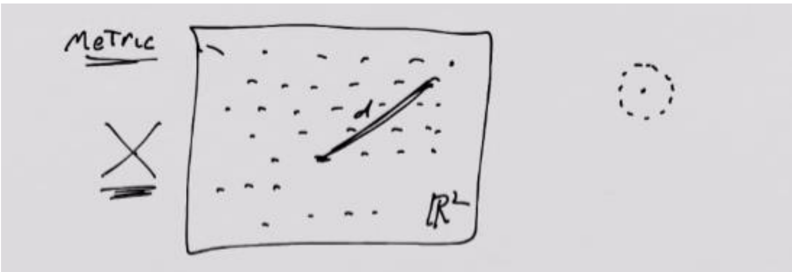
Lesson 1: Point Set Topology and Topological Spaces

Topology

- We have a set X, the topology of the set X is a set containing a lot of subsets of X, the subsets in the topology T_X are called open sets (rules defining topology)
  - An open set is an element of the topology of X
    - Arbitrary union of open sets is an open set (can be union of infinite open sets)
    - Intersection of an arbitrary number of finite open sets is an element of the topology
      - I can't take an infinite intersection of open sets
  - Null set is an element of the topology of X and X itself is also an element of the topology of itself
- If I can find a collection of open sets that satisfies the above rules, then I have created a topology for set X

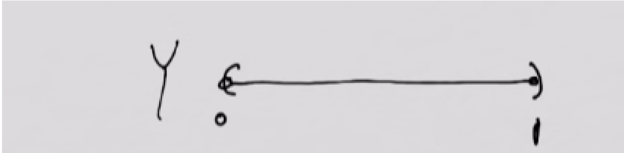
- Consider in the plane



  - A topology can be defined as a collection of open sets which are the open balls on the plane
    - An open ball will be the set of points x such that the distance of x to a center p is less than some number r (an element of a real number).
    - It doesn't include the boundary
  - Note that open balls are not the only things in the topology, union of two open balls not necessarily be an open ball

  - I can use the union of open balls to make any shape, say, a rectangle (not including its boundaries).
    - These are also part of the topology
  - Also note that, if the intersections of the open balls is the empty set, the union of the balls is still an open set

- Now consider a line



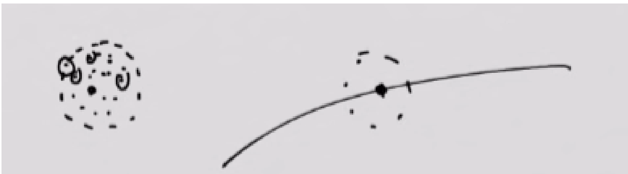
- We could define the topology in a very similar way, an open ball in a line is just an open interval

- What are some not open sets?
  - Individual points
    - Because you can only take finite intersections, and an individual points requires infinite intersections
  - But I can define the topology as every single point (every single point is an open set), this is called the discrete topology
    - Then all possible subset of points is a topology

- Note that any member of the topology can be created through the union of open balls, so the open balls are called **base** for the topology
  - A base is a subset of topology that by union of those base elements, you can create any element of the topology
  - I can declare that some rectangles that don't contain the boundary could also form some open sets
    - Because from arbitrary rectangles, you can make any shape
  - The **usual topology** is the topology that's generated by open balls

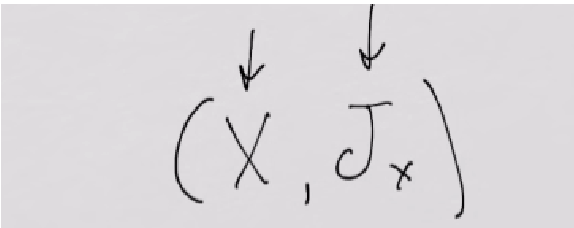
Neighborhood

- A neighborhood of the point is any open set that contains the point
  - So an open set is a neighborhood of all of its points
  - The points on the boundary wouldn't be in the neighborhood because there is no way to get another open set that is in the original open set that contains the point on the boundary
    - Every point in the neighborhood should be contained by another small neighborhood that is in the open set



Topological space

- A topological space is a combination of X and the topology of X
  - It's two sets



- ➢ The topology of X is not unique

Lesson 2: Elementary definitions

Neighborhood

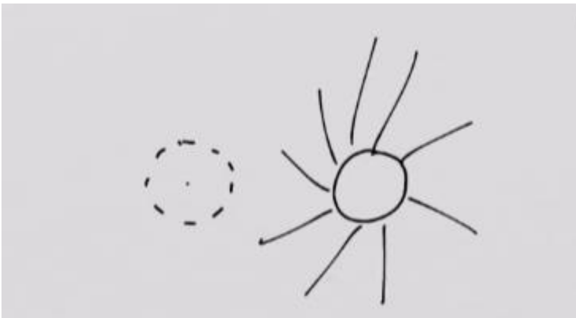
- Consider the set X
  - ➢ Consider a point p that is in X
  - ➢ Consider a set S that is a subset of X
  - ➢ If S is a neighborhood of p, then
    - ✧ p is an element of S
  - ➢ If S is a subset of N, then
    - ✧ N is also a neighborhood of p

- If S and R are neighborhoods of p, then
  - ➢ $S \cap R$ is also a neighborhood of p

- If S and T are neighborhoods of p, and S contains T, then
  - ➢ S contains a neighborhood for every point in T

Closed Sets

- It is a complement of the open set

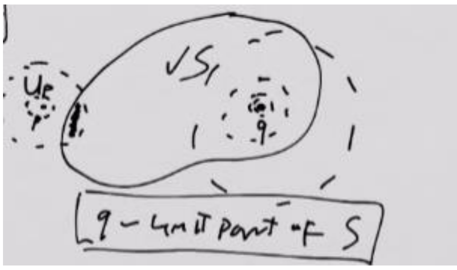


Limit Points

- An open neighborhood is simply an open set that contains a point
    - So an open neighborhood of a point p is an open set that contains the point p.

- An open set is a neighborhood of all the points contained

- Consider another subset S of X
    - A point p is called the limit point of S if every open neighborhood of p also contains a point of S
        - Point p has to be inside S or on the boundary of S



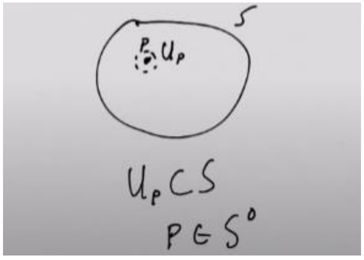
- If there is a single point that is omitted by the set S, it is still a limit point of S because at least some of points in the neighborhood of that point is in S

- If S is a sequence, then if the sequence is approaching a point p indefinitely, then p is a limit point of S

Closure, interior, and exterior

- The **closure** of the set is the set union all of its limit points

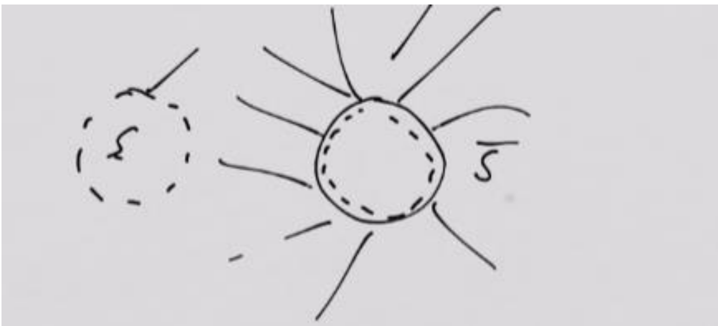
- A point p is in the **interior** of a set s if I can find an open set containing P that is a subset of S



➢ The boundary points cannot be in the interior of the set

✧ It doesn't matter if the boundary is contained in the set

- Note that the interior of a set
  ➢ It is always open
  ➢ The sum (union) of all the open sets inside S is the interior of S
  ➢ S is a topology of X (means S is open) if and only if S equals its own interior
- Note that, the interior is the complement of the closure of the complement of S



