ETF time series analysis

February 21, 2024

1 Time series

The Fred API is utilized to gather the data required for time series analysis. Specifically, the data of CBOE EuroCurrency ETF Volatility Index (EuroCurrency) and CBOE Gold ETF Volatility Index (Gold) are download with a sample size of 300 (end date at 20/02/2024) and a daily resolution. To ensure the accuracy and prevent any gaps in the plotted graphs caused by invalid data, a strategy of first removing any invalid values and then extracting the data starting from the nearest valid data point is employed. The resulting price trends of the two ETF time series are illustrated in Figure 1. It is evident from the figure that gold has maintained a consistently higher price and exhibited greater volatility compared to EuroCurrency. These characteristics will be thoroughly analyzed in our time series analysis, beginning with the application of a moving average technique with various time-windows τ.

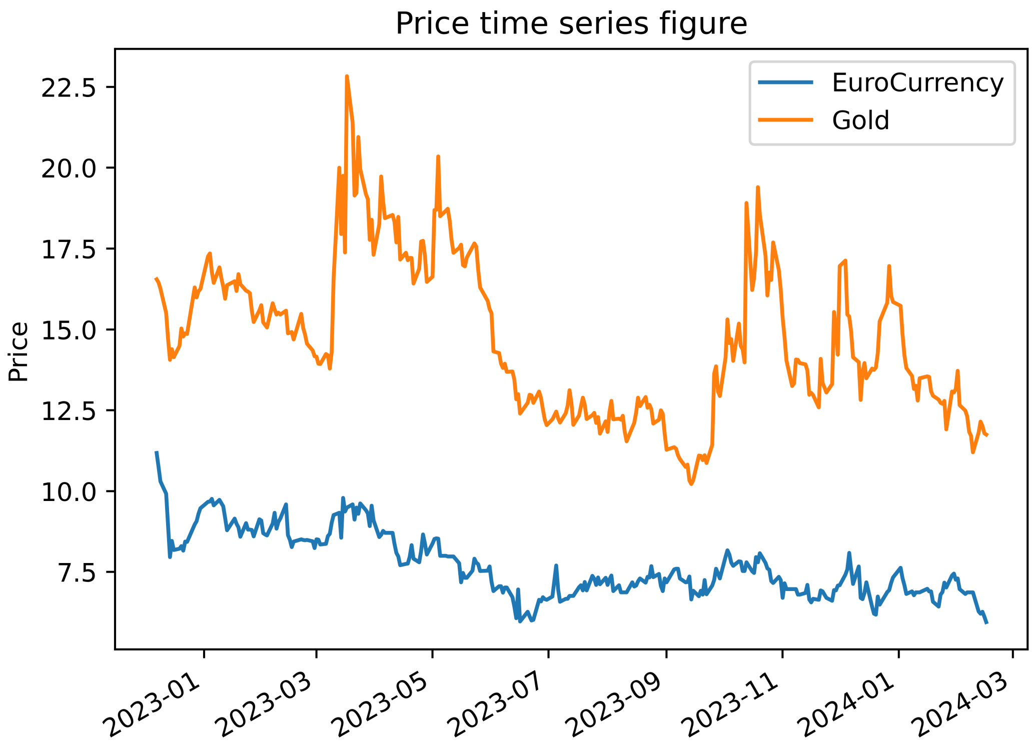


Figure 1: The price trend of EuroCurrency and Gold ETF

2 Moving averages (MA) and the returns

Moving averages are widely used in financial analysis as a smoothing technique to filter out noise and highlight longer-term trends or cycles in time series data. Let MA_t denote the moving average at time t, and P_t represent the price at time t. The moving average is calculated as:

$$MA_t = \frac{1}{\tau} \sum_{i=0}^{\tau-1} P_{t-i}.$$

Larger time-windows result in greater smoothing but they may introduce potential data lags. Moreover, crossovers between different moving averages can act as signals for trend changes, guiding trading decisions. A higher value for the shorter time window suggests an upward trend, favoring long positions, while a lower value indicates a tendency to sell assets. Additionally, the spacing between moving averages provides a measure of volatility, informing risk management strategies. Widely spaced moving averages indicate high volatility, while closely spaced moving averages suggest low volatility.

2.1 Plots and analysis

The moving averages plotted against the price time series are depicted in Figure 2.

