Assignment for the Modal Logic component of Overview of Logic COMP 6463

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Note: this assignment is worth 16.5% (= 33% * 50) of the final mark!

Deadline: 1700 Thursday 31st of October 2013 at 1700. Hand it in to me in person in the RSISE building (ring me on 61258603 to get into the building) or email it to me as a (scanned) PDF or hand it in to the main CSIT office in Building 108.

Questions and marks: There are ten questions, each is worth 20 marks.

Late Penalty: There will be a penalty of 20 marks per day for late submissions and pro-rata for parts of days.

Good excuses: If you have a good excuse then email me as soon as you know that you will be late to submit so we can work out a later submission date. But I will be the judge of whether your excuse is good enough. "I have a lot of other assignments" will certainly fail to convince me.

Questions: if you want to ask questions about this assignment or point out typos or mistakes in the questions then email me.

1. Let $\langle W, R, \vartheta \rangle$ be a Kripke model and let $w \in W$. Use the semantic clauses for the definition of $\vartheta(w, \varphi)$ around Slide 9 to prove that if $\vartheta(w, []\neg p_0) = \mathbf{t}$ then $\vartheta(w, \neg\langle\rangle p_0) = \mathbf{t}$.

   Solution: For a contradiction, assume $\vartheta(w, []\neg p_0) = \mathbf{t}$ and not $\vartheta(w, \neg\langle\rangle p_0) = \mathbf{t}$. Thus $\vartheta(w, \neg\langle\rangle p_0) = \mathbf{f}$. That is, $\vartheta(w, \langle\rangle p_0) = \mathbf{t}$. This means that there is a world v with wRv and $\vartheta(v, p_0) = \mathbf{t}$. But our first assumption, $\vartheta(w, []\neg p_0) = \mathbf{t}$, implies that every successor of w makes $\neg p_0$ true. Thus $\vartheta(v, p_0) = \mathbf{f}$. Contradiction, so if $\vartheta(w, []\neg p_0) = \mathbf{t}$ then $\vartheta(w, \neg\langle\rangle p_0) = \mathbf{t}$.

2. Show a model $\mathcal{M} = \langle W, R, \vartheta \rangle$ such that $\mathcal{M} \not\Vdash (\langle\rangle[]p) \to [][]p$. You must give concrete values of your W, R and ϑ as in my lecture notes. A diagram alone does not count but can be used to clarify your concrete values.

   Solution: let $W = \{w_4, w_1, w_2, w_3\}$ and $R = \{(w_1, w_2), (w_2, w_3,), (w_1, w_4)\}$ and $\vartheta(w_3, p) = \mathbf{f}$.

   Thus we have $w_1 R w_2 R w_3$ and hence $w_1 \Vdash \langle\rangle\langle\rangle\neg p$.

   We also have $w_1 R w_4$ with w_4 a dead-end, hence $w_4 \Vdash []p$ and hence $w_1 \Vdash \langle\rangle[]p$.

   That is, $w_1 \not\Vdash \langle\rangle[]p \to [][]p$.

3. Prove that a frame validates the shape $\langle\rangle\varphi \to \langle\rangle\langle\rangle\varphi$ if and only if the frame is dense i.e. $\forall x, z. R(x, z) \Rightarrow (\exists y. R(x, y) \& R(y, z))$.

   Solution. We have to show two things:

- if a frame validates the shape $\langle\rangle\varphi \to \langle\rangle\langle\rangle\varphi$ then the frame obeys the condition $\forall x, z.R(x,z) \Rightarrow (\exists y.R(x,y)\&R(y,z))$.

  So suppose the frame $\langle W, R\rangle$ validates the shape $\langle\rangle\varphi \to \langle\rangle\langle\rangle\varphi$. For a contradiction, suppose that R does not obey the condition $\forall x, z.R(x,z) \Rightarrow (\exists y.R(x,y)\&R(y,z))$. Thus $\exists x, z.R(x,z)$ & $(\forall y.not(R(x,y)\&R(y,z)))$. So put p_0 true at z only. Since xRz, we must have $x \Vdash \langle\rangle p_0$. For x to force $\langle\rangle\langle\rangle p_0$ we have to find a y such that there is a path $xRyRz$. That is, $R(x,y)\&R(y,z)$. But this is impossible by the fact that $\forall y.not(R(x,y)$ & $R(y,z))$. Thus $x \nVdash \langle\rangle p_0 \to \langle\rangle\langle\rangle p_0$. But this formula is an instance of $\langle\rangle\varphi \to \langle\rangle\langle\rangle\varphi$, and we know that the frame validates all instances of this shape. So we have a contradiction.

- if the frame obeys the condition $\forall x, z.R(x,z) \Rightarrow (\exists y.R(x,y)\&R(y,z))$ the frame validates the shape $\langle\rangle\varphi \to \langle\rangle\langle\rangle\varphi$.

