PH 235 SPRING 2017

ASSIGNMENT 3

Assigned Date: February 17, 2017

Due Date: 11:50pm, February 26, 2017 only via moodle (See syllabus for late policy)

Total Points: 100

All problems in this assignment are from the optional textbook for the course *Computational Physics* authored by Mark Newman.

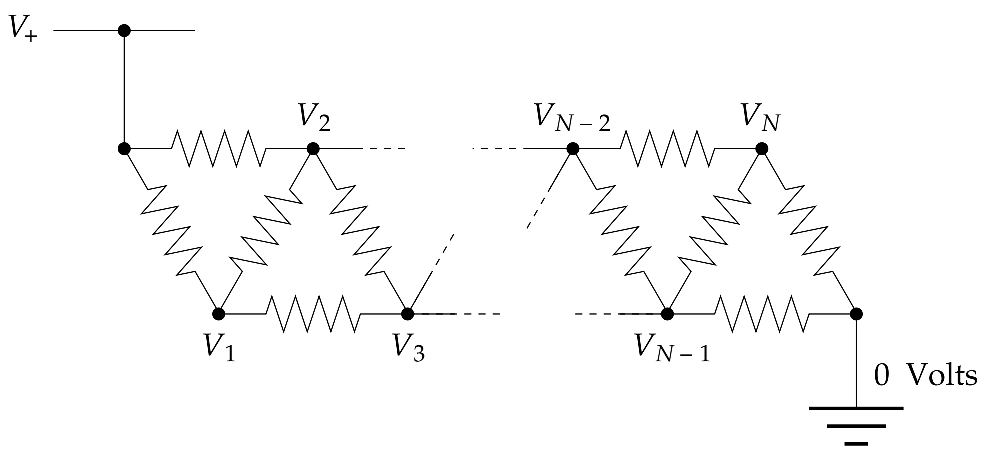
Instructions:
1. Create a folder titled YourLastName-HW3.
2. In this parent folder, create subfolders, one for each of the problems below and name each subfolder as YourLastName-ExerciseNumber for easier tracking.
3. Each file you submit should be of the format YourLastName-ExerciseNumber-Filename 4.Populate the folders with (a) the solution code for each problem, (b) screen shots of results when possible and (c) a text document with a short description of your approach, whether you successfully solved the problem. If you did not solve the problem, explain solution status (partial credit will be given for reasonable attempts).
5. Submit a zipped version of the parent folder to the moodle assignment link.

Preparation

Read the "Policy" section of new revised syllabus on moodle (first module) and pay attention to on Assignment and Project expectations with regards Academic Integrity. Also, read the document AcademicIntegrity.pdf in moodle that captures the Engineering School policy on Academic Integrity requirements and related consequences. I strictly adhere to the code - talk to me if you have any questions whether an activity could violate Academic Integrity code before you pursue the activity.

Problem 1: Ex 6.7 A chain of resistors (20 points)

Consider a long chain of resistors wired up like this:



All the resistors have the same resistance R. The power rail at the top is at voltage $V_+ = 5$V. The problem is to find the voltages $V_1 \ldots V_N$ at the internal points in the circuit.

1. Using Ohm's law and the Kirchhoff current law, which says that the total net current flow out of (or into) any junction in a circuit must be zero, show that the voltages $V_1 \ldots V_N$ satisfy the equations

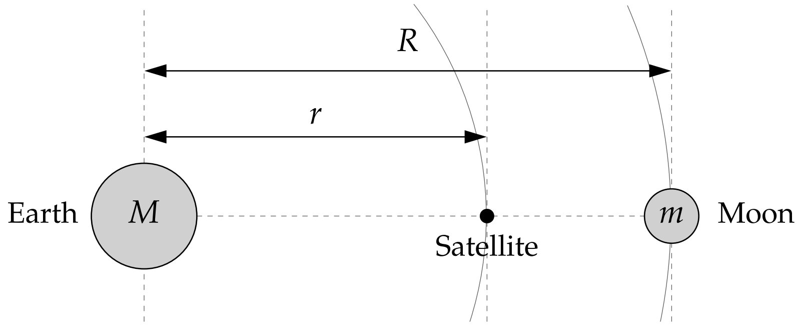
$$\begin{aligned} 3V_1 - V_2 - V_3 &= V_+, \\ -V_1 + 4V_2 - V_3 - V_4 &= V_+, \\ &\vdots \\ -V_{i-2} - V_{i-1} + 4V_i - V_{i+1} - V_{i+2} &= 0, \\ &\vdots \\ -V_{N-3} - V_{N-2} + 4V_{N-1} - V_N &= 0, \\ -V_{N-2} - V_{N-1} + 3V_N &= 0. \end{aligned}$$

   Express these equations in vector form $\mathbf{Av} = \mathbf{w}$ and find the values of the matrix $\mathbf{A}$ and the vector $\mathbf{w}$.

