# COVER PROBLEM FOR THE COMPACT SETS

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Abstract. In this work, we first address the problem of estimating the number of open circles needed to cover a compact cube. Next, we investigate whether there exist compact sets whose minimum cover number, denoted as $n(\varepsilon)$, is smaller than the number of dimensions. We show that $n(\varepsilon)$ corresponds to the Hausdorff dimension of these compact sets. Finally, we extend our analysis to functional spaces, treating them as metric spaces.

## 1. Problem statement

Now we want to discuss a covering problem, the question is: If we have a cube with side length a. Now we have a family of balls of radius $\varepsilon$ covering this cube. We want to find the smallest number of balls that can cover the cube.

The equivalent definition of this problem is that we want to find a minimum $\varepsilon$ net of this cube. It means we must figure out the minimum number of points in this net that can cover this cube.

Denote the smallest number of cover sets as $n(\varepsilon)$.

At this moment, Maybe I cannot find the precise estimation of the cover number. Just estimation.

### 1.1. More simple situation.

Now consider if we only want to use some small cubes with side length $\varepsilon$ to cover the big cube. How many cubes do we need? Maybe we can estimate like this[Figure 1]:

$$n(\varepsilon)_{cube} \sim \frac{a^3}{\epsilon^3}(\varepsilon \to 0)$$

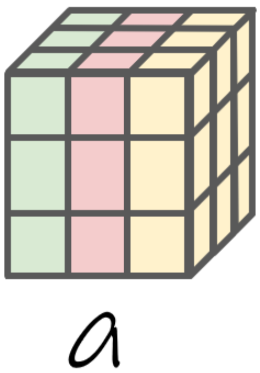


Figure 1. cover by small cube with radius $\varepsilon$

### 1.2. Estimation.

Based on the above discussion, we now want to estimate the smallest number of cover sets using small balls. [Figure 2]

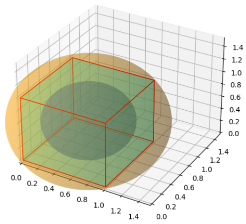

FIGURE 2. My idea

1.2.1. *Maximum estimate.* To begin with, let's consider the circumcircle of the small cube, which can easily cover the big cube, and the number of the small balls is equal to the small cube with side length $\varepsilon$. Simultaneously, the incised sphere surely cannot cover the big cube. So, We can estimate $n(\varepsilon)$ upper bound.

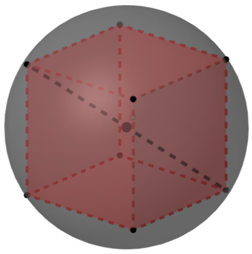

FIGURE 3. My idea

$$n(\frac{\sqrt{3}}{2}\varepsilon)_{circle} \leq n(\varepsilon)_{cube}$$

$$n(\varepsilon)_{circle} \leq n(\frac{2\sqrt{3}}{3}\varepsilon)_{cube} \tag{1}$$

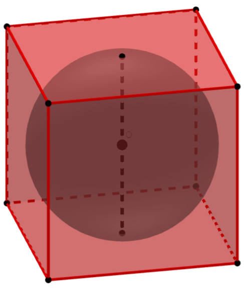

FIGURE 4. My idea

1.2.2. *Minimum estimate.*

$$n(\frac{\varepsilon}{2})_{circle} \geq n(\varepsilon)_{cube}$$

$$n(\varepsilon)_{circle} \geq n(2\varepsilon)_{cube} \tag{2}$$

1.3. **Conclution.** Finally, we have the estimation:

$$n(2\varepsilon)_{cube} \leq n(\varepsilon)_{circle} \leq n(\frac{2\sqrt{3}}{3}\varepsilon)_{cube}$$

## 2. Could we Find some Compact Set in $n$ Dimensional Space such that $n(\varepsilon) \sim \frac{1}{\varepsilon^{\alpha}}$, while $0 \leq \alpha \leq n$

We will start with the construction of the Cantor set. At each iteration step, we will consider the corresponding minimum coverage. As $n$ approaches infinity, we can obtain the minimum coverage about the Cantor set, where the order of $\varepsilon$ is less than 1, more precisely, $\log_3 2$.

