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Celestial Mechanics Applications: The Restricted 4-Body Problem and Optimization for Mission Design

Abstract and Team Roles:

The goal of the project is to create a working model for fuel efficient space travel with the ultimate goal of designing a space mission moving a rocket from an orbit around the Earth to the moon based on the model and historical space missions. The authors of the work are Jacob Maarek and Scott Jin, sophomore mechanical and electrical engineering students at the Cooper Union. Jacob Maarek will be responsible for design planning and research, mathematical analysis, half of the code, and acting as first author for the final paper. Scott Jin was be responsible half of the code, creating animations and other demonstrations of results, updating the project page, and acting as first author on the weekly progress reports of the project.

Problem History and Description:

Launching and controlling a spacecraft have long been major topics of the celestial mechanics community. Since the late 20th Century, scientists and applied mathematicians have studied how to control the spacecraft and minimize fuel consumption, flight time and radiation dose, using observational geometry and usage of natural dynamics of the solar systems. There are generally two ways to control the spacecraft into the desired trajectory; one being continuous control where acceleration is continuously given and another being several instantaneous high thrust where high power is given in a short time to change the trajectory of the spacecraft. The first is only ever really used for escaping the gravitational pull of planets as the fuel cost is very expensive. The second is used for interplanetary travel and relies on exploiting the gravitational pulls of local masses for most of the transfer of energy and change of trajectory, consequently making the space design more efficient.

Figure 1: Mission Design Trajectory

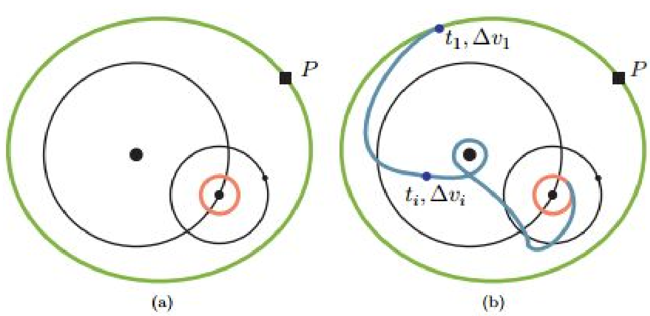

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Figure 1 demonstrates the trajectory of a rocket in a 4-body system with the design goal of moving from the red orbit to the green orbit. The dots on the blue trajectory represent

quantized points of instantaneous acceleration of the rocket needed to accomplish the design goal.

For early mission designs like the Voyager and Galileo, a model called the Patched two-body approximation seems to suffice the need and produce pretty accurate extrapolations. Essentially, the model simulates an N-body celestial system into a series of two-body solutions; the heliocentric hyperbolic trajectory of the spacecraft were found, and the center of the trajectory was changed continuously with the spacecraft enters each planet's "sphere of influence". Scientists such as Edward Belbruno improved on the work for cases where the two body approximation failed by using three and four body solutions. Our concern here will mainly be the latter case where the fuel consumption and therefore minimization of the velocity change turns into the focus of the problem. The one drawback however of take a low energy path however is that often travel time is greatly increased. Therefore people in the field of mission design have have to consider the importance of travel time to energy when planning a trajectory. The goal of the project is to calculate the optimal trajectory from an initial parking orbit of radius R_{escape} around the earth to a final parking orbit $R_{capture}$ around the moon. To accomplish this, an accurate model of the 4-body system was created and the initial parameters were optimized.

Methods:

The project is based the research papers *New Methods in Celestial Mechanics and Mission Design* authored by Jerrold Marden of California Institute of Technology and *Design and Optimization of Body-to-Body Impulsive Trajectories in Restricted Four-Body Models* by Fady Morcos of the University of Texas at Austin.

3-Body Problem:

As a first step before tackling the 4-body problem directly, the 3-body system consisting of the Earth-Moon-Rocket system was analyzed.

