YONEDA EMBEDDING

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Lemma *(Yoneda). If $K : D \to$* **Set** *is a functor from D and r an object in D (for D a category with small hom-sets), there is a bijection*

$$y : \mathrm{Nat}(D(r, -), K) \cong Kr \tag{4}$$

which sends each natural transformation $\alpha : D(r, -) \dot{\to} K$ to $\alpha_r 1_r$, the image of the identity $r \to r$.

The proof is indicated by the following commutative diagram:

$$\begin{tikzcd}
D(r, r) \arrow[r, "\alpha_r"] \arrow[d, "f_* = D(r, f)"'] & K(r) \arrow[d, "K(f)"] & r \arrow[d, "f"] \\
{} & {} & {}
\end{tikzcd} \tag{5}$$

NATURAL ISOMORPHISM

The Yoneda map y of (4) is natural in K and r. To state this fact formally, we must consider K as an object in the functor category $\mathbf{Set}^D$, regard both domain and codomain of the map y as functors of the pair $\langle K, r\rangle$, and consider this pair as an object in the category $\mathbf{Set}^D \times D$. The codomain for y is then the evaluation functor E, which maps each pair $\langle K, r\rangle$ to the value Kr of the functor K at the object r; the domain is the functor N which maps the object $\langle K, r\rangle$ to the set $\mathrm{Nat}(D(r, -), K)$ of all natural transformations and which maps a pair of arrows $F : K \to K'$, $f : r \to r'$ to $\mathrm{Nat}(D(f, -), F)$. With these observations we may at once prove an addendum to the *Yoneda Lemma:*

Lemma. *The bijection of* (4) *is a natural isomorphism* $y : N \dot{\to} E$ *between*

PROOF IN HASKELL

```haskell
isoLeft :: (Functor f) => (forall x. (a -> x) -> f x) -> f a
isoLeft nt = nt id


isoRight :: (Functor f) => f a -> (forall x. (a -> x) -> f x)
isoRight fa g = fmap g fa
```

(CO)YONEDA EMBEDDING

The object function $r \mapsto D(r, -)$ and the arrow function

$$(f : s \to r) \mapsto D(f, -) : D(r, -) \dot{\to} D(s, -)$$

for f an arrow of D together define a full and faithful functor

$$Y : D^{\mathrm{op}} \to \mathbf{Set}^D \tag{6}$$

called the *Yoneda functor*. Its dual is another such functor

$$Y' : D \to \mathbf{Set}^{D^{\mathrm{op}}} \tag{7}$$

(also faithful) which sends $f : s \to r$ to the natural transformation

IN HASKELL

```haskell
arrowMap :: (b -> a) -> (forall x. (a -> x) -> (b -> x))

arrowMap f g = g . f
```

CHALLENGE 1

Express the co-Yoneda embedding in Haskell.

SOLUTION

```haskell
arrowMap :: (a -> b) -> (forall x. (x -> a) -> (x -> b))

arrowMap f g = f . g
```

CHALLENGE 2

Show that the bijection we established between fromY and btoa is an isomorphism (the two mappings are the inverse of each other).

CHALLENGE 3

Work out the Yoneda embedding for a monoid. What functor corresponds to the monoid's single object? What natural transformations correspond to monoid morphisms?

CHALLENGE 4

What is the application of the covariant Yoneda embedding to preorders? (Question suggested by Gershom Bazerman.)

SOLUTION

$a \leq b$ if and only if $\forall x, b \leq x \Rightarrow a \leq x$

CHALLENGE 5

Yoneda embedding can be used to embed an arbitrary functor category [C, D] in the functor category [[C, D], Set]. Figure out how it works on morphisms (which in this case are natural transformations).

THANK YOU!