数学分析 I

第 3 次讨论班

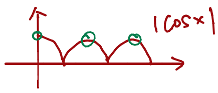
2024 年 11 月 17 日

1. 指出下列函数的间断点并说明类型.

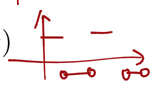
(1) $f(x) = x + \dfrac{1}{x}$ $\quad x=0 \quad f(x)\to+\infty \quad x\to 0^+$

(2) $f(x) = \dfrac{\sin x}{|x|}$ $\quad \lim\limits_{x\to 0^+} f(x) = 1 \ . \ \lim\limits_{x\to 0^-} f(x) = -1 .$

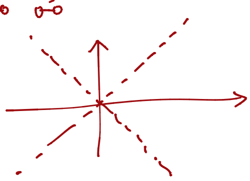
(3) $f(x) = [|\cos x|]$



(4) $f(x) = sgn\,(\cos x)$



(5) $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ -x & x \notin \mathbb{Q} \end{cases}$



$\lim\limits_{x\to 0} f(x) = 0.$ if $x \neq 0$, then $f(x)$ doesn't exist limit .

(6) $f(x) = \begin{cases} \dfrac{1}{x+7} & x < -7 \\ x & -7 \leqslant x \leqslant 1 \\ (x-1)\sin\dfrac{1}{x-1} & x > 1 \end{cases}$

$\lim\limits_{x\to -7^-} f(x) \to +\infty$

$\lim\limits_{x\to -7^+} f(x) = -7$; $\lim\limits_{x\to 1^-} f(x) = 1$

$\lim\limits_{x\to 1^+} f(x) = 0$

$\}\Rightarrow$ no-limit discontinuity

$\}\Rightarrow$ jump discontinuity

2. 求下列函数的导数. Recall : $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$ $\quad (\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$

$(\arctan x)' = \dfrac{1}{1+x^2}$ $\quad (\operatorname{arccot} x)' = -\dfrac{1}{1+x^2}$

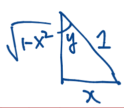
(1) $y = \dfrac{2x}{1-x^2}$

(2) $y = \dfrac{(1-x)^p}{(1+x)^q}$

How to calculate the derivative of $\arcsin x$?

Hint : $y = \arcsin x \Rightarrow x = \sin y . \Rightarrow x' = \cos y = \sqrt{1-x^2}$

(3) $y = \ln\left[\ln\left(\ln x\right)\right]$



(4) $y = \ln\left(\arccos\dfrac{1}{\sqrt{x}}\right)$

According to Derivative rule for inverse functions: $\boxed{(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}}$ $\square$

(5) $y = x^{x^x}$

$y' = (x^{x^x})'$

$\ln y = x^x \ln x$ $\leftarrow$ put this back

Firstly: $g = x^x \Rightarrow \ln g = x\ln x$

Differentiate $\dfrac{g'}{g} = \ln x + 1 \Rightarrow g'(x) = x^x(\ln x + 1)$

$\Rightarrow \dfrac{y'}{y} = x^x(\ln x+1)\cdot\ln x + x^{x-1}$

$\Rightarrow y' = (x^{x^x})' = x^{x^x}\left(x^x(\ln x+1)\ln x + x^{x-1}\right)$ $\square$

$f'''(x) \Rightarrow f''(x) \Rightarrow \cdots \Rightarrow f^{n}(x)$.

$\Downarrow$

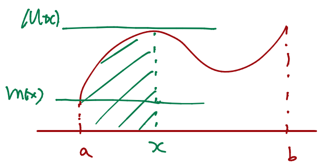
Recursion formula.

3. 设 $f(x) = (\arcsin x)^2$, 求 $f^{(n)}(0)$.

4. 设 $f(0) = 0, f'(0) = A$, 求极限 $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f\left(\frac{k}{n^2}\right)$. $\Rightarrow$ Taylor formula.

$f(\frac{k}{n^2}) = f(0) + f'(0)\frac{k}{n^2} + o(\frac{k}{n^2})$ Substitute into Taylor expansion.

5. 设函数 $f(x)$ 在区间 $[a, b]$ 上有界, 试证函数



$$m(x) = \inf_{a \leqslant t < x} f(t) \quad , \quad M(x) = \sup_{a \leqslant t < x} f(t)$$

$M(x)\uparrow$. $m(x)\downarrow$: $\forall x_0 \in [a,b]$, $M(x_0) - \varepsilon \leq f(x_1) \leq M(x_1) \leq M(x_0) \leq M(x_0) + \varepsilon$ $\exists x_1 \in [a, x_0]$ $\sup f(x)$

在 $[a, b]$ 上左连续, 并举例说明可以不右连续. $\boxed{\text{left Continuity}}$

$\forall \varepsilon > 0$, $\exists \delta = x_0 - x_1$, when $x \in (x_0 - \delta, x_0) \Rightarrow |M(x) - M(x_0)| \leq \varepsilon$

$M(x_0) - \varepsilon \leq M(x_1) \leq M(x) \leq M(x_0)$ $(x_0 \to x)$

$(x_1 \leq x \leq x_0)$

6. 设 $f(x)$ 在 $[a, +\infty)$ 上一致连续. $\varphi(x)$ 在 $[a, +\infty)$ 上连续. $\lim\limits_{x\to+\infty} [f(x) - \varphi(x)] = 0$, 证明: $\varphi(x)$ 在 $[a, +\infty)$ 上一致连续.

7. 设 $f(x)$ 在 $(-\infty, +\infty)$ 上一致连续, 则存在非负实数 a 与 b, 使得对一切 $x \in (-\infty, +\infty)$ 都有

$$|f(x)| \leqslant a|x| + b$$

试证明此结论.

When need to find the uniform $\boxed{N}$.

8. 设 $f(x)$ 在 $[0, +\infty)$ 上一致连续, 且对任意 $x > 0$ 都有 $\lim\limits_{n\to\infty} f(x+n) = 0\,(n \in \mathbb{N}^*)$, 试证: $\lim\limits_{x\to+\infty} f(x) = 0$.

若将题设条件中 $f(x)$ 一致连续改为连续, 结果如何?

$|f(x) - f(x_i+n)| < \varepsilon/2$ uniformly continuous

$|f(x_i+n)| < \varepsilon/2$ According to $\lim\limits_{n\to\infty} f(x_i) = 0$

$\mathbb{R}^+ = \bigcup\limits_{n=1}^{\infty} K_n = \{x : |f(nx)| \leq \varepsilon\}$ ① $f: \mathbb{R}^+ \to \mathbb{R}$. $\lim\limits_{n\to\infty} f(x+n) = 0$ $\xrightarrow[\text{Continuous}]{\text{uniformly}}$ $\lim\limits_{x\to+\infty} f(x) = 0$.

Baire theorem $\xrightarrow[\text{Analysis}]{\text{functional}}$ ② $f: \mathbb{R}^+ \to \mathbb{R}$ $\lim\limits_{n\to\infty} f(nx) = 0$ $\xrightarrow{\text{Continuous}}$ $\lim\limits_{x\to+\infty} f(x) = 0$.

$|f(x)| \leq \underbrace{|f(x_i+n) - f(x)|}_{\text{uniformly Continuous}} + \underbrace{|f(x_i+n)|}_{\lim\limits_{n\to\infty} f(x_i+n) = 0}$ (we use x_i because we find uniform N) finite