COVER PROBLEM

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ABSTRACT. In the first part, we are talking about how to estimate the number of open circles we need if we want to cover a compact cube. In the second part, we discuss whether there are some compacts whose the order of their smallest cover number (α) is smaller than the number of dimensions. Then, we find this α is just the Hausdorff dimension of this compact. Finally, we will consider functional space, which is a metric space.

1. PROBLEM STATEMENT

Now we want to discuss a covering problem, the question is: If we have a cube with side length a. Now we have a family of balls of radius ε covering this cube. We want to find the smallest number of balls that can cover the cube.

The equivalent definition of this problem is that we want to find a minimum ε net of this cube. It means we must figure out the minimum number of points in this net that can cover this cube.

Denote the smallest number of cover sets as $n(\varepsilon)$.

At this moment, Maybe I cannot find the precise estimation of the cover number. Just estimation.

1.1. **More simple situation.**
Now consider if we only want to use some small cubes with side length ε to cover the big cube. How many cubes do we need? Maybe we can estimate like this[Figure 1]:

$$n(\varepsilon)_{cube} \sim \frac{a^3}{\epsilon^3}(\varepsilon \to 0)$$

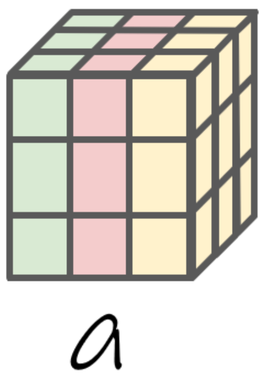

FIGURE 1. cover by small cube with radius ε

1.2. **Estimation.**
Based on the above discussion, we now want to estimate the smallest number of cover sets using small balls. [Figure 2]

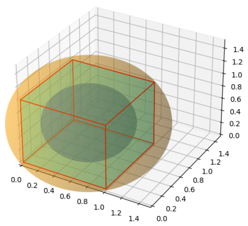

FIGURE 2. My idea

1.2.1. *Maximum estimate.* To begin with, let's consider the circumcircle of the small cube, which can easily cover the big cube, and the number of the small balls is equal to the small cube with side length ε. Simultaneously, the incised sphere surely cannot cover the big cube. So, We can estimate $n(\varepsilon)$ upper bound.

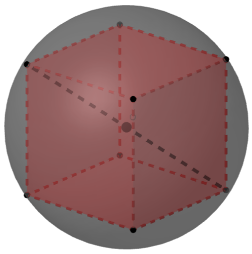

FIGURE 3. My idea

$$n(\frac{\sqrt{3}}{2}\varepsilon)_{circle} \leq n(\varepsilon)_{cube}$$

$$n(\varepsilon)_{circle} \leq n(\frac{2\sqrt{3}}{3}\varepsilon)_{cube} \tag{1}$$

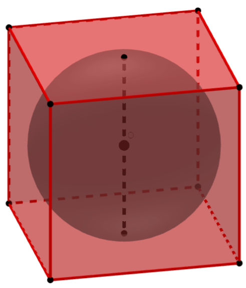

FIGURE 4. My idea

1.2.2. *Minimum estimate.*

$$n(\frac{\varepsilon}{2})_{circle} \geq n(\varepsilon)_{cube}$$

$$n(\varepsilon)_{circle} \geq n(2\varepsilon)_{cube} \tag{2}$$

1.3. **Conclution.** Finally, we have the estimation:

$$n(2\varepsilon)_{cube} \leq n(\varepsilon)_{circle} \leq n(\frac{2\sqrt{3}}{3}\varepsilon)_{cube}$$

2. Could we Find some Compact Set in n Dimensional Space such that $n(\varepsilon) \sim \frac{1}{\varepsilon^{\alpha}}$, while $0 \leq \alpha \leq n$

We will start with the construction of the Cantor set. At each iteration step, we will consider the corresponding minimum coverage. As n approaches infinity, we can obtain the minimum coverage about the Cantor set, where the order of ε is less than 1, more precisely, $\log_3 2$.

