Reading of Advanced Financial Modelling – Part 3

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- In virtually all fields in which mathematical models are used, researchers face a trade-off between how realistic a model should be and how easy it is to perform computations and get solutions in the model.
  - More complex models can be more realistic, but also harder to use.
  - In economics and finance researchers sometimes use models that look ridiculously simplistic.
  - The point is, they first try to understand the qualitative behavior of the objects being modeled (prices, investors, companies, …), rather than getting reliable quantitative conclusions (such as exactly how much an investor should invest in a given stock).
  - Once the initial understanding and interpretation are obtained, one can try to make the model more realistic and see whether the basic conclusions from the simple model change a lot or remain similar

1. Single-Period Models

- The simplest possible model assumes a security, let's call it S, starts with the value $S(0)$ at the initial time $t = 0$, of the period, and ends with the value $S(1)$ at the end of the period, $T = 1$.
  - Such a model is called the ***single-period model***.

1.1. Asset Dynamics

- Let's consider the ***single-period finite-market model***:
  - We assume that $S(1)$ is a random variable that can take K possible values, $s_1, \dots, s_K$, with probabilities $p_i = P[S(1) = s_i]$
  - More formally, consider a sample space $\Omega = \{\omega_1, \dots, \omega_k\}$ of possible states of the world at time 1, each having a positive probability.
    - If the state ω_i occurs, then $S(1) = s_i$

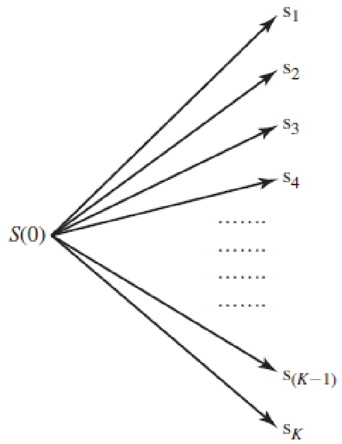

Figure 3.1
Single-period model: There are K possible values for the stock price at time 1.

- We usually assume that there is a ***risk-free asset B*** in the market, a bond or a ***bank account***, such that

  $$B(0) = 1$$

  and such that $B(1)$ is strictly positive for all states of the world. Typically, we also assume

  $$B(1) \geq B(0)$$

- The convention $B(0) = 1$, therefore, $B(1)$ is the value at time 1 of one dollar invested in the bank at time 0.
- If we denote by r the bank ***interest rate*** in this period, we have

  $$B(0) + rB(0) = B(1) \tag{3.1}$$

  or, equivalently, $r = B(1) - 1$.

1.2. Portfolio and Wealth Processes

- Consider now a market model in which there are N risky securities or stocks, $S_1, \ldots, S_N$ and bank account or bond B.
  - Our investor Taf takes positions in market assets. We will denote by $X(t)$ the value of these positions at time t, called Taf's ***wealth process***.
  - We assume that Taf starts with some ***initial wealth*** amount at time zero

    $$X(0) = x$$

    - And uses this amount to invest in the available securities.
  - Since this is a single-period model, Taf trades only once, right at time zero. Denoted by $\delta_i, i = 1, \ldots, N$ the number of security i shares that Taf holds between time zero and time one.
    - We denote by δ_0 the amount invested in the bank account at time zero.
  - We assume that there are no transaction costs when transferring funds between securities.
  - Then, Taf's wealth at the end of the period is given by

$$X(1) = \delta_0 B(1) + \delta_1 S_1(1) + \cdots + \delta_N S_N(1)$$

- Where the vector of positions is called a ***portfolio strategy***, a trading ***strategy***, or simply ***portfolio***.

  $\bar{\delta} = (\delta_0, \ldots, \delta_N)$
- Taf's portfolio has to satisfy some type of a ***budget constraint***. The typical budget constraint is the so-called ***self-financing condition***:

$$X(0) = \delta_0 B(0) + \delta_1 S_1(0) + \cdots + \delta_N S_N(0) \tag{3.2}$$

- The ***profit/loss***, P&L, of a portfolio strategy δ, denoted

$$G = X(1) - X(0)$$

will be called the **gains process.** If we denote

$$\Delta S_i = S_i(1) - S_i(0)$$

the gain of one share of security S_i, it follows from the above definitions and equation (3.1) that

$$G = \delta_0 r + \delta_1 \Delta S_1 + \cdots + \delta_N \Delta S_N$$

We have the following unsurprising relationship between the gains and wealth processes:

$$X(1) = X(0) + G$$

- We introduce the ***discount prices***:

$$\bar{S}_i(t) := S_i(t)/B(t)$$

  - We define the discounted gains process as

$$\bar{G} := \delta_1 \Delta \bar{S}_1 + \cdots + \delta_N \Delta \bar{S}_N$$

where

$$\Delta \bar{S}_i = \bar{S}_i(1) - \bar{S}_i(0)$$

The reader is asked in Problem 2 to prove these intuitive relationships:

$$\bar{X}(1) = \delta_0 + \sum_{i=1}^{N} \delta_i \bar{S}_i(1) = X(0) + \bar{G} \tag{3.3}$$

