Dr. P. Baumgartner (`Peter.Baumgartner@nicta.com.au`) August 14, 2013
NICTA and ANU
CPMP 4630/COMP 6463
Assignment "Automated Reasoning in First-Order Logic"

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**Due date: August 30, 2013, 5pm.**

Marking scheme: Full marks for a formulation that correctly answers the question and clearly shows the steps to obtain the solution.

Solutions to be submitted electronically by Email to `Peter.Baumgartner@nicta.com.au`, or on paper to a lecturer of this course. Neatly hand-written solutions are of course acceptable.

**Question 1** (Structural induction, 8 pts). Let $T$ be a tautology, $B$ an atom, $\mathcal{A}$ an assignment, and $F$ a propositional formula that contains the connectives $\wedge$ and $\vee$ only (no $\neg$). Prove by structural induction the following property:

> If $\mathcal{A} \models F$ then $\mathcal{A} \models F[B/T]$, where $F[B/T]$ is the formula obtained from $F$ by replacing every occurrence of $B$ by $T$.

**Answer**.

**Base case.** The formula $F$ is an atom. We distinguish two cases. If $F = B$ then $F[B/T] = T$. As $T$ is a tautology it holds $\mathcal{A} \models T$. If $F \neq B$ then $F = F[B/T]$ and so the claim follows trivially.

**Induction step.** $F$ is of the form $F_1 \wedge F_2$ or $F_1 \vee F_2$. If $\mathcal{A} \not\models F$ then the claim holds trivially. Hence assume $\mathcal{A} \models F$ from now on. As the induction hypothesis assume the claim holds for both $F_1$ and $F_2$. More precisely, if $\mathcal{A} \models F_i$ then $\mathcal{A} \models F_i[B/T]$, for $i \in \{1, 2\}$

Let us consider the case $F = F_1 \wedge F_2$ first. From $\mathcal{A} \models F$ it follows $\mathcal{A} \models F_1$ and $\mathcal{A} \models F_2$. With this and by the induction hypothesis conclude $\mathcal{A} \models F_1[B/T]$ and $\mathcal{A} \models F_2[B/T]$. In other words, $\mathcal{A} \models F_1[B/T] \wedge F_2[B/T]$. With the identities $F_1[B/T] \wedge F_2[B/T] = (F_1 \wedge F_2)[B/T] = F[B/T]$ it follows $\mathcal{A} \models F[B/T]$.

The case $F = F_1 \vee F_2$ is proven analogously.

**Question 2** (Semantics, 6 pts). Is the following propositional clause set $M$ satisfiable? Justify your answer by a proof. (*Hint:* consider finite subsets and the use of Theorem 25 (Compactness).)

$$M = \{A_1 \vee B_1\} \cup \{\neg A_1 \vee A_2,\ \neg B_1 \vee B_2\ \neg A_2 \vee A_3,\ \neg B_2 \vee B_3, \ldots\}\ .$$

**Answer**. The clause set $M$ is satisfiable. Proof: let $N \subset M$ be any finite subset. Observe that every clause in $N$ contains an unnegated atom. Let $\mathcal{A}_N$ be the (suitable) assignment for $N$ that assigns $\mathbf{T}$ to each of these atoms, and $\mathbf{F}$ to all remaining atoms. It follows $\mathcal{A}_N \models N$. That is, $N$ is satisfiable. Because $N$ was chosen arbitrarily we can now use the compactness theorem to conclude that $M$ is satisfiable. (qed)

(There is even a simpler proof without appeal to compactness: take, e.g., the assignment $\mathcal{A}$ such that $\mathcal{A}(A_i) = \mathbf{T}$ and $\mathcal{A}(B_i) = \mathbf{F}$, for all $i \geq 1$. It follows that $\mathcal{A}$ satisfies every clause in $M$. In other words, $M$ is satisfiable.)

**Question 3** (Clause normal form, 6 pts). Transform the formula $\forall x\ \exists y\ ((R(y,x) \wedge \neg R(x,y)) \vee \exists y\ Q(x,y))$ into clause normal form. All intermediate formulas (Prenex normal form, Skolemized form, CNF) must be given explicitly.