2.1. **How to Structure It?** We can structure it as follows:

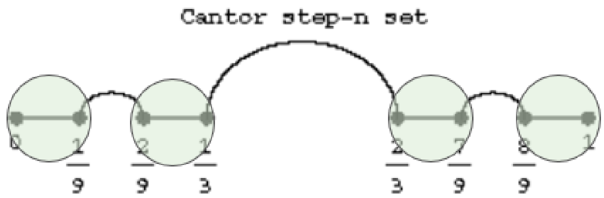

Figure 5. Cantor Set

In this picture, just at this step, the measure of intervals in the cantor set equals $\frac{1}{9}$. After removing some subsets, leaving two line segments: $\left[\frac{1}{9}, \frac{2}{9}\right], \left[\frac{2}{9}, \frac{3}{9}\right], \left[\frac{6}{9}, \frac{7}{9}\right], \left[\frac{8}{9}, \frac{1}{9}\right]$.

We can cover this set by 4 circles(we draw it as a circle, but in 1-dimensional that is an interval), whose diameter belongs to $\left(\frac{1}{9}, \frac{1}{3}\right]$. Obviously, this circle number is the smallest, to cover this set. So $n(\varepsilon) = 4$.Where the ε is the radium of the circle. (Remark: The ball that covers the compact is open.)

Indeed, for the cantor set, we will remove some interval on the n^{th} time and remain 2^n intervals:

$$\left[1, \frac{1}{3^n}\right], \left[\frac{2}{3^n}, \frac{3}{3^n}\right] \ldots\ldots \left[\frac{3^n - 1}{3^n}, 1\right].$$

When:

$$\frac{1}{3^n} < 2\varepsilon \leq \frac{1}{3^{n-1}}$$

$n(\varepsilon)$ will equals 2^n.

2.2. **Estimation.** Firstly, when $2\varepsilon > \frac{1}{3^n}$:

$$\frac{1}{\varepsilon} < 2 \cdot 3^n$$

$$\frac{1}{\varepsilon^{\alpha}} < 2^{\alpha} \cdot 3^{n\alpha}$$

Let $\alpha = \log_3 2 < 1$.

$$\varepsilon^{\log_3 2} < 2^{\log_3 2} \cdot 3^{n \log_3 2} = 2^{\log_3 2} \cdot 2^n = 2^{\log_3 2} \cdot n(\varepsilon)$$

$$\frac{1}{2^{log_3 2} \cdot \varepsilon^{log_3 2}} < n(\varepsilon)$$

Then, when $2\varepsilon \le \frac{1}{3^{n-1}}$:

$$\frac{1}{3^{n-1}} \ge 2\varepsilon$$

$$\left(\frac{1}{\varepsilon}\right)^{\alpha} \ge \left(\frac{2}{3}\right)^{\alpha} \cdot 3^{n\alpha}$$

Let $\alpha = log_3 2$:

$$\frac{1}{\varepsilon^{log_3 2}} \ge \left(\frac{2}{3}\right)^{log_3 2} \cdot n(\varepsilon)$$

$$n(\varepsilon) \le \frac{1}{\left(\frac{2}{3}\right)^{log_3 2} \cdot \varepsilon^{log_3 2}}$$

In conclusion:

$$\frac{1}{2^{log_3 2} \cdot \varepsilon^{log_3 2}} < n(\varepsilon) \le \frac{1}{\left(\frac{2}{3}\right)^{log_3 2} \cdot \varepsilon^{log_3 2}}$$

$$n(\varepsilon) \asymp \frac{1}{\varepsilon^{log_3 2}}$$

□

2.3. **In D-dimensional.** Similarly, we construct a D-dimensional cantor set. Indeed, for the cantor set, we will remove some interval on the n^{th} time and remain $2^{d \cdot n}$ intervals:

When:

$$\frac{\sqrt{d}}{3^n} < 2\varepsilon \le \frac{\sqrt{d}}{3^{n-1}}$$

Then, in the same way, we can get that:

$$\frac{C_1}{\varepsilon^{\alpha}} \le n(\varepsilon) < \frac{C_2}{\varepsilon^{\alpha}}$$

Where $\alpha = d \log_3 2 < d$, C_1 and C_2 are constant. In conclusion:

$$n(\varepsilon) \asymp \frac{1}{\varepsilon^{d log_3 2}}$$

□

2.4. **Can α Equal to Arbitrary value in (0,n) in n-Dimensional?** Let us consider the p-cantor set $p \in (0, \frac{1}{2})$. The p-cantor set means that: The open middle $\left(\frac{1}{p}\right)^{th}$ of each of these remaining segments is deleted every time not just a third of them.

FIGURE 6. P-Cantor Set in 1-dimensional (first step)

Considering N-dimensional, as the same for the p-cantor set, we will remove some interval on the n^{th} time and remain $2^{d \cdot n}$ intervals:

$$n(\varepsilon) = 2^{N \cdot n}$$

Where: $\sqrt{N} \cdot p^n < 2\varepsilon \le \sqrt{N} \cdot p^{n-1}$.

$$\frac{C_1}{\varepsilon^{\alpha}} \le n(\varepsilon) < \frac{C_2}{\varepsilon^{\alpha}}$$

Where C_1 and C_2 are constant and $\alpha = n \cdot log_{\frac{1}{p}} 2 \in (0, N)$.

Because $\alpha(p)$ is a continuous function.

$$\lim_{p\to\frac{1}{2}^-} N\cdot log_{\frac{1}{p}}2 = N$$

$$\lim_{p\to 0^+} N\cdot log_{\frac{1}{p}}2 = 0$$

For every $\alpha\in(0,N)$, Let $p=2^{-\frac{n}{\alpha}}$, then:

$$n(\varepsilon)\asymp\frac{1}{\varepsilon^{\alpha}}$$

□

2.5. **Conclusion.** In conclusion, we did find that there exist some specific compact structures in which the order of $n(\varepsilon)$ is smaller than their dimensional. Even more, we can construct compact with any order of the $n(\varepsilon)$ which is smaller than their dimensional. The order of $n(\varepsilon)$ can be called the 'Hausdorff dimension' in the field of fractal structures.

3. FUNCTIONAL SPACE

3.1. **Metric Space.** Now, let's consider a functional space. We define the distance function like: $d(f,g)=\max\limits_{x\in[0,1]}|f(x)-g(x)|$, then we have a metric space. Because:

Firstly, $d(f,g)=0\Leftrightarrow f=g$:

$$\begin{aligned} d(f,g)&\equiv 0\\ \Leftrightarrow \max_{x\in[0,1]}|f-g|&\equiv 0\\ \Leftrightarrow f(x)&\equiv g(x)\end{aligned}$$

Then, we need to prove $d(f,g)=d(g,f)$:

$$d(f,g)=\max_{x\in[0,1]}|f-g|=\max_{x\in[0,1]}|g-f|=d(g,f)$$

Finally, the triangle inequality:

$$\begin{aligned} d(f,g)&=\max_{x\in[0,1]}|f-g|\\ &=\max_{x\in[0,1]}|f-h-(g-h)|\\ &\le\max_{x\in[0,1]}|f-h|+\max_{x\in[0,1]}|g-h|\\ &=d(f,h)+d(h,g)\end{aligned}$$

So, this is a metric space. □

Now let's consider a subspace in functional space.

$$K=\{f;f(0)=0,\ |f(t)-f(s)|\le L|t-s|\quad t,s\in[0,1]\}$$

3.2. **The second question: Is K a compact set?** The answer is, yes.