1.3. Arrow-Debreu Securities

- Consider the basic model where there are K possible states, or outcomes, at $t = 1$
  - A security S can be described as a K-dimensional vector $(s_1, \ldots, s_K)$ of payoffs in these states, where all the components of the vector are nonnegative.
  - In this setting, consider risky securities that pay one unit of currency in a given state and zero in every other state. These securities are called ***Arrow-Debreu securities***

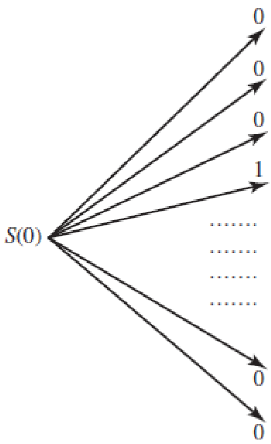

Figure 3.2
Arrow-Debreu security paying one at state 4 and paying zero at all other states.

- There are K different Arrow-Debreu securities.
- A portfolio formed by holding one share of each K Arrow-Debreu securities is a risk-free security that pays one unit of currency regardless of the state.
  - It represents a pure discount bond that pays interest determined by the difference between the payoff of one unit at time $t = 1$ and the price at time $t = 0$.
  - Arrow-Debreu securities are simple securities that can be used to describe aby other security in a single-period model.

2. Multiperiod Models

2.1. General Model Specifications

Interest rate in the period $t-1$ to t (assumed to be known already by time $t-1$):

$$r(t) = \frac{B(t) - B(t-1)}{B(t-1)}$$

Wealth process:

$$X(t) = \delta_0(t)B(t) + \delta_1(t)S_1(t) + \cdots + \delta_N(t)S_N(t), \quad t = 1, \ldots, T$$

Self-financing condition:

$$X(t) = \delta_0(t+1)B(t) + \delta_1(t+1)S_1(t) + \cdots + \delta_N(t+1)S_N(t), \quad t = 0, \ldots, T-1 \tag{3.4}$$

Gains process: We again denote by $G(t)$ the total profit or loss at time t, namely,

$$G(t) = \sum_{s=1}^{t} \delta_0(s)\Delta B(s) + \sum_{s=1}^{t} \delta_1(s)\Delta S_1(s) + \cdots + \sum_{s=1}^{t} \delta_N(s)\Delta S_N(s), \quad t = 1, \ldots, T \tag{3.5}$$

2.2. Cox-Ross-Rubinstein Binomial Model

- We want to model a market with only one risky asset S, the *stock*, and a *bank account*, with constant interest rate $r > 0$ for all the periods.
  - A simple but still stochastic multiperiod model is a model in which the risky security S can jump to only tow possible values at each point in time.
  - To make it even simpler, consider the value $u > 1 + r$ and $d < 1 + r$, and assume that the probabilities of the up and down moves are denoted by

$$p := P[S(t+1) = uS(t)], \qquad q := 1 - p = P[S(t+1) = dS(t)]$$

  - We also assume that the price changes between different time periods are independent. This model is called the ***binomial model*** or ***CRR*** model

3. Continuous-Time Models

- Continuous-time require more advanced mathematical tools than discrete-time models.
  - In return, they allow for explicit solutions of many standard pricing and investment problems, mainly thanks to the possibility of applying differential calculus.

