- Any variable whose value changes over time in an uncertain way is said to follow a ***stochastic process***
  - It should be noted that, in practice, we do not observe stock prices following continuous-variable, continuous-time processes.
  - Stock prices are restricted to discrete values (e.g., multiples of a cent) and changes can be observed only when the exchange is open for trading.
  - Nevertheless, the continuous-variable, continuous-time process proves to be a useful model for many purposes

1. The Markov property

- A ***Markov process*** is a particular type of stochastic process where only the current value of a variable is relevant for predicting the future.
  - Predictions for the future are uncertain and must be expressed in terms of probability distributions.
  - The Markov property implies that the probability distribution of the price at any particular future time is not dependent on the particular path followed by the price in the past.

- The Markov property of stock prices is consistent with the weak form of market efficiency.
  - This states that the present price of a stock impounds all the information contained in a record of past prices.
  - It is competition in the marketplace that tends to ensure that weak-form market efficiency and the Markov property hold.

2. Continuous-time stochastic processes

- Consider a variable that follows a Markov stochastic process. Suppose that its current value is 10 and that the change in its value during a year is $\phi(0,1)$, where $\phi(m, v)$ denotes a probability distribution that is normally distributed with mean m and variance v^2.
  - The change in 2 years is the sum of two normal distributions, each of which has a mean of zero and variance of 1
    - The mean of the change during 2 years in the variable we are considering is zero and the variance of this change is 2
    - Hence, the change in the variable over 2 years has the distribution $\phi(0,2)$. The standard deviation of the change is $\sqrt{2}$
  - Consider next the change in the variable during 6 months.
    - The variance of the change in the value of the variable during 1 year equals the variance of the change during the first 6 months plus the variance of the change during the second 6 months.
    - It follows that the variance of the change during a 6-month period must be 0.5.
    - Equivalently, the standard deviation of the change is $\sqrt{0.5}$.

- The probability distribution for the change in the value of the variable during 6 months is $\phi(0,0.5)$

- More generally, the change during any time period of length T is $\phi(0,T)$
  - Note that, when Markov processes are considered, the variances of the changes in successive time periods are additive. The standard deviation of the changes in successive time periods are not additive.

Wiener Process

- The process followed by the variable we have been considering is known as a ***Wiener process***.
  - It is a particular type of Markov stochastic process with a mean change of zero and a variance rate of 1 per year.
  - It is subject to a large number of small shocks and is sometimes referred to as ***Brownian motion***.

- Expressed formally, a variable z follows a Wiener process if it has the following two properties

Property 1. *The change Δz during a small period of time Δt is*

$$\Delta z = \epsilon\sqrt{\Delta t} \tag{14.1}$$

where ϵ has a standard normal distribution $\phi(0,1)$.

Property 2. *The values of Δz for any two different short intervals of time, Δt, are independent.*

  - It follows from the first property that Δz itself has a normal distribution with

$$\text{mean of } \Delta z = 0$$
$$\text{standard deviation of } \Delta z = \sqrt{\Delta t}$$
$$\text{variance of } \Delta z = \Delta t$$

  - The second property implies that z follows a Markov process.

- Consider the change in the value of z during a relatively long period of time, T.
  - This can be denoted as $z(T) - z(0)$. It can be regarded as the sum o the changes in z in N small time intervals of length Δt, where

$$N = \frac{T}{\Delta t}$$

Thus,

$$z(T) - z(0) = \sum_{i=1}^{N} \epsilon_i \sqrt{\Delta t} \tag{14.2}$$

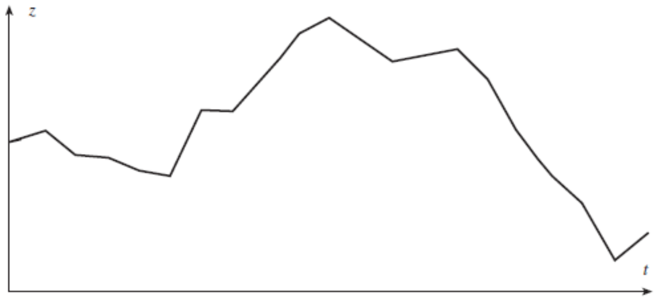
  - It follows with

$$\text{mean of } [z(T) - z(0)] = 0$$
$$\text{variance of} [z(T) - z(0)] = N\,\Delta t = T$$
$$\text{standard deviation of } [z(T) - z(0)] = \sqrt{T}$$