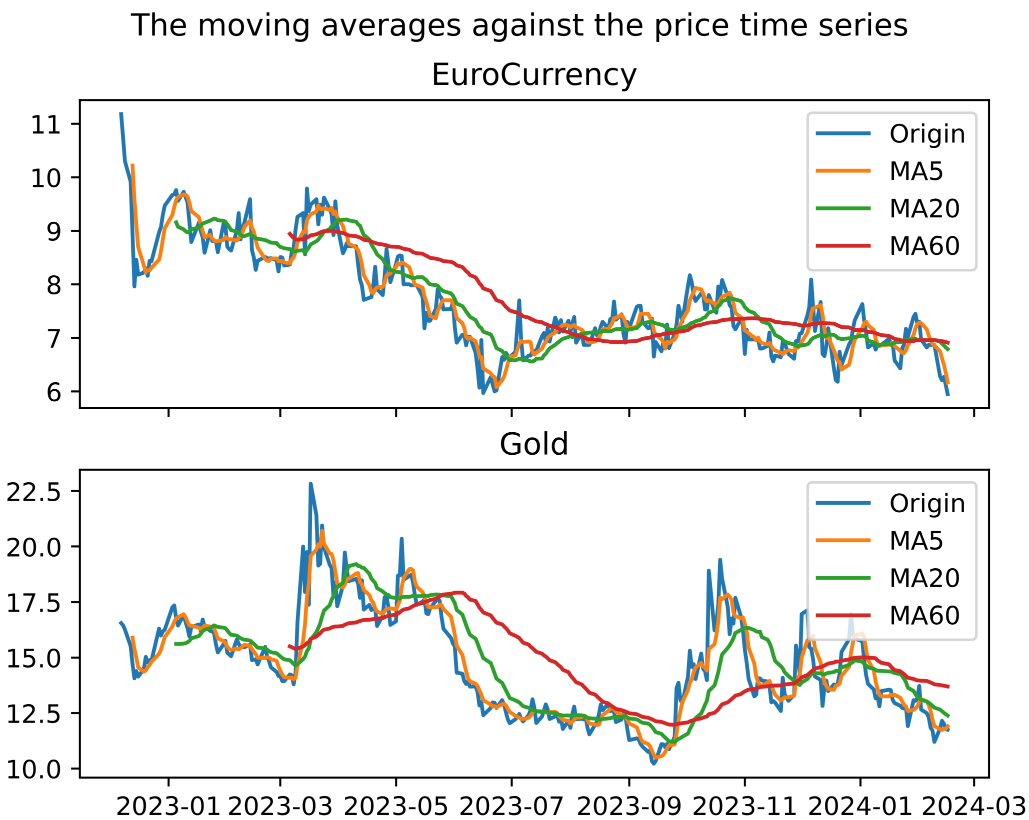


Figure 2: MA against the price time series with a shared x-axis

It is evident from the figure that the EuroCurrency exhibits lower volatility but a declining trend, whereas Gold displays a slightly oscillatory trend with greater fluctuations. Notably, focusing on the end of the figure, for both the EuroCurrency and the Gold, a short time-window yields a lower value, hence suggesting to hold a short position.

2.2 Linear and log returns

Moreover, returns play a crucial role in time series analysis. Let p_t denote the price at time t. Returns are calculated as follows:

$$Linear\ return_t = \frac{p_t - p_{t-1}}{p_{t-1}} = \frac{p_t}{p_{t-1}} - 1$$

$$Log\text{-}return_t = log(\frac{p_t}{p_{t-1}})$$

Figure 3 displays the plots of linear returns against log-returns. The plots demonstrate a close tracking relationship between the two types of returns, with only minor differences observed. The curve depicting the rate of return appears to exhibit oscillatory behavior, fluctuating both positively and negatively around a mean of zero. This observation prompts speculation regarding the stability of the data, a hypothesis that will be subjected to rigorous testing in subsequent sections.

