Supplemantary question:

$\lim_{n\to\infty} a_n = a$. $\lim_{n\to\infty} b_n = b$ prove:

$$\lim_{n\to\infty} \frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n} = ab.$$

Assume $a=b=0$. (or let $a_n = a_n - b$ and $b_n = b_n - b$)

$\Rightarrow$ $|a_n| \le M$. $|b_n| \le M$.

$\forall \varepsilon>0$. $\exists N \in \mathbb{N}^+$. $|a_n| \le \varepsilon$. $|b_n| \le \varepsilon$. let $n > 2N$.

Origin formula $= \left|\frac{\sum_{k=1}^{n} a_k b_{n-k+1}}{n}\right| \le \left|\frac{\sum_{k=1}^{N} a_k b_{n-k+1}}{n}\right| + \left|\frac{\sum_{k=N+1}^{n} a_k b_{n-k+1}}{n}\right|$ ($n-k+1>N$; $k>N$)

$\le \frac{N\cdot M\cdot\varepsilon}{n} + \frac{M(n-N-1)\varepsilon}{n} \to 0$ $(n\to\infty)$ □

数学分析I

第1次讨论班

2024 年 11 月 6 日

1. What is the "mathematical induction".

2. Sum to Product formula.

$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$.

1. 试用$\varepsilon - N$语言证明下列极限:

(1) $\lim_{n\to\infty} \frac{3}{\sqrt[3]{n}+1} = 0$ find an appropriate N.

(2) $\lim_{n\to\infty} \frac{2n^3+1}{n^3+2n^2+4} = 2$; find an appropriate N.

2. 判断以下数列是否收敛，若收敛则求出极限，若发散则给出证明:

(1)$x_n = \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{\ln n}$ Stolz theorem.

(2)$x_n = \sum_{k=n^2}^{(n+1)^2} \frac{1}{\sqrt{k}}$ Squeeze theorem.

(3)$x_n = \frac{1!+2!+\cdots+n!}{n!}$ Stolz theorem.

(4)$x_n = \sqrt[n]{n!}$ $\to +\infty$

(4). $\frac{n}{\sqrt[n]{n!}} \sim e$ $(n\to\infty)$

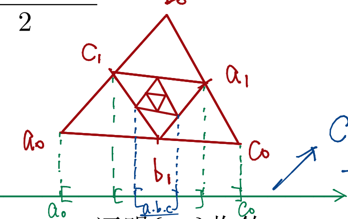
$\boxed{n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \to$ Stirling's approximation

(5)$x_n = \frac{1+\sqrt{2}+\cdots+\sqrt{n}}{n\sqrt{n}}$ Stolz $= \frac{\sqrt{n}}{n\sqrt{n}-(n-1)\sqrt{n-1}} = \frac{\sqrt{n}\,(n\sqrt{n}+(n-1)\sqrt{n-1})}{n^3-(n-1)^3} = \frac{n^2+(n-1)^{\frac{3}{2}}\sqrt{n}}{n^3-(n-1)^3} \sim \frac{2}{C_3^2}$

(6)$a_n = \frac{(2n-1)!!}{(2n)!!} = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}$

$\frac{1\cdot\sqrt{3}}{2}\cdot\frac{\sqrt{3}\cdot\sqrt{5}}{4}\cdots\frac{\sqrt{2n-1}}{2n} \le \frac{\sqrt{2n-1}}{2n} \to 0$.

3. (1)设 a, b, c 是三个给定的实数,令$a_0 = a, b_0 = b, c_0 = c$,定义
$$\begin{cases} a_n = \frac{b_{n-1}+c_{n-1}}{2}, \\ b_n = \frac{c_{n-1}+a_{n-1}}{2}, \\ c_n = \frac{a_{n-1}+b_{n-1}}{2} \end{cases} \quad (n = 1, 2, 3, \cdots).$$
证明: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = \frac{1}{3}(a+b+c)$.



Cantor theorem.

$x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$

$\Downarrow$

$x_{n+1} - x_n = -\frac{1}{2}(x_n - x_{n-1})$

$\Downarrow$

$Q_n = -\frac{1}{2} Q_{n-1}$

$= (-\frac{1}{2})^{n-1}(B-A)$

$\sum Q_n = A + \frac{2}{3}B - \frac{2}{3}A$

$= \frac{1}{3}A + \frac{2}{3}B$.

(2)已知$x_1 = A, x_2 = B$，且$x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$ $(x \ge 2)$，求$\lim_{n\to\infty} x_n$.

A B $\lim_{n\to\infty} x_{2k+1} = C$ $\lim_{n\to\infty} x_{2k} = D$ Because $|x_{2k+1} - x_{2k}| \to 0$

1 3 5 4 2

4. (1)若数列$\{x_n\}$满足$|x_{n+1} - x_n| \le k|x_n - x_{n-1}|$，其中$0 < k < 1$为常数，$n \ge 2$，证明$\{x_n\}$收敛

(2)设$0 < x_1 < \frac{\pi}{2}$，$x_{n+1} = \cos x_n$ $(n \ge 1)$，判断$\{x_n\}$的敛散性 $f(x_n) = x_{n+1}$

Contracting maps theorem. $\boxed{\text{If I have Time}}$

5. 设$a > 0$, $x_1 \in (0,1)$, $x_{n+1} = ax_n(1-x_n)$, $n = 1, 2, \ldots$

Monotonic bounded principle.

(1)若$0 < a < 1$,证明$\{x_n\}$收敛; $\Rightarrow$ $x_n \in (0,1)$ $\frac{x_{n+1}}{x_n} < 1$. $x_n \uparrow$.

$(\le)$ $(\le)$ $\le$ we can let it be equal.

(2)若$a = 1$,证明$\lim_{n\to\infty} nx_n = 1$; $A = A(1-A) \Rightarrow A = 0$.

$\boxed{b_n = \frac{1}{x_n}}$ $\frac{n}{b_n}$ $= \frac{1}{\frac{1}{x_{n+1}} - \frac{1}{x_n}} = \frac{x_n \cdot x_{n+1}}{x_n - x_{n+1}} = \frac{x_n^2(1-x_n)}{x_n - x_n + x_n^2} = 1(n\to\infty)$

Fibonacci sequence $\{1, 1, 2, 3, 5, 8, 13, 21, \cdots\}$

$F_{n+1} = F_n + F_{n-1} \ge 2F_{n-1} \Rightarrow \frac{1}{F_{n+1}} \le \frac{1}{2F_{n-1}}$

6. (1)$\{F_n\}$为斐波那契数列，即$F_1 = 1, F_2 = 1, F_{n+1} = F_n + F_{n-1}$ $(n \ge 2)$，判断$x_n = \sum_{k=1}^{n} \frac{1}{F_k}$的敛散性

$\frac{1}{F_3} + \frac{1}{F_4} \le \frac{1}{2}\left(\frac{1}{F_1} + \frac{1}{F_2}\right)$

$\sum_{k=1}^{n} \frac{1}{F_k} = \frac{1}{F_1} + \frac{1}{F_2} + \cdots + \frac{1}{F_k} + \cdots$

$= \left(\frac{1}{F_1} + \frac{1}{F_2}\right)\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) < +\infty$

$= 2 \cdot 2 = 4$ □

(2)已知$p > 0$，讨论$y_n = \sum_{k=1}^{n} \frac{1}{n^p}$的敛散性