Lemma 1. If K is in a subspace of the complete metric space, then the property of closedness is equivalent to completeness.

According to the Lemma 1, we need to prove three things (K in a subspace of the complete space):

- Totally bounded
- Boundedness
- Closedness (Because K is a subspace of the complete metric space, closedness is equivalent to completeness, so we don't need to prove K is a complete subspace).

3.2.1. *Boundedness.* Let the function $f \equiv 0$ be the circle's center and ε be the circle's radius. Let the $\varepsilon > L$, then we can easily find that this circle can cover the K.

3.2.2. *Closedness.*

- $f_n \to f, \quad f_n(0) = 0 \quad \Rightarrow \quad f(0) = 0.$
- $|f_n(s) - f_n(t)| < L|s-t| \quad \Rightarrow \quad |f(s) - f(t)| \leq L|s-t| \quad n \to \infty$

So, $K' \subseteq K$, K is a closed set and also a complete set.

3.2.3. *Totally bounded.* Now we give a cover method. K is a Lipchitz continuous function space with uniformly L. In this figure 7, the largest triangle represents K in $\mathbb{R}^2$. Then, the idea comes from the option of binary tree pricing. Every blue curve in the largest triangle is a point in ε-net.

In this figure, there are 4 periods, in the first period, there are only 2^1 cover functions. But when at the 4-period, there are 2^4 functions. After this observation, we can imply we will have 2^n functions, at n-period.

Remark: Cover functions represent the blue curves in figure 7, which are the center of the circles. These circles are then used to cover K.

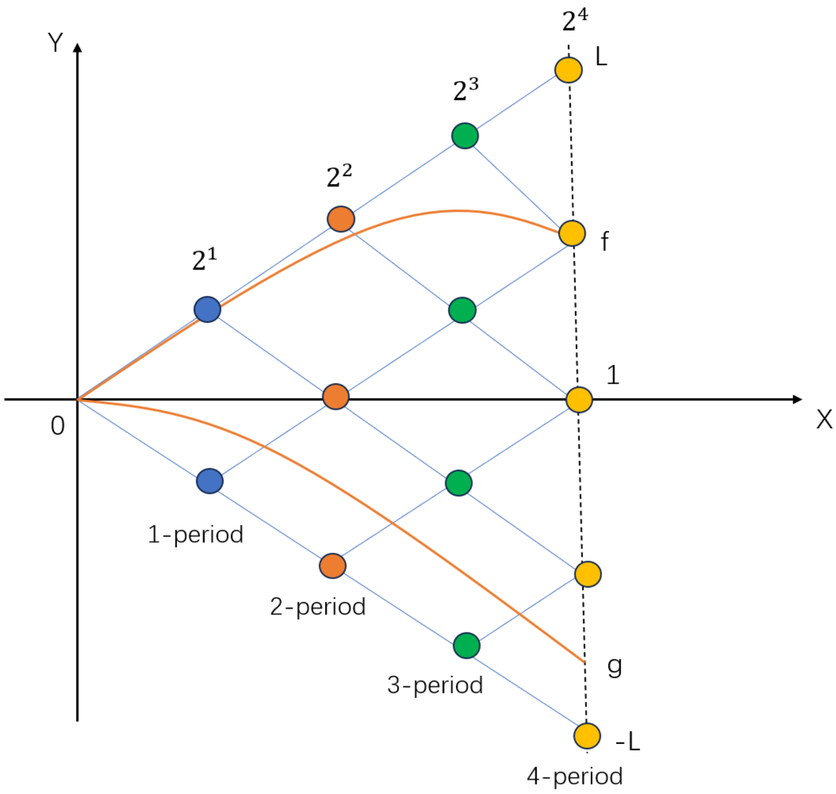

FIGURE 7. Cover method

If we look more closely, we may find that the maximum distance between functions in K and cover functions equals $\frac{L}{n}$(when we have n periods). So when the radius $\varepsilon > \frac{L}{n}$, then we can make sure this cover method can cover all functions in K.