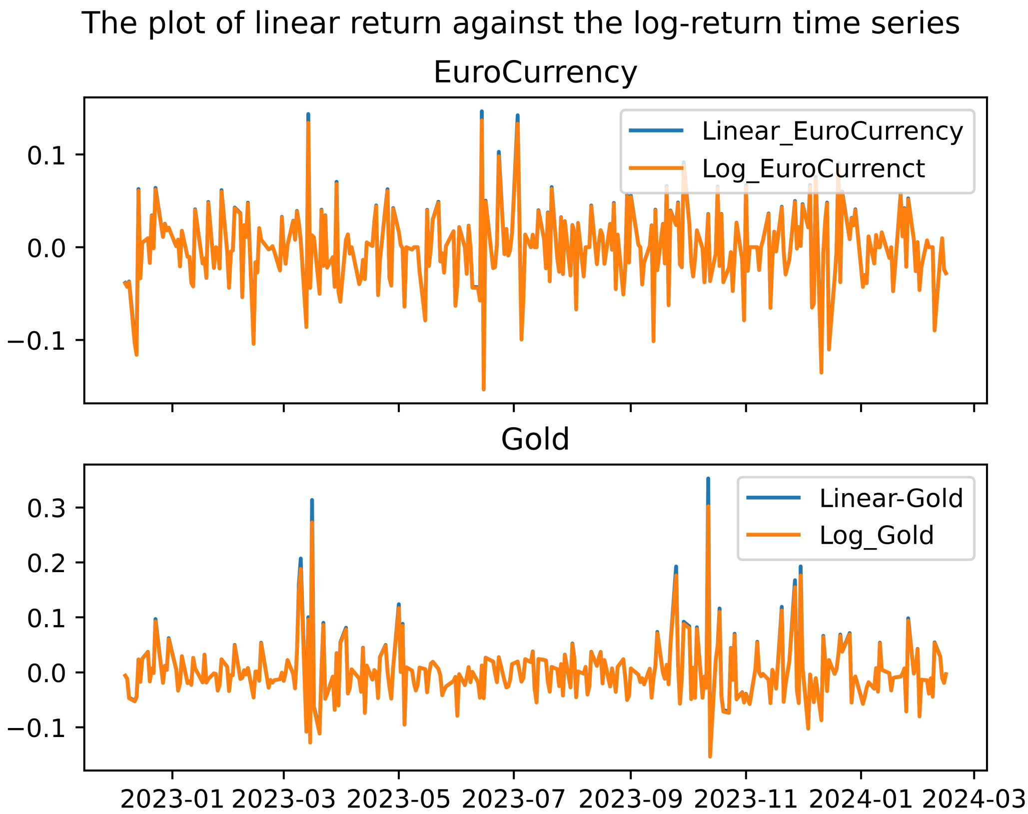


Figure 3: Plots of returns with a shared x-axis

3 Correlation analysis

3.1 Auto-correlation function (ACF)

Initiating the correlation analysis, it is imperative to consider the ACF. In essence, the ACF reflects the correlation of different lags with the series itself. Additionally, the partial auto-correlation function (PACF) represents the correlation of different lags with the series itself while removing the effects of the previous lags. To define the auto-correlation function, let's introduce the auto-covariance function $Cov(y_t, y_{t+k})$ at lag k for time series y:

$$Cov(y_t, y_{t+k}) = E[(y_t - \mu)(y_{t+k} - \mu)],$$

where μ denotes the mean value. Then the ACF $\rho(k)$ at lag k is given by:

$$\rho(k) = \frac{Cov(y_t, y_{t+k})}{\sqrt{Var(y_t)Var(y_{t+k})}}.$$

For a stationary time series, this simplifies to:

$$\rho(k) = \frac{Cov(y_t, y_{t+k})}{\sigma^2},$$

since $Var(y_t) = Var(y_{t+k}) = \sigma^2$[1]. Hence indicates the PACF $\phi(k)$ of lag k is defined as:

$$\phi(k) = Corr(y_t - \hat{y}_t, y_{t+k} - \hat{y}_{t+k}),$$

where $\hat{y}_t$ and $\hat{y}_{t+k}$ are linear combinations of previous lags that minimise the mean squared error ($MSE = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$), and $Corr(\cdot)$ denotes the correlation coefficient.

3.2 Price time series

The plots of the ACF and PACF are displayed separately in Figures 4 and 5. In the ACF plots, the blue areas depict the 95% confidence interval. Similarly, in the PACF plots, the blue areas denote the 95% confidence intervals.