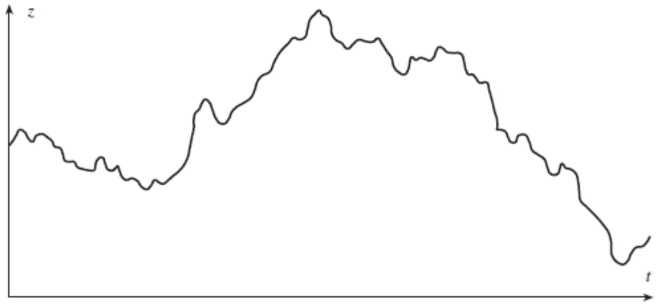
- In ordinary calculus, it is usual to proceed from small changes to the limit as the small changes become closer to zero
  - Thus, $dx = a\,dt$ is the notation used to indicate that $\Delta x = a\,\Delta t$ in the limit as $\Delta t \to 0$.

- ➢ We use similar notational conventions in stochastic calculus. So when we refer to dz as a Wiener process, we mean it has the properties for Δz given above in the limit as $\Delta t \to 0$

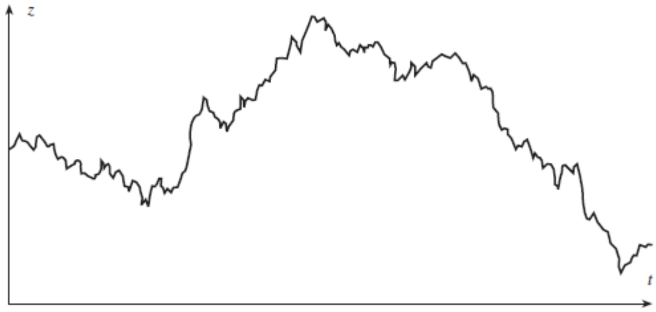
- Consider the following path followed by z as the limit $\Delta t \to 0$ is approached



Relatively large value of Δt



Smaller value of Δt



The true process obtained as $\Delta t \to 0$

- ➢ Note that the path is quite "jagged". This is because the standard deviation of the movement in z in time Δt equals $\sqrt{\Delta t}$, and when Δt is small, $\sqrt{\Delta t}$ is much bigger than Δt.
- ➢ Two intriguing properties of Wiener processes, related to this $\sqrt{\Delta t}$ property, as follows
  - ✧ 1. The expected length of the path followed by z in any time interval is infinite
  - ✧ 2. The expected number of times z equals any particular value in any time interval is infinite

Generalized Wiener Process

- The mean change per unit time for a stochastic process known as the ***drift rate*** and the variance per unit time is known as the ***variance rate***.

- The drift rate of zero means that the expected value of z at any future time is equal to its current value. The variance rate of 1 means that the variance of the change in z in a time interval of length T equals T.

- A ***generalized Wiener process*** for a variable x can be defined in terms of dz as

$$dx = a\,dt + b\,dz$$

  - Where a, b are constants
  - The $a\,dt$ term implies that x has an expected drift rate of a per unit of time.
  - Without the $b\,dz$ term, the equation is $dx = a\,dt$, which implies that $dx/dt = a$
    - Integrating with respect to time, gives

$$x = x_0 + at$$

    - Where x_0 is the value of x at time 0
    - In a period of time of length T, the variable x increases by an amount aT.
  - The $b\,dz$ term on the right side can be regarded as adding noise or variability to the path followed by x.
    - The amount of this noise or variability is b times a Wiener process.
    - It follows that b times a Wiener process has a variance rate per unit time of b^2.
  - In a small time interval Δt, the change Δx in the value of x is given by

