Mathematical Finance I

Zhekai Liu, Cenning Wang, Ang Li, Boyuan Zhang, Zichang Liu

International Undergraduate Summer Enrichment Program

University of Alberta & Jilin

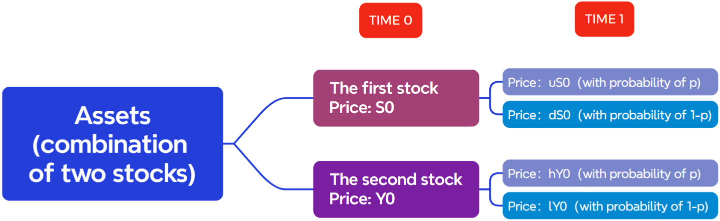
August 9, 2024

Outline

1. Introduction
2. No-arbitrage Condition
3. Pricing A Financial Derivative
4. Replicating Strategy
5. Multiperiod Model
6. Summary

Background

In Section 1 of the lecture on mathematical finance, we studied a model with one stock (risky asset) and one bank account (risk-free asset). Such a risk-free asset may not always be available. In this project, we consider a model with two stocks, but no bank account.The initial values of the stocks are S_0 and Y_0. Their values at time 1 are either uS_0 and hY_0 with probability p, or dS_0 and lY_0 with probability $1 - p$. For the project, we can consider the following different aspects of this model.



TIME 0

TIME 1

Assets (combination of two stocks)

The first stock
Price: S0

Price： uS0 (with probability of p)

Price： dS0 (with probability of 1-p)

The second stock
Price: Y0

Price： hY0 (with probability of p)

Price： lY0 (with probability of 1-p)

Outline

1 Introduction

2 No-arbitrage Condition

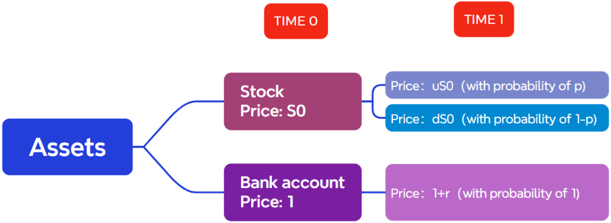
3 Pricing A Financial Derivative

4 Replicating Strategy

5 Multiperiod Model

6 Summary

No-arbitrage Condition With Risk-free Asset

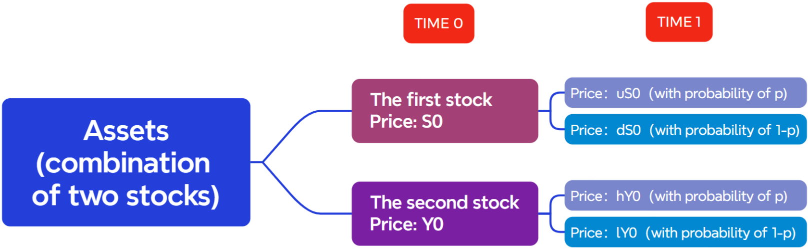

TIME 0
TIME 1
Assets
Stock
Price: S0
Price：uS0（with probability of p)
Price：dS0（with probability of 1-p)
Bank account
Price: 1
Price：1+r（with probability of 1)

In the previous lecture, we discussed the No-arbitrage condition in two assets, one is stock and the other is a risk-free asset. The No-arbitrage condition means that no investor can lock in a profit without risk and with no initial endowment:

$$d < 1 + r < u.$$

No-arbitrage Condition Without Risk-free Asset

Now, in this project, we consider a model with two stocks without risk-free assets.



TIME 0

TIME 1

Assets (combination of two stocks)

The first stock
Price: S0

Price： uS0 (with probability of p)

Price： dS0 (with probability of 1-p)

The second stock
Price: Y0

Price： hY0 (with probability of p)

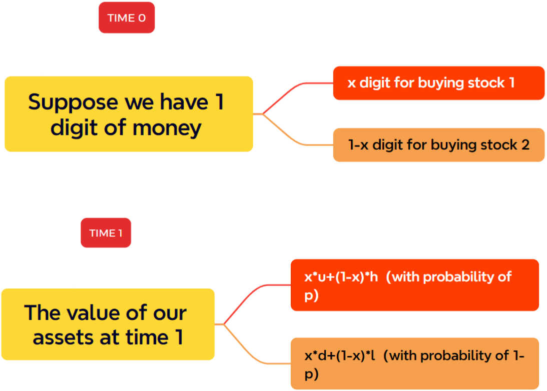
Price： lY0 (with probability of 1-p)

No-arbitrage Condition Without Risk-free Asset

The No-arbitrage condition is that:

$$d < l < h < u \quad or \quad l < d < u < h.$$

Proof.