- The **exterior** of a set is the complement of its closure

  ➢ The boundary is not in the exterior of the set S

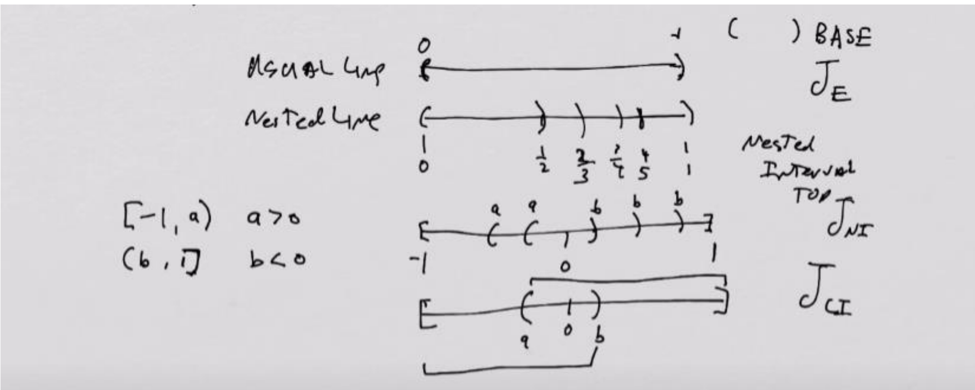
- Thus the **boundary** of a set is neither in the interior or the exterior
  ➢ Or, it is a set of points that are in the closure but not in the interior

Density

- S is **dense** in X if
    - For every point p in X, either
        - Point p is also an element of S itself or
        - Point p is an element of the closure of S

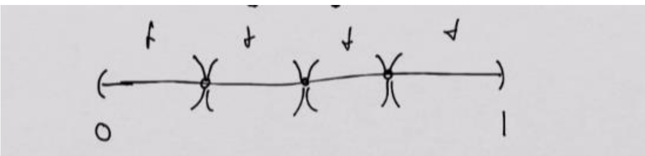
Lesson 3: Separation

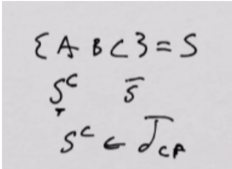
3 topologies



- Consider defining a topology base on closed interval [-1, 1] as above
    - Although each base interval is half open interval, they are open sets, because they are members of the topology, and by definition, they are open sets
- A set could be open because
    - 1. They don't include end points
    - 2. They are members of the topology

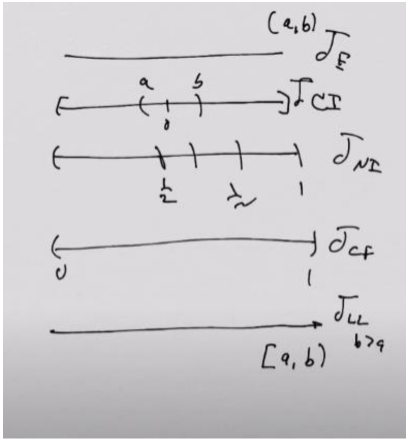
4-th topology





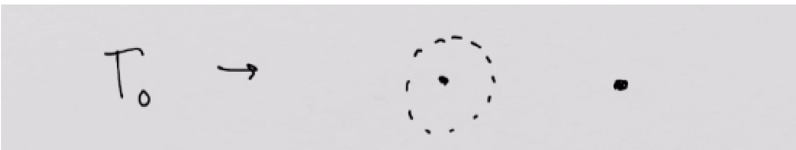
- Cofinite topology
  - ➢ Pick a finite number of points (e.g. 3 points), then you take all of the points in the set except those 3 points, and then the open set is the union of those open intervals

Lower limit topology



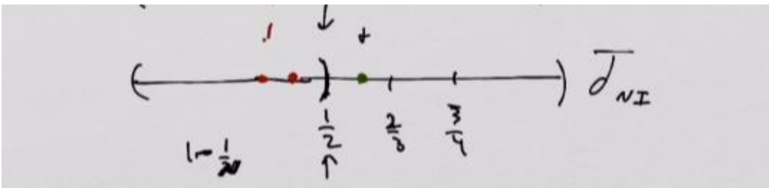
Separability

- T0
  - ➢ Take any two points in the set, I need to be able to at least find an open set that contains one of the points and not the other

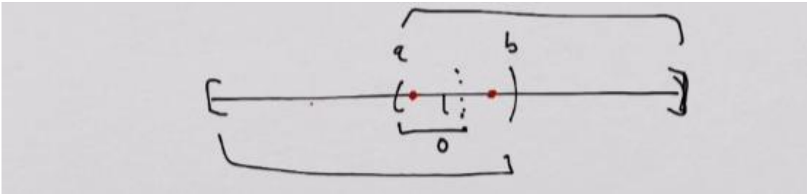


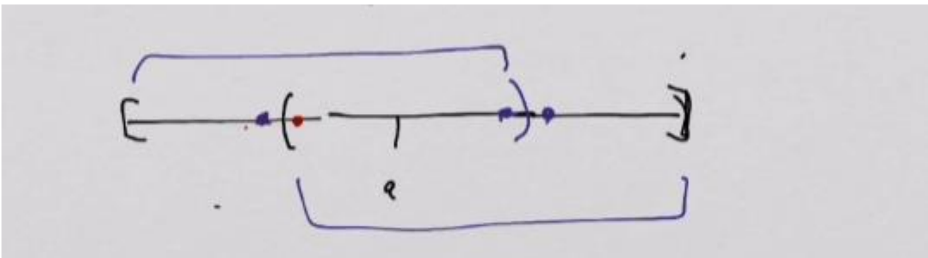
  - ➢ Example: Euclidean line is T0

  - ➢ Example: nested line is not T0



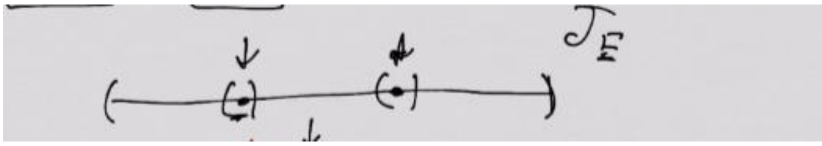
- Example: closed interval topology is T0



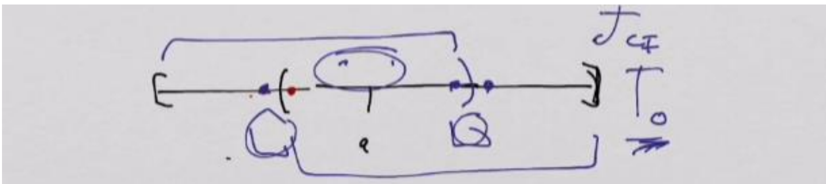


- Separation means can I find open sets that can distinguish between these two points
  - Distinguish means find different open sets that contain the points and how different are the open sets

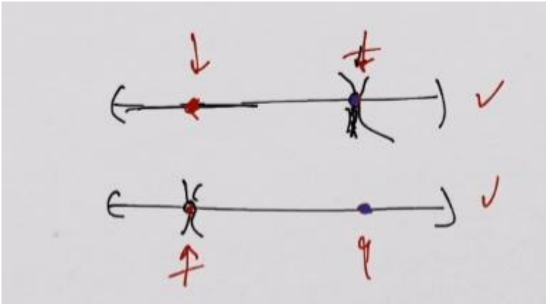
- T1
  - Any two points can be found in two different open sets
  - Example: Euclidean line is T1