$$\Delta x = a\,\Delta t + b\epsilon\sqrt{\Delta t}$$

    - Δx has a normal distribution with

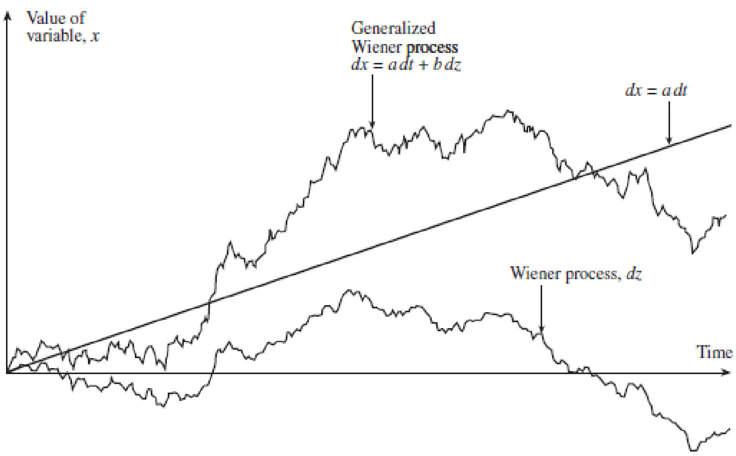
$$\text{mean of } \Delta x = a\,\Delta t$$
$$\text{standard deviation of } \Delta x = b\sqrt{\Delta t}$$
$$\text{variance of } \Delta x = b^2\,\Delta t$$

  - Similar arguments to those given for a Wiener process show that the change in the value of x in any time interval T is normally distributed with

$$\text{mean of change in } x = aT$$
$$\text{standard deviation of change in } x = b\sqrt{T}$$
$$\text{variance of change in } x = b^2 T$$

- The generalized Wiener process is illustrated in Figure 14.2

Figure 14.2 Generalized Wiener process with $a = 0.3$ and $b = 1.5$.



- Example
  - Consider the situation where the cash position of a company, measured in thousands of dollars, follows a generalized Wiener process with a drift of 20 per year and a variance rate of 900 per year. Initially, the cash position is 50
    - At the end of 1 year the cash position will have a normal distribution with mean 70, variance of 900, standard deviation of $\sqrt{900} = 30$
    - At the end of 6 months, it will have a normal distribution with a mean of 60 and a standard deviation of $30\sqrt{0.5} = 21.21$

Ito Process

- A further type of stochastic process, known as an ***Ito process*** is a generalized Wiener process in which the parameters a and b are functions of the value of the underlying variable x, and time t.
  - An Ito process can be written as

$$dx = a(x, t)\,dt + b(x, t)\,dz \qquad (14.4)$$

  - Both the expected drift rate and variance rate of an Ito process are liable to change over time. In a small time interval between t and $t + \Delta t$, the variable changes form x to $x + \Delta x$, where

$$\Delta x = a(x, t)\Delta t + b(x, t)\epsilon\sqrt{\Delta t}$$

    - This equation involves a small approximation. It assumes that the drift and variance rate of x remain constant, equal to their values at time t, during the time interval between t and $t + \Delta t$
  - Note that the process is Markov because the change in x at time t depends only on the value of x at time t, not on its history
    - A non-Markov process could be defined by letting a and b depend on values of x prior to time t.

3. The process for a stock price

- Clearly, the assumption of constant expected drift rate is inappropriate and needs to be replaced by the assumption that the expected return is constant.
  - If S is the stock price at time t, then the expected drift rate in S should be assumed to be μS for some constant parameter μ.
  - This means that in a short interval of time, Δt, the expected increase in S is $\mu S \Delta t$.
  - The parameter μ is the expected rate of return on the stock

- If the coefficient of dz is zero, so that there is no uncertainty, then this model implies that

$$\Delta S = \mu S \, \Delta t$$

In the limit, as $\Delta t \to 0$,

$$dS = \mu S \, dt$$

or

$$\frac{dS}{S} = \mu \, dt$$

➢ Integrating between time 0 and time T, we get

$$S_T = S_0 e^{\mu T} \quad (14.5)$$

➢ When there is not uncertainty, the stock price grows at a continuously compounded rate of μ per unit of time

- In practice, of course, there is uncertainty.
  ➢ A reasonable assumption is that the variability of the return in a short period of time, Δt, is the same regardless of the stock price
  ➢ This suggests that the standard deviation of the change in a short period of time Δt should be proportional to the stock price and leads to the model

$$dS = \mu S \, dt + \sigma S \, dz$$

or

$$\frac{dS}{S} = \mu \, dt + \sigma \, dz \quad (14.6)$$

  ✧ The variable μ is the stock's expected rate of return.
  ✧ The variable σ is the volatility of the stock price. The variable σ^2 is referred to as its variance rate.