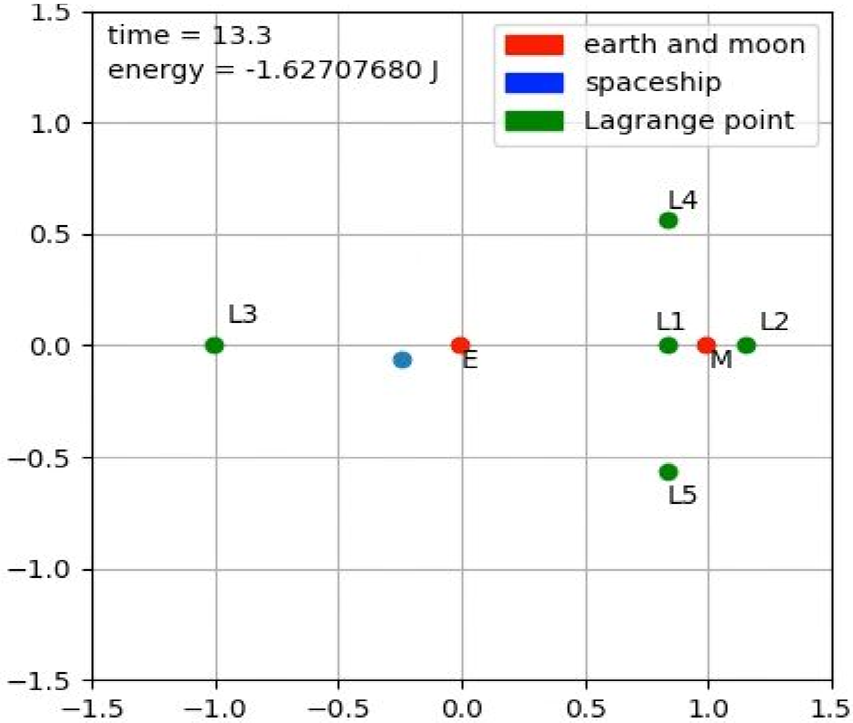
Problem Setup

First the problem was constrained to have all bodies in the x-y plane. This is done to have the equations of motion only described as changes in x and y. As seen in Figure 2, The general solution to to the rocket trajectory was solved in the normalized rotating frame of Earth-Moon system, with the origin at the barycenter of the system. To normalize the system, the combined mass of the primaries and the distance between the primaries was normalized. The normalized mass μ_i for primary i is calculated by the formula.

$$\mu_i = \frac{m_i}{m_{earth} + m_{moon}}$$

Consequently, the positions of the earth and moon in the frame are $-\mu_{moon}$ and $1-\mu_{moon}$,respectively. The other effect of normalizing the system mass and distance is that frame rotates at unity.

Figure 2: 3-Body System in Normalized Rotating Frame (Non-Dimensional Units)



Effective Potential

Given the spaceship position in the frame, the effective gravitational potential the rocket in the system can be calculated. The effective potential represents the gravitational potential energy of a point in the rotating frame. From the effective potential and the instantaneous velocity of the rocket one can calculate the equations of motion of the rocket from the Euler Lagrange Equation. Additionally, one can calculate the Lagrange Points of the system, located at the local maximums and minimums of the system. The L1 and L2 points will be important when optimizing a low energy trajectory as they are at the minimums of the effective potential.

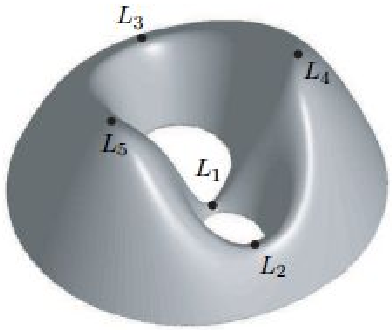
Definition of Effective Potential of a 3 body system:

$$V(x,y) = \frac{\mu_{moon}}{r_{moon}} + \frac{\mu_{earth}}{r_{earth}}$$

$$\overline{V}(x,y) = V(x,y) + \frac{x^2+y^2}{2}$$

$V(x,y)$ = potential energy of system
R = distance from spaceship to primary in normalized frame
$\overline{V}(x,y)$ = effective potential

