Chapter 5

System Realization Theory

Many different system identification methods and techniques in the field of structures have been developed, analyzed and tested for modal parameter identification. The question naturally arises whether there exists relationships among these methods. In this chapter, a unified mathematical framework based on the theory of system realization is presented to discuss the inter-relationship of these methods. The time-domain methods for modal parameter identification in the field of structures are based on the transfer function matrix, which yields Markov parameters (i.e., pulse response). The knowledge of Markov parameters makes it possible to construct a Hankel matrix as the basis for the realization of a discrete-time state-space model.

To construct a model, a fundamental question immediately arises as to whether or not all the system states of interest can be excited (controlled) and/or observed. To answer this question, one must first understand the theories of controllability and observability. This chapter will thus begin with a discussion on the theories of controllability and observability. The basic concept of minimum realization which was developed by Ho and Kalman [26] will then be described. Next, the Eigensystem Realization Algorithm (ERA) [33, 39] for modal parameter identification, which was developed using minimum realization theory, will be presented and discussed. The Polyreference Technique [85, 86] and the Least-Squares Regression method will then be derived using the mathematical framework developed in the ERA.

5.1 Controllability and Observability

Can all states of a system be controlled and/or observed? This fundamental question arises surprisingly often in both practical and theoretical investigations in the fields of control and system identification. It is most easily investigated using state-space techniques.

As shown in Chapter 2, time-invariant systems can be represented by

$$\dot{x} = A_c x + B_c u \tag{5.1}$$

$$y = Cx + Du \tag{5.2}$$

or by their discrete-time representation

$$x(k+1) = Ax(k) + Bu(k) \tag{5.3}$$

$$y(k) = Cx(k) + Du(k) \tag{5.4}$$

where x, u, and y are the state $(n \times 1)$, control or input $(r \times 1)$, and output $(m \times 1)$ vectors, respectively. The constant matrices $\{A_c,\ B_c,\ C,\ D\}$ and $\{A,\ B,\ C,\ D\}$ with appropriate dimensions represent the internal operation of the linear system, and are used to determine the system's response to any input.

The solution at time t_f to Eq. (5.1) has previously been presented in Chapter 2 as

$$x(t_f) = e^{A_c(t_f - t_0)}x(t_0) + \int_{t_0}^{t_f} e^{A_c(t_f-\tau)}B_c u(\tau)d\tau \tag{5.5}$$

for $t \geq t_0$. The solution to the discrete representation, Eq. (5.3), at time, $t_f = p\Delta t$ where Δt is the sampling time period, has also been shown in Chapter 2 as

$$x(p) = A^p x(0) + \sum_{i=1}^{p} A^{i-1}Bu(p-i) \tag{5.6}$$

or in a compact matrix form

$$x(p) = A^p x(0) + [\,B \quad AB \quad A^2B \quad \cdots \quad A^{p-1}B\,] \begin{bmatrix} u(p-1) \\ u(p-2) \\ u(p-3) \\ \vdots \\ u(0) \end{bmatrix} \tag{5.7}$$

where $x(0) = x(t_0)$ is the initial state at $t = t_0$.

Since the derivation of controllability and observability theories for the continuous-time case is similar to the discrete case, we show only the discrete case because continuous-in-time becomes discrete-in-time when sampled.

5.1.1 Controllability in the Discrete-Time Domain

A state $x(k)$ of a system is said to be *controllable* or *state-controllable* if this state can be reached from any initial state of the system in a finite time interval by some control action. If all states are controllable, the system is called *completely controllable* or simply *controllable.*

To determine complete controllability at the sampling time k, it is sufficient and necessary to determine whether the zero state, instead of all initial states, can be transferred to all final states. To justify this statement, let us rewrite Eq. (5.7) to yield

$$\begin{aligned} \hat{x}(p) &= x(p) - A^p x(0) \\ &= [B \quad AB \quad A^2B \quad \cdots \quad A^{p-1}B] \begin{bmatrix} u(p-1) \\ u(p-2) \\ u(p-3) \\ \vdots \\ u(0) \end{bmatrix} \end{aligned} \tag{5.8}$$

which is equivalent to starting from the zero state and going to a final state $\hat{x}(p) = x(p) - A^p x(0)$. Therefore, if we can show that the system can go from zero state to any $\hat{x}(p)$, then it can go from any $x(0)$ to any $x(p)$. Examination of Eq. (5.8) reveals that the matrix $[B \quad AB \quad A^2B \quad \cdots \quad A^{p-1}B]$ should play a major role in determining the controllability of the system. Indeed, from linear system theory, it is known that only those vectors $\hat{x}(p)$ can be reached which are in the space $\mathcal{S}_c$ generated by the columns of the matrix $[B \quad AB \quad A^2B \quad \cdots \quad A^{p-1}B]$. In other words, there always exists some control series $u(0)$, $u(1)$, $u(2)$, $\cdots$, $u(p)$ to transfer the zero state to the final state $\hat{x}(p)$ which is in the column space $\mathcal{S}_c$.

Theorem 5.1 *The linear, finite-dimensional, discrete-time, constant dynamical system $x(k+1) = Ax(k) + Bu(k)$ of order n is controllable if and only if the $n \times pr$ block controllability matrix $\mathcal{Q}_p$ has rank n, where*

$$\mathcal{Q}_p = [B \quad AB \quad A^2B \quad \cdots \quad A^{p-1}B] \tag{5.9}$$

Proof: To prove this theorem, first form the singular value decomposition of $\mathcal{Q}_p$ to yield

$$\mathcal{Q}_p = R \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix} S^T \triangleq [\, R_k \quad R_0 \,] \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_k^T \\ S_0^T \end{bmatrix} \tag{5.10}$$

where $R^T R = I_n$, $S^T S = I_{pr}$ and Σ_k =diag$[\sigma_1, \sigma_2, \ldots, \sigma_k]$ with $\sigma_1 \geq \sigma_2 \cdots \geq \sigma_k > 0$. Note also that I_n and I_{pr} mean identity matrices of order n and pr, respectively. The matrices R_k and R_0, and S_k and S_0 are obtained with appropriate partitioning of R and S, respectively. Note that $R^T R = I_n$ and $S^T S = I_{pr}$ imply $R_k^T R_k = I_k$ and $S_k^T S_k = I_k$, respectively. In this decomposition, it is assumed that the number k of nonzero singular values is less than n, the order of the system.

