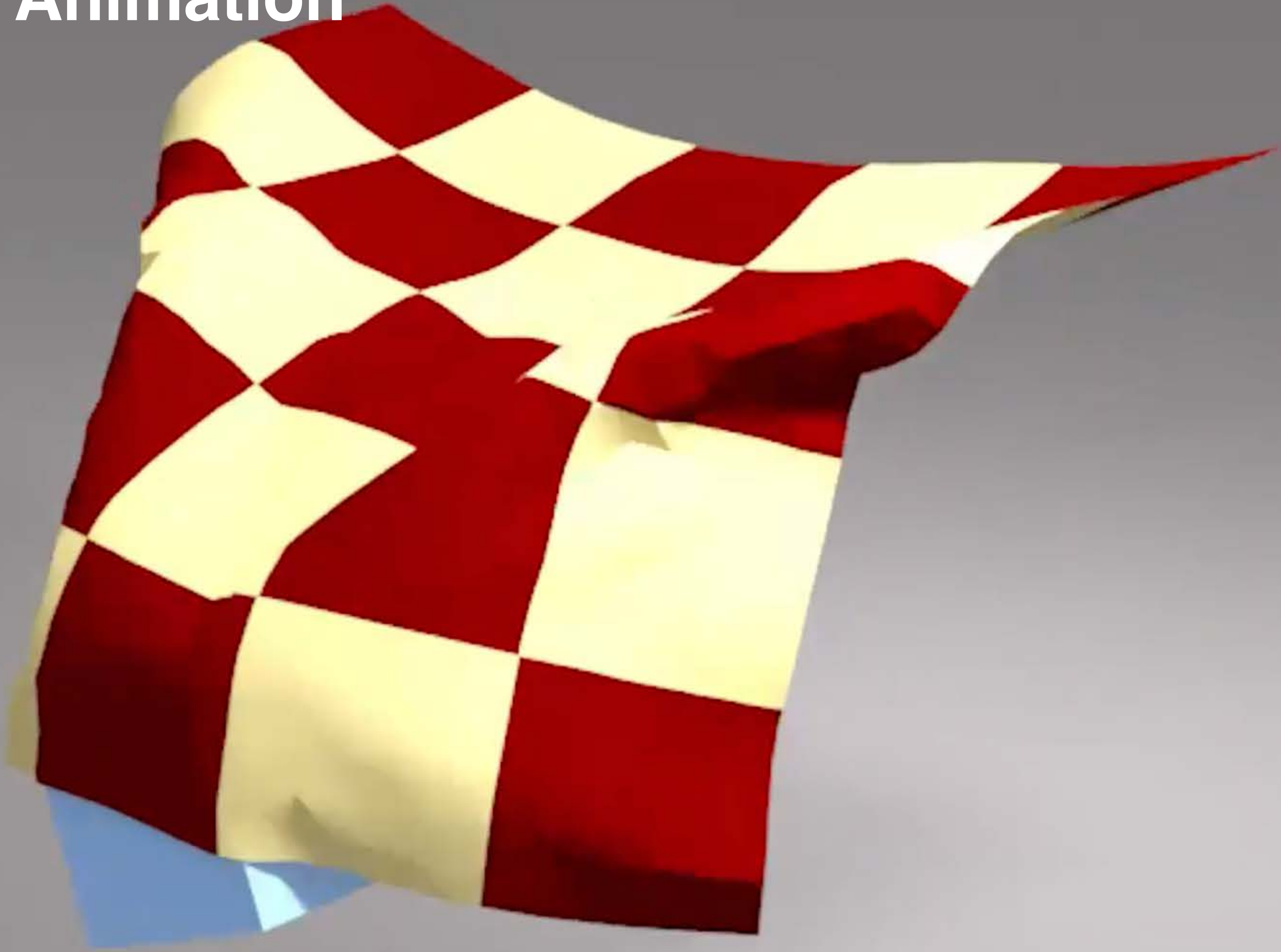
CSC418/2504 Computer Graphics

Some Slides/Images adapted from Marschner and Shirley

Physics-Based Animation



Announcements

Assignment 4 graded (Avg 85.6% -- Very Good!)

If you want to see your midterm come to office hours

Handle midterm remark requests through MarkUs

Assignment 8 out soon, due March 20th

What are we doing about COVID-19

1. All assignments and lecture notes will, by tomorrow, be online. Continue to use github issues/email for online help

2. This is the last lecture on testable material

3. No online office hours for now (if there is strong demand I will figure out how to set this up).

4. If you miss the final test for a valid reason (with documentation) your grade will be redistributed 50-50 to assignments and the previous test

Physics-Based Animation

Newton's Laws of Motion

The Mass-Spring System

Implicit Integration via Optimization

A Local-Global Solver for Fast-Mass Springs

Newton’s Laws

1. Every object will remain at rest or in uniform motion in a straight line unless compelled to change its state by the action of an external force