  - Example: Closed interval is not T1



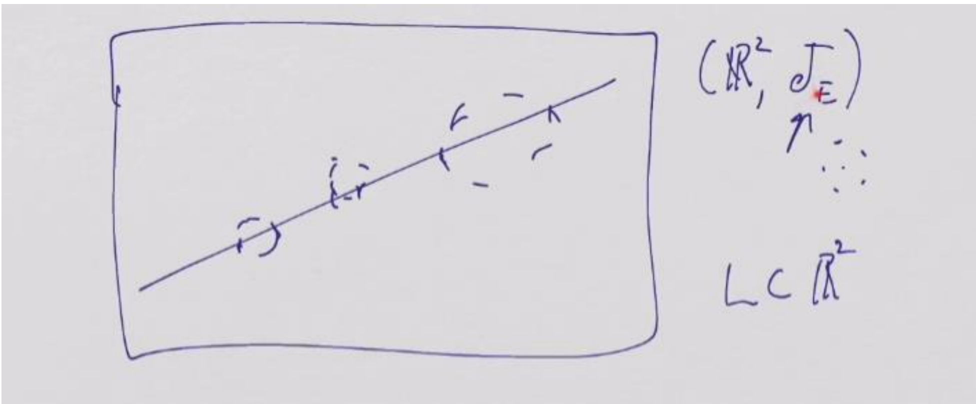
  - Example: Cofinite topology is T1 and T0



- T2: Hausdorff
  - Not only picking any two points and can put them in their own open sets, those open sets have to be disjoint

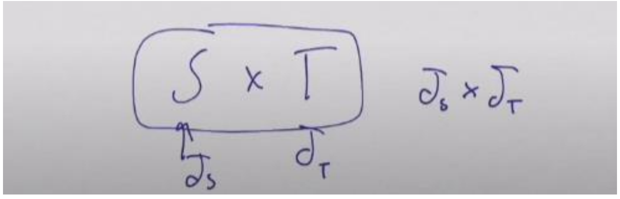
  - Further separate points

General Picture



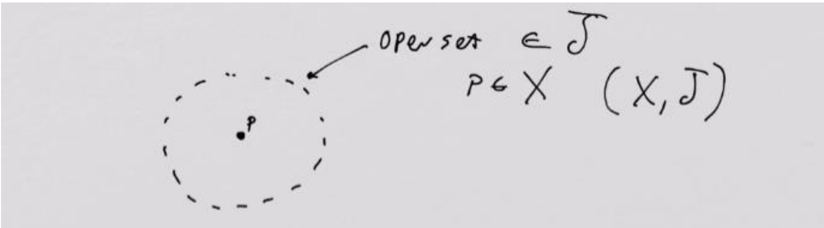
- L inherits its own topology
  - Intersections of every open set on the Euclidean plane with L will induce a topology on L, and that topology will be the standard open Euclidean topology
  - So you take a larger space, you cut out a subspace to that larger space, and you accept the induced topology on the subspace by the intersection that subspace with all of the open sets of topology

- In addition, Cartesian product of two sets with their own topology will have

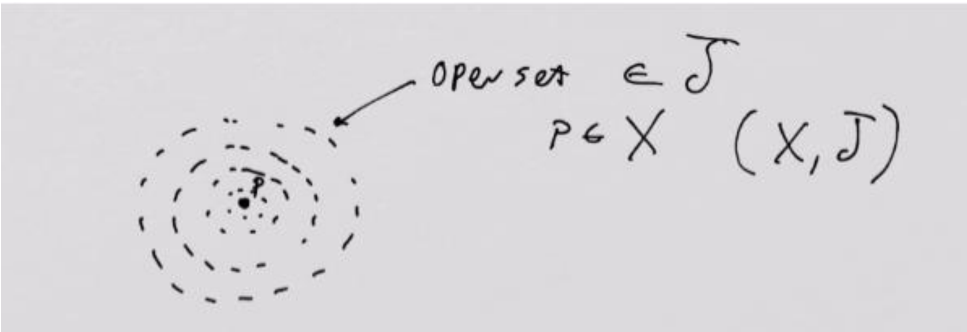


Lesson 4: Countability and Continuity

- So, I have an open set – belongs to some topology on set X – which contains a point p



  - ➢ Now the question is, is there another open set contained in the first one that has p in it? Ask the question again and again, …



  - ➢ Then you'll have this nested collection of open sets that contain the point p.
    - ✧ It might be just 1, 2, 3 nested sets and then stop, there is no other.
    - ✧ But of course, generally speaking, certainly in this open ball topology, there's an infinite number of these nested open sets and they just go down forever, and as long as you don't have the discrete topology, there always be more than one point in the set
  - ➢ Then the question became: is there a countable number of them?
    - ✧ For the open ball topology, clearly, there is an uncountable number because every diameter of these open sets in this open ball could be of just some element of the real numbers.
    - ✧ But you can always create a countable bunch
      - ▫ E.g. we restrict the diameter to be from rational numbers, then you have a countable number of open sets

- If for every **point**, you can create a countable sort of nested collection of open sets, then we are dealing with something called nested open neighborhoods
  - ➢ The space is called **first countable**

- ➢ First countability is about the properties of a space, that at every point, you can create what we call a countable basis

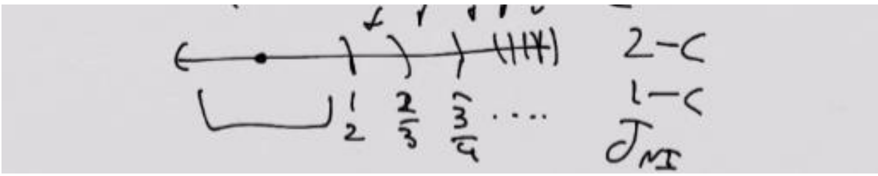
- **Second countable** asks the question:
  - ➢ Is every open **set** in the topology, the union of a countable number of members of the basis element?
    - ✧ In this open ball topology, I can.
    - ✧ I can restrict the diameter to rational numbers, I can still count all of the open balls that are building the shapes

- Note that all of the **manifold** must be second countable
  - ➢ The reason is because there are many topological properties that are important that second count ability will guarantee and it has to do with stitching together manifolds and having manifolds that behave well

- Example: Euclidean line topology

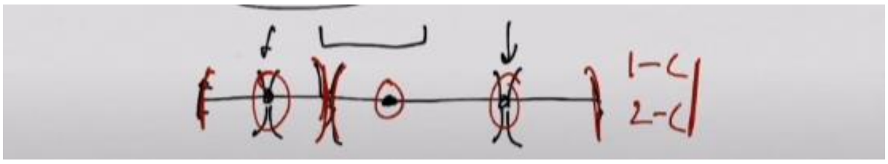
  - ➢ It is first countable since every point have a potentially countable open basis
  - ➢ It is second countable since every open set can be built with union of countable open set basis

- Example: Nested interval line topology



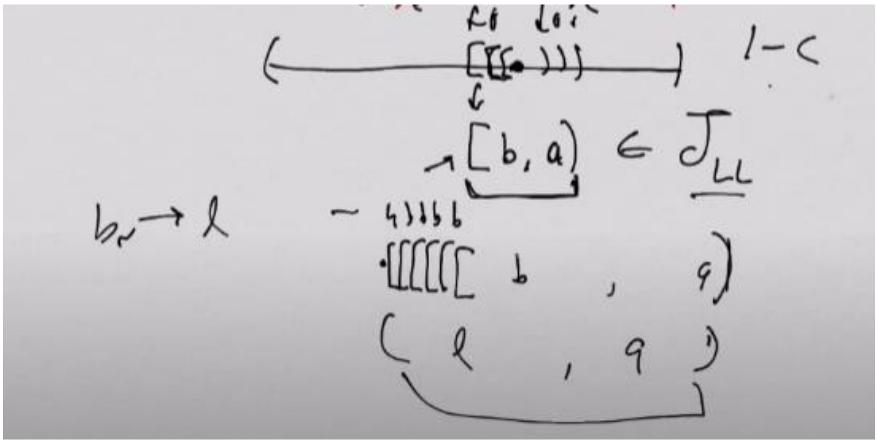
  - ➢ It is first countable since every point have countable open set basis
  - ➢ It is second countable since every open set can be built with union of countable open set basis