Since R is a nonsingular matrix, we can pre-multiply Eq. (5.8) by R^T to yield

$$\begin{aligned} R^T \hat{x}(p) &= R^T [\, B \quad AB \quad A^2B \quad \cdots \quad A^{p-1}B \,] \begin{bmatrix} u(p-1) \\ u(p-2) \\ u(p-3) \\ \vdots \\ u(0) \end{bmatrix} \\ &= R^T R \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix} S^T \begin{bmatrix} u(p-1) \\ u(p-2) \\ u(p-3) \\ \vdots \\ u(0) \end{bmatrix} \end{aligned} \tag{5.11}$$

Now define $u_p^T = [\, u(p-1)^T \quad u(p-2)^T \quad u(p-3)^T \quad \cdots \quad u(0)^T \,]$. The above equation becomes

$$\begin{bmatrix} R_k^T \hat{x}(p) \\ R_0^T \hat{x}(p) \end{bmatrix} = \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_k^T \\ S_0^T \end{bmatrix} u_p = \begin{bmatrix} \Sigma_k S_k^T u_p \\ 0 \end{bmatrix} \tag{5.12}$$

From this equation, it is seen that the lower part of the vector on the left-hand side cannot be reached by any force vector u_p from the zero state. This in turn implies that only those vectors $\hat{x}(p)$ which make the component $R_0^T \hat{x}(p) = 0$ can be reached from the zero state, i.e., the system is not controllable.

To make the system controllable, it is sufficient and necessary to eliminate the zero portion on the right-hand side of the above equation. The only way

to do that is to increase the number of nonzero singular values, k, until $k = n$ (the order of the system). If $k = n$, the above equation becomes

$$R^T\hat{x}(p) = [\Sigma_n \quad 0]\begin{bmatrix} S_n^T \\ S_0^T \end{bmatrix} u_p = \left[\Sigma_n S_n^T u_p\right] \tag{5.13}$$

Since Σ and S_n are both rank n, the control vector u_p can be solved as

$$\begin{aligned} u_p &= (S_n^T)^\dagger \Sigma_n^{-1} R^T \hat{x}(p) + \left[I - (S_n^T)^\dagger S_n^T\right]\alpha \\ &= S_n \Sigma_n^{-1} R^T \hat{x}(p) + \left[I - S_n S_n^T\right]\alpha \end{aligned} \tag{5.14}$$

where $\dagger$ means the pseudo-inverse and α is an $nr \times 1$ arbitrary vector. Note that $(S_n^T)^\dagger = S_n(S_n^T S_n)^{-1} = S_n$ since $S_n^T S_n = I_n$, which is the result of $S^T S = I_{pr}$ as indicated before. This equation simply implies that, for any desired vector $\hat{x}(p)$ and an arbitrarily chosen vector α, there exists a control vector u_p which will transfer the zero state to the desired vector $\hat{x}(p)$. Because α is an arbitrary vector, the control vector is not uniquely determined. In other words, there exists more than one control vector which can transfer the zero state to the final desired state $\hat{x}(p)$. The norm of the control vector u_p becomes minimum when $\alpha = 0$. We thus conclude that *a linear, finite-dimensional, discrete-time, constant dynamical system is controllable if and only if its controllability matrix has rank equal to the order of the system.*

Although the proof of Theorem 5.1 is mathematically rigorous, its physical insight is not obvious. Let us look at one special case in the following theorem.

Theorem 5.2 *Consider a scalar control force $u(k)$ to the discrete-time system $x(k+1) = Ax(k) + bu(k)$ where b is a column vector. Assume that the constant matrix A has distinct eigenvalues λ_i. The system is controllable if and only if the constant vector $b_m = \Psi^{-1}b$ has no zero elements, where Ψ is the modal matrix with eigenvectors of A as its column vectors.*

Proof: To prove this theorem, first form the following controllability matrix:

$$\begin{aligned} Q_p &= [b \quad Ab \quad \cdots \quad A^{p-1}b] \\ &= [b \quad A\Psi\Psi^{-1}b \quad \cdots \quad A^{p-1}\Psi\Psi^{-1}b] \\ &= \Psi\Psi^{-1}[b \quad A\Psi\Psi^{-1}b \quad \cdots \quad A^{p-1}\Psi\Psi^{-1}b] \\ &= \Psi[\Psi^{-1}b \quad \Psi^{-1}A\Psi\Psi^{-1}b \quad \cdots \quad \Psi^{-1}A^{p-1}\Psi\Psi^{-1}b] \end{aligned} \tag{5.15}$$

Now, note that

$$\begin{aligned}
\Psi^{-1}b &= b_m \\
\Psi^{-1}A\Psi &= \text{diag}[\lambda_1,\ \lambda_2, \cdots \lambda_n] = \Lambda \\
\Psi^{-1}A^2\Psi &= [\Psi^{-1}A\Psi][\Psi^{-1}A\Psi] \\
&= \text{diag}[\lambda_1^2,\ \lambda_2^2, \cdots \lambda_n^2] = \Lambda^2 \\
&\vdots \\
\Psi^{-1}A^p\Psi &= [\Psi^{-1}A\Psi]\cdots[\Psi^{-1}A\Psi] \\
&= \text{diag}[\lambda_1^p,\ \lambda_2^p, \cdots \lambda_n^p] = \Lambda^p
\end{aligned} \tag{5.16}$$

Application of Eq. (5.16) to Eq. (5.15) thus yields

$$\begin{aligned}
\mathcal{Q}_p &= \Psi\,[b_m \quad \Lambda b_m \quad \cdots \quad \Lambda^{p-1}b_m] \\
&= \Psi \begin{bmatrix} b_1 & \lambda_1 b_1 & \cdots & \lambda_1^{p-1}b_1 \\ b_2 & \lambda_2 b_2 & \cdots & \lambda_2^{p-1}b_2 \\ \vdots & \vdots & \ddots & \vdots \\ b_n & \lambda_n b_n & \cdots & \lambda_n^{p-1}b_n \end{bmatrix}
\end{aligned} \tag{5.17}$$

where

$$b_m = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If any element b_i of the b_m vector is zero, say for example $b_1 = 0$, then the controllability matrix becomes

$$\mathcal{Q}_p = \Psi \begin{bmatrix} 0 & 0 & \cdots & 0 \\ b_2 & \lambda_2 b_2 & \cdots & \lambda_2^{p-1}b_2 \\ \vdots & \vdots & \ddots & \vdots \\ b_n & \lambda_n b_n & \cdots & \lambda_n^{p-1}b_n \end{bmatrix}$$

This matrix obviously has rank less than n, because the second matrix on the right-hand side has only $n - 1$ nonzero rows. Therefore, if the system is controllable, then all elements b_i for $i = 1,\ 2,\ \ldots,\ n$ must be nonzero. In the proof, it is noted that the eigenvector matrix Ψ has a full-rank n because a system with distinct eigenvalues has all independent eigenvectors. In structures, $b_1 = 0$ happens only when the control force u is applied at a node of the mode having the eigenvalue λ_1. This means that if the control force u is located at a node of a mode, then the mode is not controllable.