Figure 3: Effective Potential and Lagrange Points



Effective Potential

Equations of Motion for 3-Body System:
The equations of motion for the 3-body system are derived using the Lagrangian method, using the derivative of the Lagrangian, defined as the difference of the Kinetic and Potential energies, to model how energy flows from kinetic to potential energy. From the transfer of energy, one can calculate the equations of motion of the system.

$$V(x,y) = \frac{\mu_{moon}}{r_{moon}} + \frac{\mu_{earth}}{r_{earth}}$$
$$K(x,y,x',y') = \frac{1}{2}[(x'-y)^2 + (y'+x)^2]$$
$$L(x,y,x',y') = K(x,y,x',y') - V(x,y)$$

Note: As mentioned before all equations are being calculated in the normalized rotating frame. As such the kinetic energy with respect to an inertial frame needs to be corrected from the ½*v^2 term to the equation above.

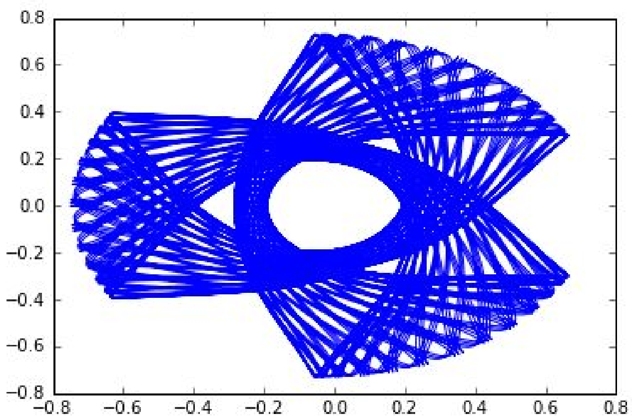
From the Euler-Lagrange Equations:
$$x'' - 2y' = \frac{d\overline{V}(x,y)}{dx}$$
$$y'' + 2x' = \frac{d\overline{V}(x,y)}{dy}$$

Equations of Energy for 3-Body System:
From the governing equations, an equation for the energy of the system can be derived. This was used to show that energy was conserved in the system, demonstrating the validity of the governing equations of motion.
$$E(x,y,x',y') = \frac{1}{2}[x'^2 + y'^2] - \overline{V}(x,y)$$

Results for 3-Body System:

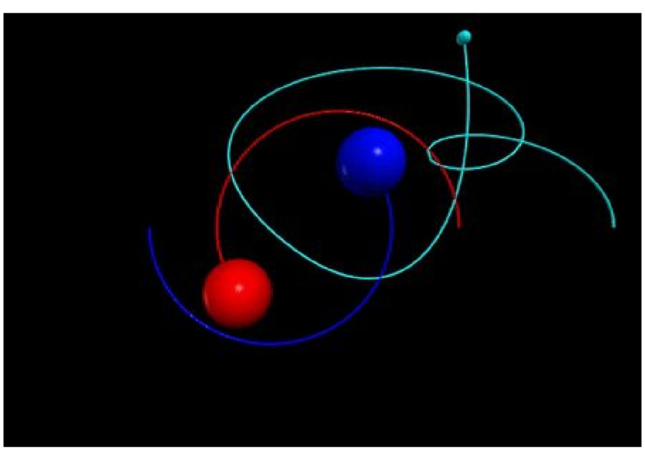
From the governing equations, one knows that the trajectory of the spaceship is governed by two second order equations. While it might not seem intuitive due to use the gradient of the effective potential, the effective potential itself was shown to be a function of the x and y positions. To solve the system of ODEs, The two second order differential equations were turned into 4-first order equations with x' and y' defined as vx and vy and x'' and y'' defined as vx' and vy'. Next an approximation for the gradient of the effective potential was created using central difference theorem. Finally the four first order differential equations were solved using a fixed step RK4 solver. Figure 4 demonstrates the calculated trajectory for the 3-body for a given initial position and velocity.

