Applied and Numerical Aspects for Nonlocal Initial Value Problems

Anders Bahrami Diego Rubio Garcia Dianlun (Jennifer) Luo Zhekai Liu

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Abstract

Nonlocal operators have recently been used successfully to approximate classical derivatives. A natural question is the formulation of a well-posed nonlocal counterpart to initial value problems. Constructing such a system which admits unique solutions is nontrivial and it is highly dependent on the choice of the interaction kernel. This poster is a companion to the poster "Theoretical aspects for nonlocal initial value problems."

Problem Setup

Problem 1: How can nonlocal models successfully approximate local (classical) models? Consider the nonlocal operator with kernel K_n:

$$\mathcal{D}_{\mathcal{K}n} u(x_0) = \int_{-\infty}^{+\infty} [u(x_0+y) - u(x_0)]\, \mathcal{K}_n(y) \mathrm{d}y$$

Goal: Find a proper kernel function $\mathcal{K}_n(x)$ such that $\mathcal{D}_{\mathcal{K}n} u(x_0) \to u'(x_0)$.

Problem 1: Properties for the kernel functions

We seek a kernel function $\mathcal{K}$ such that $\int_{-\infty}^{+\infty} \mathcal{K}(y)dy = 0$. Suppose we have the sequence $\{\mathcal{K}_i\}_{i\in\mathbb{N}}$ and $u(x)$ is a (sufficiently) smooth function. Perform a Taylor expansion for the function u (of order 3) near the point x (assume y is small):

$$\begin{aligned}\mathcal{D}_{\mathcal{K}_n} u(x) &= \int_{-\infty}^{+\infty} \left[u(x) + u'(x)y + \frac{u''(x)}{2!}y^2 + \frac{u'''(\xi_x)}{3!}y^3 - u(x) \right] \mathcal{K}_n(y) \mathrm{d}y \\ &= u'(x) \int_{-\infty}^{+\infty} y\mathcal{K}_n(y)\mathrm{d}y + \frac{u'''(\xi_n)}{3!} \int_{-\infty}^{+\infty} y^3 \mathcal{K}_n(y)\mathrm{d}y \to u'(x) \quad (n \to +\infty)\end{aligned}$$

These integral operators will successfully approximate the derivative (the last convergence holds) if and only if the kernels satisfy these properties:

- $\lim_{n\to\infty} \int_{-\infty}^{+\infty} y\mathcal{K}_n(y)\mathrm{d}y = 1$
- $\lim_{n\to\infty} \int_{-\infty}^{+\infty} y^3\mathcal{K}_n(y)\mathrm{d}y = 0$
- $\int_0^{+\infty} y^2\mathcal{K}_n(y)\mathrm{d}y < +\infty$

Example of kernel functions

We plot the nonlocal derivatives with kernel functions $\mathcal{K}(y) = \frac{1}{y^{0.5}}\chi_{(0,\alpha]} - \frac{1}{|y|^{0.5}}\chi_{[-\alpha,0)}$ (where $\alpha = (0.75)^{\frac{2}{3}}$ and with the kernel function $\mathcal{K}_n(y) = \frac{1}{h^2}\chi_{[0,h]} - \frac{1}{h^2}\chi_{[-h,0]}$ (where $n=2,\ h=\frac{1}{n}$). The input function $u(x) = \sin(x)$ whose exact (classical) derivative is $cos(x)$:

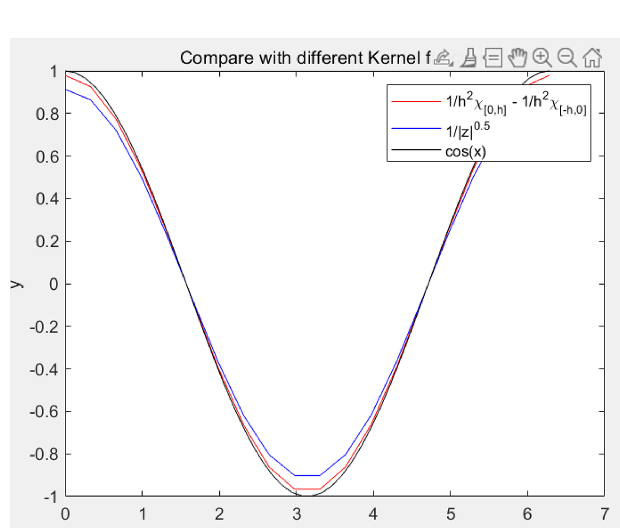


Figure 1. Approximation with different kernels

Problem Setup

Problem 2: With an appropriate kernel function, we want to **numerically** study solutions to the *nonlocal initial value problem* below and see if they approach the *classical solutions*.

$$\begin{cases} \mathcal{D}_{\mathcal{K}} u(x) = f(x, u(x)) \\ \text{Appropriate Initial Conditions} \end{cases}$$