5.1.2 Observability in the Discrete-Time Domain

A state $x(p)$ at the given sample time p of a system is *observable* if knowledge of the input $u(k)$ and output $y(k)$ over a finite time interval $0 < k \leq p$ completely determines the state $x(p)$. If all states are observable, the system is called *completely observable* or simply *observable.*

To determine complete observability, it is necessary and sufficient to see whether the initial state $x(0)$ of the zero input system, i.e., $u(k) = 0$, can be completely determined from the output $y(k)$ ($k = 0, 1, \ldots, p-1$), because knowledge of the initial state $x(0)$ and input $u(k)$ ($k = 0, 1, \ldots, p-1$) permits the calculation of the state $x(p-1)$ from Eq. (5.7).

Theorem 5.3 *The linear, finite-dimensional, discrete-time, constant dynamical system $x(k+1) = Ax(k) + Bu(k)$ of order n with the measurement equation $y(k) = Cx(k) + Du(k)$ of order m is observable if and only if the $mp \times n$ block observability matrix $\mathcal{P}_p$ has rank n, where*

$$\mathcal{P}_p = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{p-1} \end{bmatrix} \tag{5.18}$$

Proof: To prove this theorem, it is sufficient to see if the initial state $x(0)$ can be reconstructed from knowledge of $y(k)$ for $k = 0, 1, \ldots, p-1$, in the case where $u(k) = 0$. From the discrete-time model, Eqs. (5.3) and (5.4) with $u(k) = 0$,

$$\begin{aligned} y(0) &= Cx(0) \\ y(1) &= Cx(1) = CAx(0) \\ &\vdots \\ y(p-1) &= Cx(p-1) = CA^{p-1}x(0) \end{aligned} \tag{5.19}$$

This equation can be rewritten as

$$Y_p \triangleq \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(p-1) \end{bmatrix} = \mathcal{P}_p x(0) \tag{5.20}$$

where a unique solution exists if and only if $\mathcal{P}_p$ has rank n. The unique solution should be $x(0) = \mathcal{P}_p^{\dagger} Y_p$ where $\dagger$ means the pseudo-inverse.

Based on Theorem 5.3, a special theorem for the discrete-time model in modal coordinates can be derived providing some physical insight into the general theory of observability.

Theorem 5.4 *Consider the discrete-time system*

$$x(k+1) = Ax(k) + Bu(k)$$

and a scalar measurement vector

$$y(k),\ k = 1,\ 2,\ \ldots$$

from the measurement equation $y(k) = cx(k) + du(k)$ *where* c *and* d *are constant row vectors, indicating that the system has a single output sensor. Assume that the constant matrix* A *has distinct eigenvalues* λ_i. *The system is observable if and only if the constant vector* $c_m = c\Psi$ *has no zero elements, where* Ψ *is the modal matrix with eigenvectors of* A *as its column vectors.*

Proof: This Theorem is dual in nature to Theorem 5.2. Because it can be proved by the approach used for Theorem 5.2, the proof is omitted.

Physically this theorem implies that if there is a single sensor and it is located at a node of a mode, then the mode becomes unobservable. This statement can be easily justified by transforming the discrete-time system

$$x(k+1) = Ax(k) + Bu(k) \quad \text{and} \quad y(k) = cx(k) + du(k)$$

into

$$x_m(k+1) = \Lambda x_m(k) + B_m u(k) \quad \text{and} \quad y(k) = c_m x(k) + du(k)$$

in modal coordinates where

$$x = \Psi x_m,\ B_m = \Psi^{-1}B,\ c_m = c\Psi$$

$$\Lambda = \text{diag}[\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n]$$

If any element c_i of the vector c_m is zero, say for example $c_1 = 0$, then the corresponding coordinate x_{m1} is not observable in the sense that the measurement

$$y(k) = [0\ c_2\ \cdots\ c_n] \begin{bmatrix} x_{m1} \\ x_{m2} \\ \vdots \\ x_{mn} \end{bmatrix}$$

does not contain any contribution from the modal coordinate x_{m1}.

Example 5.1

Consider a rigid body of mass m with an applied force f acting along the direction of motion. The equation of motion is

$$m\ddot{w}(t) = f(t)$$

where w is the displacement of the mass. This equation can be written in the first-order state-space form

$$\begin{bmatrix} \dot{w} \\ \ddot{w} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ \dot{w} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{f}{m}$$

or

$$\dot{x} = A_c x + B_c u$$

where

$$x = \begin{bmatrix} \dot{w} \\ \ddot{w} \end{bmatrix}, \quad A_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u = \frac{f}{m}$$

Assume that the displacement w is sampled on every time interval Δt and the applied force does not vary over the interval Δt. The continuous-time model can be converted into the following discrete-time model

$$x(k+1) = Ax(k) + Bu(k)$$

where

$$\begin{aligned} A &= e^{A_c \Delta t} = [I + A_c \Delta t + \frac{1}{2}(A_c \Delta t)^2 + \cdots] = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \\ B &= \int_0^{\Delta t} e^{A_c \Delta \tau} d\tau B_c = \Delta t[I + \frac{1}{2} A_c \Delta t + \frac{1}{3}(A_c \Delta t)^2 + \cdots] B_c \\ &= \begin{bmatrix} \Delta t & \frac{1}{2}\Delta t^2 \\ 0 & \Delta t \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix} \end{aligned}$$

Note that $A_c^2 = A_c^3 = \ldots = A_c^\infty = 0$.

The discrete-time model is controllable because the controllability matrix

$$\mathcal{Q}_2 = [B \quad AB] = \begin{bmatrix} \frac{1}{2}\Delta t^2 & \frac{3}{2}\Delta t^2 \\ \Delta t & \Delta t \end{bmatrix} = \Delta t \begin{bmatrix} \frac{1}{2}\Delta t & \frac{3}{2}\Delta t \\ 1 & 1 \end{bmatrix}$$

always has rank 2 unless $\Delta t = 0$.

Let y be the direct measurement of the displacement w. The measurement equation becomes

$$y(t) = w(t) = [1 \quad 0] \begin{bmatrix} w \\ \dot{w} \end{bmatrix} \triangleq Cx(t)$$

Assume that the displacement w is sampled on every time interval Δt. The discrete measurement equation becomes

$$y(k) = Cx(k) \quad \text{with} \quad C = [1 \quad 0]$$

The discrete-time model is observable because the observability matrix

$$\mathcal{P}_2 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & \Delta t \end{bmatrix}$$

has rank 2 unless $\Delta t = 0$. Physically, this implies that knowledge of displacement alone determines the state x which includes the displacement and the velocity. The velocity can be obtained by differentiating the displacement.

