L — Smooth condition: $|f(x)-f(y)-\nabla f(y)^T(x-y)| \le \frac{\beta}{2}\|x-y\|^2$.

Let $g(t) =: f(y+t(x-y)) \Rightarrow \begin{cases} f(x) = g(1) \\ f(y) = g(0) \end{cases}$

$f(x)-f(y) = g(1) - g(0) = \int_0^1 g'(t)\,dt$.

$g'(t) = \nabla f(y+t(x-y))^T \cdot (x-y)$

$|f(x)-f(y)-\nabla f(y)^T(x-y)| = |\int_0^1 (\nabla f(y+t(x-y))-\nabla f(y))^T (x-y)\,dt|$

$\le \int_0^1 \|\nabla f(y+t(x-y))-\nabla f(y)\|\cdot\|x-y\|\,dt$

$\le \int_0^1 \beta t \|x-y\|^2\,dt$

$= \frac{\beta}{2}\|x-y\|^2$ $\square$

数学分析 II

第 2 次讨论班

2025 年 3 月 11 日

1. 复习书 p158 的不定积分表及本章例题.

   integration

   解答. 略.

2. 计算下列不定积分.

   $\tan^2 x = \sec^2 x - 1 = \frac{1}{\cos^2 x} - 1$

   (a) $\int x\tan x\sec^2 x\mathrm{d}x = \frac{1}{2}x\tan^2 x - \int \frac{1}{2}\tan^2 x\mathrm{d}x = \frac{1}{2}x\tan^2 x + \frac{1}{2}(x-\tan x) + C$

   $\tan x$

   (b) $\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}} = \int \frac{\mathrm{d}x}{x^2\sqrt{1-\left(\frac{1}{x}\right)^2}} = -\int \frac{\mathrm{d}\left(\frac{1}{x}\right)}{\sqrt{1-\left(\frac{1}{x}\right)^2}} = -\arctan\frac{1}{x} + C$

   (c) $\int(\arcsin x)^2\mathrm{d}x = x(\arcsin x)^2 - \int \frac{2x\arcsin x}{\sqrt{1-x^2}}\mathrm{d}x = x(\arcsin x)^2 + 2\int \arcsin x\mathrm{d}\sqrt{1-x^2} = x(\arcsin x)^2 + 2(\arcsin x\sqrt{1-x^2} - x) + C$

   (d) $I = \int \csc^4 x\mathrm{d}x = \int \csc^2 x\mathrm{d}(-\cot x) = -\cot x\csc^2 x + \int \cot x\mathrm{d}(\csc^2 x) = -\cot x\csc^2 x - 2\int(\csc^4 x - \csc^2 x)\mathrm{d}x = -\cot x\csc^2 x - 2I - 2\cot x$

   $\cot^2 x\cdot\csc^2 x$

   $(\csc^2 x - 1)$

   解得: $I = -\frac{1}{3}\cot x\csc^2 x - \frac{2}{3}\cot x + C$

   (e) $\int \frac{x\arctan x}{(1+x^2)^2}\mathrm{d}x$

   首先用分部积分法:

   $$\int \frac{x\arctan x}{(1+x^2)^2}\mathrm{d}x = -\frac{1}{2}\int \arctan x\mathrm{d}\left(\frac{1}{1+x^2}\right)$$
   $$= -\frac{\arctan x}{2(1+x^2)} + \frac{1}{2}\int \frac{\mathrm{d}\arctan x}{1+x^2}$$

   然后对后一项做代换$\boxed{\arctan x = t,}$有 $\begin{cases} \arctan x = t \\ \tan t = x \end{cases}$.

   $$\frac{1}{2}\int \frac{\mathrm{d}\arctan x}{1+x^2} = \frac{1}{2}\frac{\mathrm{d}t}{1+\tan^2 t}$$
   $$= \frac{1}{2}\int \cos^2 t\mathrm{d}t = \frac{1}{4}\int(\cos 2t + 1)\,\mathrm{d}t$$
   $$= \frac{1}{8}\sin 2t + \frac{1}{4}t + C$$

   $\cos 2t = 2\cos^2 t - 1$

   代入并合并结果后就得到

   $$\int \frac{x\arctan x}{(1+x^2)^2}\mathrm{d}x = \frac{1}{4}\arctan x\left(\frac{x^2-1}{x^2+1}\right) + \frac{1}{4}\frac{x}{(1+x^2)} + C$$

3. 计算有理函数不定积分.