2.1. **How to Structure It?** We can structure it as follows:

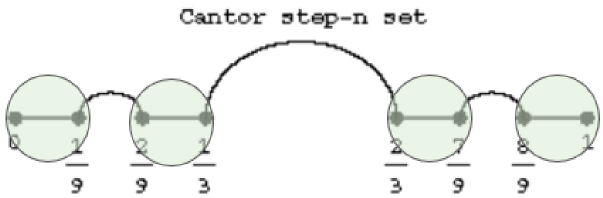


Figure 5. Cantor Set

In this picture, just at this step, the measure of intervals in the cantor set equals $\frac{1}{9}$. After removing some subsets, leaving two line segments: $\left[\frac{1}{9}, \frac{2}{9}\right], \left[\frac{2}{9}, \frac{3}{9}\right], \left[\frac{6}{9}, \frac{7}{9}\right], \left[\frac{8}{9}, \frac{1}{9}\right]$.

We can cover this set by 4 circles(we draw it as a circle, but in 1-dimensional that is an interval), whose diameter belongs to $\left(\frac{1}{9}, \frac{1}{3}\right]$. Obviously, this circle number is the smallest, to cover this set. So $n(\varepsilon) = 4$.Where the $\varepsilon$ is the radium of the circle. (Remark: The ball that covers the compact is open.)

Indeed, for the cantor set, we will remove some interval on the $n^{th}$ time and remain $2^n$ intervals:

$$\left[1, \frac{1}{3^n}\right], \left[\frac{2}{3^n}, \frac{3}{3^n}\right] ...... \left[\frac{3^n - 1}{3^n}, 1\right].$$

When:

$$\frac{1}{3^n} < 2\varepsilon \leq \frac{1}{3^{n-1}}$$

$n(\varepsilon)$ will equals $2^n$.

2.2. **Estimation.** Firstly, when $2\varepsilon > \frac{1}{3^n}$ :

$$\frac{1}{\varepsilon} < 2 \cdot 3^n$$

$$\frac{1}{\varepsilon^{\alpha}} < 2^{\alpha} \cdot 3^{n\alpha}$$

Let $\alpha = \log_3 2 < 1$.

$$\varepsilon^{\log_3 2} < 2^{\log_3 2} \cdot 3^{n \log_3 2} = 2^{\log_3 2} \cdot 2^n = 2^{\log_3 2} \cdot n(\varepsilon)$$

$$\frac{1}{2^{log_3 2} \cdot \varepsilon^{log_3 2}} < n(\varepsilon)$$

Then, when $2\varepsilon \leq \frac{1}{3^{n-1}}$ :

$$\frac{1}{3^{n-1}} \geq 2\varepsilon$$
$$\left(\frac{1}{\varepsilon}\right)^{\alpha} \geq \left(\frac{2}{3}\right)^{\alpha} \cdot 3^{n\alpha}$$

Let $\alpha = log_3 2$ :

$$\frac{1}{\varepsilon^{log_3 2}} \geq \left(\frac{2}{3}\right)^{log_3 2} \cdot n(\varepsilon)$$
$$n(\varepsilon) \leq \frac{1}{\left(\frac{2}{3}\right)^{log_3 2} \cdot \varepsilon^{log_3 2}}$$

In conclusion:

$$\frac{1}{2^{log_3 2} \cdot \varepsilon^{log_3 2}} < n(\varepsilon) \leq \frac{1}{\left(\frac{2}{3}\right)^{log_3 2} \cdot \varepsilon^{log_3 2}}$$
$$n(\varepsilon) \asymp \frac{1}{\varepsilon^{log_3 2}}$$

□

2.3. **In D-dimensional.** Similarly, we construct a D-dimensional cantor set. Indeed, for the cantor set, we will remove some interval on the $n^{th}$ time and remain $2^{d \cdot n}$ intervals:

When:

$$\frac{\sqrt{d}}{3^n} < 2\varepsilon \leq \frac{\sqrt{d}}{3^{n-1}}$$

Then, in the same way, we can get that:

$$\frac{C_1}{\varepsilon^{\alpha}} \leq n(\varepsilon) < \frac{C_2}{\varepsilon^{\alpha}}$$

Where $\alpha = d \log_3 2 < d$, $C_1$ and $C_2$ are constant. In conclusion:

$$n(\varepsilon) \asymp \frac{1}{\varepsilon^{d log_3 2}}$$

□

2.4. **Can $\alpha$ Equal to Arbitrary value in (0,n) in n-Dimensional?** Let us consider the p-cantor set $p \in (0, \frac{1}{2})$. The p-cantor set means that: The open middle $\left(\frac{1}{p}\right)^{th}$ of each of these remaining segments is deleted every time not just a third of them.