TIME 0

Suppose we have 1 digit of money

- x digit for buying stock 1
- 1-x digit for buying stock 2

TIME 1

The value of our assets at time 1

- x*u+(1-x)*h (with probability of p)
- x*d+(1-x)*l (with probability of 1-p)

No-arbitrage Condition Without Risk-free Asset

For situation with probability of p or 1-p(as the picture above shows),the derivative of value can't be positive or negative at the same time,otherwise we can only invest our money into stock1(when the derivatives are all positive)or into the stock2(when the derivatives are all negative),in order to maximize our assets at time 1.Howeverthat kind of financial market is not what we are hoping for.

If we suppose $u <$d, $h < l$(it's reasonable because the other situations are exactly the sameso we can feel free to make this assumption)therefore when the derivatives mentioned above are neither all postive nor negativewe can infer that:

$$h < u < d < l \quad or \quad u < h < l < d.$$

Outline

1. Introduction
2. No-arbitrage Condition
3. **Pricing A Financial Derivative**
4. Replicating Strategy
5. Multiperiod Model
6. Summary

For the second question, let us consider there is a bank that sells the options. The bank never wants to lose money, so it will use the option's fee to construct a portfolio in the market. The bank needs to make sure whether the stocks prices go up or down, it will never lose money.

Equation:

$$f_u = \Delta_1 u S_0 + \Delta_2 h Y_0.$$

$$f_d = \Delta_1 d S_0 + \Delta_2 l Y_0.$$

Answer:

$$\Delta_1 = \frac{l f_u - h f_d}{S_0(lu - dh)}.$$

$$\Delta_2 = \frac{d f_u - u f_d}{Y_0(dh - ul)}.$$

Process:

$$\begin{cases} f_u = \Delta_1 u S_0 + \Delta_2 h Y_0 \\ f_d = \Delta_1 d S_0 + \Delta_2 l Y_0 \end{cases}$$

$$\begin{gathered} \mathrm{d}\mathrm{f}_u = \Delta_1 du S_0 + \Delta_2 dh Y_0 \\ \mathrm{u}\mathrm{f}_d = \Delta_1 ud S_0 + \Delta_2 ul Y_0 \\ \mathrm{d}\mathrm{f}_u - uf_d = (dh - ul) Y_0 \Delta_2 \end{gathered}$$

$$\begin{gathered} \mathrm{l}\mathrm{f}_u = \Delta_1 lu S_0 + \Delta_2 lh Y_0 \\ \mathrm{h}\mathrm{f}_d = \Delta_1 hd S_0 + \Delta_2 hl Y_0 \\ \mathrm{l}\mathrm{f}_u - hf_d = (lu - dh) S_0 \Delta_1 \end{gathered}$$

$$\begin{cases} \Delta_1 = \frac{lf_u - hf_d}{S_0(lu - dh)} \\ \Delta_2 = \frac{df_u - uf_d}{Y_0(dh - ul)} \end{cases}$$

Consider a derivative with payoff $f_u = 2uS_0$ and $f_d = 0$. The parameters of the stocks are given by $S_0 = \$20$, $Y_0 = \$25$, $u = 1.4$, $d = 0.8$, $h = 1.5$ and with different values for l. Create a plot of the derivative price as a function of l = 0, 0.01, 0.02, . . . , 0.6.