2. Write a program to solve for the values of the V_i when there are $N = 6$ internal junctions with unknown voltages. (Hint: All the values of V_i should lie between zero and 5V. If they don't, something is wrong.)

3. Now repeat your calculation for the case where there are $N = 10\,000$ internal junctions. This part is not possible using standard tools like the `solve` function. You need to make use of the fact that the matrix $\mathbf{A}$ is banded and use the `banded` function from the file `banded.py`, discussed in class and available via moodle.

Problem 2: Ex 6.16 The Lagrange point (15 points)
here is a magical point between the Earth and the Moon, called the L_1 Lagrange point, at which a satellite will orbit the Earth in perfect synchrony with the Moon, staying always in between the two. This works because the inward pull of the Earth and the outward pull of the Moon combine to create exactly the needed centripetal force that keeps the satellite in its orbit. Here's the setup:



1. Assuming circular orbits, and assuming that the Earth is much more massive than either the Moon or the satellite, show that the distance r from the center of the Earth to the L_1 point satisfies

$$\frac{GM}{r^2} - \frac{Gm}{(R-r)^2} = \omega^2 r,$$

where M and m are the Earth and Moon masses, G is Newton's gravitational constant, and ω is the angular velocity of both the Moon and the satellite.

2. The equation above is a fifth-order polynomial equation in r (also called a quintic equation). Such equations cannot be solved exactly in closed form, but it's straightforward to solve them numerically. Write a program that uses either Newton's method or the secant method to solve for the distance r from the Earth to the L_1 point. Compute a solution accurate to at least four significant figures.

The values of the various parameters are:

$$\begin{aligned} G &= 6.674 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}, \\ M &= 5.974 \times 10^{24}\,\mathrm{kg}, \\ m &= 7.348 \times 10^{22}\,\mathrm{kg}, \\ R &= 3.844 \times 10^{8}\,\mathrm{m}, \\ \omega &= 2.662 \times 10^{-6}\,\mathrm{s^{-1}}. \end{aligned}$$

You will also need to choose a suitable starting value for r, or two starting values if you use the secant method.

Problem 3: Ex 6.18 Temperature of a Light Bulb (20 points)

An incandescent light bulb is a simple device—it contains a filament, usually made of tungsten, heated by the flow of electricity until it becomes hot enough to radiate thermally. Essentially all of the power consumed by such a bulb is radiated as electromagnetic energy, but some of the radiation is not in the visible wavelengths, which means it is useless for lighting purposes.

Let us define the efficiency of a light bulb to be the fraction of the radiated energy that falls in the visible band. It's a good approximation to assume that the radiation from a filament at temperature T obeys the Planck radiation law, meaning that the power radiated per unit wavelength λ obeys

$$I(\lambda) = 2\pi A h c^2 \frac{\lambda^{-5}}{e^{hc/\lambda k_B T} - 1},$$

where A is the surface area of the filament, h is Planck's constant, c is the speed of light, and k_B is Boltzmann's constant. The visible wavelengths run from $\lambda_1 = 390\,\text{nm}$ to $\lambda_2 = 750\,\text{nm}$, so the total energy radiated in the visible window is $\int_{\lambda_1}^{\lambda_2} I(\lambda)\,\mathrm{d}\lambda$ and the total energy at all wavelengths is $\int_0^\infty I(\lambda)\,\mathrm{d}\lambda$. Dividing one expression by the other and substituting for $I(\lambda)$ from above, we get an expression for the efficiency η of the light bulb thus:

$$\eta = \frac{\int_{\lambda_1}^{\lambda_2} \lambda^{-5}/(\mathrm{e}^{hc/\lambda k_B T} - 1)\,\mathrm{d}\lambda}{\int_0^\infty \lambda^{-5}/(\mathrm{e}^{hc/\lambda k_B T} - 1)\,\mathrm{d}\lambda},$$

where the leading constants and the area A have canceled out. Making the substitution $x = hc/\lambda k_B T$, this can also be written as

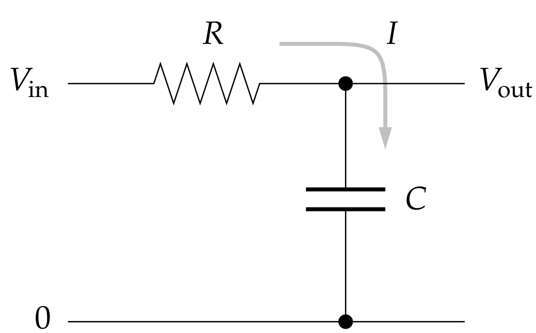
$$\eta = \frac{\int_{hc/\lambda_2 k_B T}^{hc/\lambda_1 k_B T} x^3/(\mathrm{e}^x - 1)\,\mathrm{d}x}{\int_0^\infty x^3/(\mathrm{e}^x - 1)\,\mathrm{d}x} = \frac{15}{\pi^4}\int_{hc/\lambda_2 k_B T}^{hc/\lambda_1 k_B T} \frac{x^3}{\mathrm{e}^x - 1}\,\mathrm{d}x,$$

where we have made use of the known exact value of the integral in the denominator.