FIGURE 6. P-Cantor Set in 1-dimensional (first step)

Considering N-dimensional, as the same for the p-cantor set, we will remove some interval on the $n^{th}$ time and remain $2^{d \cdot n}$ intervals:

$$n(\varepsilon) = 2^{N \cdot n}$$

Where: $\sqrt{N} \cdot p^n < 2\varepsilon \leq \sqrt{N} \cdot p^{n-1}$.

$$\frac{C_1}{\varepsilon^{\alpha}} \leq n(\varepsilon) < \frac{C_2}{\varepsilon^{\alpha}}$$

Where $C_1$ and $C_2$ are constant and $\alpha = n \cdot log_{\frac{1}{p}} 2 \in (0, N)$.

Because $\alpha(p)$ is a continuous function.

$$\lim_{p\to\frac{1}{2}^-} N\cdot log_{\frac{1}{p}}2 = N$$

$$\lim_{p\to 0^+} N\cdot log_{\frac{1}{p}}2 = 0$$

For every $\alpha\in(0,N)$, Let $p=2^{-\frac{n}{\alpha}}$, then:

$$n(\varepsilon)\asymp\frac{1}{\varepsilon^\alpha}$$

□

2.5. **Conclusion.** In conclusion, we did find that there exist some specific compact structures in which the order of $n(\varepsilon)$ is smaller than their dimensional. Even more, we can construct compact with any order of the $n(\varepsilon)$ which is smaller than their dimensional. The order of $n(\varepsilon)$ can be called the 'Hausdorff dimension' in the field of fractal structures.

## 3. FUNCTIONAL SPACE

3.1. **Some basic properties about Compact set in a Hilbert($L^2$) metric space.** Suppose we have some assumptions here:

$$h=\sum_{n=1}^{+\infty}a_ne_n,\quad \|h\|^2=\sum_{n=1}^{+\infty}a_n^2$$

**Lemma 1.** *If we have a convergent sequence $v_n$ in Hilbert converging to $v$, then the subset $K=\{v_n:n\in\mathbb{N}\}\cup\{v\}$ is compact, and it has* ***equi-small tails*** *with respect to any orthonormal subset*

*Proof.* For **compactness**, $K$ is also a compact set because every sub-sequence in $\{v_m\}\cup\{v\}$ is convergent in $\{v_m\}\cup\{v\}$.

For **equi-small tails**, for every $\varepsilon>0$, since $v_n\to v$, there will exist $M\in\mathbb{N}$ so that, for all $n\geq M$, we have $\|v_n-v\|^2\leq\frac{\varepsilon}{2}$. Then we can choose $N$ large enough so that, for fixed $v$,

$$\sum_{k>N}|\langle v,e_k\rangle|^2+max_{1\leq n\leq M-1}\sum_{k>N}|\langle v_n,e_k\rangle|^2\leq\frac{\varepsilon^2}{4}$$

This only include finite terms, so we can find uniform $N$ here. If it can satisfy the formula above, then:

For the first part:

$$\sum_{k>N}|\langle v,e_k\rangle|^2\leq\frac{\varepsilon^2}{4}\tag{3}$$

For the second part $1\leq n\leq M-1$:

$$\sum_{k>N}|\langle v_n,e_k\rangle|^2\leq\frac{\varepsilon^2}{4}$$

So we just need to check when $n\geq M$:

$$\begin{aligned}\left(\sum_{k\geq N}|\langle v_n,e_k\rangle|^2\right)^{\frac{1}{2}}&=\left(\sum_{k>N}|v_n-v,e_k|^2+\sum_{k>N}|\langle v,e_k\rangle|^2\right)^{\frac{1}{2}}\\&\leq\left(\sum_{k>N}|v_n-v,e_k|^2\right)^{\frac{1}{2}}+\left(\sum_{k>N}|\langle v,e_k\rangle|^2\right)^{\frac{1}{2}}\end{aligned}$$