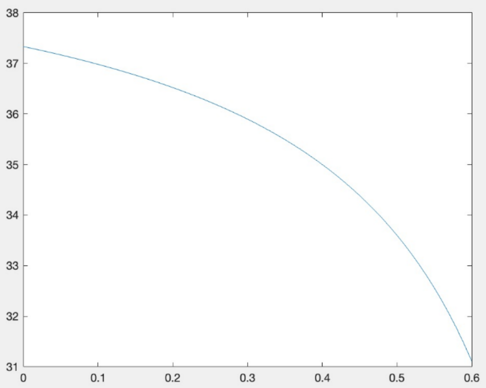
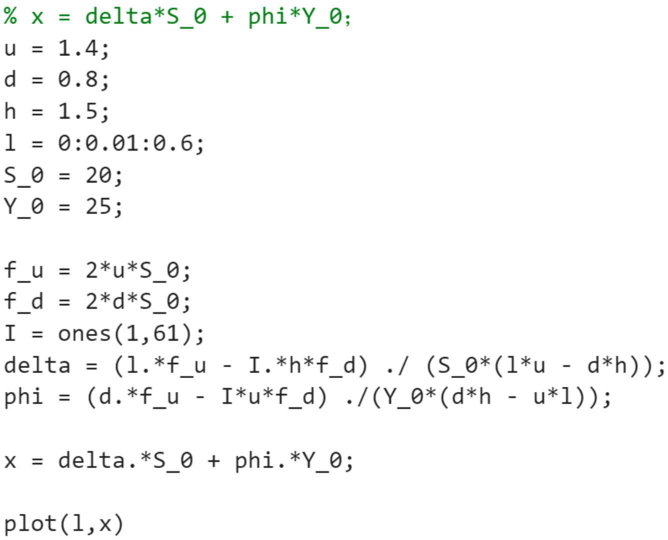


Figure 1: Fair Frice Chart of $f_d = 0$

Fair Frice Chart of $f_d = 2dS_0$



```
% x = delta*S_0 + phi*Y_0;
u = 1.4;
d = 0.8;
h = 1.5;
l = 0:0.01:0.6;
S_0 = 20;
Y_0 = 25;

f_u = 2*u*S_0;
f_d = 2*d*S_0;
I = ones(1,61);
delta = (l.*f_u - I.*h*f_d) ./ (S_0*(l*u - d*h));
phi = (d.*f_u - I*u*f_d) ./(Y_0*(d*h - u*l));

x = delta.*S_0 + phi.*Y_0;

plot(l,x)
```

Consider a derivative with payoff $f_u = 2uS_0$ and $f_d = 2dS_0$. The parameters of the stocks are given by $S_0 = \$20$, $Y_0 = \$25$, $u = 1.4$, $d = 0.8$, $h = 1.5$ and with different values for l. Create a plot of the derivative price as a function of $l = 0, 0.01, 0.02, \ldots, 0.6$.

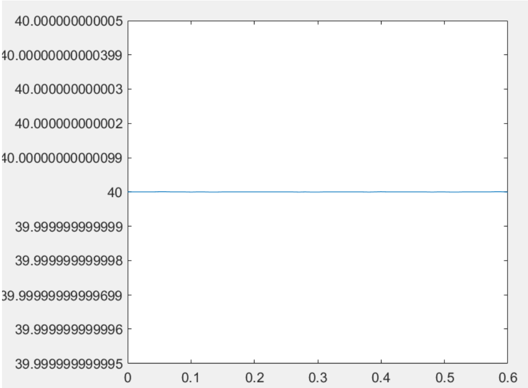


Figure 2: Fair Frice Chart of $f_d = 2dS_0$

- Firstly, when $f_d = 0$, it's really easy to prove the fair price will decline when I increases by the solutions of these equations.
- When $f_d = 2dS_0$, you can easily find that $\Delta_2 \equiv 0$, which means that all the money will be invested in the first stock.
- But for the first figure, it isn't really intuitive(intuitive means: obtained by using your feelings rather than by considering the facts)

We have some idea about why the fair price curve is a monotonic decrease function. If we look more closely, you may find that the Δ_1 is always negative, so it means that the bank needs a margin financing to stock 1 which means going short on the asset. And to balance it, we need to buy stock 2 which shows that the bank going long on this asset. Now if I increases, then the expectation or probability of the Second stock has grown. So we need more security loans on stock 2(Increase Δ_2), which means we can get more money from security loans. So the fair price decreases.

Outline

1. Introduction
2. No-arbitrage Condition
3. Pricing A Financial Derivative
4. **Replicating Strategy**
5. Multiperiod Model
6. Summary

(a) Using 2(a) i. and ii., write a MATLAB function that returns the replicating strategy $(\Delta_1; \Delta_2)$ for a derivative $\mathrm{f} = (\mathrm{f}_u; f_d)$.

(b) Use your previous MATLAB function to write a MATLAB function that returns the replicating strategy (Δ_1, Δ_2) for a call option on the first stock with strike price K.