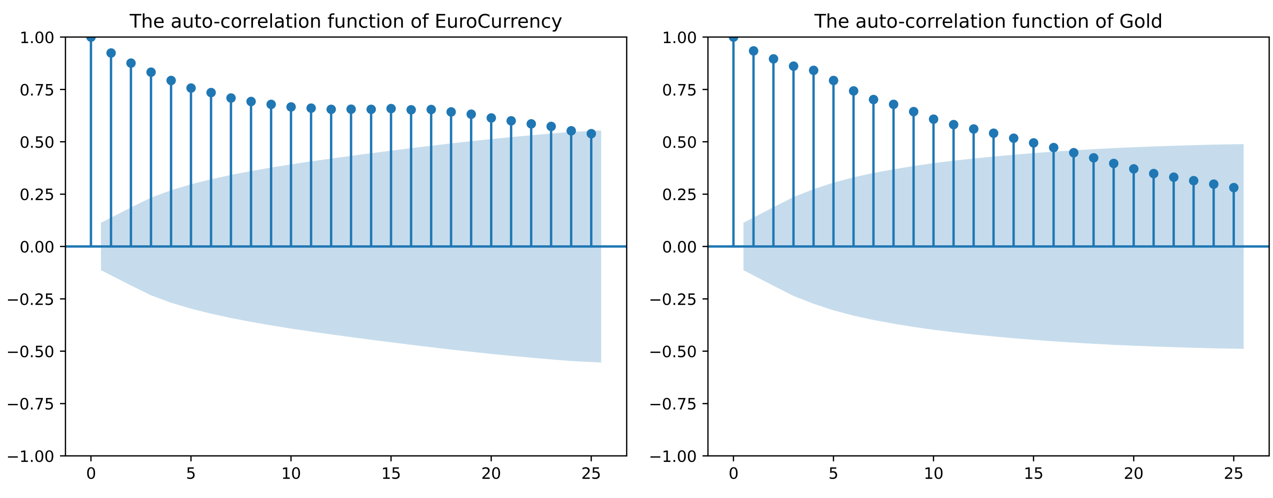


Figure 4: The ACF of the price time series

The ACF graphs indicate a clear non-stationary signature for gold price, while EuroCurrency price exhibits no such discernible pattern, a hypothesis that will be subjected to more rigorous verification in subsequent analyses.

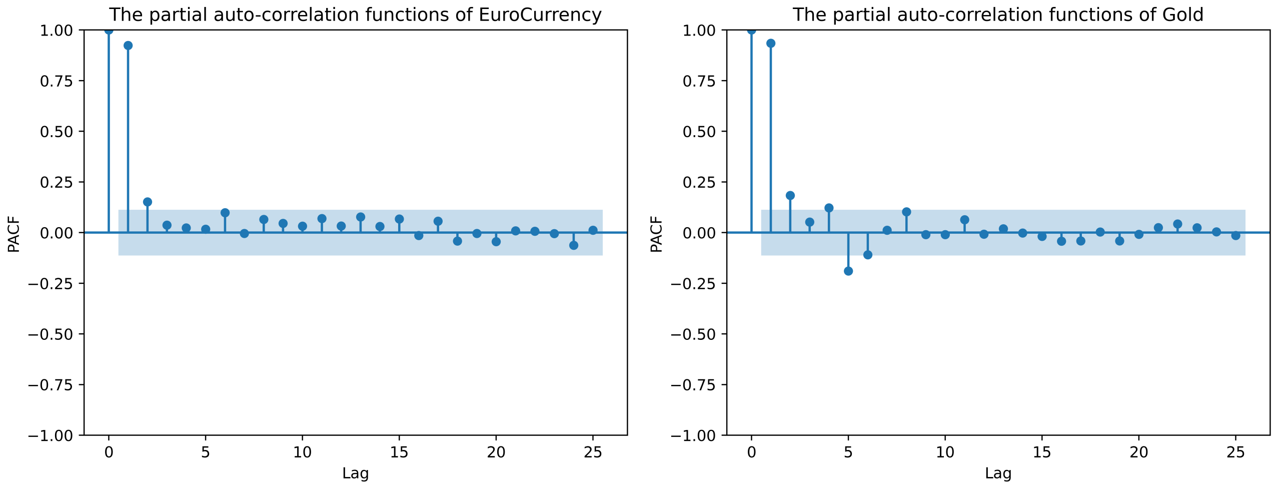


Figure 5: The PACF of the price time series

Based on the information provided in the following table[2], a suitable model could be identified to fit the data.