Figure 4: Trajectory of Spaceship in the normalized rotating frame for a set initial parameters in the 3-body system



Lastly the position of the system in Euclidian space was calculated by rotating the frame by a unit angular velocity and scaling the system by the distance from the Earth to the Moon. Figure 5 demonstrates the travel of the spaceship in Euclidian Space

Figure: 5 Trajectory of 3-body system in Euclidian Space



Evaluation:
To demonstrate correctness of the RK4-solver a conservation of energy test and a time-step reduction test was used. Figure 6 demonstrates the Energy of the system over the time interval, demonstrating that conservation of energy is held. Next the time-step was halved and the error of the system was calculated, testing for the order of the error of the RK-solver. Initially the error was only shown to decrease by a factor of four due to using a second order central difference theorem calculate the gradient of the effective potential. As such the gradient calculator was upgraded to a fourth order approximation. Figure 7 demonstrates the validity of the RK solver.

Figure 6: Energy of the System over the time interval for the 3-body system

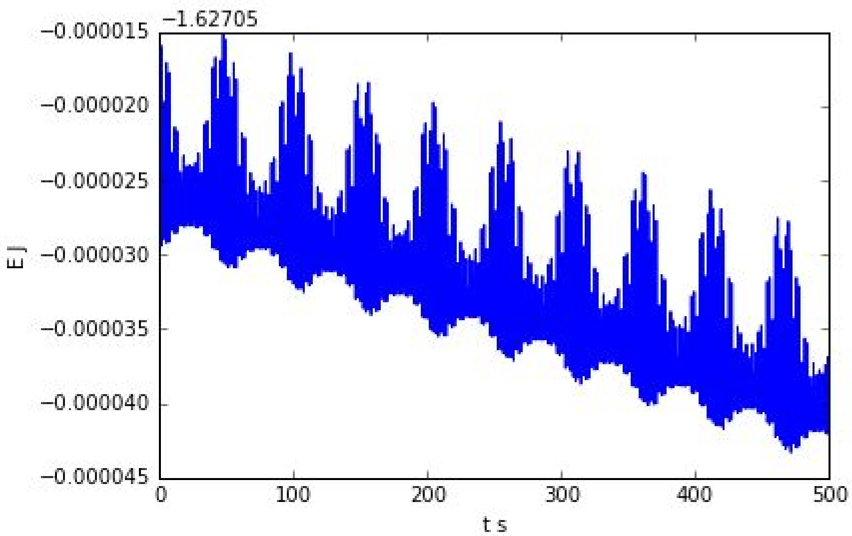


Figure 7: Justification of RK-4 solver

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>>>
 RESTART: /Users/scott/Celestial-Mechanics-Application/Optimum Trajectory in ear
th_moon system/Justification of RK4(error drop).py