Discrete-Time Model

- The model of stock price behavior we have developed is known as ***geometric Brownian motion***. The discrete-time version of the model is

$$\frac{\Delta S}{S} = \mu \, \Delta t + \sigma \epsilon \sqrt{\Delta t} \quad (14.7)$$

or

$$\Delta S = \mu S \, \Delta t + \sigma S \epsilon \sqrt{\Delta t} \quad (14.8)$$

  ➢ The equation shows that $\Delta S / S$ is approximately normally distributed with mean $\mu \Delta t$ and standard deviation $\sigma \sqrt{\Delta t}$.

$$\frac{\Delta S}{S} \sim \phi(\mu \, \Delta t, \, \sigma^2 \Delta t) \quad (14.9)$$

- Example
  ➢ Consider a stock that pays no dividends, has a volatility of 30% per annum, and provides an expected return of 15% per annum with continuous compounding.
  ➢ In this case, the process for the stock price is

$$\frac{dS}{S} = 0.15 \, dt + 0.30 \, dz$$

  ➢ If S is the stock price at a particular time and ΔS is the increase in the stock price in the next small interval of time, the discrete approximation to the process is

$$\frac{\Delta S}{S} = 0.15\Delta t + 0.30\epsilon\sqrt{\Delta t}$$

- Where ϵ has a standard normal distribution.
- Consider a time interval of 1 week, or 0.0192 year, so that Δt = 0.0192. Then the approximation gives

$$\frac{\Delta S}{S} = 0.15 \times 0.0192 + 0.30 \times \sqrt{0.0192}\,\epsilon$$

or

$$\Delta S = 0.00288S + 0.0416S\epsilon$$

Monte Carlo Simulation

- A Monte Carlo simulation of a stochastic process is a procedure for sampling random outcomes for the process.
  - We will use it as a way of developing some understanding of the nature of the stock price process

- Consider the situation where the expected return from a stock is 15% per annum and the volatility is 30% per annum. The stock price change over 1 week was shown to be approximately

$$\Delta S = 0.00288S + 0.0416S\epsilon \qquad (14.10)$$

  - A path for the stock price over 10 weeks can be simulated by sampling repeatedly for ϵ from $\phi(0,1)$ and substituting into equation 14.10
  - The following table shows one path for a stock price that was sampled in this way

Table 14.1 Simulation of stock price when $\mu = 0.15$ and $\sigma = 0.30$ during 1-week periods.

Stock price at start of period	*Random sample for ϵ*	*Change in stock price during period*
100.00	0.52	2.45
102.45	1.44	6.43
108.88	−0.86	−3.58
105.30	1.46	6.70
112.00	−0.69	−2.89
109.11	−0.74	−3.04
106.06	0.21	1.23
107.30	−1.10	−4.60
102.69	0.73	3.41
106.11	1.16	5.43
111.54	2.56	12.20

    - The initial stock price is assumed to be $100.
    - For the first period, ϵ is sampled as 0.52. From equation 14.10, the change during the first time period is

$$\Delta S = 0.00288 \times 100 + 0.0416 \times 100 \times 0.52 = 2.45$$

    - Therefore, at the beginning of the second time period, the stock price is $102.45.
  - The final stock price of 111.54 can be regarded as a random sample from the distribution of stock prices at the end of 10 weeks.
  - By repeatedly simulating movements in the stock price, a complete probability distribution of the stock price at the end of this time is obtained.

4. The parameters

- The parameter μ is the expected return (annualized) earned by an investor in a short period of time
  - ➢ The value of μ should depend on the risk of the return from the stock.
  - ➢ It should also depend on the level of interest rates in the economy. The higher the level of interest rates, the higher the expected return required on any given stock.

- Fortunately, we do not have to concern ourselves with the determinants of μ in any detail because the value of a derivative dependent on a stock is, in general, independent of μ.
  - ➢ The parameter σ, the stock price volatility, is, by contrast, critically important to the determination of the value of many derivatives
  - ➢ Typical values of σ for a stock are in the range 0.15 to 0.6

- The standard deviation of the proportional change in the stock price in a small interval of time Δt is $\sigma\sqrt{\Delta t}$.
  - ➢ As a rough approximation, the standard deviation of the proportional change in the stock price over a relatively long period of time T is $\sigma\sqrt{T}$
  - ➢ This means that, as an approximation, volatility can be interpreted as the standard deviation of the change in the stock price in 1 year.