2. The force acting on an object is equal to the time rate-of-change of the momentum

3. For every action there is an equal and opposite reaction

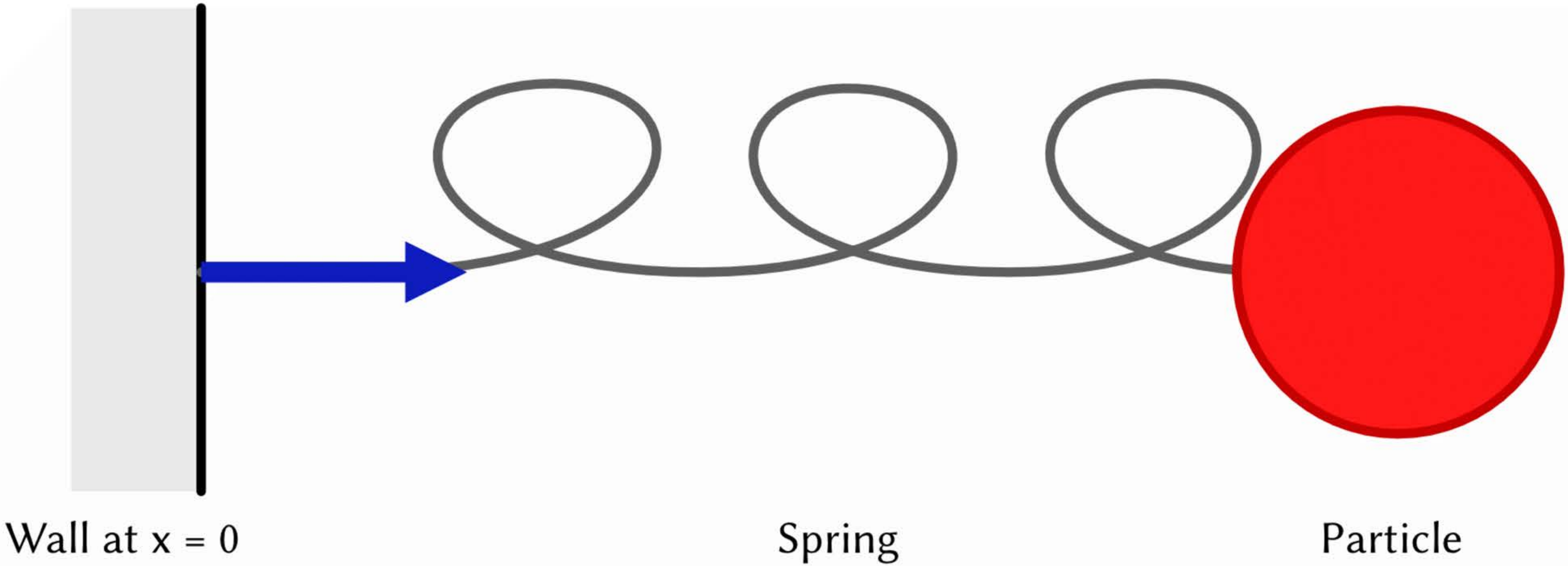
Newton’s Second Law

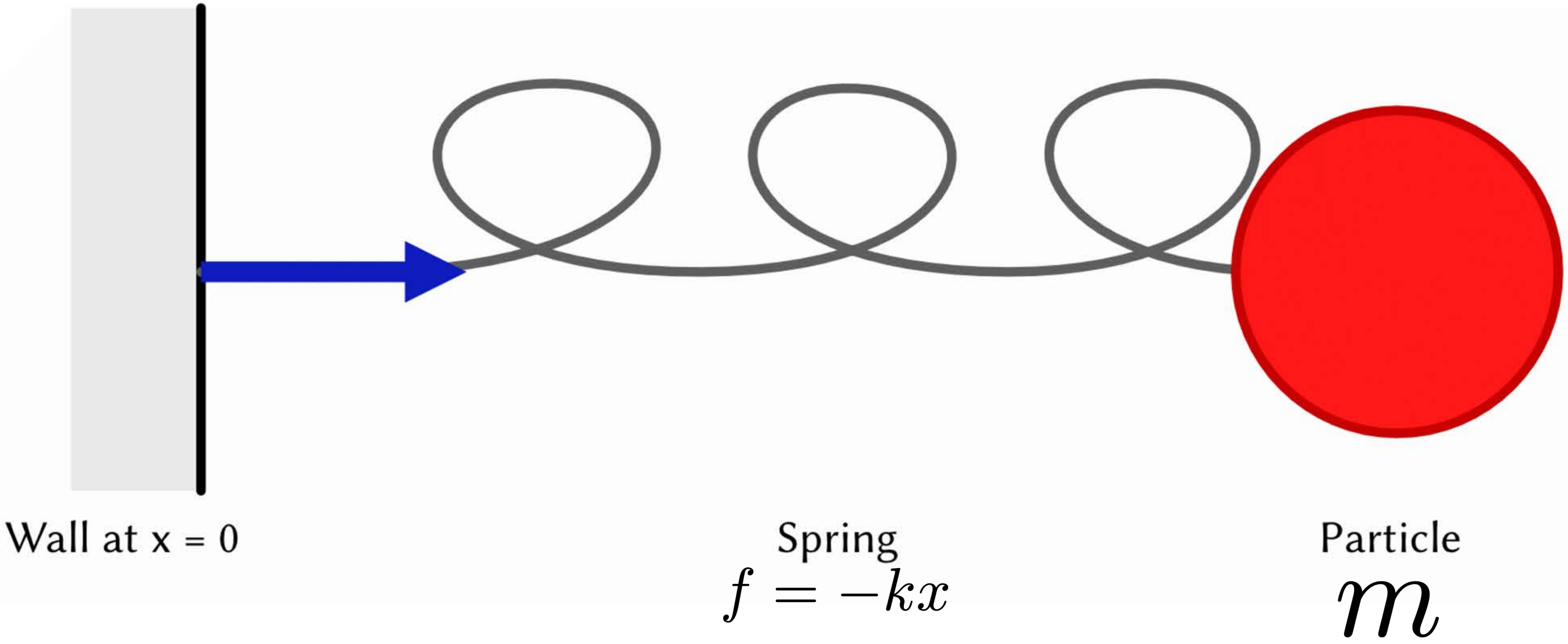
Acceleration

$$ma = f$$

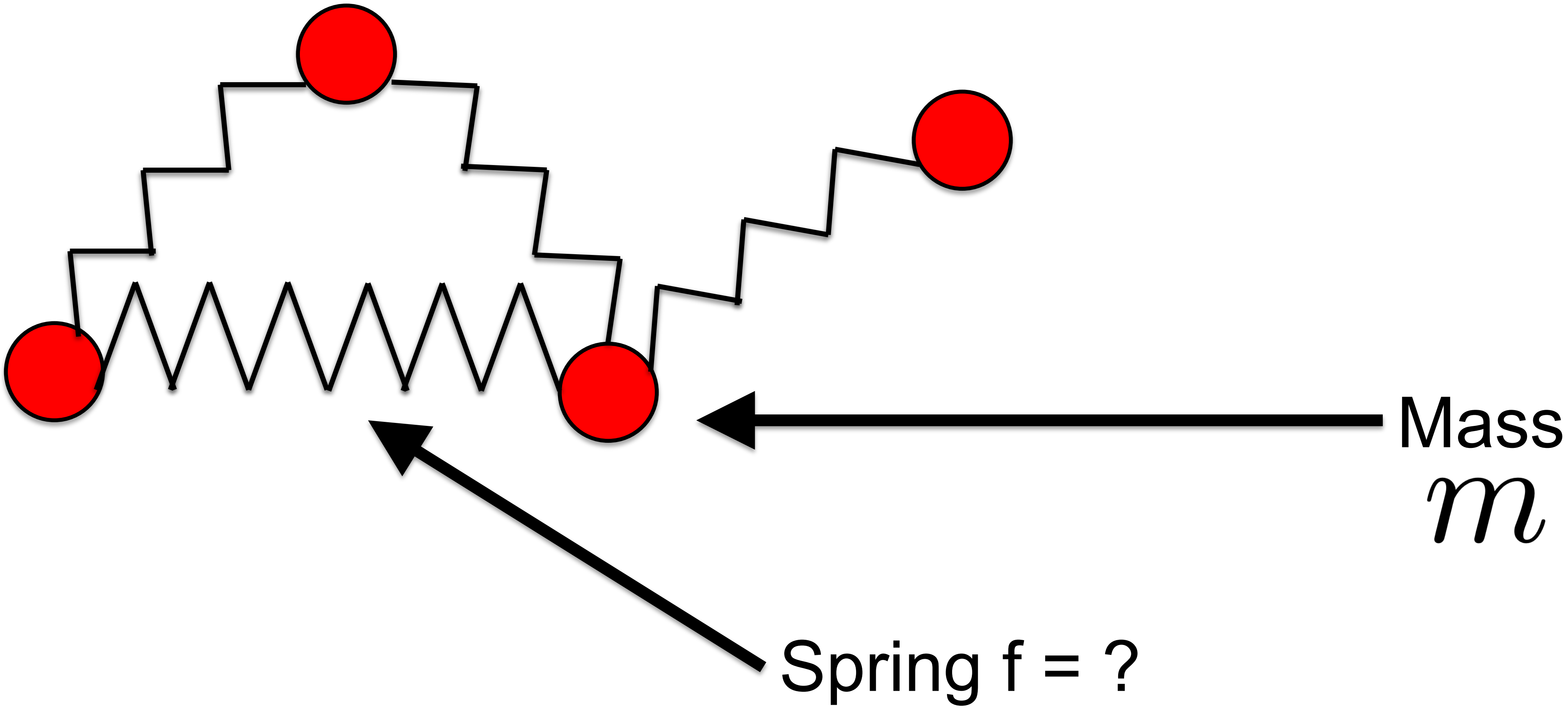
Mass

force

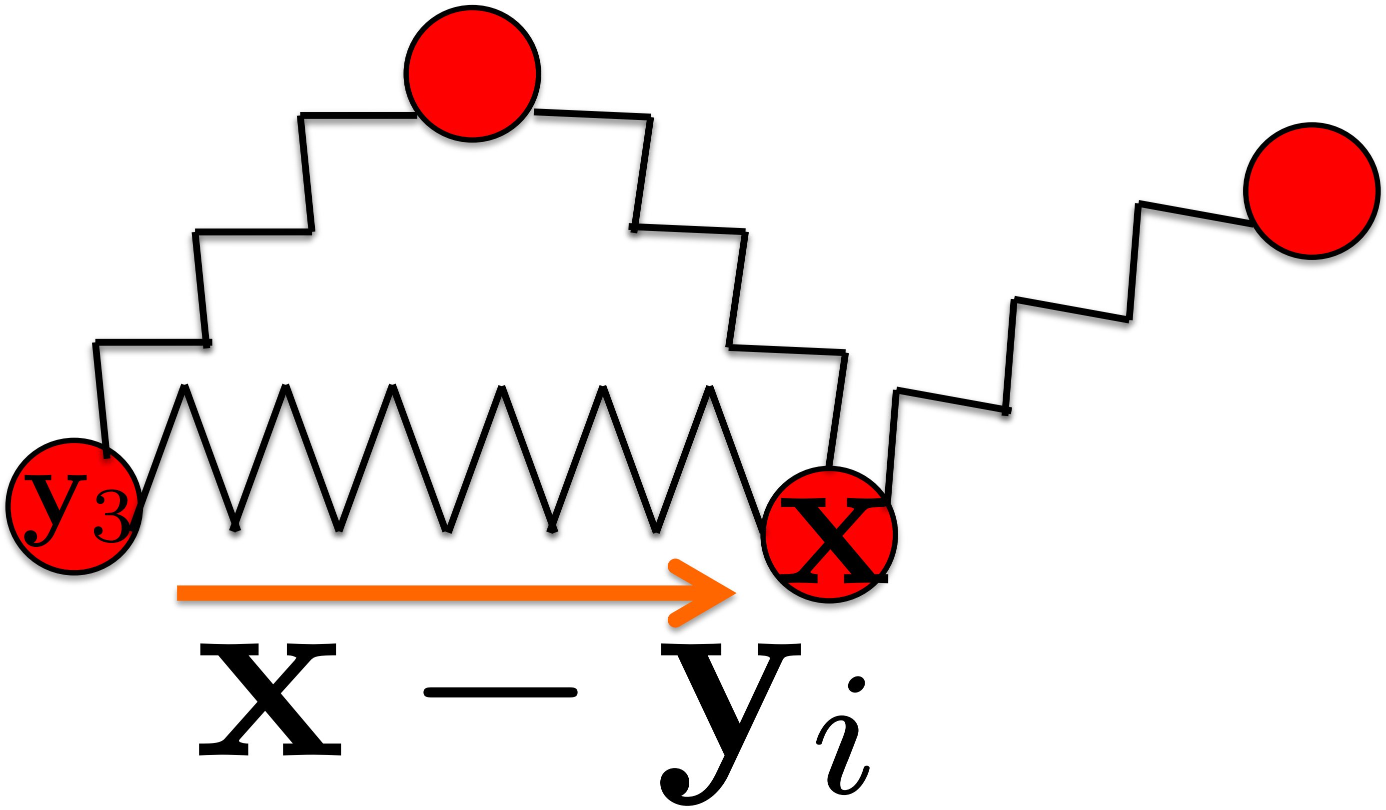

Wall at x = 0
Spring
Particle


Wall at x = 0
Spring
$f = -kx$
Particle
m

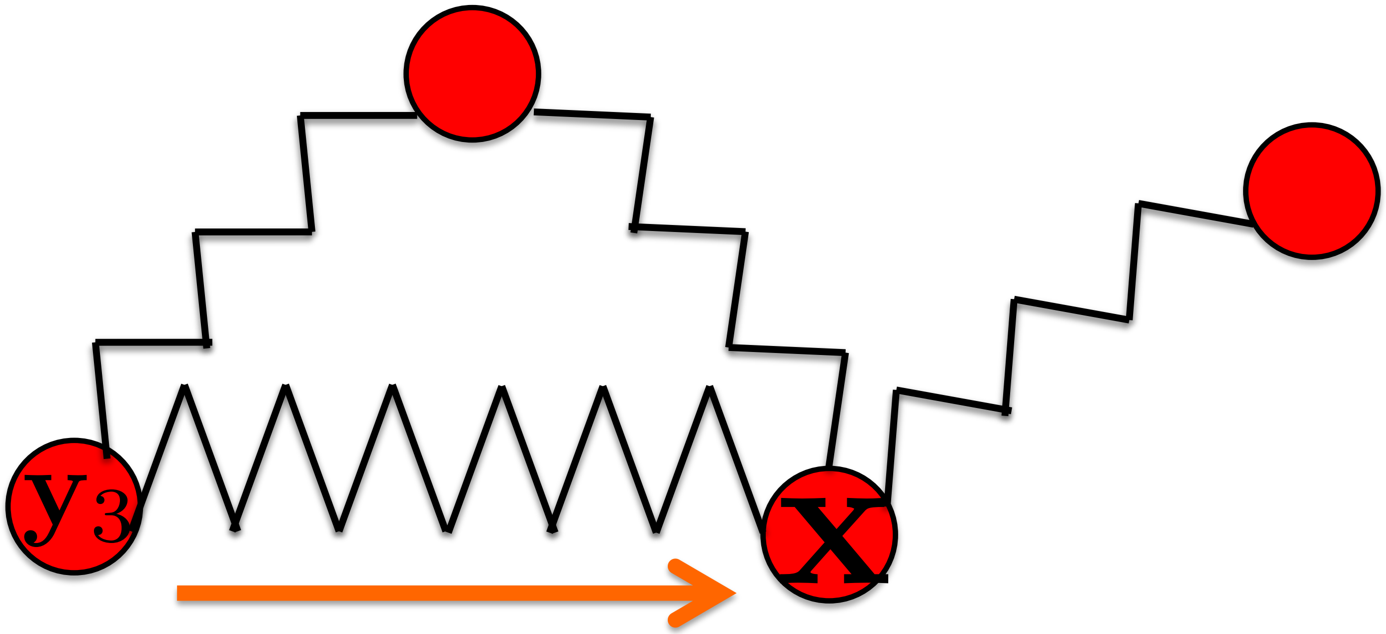
The Mass-Spring System



The Mass-Spring System



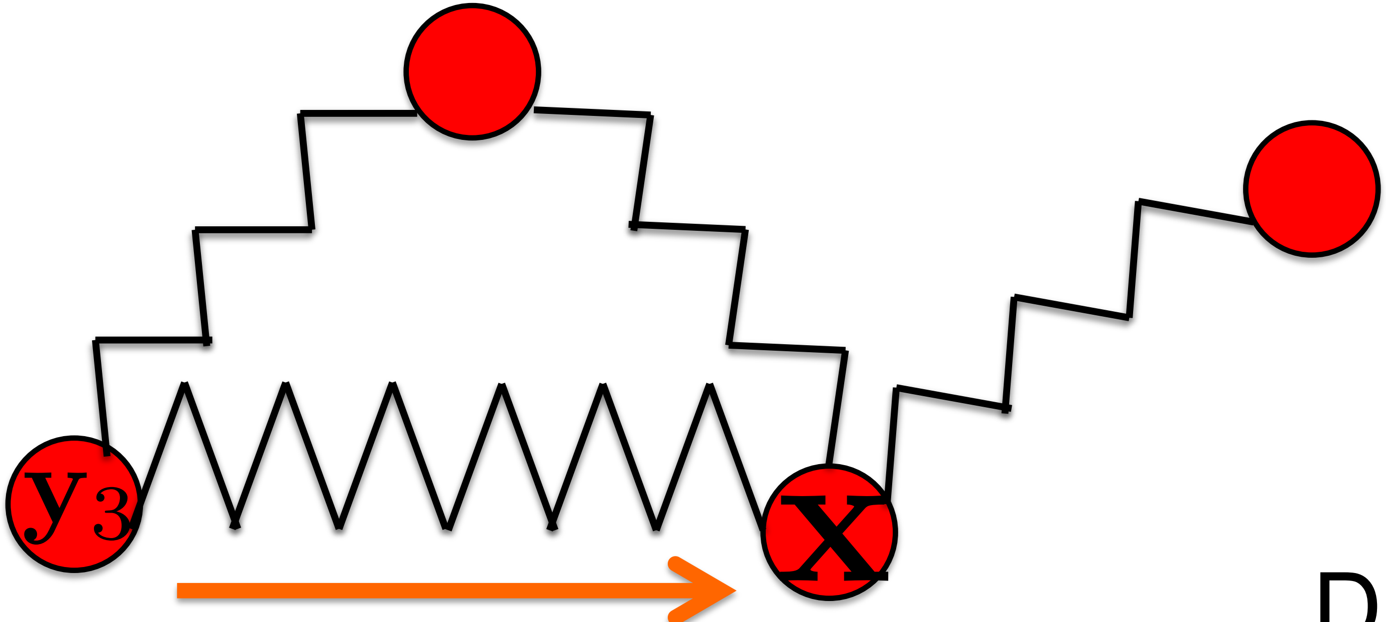
The Mass-Spring System



$$\mathbf{x} - \mathbf{y}_i$$

$$\mathbf{f}_x = -k\left(\|\mathbf{x} - \mathbf{y_i}\| - r_{ix}\right)\frac{\mathbf{x} - \mathbf{y_i}}{\|\mathbf{x} - \mathbf{y_i}\|}$$

The Mass-Spring System



$$\mathbf{x} - \mathbf{y}_i$$

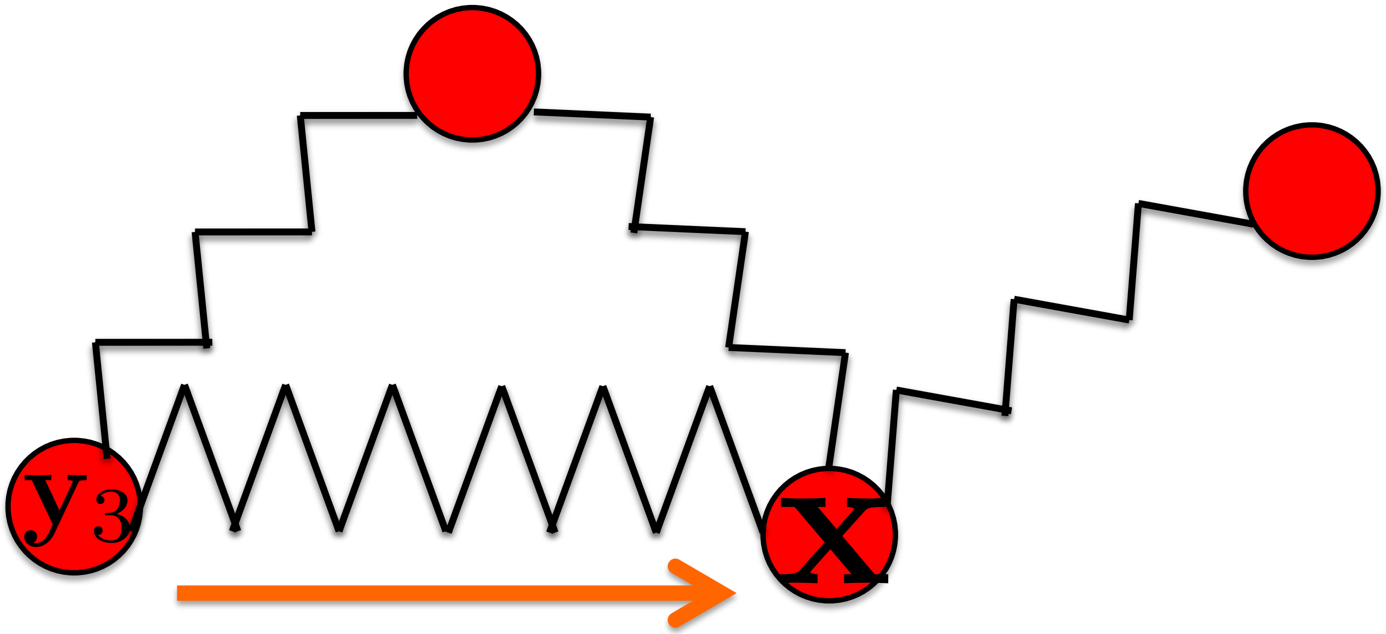
Direction of Spring

Rest Length

$$\mathbf{f}_x = -k\left(\|\mathbf{x} - \mathbf{y_i}\| - r_{ix}\right)\frac{\mathbf{x} - \mathbf{y_i}}{\|\mathbf{x} - \mathbf{y_i}\|}$$