for h= 0.5 error x = 0.555285542643 error y = 0.413837173785

for h= 0.25 error x = 0.0963736776923 error y = 0.120269872706

for h= 0.125 error x = 0.00601936669158 error y = 0.0101683736717

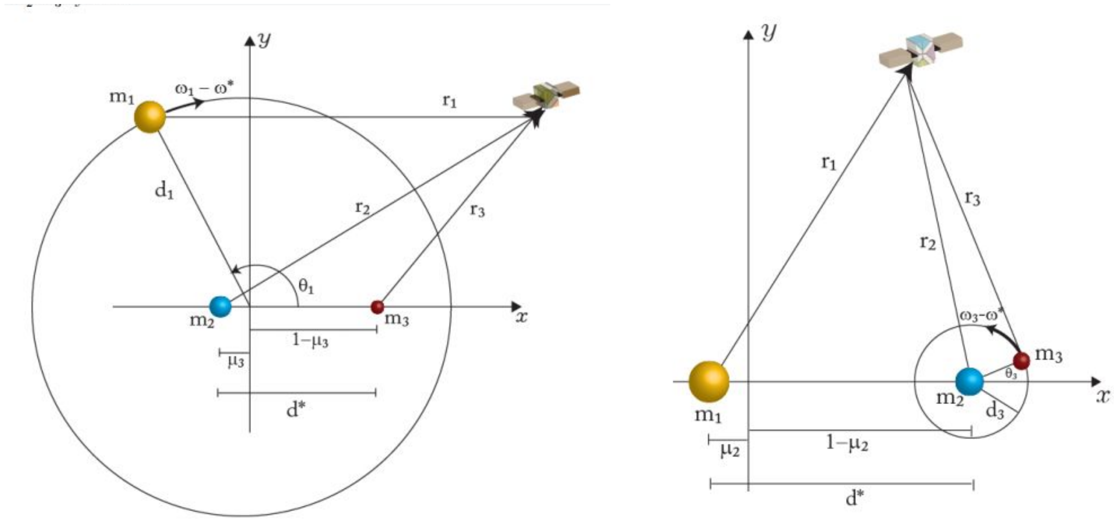
for h= 0.0625 error x = 0.000433123131858 error y = 0.000971415889958
>>>
```

4-Body Problem:

An approximation for the 4-body problem can be developed using the Patched 3-body approximation, which uses solutions developed from the 3-body problem for a numerical procedure which converges to a full four-body solution.

Problem Setup:
As an introduction, the four body problem solved is known as the bicircular restricted 4-body problem, which consists of the earth orbiting around the sun and the moon orbiting around the Earth. like the 3-body problem, the problem is first constrained to have all the objects to be coplanar. However due to having three primaries rather than two a rotating frame of reference needs to be chosen, either the Earth-Moon rotating or the Earth-Sun rotating frame. The Earth-Moon frame has the barycenter of the Earth Moon system at the origin, the earth and the moon fixed on the x-axis, and the sun rotating with a scalar factor of the normalized angular velocity of the Earth Moon velocity. The Earth-Sun frame has the barycenter of the Earth Sun system at the origin, the earth and the sun fixed on the x-axis, and the moon rotating with a scalar factor of the normalized angular velocity of the Earth Sun velocity. After choosing a frame of reference, the process of normalizing the mass and orbit radius described for the 3-body system is used. Figure 8 demonstrates the two frames.

Figure 8: Earth-Moon Rotating Frame versus the Earth-Sun Rotating Frame



Equations of Motion for 4-Body System:
Following choosing a frame, the equations of motion for the system can be approximated This is done using the patched-3 body approximation, which approximates the 4-body system as two three body systems and using perturbation theory to create an approximation for the 4-body system that converges to the general result. The equations of motion in the earth moon frame are described below.

$$x'' - 2y' = \frac{d\overline{V}(x,y)}{dx} + I_x$$

$$y'' + 2x' = \frac{d\overline{V}(x,y)}{dy} + I_y$$

$$\overline{V}(x,y) = \frac{\mu_{moon}}{r_{moon}} + \frac{\mu_{earth}}{r_{earth}} + \frac{\mu_{sun}}{r_{sun}} + \frac{x^2+y^2}{2}$$

$$I = -\mu_{sun}\frac{\widehat{r}_{sun}}{|\widehat{r}_{sun}|^3}$$

$$\widehat{r}_{sun} = \left[\frac{d_{earth-sun}}{d_{earth-moon}}cos((\frac{\omega_{earth-sun}}{\omega_{earth-moon}} - 1)\cdot t)), \frac{d_{earth-sun}}{d_{earth-moon}}sin((\frac{\omega_{earth-sun}}{\omega_{earth-moon}} - 1)\cdot t))\right]$$

Equations of Energy for 4-Body System:

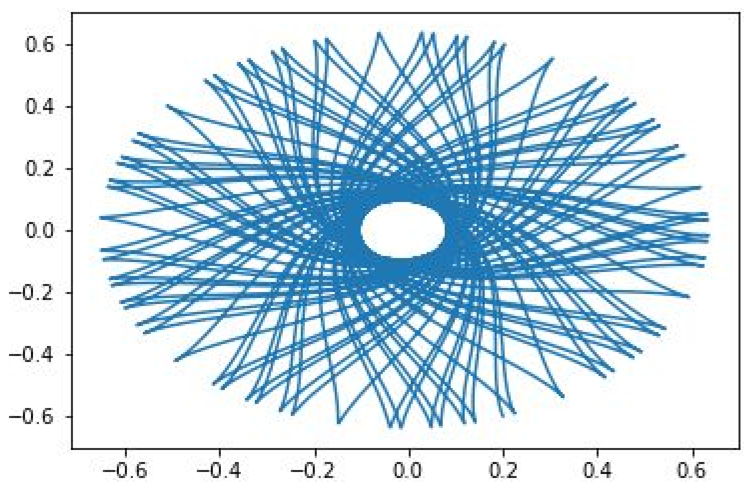