Now, we can prove that K is totally bounded. Therefore, K is a compact set. $\square$

3.3. **what is the $n(\varepsilon)$ of K?.** Let's consider figure 7 again, the idea comes from the binary tree. In the first period, when the $\frac{L}{n} < \varepsilon < L$, we need 2^1 functions at least (Analogy figure 2.1 cover method). If we use three or more than 2^1 functions in the first period, then we will need more functions in future periods. For instance, if we have 3 functions in the first period then we need three nodes(states) in the second period if we still want to cover the K in these two periods with radius $\frac{L}{n} < \varepsilon < L$. Now we may notice that we will have the cover functions more than 2^2.

Of course, you may confused, the more notes we have, the less cover function radius we will use, Which means that if we consider figure 8, then we will get the maximum distance between cover functions and functions in K equals $\frac{L}{2n}$. We will reach this accuracy by using $2^{2n} = 4^n$ cover functions in 7 method and 3^n cover functions in 8 method. [here are still some problems, because I want to prove binary tree is the best method]

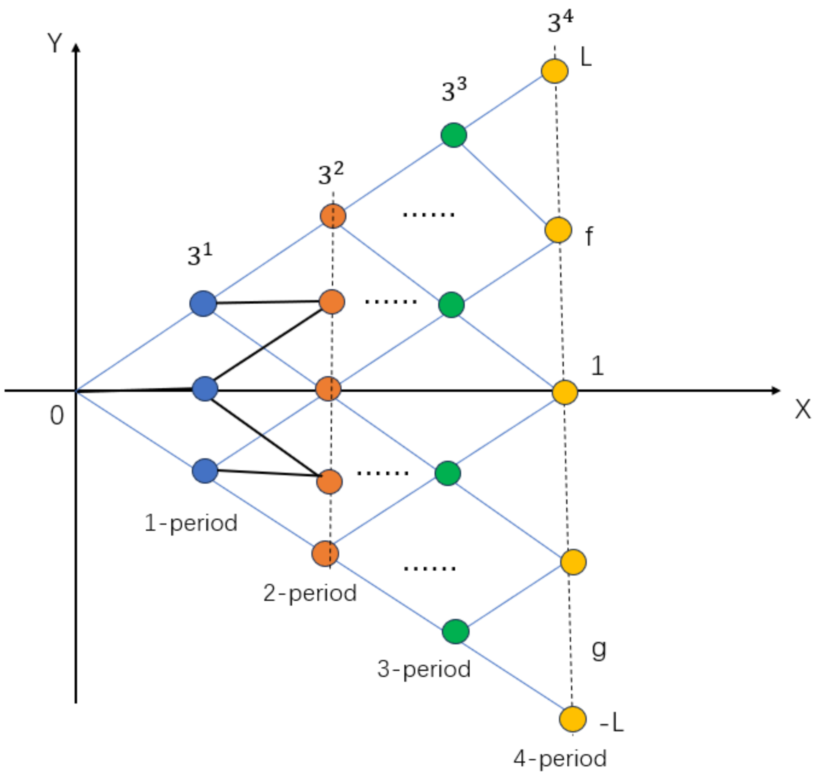

FIGURE 8. Cover method more notes

Now, according to figure 7 the binary tree method to cover compact K, we can know that when the radius $\frac{L}{n} < \varepsilon < \frac{2L}{n}$, number of cover functions equals 2^n.

$$\frac{L}{\varepsilon} < n < \frac{2L}{\varepsilon}$$
$$2^{\frac{L}{\varepsilon}} < 2^n < 2^{\frac{2L}{\varepsilon}}$$

So we can get that:

$$n(\varepsilon) < C_1 2^{\frac{L}{\varepsilon}}$$

Now we still need to prove $n(\varepsilon) > C_2 2^{\frac{L}{\varepsilon}}$.