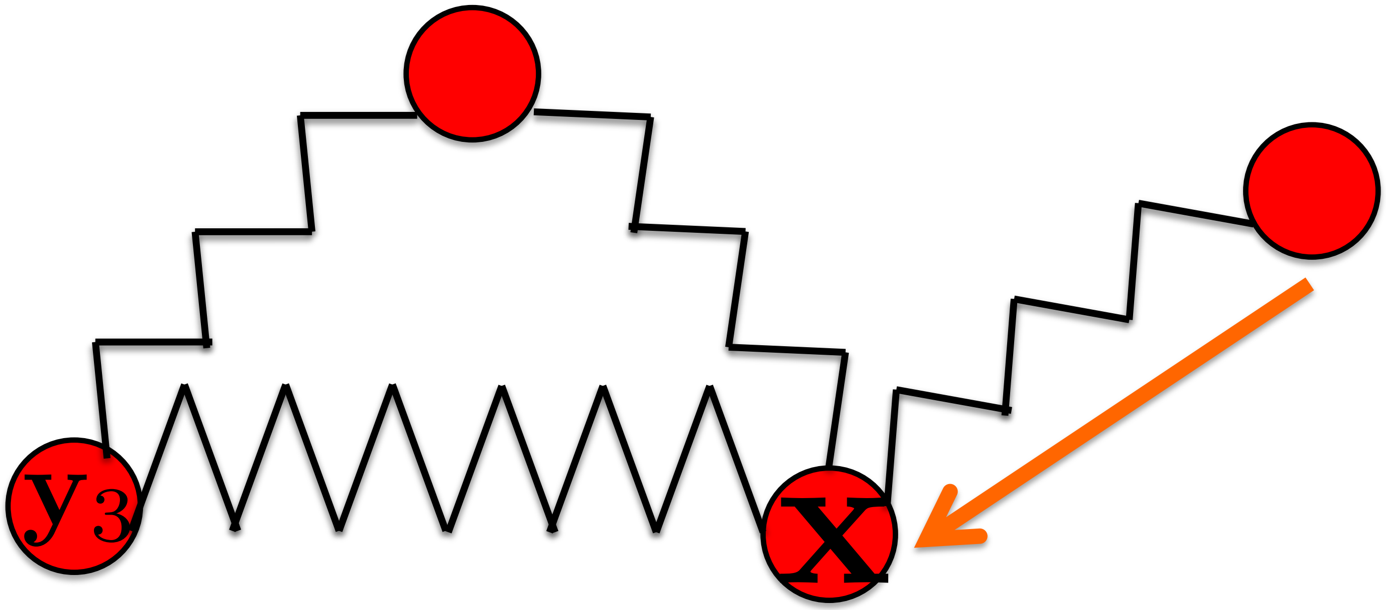
Change in Length

Newton's Second Law for Each Particle



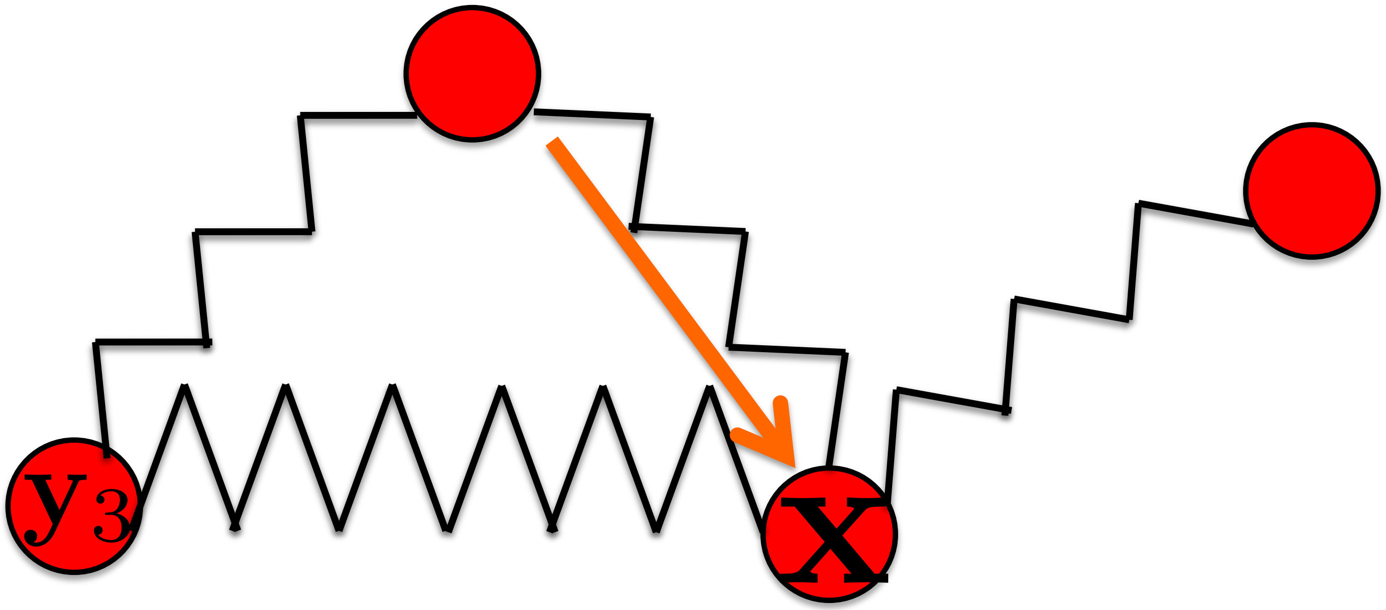
$$m_x \mathbf{a}_x = \sum_i \mathbf{f}_x \left(\mathbf{y}_i \right)$$

Newton's Second Law for Each Particle



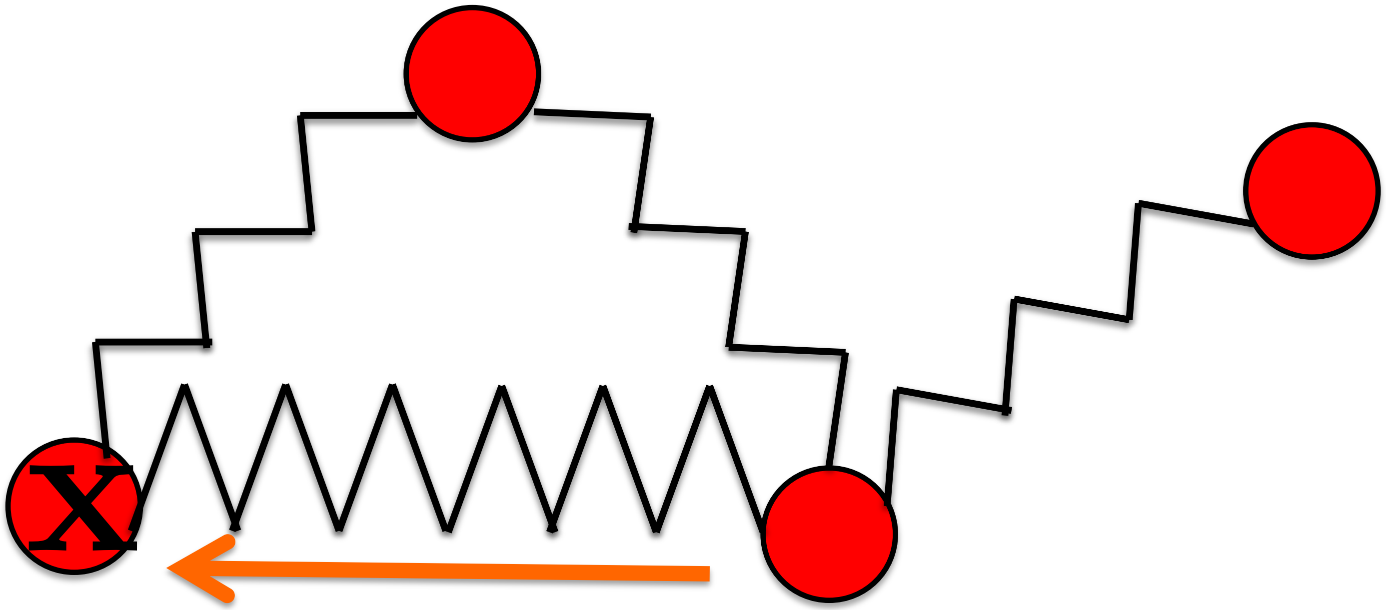
$$m_x \mathbf{a}_x = \sum_i \mathbf{f}_x\left(\mathbf{y}_i\right)$$

Newton's Second Law for Each Particle



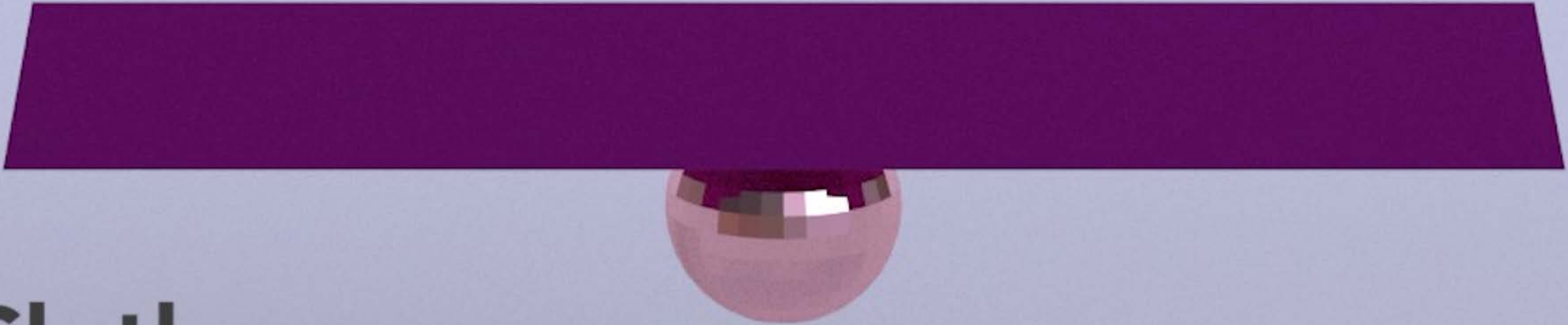
$$m_x \mathbf{a}_x = \sum_i \mathbf{f}_x(\mathbf{y}_i)$$

Newton's Second Law for Each Particle



$$m_x \mathbf{a}_x = \sum_i \mathbf{f}_x\left(\mathbf{y}_i\right)$$

One Equation for each particle
We will solve them all together



Cloth

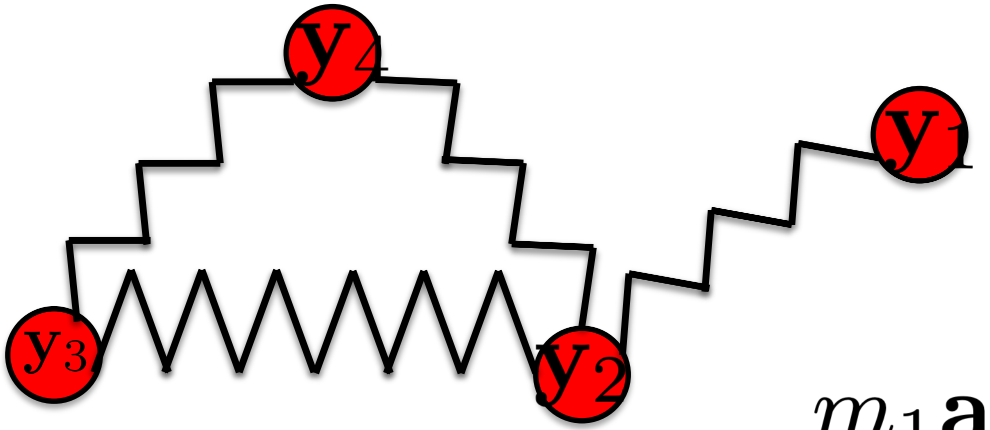
SIMIT GPU

15,630 Triangles

7,988 Verts

14 FPS

Newton's Second Law: System of Equations



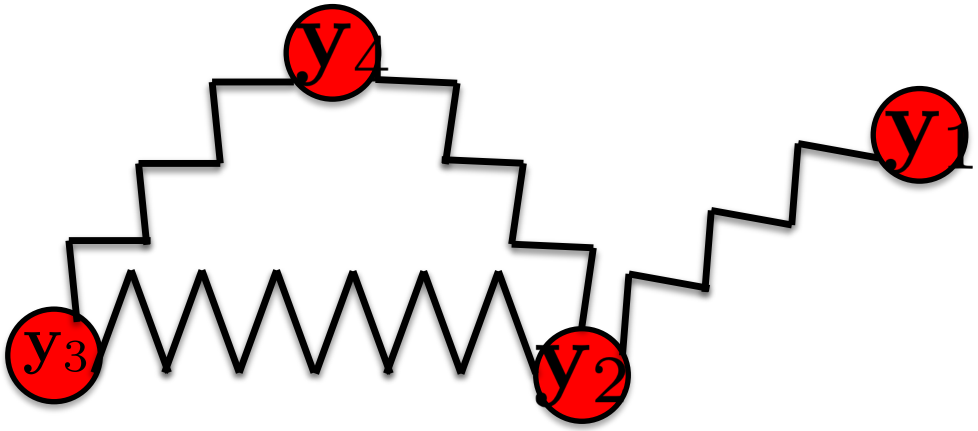
$$m_1\mathbf{a}_1 = \sum_i \mathbf{f}_1\left(\mathbf{y}_i\right)$$

$$m_2\mathbf{a}_2 = \sum_i \mathbf{f}_2\left(\mathbf{y}_i\right)$$

$$m_3\mathbf{a}_3 = \sum_i \mathbf{f}_3\left(\mathbf{y}_i\right)$$

$$m_4\mathbf{a}_4 = \sum_i \mathbf{f}_4\left(\mathbf{y}_i\right)$$

Newton's Second Law: System of Equations



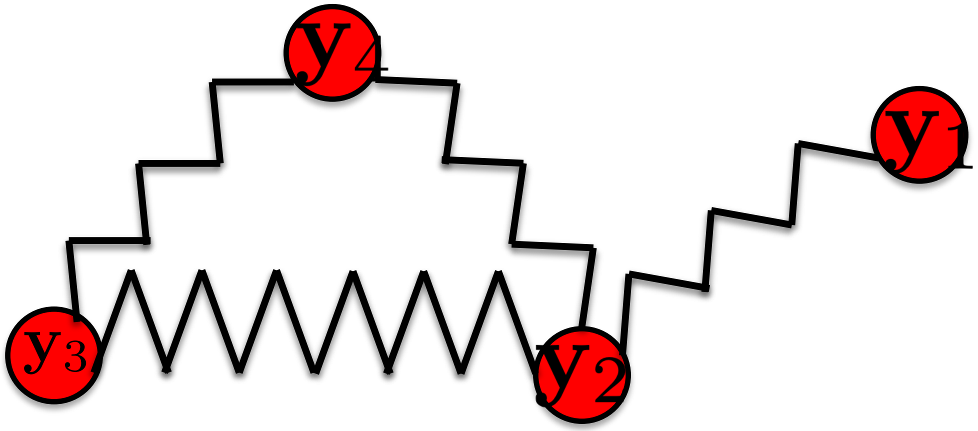
$$\begin{pmatrix} m_1 \cdot I & 0 & 0 & 0 \\ 0 & m_2 \cdot I & 0 & 0 \\ 0 & 0 & m_3 \cdot I & 0 \\ 0 & 0 & 0 & m_4 \cdot I \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \end{pmatrix}$$