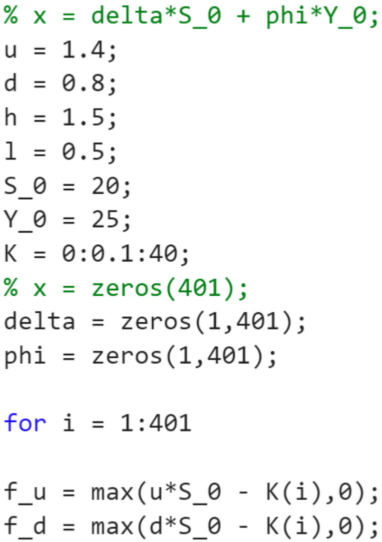
For the call options, let us consider S_0 will turn into:

$$\begin{cases} f_u = max\{uS_0 - K, 0\} \\ f_d = min\{dS_0 - K, 0\} \end{cases}$$

at time 1.

(c) Consider a call option on the first stock. The parameters of the stocks are given by $S_0 = \$20$, $Y_0 = \$25$, $u = 1.4$, $d = 0.8$, $h = 1.5$ and $l = 0.5$. Create a plot of the replicating portfolio (Δ_1, Δ_2) as a function of K = 0, 0.1, 0.2, . . . , 40.

Matlab Function

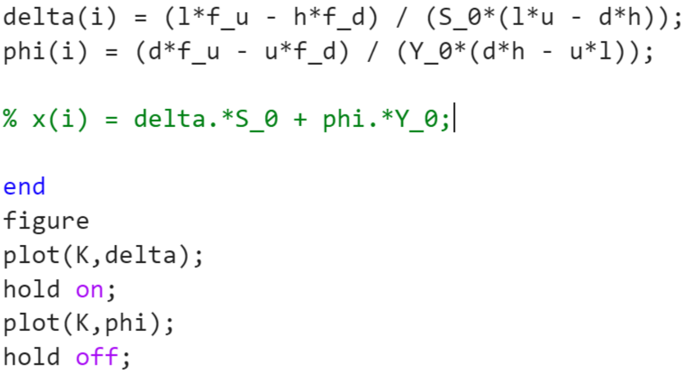


```
% x = delta*S_0 + phi*Y_0;
u = 1.4;
d = 0.8;
h = 1.5;
l = 0.5;
S_0 = 20;
Y_0 = 25;
K = 0:0.1:40;
% x = zeros(401);
delta = zeros(1,401);
phi = zeros(1,401);

for i = 1:401

f_u = max(u*S_0 - K(i),0);
f_d = max(d*S_0 - K(i),0);
```

Matlab Function



```
delta(i) = (l*f_u - h*f_d) / (S_0*(l*u - d*h));
phi(i) = (d*f_u - u*f_d) / (Y_0*(d*h - u*l));

% x(i) = delta.*S_0 + phi.*Y_0;

end
figure
plot(K,delta);
hold on;
plot(K,phi);
hold off;
```

(d) When K $\geq$ 30. There is no probability that the assets can reach this strike price K. For call options, there is no chance to win money, so the fair price will be zero.
(e) When K $=$ 0, the f_u and f_d can perfectly match the equations. The bank can buy the first stock directly instead of the second one.

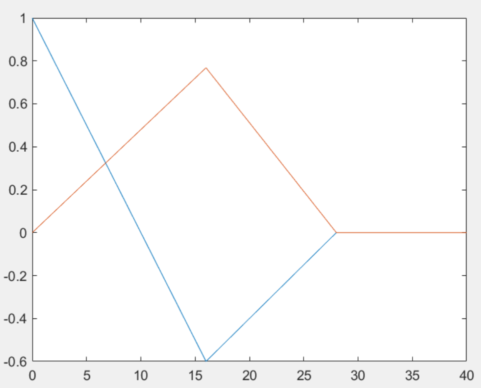


Figure 3: Replicating Strategy

Outline

1. Introduction
2. No-arbitrage Condition
3. Pricing A Financial Derivative
4. Replicating Strategy
5. **Multiperiod Model**
6. Summary

- Expand the model to n periods. What will be the fair price of the financial derivative at time zero?
- Can this price be considered as the discounted expectation of the derivative payoff under a risk-neutral probability measure?