	AR(p)	MA(q)	ARMA(p,q)
ACF	Tails off (Geometric decay)	Significant at lag q / Cuts off after lag q	Tails off (Geometric decay)
PACF	Significant at lag q / Cuts off after lag q	Tails off (Geometric decay)	Tails off (Geometric decay)

3.3 Return time series

Figures 6 and 7 illustrate the ACF and PACF for the linear return time series. In the ACF plots, the horizontal solid lines represent the 95% confidence intervals, while the dashed line indicates the 99% confidence interval.

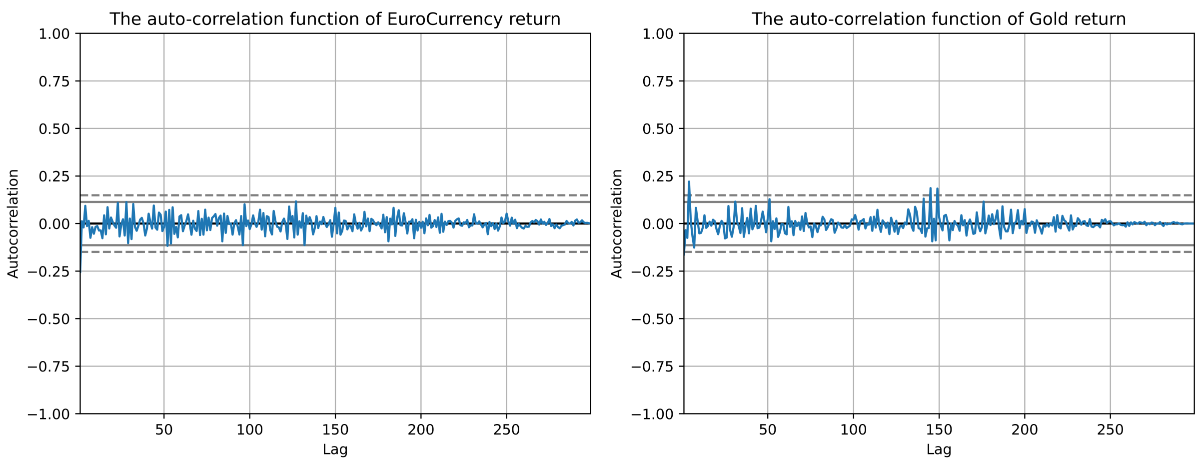


Figure 6: The ACF of the linear return time series

Contrary to previous plots, the ACF plots for returns exhibit insignificant autocorrelations, consistent with the findings outlined in Rama Cont's research on stylized facts [3]. This property, commonly referred to as the 'absence of autocorrelation,' aligns with the empirical fact that the autocorrelation function for price movements rapidly diminishes to zero within a few of minutes.