Mass Matrix $\quad \mathbf{a}(t) \quad \mathbf{f}(t)$

Time Integration

Time Integration Converts Accelerations to Positions

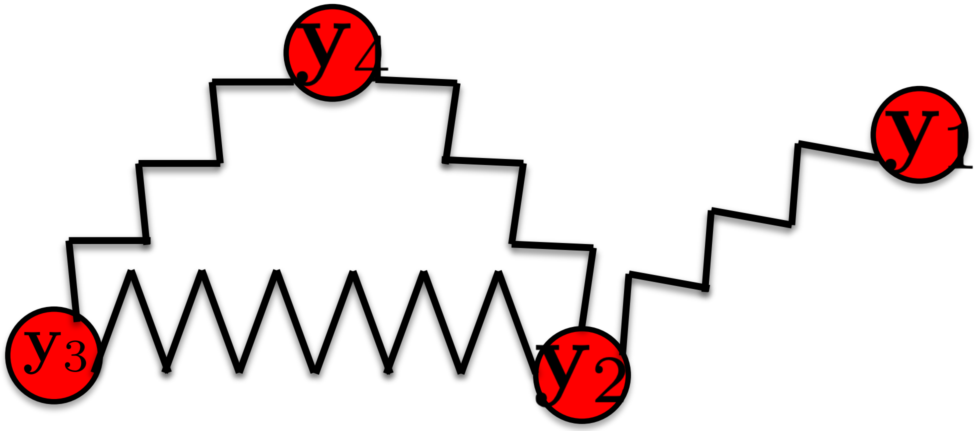
Time Integration



$$\begin{pmatrix} m_1 \cdot I & 0 & 0 & 0 \\ 0 & m_2 \cdot I & 0 & 0 \\ 0 & 0 & m_3 \cdot I & 0 \\ 0 & 0 & 0 & m_4 \cdot I \end{pmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \end{pmatrix}$$

Mass Matrix $\quad \mathbf{a}(t) \quad \mathbf{f}(t)$

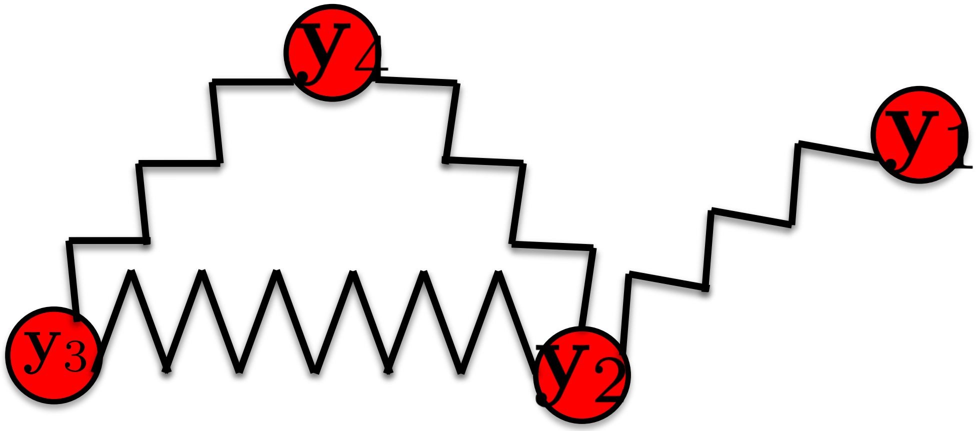
Time Integration



$$M\mathbf{a}(t) = \mathbf{f}(\mathbf{y}(t))$$

Mass Matrix

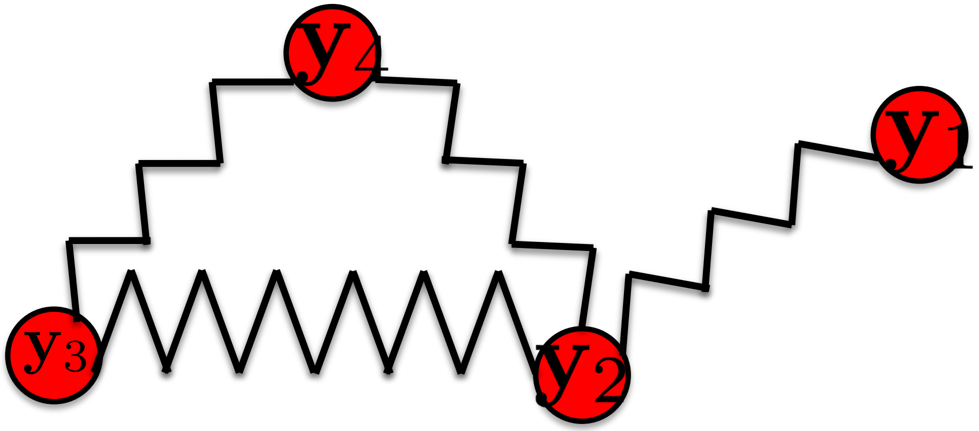
Time Integration



$$M\frac{d^2\mathbf{y}(t)}{dt^2} = \mathbf{f}(\mathbf{y}(t))$$

Use Finite Differences: $\frac{d^2\mathbf{y}(t)}{dt^2} \approx \frac{1}{\Delta t^2}\left(\mathbf{y}^{t+1} - 2\mathbf{y}^{t} + \mathbf{y}^{t-1}\right)$

Time Integration

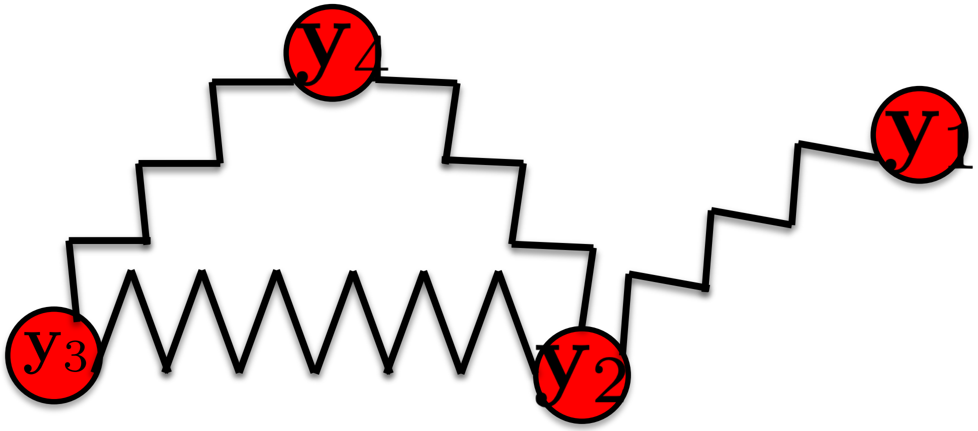


Need to Discretize

$$M\frac{d^2\mathbf{y}(t)}{dt^2} = \mathbf{f}(\mathbf{y}(t))$$

Use Finite Differences: $\frac{d^2\mathbf{y}(t)}{dt^2} \approx \frac{1}{\Delta t^2}\left(\mathbf{y}^{t+1} - 2\mathbf{y}^{t} + \mathbf{y}^{t-1}\right)$

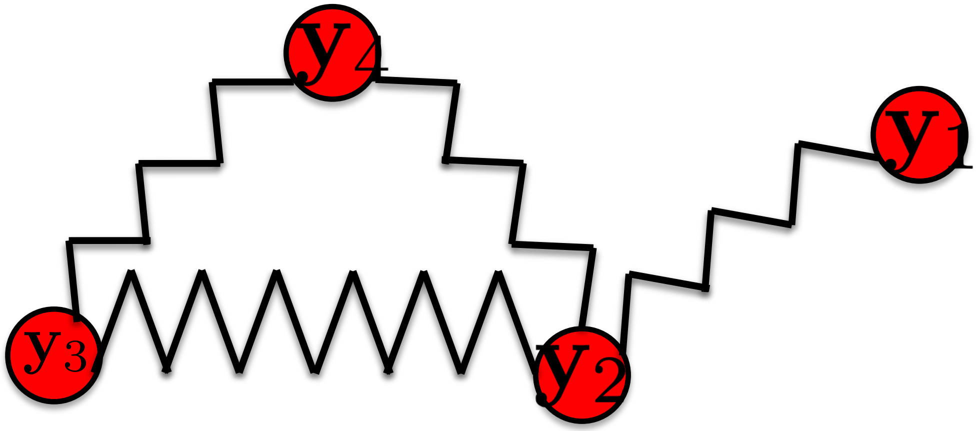
Implicit Time Integration



$$M\frac{d^2\mathbf{y}}{dt^2}(t) = \mathbf{f}\left(\mathbf{y}^{t+1}\right)$$

Use Finite Differences: $\frac{d^2\mathbf{y}(t)}{dt^2} \approx \frac{1}{\Delta t^2}\left(\mathbf{y}^{t+1} - 2\mathbf{y}^{t} + \mathbf{y}^{t-1}\right)$

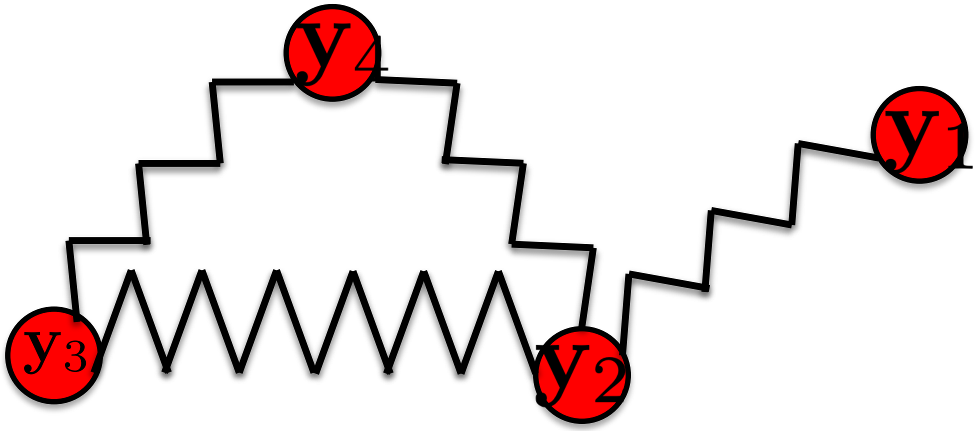
Implicit Time Integration



$$M\mathbf{y}^{t+1} = M\left(2\mathbf{y}^{t} - \mathbf{y}^{t-1}\right) + \Delta t^{2}\mathbf{f}\left(\mathbf{y}^{t+1}\right)$$

Goal: Solve for $\mathbf{y}^{t+1}$

Implicit Time Integration



$$M\mathbf{y}^{t+1} - M\left(2\mathbf{y}^{t} - \mathbf{y}^{t-1}\right) - \Delta t^2 \mathbf{f}\left(\mathbf{y}^{t+1}\right) = \mathbf{0}$$

How to find when some equation = 0?

Goal: Solve for $\mathbf{y}^{t+1}$

Implicit Integration as Optimization

If we can find a function E(*q)* such that:

$$\nabla_{\mathbf{q}} E\left(\mathbf{y}^{t+1}\right) = 0$$

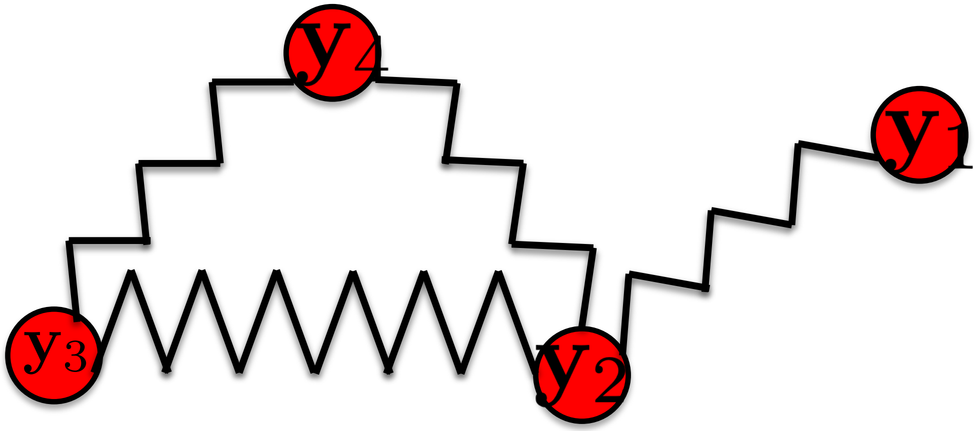
then, rather than solve

$$M\mathbf{y}^{t+1} - M\left(2\mathbf{y}^{t} - \mathbf{y}^{t-1}\right) - \Delta t^{2}\mathbf{f}\left(\mathbf{y}^{t+1}\right) = \mathbf{0}$$

we can solve

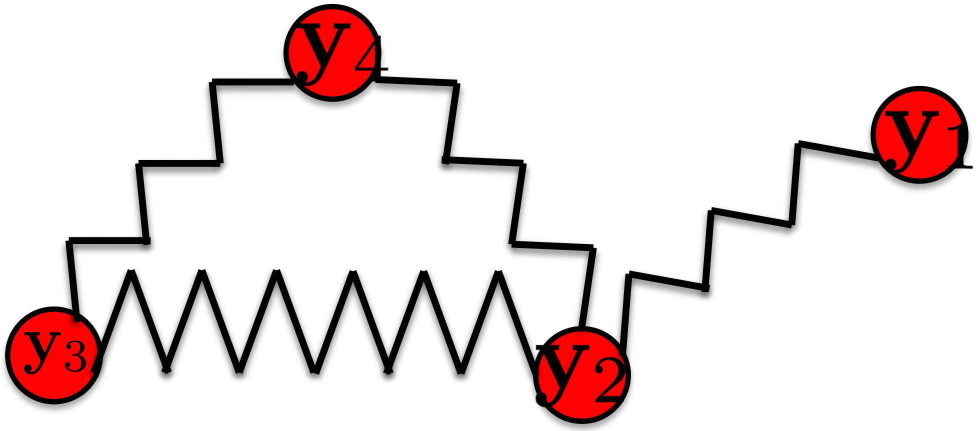
$$\mathbf{y}^{t+1} = \arg\min_{\mathbf{q}} E\left(\mathbf{q}\right)$$