3.1. Brownian Motion Process

- The basic continuous-time model is driven by a random process called ***Brownian motion***, or the ***Wiener process***, whose value at time t is denoted $W(t)$. We introduce it as a limit of discrete-time model

$$W(t_{k+1}) = W(t_k) + z(t_k)\sqrt{\Delta t}, \qquad W(0) = 0 \tag{3.11}$$

  - Here, $z(t_k)$ are standard normal random variables (that is, with mean zero and variance one), independent of each other.
  - It follows that the differences $W(t_{k+1}) - W(t_k)$ are normally distributed random variables with mean zero and variance Δt. More generally, for $k < l$

$$W(t_l) - W(t_k) = \sum_{i=k}^{l-1} z(t_i)\sqrt{\Delta t}$$

- It follows that $W(t_l) - W(t_k)$ is normally distributed with mean 0 and variance $t_l - t_k$. The process W is called a ***random walk*** process
  - In the limit when Δt goes to zero, it can be shown that the random walk converges to a process called Brownian motion, also denoted W, with the following properties

DEFINITION 3.1 **(Brownian Motion)**

a. $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$, for $s < t$.

b. The process W has independent increments: for any set of times $0 \le t_1 < t_2 < \cdots < t_n$, the random variables

$$W(t_2) - W(t_1), W(t_3) - W(t_2), \ldots, W(t_n) - W(t_{n-1})$$

are independent.

c. $W(0) = 0$.

d. The sample paths $\{W(t); t \ge 0\}$ are continuous functions of t.

- Despite their continuity, that is, condition d, Brownian motion paths are not differentiable anywhere

- It can be shown that Brownian motion is a ***Markov process***
  - Using properties of conditional expectations and the fact that $W(t) - W(s)$ is independent of $W(s)$, we get

$$\begin{aligned} E[W(t) \mid W(s)] &= E[W(t) - W(s) \mid W(s)] + E[W(s) \mid W(s)] \\ &= E[W(t) - W(s)] + W(s) \\ &= W(s) \end{aligned}$$

  - Therefore, the expected value of a future outcome of Brownian motion, conditional on the past and present information, is exactly equal to the present value.
  - That is, the following so-called ***martingale property*** is satisfied

$$E[W(t) \mid W(s)] = W(s)$$

  - We say that Brownian motion is a ***martingale process***.

- A more precise definition of a martingale process in continuous time requires the notions of a ***σ-algebra*** or ***σ-field*** and conditional expectations with respect to a σ-algebra
  - Intuitively, a σ-algebra $F(t)$ represents the information available up to time t.
  - A collection of such σ-algebras $\boldsymbol{F} := \{F(t)\}_{t\ge0}$ is called a ***filtration***.
  - We say that a process $X(.)$ Is ***adapted*** to the filtration **F** if the random variable $X(t)$ is ***measurable*** with respect to σ-algebra F_t, for every $t \ge 0$
  - Intuitively, $X(t)$ is known if the information given by F_t is known. Then we say that a process M **is a martingale** with respect to information filtration F if it is adapted to that filtration and if

$$E[X(t) \mid \mathcal{F}(t)] = X(s), \quad s \le t \tag{3.12}$$

  - In words, the conditional expectation of the future value given the current information is equal to the present value.

3.2. Diffusion Process, Stochastic Integrals

- Brownian motion in itself is not sufficiently flexible for modeling various asset price processes.
  - With this in mind, we generalize the approach of the previous section by considering processes of the form

$$X(t_{k+1}) = X(t_k) + \mu(t, X(t_k))\Delta t + \sigma(t, X(t_k))\sqrt{\Delta t} \cdot z(t_k), \qquad X(0) = x \tag{3.13}$$

- Where μ, σ are deterministic functions of the ***time variable*** t and the ***state variable*** x

- In the limit, when Δt goes to zero, we would expect to get

$$X(t) = x + \int_0^t \mu(u, X(u))\,du + \int_0^t \sigma(u, X(u))\,dW(u) \tag{3.14}$$

This relation is more often written as

$$dX(t) = \mu(t, X(t))\,dt + \sigma(t, X(t))\,dW(t), \qquad X(0) = x \tag{3.15}$$

- This expression is called a ***stochastic differential equation***, or SDE, for the process, X, which is called a ***diffusion process***. We call μ the ***drift*** and σ the ***diffusion function*** of process X

- The integral

$$\int_0^t Y(u)\,dW(u)$$

- For a given process Y is called a ***stochastic integral*** or ***Ito integral***.
- We will argue that, under some conditions on Y, the Ito integral, as process in time, is a martingale

3.3. Technical Properties of Stochastic Integral

- A meaningful definition of the integral $\int_0^t Y(u)dW(u)$ is possible for a process Y which is such that $Y(t)$ is known if the past and present values $W(u),\ u \le t$, of Brownian motion are known.
  - We say that Y is a process ***adapted*** to the information generated by the Brownian motion processes

