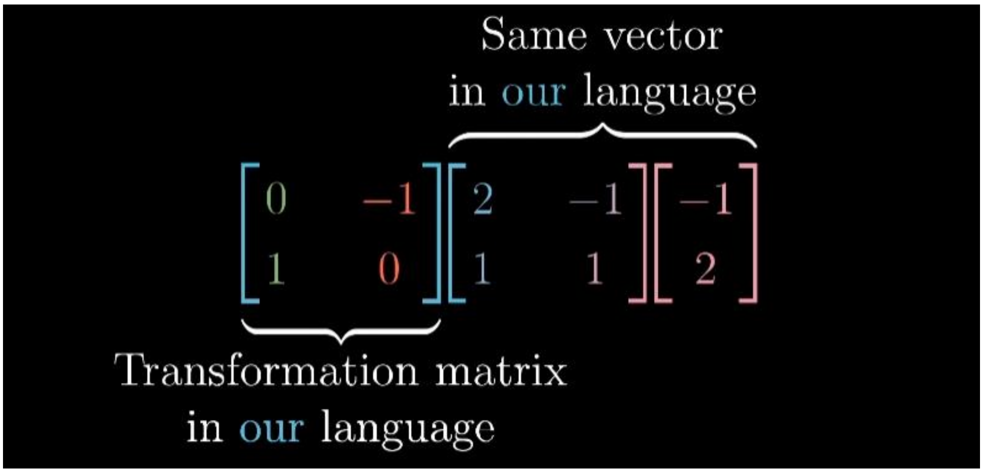
- This is the same thing as taking a dot product between x, y, z and a vector that's perpendicular to v and w with the length equal to the area of that parallelogram