Implicit Integration as Optimization



$$\underbrace{M\mathbf{y}^{t+1} - M\left(2\mathbf{y}^{t} - \mathbf{y}^{t-1}\right)}_{\text{find } \mathbf{E}_1\left(\mathbf{y}^{t+1}\right)} - \Delta t^2 \mathbf{f}\underbrace{\left(\mathbf{y}^{t+1}\right)}_{\text{find } \mathbf{E}_2\left(\mathbf{y}^{t+1}\right)} = \mathbf{0}$$

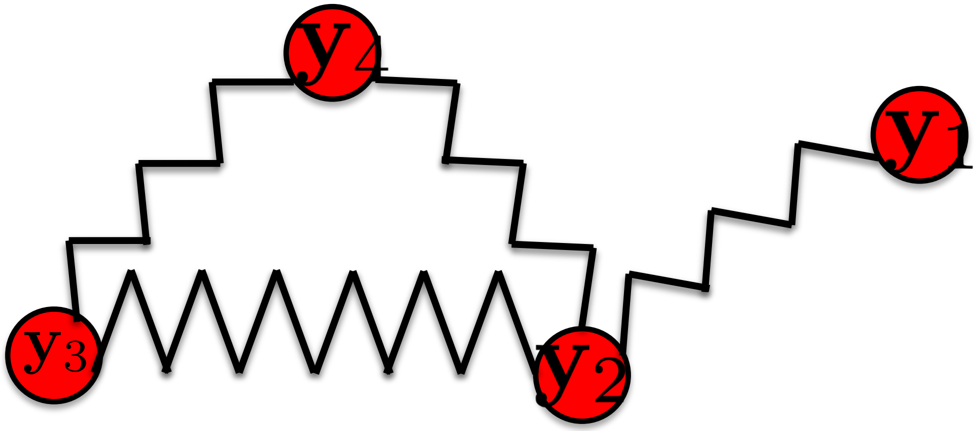
Implicit Integration as Optimization



$$\mathbf{E}_1\left(\mathbf{y}^{t+1}\right) = \frac{1}{2}\left(\mathbf{y}^{t+1}\right)^T M\mathbf{y}^{t+1} - \left(\mathbf{y}^{t+1}\right)^T M\mathbf{b}$$

$$\mathbf{b} = 2\mathbf{y}^t - \mathbf{y}^{t-1}$$

Implicit Integration as Optimization



$$\underbrace{M\mathbf{y}^{t+1} - M\left(2\mathbf{y}^{t} - \mathbf{y}^{t-1}\right)}_{\text{find } \mathbf{E}_1\left(\mathbf{y}^{t+1}\right)} - \Delta t^2 \mathbf{f}\underbrace{\left(\mathbf{y}^{t+1}\right)}_{\text{find } \mathbf{E}_2\left(\mathbf{y}^{t+1}\right)} = \mathbf{0}$$

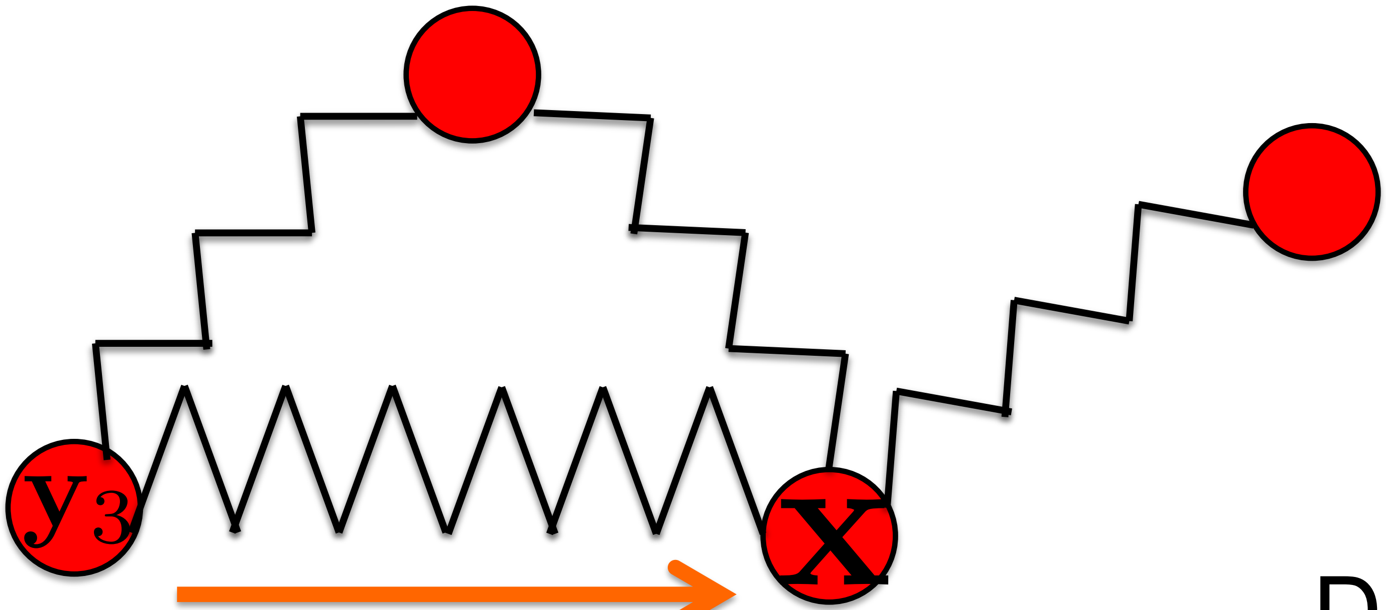
Potential energy

We are going to introduce a special type of energy called potential energy

If $\mathbf{E}_2$(q) is a potential energy then

$$\nabla_{\mathbf{q}} E_2 = -\mathbf{f}\left(\mathbf{q}\right)$$

Potential Energy of a Spring



$$\mathbf{x} - \mathbf{y}_i$$

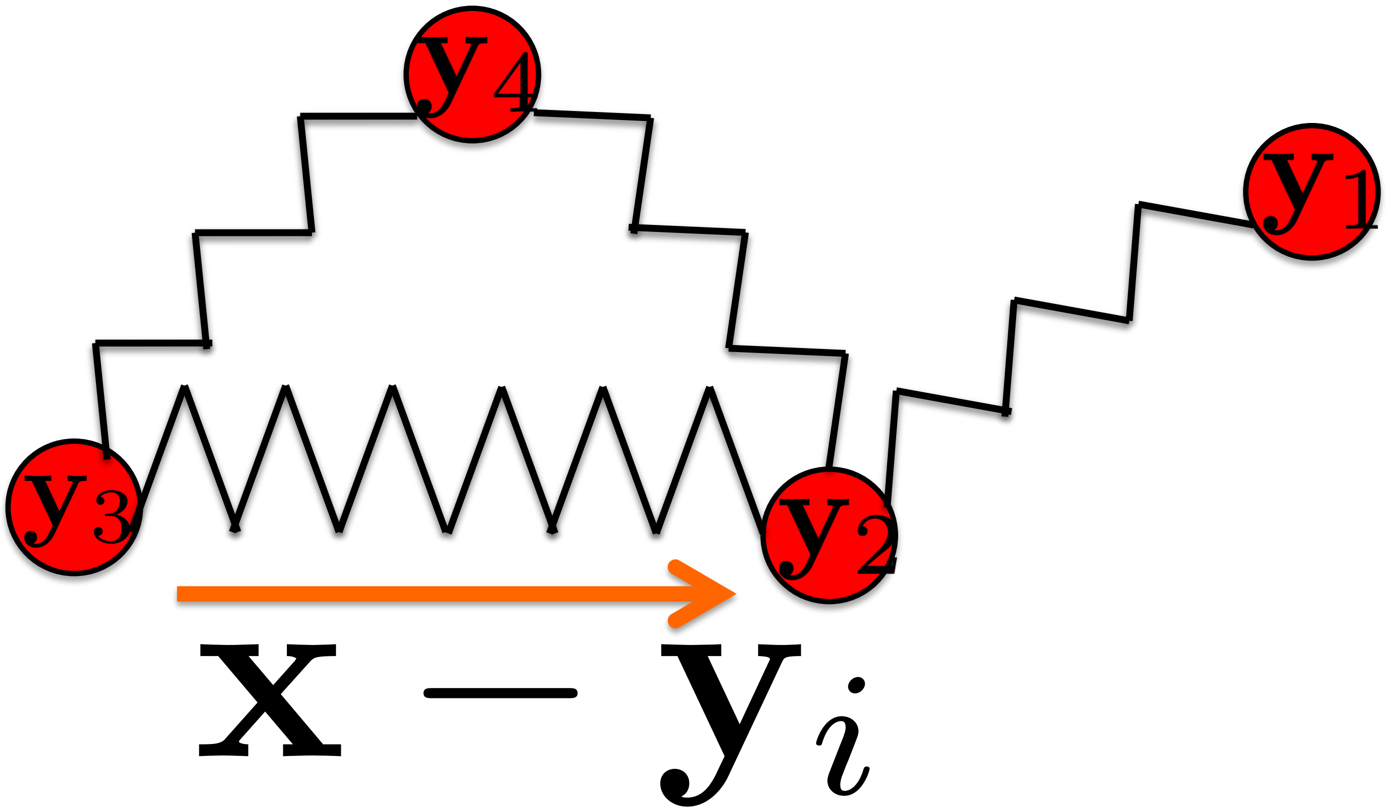
Rest Length

Direction of Spring

$$\mathbf{f}_x = -k\left(\|\mathbf{x} - \mathbf{y_i}\| - r_{ix}\right)\frac{\mathbf{x} - \mathbf{y_i}}{\|\mathbf{x} - \mathbf{y_i}\|}$$

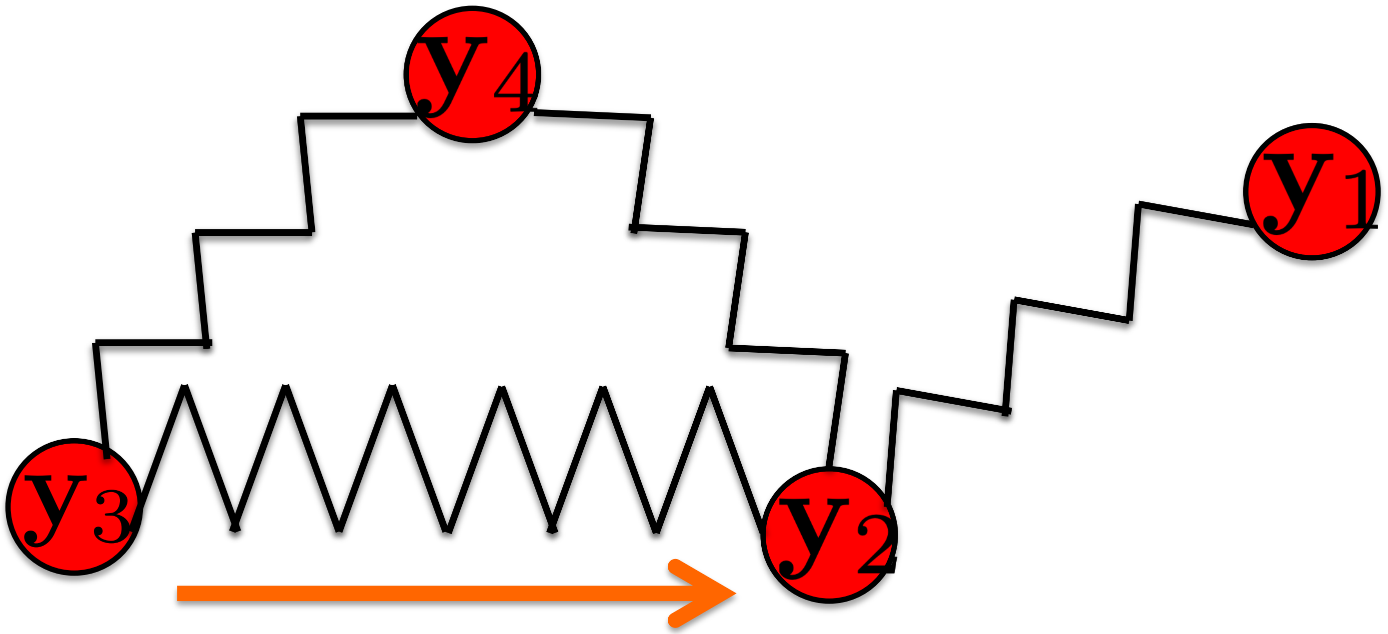
Change in Length

Potential Energy of a Spring



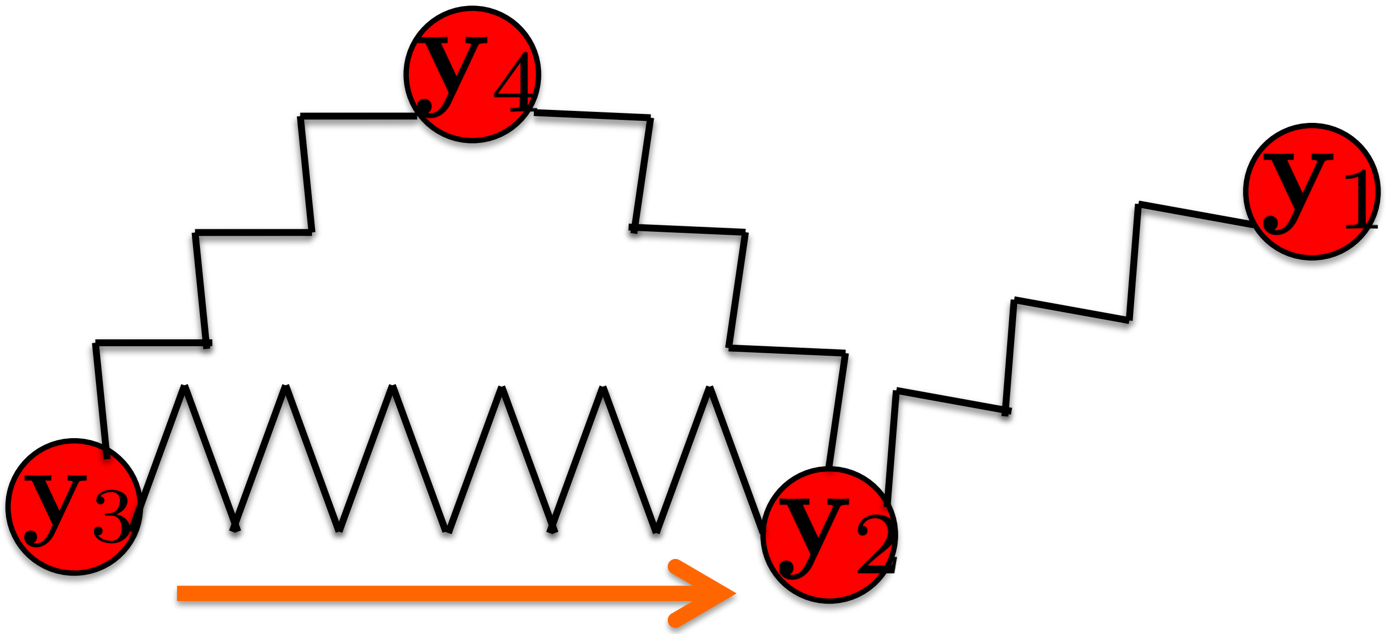
$$\mathbf{f}_{\mathbf{y}_j} = -k\left(\|\mathbf{y}_j - \mathbf{y_i}\| - r_{ij}\right)\frac{\mathbf{y}_j - \mathbf{y_i}}{\|\mathbf{y}_j - \mathbf{y_i}\|}$$