- We now state the most important properties of the Ito integral

THEOREM 3.1 (ITÔ INTEGRAL PROPERTIES) Let Y be a process adapted to the information given by Brownian motion W. Let $T > 0$ be a given time horizon. Assume that

$$E\left[\int_0^T Y^2(u)\,du\right] < \infty \tag{3.16}$$

Then, for $t \le T$, the integral process

$$M(t) := \int_0^t Y(u)\,dW(u)$$

is well defined. Moreover, it is a martingale with mean zero and variance process $E[\int_0^t Y^2(u)\,du]$. In other words, for all $s < t$,

$$E\left[\int_0^t Y(u)\,dW(u)\right] = 0$$

$$E\left[\int_0^t Y(u)\,dW(u)\,\middle|\,W(u), 0 \le u \le s\right] = \int_0^s Y(u)\,dW(u)$$

$$E\left[\left(\int_0^t Y(u)\,dW(u)\right)^2\right] = E\left[\int_0^t Y^2(u)\,du\right]$$

- You can think of Y as a portfolio process and $M(t)$ as the corresponding gains process.
  - The fact that $E[M(t)] = 0$ can be interpreted as saying that if we invest in process W with mean zero, then our gains will also have mean zero.

- We will only provide a discrete-time intuition for ***Ito Integral Properties***:

$$M(t_l) := \sum_{j=0}^{l-1} Y(t_j)[W(t_{j+1}) - W(t_j)], \qquad M(0) = 0$$

Note that by properties of conditional expectations we have, for all $k < l$,

$$\begin{aligned} E\left[\sum_{j=k}^{l} Y(t_j)\{W(t_{j+1}) - W(t_j)\}\right] &= \sum_{j=k}^{l} E[E\{Y(t_j)[W(t_{j+1}) - W(t_j)] \mid W(0), \ldots, W(t_j)\}] \\ &= \sum_{j=k}^{l} E[Y(t_j)E\{W(t_{j+1}) - W(t_j)\}] = 0 \qquad (3.17) \end{aligned}$$

Therefore, $E[M(t_l)] = 0$, for all t_l. Next, using equation (3.17), we show the martingale property:

$$\begin{aligned} E[M(t_l) \mid W(0), \ldots, W(t_k)] &= E\left[\sum_{j=0}^{k-1} Y(t_j)\{W(t_{j+1}) - W(t_j)\} \,\middle|\, W(0), \ldots, W(t_k)\right] \\ &\quad + E\left[\sum_{j=k}^{l-1} Y(t_j)\{W(t_{j+1}) - W(t_j)\} \,\middle|\, W(0), \ldots, W(t_k)\right] \\ &= \sum_{j=0}^{k-1} Y(t_j)[W(t_{j+1}) - W(t_j)] \\ &\quad + E\left[\sum_{j=k}^{l-1} Y(t_j)\{W(t_{j+1}) - W(t_j)\}\right] \\ &= M(t_k) \end{aligned}$$

  - As for the variance, we only look at two periods:

$$\begin{aligned} &E[\{Y(t_0)[W(t_1) - W(t_0)] + Y(t_1)[W(t_2) - W(t_1)]\}^2] = E[Y^2(t_0)\{W(t_1) - W(t_0)\}^2] \\ &\quad + E[Y^2(t_1)\{W(t_2) - W(t_1)\}^2] + E[Y(t_0)Y(t_1)\{W(t_1) - W(t_0)\}\{W(t_2) - W(t_1)\}] \end{aligned}$$

The second term on the right-hand side (and similarly the first term) can be computed as

$$\begin{aligned} E[Y^2(t_1)\{W(t_2) - W(t_1)\}^2] &= E[E\{Y^2(t_1)[W(t_2) - W(t_1)]^2 \mid W(t_0), W(t_1)\}] \\ &= E[Y^2(t_1)E\{[W(t_2) - W(t_1)]^2\}] = E[Y^2(t_1)\Delta t] \end{aligned}$$

The last term is zero, as in equation (3.17):

$$\begin{aligned} &E[Y(t_0)Y(t_1)\{W(t_1) - W(t_0)\}\{W(t_2) - W(t_1)\}] \\ &\quad = E[Y(t_0)Y(t_1)\{W(t_1) - W(t_0)\}E\{W(t_2) - W(t_1)\}] = 0 \end{aligned}$$

Combining these, and extending the process to more than two periods, we get

$$E[M^2(t_l)] = E\left[\sum_{j=0}^{l-1} Y^2(t_j)\Delta t\right]$$

This equation corresponds to the integral version in the continuous-time model, as in the statement of the theorem.