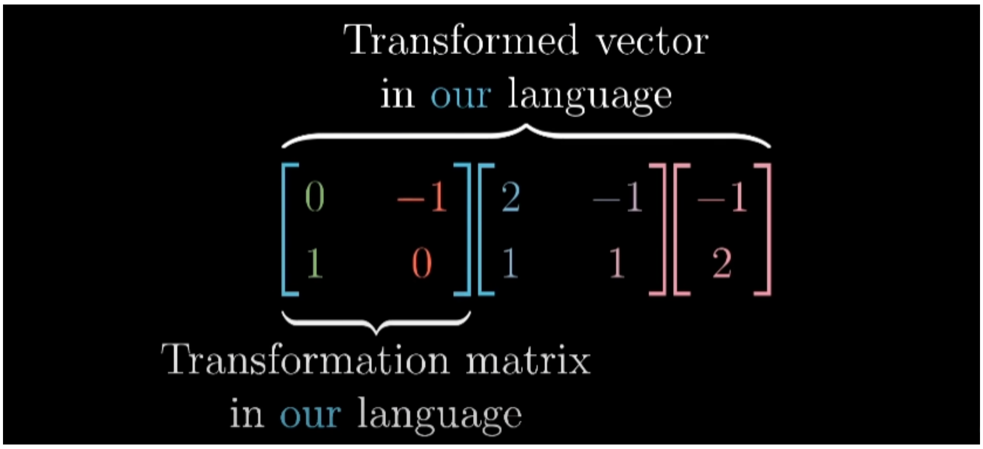
Change of basis

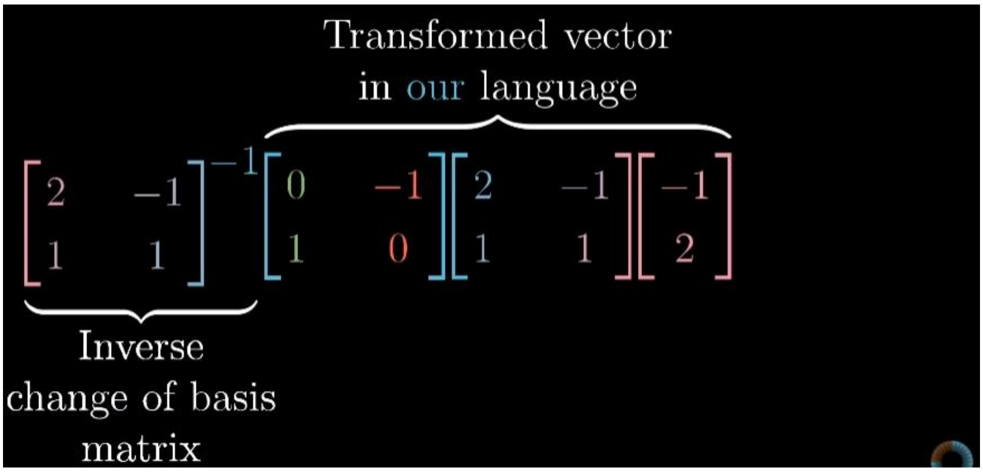


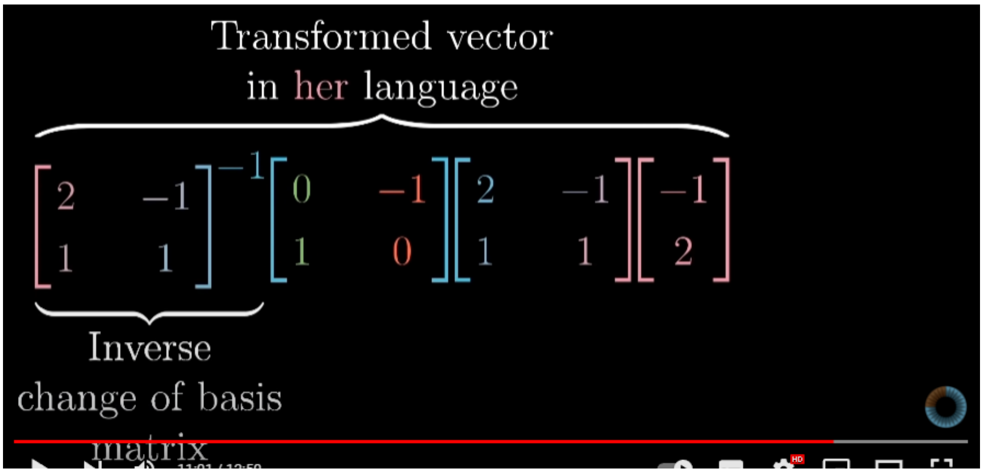
- How to translate a matrix: how to perform 90 rotation to vectors in Jennifer's system?



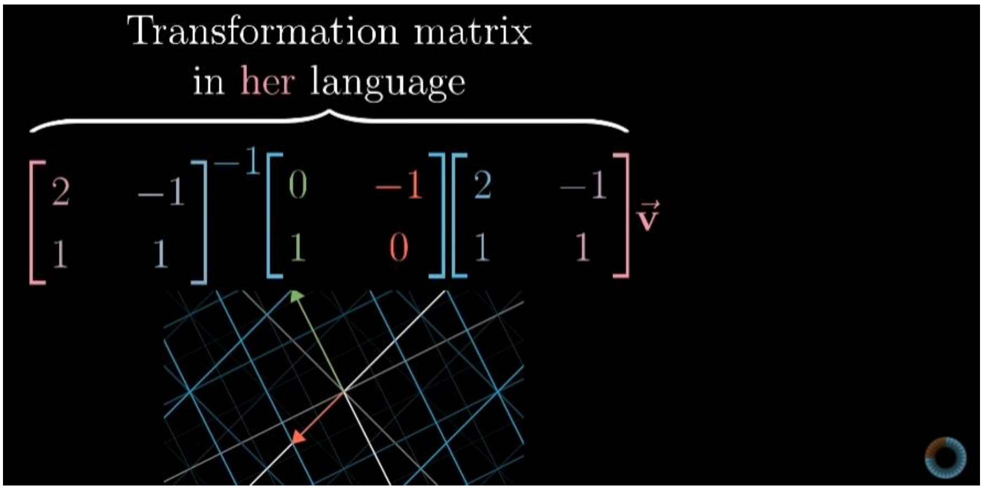

Same vector
in our language
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$
Transformation matrix
in our language


Transformed vector
in our language
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$
Transformation matrix
in our language


Transformed vector
in our language
$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$
Inverse
change of basis
matrix



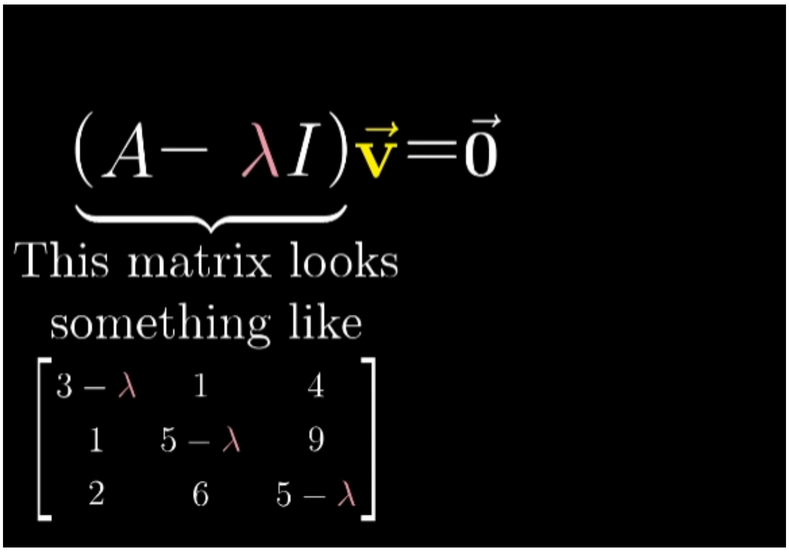
- We can perform this for any vector in her system