With direct velocity measurement, the measurement equation becomes

$$y(t) = \dot{w}(t) = [0 \quad 1] \begin{bmatrix} w \\ \dot{w} \end{bmatrix} \triangleq Cx(t)$$

Assume that the velocity $\dot{w}$ is sampled on every time interval Δt. The discrete measurement equation becomes

$$y(k) = Cx(k) \quad \text{with} \quad C = [0 \quad 1]$$

The discrete-time model is not observable because the observability matrix

$$\mathcal{P}_2 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

has rank 1. Physically this means that knowledge of the velocity alone cannot determine the state x which includes the displacement w and the velocity $\dot{w}$. Indeed, integrating the velocity does not provide an accurate displacement unless the initial displacement is known.

With direct acceleration measurement, the measurement equation becomes

$$y(t) = \ddot{w}(t) = \frac{f(t)}{m}$$

Assume that the acceleration $\ddot{w}$ is sampled on every time interval Δt. The discrete measurement equation becomes

$$y(k) = Cx(k) + Du(k) \quad \text{with} \quad C = [0 \quad 0], \; D = 1, \; u(k) = \frac{f(k)}{m}$$

The discrete-time model is not observable because the observability matrix

$$\mathcal{P}_2 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has rank 0. This indicates that knowledge of the acceleration alone cannot determine the state x which includes the displacement w and the velocity $\dot{w}$. Indeed, integrating the acceleration does not provide an accurate displacement and velocity unless the initial displacement and velocity are known.

5.2 Basic Concepts of Realization

Let us again recall Eqs. (2.10) and (2.11), which represent the system characteristics in the discrete domain, i.e.,

$$x(k+1) = Ax(k) + Bu(k) \tag{5.21}$$

and

$$y(k) = Cx(k) + Du(k) \tag{5.22}$$

Let $u_i(0) = 1$ $(i = 1, 2, \ldots, r)$ and $u_i(k) = 0$ $(k = 1, 2, \ldots)$ be substituted into Eqs. (5.21) and (5.22). When the substitution is performed for each input element, the results can be assembled into a pulse-response matrix Y with dimensions m by r as follows:

$$Y_0 = D, \quad Y_1 = CB, \quad Y_2 = CAB, \quad \cdots, \quad Y_k = CA^{k-1}B \tag{5.23}$$

The constant matrices in the sequence are known as **Markov parameters**. Computation of Markov parameters from experimental data will be shown later. The Markov parameters are used here as the basis for identifying the discrete-time model, Eqs. (5.21) and (5.22), represented by the four constant matrices A, B, C, and D. Since $D = Y_0$, only the three matrices A, B, and C need to be determined.

A *realization* is the computation of a triplet $[A, B, C]$ from the Markov parameters shown in Eq. (5.23), for which the discrete-time model, Eqs. (5.21) and (5.22), is satisfied. As shown in Eqs. (2.12) through Eq. (2.14), any system has an infinite number of realizations which will predict the identical response for any particular input. Minimum realization means a model with the smallest state-space dimensions among all realizable systems that have the same input-output relations. All minimum realizations have the same set of eigenvalues, which are modal parameters of the system itself.

Assume that the state matrix A of order n has a complete set of linearly independent eigenvectors $(\psi_1, \psi_2, \cdots, \psi_n)$ with corresponding eigenvalues $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ which are not necessarily distinct. Define Λ as the diagonal matrix of eigenvalues and Ψ as the matrix of eigenvectors, i.e.,

$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

and

$$\Psi = [\psi_1, \psi_2, \cdots, \psi_n].$$

The realization $[A,\ B,\ C]$ can then be transformed to the realization $[\Lambda,\ \Psi^{-1}B,\ C\Psi]$. The diagonal matrix Λ contains the information of modal damping rates and damped natural frequencies. The matrix $\Psi^{-1}B$ defines the initial modal amplitudes and the matrix $C\Psi$ the mode shapes at the sensor points. All the modal parameters of a dynamic system can thus be identified by the triplet $[\Lambda,\ \Psi^{-1}B,\ C\Psi]$. The desired modal damping rates and damped natural frequencies are simply the real and imaginary parts of the eigenvalues Λ_c, after transformation from the discrete-time domain to the continuous-time domain using the relation $\Lambda_c = \ln(\Lambda)/\Delta t$.

It is important to note that the transformation from the discrete-time model to the continuous-time model is not unique. The imaginary part of the natural logarithm of a complex number can be adjusted by adding any multiple of 2π, which allows Λ_c to take on different values. This corresponds to the fact that any two frequencies that differ by a multiple of $2\pi/\Delta t$ are indistinguishable when observed at the sample times. Therefore in practice, if one wishes to interpret the natural frequencies of the physical system, either the sample time interval Δt must be sufficiently short or a filter must be added to prevent the frequencies beyond the Nyquist frequency from being folded into a lower frequency in the realization.

System realization begins by forming the generalized $\alpha m \times \beta r$ Hankel matrix [6, 7, 26, 44, 78, 88], composed of the Markov parameters from Eq. (5.23):

$$H(k-1) = \begin{bmatrix} Y_k & Y_{k+1} & \cdots & Y_{k+\beta-1} \\ Y_{k+1} & Y_{k+2} & \cdots & Y_{k+\beta} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{k+\alpha-1} & Y_{k+\alpha} & \cdots & Y_{k+\alpha+\beta-2} \end{bmatrix} \tag{5.24}$$

For the case when $k = 1$,

$$H(0) = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_\beta \\ Y_2 & Y_3 & \cdots & Y_{1+\beta} \\ \vdots & \vdots & \ddots & \vdots \\ Y_\alpha & Y_{1+\alpha} & \cdots & Y_{\alpha+\beta-1} \end{bmatrix}$$

Note that $Y_0 = D$ is not included in $H(0)$. If $\alpha \geq n$ and $\beta \geq n$ (the order of the system), the matrix $H(k-1)$ is of rank n. To confirm this point, substituting the Markov parameters from Eq. (5.23) into Eq. (5.24) and decomposing $H(k-1)$ into three matrices yield

$$H(k-1) = \mathcal{P}_\alpha A^{k-1} \mathcal{Q}_\beta \tag{5.25}$$

where $\mathcal{P}_\alpha$ and $\mathcal{Q}_\beta$ are

$$\mathcal{P}_\alpha = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{\alpha-1} \end{bmatrix} \quad \text{and} \quad \mathcal{Q}_\beta = [B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{\beta-1}B].$$

The block matrix $\mathcal{P}_\alpha$ is the observability matrix, whereas the block matrix $\mathcal{Q}_\beta$ is the controllability matrix. If the order of the system is n, then the minimum dimension of the state matrix is $n \times n$. If the system is controllable and observable, the block matrices $\mathcal{P}_\alpha$ and $\mathcal{Q}_\beta$ are of rank n. Therefore the Hankel matrix is of rank n by Eq. (5.24). Based on the properties of the Hankel matrix composed of the Markov parameters (pulse response samples), several methods for modal parameter identification are discussed in the following sections.