For the first part because $\|v_n - v\|^2 < \dfrac{\varepsilon}{2}$:

$$\sum_{k>N} |v_n - v, e_k|^2 \leq \frac{\varepsilon}{2}$$

and for the second part because of the 3:

$$\sum_{k>N} |\langle v, e_k \rangle| \leq \frac{\varepsilon}{2}$$

In conclusion, $K = \{v_n\} \cup \{v\}$ will fit the **equi-small tails**. $\square$

**Proposition 1.** *$\mathcal{F}$ is a closed and bounded set in Hilbert space, then $\mathcal{F}$ is a compact set* ***if and only if*** *for arbitrary $\varepsilon > 0$ there exists an $N$ then for any $v \in \mathcal{F}$ the series:*

$$\sum_{n=N+1}^{+\infty} |a_n e_n|^2 = \sum_{n=N+1}^{+\infty} |\langle v, e_k \rangle|^2 \leq \varepsilon^2 \tag{4}$$

*converges uniformly* ***(equial-tail small)****. Where $\{e_k\}_{n=1}^{\infty}$ is a countable orthogonal normal basis in the Hilbert space.*

*Proof.* For the forward direction, suppose that $\mathcal{F}$ is compact. According to the compact theory, we can easily know that $K$ is closed and bounded. Now let's prove $\mathcal{F}$ also has **equial-samll tails** (4).

Suppose otherwise: then there exists some $\varepsilon_0$ such that for each natural $N$, there is some $u_N \in K$ such that:

$$\sum_{n=N+1}^{+\infty} |\langle u_N, e_k \rangle|^2 \geq \varepsilon_0^2 \tag{5}$$

This then gives us a sequence $\{u_n\} \subseteq \mathcal{F}$ by picking such a $u_N$ for every natural number $N$. Because $\mathcal{F}$ is a compact set, there must exist a sub-sequence $\{v_m\}$ such that, $v_m \to v$ where $v \in \mathcal{F}$. According to the lemma 1, $\{v_m\} \cup \{v\}$ is also a compact set because every sub-sequence in $\{v_m\} \cup \{v\}$ is convergent in $\{v_m\} \cup \{v\}$. What's more this compact set also satisfies the **equial-tail small** property. Now we have a contradiction because of the equation 5, this equation tells us that every component in $\{u_n\}$ does not have **equial-tail small** property. So we can know that $\mathcal{F}$ also has the **equial-tail small** property.

On the other hand, suppose $\mathcal{F}$ is closed, bounded, and has **equi-small tails**. We wish to show that any sequence $u_n$ has a convergent sub-sequence in $\mathcal{F}$. Because $\mathcal{F}$ is closed, any sequence that converges will converge in $\mathcal{F}$, so we just need to show that there is some convergent sub-sequence.

Because, $\mathcal{F}$ is bounded, so $\{u_n\}$ is bounded on every dimension. When $k = 1$, the sub-sequence in $\{\langle u_n, e_1 \rangle\}$:

$$\{\langle u_{n_1(i), e_1} \rangle\}$$

will converges in $\mathbb{C}$. Then extract a sub-sub-sequence from $\{\langle u_{n_1(i), e_1} \rangle\}$, which can converge in $e_2$ $(k = 2)$.

$$\{u_{n_2(j)}, e_2\}$$

Then, we can repeat this argument, for arbitrary $N < +\infty$, we can get a sub-sequence:

$$\{\langle u_{N_k(l)}, e_N \rangle\}$$

which can converge in the first $l$ entries. If now we define:

$$v_l = u_{n_N(l)} \quad \forall l \in \mathbb{N}$$

Now, $\{v_l\}$ formed by $\{u_n\}$ will converge in the first $N^{th}$ dimension as $l \to \infty$. At this moment, we cannot prove that $\{v_l\}$ is convergent. But because of the **equi-small trails** assumption, for all $\varepsilon > 0$, there is some $n \geq N$ so that for all $v \in K$, we have

$$\sum_{k>N} |\langle v, e_k \rangle|^2 < \varepsilon^2$$

So, there exists $N$ such that:

$$\sum_{k>N} |\langle v_l, e_k \rangle|^2 < \frac{\varepsilon^2}{16}$$

for all $l \in \mathbb{N}$. Now because $v_l$ converges in the first $N^{th}$ dimension, so we can find a $M$ such for $l, m \geq M$:

$$\sum_{k=1}^{N} |\langle v_l, e_k \rangle - \langle v_m, e_k \rangle|^2 < \frac{\varepsilon^2}{4}$$