The equation of energy for the 4-body system is identical for that of the 3-body system, using the effective potential function of the 4-body system. .

$$E(x, y, x', y') = \frac{1}{2}[x'^2 + y'^2] - \overline{V}(x, y)$$

Results for 4-Body System:

The same process of approximating the gradient of the effective potential, transforming the two second order ordinary differential equations to a system of four first order ordinary differential equations, and solving the first order differential equations simultaneously using a fixed step RK4 method was used. Figure 9 demonstrates the calculated trajectory for the 3-body for a given initial position and velocity.

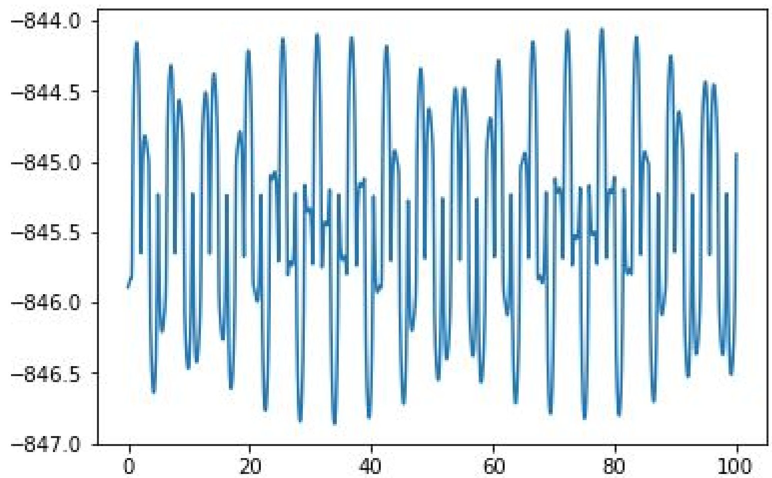
Figure 9: Trajectory of Spaceship in the normalized Earth-Moon rotating frame for a set initial parameters in the 4-body system



Energy Conservation for 4-Body System

To validate the 4-body model, conservation of energy was again used. Figure 10 shows the Energy-time graph of the 4-body model. What is interesting to compare is the difference in the two graphs. The variation in energy over a 100 second time interval for the 3-body system is about 0.001 percent compared to 0.1 percent for the 4-body model. This would lead to believing that the 3-body model is more accurate, which is intuitive as the equations of motion for the 3-body model come from the actual derivation while the equations of motions for the 4-body model come from the patched 3-body approximation. However the energy to the 3-body model always seems to be growing due to machine error whereas the error in the 4-body model seems to have error bounds. As such, one could argue that over long time intervals, the 4-body model would become more accurate.

Figure 10: Energy of the System over the time interval for the 4-body system



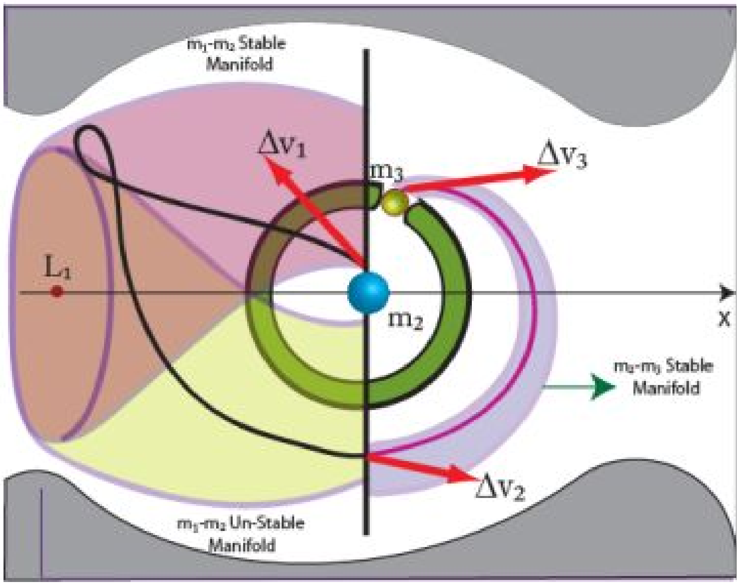
Optimization for Fuel-Efficient Trajectory:

The calculation for the optimal trajectory from an orbit to orbit transfer relies on using optimal trajectory theory from space dynamics literature and then varying the initial parameters in an optimization algorithm to minimize the delta-V parameter that meets the end conditions.

Theory:

As a way to confine the search for a fuel efficient trajectory, space dynamics literature is used. First the escape and capture delta-Vs are confined to be tangent to the orbit, known to be the best way to escape the orbit of a primary.This is known as the Hohmann transfer. As such the only initial parameters are the magnitude of delta-V and the state of the 4-body system at the escape and capture points. Second, one uses knowledge that traveling in the stable manifolds surrounding the L1 and L2 points will produce an optimal trajectory. Figure 11 shows an example trajectory in the realm of the lagrange points and using the tangent delta-Vs described. Lastly, theory states that the orbit transfer can be done in solely three changes in velocity, delta-$V_{escape,}$ delta-$V_{capture}$, and delta-$V_{transient}$. Furthermore, it describes delta-V_{escape} and delta-$V_{capture}$ as dominant, meaning that if those are minimized, the delta-V of the system is minimized.