- Example: Cofinite topology

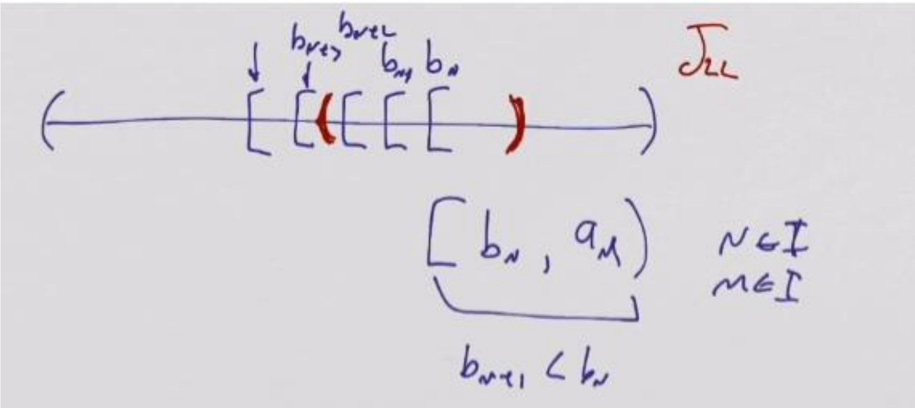


  - ➢ This is neither first countable nor second countable because we cannot ever create nested series of open sets.
  - ➢ And this is where space has become like dysfunctional and odd, and that’s exactly what we ant to avoid when we build manifolds

- Note for lower limit topology

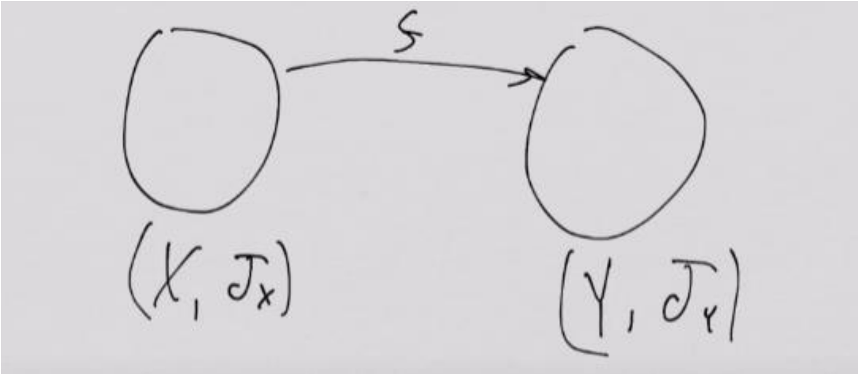


- We can build an open “open” set with left boundary infinitely close to a lower limit, but never touches it.
- There are more sets can be constructed in lower limit topology than Euclidean topology
  - **Euclidean topology (coarser) is a subset of lower limit topology (finer)**
- But lower limit topology is not second countable
  - If I take any countable basis, I can always find little open intervals between these countable elements of the basis that I can’t construct and therefore there are open sets that I cannot build out of the countable basis



- When build manifold, require 1) Hausdorff (T2) and 2) second countable
  - We are looking for coordinates, looking for things that we can do calculus on, things were we can take derivatives, and you need these properties to be able to do that

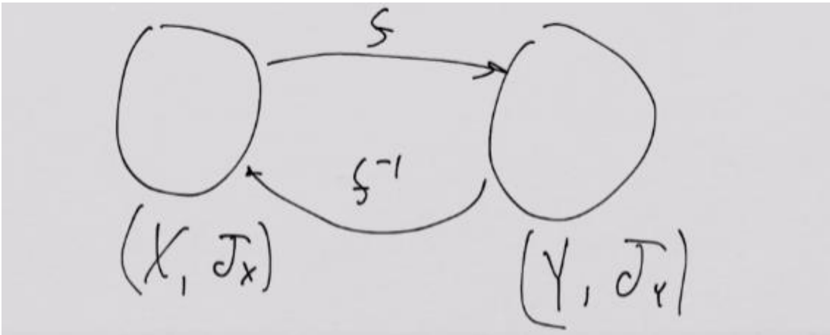
Continuity



- This idea of a function works as: from one set to another, from the domain to the range
  - ➢ But now, domain is endowed with the topology and the range is endowed with the topology
  - ➢ The notion of continuity wouldn't make any sense without a topology without some concept of these open sets which we now understand, which collect the points together in ways that are perhaps separable, perhaps not, with different degrees of separability

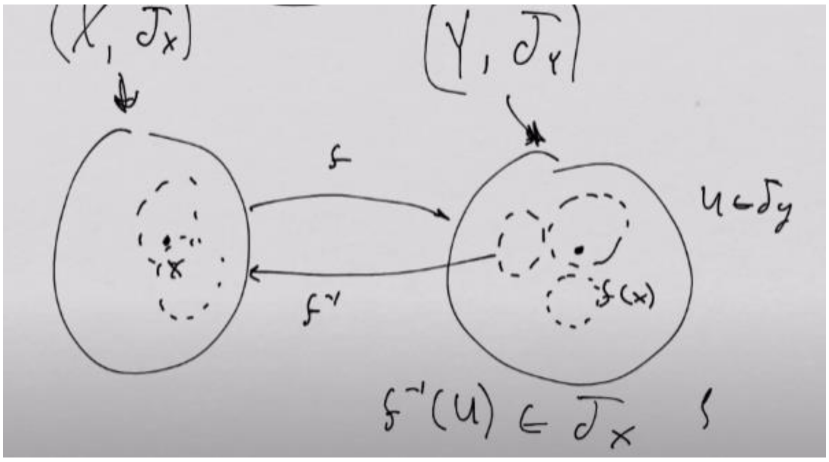
- Recall that we used to define limits, continuity using epsilon and delta sort of proof.
  - ➢ You can do those proof because we have distance functions and we know the distance between different points and we have a lot of extra information that allows us to work with this
  - ➢ But when you don't have that extra information, you can still define continuity, but you have to use topology, then you have to have a well understood topology of your two sets

- How does this work?
  - ➢ Recall that there's an inverse mapping
    - ✧ Which may not be a function



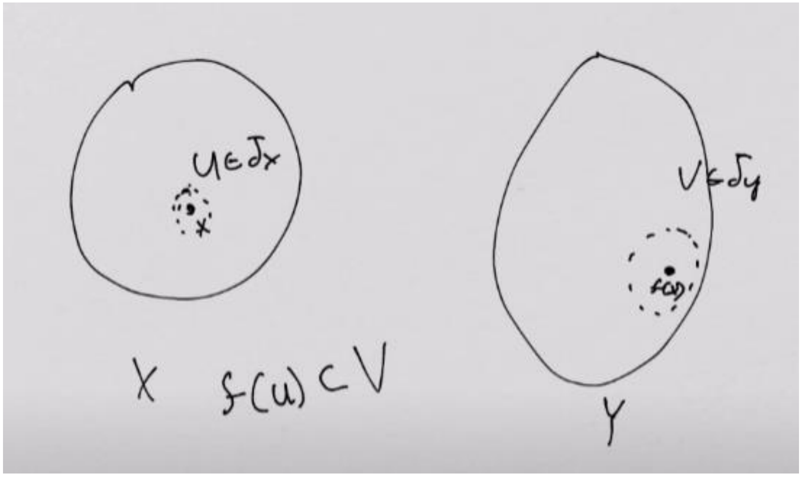
  - ➢ What makes a function a function is that: for every element in the domain, there is a unique value in the range that corresponds to it (for a x, there is only 1 y)
    - ✧ But for 1 y, there could be more than one x correspond to it, hence make the inverse map not a function

- **Definition**



- For any open set in y, if the preimage under the inverse is an open set in x, then f is continuous.
  - So the continuity of f depends critically on how the inverse mapping works on these open sets.