5.3 The Eigensystem Realization Algorithm (ERA)

The basic development of the state-space realization is attributed to Ho and Kalman [26] who introduced the important principles of minimum realization theory. The Ho-Kalman procedure uses the generalized Hankel matrix, Eq. (5.24), to construct a state-space representation of a linear system from *noise-free data.* The methodology has been modified and substantially extended to develop the Eigensystem Realization Algorithm [33] to identify modal parameters from noisy measurement data.

In contrast to classical system realization methods which use the generalized Hankel matrix given in Eq. (5.24), the ERA algorithm begins by forming a block data matrix which is obtained by deleting some rows and columns of the generalized Hankel matrix of Eq. (5.24), but maintaining the first block matrix intact, i.e., Y_k. Furthermore, the standard ordering of entries in the generalized Hankel matrix does not need to be maintained.

Recall that r is the number of inputs and m the number of outputs. Let the input and output matrices be defined as

$$B = [b_1 \;\; b_2 \;\; \cdots \;\; b_r] \quad \text{and} \quad C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$

where the column vector b_i is the control influence vector for the ith control input and the row vector c_j is the measurement influence vector for the jth measurement sensor. Denote column submatrices of B by B_i $(i = 0, 1, \cdots, \eta)$, and row submatrices of C by C_j $(j = 0, 1, \cdots, \xi)$. The matrices B_i and C_j^T are subsets of B and C, $[b_1, b_2, \cdots, b_r]$ and $[c_1^T, c_2^T, \ldots, c_m^T]^T$, respectively. The ERA data block matrix can then be expressed by

$$H(k-1) = [Y_{s_i+k+t_j}]; \quad Y_{s_i+k+t_j} = C_j A^{s_i+k-1+t_j} B_i \tag{5.26}$$

where $s_0 = t_0 = 0$, and s_i and t_j are arbitrary integers. When $i = j = 0$, $Y_k = CA^{k-1}B$. This matrix mathematically looks more complex in shape than that shown in Eq. (5.24). Nevertheless, it represents the matrix shown in Eq. (5.24) where some rows and columns are deleted but the first block is maintained intact.

The ERA block data matrix, Eq. (5.26), allows one to include only good or strongly measured signals without losing any capability. This is useful since some measurement data may be noisier than others or sensors may malfunction during the test. The advantage of this capability is the potential to minimize the distortion of the identified parameters caused by noise. A judicious choice of data and its proper arrangement in the block matrix $H(k)$ can also be used to minimize the computational requirements of the method. For example, the columns of $H(k)$ may be made as independent as possible by properly selecting the data samples to use as entries of the matrix. This effort could substantially reduce the order of the matrix for large problems. For sufficiently low noise data, the order can be the same as that of the true system state matrix A. This fact results from examination of the controllability and observability matrices, to be discussed next.

From Eq. (5.26), it can be shown that

$$H(k) = \mathcal{P}_\alpha A^k \mathcal{Q}_\beta \tag{5.27}$$

where

$$\mathcal{P}_\alpha = \begin{bmatrix} C \\ C_1 A^{s_1} \\ \vdots \\ C_\alpha A^{s_\alpha} \end{bmatrix}$$

and

$$\mathcal{Q}_\beta = [B \;\; A^{t_1} B_1 \;\; \cdots \;\; A^{t_\beta} B_\beta]$$

Again, s_i and t_j are arbitrary integers. Note that $\mathcal{P}_\alpha$ and $\mathcal{Q}_\beta$ are generalized observability and controllability matrices, which are somewhat different

from those shown in Eqs. (5.9) and (5.18). However, they possess the same properties of controllability and observability as discussed before.

Assume that there exists a matrix $H^\dagger$ satisfying the relation

$$\mathcal{Q}_\beta H^\dagger \mathcal{P}_\alpha = I_n \tag{5.28}$$

where I_n is an identity matrix of order n. It will be shown that the matrix $H^\dagger$ plays a major role in deriving the ERA. What is $H^\dagger$? Observe that,

$$H(0)H^\dagger H(0) = \mathcal{P}_\alpha \mathcal{Q}_\beta H^\dagger \mathcal{P}_\alpha \mathcal{Q}_\beta = \mathcal{P}_\alpha \mathcal{Q}_\beta = H(0) \tag{5.29}$$

The matrix $H^\dagger$ is thus the pseudo-inverse of the matrix $H(0)$ in a general sense.

The ERA process starts with the factorization of the block data matrix, Eq. (5.26), for $k = 1$, using singular value decomposition,

$$H(0) = R\,\Sigma\,S^T \tag{5.30}$$

where the columns of matrices R and S are orthonormal and Σ is a rectangular matrix

$$\Sigma = \begin{bmatrix} \Sigma_n & 0 \\ 0 & 0 \end{bmatrix}$$

with

$$\Sigma_n = \text{diag}[\sigma_1,\ \sigma_2, \cdots, \sigma_i,\ \sigma_{i+1}, \cdots, \sigma_n]$$

and monotonically non-increasing σ_i $(i = 1,\ 2,\ \ldots,\ n)$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_i \geq \sigma_{i+1} \geq \cdots \sigma_n \geq 0.$$

Next, let R_n and S_n be the matrices formed by the first n columns of R and S, respectively. Hence, the matrix $H(0)$ and its pseudo-inverse become

$$H(0) = R_n \Sigma_n S_n^T \quad \text{where} \quad R_n^T R_n = I_n = S_n^T S_n \tag{5.31}$$

and

$$H^\dagger = S_n \Sigma_n^{-1} R_n^T. \tag{5.32}$$

Equation (5.32) can be readily proved by observing Eq. (5.29).

Observation: Comparison of Eqs. (5.31) and (5.27) with $k = 0$ suggests that $\mathcal{P}_\alpha$ is related to R_n and $\mathcal{Q}_\beta$ is related to S_n^T. Indeed, one possible choice is $\mathcal{P}_\alpha = R_n \Sigma_n^{1/2}$ and $\mathcal{Q}_\beta = \Sigma_n^{1/2} S_n^T$. This choice appears to make both $\mathcal{P}_\alpha$ and $\mathcal{Q}_\beta$ balanced. From Eq. (5.27), it is clear that the first r columns form

the input matrix B whereas the first m rows form the output matrix C. With $k = 1$ in Eq. (5.27), one obtains that

$$H(1) = \mathcal{P}_\alpha A \mathcal{Q}_\beta = R_n \Sigma_n^{1/2} A \Sigma_n^{1/2} S_n^T$$

One obvious solution for the state matrix A becomes

$$A = \Sigma_n^{-1/2} R_n^T H(1) S_n \Sigma_n^{-1/2}$$

Although the above observation is somewhat valid, a rigorous mathematical proof is necessary to support it.