  So suppose that the frame $\langle W, R\rangle$ obeys the condition $\forall x, z.R(x,z) \Rightarrow (\exists y.R(x,y)\&R(y,z))$ but that it does not validate the shape $\langle\rangle\varphi \to \langle\rangle\langle\rangle\varphi$. That is, there is some valuation ϑ, some world $x \in W$, and some instance $\langle\rangle\psi \to \langle\rangle\langle\rangle\psi$ such that $x \Vdash \langle\rangle\psi$ and $x \nVdash \langle\rangle\langle\rangle\psi$. Since $x \Vdash \langle\rangle\psi$ there must be a world z such that xRz and $z \Vdash \psi$. By the condition $\forall x, z.R(x,z) \Rightarrow (\exists y.R(x,y)\&R(y,z))$, there must exist a world y such that $R(x,y)\&R(y,z)$. That is, $xRyRz$. That is, $y \Vdash \langle\rangle\psi$ and hence $x \Vdash \langle\rangle\langle\rangle\psi$. Contradiction since we already have that $x \nVdash \langle\rangle\langle\rangle\psi$.

4. Use the tableau method for **K** with global assumptions to decide whether the formula $\langle\rangle p_1 \to [\,][\,]p_0$ is a logical consequence of $\{p_0, p_0 \to p_1\}$. Show the tableau(x).

   The shortest closed tableau is this one, but it does not use the strategy that I told you about. It is still a legal tableau.

$$
\begin{array}{c}
p_0; p_0 \to p_1; \neg(\langle\rangle p_1 \to [\,][\,]p_0) \\
\big\downarrow{\scriptstyle nnf} \\
p_0; \neg p_0 \vee p_1; \langle\rangle p_1 \wedge \langle\rangle\langle\rangle\neg p_0 \\
\big\downarrow{\scriptstyle \wedge} \\
p_0; \neg p_0 \vee p_1; \langle\rangle p_1; \langle\rangle\langle\rangle\neg p_0 \\
\big\downarrow{\scriptstyle \langle\rangle K} \\
p_0; \neg p_0 \vee p_1; \langle\rangle\neg p_0 \\
\big\downarrow{\scriptstyle \langle\rangle K} \\
p_0; \neg p_0 \vee p_1; \neg p_0
\end{array}
$$

A tableau that follows the strategy is as follows:

$$
\begin{tikzcd}
p_0; p_0 \to p_1; \neg(\langle\rangle p_1 \to [][]p_0) \arrow[d, "nnf"'] & \\
p_0; \neg p_0 \vee p_1; \langle\rangle p_1 \wedge \langle\rangle\langle\rangle\neg p_0 \arrow[d, "\wedge"'] & \\
p_0; \neg p_0 \vee p_1; \langle\rangle p_1; \langle\rangle\langle\rangle\neg p_0 \arrow[d, "\vee"'] \arrow[dr, "\vee"'] & \\
p_0; \neg p_0; \langle\rangle p_1; \langle\rangle\langle\rangle\neg p_0 & p_0; p_1; \langle\rangle p_1; \langle\rangle\langle\rangle\neg p_0 \arrow[d, "K\langle\rangle"'] \\
\times & p_0; \neg p_0 \vee p_1; \langle\rangle\neg p_0 \arrow[dl, "\vee"'] \arrow[d, "\vee"'] \\
p_0; p_1; \langle\rangle\neg p_0 \arrow[d, "K\langle\rangle"'] & p_0; \neg p_0; \langle\rangle\neg p_0 \\
p_0; \neg p_0 \vee p_1; \neg p_0 & \times \\
\times &
\end{tikzcd}
$$

5. Give a tense logic model which satisfies $(\langle F\rangle\langle P\rangle p_0) \wedge (\langle F\rangle\langle P\rangle\neg p_0)$. You have to give a concrete W, R and ϑ. A diagram alone does not count but can be used to clarify your concrete values.

   Solution: $W = \{w_1, w_2, w_3, w_4\}$ and $R = \{(w_1, w_2), (w_3, w_2), (w_4, w_2)\}$ and $\vartheta(w_3, p_0) = \mathbf{t}$ and $\vartheta(w_4, p_0) = \mathbf{f}$.

   Since $w_4 R w_2$ and $w_4 \Vdash \neg p_0$, we must have $w_2 \Vdash \langle P\rangle\neg p_0$.

   Since $w_3 R w_2$ and $w_3 \Vdash p_0$, we must have $w_2 \Vdash \langle P\rangle p_0$.

   Since $w_1 R w_2$ and $w_2 \Vdash \langle P\rangle p_0$, we must have $w_1 \Vdash \langle F\rangle\langle P\rangle p_0$.

   Since $w_1 R w_2$ and $w_2 \Vdash \langle P\rangle\neg p_0$, we must have $w_1 \Vdash \langle F\rangle\langle P\rangle\neg p_0$.