3.4. Ito's Lemma (Rule)

THEOREM 3.2 (ITÔ'S RULE) Let $f(t, x)$ be a function that has a continuous derivative with respect to t and two continuous derivatives with respect to x. Then, given the diffusion process

$$dX(t) = \mu(t, X(t))\,dt + \sigma(t, X(t))\,dW(t)$$

the process $f(t, X(t))$ satisfies

$$df(t, X(t)) = \left[f_t + \frac{1}{2}\sigma^2 f_{xx}\right](t, X(t))\,dt + f_x(t, X(t))\,dX(t)$$

We see that the extra term involves the second derivative f_{xx}. If we substitute for $dX(t)$ and suppress in the notation the dependence on $(t, X(t))$, we get the following **useful form of Itô's rule:**

$$df = \left[f_t + \mu f_x + \frac{1}{2}\sigma^2 f_{xx}\right] dt + \sigma f_x\,dW \tag{3.18}$$

- As a good way to remember it, we could write the Ito's rule in an informal way

$$df = f_t\,dt + f_x\,dX + \tfrac{1}{2} f_{xx}\,dX \cdot dX \tag{3.19}$$

using the following informal rules:

$$dt \cdot dt = 0, \qquad dt \cdot dW = 0, \qquad dW \cdot dW = dt \tag{3.20}$$

The last equality is due to the fact that a sum of small squared changes in W over a given interval $[0, t]$ is approximately t. The informal rules imply

$$dX \cdot dX = (\mu\,dt + \sigma\,dW) \cdot (\mu\,dt + \sigma\,dW) = \sigma^2\,dt$$

- In order to motivate a need for a rule like this, the reader can think of $X(t)$ as a stock-price process, and $f(t, X(t))$ as the price of an option written on the stock.
  - ➢ Ito's rule gives us a connection between the two

- Suppose we have a ***two-dimensional Brownian motion*** $(W_1(t), W_2(t))$, meaning that W_1 and W_2 are two independent one-dimensional Brownian motions.
  - ➢ Consider two processes given by

$$dX = \mu_X\,dt + \sigma_{X,1}\,dW_1 + \sigma_{X,2}\,dW_2$$
$$dY = \mu_Y\,dt + \sigma_{Y,1}\,dW_1 + \sigma_{Y,2}\,dW_2$$

  - ➢ Then, for a sufficiently smooth function $f(x, y)$ (that is, a function that has at least two continuous derivatives with respect to x and y, plus continuous mixed partial derivatives between x and y), the process $f(X(t), Y(t))$ satisfies the ***two-dimensional Ito's rule***:

$$df(X,Y) = f_x\,dX + f_y\,dY + \left[\tfrac{1}{2}\left(\sigma_{X,1}^2 + \sigma_{X,2}^2\right)f_{xx} + \tfrac{1}{2}\left(\sigma_{Y,1}^2 + \sigma_{Y,2}^2\right)f_{yy} + f_{xy}(\sigma_{X,1}\sigma_{Y,1} + \sigma_{X,2}\sigma_{Y,2})\right]dt \quad (3.22)$$

Again, as a way to remember it, we could write the **two-dimensional Itô's rule in an informal way** as

$$df(X,Y) = f_x\,dX + f_y\,dY + \tfrac{1}{2}[f_{xx}\,dX\cdot dX + f_{yy}\,dY\cdot dY] + f_{xy}\,dX\cdot dY \quad (3.23)$$

where, in addition to the informal rules of equations (3.20), we also use

$$dW_1 \cdot dW_2 = 0$$

- Which is a consequence of W_1 and W_2 being independent of each other

- In come applications it is more convenient that W_1, and W_2 are not independent, but correlated, with **instantaneous correlation** equal to ρ, in the sense that

$$E[W_1(t)W_2(t)] = \rho t$$

for every t. Then the informal rule is

$$dW_1 \cdot dW_2 = \rho\,dt$$

and Itô's rule is

$$df(X,Y) = f_x\,dX + f_y\,dY + \left[\tfrac{1}{2}\left(\sigma_{X,1}^2 + \sigma_{X,2}^2\right)f_{xx} + \tfrac{1}{2}\left(\sigma_{Y,1}^2 + \sigma_{Y,2}^2\right)f_{yy} + f_{xy}(\sigma_{X,1}\sigma_{Y,1} + \sigma_{X,2}\sigma_{Y,2} + \rho\sigma_{X,1}\sigma_{Y,2} + \rho\sigma_{X,2}\sigma_{Y,1})\right]dt \quad (3.24)$$