Potential Energy of a Spring



$$E_{ij} = \frac{k}{2}\left(\|\mathbf{y}_j - \mathbf{y_i}\| - r_{ij}\right)^2$$

Potential Energy for a Mass-Spring System



$$E_2 = \sum_{ij} E_{ij} = \sum_{ij} \frac{k}{2} \left(\|\mathbf{y}_i - \mathbf{y_j}\| - r_{ij} \right)^2$$

Implicit Integration as Optimization

If we can find a function E(*q)* such that:

$$\nabla_{\mathbf{q}} E\left(\mathbf{y}^{t+1}\right) = 0$$

then, rather than solve

$$M\mathbf{y}^{t+1} - M\left(2\mathbf{y}^{t} - \mathbf{y}^{t-1}\right) - \Delta t^{2}\mathbf{f}\left(\mathbf{y}^{t+1}\right) = \mathbf{0}$$

we can solve

$$\mathbf{y}^{t+1} = \arg\min_{\mathbf{q}} E_1(\mathbf{q}) + \Delta t E_2(q)$$

Local-Global Solvers for Mass-Spring Systems

WHILE Not done

For Each Spring

Local Optimization

Global Optimization

END

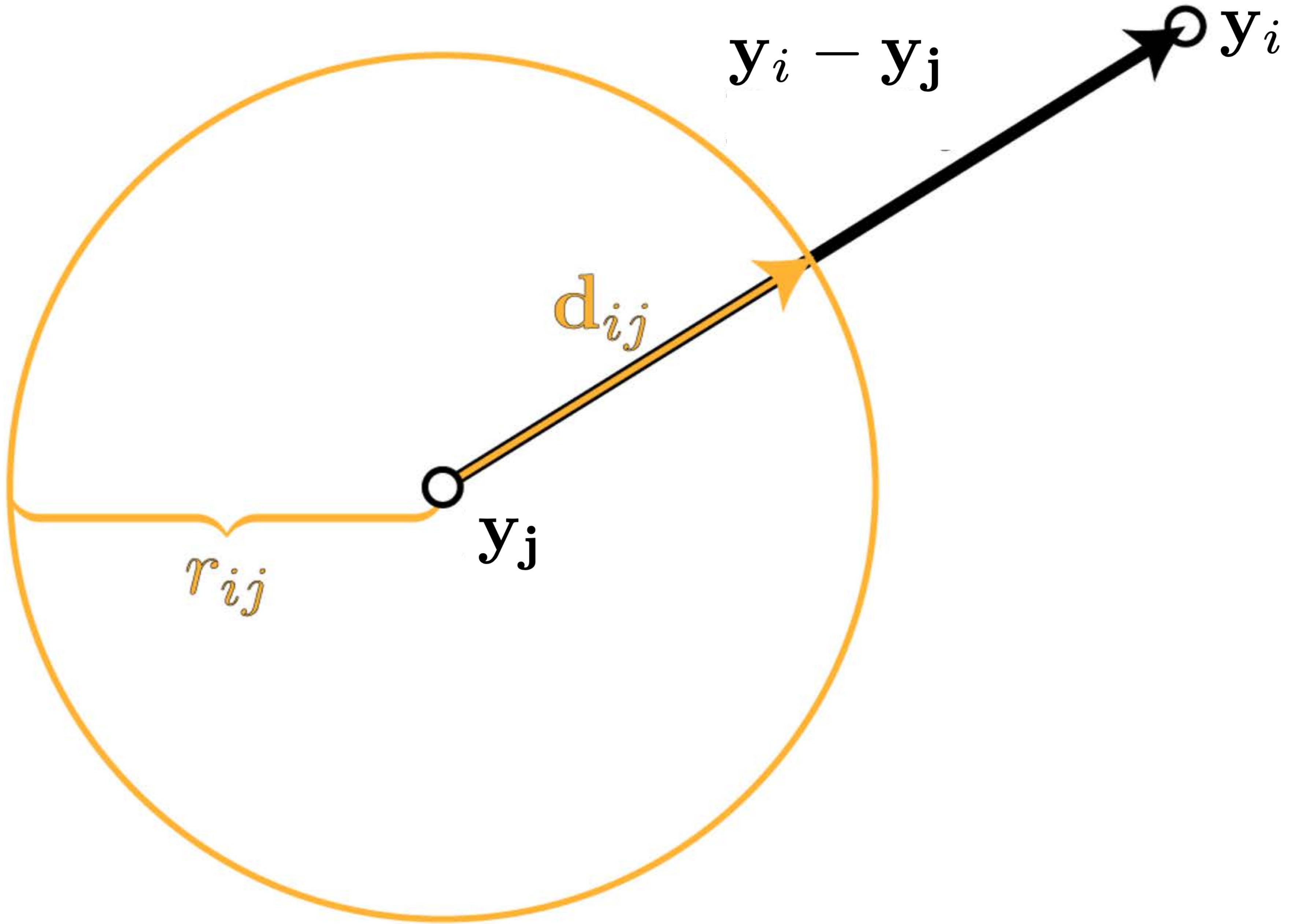
Now we can start defining these steps for mass-springs

Rethinking Potential Energy

$$E_{ij} = \frac{k}{2}\left(\|\mathbf{y}_j - \mathbf{y_i}\| - r_{ij}\right)^2$$ is equivalent to

$$E_{ij} = \arg\min_{\mathbf{d}_{ij}, |\mathbf{d}_{ij}| = r_{ij}} \frac{k}{2}\|\mathbf{y}_i - \mathbf{y_j} - \mathbf{d}_{ij}\|^2$$

Given $\mathbf{y}_i - \mathbf{y_j}$ we can quickly find $\mathbf{d}_{ij}$



$\mathbf{y}_i$
$\mathbf{y}_i - \mathbf{y_j}$
$\mathbf{d}_{ij}$
$\mathbf{y_j}$
r_{ij}

Why Do This ?

$$E_{ij} = \frac{k}{2}\left(\|\mathbf{y}_j - \mathbf{y_i}\| - r_{ij}\right)^2$$ is equivalent to

$$E_{ij} = \arg\min_{\mathbf{d}_{ij}, |\mathbf{d}_{ij}| = r_{ij}} \frac{k}{2}\|\mathbf{y}_i - \mathbf{y_j} - \mathbf{d}_{ij}\|^2$$

We can expand a bit more …

$$E_{ij} = \arg\min_{\mathbf{d}_{ij}, |\mathbf{d}_{ij}| = r_{ij}} \frac{k}{2}\left(\|\mathbf{y}_i - \mathbf{y_j}\|^2 - (\mathbf{y}_i - \mathbf{y_j})^T\mathbf{d}_{ij} + \mathbf{d}_{ij}^T\mathbf{d}_{ij}\right)$$

Aside from the constraints, this is a nice quadratic energy

Local-Global Solvers for Mass-Spring Systems

$$\mathbf{E}_1\left(\mathbf{y}^{t+1}\right) = \frac{1}{2}\left(\mathbf{y}^{t+1}\right)^T M\mathbf{y}^{t+1} - \left(\mathbf{y}^{t+1}\right)^T M\mathbf{b}$$

$$E_2 = \sum_{ij}\frac{k}{2}\left(\|\mathbf{y}_i - \mathbf{y_j}\|^2 - 2(\mathbf{y}_i - \mathbf{y}_j)^T\mathbf{d}_{ij} + \mathbf{d}_{ij}^T\mathbf{d}_{ij}\right)$$

Both energies are quadratic now. This will let us build a fast algorithm

We will do this using block coordinate descent. First optimize over one set of variables (the d’s) then the second set (the y’s) …. Rince and repeat!

Local-Global Solvers for Mass-Spring Systems

$$\mathbf{y}^{t+1} = \arg \min_{\mathbf{y}, \mathbf{d}_{ij}, \|\mathbf{d}_{ij}\| = r_{ij}} E_1\left(\mathbf{y}\right) + \Delta t E_2(\mathbf{y}, \mathbf{d}_{ij})$$

For **step 1** we will hold y constant and minimize with respect to d and its constraints

Note that this recovers the problem

$$\arg\min_{\mathbf{d}_{ij}, |\mathbf{d}_{ij}| = r_{ij}} \sum_{ij} \frac{k}{2} \underbrace{\left(\|\mathbf{y}_i - \mathbf{y}_\mathbf{j}\|^2 - 2(\mathbf{y}_i - \mathbf{y}_j)^T \mathbf{d}_{ij} + \mathbf{d}_{ij}^T \mathbf{d}_{ij} \right)}_{\text{Each d acts on a spring independently!}}$$

The Local Step

This gives us our local step:

$$\arg\min_{\mathbf{d}_{ij},|\mathbf{d}_{ij}|=r_{ij}} \sum_{ij} \frac{k}{2}\|\mathbf{y}_i - \mathbf{y_j}\|^2 - 2(\mathbf{y}_i - \mathbf{y}_j)^T \mathbf{d}_{ij} + \mathbf{d}_{ij}^T \mathbf{d}_{ij}$$

Can be minimized by visiting each spring and finding d such that

$$E_{ij} = \arg\min_{\mathbf{d}_{ij},|\mathbf{d}_{ij}|=r_{ij}} \underbrace{\frac{k}{2}\|\mathbf{y}_i - \mathbf{y_j} - \mathbf{d}_{ij}\|^2}_{\text{No sum anymore!}}$$

Local-Global Solvers for Mass-Spring Systems

$$\mathbf{E}_1\left(\mathbf{y}^{t+1}\right) = \frac{1}{2}\left(\mathbf{y}^{t+1}\right)^T M\mathbf{y}^{t+1} - \left(\mathbf{y}^{t+1}\right)^T M\mathbf{b}$$

$$E_2 = \sum_{ij} \frac{k}{2}\left(\|\mathbf{y}_i - \mathbf{y_j}\|^2 - 2(\mathbf{y}_i - \mathbf{y}_j)^T\mathbf{d}_{ij} + \mathbf{d}_{ij}^T\mathbf{d}_{ij}\right)$$

Both energies are quadratic now. This will let us build a fast algorithm

We will do this using block coordinate descent. First optimize over one set of variables (the d's) then the second set (the y's) …. Rince and repeat!

The Global Step

Minimizing wrt to y requires us to find

$$\mathbf{y}^{t+1} \text{ s.t. } \nabla_{\mathbf{y}}(E_1(\mathbf{y}) + \Delta t E_2(\mathbf{y}, \mathbf{d}_{ij})) = \mathbf{0}$$

Recall $\mathbf{E}_1(\mathbf{y}) = \frac{1}{2}\mathbf{y}^T M \mathbf{y} - \mathbf{y}^T M \mathbf{b}$

$$\nabla \mathbf{E}_1 = M\mathbf{y} - M\mathbf{b}$$

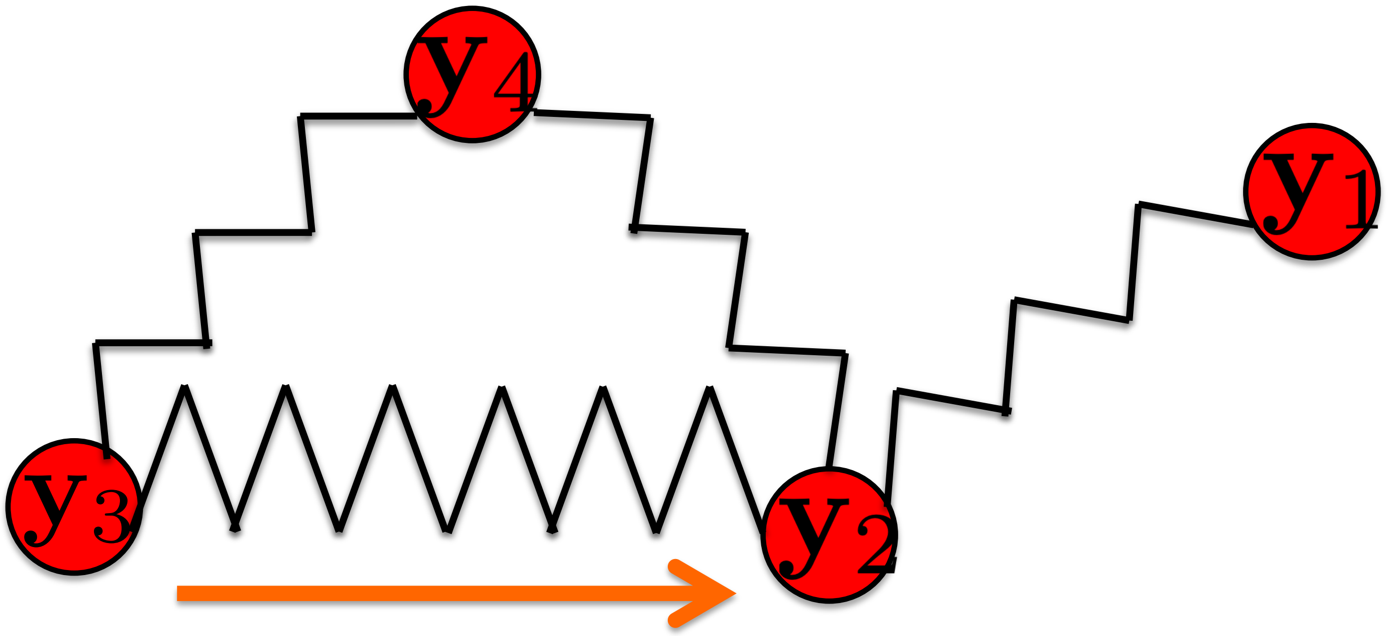
Not so bad …

The Global Step

$$E_2 = \sum_{ij} \frac{k}{2}\left(\|\mathbf{y}_i - \mathbf{y}_\mathbf{j}\|^2 - 2(\mathbf{y}_i - \mathbf{y}_j)^T \mathbf{d}_{ij} + \mathbf{d}_{ij}^T \mathbf{d}_{ij}\right)$$

This is a little trickier.