(a)

$$\begin{aligned}&\int \frac{2x^4+x^3+3x^2-2x+1}{x^3+2x-3}\mathrm{d}x\\&=\int\left(2x+1+\frac{-x^2+2x+4}{x^3+2x-3}\right)\mathrm{d}x\\&=x^2+x+\int\left(\frac{1}{x-1}-\frac{2x+1}{x^2+x+3}\right)\mathrm{d}x\\&=x^2+x+\ln|x-1|-\ln(x^2+x+3)+C\end{aligned}$$

(b)

$$\begin{aligned}&\int \frac{x}{(x^2+2x-3)(x^2+2x+2)}\mathrm{d}x\\&=\frac{1}{5}\int\left(\frac{x}{x^2+2x-3}-\frac{x}{x^2+2x+2}\right)\mathrm{d}x\\&=\frac{1}{20}\int\left(\frac{3}{x+3}+\frac{1}{x-1}\right)\mathrm{d}x-\frac{1}{5}\int\frac{\frac{1}{2}(2x+2)-1}{x^2+2x+2}\mathrm{d}x\\&=\frac{3}{20}\ln|x+3|+\frac{1}{20}\ln|x-1|-\frac{1}{10}\ln(x^2+2x+2)+\frac{1}{5}\arctan(x+1)+C\end{aligned}$$

4. 计算无理函数不定积分.

(a) 令 $u=\sqrt{2x^2+3}$, 则

$$\begin{aligned}&\int\frac{1}{x\sqrt{2x^2+3}}\mathrm{d}x\\&=\int\frac{1}{u^2-3}\mathrm{d}u\\&=\frac{1}{2\sqrt{3}}\ln\left|\frac{u-\sqrt{3}}{u+\sqrt{3}}\right|+C\\&=\frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{2x^2+3}-\sqrt{3}}{\sqrt{2x^2+3}+\sqrt{3}}\right|+C\end{aligned}$$

(b) 令 $\sqrt{x^2-x+1}=t-x$, 则 $x=\dfrac{t^2-1}{2t-1}$, $\mathrm{d}x=\dfrac{2(t^2-t+1)}{(2t-1)^2}\mathrm{d}t$, 从而

$\cancel{x^2}-x+1=t^2-2xt+\cancel{x^2}$

$x(2t-1)=t^2-1$

$$\begin{aligned}&\int\frac{1}{x+\sqrt{x^2-x+1}}\mathrm{d}x\\&=\int\frac{2(t^2-t+1)}{t(2t-1)^2}\mathrm{d}t\\&=\int\left(\frac{2}{t}-\frac{3}{2t-1}+\frac{3}{(2t-1)^2}\right)\mathrm{d}t\\&=-\frac{3}{2(2t-1)}+2\ln|t|-\frac{3}{2}\ln|2t-1|+C\\&=-\frac{3}{2(2\sqrt{x^2-x+1}+2x-1)}+2\ln|\sqrt{x^2-x+1}+x|\\&-\frac{3}{2}\ln|2\sqrt{x^2-x+1}+2x-1|+C\end{aligned}$$

(c) 令 $t = \ln(x + \sqrt{1 + x^2})$, 则 $x = \sinh t$, 再由双曲函数的定义可得

$$
\begin{aligned}
&\int \frac{\ln x + \sqrt{1 + x^2}}{(1 + x^2)^{\frac{3}{2}}} \mathrm{d}x \\
= &\int \frac{t}{\cosh^2 t} \mathrm{d}t \\
= &\int t \mathrm{d} \tanh t \\
= &\ t \cdot \tanh t - \ln \cosh t + C \\
= &\ \frac{x}{\sqrt{1 + x^2}} \ln(x + \sqrt{1 + x^2}) - \frac{1}{2} \ln(1 + x^2) + C
\end{aligned}
$$

5. 计算三角函数有关不定积分.

