数学分析 I

第 4 次讨论班

2024 年 11 月 21 日

6. $f(x)=x^n\ln x$, $n\in\mathbb{N}^+$. Please calculate $\lim\limits_{n\to\infty}\frac{f^{(n)}(\frac{1}{n})}{n!}$.

$f'(x)=nx^{n-1}\ln x+x^{n-1}$

denote: $f_n(x)=x^n\ln x$. then: $f_n'(x)=nf_{n-1}(x)+x^{n-1}$ $(n=0,1,\cdots)$

$\Downarrow$ differentiate $(n-1)$ times again.

$f_n^{(n)}(x)=nf_{n-1}^{(n-1)}(x)+(n-1)!$

$=: g_n(x)$

$\frac{f_n^{(n)}}{n!}=\frac{f_n^{(n-1)}}{(n-1)!}+\frac{1}{n}.$

$g_n=g_{n-1}+1=g_{n-2}+1+\frac{1}{2}=\cdots=\overset{\ln x}{g_0}+(1+\frac{1}{2}+\cdots+\frac{1}{n})\Big|_{x=\frac{1}{n}}$

$=1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\longrightarrow$ Euler Constant

□

1. 写出下列高阶导数公式.

(1) $(a^x)^{(n)}$

(2) $(e^x)^{(n)}$

(3) $(\sin x)^{(n)}$

(4) $(\cos x)^{(n)}$

(5) $(x^m)^{(n)}$

(6) $(\ln x)^{(n)}$

2. 求下列函数的微分.

(1) $y=x^2\cos 2x$

(2) $y=\dfrac{x}{1-x^2}$

(3) $y=e^{ax}\sin bx$

(4) $y=\arcsin\sqrt{1-x^2}$

$\lim\limits_{x\to\frac{1}{2}^-}\frac{f(x)-\frac{1}{2}}{x-\frac{1}{2}}\le\frac{q(x)-\frac{1}{2}}{x-\frac{1}{2}}=\frac{-x+\frac{1}{2}}{x-\frac{1}{2}}=-1$

3. 设 $p(x)=x, q(x)=1-x$, $f(x)$ 为多项式, 又有 $f(x)\geqslant p(x), f(x)\ \cancel{\geqslant}\ \leqslant q(x)\ (\forall x\in(-\infty,+\infty))$. 试证: $f\left(\dfrac{1}{2}\right)>\dfrac{1}{2}$.

$f'(x)\big|_{x=\frac{1}{2}}\Rightarrow\lim\limits_{x\to\frac{1}{2}^+}\frac{f(x)-\frac{1}{2}}{x-\frac{1}{2}}\le\frac{p(x)-\frac{1}{2}}{x-\frac{1}{2}}=1$

Contradiction!

4. 证明: 若函数

$$f(x)=\begin{cases}e^{-\frac{1}{x^2}}, & x\neq 0\\ 0, & x=0\end{cases}$$

Use the definition of the derivative.

$\left(e^{-\frac{1}{x^3}}\right)'=\cancel{e^{-\frac{1}{x^2}}\cdot x^{-3}\cdot 2}$ ✗.

Because we don't know whether $f(x)$ can differentiate at $x=0$ or not.

则在 $x=0$ 处 $f^{(n)}(0)=0,(n=1,2,\cdots)$.

5. 设 $f(x)$ 在 $x=0$ 处可导, $f(0)\neq 0, f'(0)\neq 0$. 又有

$$\frac{af(h)+bf(2h)-f(0)}{h}=\frac{o(h)}{h}=o(1)\quad(h\to 0)$$

求 a,b 的值. $\begin{cases}a+2b=0\\ a+b-1=0\end{cases}\Rightarrow\begin{cases}a=2\\ b=-1\end{cases}$ □

6. 设 $f(x)=x^n\ln x, n\in\mathbb{N}^*$, 求 $\lim\limits_{n\to\infty}\dfrac{f^{(n)}(1/n)}{n!}$.

7. 设 $f(x)$ 在 $x=x_0$ 处可微, $\alpha_n<x_0<\beta_n,(n=1,2,\cdots)$. 又有 $\lim\limits_{n\to\infty}\alpha_n=\lim\limits_{n\to\infty}\beta_n=x_0$. 证明: $\lim\limits_{n\to\infty}\dfrac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}=f'(x_0)$.

$\left|\frac{f(\beta_n)-f(x_0)+f(x_0)-f(\alpha_n)}{\beta_n-\alpha_n}-f'(x_0)\right|$

$=\overset{\lambda_n}{\boxed{\frac{\beta_n-x_0}{\beta_n-\alpha_n}}}\cdot\underbrace{\left|\frac{f(\beta_n)-f(x_0)}{\beta_n-x_0}-f'(x_0)\right|}_{\le\varepsilon}+\overset{1-\lambda_n}{\boxed{\frac{x_0-\alpha_n}{\beta_n-\alpha_n}}}\cdot\underbrace{\left|\frac{f(x_0)-f(\alpha_n)}{x_0-\alpha_n}-f'(x_0)\right|}_{\le\varepsilon}\xrightarrow{n\to+\infty}\lambda\varepsilon+(1-\lambda)\varepsilon=\varepsilon.$

□