Define O_i as a null matrix of order i, I_i as an identity matrix of order i, $E_m^T = [I_m \; O_m \cdots O_m]$ where m is the number of outputs, and $E_r^T = [I_r \; O_r \cdots O_r]$ where r is the number of inputs. Using Eqs. (5.26), (5.27), (5.28), (5.31), and (5.32), a minimum order realization can be obtained as follows:

$$\begin{aligned}
Y_k &= E_m^T H(k-1) E_r \quad \text{(from Eq.(5.26))} \\
&= E_m^T \mathcal{P}_\alpha A^{k-1} \mathcal{Q}_\beta E_r \quad \text{(use Eqs.(5.26) and (5.27))} \\
&= E_m^T \mathcal{P}_\alpha [\mathcal{Q}_\beta H^\dagger \mathcal{P}_\alpha] A^{k-1} [\mathcal{Q}_\beta H^\dagger \mathcal{P}_\alpha] \mathcal{Q}_\beta E_r \quad \text{(use Eq. (5.28))} \\
&= E_m^T H(0) [S_n \Sigma_n^{-1} R_n^T] \mathcal{P}_\alpha A^{k-1} \mathcal{Q}_\beta [S_n \Sigma_n^{-1} R_n^T] H(0) E_r \\
&\quad \text{(use Eqs. (5.27) and (5.32))} \\
&= E_m^T H(0) S_n \Sigma_n^{-1/2} [\Sigma_n^{-1/2} R_n^T H(1) S_n \Sigma_n^{-1/2}]^{k-1} \Sigma_n^{-1/2} R_n^T H(0) E_r \\
&\quad \text{(use Eq. (5.31))} \\
&= E_m^T R_n \Sigma_n^{1/2} [\Sigma_n^{-1/2} R_n^T H(1) S_n \Sigma_n^{-1/2}]^{k-1} \Sigma_n^{1/2} S_n^T E_r \qquad (5.33)
\end{aligned}$$

This is the basic formulation of realization for the ERA. The triplet

$$\hat{A} = \Sigma_n^{-1/2} R_n^T H(1) S_n \Sigma_n^{-1/2}, \quad \hat{B} = \Sigma_n^{1/2} S_n^T E_r, \quad \hat{C} = E_m^T R_n \Sigma_n^{1/2} \qquad (5.34)$$

is a minimum realization. Here the quantities with $\hat{}$ mean estimated quantities to distinguish from the true quantities. The order of the matrix $\hat{A}$ is n which is the order of the system for sufficiently low-noise data. When the matrices R_n and S_n are obtained through other factorization methods such that $R_n \, R_n^T \neq I_n$ and $S_n \, S_n^T \neq I_n$, Eq. (5.34) is still valid if the matrices R_n^T and S_n^T are replaced by $R_n^\dagger$ and $S_n^\dagger$, respectively.

The realized discrete-time model represented by the matrices $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ can be transformed to the continuous-time model. The system frequencies and dampings may then be computed from the eigenvalues of the estimated continuous-time state matrix. The eigenvectors allow a transformation of the realization to modal space and hence the determination of the complex (or damped) mode shapes and the initial modal amplitudes (or modal participation factors).

Example 5.2

Consider a rigid body of mass m with a force f acting along the direction of motion. As shown in Example 5.1, the discrete-time model with the sampling time interval Δt for this system can be written as

$$x(k+1) = Ax(k) + Bu(k)$$

where

$$A = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix}, \quad u(k) = \frac{f(k)}{m}$$

Assume that the displacement w is sampled on every time interval Δt. The discrete measurement equation becomes

$$y(k) = Cx(k) \quad \text{with} \quad C = [1 \quad 0]$$

The pulse response samples for this model are

$$\begin{aligned}
Y_0 &= D = 0 \\
Y_1 &= CB = [1 \quad 0] \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix} = \frac{1}{2}\Delta t^2 \\
Y_2 &= CAB = [1 \quad 0] \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix} = \frac{3}{2}\Delta t^2 \\
Y_3 &= CA^2B = [1 \quad 0] \begin{bmatrix} 1 & 2\Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix} = \frac{5}{2}\Delta t^2 \\
Y_4 &= CA^3B = [1 \quad 0] \begin{bmatrix} 1 & 3\Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix} = \frac{7}{2}\Delta t^2 \\
Y_5 &= CA^4B = [1 \quad 0] \begin{bmatrix} 1 & 4\Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix} = \frac{9}{2}\Delta t^2
\end{aligned}$$

Form the Hankel matrix $H(0)$

$$H(0) = \begin{bmatrix} Y_1 & Y_2 \\ Y_2 & Y_3 \end{bmatrix} = \frac{\Delta t^2}{2} \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$$

and the shifted Hankel matrix $H(1)$

$$H(1) = \begin{bmatrix} Y_2 & Y_3 \\ Y_3 & Y_4 \end{bmatrix} = \frac{\Delta t^2}{2} \begin{bmatrix} 3 & 5 \\ 5 & 7 \end{bmatrix}$$

For a noise-free system of order two, a 2×2 Hankel matrix is sufficient for system identification. Taking singular value decomposition of the Hankel matrix $H(0)$ yields

$$\begin{aligned} H(0) &= R\Sigma S^T \\ &= \begin{bmatrix} 0.3346 & 0.9424 \\ 0.9424 & -0.3346 \end{bmatrix} \begin{bmatrix} 4.5616\Delta t^2 & 0 \\ 0 & 0.4384\Delta t^2 \end{bmatrix} \begin{bmatrix} 0.5531 & -0.8331 \\ 0.8331 & 0.5531 \end{bmatrix}^T \end{aligned}$$

Let $H(0)$ be written as

$$\begin{aligned} H(0) &= [R\Sigma^{1/2}][\Sigma^{1/2}S^T] \\ &= \left\{ \begin{bmatrix} 0.7146 & 0.6240 \\ 2.0127 & -0.2215 \end{bmatrix} \Delta t \right\} \left\{ \begin{bmatrix} 1.1814 & 1.7793 \\ -0.5516 & 0.3663 \end{bmatrix} \Delta t \right\} \\ &\triangleq \{\mathcal{P}\}\{\mathcal{Q}\} \end{aligned}$$

The first row of $\mathcal{P}$ becomes the identified output matrix $\hat{C}$

$$\hat{C} = [0.7146 \quad 0.6240]\,\Delta t$$

and the first column of $\mathcal{Q}$ becomes the identified input matrix $\hat{B}$

$$\hat{B} = \begin{bmatrix} 1.1814 \\ -0.5516 \end{bmatrix} \Delta t$$

The state matrix can then be identified by

$$\hat{A} = \Sigma^{-1/2} R^T H(1) S \Sigma^{-1/2} = \begin{bmatrix} 1.3881 & -0.2528 \\ 0.5958 & 0.6119 \end{bmatrix}$$

The identified state matrix has two repeated eigenvalues $\lambda = 1$, which mean zero (ln(1)=0) in the continuous-time model; i.e., rigid body motion. The identified matrices $\hat{A}$, $\hat{B}$, and $\hat{C}$ appear different from the original discrete matrices A , B and C. Nevertheless, both $\hat{A}$, $\hat{B}$, $\hat{C}$ and A, B, C represent the same model in the sense that they provide identical pulse response samples.