From the second question, the portfolio is that:

$$\begin{cases}\Delta_1 = \frac{lf_u - hf_d}{S_0(lu-dh)} \\ \Delta_2 = \frac{df_u - uf_d}{Y_0(dh-ul)}\end{cases}$$

Fair price x:

$$\begin{aligned} x &= \frac{lf_u - hf_d}{S_0(lu-dh)}S_0 + \frac{df_n - uf_d}{Y_0(dh-ul)}Y_0 \\ &= \frac{lf_u - hf_d}{lu-dh} + \frac{df_u - uf_d}{dh-ul} \\ &= \frac{l}{lu-dh}f_u - \frac{u-h}{lu-dh}f_d + \frac{d}{dh-ul}f_u - \frac{u}{dh-ul}f_d \\ &= \frac{l-d}{lu-dh}f_u + \frac{u-h}{lu-dh}f_d \end{aligned}$$

$$q_u = \frac{l-d}{lu-dh} \qquad q_d = \frac{u-h}{lu-dh}$$

$$x = E^Q[f]$$

$Q = (q_u, q_d)$ **is not a probability measure!**

$$\begin{aligned} x &= q_u f_u + q_d f_d \quad (1\ period) \\ &= q_u^2 f_{uu} + 2q_u q_d f_{ud} + q_d^2 f_{dd} \quad (2\ periods) \\ &= E^{Q_2}[f\ ] \end{aligned}$$

$$Denote : Q_2 = [q_u^2, 2q_u q_d, q_d^2]$$

$$\begin{aligned} E^{Q_n}[f\ ] = q_u^n u^n S_0 &+ C_n^1 q_n^{n-1} q_d u^{n-1} dS_0 \\ &+ \cdots \\ &+ C_n^{n-1} q_u q_d^{n-1} ud^{n-1} S_0 \\ &+ q_d^n d^n S_0 = S_0 \end{aligned}$$

The last equation works because if we rearrange the equations you will find that the coefficients of S_0 are the coefficients of binomial expansion. (put u in q_u and d in q_d).

So, this price can be considered as a risk-neutral price.

Outline

1 Introduction

2 No-arbitrage Condition

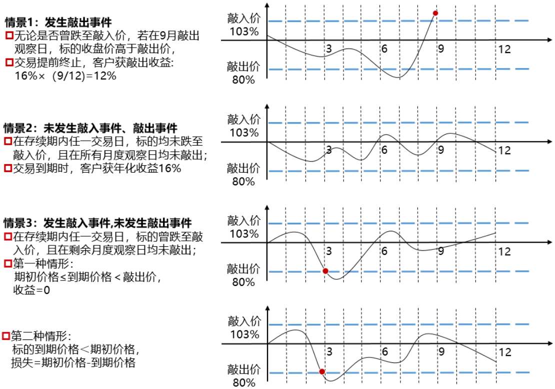
3 Pricing A Financial Derivative

4 Replicating Strategy

5 Multiperiod Model

6 Summary

'Snowball' Derivative



情景1：发生敲出事件
□无论是否曾跌至敲入价，若在9月敲出观察日，标的收盘价高于敲出价，
□交易提前终止，客户获敲出收益:
16%×（9/12)=12%

情景2：未发生敲入事件、敲出事件
□在存续期内任一交易日，标的均未跌至敲入价，且在所有月度观察日均未敲出;
□交易到期时，客户获年化收益16%

情景3：发生敲入事件,未发生敲出事件
□在存续期内任一交易日，标的曾跌至敲入价，且在剩余月度观察日均未敲出;
□第一种情形:
期初价格≤到期价格＜敲出价，
收益=0

□第二种情形:
标的到期价格＜期初价格，
损失=期初价格-到期价格

The money you earn is just like a snowball in the snow. The more time the stock price is in the boundary area, the more money you will make. It's just like a snowball gathers as it goes on.

How does it works?

$$\text{The customers}\xrightarrow{\text{Put options}}\text{The securities company}$$

$$\text{The securities company}\xrightarrow{\text{Make a protfolio}}\text{Markets}$$

$$\text{Markets}\xrightarrow{\text{Throw high and suck low}}\text{The securities company}$$

$$\text{The securities company}\xrightarrow{\text{Throw high and suck low}}\text{The customers}$$

We do the same thing that the securities company does in some aspects. Binary tree options pricing without risk-free assets is an important method if you want to do a financial job in the future, and it has been used in many ways.

Thank you!