**Answer**.

$$
\begin{array}{ll}
\forall x\ \exists y\ ((R(y,x) \wedge \neg R(x,y)) \vee \exists y\ Q(x,y)) & \text{(given formula)} \\
\Rightarrow \forall x\ \exists y\ ((R(y,x) \wedge \neg R(x,y)) \vee \exists z\ Q(x,z)) & \text{(bound renaming)} \\
\Rightarrow \forall x\ \exists y\ \exists z\ ((R(y,x) \wedge \neg R(x,y)) \vee Q(x,z)) & \text{(pull out } \exists\text{)} \\
\Rightarrow \forall x\ \exists z\ ((R(f(x),x) \wedge \neg R(x,f(x))) \vee Q(x,z)) & \text{(Skolemize } y\text{)} \\
\Rightarrow \forall x\ ((R(f(x),x) \wedge \neg R(x,f(x))) \vee Q(x,g(x))) & \text{(Skolemize } z\text{)} \\
\Rightarrow \forall x\ ((R(f(x),x) \vee Q(x,g(x))) \wedge (\neg R(x,f(x)) \vee Q(x,g(x)))) & \text{(matrix in CNF)} \\
\Rightarrow \{R(f(x),x) \vee Q(x,g(x)),\ \neg R(x,f(x)) \vee Q(x,g(x))\} & \text{(clause set)}
\end{array}
$$

**Question 4** (Herbrand interpretation, 2 + 4 pts). Let $F = \forall x\ (P(a) \wedge \neg P(f(b)) \wedge (\neg P(x) \vee P(f(x))))$ be a sentence in Skolem normal form.

1. Describe the Herbrand universe for $F$ and the Herbrand expansion of $F$.
2. Find a Herbrand structure $\mathcal{A}$ such that $\mathcal{A} \models F$.

**Answer**.

1. The Herbrand universe is $D(F) = \{a, b, f(a), f(b), f(f(a)), f(f(b)), \ldots\}$ and the Herbrand expansion is

$$
\begin{aligned}
E(F) = \{&P(a) \wedge \neg P(f(b)) \wedge (\neg P(a) \vee P(f(a))), \\
&P(a) \wedge \neg P(f(b)) \wedge (\neg P(b) \vee P(f(b))), \\
&P(a) \wedge \neg P(f(b)) \wedge (\neg P(f(a)) \vee P(f(f(a)))), \\
&P(a) \wedge \neg P(f(b)) \wedge (\neg P(f(b)) \vee P(f(f(b)))), \\
&P(a) \wedge \neg P(f(b)) \wedge (\neg P(f(f(a))) \vee P(f(f(f(a))))), \\
&P(a) \wedge \neg P(f(b)) \wedge (\neg P(f(f(b))) \vee P(f(f(f(b))))), \ldots\}
\end{aligned}
$$

2. To specify the desired Herbrand structure $\mathcal{A} = (U_{\mathcal{A}}, I_{\mathcal{A}})$ it suffices to specify the interpretation function $I_{\mathcal{A}}$ for all predicate symbols occurring in $F$, which is only $P$. By convention, $I_{\mathcal{A}}$ is specified by a set of ground atoms, those that are true in $I_{\mathcal{A}}$. We take

$$I_{\mathcal{A}} = \{P(a), P(f(a)), P(f(f(a))), \ldots\}$$

and it holds $\mathcal{A} \models E(F)$ and hence $\mathcal{A} \models F$.

This can be seen as follows: every element in $E(F)$ is of the form

(a) $P(a) \wedge \neg P(f(b)) \wedge (\neg P(f^i(a)) \vee P(f(f^i(a))))$, or

(b) $P(a) \wedge \neg P(f(b)) \wedge (\neg P(f^i(b)) \vee P(f(f^i(b))))$, for some $i \geq 0$. In case (a), $\mathcal{A}$ satisfies the first conjunct because of $P(a) \in I_{\mathcal{A}}$ and $P(f(b)) \notin I_{\mathcal{A}}$, and $\mathcal{A}$ satisfies the second conjunct because of $P(f^i(a)) \in I_{\mathcal{A}}$, for all $i \geq 0$. In case (b), $\mathcal{A}$ satisfies the first conjunct for the same reason as in case (a), and $\mathcal{A}$ satisfies the second conjunct because of $P(f^i(b)) \notin I_{\mathcal{A}}$, for all $i \geq 0$.