Problem 2: Solve the differential equations

By formulating the nonlocal equation $\mathcal{D}_{\mathcal{K}} u(x) = f(x,u)$ with the (discrete version) $\mathcal{D}_{\mathcal{K}} u_i = \sum_{j=-m}^{m} k_j u_{i+j}$, renders the system of N equations:

$$\begin{cases} \mathcal{D}_{\mathcal{K}} u_0 = f(u_0, x_0) & = k_{-m,0}u_{-m} \cdots + k_{0,0}u_0 \cdots + k_{m,0}u_m \\ \mathcal{D}_{\mathcal{K}} u_1 = f(u_1, x_1) & = k_{-m,1}u_{-m+1} \cdots + k_{0,1}u_1 \cdots + k_{m,1}u_{m+1} \\ & \vdots \\ \mathcal{D}_{\mathcal{K}} u_{N-1} = f(u_{N-1}, x_{N-1}) & = k_{-m,N-1}u_{-m+N} \cdots + k_{0,N-1}u_N \cdots + k_{m,N-1}u_{m+N-1} \end{cases}$$

The solution to the above system "approximates" the classical ODE solution, and we need to provide "initial conditions" (IC) for solvability. In particular, we may provide IC values: $u_{-m}, \ldots, u_{-1}, u_N, \ldots, u_{N-1+m})$

Question: What kernel/coefficients $\{\{k_{i,j}\}_{i=-m}^{m+N-1}\}_{j=0}^{N-1}$ should we use?

Nonlocal method

For this example, we use approximations for the kernel:

$$k(x-y) = \frac{1}{\delta^2}\chi_{[0,\delta]} - \frac{1}{\delta^2}\chi_{[-\delta,0]} \approx \{\{k_{i,j}\}_{i=-m}^{m+N-1}\}_{j=0}^{N-1},$$

where: $\alpha = 0.4$, fixed $\delta = (0.75)^{\frac{2}{3}}$, and $m = 2$. The results:

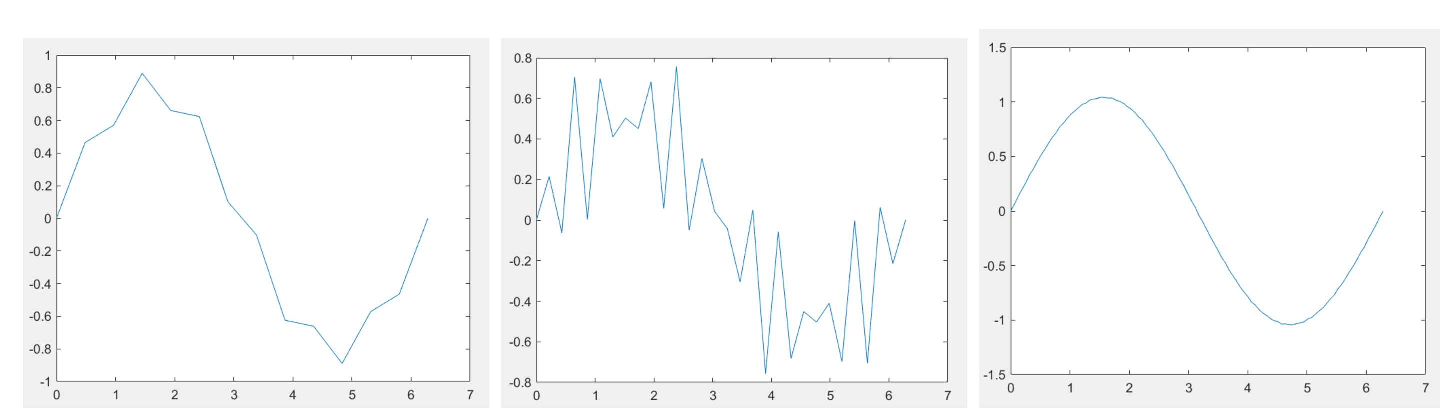


Figure 2. $N = 14, 30, 100$ from left to right

Finite Difference Method

However, you may find that the nonlocal method may not be stable, we find it is equivalent to finding the proper kernel and coefficients, so why not find the coefficients directly? We have proposed new method:

System of Equations:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ -m & -(m-1) & \cdots & m \\ m^2 & (m-1)^2 & \cdots & m^2 \\ \vdots & \vdots & \ddots & \vdots \\ m^{2m} & (m-1)^{2m} & \cdots & m^{2m} \end{pmatrix} \begin{pmatrix} a_{-m} \\ a_{-(m-1)} \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Solvability: The matrix is a square Vandermonde matrix and is invertible if all entries are distinct.

Accuracy: The method's theoretical order of accuracy is $O(h^{2m})$.