- The equivalent way of describing this in terms of neighborhoods

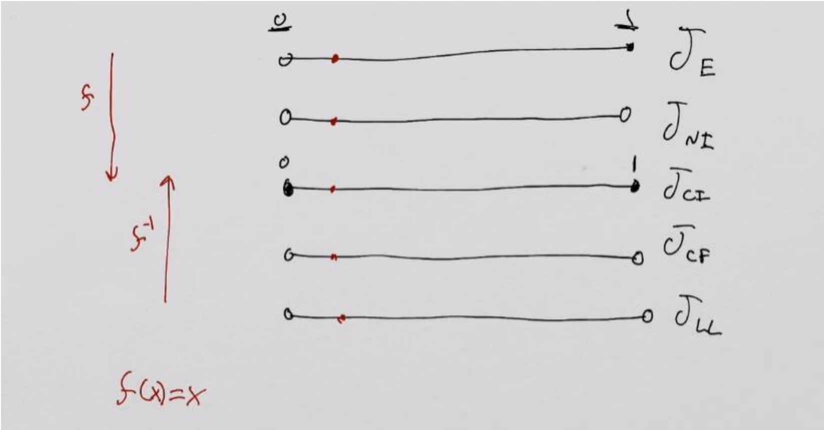


- For any open neighborhood here, V, as an element of the topology of y, which is the f of x, the inverse mapping of V must be an open subset of x
- That's where you can get a sense of the epsilon-delta
  - Recall that

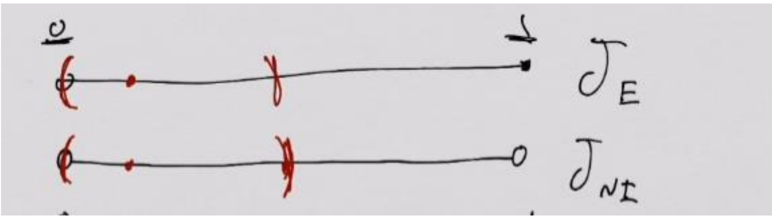
We recall the definition of continuity: Let $f : [a, b] \to \mathbb{R}$ and $x_0 \in [a, b]$. f is *continuous at* x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.

  - Here, if the open set V represents the measurement error of f(x), in which the function is the measurement, now, the question is that, can you find a neighborhood of x that's so tight that you can get so close to x that you can get inside that measurement error, the neighborhood of the set y.

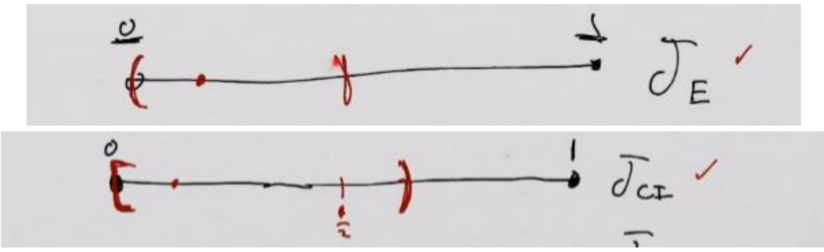
- Example: Euclidean, nested interval, closed interval, Cofinite lower limit topology



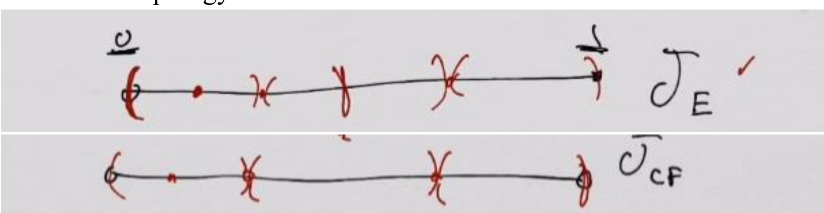
- Consider, Euclidean is the domain, the rest is range, is the function continuous?
  - Nested interval line:
    - We can look at every single open set, first one (0, 1/2), (0, 2/3), …
    - Each of these is an open interval on the line segment, the inverse image is still x,
    - So the interval here is going to map to the interval as follows



    - This certainly is an element of the topology of Euclidean line
    - And this is true for every single open interval for nested interval line
    - The function f is continuous
  - Consider closed interval topology

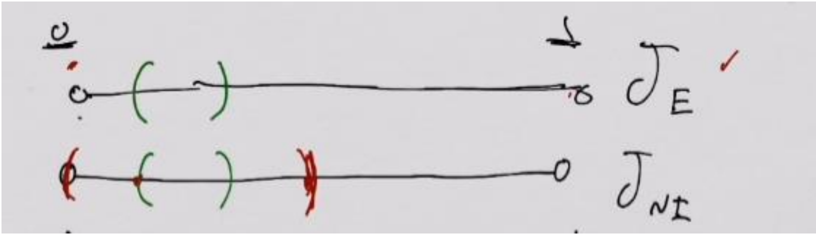


    - Here, the inverse mapping of y, is not an open subset of x, (can't have half closed interval in Euclidean topology)
    - Thus, f is not continuous going from Euclidean to closed interval
  - Consider Cofinite topology

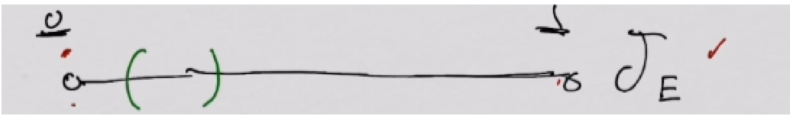


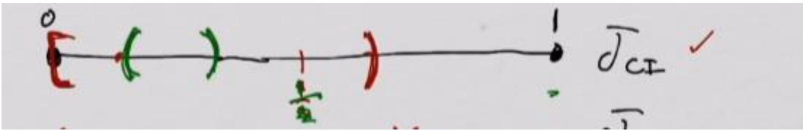
- ✧ Function f is continuous going for Euclidean line to Cofinite line
- ➢ For lower limit topology, similar to closed interval topology, f is not continuous

- If we look at things in reverse
    - ➢ Nested interval to Euclidean

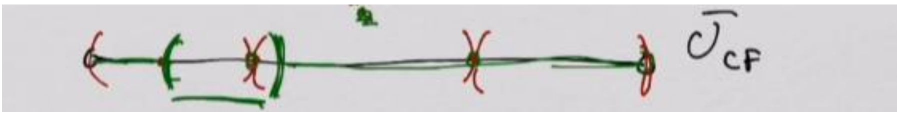


    - ✧ Nope, f not continuous
    - ➢ Closed interval topology to Euclidean topology





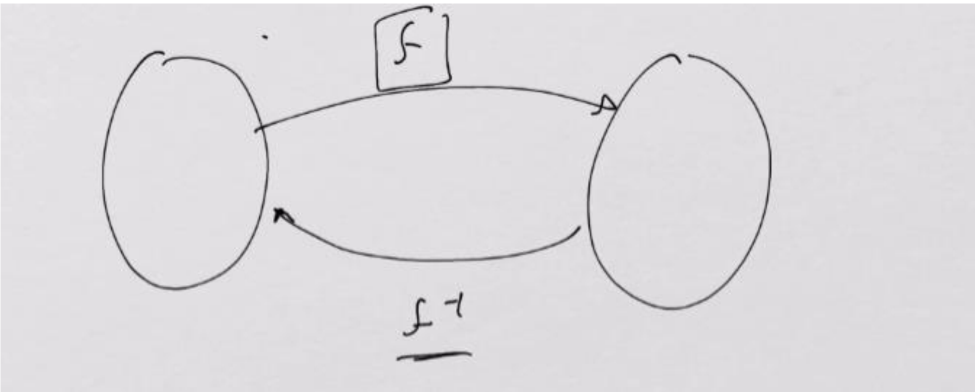
    - ✧ Nope, the only open set can be constructed must contain 1/2
    - ✧ Function f is not continuous
    - ➢ Cofinite topology