**Question 5** (Unification, 3 + 3 pts). Apply the unification algorithm presented in class to these sets of equations and read off the result, i.e., either FAIL or the unifier ($a$ is a constant, $x$ and $y$ are variables):

1. $U = \{x = y,\ g(f(x)) = g(y)\}$
2. $U = \{a = x,\ g(f(x, z)) = y,\ y = g(f(z, x))\}$

**Answer**.

1. The set $U$ is processed as follows:

$$\begin{aligned}
U = \{&\underline{x = y},\ g(f(x)) = g(y)\} && \text{(Binding)}\\
\Rightarrow\{&x = y,\ \underline{g(f(x)) = g(x)}\} && \text{(Decomposition)}\\
\Rightarrow\{&x = y,\ \underline{f(x) = x}\} && \text{(Orientation)}\\
\Rightarrow\{&x = y,\ \underline{x = f(x)}\} && \text{(Occur check)}\\
\Rightarrow&\mathsf{FAIL}
\end{aligned}$$

2. The set $U$ is processed as follows:

$$\begin{aligned}
U = \{&\underline{a = x},\ g(f(x, z)) = y,\ y = g(f(z, x))\} && \text{(Orientation)}\\
\Rightarrow\{&\underline{x = a},\ g(f(x, z)) = y,\ y = g(f(z, x))\} && \text{(Binding)}\\
\Rightarrow\{&x = a,\ g(f(a, z)) = y,\ \underline{y = g(f(z, a))}\} && \text{(Binding)}\\
\Rightarrow\{&x = a,\ \underline{g(f(a, z)) = g(f(z, a))},\ y = g(f(z, a))\} && \text{(Decomposition)}\\
\Rightarrow\{&x = a,\ \underline{f(a, z) = f(z, a)},\ y = g(f(z, a))\} && \text{(Decomposition)}\\
\Rightarrow\{&x = a,\ a = z,\ \underline{z = a},\ y = g(f(z, a))\} && \text{(Binding)}\\
\Rightarrow\{&x = a,\ \underline{a = a},\ z = a,\ y = g(f(a, a))\} && \text{(Trivial)}\\
\Rightarrow\{&x = a,\ z = a,\ y = g(f(a, a))\} && \text{(Trivial)}
\end{aligned}$$

No more rule is applicable at this stage. The resulting unifier is $[x/a,\ z/a,\ y/g(f(a, a))]$.

**Question 6** (First-order resolution, 8 pts). Find a Resolution refutation of the following clause set. As for the mgus used, it suffices to only state them, you do not need to show the details of the runs of the unification algorithm.

| | | | |
|---|---|---|---|
| (1) | $\neg P(x) \vee Q(f(x)) \vee Q(y)$ | (3) | $P(a)$ |
| (2) | $\neg Q(x) \vee R(x)$ | (4) | $\neg R(f(a))$ |

**Answer**. A Resolution refutation is as follows:

| | | |
|---|---|---|
| (1) | $\neg P(x) \vee Q(f(x)) \vee Q(y)$ | (given) |
| (2) | $\neg Q(x) \vee R(x)$ | (given) |
| (3) | $P(a)$ | (given) |
| (4) | $\neg R(f(a))$ | (given) |
| (5) | $Q(f(a)) \vee Q(y)$ | (by Resolution from (1).1 and (3).1 with mgu $[x/a]$) |
| (6) | $Q(f(a))$ | (by Factoring from (5).1 and (5).2 with mgu $[y/f(a)]$) |
| (7) | $R(f(a))$ | (by Resolution from (2).1 and (6).1 with mgu $[x/f(a)]$) |
| (8) | $\Box$ | (by Resolution from (4).1 and (7).1 with mgu $[]$) |