Figure 11: Optimal Trajectory for Earth to Moon Orbit Transfer in Earth-Sun Frame



Optimization Algorithm:

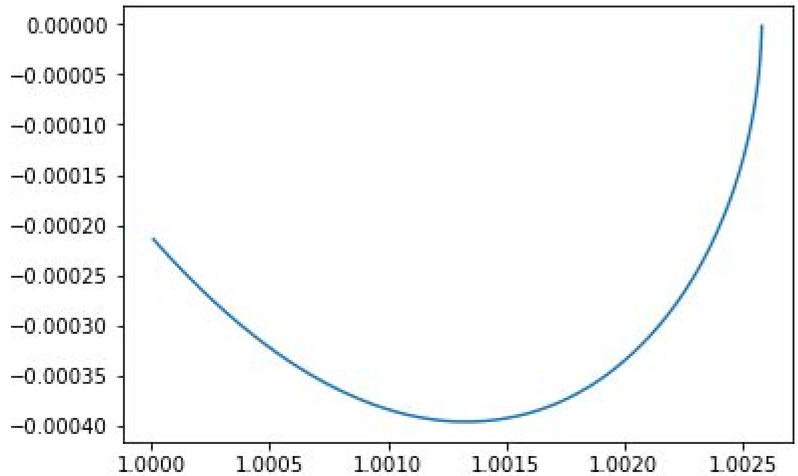
The algorithm described will work backwards, starting from the capture point and working backwards from the capture point and calculating the minimum delta-$V_{capture}$, then the minimum delta-V_{escape} of the system that creates a continuous path with the capture trajectory, and lastly calculating delta-$V_{transient}$ using simple vector subtraction.

Delta-$V_{capture}$:
Given knowledge of trajectory optimization theory, one knows that the delta-V is tangent to the orbit, meaning that the initial parameters is the magnitude of delta-V and the state of the system at capture. Using knowledge that if the spaceship passes the Earth-Moon L1 point it will transfer to the other side, one can create an optimization algorithm with the following initial parameters and end conditions:

$$z = \left[\tau,\ deltaV_{escape}\right]$$
$$c = \left\{x_f \le x_{L1},\ x'_f < 0\right\}$$

To create the optimization algorithm, the tau is varied over a certain interval and the minimum delta-V_{escape} that satisfies the end conditions is met. Finally the minimum of these calculated delta-V_{escape} is chosen. Next one uses the minimized initial parameters in the Earth Sun Frame to calculate the state of the 4-body system when the spaceship passes the Earth. Figure 12 demonstrates the optimized capture trajectory in the Earth Sun frame. The left endpoint represents the beginning of the trajectory and the right endpoint represents the capture point. This compares favorably to the violet line in the Figure 11.

Figure 12: Optimized Capture Trajectory:



Delta-V$_{escape}$:

The same technique described is applied for the Delta-V$_{escape}$, but the initial parameters end conditions are slightly different. First one must choose an initial time as well as an initial tau, together defining the state of the system at the start. At the end conditions the state of the system must match the start state of the capture trajectory, signify that the path created is a continuous path. As such initial and end conditions are defined by:

$$z = \left[\tau,\ deltaV_{escape},\ t_{init}\right]$$

$$c = \left\{x_f = x_{earth},\ y_f = y_{transient},\ x'_f < 0\right\}$$

One would then again minimize for Delta-V$_{escape}$ by varying varying the other two initial conditions and finding the necessary Delta-V$_{escape}$ that satisfies the end conditions.