Problem Setup

Problem 3: Definition (Nonlocal Initial Value Problem). Let $\delta > 0$. Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ and $u_0 : [-2\delta, 0] \to \mathbb{R}$ are given functions. A *nonlocal initial value problem* (IVP) is defined as:

$$\mathcal{D}_k u(x) = \int_{-\delta}^{\delta} [u(x+y) - u(x)]k(y)\,dy = f(x, u(x)), \quad x \in [-\delta, \delta],$$

$$u(x) = u_0(x), \quad x \in [-2\delta, 0].$$

Theorem (Continuation Argument). Suppose $u(x)$ is a solution to the nonlocal IVP described above. Then, for $x \in [0,\delta)$, $u(x)$ satisfies the following integral equation:

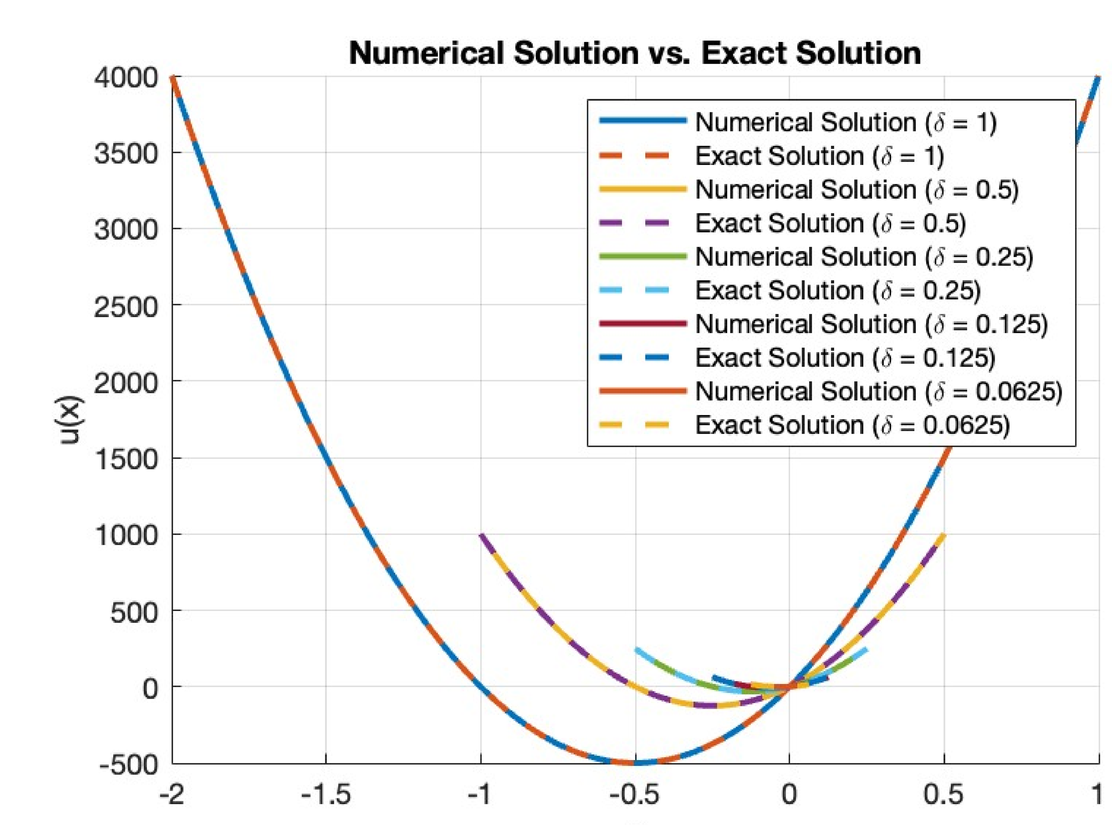
$$u(x)k(\delta) - \int_0^x u(y)k'(y-x+\delta)\,dy = V_\delta'(x),$$

where $V_\delta(x)$ is a known function that depends on the initial data (k, δ, f, u_0).

Numerical Example

We perform numerical tests using the exact solution: $u_{\text{exact}}(x) = A \cdot x^2 + A \cdot \delta \cdot x$, and the source function: $f(x,u) = u - Ax^2 - A\cdot\delta\cdot x + 2Ax + A\cdot\delta$. The initial condition $u_0(x) = u_{\text{exact}}(x)$ for $x \in [-2\delta, 0]$. The kernel $k_\delta(y) = \begin{cases} \frac{3}{2\delta^3}y & \text{if } |y| \le \delta, \\ 0 & \text{if } |y| > \delta. \end{cases}$

For convergence with respect to δ, we fix the step size $h = 1.e-5$ and vary δ. The parameter A is set to $A = 2000$.



δ	L^2 Error	Rate
1.0	$2.981e-05$	–
0.5	$7.454e-06$	2.000
0.25	$1.864e-06$	2.000
0.125	$4.674e-07$	1.996
0.0625	$1.198e-07$	1.964

References

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Advisors: Petronela Radu <pradu@unl.edu> and Mikil Foss <mfoss@unl.edu>, University of Nebraska-Lincoln