    - ✧ Nope, works only for some open set in Euclidean topology, not all of them without changing the set definition in Cofinite topology
    - ➢ Lower limit topology
    - ✧ Yes, can artificially create open “open” intervals

- Consider discrete topology
    - ➢ Since any subset in the topology is open set, any open set from Euclidean topology can be mapped to discrete topology → f continuous going from discrete to Euclidean
    - ➢ A closed interval is open in discrete topology, but it is not open in Euclidean topology, so cannot map every open set from discrete topology to Euclidean topology, so f not continuous from Euclidean to discrete topology

Homeomorphic



- If the inverse image is also a function, that would basically mean that to check the continuity of the inverse image, you would have to use the function itself
  - ➢ And if both functions are continuous on both directions (i.e. one-to-one and ONTO), then this is something called a **homeomorphism**
  - ➢ It is a very important kind of function because there are lots of topological properties that are actually important that homeomorphisms preserve.
    - ✧ For example, compactness, separability, connectedness, countability
  - ➢ So the question is that if I have a homeomorphism, does it preserve the compactness property?
    - ✧ If X space is compact in a way, is Y space compact in the same way?
  - ➢ And the answer is YES

Lesson 5: Compactness, Connectedness, and Topological Properties

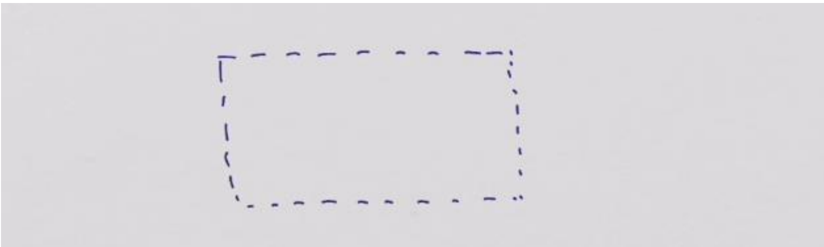
- We could argue that topology is really the search and the study of things that are invariant between homeomorphic spaces
  - ➢ If I could find any single function that's one-to-one and onto which map one topology space to another topology space, then the two spaces are homeomorphic
    - ✧ Then, the same topological property of the first space is going to be the same as the topological properties of the second space.
  - ➢ However, if I can find a topological property of the second space that is not the same as the first space, then no matter how hard I look, I will never find a homeomorphic function connecting the two

- This is important because open sets of X are mutated into open sets of Y by the function, and open sets of Y are mutated into open sets of X by the inverse function and this is why homeomorphism is so interesting

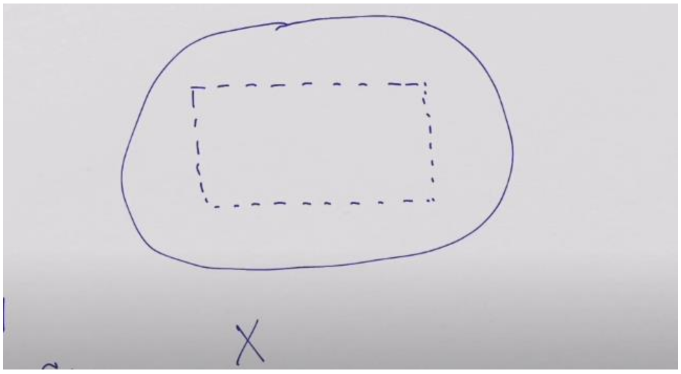
- Since the direction goes both ways, you will never loss sets, or two topological sets get squeezed into one or some weird thing
- The topology of Y gets morphed into the topology of X through the inverse function
- What you really doing is sort of changing the shape of things
- And this is the point-set topology foundation for the old famous fact that a doughnut equals a coffee cup
  - They both have a single whole

  - You can turn the doughnut into a cup with some function, but you can't turn the cup back to doughnut unless the inverse function is also continuous
- Note that, there is no geometric or affine transformation here, that's why it is so fundamental, no restrictions and additional notions

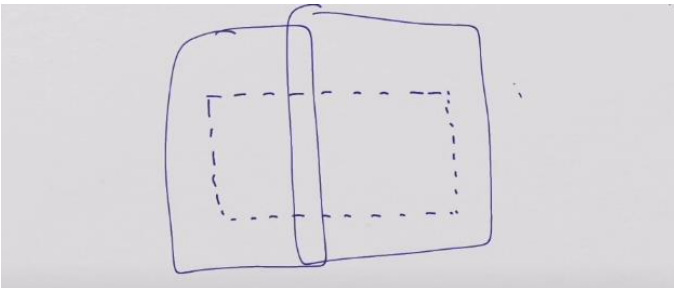
Compactness



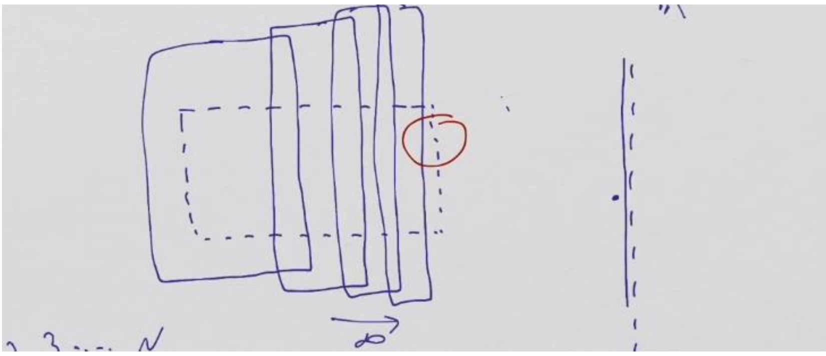
- Is this compact?
  - We need to talk about open covers
    - It's a set of open sets in the topology of the plane that literally covers the set in question



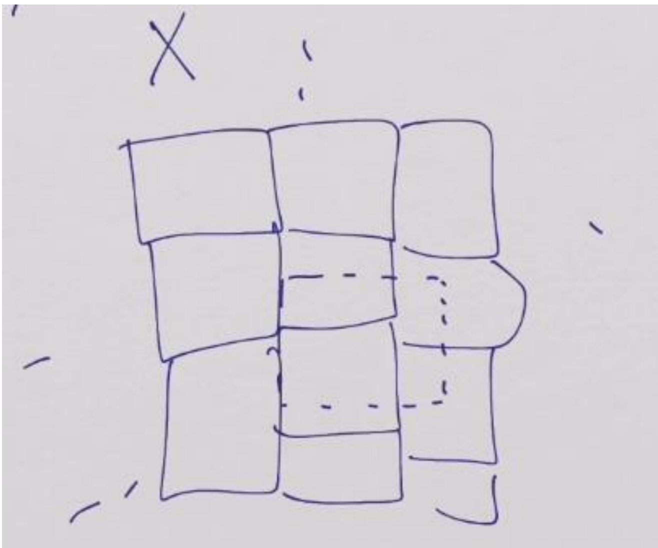
  - Consider another open set that covers X
    - Note that this is an open set that includes its boundaries, "open" in the sense that it's a member of the topology
  - In the above example, I covered X with only 1 open set
  - Consider using two overlapping open sets to do the job



- ➢ In the above, there are a finite number of sets that covers X
- ➢ I can also have infinite number of sets to complete the cover