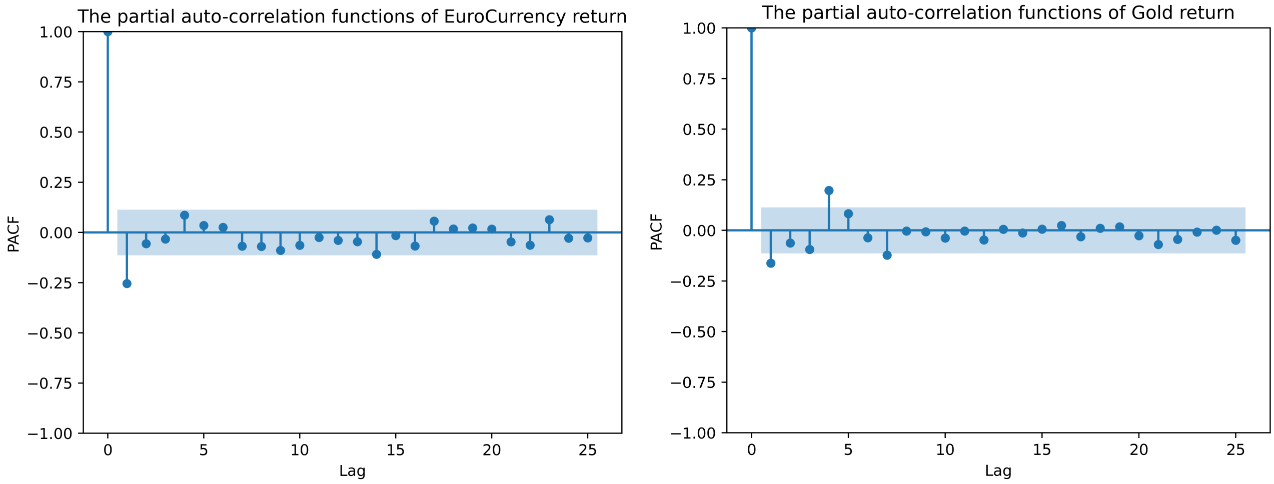
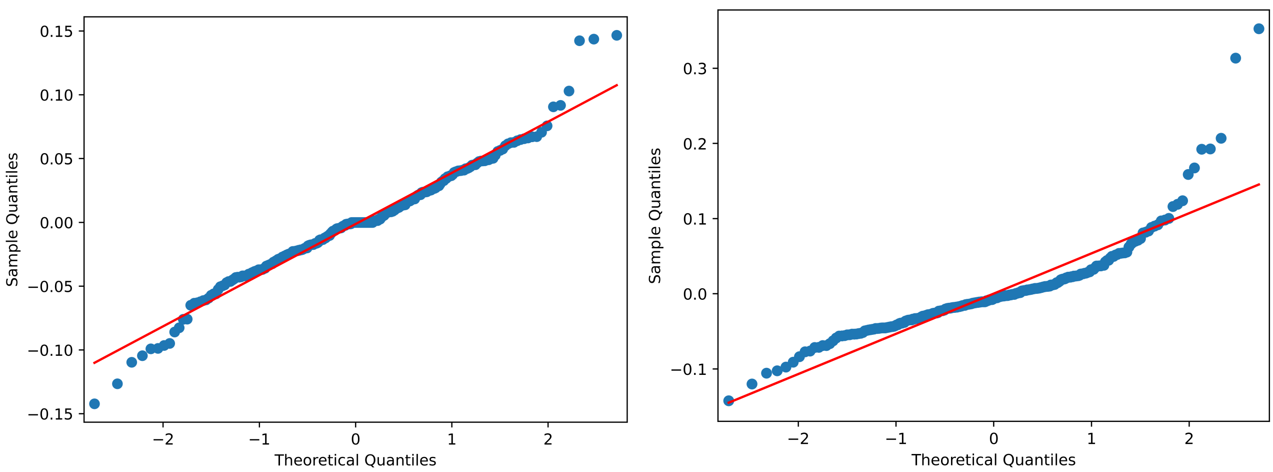


Figure 7: The PACF of the linear return time series

Mandelbrot [4] stating that "arbitrage tends to whiten the spectrum of price changes", this notion is corroborated by the ACF and PACF plots, which both exhibit characteristics akin to white noise.

4 Gaussian and Stationary tests

4.1 Gaussian tests



(a) Q-Q plot for EuroCurrency return

(b) Q-Q plot for Gold return

Figure 8: Q-Q plots for linear returns

The Gaussian test, also known as a normality test, aims to determine whether the sample data has been drawn from a normally distributed population, a crucial assumption for many statistical analyses. Several studies have investigated the performance of various normality tests[5, 6], and considering the sample size of 300, the Shapiro-Wilk test is used.

Before performing the normality test, it is valuable to visually assess the distribution of the sample data. One effective method is generating a Q-Q plot, which compares the quantiles of the sample data to the quantiles of a theoretical normal distribution. However, as illustrated in Figure 8, with the red line representing the standardized line, the expected order statistics are scaled by the standard deviation of the given sample and have the mean added to them[7]. There is no evident feature of a normal distribution, and the plot exhibits the characteristics of heavy tails, consistent with the findings of Rama Cont[3].

After generating the Q-Q plot, the Shapiro-Wilk test is conducted. The null hypothesis of this test is that the sample data $x_1, \ldots, x_n$ came from a normally distributed population. Let W denote the test statistic, $\bar{x}$ denote the sample mean and s^2 denote the sample variance. Arranging the sample values in ascending order, denoeted by $x_{(1)}, \ldots, x_{(n)}$, the test statistic which is compared to critical values from the Shapiro-Wilk distribution is calculated by:

$$W = \frac{(\sum_{i=1}^{n} a_i x_{(i)})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2},$$

where the coefficients a_i are calculated by:

$$(a_1, ..., a_n) = \frac{m^T V^{-1}}{(m^T V^{-1} V^{-1} m)^{1/2}},$$

and vector $m = (m_1, ..., m_n)^T$[5], which consists of the expected values of the order statistics derived from independent and identically distributed random variables sampled from the standard normal distribution. Additionally, V represents the covariance matrix of these normal order statistics.