- As a particular case, we have the following ***Ito's rule for product*** of two processes

$$d(XY) = X\,dY + Y\,dX + [\sigma_{X,1}\sigma_{Y,1} + \sigma_{X,2}\sigma_{Y,2} + \rho\sigma_{X,1}\sigma_{Y,2} + \rho\sigma_{X,2}\sigma_{Y,1}]\,dt \quad (3.25)$$

Example 3.1 (Brownian Motion Squared)

- We want to find the dynamics of the process $Y(t) \coloneqq W^2(t)$.
  - We can think of process Y as a function $Y(t) = f\big(W(t)\big)$ of Brownian motion, with $f(x) = x^2$, $f_x(x) = 2x$, and $f_{xx}(x) = 2$.
  - Given

$$dW = 0\cdot dt + 1\cdot dW$$

  - Brownian motion is a diffusion process with the drift equal to zero and the diffusion term equal to one.
  - Applying Ito's rule, we get

$$d(W^2(t)) = 2W(t)\,dW(t) + dt$$

- This is the same as

$$2\int_0^t W(u)\,dW(u) = W^2(t) - t$$

- We see that the process $W^2(t) - t$ is a stochastic integral, hence a **martingale**.
- In general, if, for a given process Y there exists a process Q such that the process $Y^2 - Q$ is a martingale, we say that Q is a quadratic variation process of Y.

Example 3.2 (Exponential of Brownian Motion)

- In this example, we want to find the dynamics of the process

$$Y(t) = e^{aW(t)+bt}$$

We can think of process Y as a function $Y(t) = f(t, W(t))$ of Brownian motion and time, with

$$f(t,x) = e^{ax+bt}, \qquad f_t(t,x) = bf(t,x), \qquad f_x(t,x) = af(t,x),$$
$$f_{xx}(t,x) = a^2 f(t,x)$$

Applying Itô's rule we get

$$dY = \left[b + \frac{1}{2}a^2\right] Y\,dt + aY\,dW$$

If we set

$$b = -\tfrac{1}{2}a^2$$

the drift term of Y will disappear, making it a stochastic integral $Y(t) = Y(0) + \int_0^t aY(u)\,dW(u)$, hence a martingale. In other words, the process

$$Y(t) = e^{aW(t) - \frac{1}{2}a^2 t}$$

is a martingale. In particular, this fact means that $E[Y(t)] = Y(0) = 1$, or

$$E\left[e^{aW(t)}\right] = e^{\frac{1}{2}a^2 t} \tag{3.21}$$

- $W(t)$ is normally distributed with variance t as this is the expression of the moment-generating function of a normal distribution

Example 3.3 (Product Rule)

- We want to find the dynamics of the process

$$Y(t) = W^2(t) \cdot e^{aW(t)}$$

- We could do so using Ito's rule directly, or we can consider this process as a product of the processes from previous examples, then use the product rule

$$dY = e^{aW} \cdot d(W^2) + W^2 \cdot d(e^{aW}) + d(W^2) \cdot d(e^{aW})$$

Using Examples 3.1 and 3.2, this gives us

$$dY = e^{aW}\left[t + \tfrac{1}{2}a^2W^2 + 2Wa\right] dt + ae^{aW}W^2\, dW$$

3.5. Wealth Processes and Portfolio Process

- We denote by $\pi(t)$ the amount of money (not the number of shares). We call π a ***portfolio process***.
  - It is assumed that the agent cannot see the future, so $\pi(t)$ has to be determined from the information up to time t
  - In other words π is an adapted process.
  - We also require the technical condition

$$E\left[\int_0^T \pi^2(u)\, du\right] < \infty$$

  - Which implies that the integral is a well-defined martingale process. Such a portfolio is called ***admissible***.

- We denote by $X = X^{x,\pi}$ the ***wealth process*** corresponding to the initial investment $x > 0$ and the portfolio strategy π.
  - Typically, we require the portfolio strategy π to be ***self-financing***, by which we mean that the amount held in the bank at time t is equal to $X(t) - \pi(t)$ and that X satisfies

$$dX = \frac{\pi}{S}\, dS + \frac{X - \pi}{B}\, dB \tag{3.34}$$

  - The integral form of this equation is

$$X(t) = X(0) + \int_0^t \frac{\pi(u)}{S(u)}\, dS(u) + \int_0^t \frac{X(u) - \pi(u)}{B(u)}\, dB(u)$$