Global Step



$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \mathbf{y}_4 \end{pmatrix}$$

$$\Delta \mathbf{y} = \underbrace{\begin{pmatrix} I & -I & 0 & 0 \\ 0 & I & -I & 0 \\ 0 & I & 0 & -I \\ 0 & -I & I & -I \end{pmatrix}}_{\color{red}{G}} \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \mathbf{y}_4 \end{pmatrix}$$

Each row is a spring

Global Step

Using this we can rewrite the second energy as

$$E_2 = \frac{k}{2}\left(\mathbf{y}G^T G\mathbf{y} - 2\mathbf{y}^T G^T\mathbf{d} + \mathbf{d}^T\mathbf{d}\right) \qquad \mathbf{d} = \begin{pmatrix} \mathbf{d}_{12} \\ \mathbf{d}_{23} \\ \mathbf{d}_{24} \\ \mathbf{d}_{34} \end{pmatrix}$$

So the gradient becomes

$$\nabla E_2 = kG^T G\mathbf{y} - k\mathbf{G}^T\mathbf{d}$$

And the total global step finds y so that

$$\nabla(E_1 + \Delta t^2 E_2) = (M + \Delta t^2 kG^T G)\mathbf{y} - (M\mathbf{b} - \Delta t^2 k\mathbf{G}^T\mathbf{d}) = 0$$

Global Step

And the total global step finds y so that

$$\nabla(E_1 + \Delta t^2 E_2) = (M + \Delta t^2 k G^T G)\mathbf{y} - (M\mathbf{b} + \Delta t^2 k \mathbf{G}^T \mathbf{d}) = 0$$

Or

$$(M + \Delta t^2 k G^T G)\mathbf{y} = (M\mathbf{b} + \Delta t^2 k \mathbf{G}^T \mathbf{d})$$

You can solve this linear system using the Cholesky Solver in Eigen

Local-Global Solvers for Mass-Spring Systems

WHILE Not done

//Local Steps

For Each Spring

$$E_{ij} = \arg\min_{\mathbf{d}_{ij}, |\mathbf{d}_{ij}|=r_{ij}} \frac{k}{2} \|\mathbf{y}_i - \mathbf{y_j} - \mathbf{d}_{ij}\|^2$$

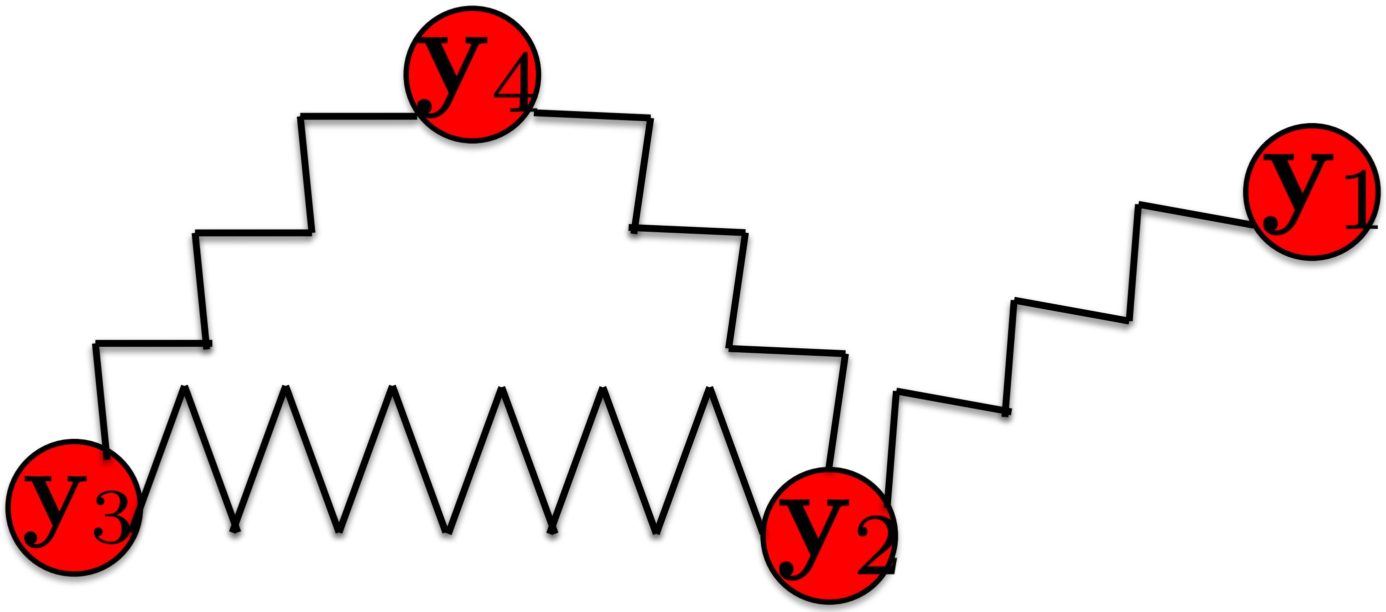
//Global Step

Solve $(M + \Delta t^2 k G^T G)\mathbf{y} = (M\mathbf{b} + \Delta t^2 k \mathbf{G}^T \mathbf{d})$

END


libigl viewer

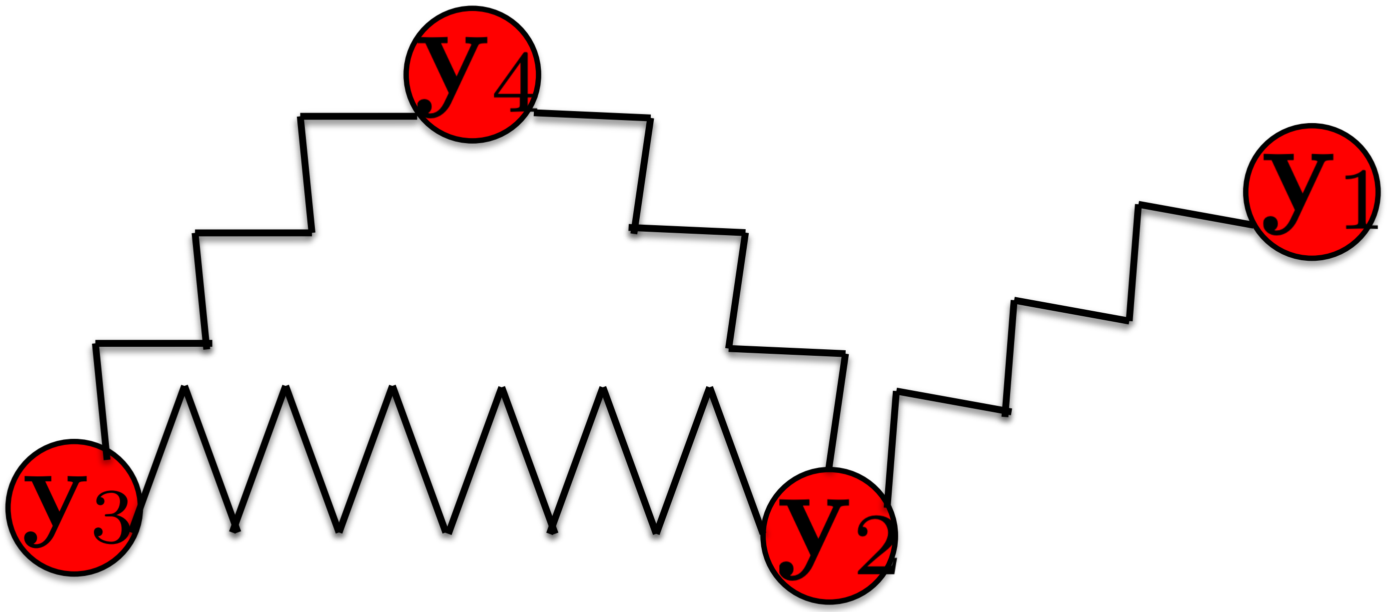
Fixed Points



$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \mathbf{y}_4 \end{pmatrix}$$

Let's say we never want y3 to move

Fixed Points

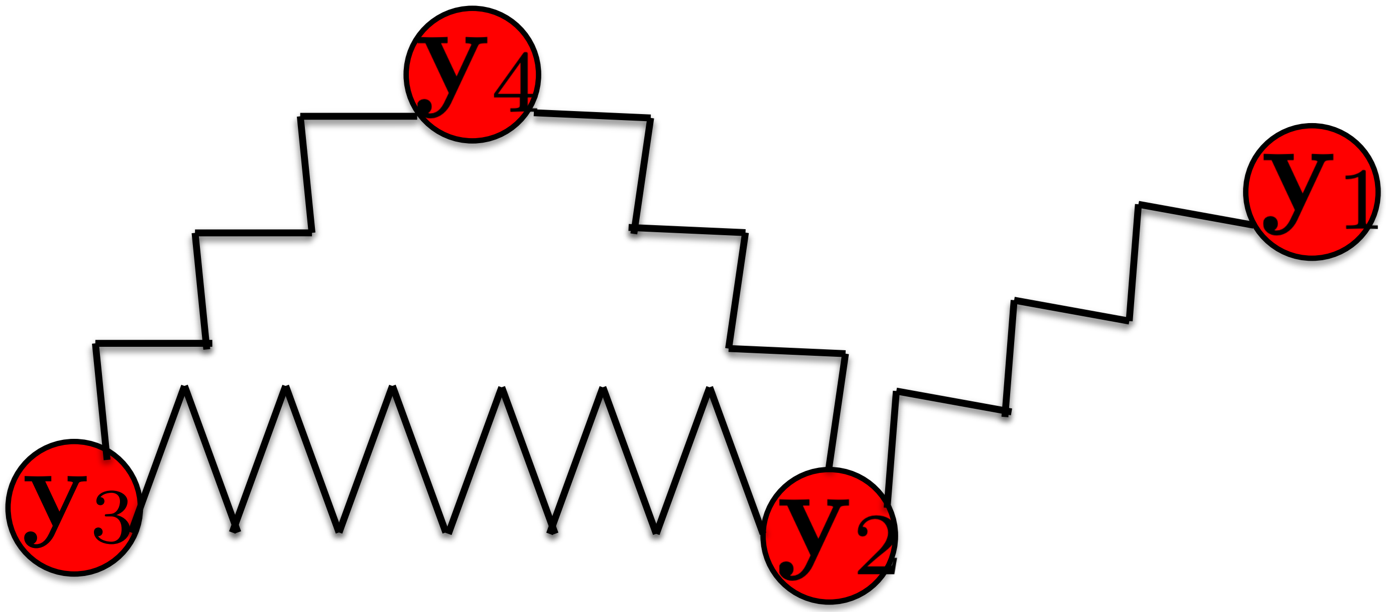


$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \mathbf{y}_4 \end{pmatrix}$$

Let’s say we never want y3 to move

i.e $\mathbf{y}_3 = \mathbf{c}$ forever and always

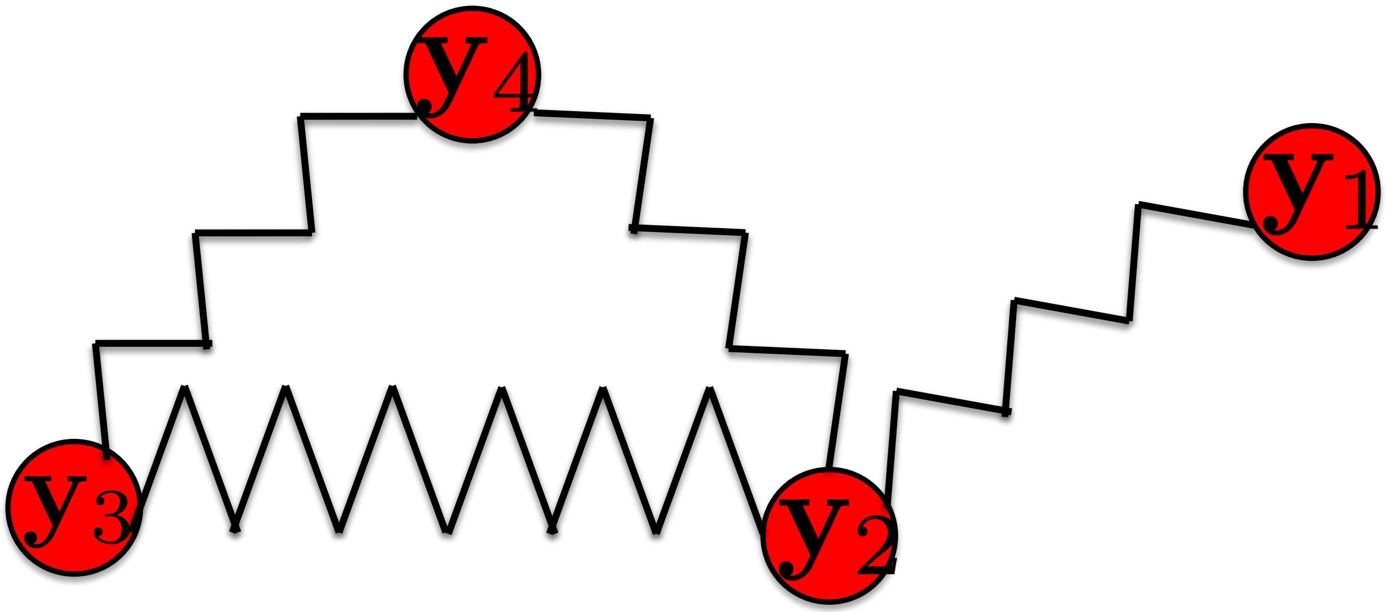
Fixed Points via Projection



$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \mathbf{y}_4 \end{pmatrix}$$

$$\mathbf{y} = \underbrace{\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}}_{\mathrm{P}} \underbrace{\begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_4 \end{pmatrix}}_{\tilde{\mathrm{y}}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ \mathbf{c} \\ 0 \end{pmatrix}}_{\mathrm{c}}$$