Since the null hypothesis is that the population is normally distributed, if the p-value is less than the significant level α (usually 0.05), the null hypothesis is rejected, indicating that the data is not from a normally distributed population.

The calculated p-values for both linear return time series are small, 5.76×10^{-5} and 2.63×10^{-17} respectively. Hence neither of their populations fits the normal distribution.

4.2 Stationary tests

Since stationary is also a crucial property to confirm and used to simplify the auto-correlation formula, stationaty tests are performed. The Augmented Dickey-Fuller (ADF) test is a common test where the null hypothesis is that a unit root is present in a time series sample (i.e., the data is non-stationary). On the other hand, another common test, the Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test has the opposite null hypothesis, stating that an observable time series is stationary around a deterministic trend.

The ADF test is based on the Dickey-Fuller test applied to the model:

$$\triangle y_t = \alpha + \beta t + \gamma y_{t-1} + \delta_1 \triangle y_{t-1} + \cdots + \delta_p \triangle y_{t-p+1} + \epsilon_t,$$

where α is a constant, β is the coeffient on a time trend, $\gamma = \sum_{i=1}^{p} \delta_i - 1$, and p is the lag usually chosen by Akaike Information Criterion or Bayes Information Criterion. Thus, the null hypothesis is $\gamma = 0$ (while the alternative hypothesis is $\gamma < 0$), which indicates that the process y_t is a random walk. The test statistic is given by $t_\tau = \frac{\hat{\gamma}}{SE(\hat{\gamma})}$, where $SE(\hat{\gamma})$ is the standard error of the estimated coefficient. The null hypothesis is rejected if the test statistic is less than the critical value, indicating that the time series is stationary.
Conversely, for the KPSS test, the null hypothesis is rejected if the test statistic is greater than the critical value, indicating that the time series is non-stationary. The results are shown in the following table.

Test Method	Test Statistic	p-value	1%	10%
ADF(EuroCurrency linear return)	-11.516	4.147×10^{-21}	-3.453	-2.572
ADF(Gold linear return)	-6.791	2.368×10^{-09}	-3.453	-2.572
KPSS(EuroCurrency linear return)	0.064	≥ 0.1	0.739	0.347
KPSS(Gold linear return)	0.047	≥ 0.1	0.739	0.347

The ADF test statistics for both the EuroCurrency and Gold linear returns are highly negative, indicating strong evidence against the null hypothesis of a unit root (i.e., non-stationarity). The corresponding p-values are extremely small, further supporting the rejection of the null hypothesis. Thus, both time series are stationary based on the ADF test. The KPSS test statistics for both time series are small and positive, while the p-values are both greater than or equal to 0.1. Therefore, there is no evidence to suggest non-stationarity based on the KPSS test.
Combining the results from both the ADF and KPSS tests, it is concluded that both the EuroCurrency and Gold linear returns are stationary time series.

5 Cointegration tests

Cointegration demonstrates a long-term equilibrium relationship between the series, and mathematically, there exists a linear combination that exhibits a lower level of integration, even though each of these series may be non-stationary, the Engle-Granger cointegration test is a common method used to assess this relationship.
This test examines the scenario in which there is just one cointegrating vector and is based on the idea that the residual of the cointegrating regression should be stationary. Using Ordinary Least Squares (OLS) to estimate the cointegrating vector from the regression $y_t = \alpha + \beta x_t + u_t$, the residuals are calculated by $\hat{u}_t = y_t - \hat{\alpha} - \hat{\beta} x_t$. Performing a unit root test on the residuals, such as the ADF test mentioned before, a stationary residual indicates a cointegrating relationship between x_t and y_t. Since the null hypothesis assumes no cointegration, it is rejected when the p-value is smaller than the confidence level, implying that the two series are cointegrated.
It is important to note that before performing a cointegration test, both time series should have the same order of integration ($I(m)$). The order of integration represents the minimum number of differences required to make a non-stationary series stationary. To visually illustrate the process, the flow

chart shown in Figure 9 is provided: Since the return time series have already been tested for sta-