   Thus $w_1 \Vdash (\langle F\rangle\langle P\rangle p_0) \wedge (\langle F\rangle\langle P\rangle\neg p_0)$.

6. Prove that the following formula $(\langle F\rangle\langle F\rangle[P]p_1) \to \langle F\rangle p_1$ is Kt-valid.

   Solution: Suppose for a contradiction that $(\langle F\rangle\langle F\rangle[P]p_1) \to \langle F\rangle p_1$ is not Kt-valid.

   Thus there is some model M and some world w such that $w \nVdash (\langle F\rangle\langle F\rangle[P]p_1) \to \langle F\rangle p_1$. That is, $w \Vdash \langle F\rangle\langle F\rangle[P]p_1$ and $w \nVdash \langle F\rangle p_1$.

   Since $w \Vdash \langle F\rangle\langle F\rangle[P]p_1$, there must be two worlds u, v such that $wRuRv$ and $v \Vdash [P]p_1$.

   Since uRv we have $u \Vdash p_1$.

   Since wRu we have $w \Vdash \langle F\rangle p_1$.

   But this contradicts the assumption that $w \nVdash \langle F\rangle p_1$. Hence no such Kt-model can exist. That is, every Kt-model and every world in that model must make the formula $(\langle F\rangle\langle F\rangle[P]p_1) \to \langle F\rangle p_1$ true. That is, $(\langle F\rangle\langle F\rangle[P]p_1) \to \langle F\rangle p_1$ is Kt-valid.

7. Use the tableau method from the lecture notes for PLTL to decide whether the formula $(\top \mathcal{U} p) \to \langle F \rangle p$ is PLTL-valid. Show the tableau.

$$\begin{tikzcd}
\neg((\top \mathcal{U} p) \to \langle F \rangle p) \arrow[d, "rewrite"'] & \\
(\top \mathcal{U} p) \wedge \neg \langle F \rangle p \arrow[d, "\wedge"'] & \\
\top \mathcal{U} p ; \neg \langle F \rangle p \arrow[d, "rewrite"'] & \\
\top \mathcal{U} p ; [F] \neg p \arrow[d, "rewrite \times 2"'] & \\
p \vee \oplus \; (\top \mathcal{U} p) ; \neg p \wedge \oplus \; [F] \neg p \arrow[d, "\wedge"'] & \\
p \vee \oplus \; (\top \mathcal{U} p) ; \neg p ; \oplus \; [F] \neg p \arrow[d, "\vee"'] \arrow[dr, "\vee"'] & \\
p ; \neg p ; \oplus \; [F] \neg p \arrow[d, phantom] & \oplus \; (\top \mathcal{U} p) ; \neg p ; \oplus \; [F] \neg p \arrow[uuul, bend right, "loop"'] \\
\times &
\end{tikzcd}$$

Delete the node $\top \mathcal{U} p ; [F] \neg p$ since it contains an unfulfilled eventuality $\top \mathcal{U} p$.

Delete its parent, since the parent has no child any more.

Delete the parent of this parent, since the parent has no child any more.

Delete the root since the root has no child any more.

The root is deleted, so the given formula set is pltl-unsatisfiable.

Hence the original formula is pltl-valid.

8. Show that $([\alpha]q) \leftrightarrow (\neg\langle\alpha\rangle\neg q)$ is true in all PDL-models.

Hint: You have to show two parts, one for each direction of the bi-implication. The easiest way is to argue by contradiction for each part. Alternatively, use the definitions of $\tau_M(.)$ to show the equivalence directly.

9. Is the formula $\langle \gamma * \rangle q \rightarrow \langle \gamma ; (\gamma *) \rangle q$ PDL-valid? Give your reasoning.

Hint: Use a proof by contradiction and use the definitions of $\tau_M(.)$ if it is PDL-valid or else give a PDL-model that falsifies this formula if it is not PDL-valid.

10. Let us call a tense-logical formula "purely modal" if it contains no occurrences of $\langle P \rangle$ nor $[P]$. Prove the following theorem: a purely modal formula φ is **Kt**-valid if and only if it is **K**-valid.

    Solution: if a Kt-formula is "purely modal" then it can contain all the classical connectives and $[F]$ and $\langle F \rangle$.

    The problem is that $[F]$ and $\langle F \rangle$ are not part of the syntax of **K**.

    The point is that the syntax is not set in stone. That is, we know that we can just as well use & as the symbol for conjunction instead of $\wedge$. So we can just define the syntax of Kt to use $\langle\rangle$ and $[]$ instead of $\langle F \rangle$ and $[F]$, while leaving $\langle P \rangle$ and $[P]$ as they are.

    The resulting logic is still Kt since the semantics remain the same.

    Now, every "purely modal" Kt-formula is also a K-formula since it can contain all the classical connectives and $\langle\rangle$ and $[]$ only.

    The theorem follows since the class of Kt-frames is exactly the same as the class of K-frames: that is they are both the class of all Kripke frames.