Delta-V$_{transient}$:

Delta-V$_{transient}$ represents the change in velocity necessary to put the space ship from the escape trajectory onto the capture trajectory. Therefore it is simply the vector subtraction:

$delta\ V_{transient} = deltaV_{capture,f} - deltaV_{escape,f}$

Physics 235 Related Topics:

The model for celestial mechanics problems is based on the transfer of energy from kinetic and gravitational potential energy of a rocket, as well as changes in trajectory caused rocket acceleration and gravitational forces caused by local large masses. This relies on solving and manipulating 2nd order ordinary and partial differential equations. Fourth-order Runge Kutta was be used upon adaptive step size to give a reasonable running time to solve the ordinary differential equation. While the method of finite difference will be employed for the partial differential equations involved with the motions. Furthermore the system is a chaotic system and as such the analysis of the results will have to be treated carefully. Lastly demonstrations of the results will be created using visualizations techniques learned in the class. While optimization techniques have not be a focus of the class, they will be required to create the model.

Project Evaluation:

As a whole the problem got progressively more challenging as it progressed. Creating the 3-body model was relatively simple given a simple set-up and having been already given the equations of motions. The 4-body model was more challenging as it involved choosing a frame of reference and then relating the motion of one 3-body system to the other. This made the setup of the problem challenging, but the method for solving for the trajectory remain the same. The most challenging part of the project was the optimization, as it was a boundary value problem as well as an optimization question. The optimization was also for an initial parameter and not for the output, which posed a significant challenge. In the end, a brute force algorithm was created which was able to solve for the most efficient trajectory by varying the initial conditions and running the RK solver to test if the boundary conditions were met. The problem with this method was that it was extremely computationally intense. For the capture trajectory, there was only two initial conditions and end conditions and would take minutes to run for even low accuracy. The escape trajectory which involved three initial conditions and three end conditions was never solved. To be able to complete the problem machine learning technique would need to be included in the algorithm to produce high accuracy results in a reasonable amount of time.

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