5. Correlated Processes

- So far, we have considered how the stochastic process for a single variable can be represented. We now extend the analysis to the situation where there are two or more variables following correlated stochastic processes. Suppose

$$dx_1 = a_1\,dt + b_1\,dz_1 \quad \text{and} \quad dx_2 = a_2\,dt + b_2\,dz_2$$

  - ➢ Where dz_1 and dz_2 are Wiener processes.

- The discrete-time approximations for these processes are

$$\Delta x_1 = a_1\,\Delta t + b_1\,\epsilon_1\sqrt{\Delta t} \quad \text{and} \quad \Delta x_2 = a_2\,\Delta t + b_2\,\epsilon_2\sqrt{\Delta t}$$

  - ➢ Where ϵ_1 and ϵ_2 are samples from a standard normal distribution $\phi(0,1)$.

- If they are uncorrelated with each other, the random samples ϵ_1 and ϵ_2 that are used to obtain movements in a particular period of time Δt should be independent of each other.

- If x_1, and x_2 have nonzero correlation ρ. In this situation, we would refer to the Wiener processes ϵ_1 and ϵ_2 that are used to obtain movements in a particular period of time should be sampled from a bivariate normal distribution.
- Each variable in the bivariate normal distribution has a standard normal distribution and the correlation between the variables is ρ.
- In this situation, we would refer to the Wiener processes dz_1 and dz_2 as having a correlation ρ.

- To sample standard normal variables ϵ_1 and ϵ_2 with correlation ρ, we can set

$$\epsilon_1 = u \quad \text{and} \quad \epsilon_2 = \rho u + \sqrt{1-\rho^2}\, v$$

  - Where u and v are sampled as uncorrelated variables with standard normal distribution

- Note that, in the processes we have assumed for x_1, and x_2, the parameters a_1, a_2, b_1, b_2 can be functions of x_1, x_2, and t.
  - In particular, a_1, b_1 can be functions of x_2 as well as x_1 and t

6. Ito's Lemma

- Generally, we can say that the price of any derivative is a function of the stochastic variables underlying the derivative and time.
  - An important result in this area is known as ***Ito's Lemma***.

- Suppose that the value of a variable x follows the Ito's process

$$dx = a(x,t)\,dt + b(x,t)\,dz \qquad \textbf{(14.11)}$$

  - Where dz is a Wiener process and a, and b are functions of x and t.
  - Ito's lemma shows that a function G of x and t follows the process

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz \qquad \textbf{(14.12)}$$

  - Thus, G also follows an Ito process, with a drift rate of

$$\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2$$

and a variance rate of

$$\left(\frac{\partial G}{\partial x}\right)^2 b^2$$

- Earlier, we argued that

$$dS = \mu S\,dt + \sigma S\,dz \qquad \textbf{(14.13)}$$

  - Where μ, σ are constant. This is a reasonable model of stock price movements.
  - From Ito's lemma, it follows that the process followed by a function G of S and t is

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G}{\partial S}\sigma S\,dz \qquad \textbf{(14.14)}$$

  - Note that both S and G are affected by the same underlying source of uncertainty, dz.
  - This proves to be very important in the derivation of the Black-Sholes-Merton results.

Application to Forward Contract

- To illustrate Ito's lemma, consider a forward contract on a non-dividend-paying stock
  - We are interested in what happens to the forward price as time passes. We define F as the forward price at general t, and S as the stock price at time t, with t < T. the relationship between F and S is given by

$$F = Se^{r(T-t)} \tag{14.15}$$

  - We can use Ito's lemma to determine the process for F

$$\frac{\partial F}{\partial S} = e^{r(T-t)}, \qquad \frac{\partial^2 F}{\partial S^2} = 0, \qquad \frac{\partial F}{\partial t} = -rSe^{r(T-t)}$$

From equation (14.14), the process for F is given by

$$dF = \left[e^{r(T-t)}\mu S - rSe^{r(T-t)}\right]dt + e^{r(T-t)}\sigma S\,dz$$

Substituting F for $Se^{r(T-t)}$ gives

$$dF = (\mu - r)F\,dt + \sigma F\,dz \tag{14.16}$$

  - Like S, the forward price F follows geometric Brownian motion. It has an expected growth rate of $\mu - r$ rather than μ.
  - The growth rate in F is the excess return of S over the risk-free rate.