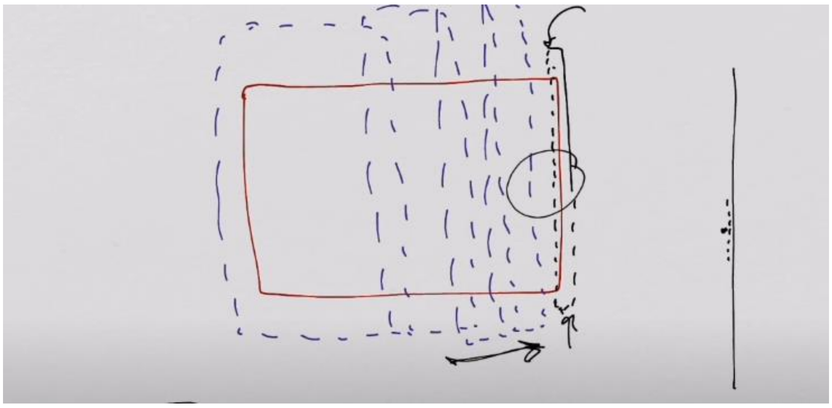
- ➢ For any point arbitrarily close to the boundary of X, I can find an open set that covers it
  - ✧ I don't need to worry about the boundary points, because X doesn't contain its boundaries



- ➢ Alternatively, I can use infinite many rectangular open sets to cover X

- Consider the difference between the two infinite covers cases
  - ➢ For the second case, there are finite subcovers, but for the first case, there are infinite subcovers (covers that intercept with X)
  - ➢ That's what defines **compactness**
    - ✧ If I can find an infinite subcovers (first case), then X is **not compact**.

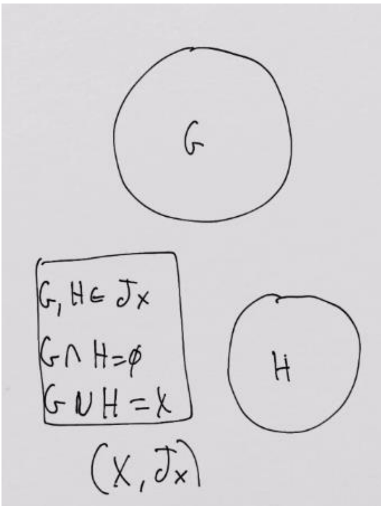
- Now, let's consider a closed rectangle as X



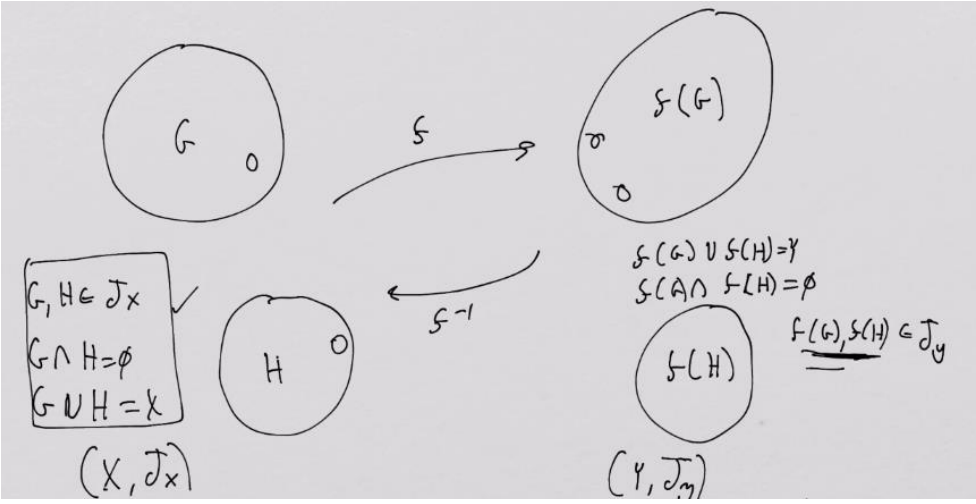
- ➢ By construction, the points on the boundary can never be covered in this case
- ➢ So in order the cover X, I have to include a rectangle that covers the boundary line, which would merge with the infinitely smaller rectangles we are constructing. Thus, not infinite covers
- ➢ So X is compact

Connectedness

- Consider now topology X has two open sets that do not intersect



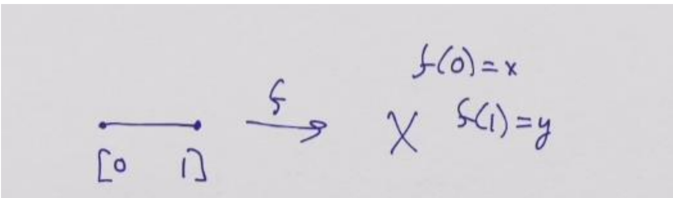
- ➢ Since X is the union of G and H, and G, H has no intersection, and also G and H are open sets – elements of the topology of X, so X is not connected
- ➢ Now, if there is a function f that is continuous from X to Y and from Y to X (homeomorphism), then we can conclude the same level of connectedness of Y



Path Connectedness

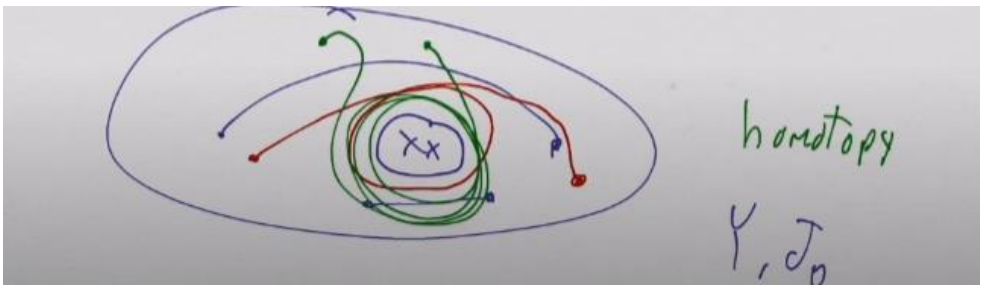
- Consider two points x, and y in $\mathbb{R}^2$, if I can draw a line from x to y and that line is inside the set $\mathbb{R}^2$, we say x, and y are path connected

- It's more abstract for arbitrary set
  - ➢ What I have to prove, is that there's a continuous map (i.e. f) from the line segment 0 to 1 into the topological space X. And f has to have a few properties



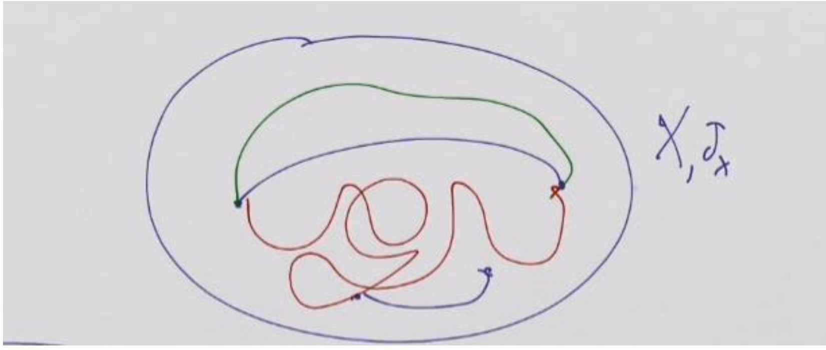
    - ✧ Note that, the two topological spaces don't have to be homeomorphic
  - ➢ Then, I can say that x and y are path connected
- And, if I can do this for any x, and any y in the space X, then X is path connected.
  - ➢ Then it inherits connectedness that we talked about before

