

$$\lim_{n \rightarrow \infty} a_n = a, \quad \lim_{n \rightarrow \infty} b_n = b \quad \text{prove:}$$

$$\lim_{n \rightarrow \infty} \frac{a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1}{n} = ab.$$

$$\Rightarrow |a_n| \leq M, \quad |b_n| \leq M.$$

$\forall \varepsilon > 0. \exists N \in \mathbb{N}. |a_n| \leq \varepsilon. |b_n| \leq \varepsilon. \text{ let } n > 2N.$

$$\begin{aligned} \text{origin formula} &= \left| \frac{\sum_{k=1}^n a_k b_{n-k+1}}{n} \right| \leq \left| \frac{\sum_{k=1}^N \underbrace{a_k b_{n-k+1}}_{n-k+1 > n}}{n} \right| + \left| \frac{\sum_{k=N+1}^n \underbrace{a_k b_{n-k+1}}_{k > n}}{n} \right| \\ &\leq \frac{N \cdot M \cdot \varepsilon}{n} + \frac{M(n-N+1)\varepsilon}{n} \rightarrow 0 \quad (n \rightarrow \infty) \quad \square \end{aligned}$$

↓ ↗ find an appropriate N .
 ↖ ↘
 ↓ find an appropriate N .

Stolz theorem.

Squeeze theorem.

→ $+\infty$  Stolz theorem.

$$Sto_{\text{tot}} = \frac{\sqrt{n}}{n\sqrt{n} - (n-1)\sqrt{n-1}} = \frac{\sqrt{n} (n\sqrt{n} + (n-1)\sqrt{n-1})}{n^3 - (n-1)^3} = \frac{n^2 + (n-1)^{\frac{3}{2}}\sqrt{n}}{n^3 - (n-1)^3} \sim \frac{2}{C_3}$$

$$\frac{1 \cdot \sqrt{3}}{2} \cdot \frac{\sqrt{3} \cdot \sqrt{3}}{4} \cdot \dots \cdot \frac{\sqrt{n-1}}{2n} \leq \frac{\sqrt{n-1}}{2n} \rightarrow 0$$

$$\begin{aligned} x_{n+1} &= \frac{1}{2}(x_n + x_{n-1}) \\ &\Downarrow \\ x_{n+1} - x_n &= -\frac{1}{2}(x_n - x_{n-1}) \\ &\Downarrow \\ \frac{Q_n}{T} &= -\frac{1}{2} Q_{n-1} \\ &\Downarrow \\ \frac{2}{n} Q_n &= A + \frac{2}{3} B - \frac{2}{3} A \\ &= \frac{1}{3} A \\ &\quad + \frac{2}{3} B. \end{aligned}$$

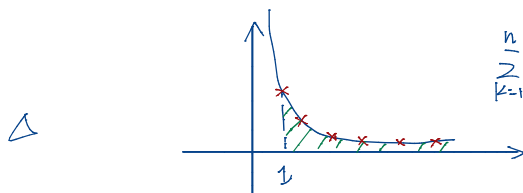
★ Contracting maps theorem. If I have time

$(\leq) \quad (\leq)$

$$\Rightarrow x_n \in (0,1) \quad \frac{x_{n+1}}{x_n} \underset{!}{\prec} 1 \quad \underbrace{x_n \uparrow}_{!}$$

$$\underline{A = A(A - A)} \Rightarrow A = 0.$$

Fibonacci sequence $\{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$



$$\begin{aligned} \sum_{k=1}^n \frac{1}{F_k} &= \frac{1}{F_1} + \frac{1}{F_2} + \dots + \frac{1}{F_k} + \dots \\ &= \underbrace{\left(\frac{1}{F_1} + \frac{1}{F_2} \right)}_2 \cdot \underbrace{\left(1 + \frac{1}{2} + \frac{1}{4} + \dots \right)}_{\delta} \propto +\infty \\ &= 2 \cdot 2 = 4 \end{aligned}$$

1. What is the "mathematical induction".

2. Sum to Product formula:

$$\sin \alpha + \sin \beta = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$$

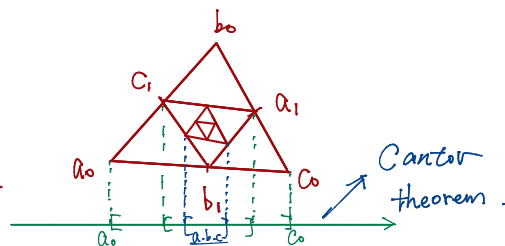
$$\sin \alpha - \sin \beta = 2 \cos \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right)$$

$$\cos \alpha + \cos \beta = 2 \cos \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$$

$$\cos \alpha - \cos \alpha = -2 \sin \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right)$$

(4). $\frac{n}{n \sqrt{n!}} \sim e \quad (n \rightarrow \infty)$

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \rightarrow \text{Stirling's approximation}$$



$$f(x_n) = x_{n+1}$$

Monotonic bounded principle.