Due to measurement noise, nonlinearity, and computer round-off, the block matrix $H(k)$ will usually be of full rank which does not, in general, equal the true order of the system under test. It should not be the aim

to obtain a system realization which exactly reproduces the noisy sequence of data. A realization which produces a smoothed version of the sequence, and which closely represents the underlying linear dynamics of the system, is more desirable.

If Eqs. (5.27) and (5.29) are examined as a whole, one can write the equality

$$H(0) = [R_n \Sigma_n^{1/2}][\Sigma_n^{1/2} S_n^T] \cong \mathcal{P}_\alpha \mathcal{Q}_\beta \tag{5.35}$$

where the approximation sign $\cong$ is used due to noise and truncation of nonzero small singular values. The controllability grammian, $\mathcal{Q}_\beta \mathcal{Q}_\beta^T$, and the observability grammian, $\mathcal{P}_\alpha^T \mathcal{P}_\alpha$, can then be computed as

$$\mathcal{Q}_\beta \mathcal{Q}_\beta^T = \Sigma_n \quad \text{and} \quad \mathcal{P}_\alpha^T \mathcal{P}_\alpha = \Sigma_n \tag{5.36}$$

The fact that the controllability and observability grammians are equal and diagonal implies that the realized system $[\hat{A},\ \hat{B},\ \hat{C}]$ is as controllable as it is observable. This property is called an internally balanced realization. It means that the signal transfer from the input to the state and then from the state to the output are similar and balanced.

Some singular values, say $\sigma_{i+1}, \cdots, \sigma_n$, may be relatively small and negligible in the sense that they contain more noise information than system information. In other words, the directions determined by the singular values $\sigma_{i+1}, \cdots, \sigma_n$, have less significant degrees of controllability and observability relative to the noise. It would be unwise to require a realization to include these directions. The reduced model of order i after deleting singular values $\sigma_{i+1}, \cdots, \sigma_n$ is then considered as the robustly controllable and observable part of the realized system. Reference [34] provides the mathematical framework for establishing the relationship between the singular values and the characteristics of the noise. Besides the singular values, several other accuracy indicators have been developed for quantitatively partitioning the realized model into principal and perturbational (noise) portions so that the noise portion can be disregarded.

5.3.1 Candidate Methods for Distinguishing True Modes from Noise Modes

Two approaches to distinguish true modes from noise modes are considered. These include the Modal Amplitude Coherence (MAC) and the Mode Singular Value (MSV). The following subsections describe each in more detail. Let us begin with some basic formulations.

Consider an identified discrete-time model in modal coordinates

$$\begin{aligned} x_m(k+1) &= \hat{\Lambda} x_m(k) + \hat{B}_m u(k) \\ y(k) &= \hat{C}_m x_m(k) + Du(k) \end{aligned} \tag{5.37}$$

with r inputs and m outputs, where $\hat{\Lambda}$ is a diagonal matrix containing the identified eigenvalues, $\hat{\lambda}_i\,(i = 1, 2, ..., n)$, of the system, and $\hat{B}_m$ and $\hat{C}_m$ are the input and output matrices in modal coordinates, respectively. Because the measurement vector y is real, all complex quantities in Eq. (5.37) including the eigenvalues occur as complex conjugate pairs.

For a linear system, the map from input u to output y can be described by the identified Markov parameter sequence

$$\begin{aligned} \hat{Y} &= [\hat{D} \quad \hat{C}_m\hat{B}_m \quad \hat{C}_m\hat{\Lambda}\hat{B}_m \quad \cdots \quad \hat{C}_m\hat{\Lambda}^{\ell-2}\hat{B}_m] \\ &\triangleq [\hat{Y}_0 \quad \hat{Y}_1 \quad \hat{Y}_2 \quad \cdots \quad \hat{Y}_{\ell-1}] \end{aligned} \tag{5.38}$$

where ℓ is the number of Markov parameters. For the noise-free case, the Markov parameter sequence is a combination of pulse response samples generated by unit pulse from different input locations. The jth column of the Markov parameter matrix $\hat{Y}_i$ represents the pulse response at the ith time step with a unit pulse applied at the jth input. The Markov parameter sequence is coordinate-independent and unique. Let the input and output matrices be partitioned as

$$\hat{B}_m = \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \vdots \\ \hat{b}_n \end{bmatrix}; \quad \hat{C}_m = [\hat{c}_1 \quad \hat{c}_2 \quad \cdots \quad \hat{c}_n] \tag{5.39}$$

where n is the number of modal coordinates, $\hat{b}_i$ $(i = 1, 2, \ldots, n)$ a row vector of length r, and $\hat{c}_i$ a column vector of length m. Each individual Markov parameter can then be written as a combination of n components contributed from different modal coordinates, for example,

$$\hat{Y}_2 = \hat{C}_m\hat{\Lambda}\hat{B}_m = \sum_{i=1}^{n} \hat{c}_i\hat{\lambda}_i\hat{b}_i \tag{5.40}$$

Therefore, each coordinate has a sequence of Markov parameters described as follows

$$[\hat{c}_i\hat{b}_i \quad \hat{c}_i\hat{\lambda}_i\hat{b}_i \quad \cdots \quad \hat{c}_i\hat{\lambda}_i^{\ell-2}\hat{b}_i]; \quad i = 1, 2, \ldots, n \tag{5.41}$$

where ℓ is the length of data. Define the sequence

$$\hat{q}_i = [\hat{b}_i \quad \hat{\lambda}_i\hat{b}_i \quad \hat{\lambda}_i^2\hat{b}_i \quad \cdots \quad \hat{\lambda}_i^{\ell-2}\hat{b}_i]; \quad i = 1, 2, \ldots, n \tag{5.42}$$

which represents the time history reconstructed from the identified eigenvalue, $\hat{\lambda}_i$, and the row vector $\hat{b}_i$. The total Markov parameter sequence becomes

$$\hat{Y} = \left[D \quad \sum_{i=1}^{n} \hat{c}_i\hat{q}_i \right] \tag{5.43}$$

The sequence $\hat{q}_i$ is called the identified modal amplitude time history for the ith mode, because it represents the temporal contribution of the ith mode associated with the output matrix $\hat{c}_i$ to the Markov parameter sequence $\hat{Y}$. Note that the output vector $\hat{c}_i$ in modal coordinates represents the ith mode shape information at the sensor points. The sequence $\hat{q}_i$ in Eq. (5.42) is computed from the identified eigenvalue λ_i and the input matrix $\hat{b}_i$.