Now, $\{v_l\}$ is what we need, indeed:

$$\begin{aligned}
\|v_l - v_m\| &= \left[ \sum_{k=1}^{N} |\langle v_l - v_m, e_k \rangle|^2 + \sum_{k>N} |\langle v_l - v_m, e_k \rangle|^2 \right]^{\frac{1}{2}} \\
&\leq [\sum_{k=1}^{N} |\langle v_l - v_m, e_k \rangle|^2]^{\frac{1}{2}} + [\sum_{k>N} |\langle v_l - v_m, e_k \rangle|^2]^{\frac{1}{2}}
\end{aligned}$$

where $l, m > M$. By the choice of M, the first term can be smaller than $\dfrac{\varepsilon}{2}$, then:

$$\begin{aligned}
\|v_l - v_m\| < \frac{\varepsilon}{2} + \underbrace{[\sum_{k>N} |\langle v_l, e_k \rangle|^2]^{\frac{1}{2}}}_{equi-smalltrails} + \underbrace{[\sum_{k>N} |\langle v_m, e_k \rangle|]^{\frac{1}{2}}}_{equi-smalltrails} &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
&= \varepsilon
\end{aligned}$$

In conclusion, $v_l$ is a Cauchy thus convergence. We have already proved that every sequence will have a sub-sequence convergent. So $\mathcal{F}$ is a Compact set.

□

3.2. **Metric Space.** Now, let's consider a functional space. We define the distance function like: $d(f, g) = \max_{x \in [0,1]} |f(x) - g(x)|$, then we have a metric space. Because:

Firstly, $d(f, g) = 0 \Leftrightarrow f = g$:

$$\begin{aligned}
d(f, g) &\equiv 0 \\
\Leftrightarrow \max_{x \in [0,1]} |f - g| &\equiv 0 \\
\Leftrightarrow f(x) &\equiv g(x)
\end{aligned}$$

Then, we need to prove $d(f, g) = d(g, f)$:

$$d(f, g) = \max_{x \in [0,1]} |f - g| = \max_{x \in [0,1]} |g - f| = d(g, f)$$

Finally, the triangle inequality:

$$\begin{aligned} d(f,g) &= \max_{x\in[0,1]} |f-g| \\ &= \max_{x\in[0,1]} |f-h-(g-h)| \\ &\le \max_{x\in[0,1]} |f-h| + \max_{x\in[0,1]} |g-h| \\ &= d(f,h)+d(h,g) \end{aligned}$$

So, this is a metric space. □

Now let's consider a subspace in functional space.

$$K = \{f; f(0)=0,\ |f(t)-f(s)| \le L|t-s| \quad t,s\in[0,1]\}$$

### 3.3. **The second question: Is K a compact set?** The answer is, yes.

**Lemma 2.** *If $K$ is in a subspace of the complete metric space, then the property of closedness is equivalent to completeness.*

According to the Lemma 1, we need to prove three things (K in a subspace of the complete space):

- Totally bounded
- Boundedness
- Closedness (Because K is a subspace of the complete metric space, closedness is equivalent to completeness, so we don't need to prove K is a complete subspace).

3.3.1. *Boundedness.* Let the function $f \equiv 0$ be the circle's center and $\varepsilon$ be the circle's radius. Let the $\varepsilon > L$, then we can easily find that this circle can cover the K.

3.3.2. *Closedness.*

- $f_n \to f, \quad f_n(0)=0 \quad \Rightarrow \quad f(0)=0.$
- $|f_n(s)-f_n(t)| < L|s-t| \quad \Rightarrow \quad |f(s)-f(t)| \le L|s-t| \quad n\to\infty$

So, $K' \subseteq K$, K is a closed set and also a complete set.

3.3.3. *Totally bounded.* Now we give a cover method. K is a Lipchitz continuous function space with uniformly L. In this figure 7, the largest triangle represents K in $\mathbb{R}^2$. Then, the idea comes from the option of binary tree pricing. Every blue curve in the largest triangle is a point in $\varepsilon$-net.