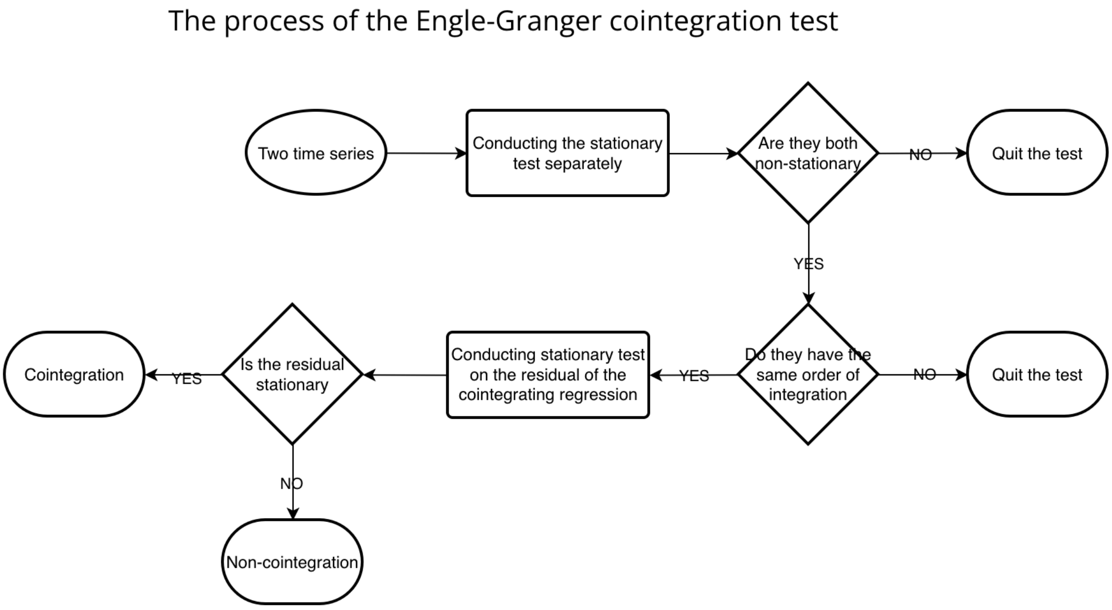


Figure 9: Flow chart

tionary, the ADF test is conducted on the price time series and the results are shown in the following table.

Series	Test Statistic	p-value	1%	5%	10%
Price time series (EuroCurrency)	-2.80	0.06	-3.45	-2.87	-2.57
Price time series (Gold)	-2.54	0.11	-3.45	-2.87	-2.57

This outcome implies that both the price of EuroCurrency and the price of Gold are non-stationary. The next step is to test their order of integration, after once differencing, the series becomes stationary, hence it is shown that both two price time series are first order integration and able to practice the Engle-Granger cointegration test. The result is:

Series	Test Statistic	p-value	1%	5%	10%
Price time series	-3.02	0.11	-3.93	-3.36	-3.06

The test statistic for the price time series is -3.02, with a corresponding p-value of 0.11, indicating insufficient evidence to reject the null hypothesis of no cointegration. This suggests that there may not be a long-term relationship between the prices. For the stationary return time series there is no necessity to conduct an Engle-Granger cointegration test, which is designed for non-stationary time series. If further analysis is required, a Granger test of causality could be employed.

References

[1] B. Walczak, *Wavelets in chemistry.* Elsevier, 2000.

[2] L. Iamleonie, “Time series: Interpreting acf and pacf,” Mar 2022. [Online]. Available: https://www.kaggle.com/code/iamleonie/time-series-interpreting-acf-and-pacf/notebook

[3] R. Cont, “Empirical properties of asset returns: stylized facts and statistical issues,” *Quantitative finance*, vol. 1, no. 2, p. 223, 2001.

[4] B. B. Mandelbrot, “When can price be arbitraged efficiently? a limit to the validity of the random walk and martingale models,” *The Review of Economics and Statistics*, pp. 225–236, 1971.

[5] N. M. Razali, Y. B. Wah *et al.*, “Power comparisons of shapiro-wilk, kolmogorov-smirnov, lilliefors and anderson-darling tests,” *Journal of statistical modeling and analytics*, vol. 2, no. 1, pp. 21–33, 2011.

[6] N. A. Ahad, T. S. Yin, A. R. Othman, and C. R. Yaacob, “Sensitivity of normality tests to non-normal data,” *Sains Malaysiana*, vol. 40, no. 6, pp. 637–641, 2011.

[7] “statsmodels.graphics.gofplots.qqplot - statsmodels 0.15.0 (+213),” Feb 2024. [Online]. Available: https://www.statsmodels.org/devel/generated/statsmodels.graphics.gofplots.qqplot.html#statsmodels.graphics.gofplots.qqplot