1. Write a Python function that takes a temperature T as its argument and calculates the value of η for that temperature from the formula above. The integral in the formula cannot be done analytically, but you can do it numerically using any method of your choice. (For instance, Gaussian quadrature with 100 sample points works fine.) Use your function to make a graph of η as a function of temperature between 300 K and 10 000 K. You should see that there is an intermediate temperature where the efficiency is a maximum.
2. Calculate the temperature of maximum efficiency of the light bulb to within 1 K using golden ratio search. (Hint: An accuracy of 1 K is the equivalent of a few parts in ten thousand in this case. To get this kind of accuracy in your calculation you'll need to use values for the fundamental constants that are suitably accurate, i.e., you will need values accurate to several significant figures.)
3. Is it practical to run a tungsten-filament light bulb at the temperature you found? If not, why not?

Problem 4: Ex 8.1 A low-pass filter (20 points)

Here is a simple electronic circuit with one resistor and one capacitor:



This circuit acts as a low-pass filter: you send a signal in on the left and it comes out filtered on the right.

Using Ohm's law and the capacitor law and assuming that the output load has very high impedance, so that a negligible amount of current flows through it, we can write down the equations governing this circuit as follows. Let I be the current that flows through R and into the capacitor, and let Q be the charge on the capacitor. Then:

$$IR = V_{\text{in}} - V_{\text{out}}, \qquad Q = CV_{\text{out}}, \qquad I = \frac{\mathrm{d}Q}{\mathrm{d}t}.$$

Substituting the second equation into the third, then substituting the result into the first equation, we find that $V_{\text{in}} - V_{\text{out}} = RC\,(\mathrm{d}V_{\text{out}}/\mathrm{d}t)$, or equivalently

$$\frac{\mathrm{d}V_{\text{out}}}{\mathrm{d}t} = \frac{1}{RC}\big(V_{\text{in}} - V_{\text{out}}\big).$$

1. Write a program (or modify a previous one) to solve this equation for $V_{\text{out}}(t)$ using the fourth-order Runge–Kutta method when in the input signal is a square-wave with frequency 1 and amplitude 1:

$$V_{\text{in}}(t) = \begin{cases} 1 & \text{if } \lfloor 2t \rfloor \text{ is even,} \\ -1 & \text{if } \lfloor 2t \rfloor \text{ is odd,} \end{cases} \tag{1}$$

   where $\lfloor x \rfloor$ means x rounded down to the next lowest integer. Use the program to make plots of the output of the filter circuit from $t = 0$ to $t = 10$ when $RC = 0.01$, 0.1, and 1, with initial condition $V_{\text{out}}(0) = 0$. You will have to make a decision about what value of h to use in your calculation. Small values give more accurate results, but the program will take longer to run. Try a variety of different values and choose one for your final calculations that seems sensible to you.

2. Based on the graphs produced by your program, describe what you see and explain what the circuit is doing.

A program similar to the one you wrote is running inside most stereos and music players, to create the effect of the "bass" control. In the old days, the bass control on a stereo would have been connected to a real electronic low-pass filter in the amplifier circuitry, but these days there is just a computer processor that simulates the behavior of the filter in a manner similar to your program.

Problem 5: Ex 8.3 The Lorenz equations (25 points)

One of the most celebrated sets of differential equations in physics is the Lorenz equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = rx - y - xz, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = xy - bz,$$

where σ, r, and b are constants. (The names σ, r, and b are odd, but traditional—they are always used in these equations for historical reasons.)

These equations were first studied by Edward Lorenz in 1963, who derived them from a simplified model of weather patterns. The reason for their fame is that they were one of the first incontrovertible examples of *deterministic chaos*, the occurrence of apparently random motion even though there is no randomness built into the equations. We encountered a different example of chaos in the logistic map of Exercise 3.6.

1. Write a program to solve the Lorenz equations for the case $\sigma = 10$, $r = 28$, and $b = \frac{8}{3}$ in the range from $t = 0$ to $t = 50$ with initial conditions $(x, y, z) = (0, 1, 0)$. Have your program make a plot of y as a function of time. Note the unpredictable nature of the motion. (Hint: If you base your program on previous ones, be careful. This problem has parameters r and b with the same names as variables in previous programs—make sure to give your variables new names, or use different names for the parameters, to avoid introducing errors into your code.)
2. Modify your program to produce a plot of z against x. You should see a picture of the famous "strange attractor" of the Lorenz equations, a lop-sided butterfly-shaped plot that never repeats itself.