(a) 令 $t = \tan x$, 则

$$
\begin{aligned}
&\int \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} \mathrm{d}x \\
= &\int \frac{1}{a^2 t^2 + b^2} \mathrm{d}t \\
= &\ \frac{1}{ab} \arctan\left(\frac{a}{b} t\right) + C \\
= &\ \frac{1}{ab} \arctan\left(\frac{a}{b} \tan x\right) + C
\end{aligned}
$$

(b) 令 $t = \tan \frac{x}{2}$, 则

$x = 2 \arctan t.$

$x' = 2 \cdot \frac{1}{1+t^2}$

$\Downarrow$

$$
\begin{cases}
dx = \frac{2}{1+t^2} dt. \\
\sin x = \frac{2t}{1+t^2} \\
\cos x = \frac{1-t^2}{1+t^2}.
\end{cases}
$$

$$
\begin{aligned}
&\int \frac{1}{\sin x + 2 \cos x + 3} \mathrm{d}x \\
= &\ 2 \int \frac{1}{t^2 + 2t + 5} \mathrm{d}t \\
= &\ 2 \int \frac{1}{(t + 1)^2 + 4} \mathrm{d}(t + 1) \\
= &\ \arctan \frac{t + 1}{2} + C \\
= &\ \arctan \frac{\tan \frac{x}{2} + 1}{2} + C
\end{aligned}
$$

raise the order

(c) 令 $I = \int \frac{\sin x}{2 \sin x + 3 \cos x} \mathrm{d}x$, $J = \int \frac{\cos x}{2 \sin x + 3 \cos x} \mathrm{d}x$, 则

$$
\begin{cases}
2I + 3J = \int \mathrm{d}x = x + C \\
2J - 3I = \int \frac{1}{2 \sin x + 3 \cos x} \mathrm{d}(2 \sin x + 3 \cos x)
\end{cases}
$$
$$
= \ln |2 \sin x + 3 \cos x| + C
$$

解得: $I = \frac{1}{13} (2x - 3 \ln |2 \sin x + 3 \cos x|) + C$. 同时我们也可以得到另一个不定积分的结果.

6. 导出求不定积分 $I_n = \int \frac{\mathrm{d}x}{(1 + x^2)^n}$ (n是自然数)的递推公式.

解答. 由分部积分法有:

$$\begin{aligned}
I_n=\int\frac{\mathrm{d}x}{(1+x^2)^n}&=\frac{x}{(1+x^2)^n}+2n\int\frac{x^2}{(1+x^2)^{n+1}}\mathrm{d}x\\
&=\frac{x}{(1+x^2)^n}+2n\int\left(\frac{1}{(1+x^2)^n}-\frac{1}{(1+x^2)^{n+1}}\right)\mathrm{d}x\\
&=\frac{x}{(1+x^2)^n}+2nI_n-2nI_{n+1}
\end{aligned}$$

因此得到递推公式:

recurrence formula.

$$I_{n+1}=\frac{1}{2n}\cdot\frac{x}{(1+x^2)^n}+\left(1-\frac{1}{2n}\right)I_n,\ n\in\mathbb{N}^*$$

7. 求不定积分 $\int\frac{1}{1+x^3}\mathrm{d}x,\int\frac{1}{1+x^4}\mathrm{d}x.$

解答.

$$\begin{aligned}
\int\frac{1}{1+x^3}&=\int\frac{1}{(1+x)(x^2-x+1)}\\
&=\frac{1}{3}\int\left(\frac{1}{1+x}+\frac{x-2}{x^2-x+1}\right)\mathrm{d}x\\
&=\frac{1}{3}\int\left(\frac{1}{1+x}+\frac{3}{2}\frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)\mathrm{d}x-\int\frac{1}{2}\frac{\mathrm{d}x^2-x+1}{x^2-x+1}\\
&=\frac{1}{3}\ln|1+x|-\frac{1}{6}\ln|x^2-x+1|+\frac{\sqrt{3}}{3}\arctan\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}
\end{aligned}$$

下面计算另一个不定积分. 首先, 令 $I=\int\frac{1}{1+x^4}\mathrm{d}x,\ J=\int\frac{x^2}{1+x^4}\mathrm{d}x$, 则

$$\begin{aligned}
I+J&=\int\frac{1+x^2}{1+x^4}\mathrm{d}x\\
&=\int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\mathrm{d}x\\
&=\int\frac{\mathrm{d}\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2}\\
&=\frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}x}+C
\end{aligned}$$

同理可以计算得:

$$I-J=-\frac{1}{2\sqrt{2}}\ln\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}+C$$

解得: $I=\frac{1}{2\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}x}-\frac{1}{4\sqrt{2}}\ln\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}+C$