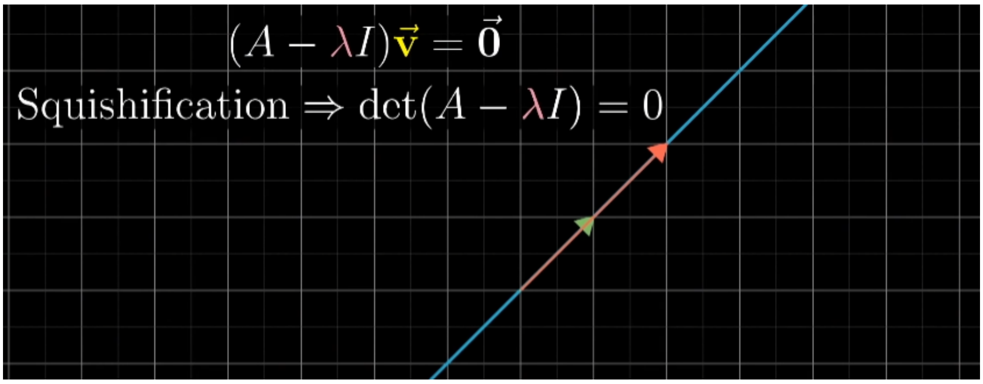
Eigenvalues and Eigenvectors

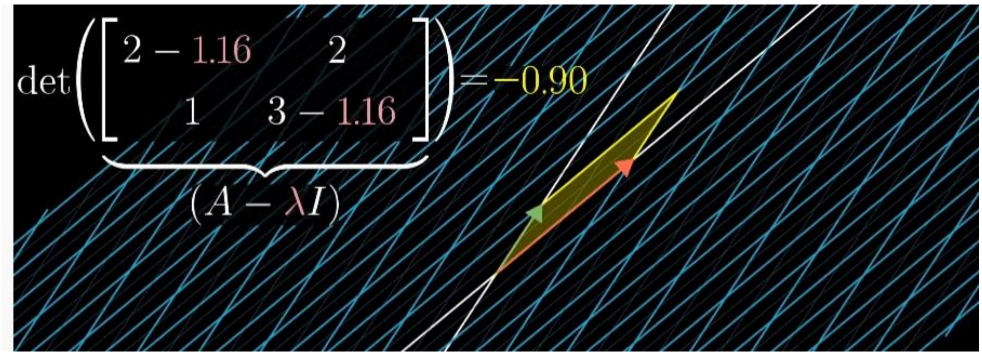
- When linear transformation is performed via matrix multiplication, the vectors that remained on its span with some stretching or squashing are called eigenvectors, and the value of the magnitudes of stretching or squashing are called eigenvalues of the transformation.

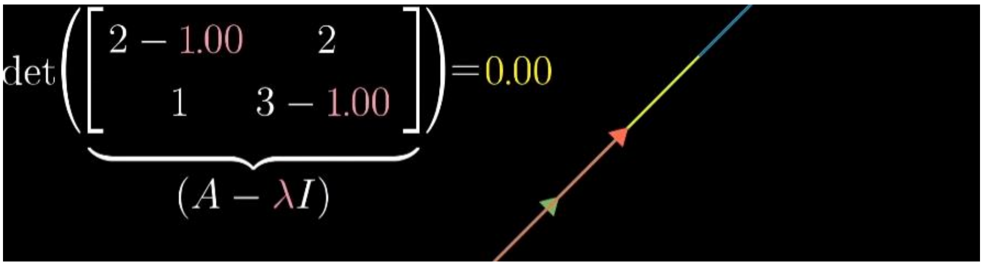
- Computation



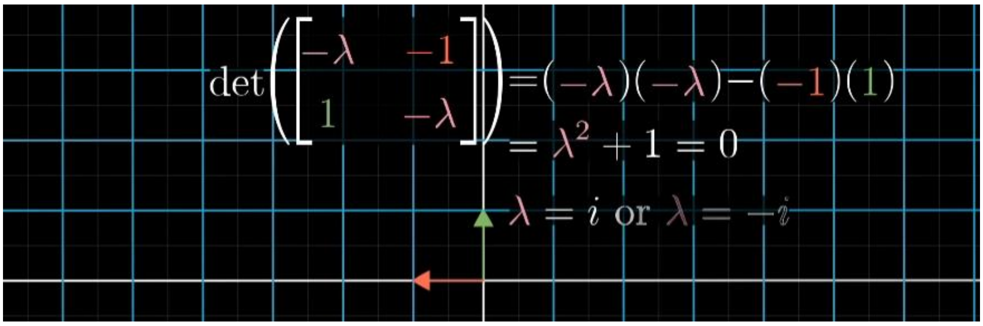
- This is only true, when the matrix transformation squashes the dimension







- There could be no eigenvalues and eigenvectors: consider 90 rotation


$\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = (-\lambda)(-\lambda) - (-1)(1)$
$= \lambda^2 + 1 = 0$
$\lambda = i$ or $\lambda = -i$