Homotopy



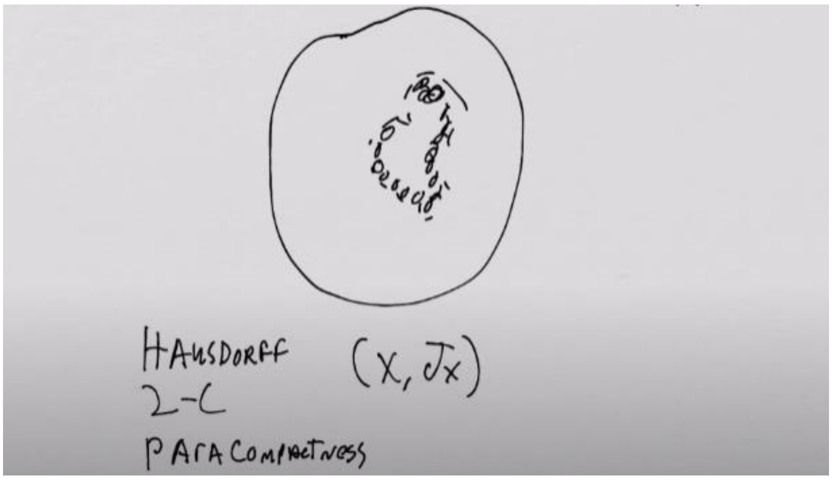
- Although the topological space of Y is path connected, there is a fundamental difference between the path in blue, than path in red, than the path in green which goes around the whole in Y
    - The paths cannot be continuously deformed into other paths
    - The distinction between these types of paths leads you down to the subject of **Homotopy**

- Homotopy leans heavily on continuous functions because you have to create a continuous function to study the theories, and the function needs to take a path and continuously deforms it into another path



    - The path without looping around a whole can always be deformed in every other path, this is called **simply connected**.

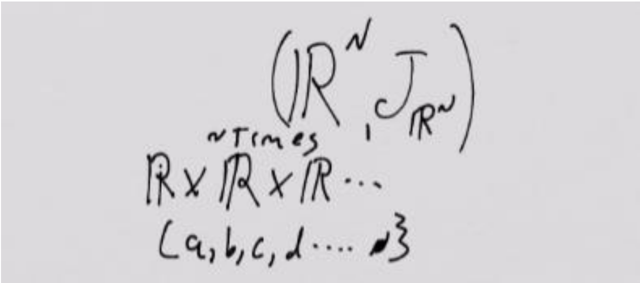
Lessen 6: Topological Manifold



- Consider a topology space that is Hausdorff, second countable, and para-compactness
    - Hausdorff means any point can be included in two non-intersect open sets inside the topology space
    - Second countable means, any open set can be produced by a countable number of open set basis
    - Para-compactness means every cover can be broken down into subsets, and you need to be able to show that any open set intersects only a finite number of members of the subsets
- There three properties ensured that the space is ultimately metrizable
    - Means we can actually create a metric on this space
    - We can invent a metric that would measure something on the space

- ➢ I can covert the Cartesian product of x in the space, to a real number
- ➢ Hausdorff lets you aggressively distinguish points from one another
- ➢ Para-compactness and second countable show you that the open sets are under some kind of control and this makes the space useful for most work.

- The other thing we're going to need is basically access to Euclidean space $\mathbb{R}^n$
  - ➢ So we need to create maps from the topological manifold to Euclidean space, which is a set of ordered tuples of real numbers that is endowed with usual topology (open ball topology) of $\mathbb{R}^n$



  - ➢ And now, this is a second topological space

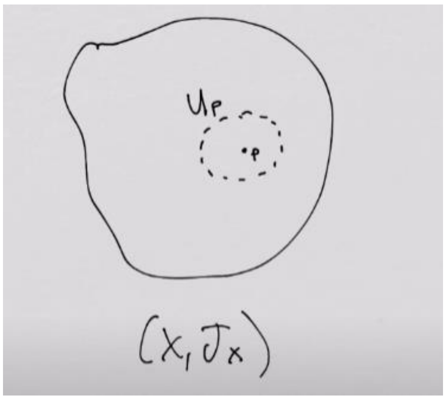
- Question is: is the topological space X and the topological space of $\mathbb{R}^n$ for a homeomorphism?
  - ➢ In other words, is the first topology rich, or fine enough?
  - ➢ Or, are all the topological properties of the first space the same as Euclidean space?

- If the answer is yes, then you can do all your work or your study in Euclidean space and then you convert back to the original topology
  - ➢ And the original topological space would be topological manifold

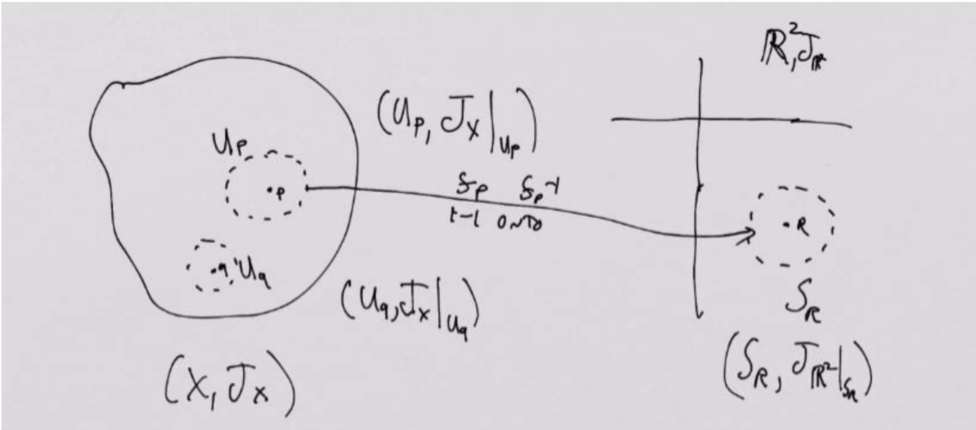
Locally Euclidean

- We are interested in finding spaces that are only locally Euclidean
  - ➢ Locally homeomorphic to Euclidean space

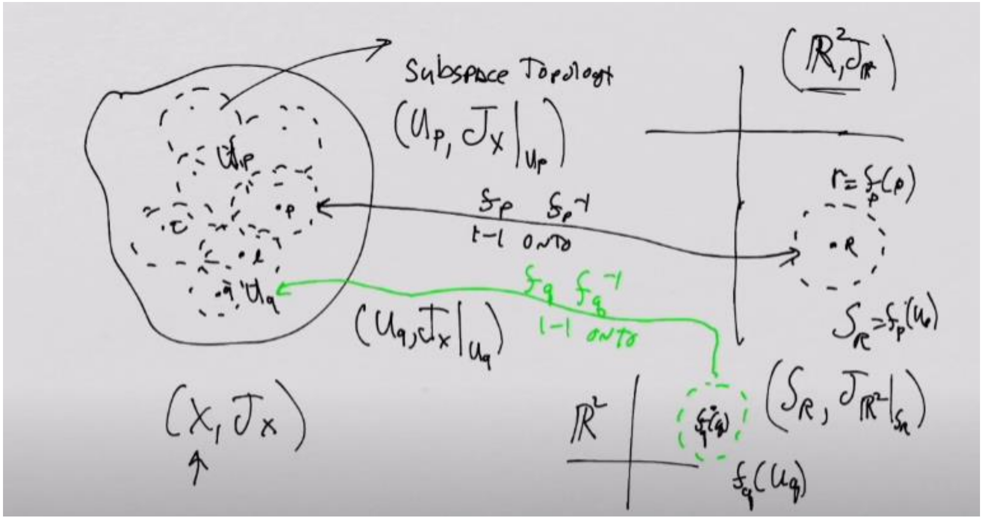
- Consider an open neighborhood of point p, which is an open set in X



  - ➢ This open set U_p is also a topological space



- ➢ Now, we get to look at the homeomorphism in a smaller scope of topological subspace in their own topological superspace. Note that subspace inherit all the things from superspace



- ➢ Note that if we have multiple mappings from different subspaces, then each destination of the mapping is in a separate $\mathbb{R}^2$ space.
  - ✧ Then, the mapping gives you coordinates, then every point in the X subspaces would have a coordinate

- If I can show that every mapping from open neighborhood of any point in X is homeomorphic to our $\mathbb{R}^n$, then the space X is locally Euclidean
  - ➢ This is another step to being a topological manifold