In this figure, there are 4 periods, in the first period, there are only $2^1$ cover functions. But when at the 4-period, there are $2^4$ functions. After this observation, we can imply we will have $2^n$ functions, at n-period.

**Remark:** Cover functions represent the blue curves in figure 7, which are the center of the circles. These circles are then used to cover $K$.

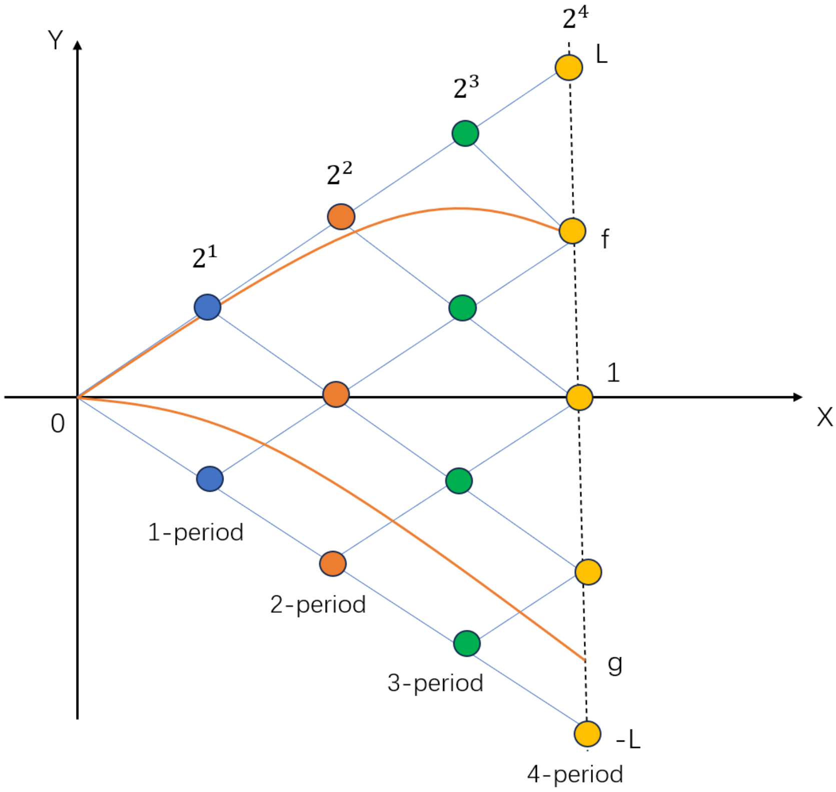


FIGURE 7. Cover method

If we look more closely, we may find that the maximum distance between functions in $K$ and cover functions equals $\frac{L}{n}$(when we have n periods). So when the radius $\varepsilon > \frac{L}{n}$, then we can make sure this cover method can cover all functions in $K$.

Now, we can prove that $K$ is totally bounded. Therefore, K is a compact set. □

3.4. **what is the $n(\varepsilon)$ of K?.** Let's consider figure 7 again, the idea comes from the binary tree. In the first period, when the $\frac{L}{n} < \varepsilon < L$, we need $2^1$ functions at least (Analogy figure 2.1 cover method).

Now, according to figure 7 the binary tree method to cover compact $K$, we can know that when the radius $\frac{L}{n} < \varepsilon < \frac{2L}{n}$, number of cover functions equals $2^n$. Because our coverage is a critical condition coverage, so if the number of coverage is lower than this value, $K$ must not be covered. In conclusion, we can estimate the $n(\varepsilon)$ like:

$$\frac{L}{\varepsilon} < n < \frac{2L}{\varepsilon}$$
$$2^{\frac{L}{\varepsilon}} < 2^n < 2^{\frac{2L}{\varepsilon}}$$

So we can get that:

$$C_1 2^{\frac{L}{\varepsilon}} < n(\varepsilon) < C_2 2^{\frac{L}{\varepsilon}}$$

Finally, our estimation is:

$$n(\varepsilon) \asymp 2^{\frac{L}{\varepsilon}}$$

□

**References:**

- https://ocw.mit.edu/courses/18-102-introduction-to-functional-analysis-spring-2021