  - Where the integral terms are interpreted as gains from trade in S and gains from trade in B. Substituting for dS and dB we obtain the ***wealth equation***

$$dX = [rX + \pi(\mu - r)]\, dt + \pi\sigma\, dW \tag{3.35}$$

For the **discounted wealth process**

$$\bar{X}(t) := e^{-rt}X(t)$$

we get, by Itô's rule for products,

$$d\bar{X} = [\bar{\pi}(\mu - r)]\, dt + \bar{\pi}\sigma\, dW \tag{3.36}$$

4. Arbitrage and Market Completeness

- We assume two fundamental notions: complete markets and absence of arbitrage
  - A market satisfying those requirements is called a ***perfect market***

4.1. Notion of Arbitrage

- We say that there is an ***arbitrage opportunity*** when someone has a positive probability of achieving a positive return with no risk of loss
  - Mathematically, we say that there is arbitrage in the market if there exists a trading strategy δ such that

$$X^{\delta}(0) = 0, \qquad X^{\delta}(T) \geq 0, \qquad P[X^{\delta}(T) > 0] > 0 \quad \text{for some } T > 0 \qquad (3.42)$$

4.2. Arbitrage in Discrete-Time models

- We first consider the single-period models.
  - We denote by $S_i^k(1)$ the payoff of security i if outcome (state) k occurs at time 1, $k = 1, \dots, K$.
  - An arbitrage opportunity exists when there is a portfolio δ such that

$$\delta_0 B(0) + \delta_1 S_1(0) + \cdots + \delta_N S_N(0) = 0$$

and

$$\delta_0 B(1) + \delta_1 S_1^k(1) + \cdots + \delta_N S_N^k(1) \geq 0, \qquad k = 1, \dots, K$$

with at least one state k such that

$$\delta_0 B(1) + \delta_1 S_1^k(1) + \cdots + \delta_N S_N^k(1) > 0$$

4.3. Arbitrage in Continuous-Time Models

- Here we present an example when arbitrage can arise in continuous-time models.
  - Consider the Merton-Black-Scholes model with two stocks that have the same volatility and are driven by the same Brownian motion W. There is an arbitrage opportunity if

$$\mu_1 \neq \mu_2$$

  - For example, suppose that $\mu_1 > \mu_2$. Then, we short $S_1(0)/S_2(0)$ shares of stock 2 and buy one share of stock 1.
  - The initial cost of this strategy is zero, according to the following equation

$$S(t) = S(0)\exp\left\{\sigma W(t) + \left(\mu - \tfrac{1}{2}\sigma^2\right)t\right\} \tag{3.28}$$

- We make the positive profit at time T equal to

$$S_1(T) - \frac{S_1(0)}{S_2(0)}S_2(T) = S_1(0)e^{\sigma W(T) - T\sigma^2/2}(e^{\mu_1 T} - e^{\mu_2 T})$$

4.4. Notion of Complete Markets

- In order to describe market completeness we introduce a notion of a ***contingent claim***.
  - A contingent claim is a financial contract with a random payoff that can be positive or negative.
  - The term "contingent" comes from the fact that the payoff is random, hence contingent (dependent) on which random outcome has occurred, that is, which state of nature is realized at the time of the payoff.
  - All the securities that we have described are contingent claims

- We say that a portfolio ***replicates*** a contingent claim when the payoff of the portfolio matches the payoff of the contingent claim in all possible states
  - We require the ***replicating portfolio*** strategy to be self-financing

- We say that a ***market is complete*** when we can replicate any contingent claim with the existing securities.
  - We now study the requirements for a market to be complete in various models

4.5. Complete Markets in Discrete-Time Models

- We first consider the single-period model.
  - Suppose that Taf has a short position in a contingent claim with a payoff $g(S(1))$, for some function g.
    - In other words, at $t = 1$, Taf will have to pay $g(s_k)$ to the holder of the contingent claim if the stock price $S(1)$ at the end of the period takes the value s_k.
    - We assume that there are K possible values for the stock, $s_1, \dots, s_k$
  - Suppose that Taf wants to find a strategy that would replicate the payoff of this option no matter what happens at time t=1.
    - Taf wants this because his objective is to construct a portfolio that guarantees that he will be able to meet the option payoff, regardless of which value the stock has at time 1.
  - Therefore, the terminal value $X(1)$ of Taf's strategy δ has to satisfy

$$X(1) = \delta_0 B(1) + \delta_1 s_k = g(s_k), \qquad k = 1, \dots, K \tag{3.44}$$

  - This is a system of K equations with two unknowns, δ_0 and δ_1. Typically, we can find a solution only if $K \leq 2$