Fixed Points via Projection

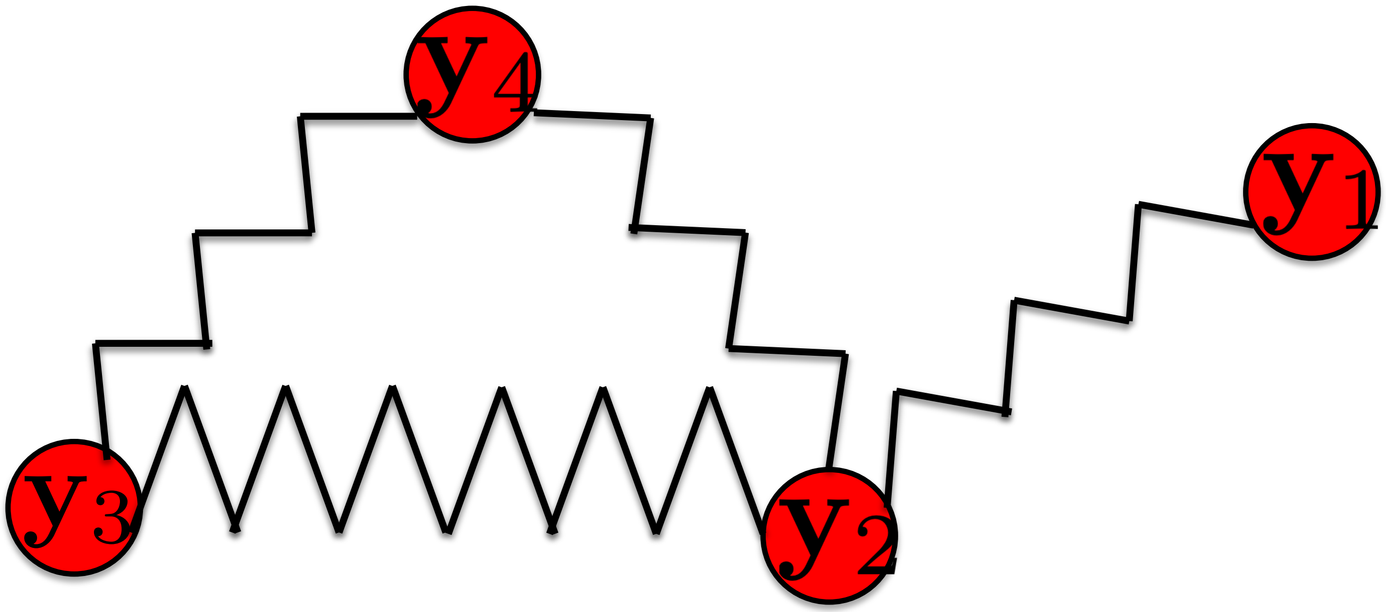


$$(M + \Delta t^2 k G^T G)\mathbf{y} = (M\mathbf{b} - \Delta t^2 k \mathbf{G}^T \mathbf{d})$$

A

f

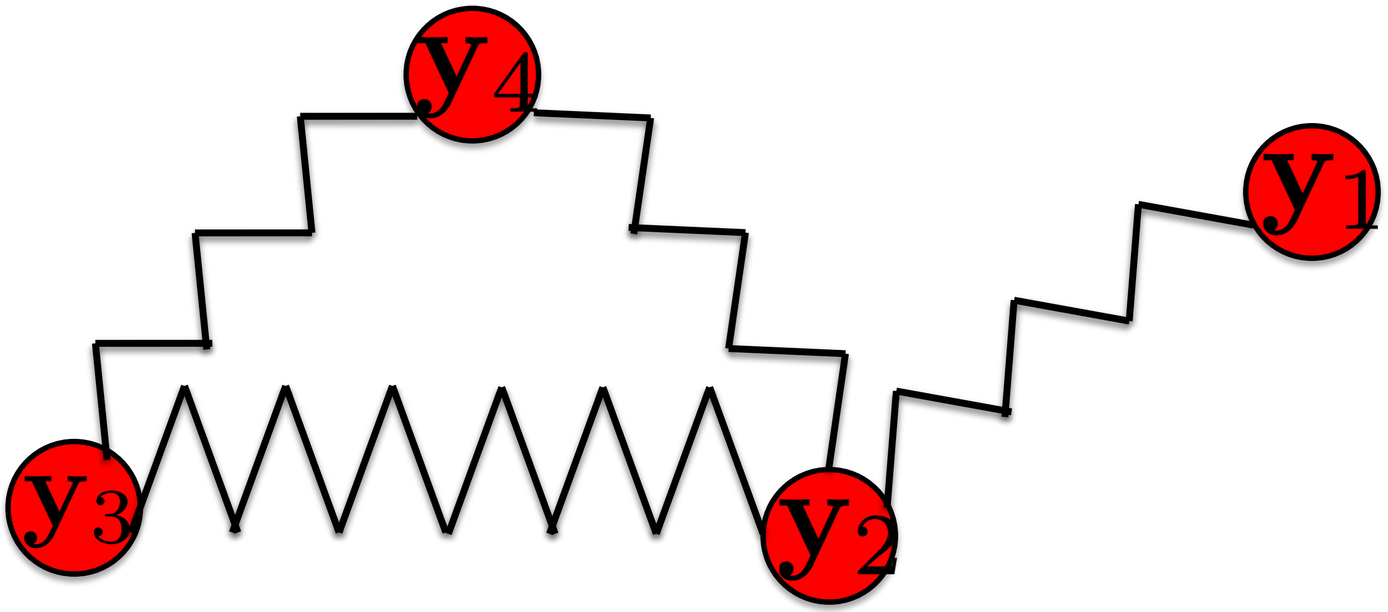
Fixed Points via Projection



$$A\mathbf{y} = \mathbf{f}$$

$$\mathbf{y} = P\tilde{\mathbf{y}} + \mathbf{c}$$

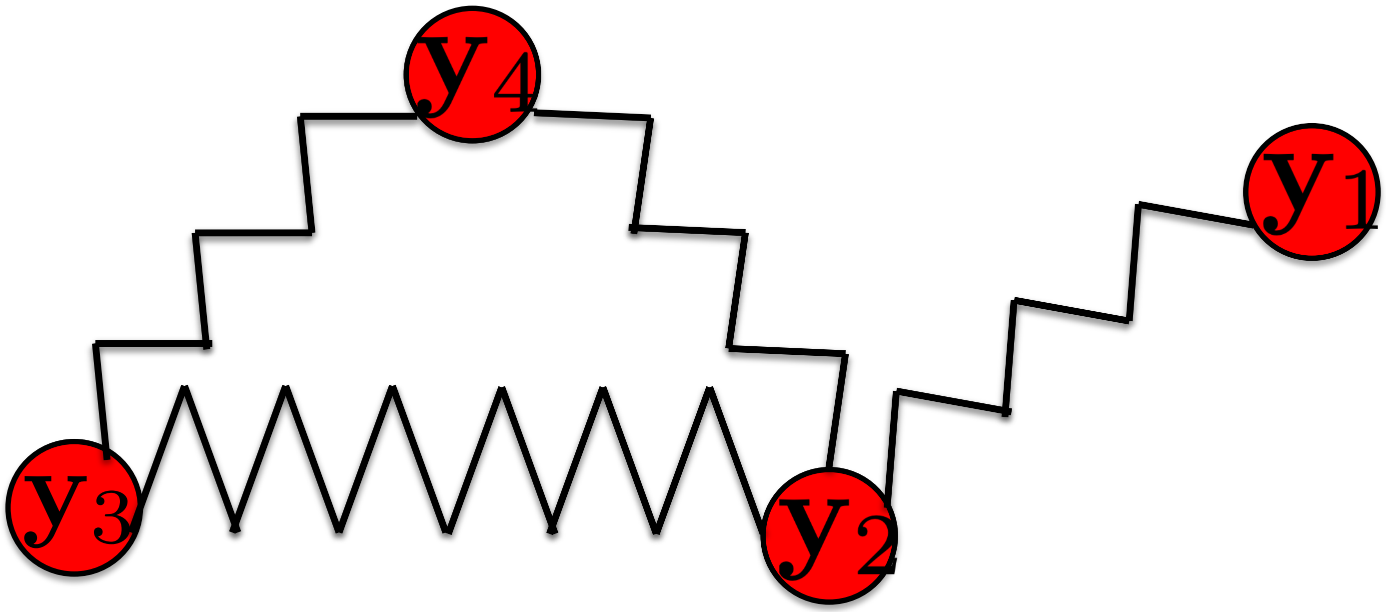
Fixed Points via Projection



$$AP\tilde{\mathbf{y}} = \mathbf{f} - A\mathbf{c}$$

Too many rows now …

Fixed Points via Projection

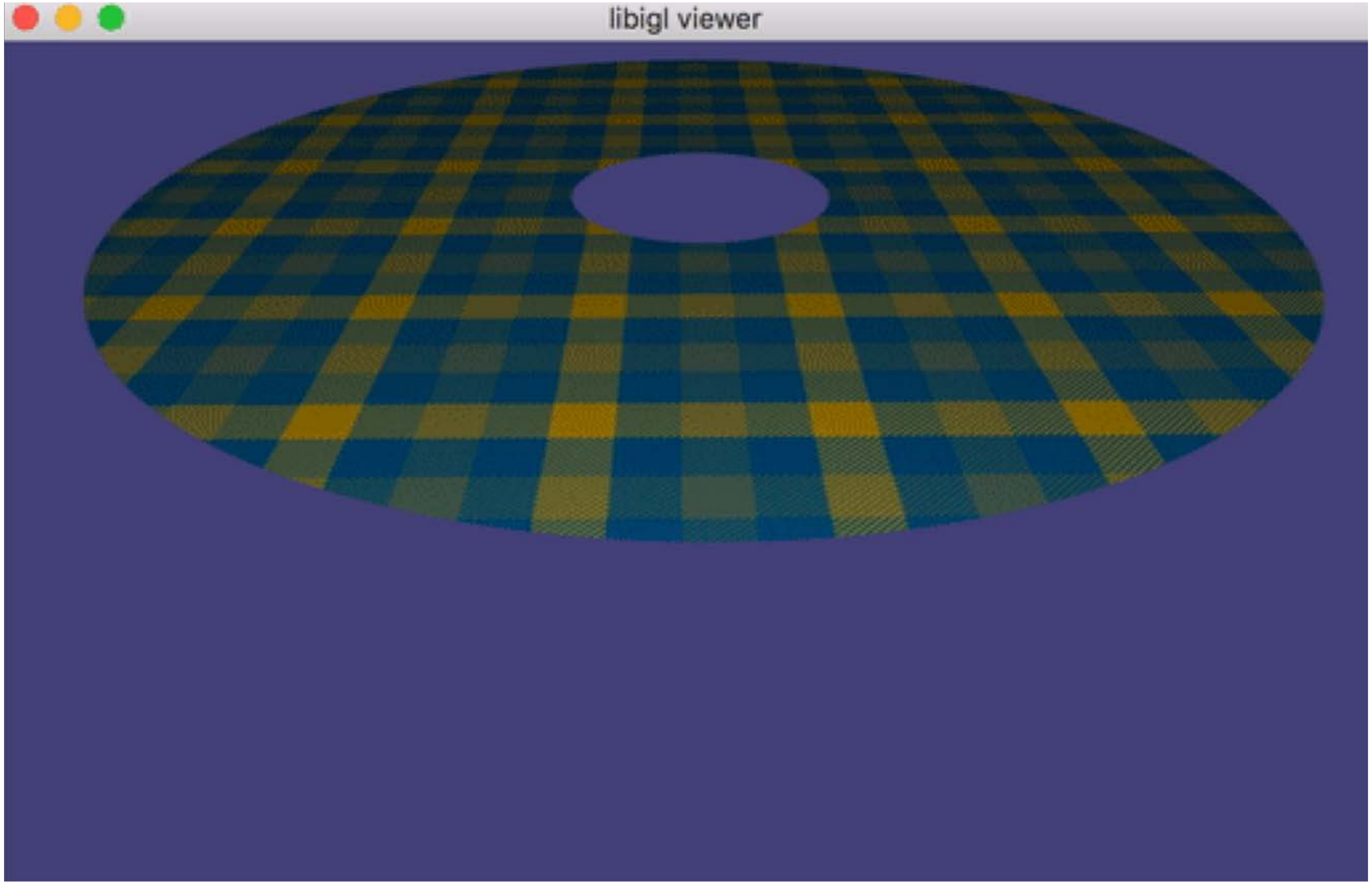


$$P^T A P \tilde{\mathbf{y}} = P^T (\mathbf{f} - A\mathbf{c})$$

Just right … but don't forget to rebuild y after solving

Lots more on the Assignment Page

So please read it carefully when doing the assignment


libigl viewer

Done for Today

Office hours: Right now! BA5268