Proof of Cantor Set of degree 3

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1 Introduction

$$I_s(P_n) = n^{\text{-2}} \sum_{p \neq p'; p, p' \in P_n; |P_n| = n} |p - p'|^{\text{-s}}$$

The structure of this Cantor Set is just each time dividing an interval [0,1] into three even parts. You pick the first and last parts and do the same process as above.

when doing the division for n time, we have $2^{\text{n+1}} points$, plug it in the discrete-s energy, the value of $|p - p'|$ can be approximated as $3^{\text{-j}}$ where j goes from 1 to n :

$$\begin{aligned} I_s(P_n) &= 2^{\text{-2n-2}} \sum_{p \neq p'; p, p' \in P_n; |P_n| = n} |p - p'|^{\text{-s}} \\ &= 2^{\text{-2n-2}} \sum_{j=1}^{n} \sum_{|p - p'| = 3^{\text{-j}}} |p - p'|^{\text{-s}} \\ &= 2^{\text{-2n-2}} \sum_{j=1}^{n} \sum_{|p - p'| = 3^{\text{-j}}} (3^{\text{-j}})^{\text{-s}} \end{aligned}$$

As described in the paper, we can count the length $|p - p'| \approx n^{\text{-j}}$ as $n^{\text{-j-1}} < |p - p'| \leq n^{\text{-j}}$. Therefore, for a given j, we fix the p and count how many p' there satisfies $|p - p'| \approx n^{\text{-j}}$: in fact, it can be regarded as the total amount of points with n steps divided by the total amount of points with j steps, that is $\frac{|p_n|}{|p_j|} = \frac{2^{\text{n+1}}}{2^{\text{j+1}}}$

$$
\begin{aligned}
I_s(P_n) &= 2^{\text{-2n-2}} \sum_{j=1}^{n} 3^{\text{j*s}} \sum_{|p-p'|=3^{\text{-j}}} 1 \\
&= 2^{\text{-2n-2}} \sum_{j=1}^{n} 3^{\text{j*s}} \sum_{a \in P_n} \frac{2^{\text{n+1}}}{2^{\text{j+1}}} \\
&= 2^{\text{-2n-2}} \sum_{j=1}^{n} 3^{\text{j*s}} * 2^{\text{n+1}} * \frac{2^{\text{n+1}}}{2^{\text{j+1}}} \\
&= 2^{\text{-1}} * \sum_{j=1}^{n} (\frac{3^s}{2})^j
\end{aligned}
$$

The summation is just a geometric sum which converges (as n goes very large) when $(\frac{3^s}{2})^j$ is less than 1.

Similar for the Cartesian product of two Cantor Sets, note the number of total points after n steps becomes $2^{\text{2n+2}}$, while now for a given j, fixed p, the number of p' $(\frac{|p_n \times p_n|}{|p_j \times p_j|})$ simply becomes $\frac{2^{\text{2n+2}}}{2^{\text{2j+2}}}$

Now the discrete-s energy becomes:

$$
\begin{aligned}
I_s(P_n) &= 2^{\text{-4n-4}} \sum_{j=1}^{n} 3^{\text{j*s}} \sum_{|p-p'|=3^{\text{-j}}} 1 \\
&= 2^{\text{-4n-4}} \sum_{j=1}^{n} 3^{\text{j*s}} \sum_{a \in P_n} \frac{2^{\text{2n+2}}}{2^{\text{2j+2}}} \\
&= 2^{\text{-4n-4}} \sum_{j=1}^{n} 3^{\text{j*s}} * 2^{\text{2n+2}} * \frac{2^{\text{2n+2}}}{2^{\text{2j+2}}} \\
&= 2^{\text{-2}} * \sum_{j=1}^{n} (\frac{3^s}{4})^j
\end{aligned}
$$

Again, it converges when $\frac{3^s}{4}$ is less than 1. Take the natural log on both sides you will get $s < 2\frac{log2}{log3}$.