- For concreteness, we consider a two-period binomial model.
  - In such a model there are three possible payoff values of a given contingent claim C maturing at the end of the second period. Let's call these values $C^{uu}, C^{ud}, and\ C^{dd}$.
    - Similarly for $S^{uu}, S^{ud}, and\ S^{dd}$
  - We denote by $C^u(1), C^d(1)$ the time-1 values of the wealth process that replicates the payoff C.
  - Also, denote by $\delta_1^u(1)$ the number of units of the risky asset the replicating portfolio holds at time 1 in the upper node
    - And by $\delta_0^u(1)$ the amount held in the risk-free asset in that node.
  - In order to have replication at maturity, we start replicating backward, from the end of the tree.
    - We need to have the sum of money in the bank and money in the stock equal to the payoff, that is

$$\begin{aligned}\delta_0^u(1)(1+r)+\delta_1^u(1)S^{uu} &= C^{uu}\\ \delta_0^u(1)(1+r)+\delta_1^u(1)S^{ud} &= C^{ud}\end{aligned} \tag{3.45}$$

This system can be solved as

$$\begin{aligned}\delta_0^u(1) &= \frac{1}{1+r}\left[C^{uu}-\delta_1^u(1)S^{uu}\right]\\ \delta_1^u(1) &= \frac{C^{uu}-C^{ud}}{S^{uu}-S^{ud}}\end{aligned} \tag{3.46}$$

Then the replicating portfolio's value at time 1 in the upper node has to be

$$C^u(1) = \delta_1^u(1)S^u + \delta_0^u(1)$$

We can similarly find the value for $C^d(1)$. Then we need to replicate these two values in the first period; that is, we need to have

$$\begin{aligned}\delta_0(0)(1+r)+\delta_1(0)S^u &= C^u(1)\\ \delta_0(0)(1+r)+\delta_1(0)S^d &= C^d(1)\end{aligned} \tag{3.47}$$

  - We can now solve this system as before. This result shows that the model is complete

- We mention that in this model the number of shares the replicating portfolio holds in the underlying asset is always of the same type – equation 3.46
  - That is, it is equal to the change in the future values of the claim divided by the change in the future values of the underlying asset, denoted

$$\delta_1 = \frac{\Delta C}{\Delta S} \tag{3.48}$$

This number is known as the **delta** of the claim, and it is different at different nodes of the tree.

Example: replication in a single-period model

- Consider a single-period model with $r = 0.005, S(0) = 100, S_1 = 101, S_2 = 99$.
  - The payoff Taf is trying to replicate is the European call option

    $$g[S(1)] = [S(1) - \tilde{K}]^+ = \max[S(1) - \tilde{K}, 0]$$

    with $\tilde{K} = 100$. According to equation (3.44), Taf has to solve the system

    $$\delta_0(1 + 0.005) + \delta_1 101 = 1$$
    $$\delta_0(1 + 0.005) + \delta_1 99 = 0$$

  - The solution is $\delta_0 = -49.2537$, $\delta_1 = 0.5$.
    - So, Taf has to borrow 49.2537 from the bank and buy half a share of the stock.
  - Note that the cost of the replicating strategy is

    $$\delta_0 B(0) + \delta_1 S(0) = 0.746$$

Example: Replication in a Trinomial Model

- Consider the single-period model in which the price of the stock today is \$100
  - In the next period, it can go to \$120, \$100, \$90.
  - The price of the European call on this stock with strike \$105 is \$5
  - The price of the European call on this stock with strike \$95 is \$10, both expiring in the next period

- We want to replicate the payoff of the risk-free security that pays \$1 after one period regardless of what happens
  - We denote by δ_1 the number of shares of the stock, by δ_2 the number of calls with strike price 105, and δ_3 the number of calls with strike 95
  - By computing the payoffs of the three securities in each of the possible three states in the next period, we see that in order to replicate the risk-free security, we need to have

    $$120\delta_1 + 15\delta_2 + 25\delta_3 = 1$$
    $$100\delta_1 + 5\delta_3 = 1$$
    $$90\delta_1 = 1$$

    The solution to this system is $\delta_1 = 1/90$, $\delta_2 = 2/135$, and $\delta_3 = -1/45$. We note that the cost of this portfolio is

    $$\frac{1}{90}100 + \frac{2}{135}5 - \frac{1}{45}10 = 0.963$$

    We will argue later that this should then be the price of the risk-free security. Since its relative return is $1/0.963 - 1 = 3.85\%$, this is the risk-free rate in this model.