7. The lognormal property

- We now use Ito's lemma to derive the process followed by $\ln S$

$$G = \ln S$$

Since

$$\frac{\partial G}{\partial S} = \frac{1}{S}, \qquad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}, \qquad \frac{\partial G}{\partial t} = 0$$

it follows from equation (14.14) that the process followed by G is

$$dG = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\,dz \tag{14.17}$$

  - Since μ, σ are constant, this equation indicates that $G = \ln S$ follows a generalized Wiener process.
  - The change in $\ln S$ between time 0 and some future time T is therefore normally distributed, with mean $(\mu - \sigma^2/2)\,T$ and variance $\sigma^2 T$. This means

$$\ln S_T - \ln S_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right] \tag{14.18}$$

or

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right] \tag{14.19}$$

- The equation above shows that $\ln S$ is normally distributed.
  - A variable has a lognormal distribution if the natural logarithm of the variable is normally distributed.

- The model of stock price behavior we have developed in this chapter therefore implies that a stock's price at time T, given its price today, is lognormally distributed.
- The standard deviation of the logarithm of the stock price is $\sigma\sqrt{T}$. It is proportional to the square root of how far ahead we are looking

Appendix: derivation of Ito's lemma

- If Δx is a small change in x and ΔG is the resulting small change in G, a well-known result from ordinary calculus is

$$\Delta G \approx \frac{dG}{dx}\Delta x \qquad (14A.1)$$

  - ΔG is approximately equal to the rate of change of G with respect to x multiplied by Δx
  - The error involves terms of order Δx^2. If more precision is required, a Taylor series expansion of ΔG can be used

$$\Delta G = \frac{dG}{dx}\Delta x + \frac{1}{2}\frac{d^2G}{dx^2}\Delta x^2 + \frac{1}{6}\frac{d^3G}{dx^3}\Delta x^3 + \cdots$$

  - For a continuous and differentiable function G of two variables x and y, the result is

$$\Delta G \approx \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y \qquad (14A.2)$$

and the Taylor series expansion of ΔG is

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x\,\partial y}\Delta x\,\Delta y + \frac{1}{2}\frac{\partial^2 G}{\partial y^2}\Delta y^2 + \cdots \qquad (14A.3)$$

In the limit, as Δx and Δy tend to zero, equation (14A.3) becomes

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy \qquad (14A.4)$$

- Now suppose that a variable x follows the Ito's process

$$dx = a(x,t)\,dt + b(x,t)\,dz \qquad (14A.5)$$

and that G is some function of x and of time t. By analogy with equation (14A.3), we can write

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x\,\partial t}\Delta x\,\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \cdots \qquad (14A.6)$$

Equation (14A.5) can be discretized to

$$\Delta x = a(x,t)\,\Delta t + b(x,t)\epsilon\sqrt{\Delta t}$$

or, if arguments are dropped,

$$\Delta x = a\,\Delta t + b\epsilon\sqrt{\Delta t} \qquad (14A.7)$$

  - Before, when limiting arguments were used as $\Delta t \to 0$, Δx^2 were ignored because they were second-order terms. However, in this case, we know that

$$\Delta x^2 = b^2\epsilon^2\Delta t + \text{terms of higher order in } \Delta t \qquad (14A.8)$$

- This shows that the term involving Δx^2 has a component that is of order Δt and cannot be ignored

- The expected value of $\epsilon^2 \Delta t$ is Δt. The variance of $\epsilon^2 \Delta t$ is $2\Delta t^2$, from the properties of standard normal distribution.
  - We know that the variance of the change in a stochastic variable in time Δt is proportional to Δt, not Δt^2.
  - The variance of $\epsilon^2 \Delta t$ is therefore too small for it to have a stochastic component.
  - As such, we treat Δx^2 as its expected value, $b^2 \Delta t$, and as $\Delta t \to 0$, it became $b^2\, dt$

- Now

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2 dt \qquad \textbf{(14A.9)}$$

This is Itô's lemma. If we substitute for dx from equation (14A.5), equation (14A.9) becomes

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz.$$