Another way of computing the sequence $\hat{q}_i$ is to decompose the Markov parameter sequence $[Y_0 \quad Y_1 \quad Y_2 \quad \cdots \quad Y_{\ell-1}]$ which are directly obtained from pulse response samples. The technique begins by forming the Hankel matrix

$$H(0) = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_{\ell-\alpha} \\ Y_2 & Y_3 & \cdots & Y_{\ell-\alpha+1} \\ \vdots & \vdots & \ddots & \vdots \\ Y_\alpha & Y_{\alpha+1} & \cdots & Y_{\ell-1} \end{bmatrix} \tag{5.44}$$

where α is chosen such that $m\alpha$ is greater than or equal to the system order n. The Hankel matrix can be decomposed by using the singular value decomposition to become

$$\begin{aligned} H(0) &= [R_n\Sigma_n^{1/2}][\Sigma_n^{1/2}S_n^T] = [R_n\Sigma_n^{1/2}][\Psi\Psi^{-1}][\Sigma_n^{1/2}S_n^T] \\ &= [R_n\Sigma_n^{1/2}\Psi][\Psi^{-1}\Sigma_n^{1/2}S_n^T] \triangleq \bar{\mathcal{P}}\bar{\mathcal{Q}} \end{aligned} \tag{5.45}$$

where Ψ is an arbitrary nonsingular matrix to be determined by the coordinates chosen for the model. If the modal coordinates are used, Ψ is the eigenvector matrix of the real system state matrix, A. However, in practice, Ψ is chosen to be the eigenvector matrix of the estimated state matrix $\hat{A}$, because the real state matrix A is unknown. Substitution of the relationship

between the Markov parameters and the system matrices shown in Eq. (5.38) into Eq. (5.45) yields

$$\begin{aligned} H(0) &= \bar{\mathcal{P}}\bar{\mathcal{Q}} \\ &\cong \begin{bmatrix} C_m B_m & C_m \Lambda B_m & \cdots & C_m \Lambda^{\ell-\alpha-1} B_m \\ C_m \Lambda B_m & C_m \Lambda^2 B_m & \cdots & C_m \Lambda^{\ell-\alpha} B_m \\ \vdots & \vdots & \ddots & \vdots \\ C_m \Lambda^{\alpha-1} B_m & C_m \Lambda^{\alpha} B_m & \cdots & C_m \Lambda^{\ell-2} B_m \end{bmatrix} \\ &= \begin{bmatrix} C_m \\ C_m \Lambda \\ \vdots \\ C_m \Lambda^{\alpha-1} \end{bmatrix} \begin{bmatrix} B_m & \Lambda B_m & \cdots & \Lambda^{\ell-\alpha-1} B_m \end{bmatrix} \end{aligned} \tag{5.46}$$

Note that Λ, B_m, and C_m represent real system matrices in modal coordinates. Therefore, this equation is only an "approximation" when some nonzero singular values due to noise have been truncated. Let $\bar{\mathcal{Q}}$ be partitioned such that

$$\bar{\mathcal{Q}} = \begin{bmatrix} \bar{q}_1 \\ \bar{q}_2 \\ \vdots \\ \bar{q}_n \end{bmatrix} = \begin{bmatrix} B_m & \Lambda B_m & \cdots & \Lambda^{\ell-\alpha-1} B_m \end{bmatrix} \tag{5.47}$$

Therefore, one obtains

$$\bar{\mathcal{Q}} = \begin{bmatrix} [b_1 & \lambda_1 b_1 & \cdots & \lambda_1^{\ell-\alpha-1} b_1] \\ [b_2 & \lambda_2 b_2 & \cdots & \lambda_2^{\ell-\alpha-1} b_2] \\ & & \vdots & \\ [b_n & \lambda_n b_n & \cdots & \lambda_n^{\ell-\alpha-1} b_n] \end{bmatrix} \tag{5.48}$$

Comparison of Eqs. (5.42) and (5.48) reveals that $\bar{q}_i$ and $\hat{q}_i$ $(i = 1, 2, 3, \ldots, n)$ are identical for the noise-free case. With noise present and some small nonzero singular values truncated, $\hat{q}_i$ is an approximation of $\bar{q}_i$.

Modal Amplitude Coherence (MAC)

The MAC developed can be thought of as a dot product between (1), the vector composed of a chosen number of time steps of the unit pulse response history associated with a mode of the identified model, and (2), the corresponding vector from the pulse response data used in the identification

$$MAC_i = |\bar{q}_i \hat{q}_i^*| / (|\bar{q}_i \bar{q}_i^*| |\hat{q}_i \hat{q}_i^*|)^{1/2} \tag{5.49}$$

where $i = 1, 2, \ldots, n$ and "*" means transpose and complex conjugate. If the two vectors coincide, then the model reproduces the pulse response data. Otherwise, the dot product can be thought of as a generalized cosine of the angle between the vectors of the measured response history and the identified model's response history. In practice, the computation time required to evaluate the MAC for each mode is quite small.

Mode Singular Value (MSV)

The MSV is a method of characterizing the contribution of each identified mode to the identified model pulse response history. The identification algorithm, such as ERA, attempts to identify a model to match the pulse response history. It is thus reasonable that a mode that has a large contribution to the identified model's pulse response history has a large contribution to the system's pulse response data and is then well identified by the algorithm. A precise statement of what is computed is as follows:

From Eq. (5.41), it is obvious that each modal coordinate contributes to the pulse response by the individual modal sequence, which can be quantified by taking its maximum singular value, i.e.,

$$\begin{aligned} MSV_i &= \sqrt{|\hat{c}_i|(1 + |\hat{\lambda}_i| + |\hat{\lambda}_i^2| + \ldots + |\hat{\lambda}_i^{\ell-2}|)|\hat{b}_i|} \\ &\approx \sqrt{\frac{|\hat{c}_i||\hat{b}_i|}{(1 - |\hat{\lambda}_i|)}} \end{aligned} \tag{5.50}$$

where the approximation sign is valid only when $|\hat{\lambda}_i|$ is less than 1 and the number of Markov parameters ℓ is sufficiently large.

5.3.2 Computational Steps of ERA

The computational steps are shown in FIG 5.1 and summarized as follows:

1) Construct a block Hankel matrix $H(0)$ by arranging the Markov parameters (pulse response samples) into blocks with given α, β, s_i ($i = 1, 2, \ldots, \alpha$) and t_j ($j = 1, 2, \ldots, \beta$), [Eq. (5.26)].

2) Decompose $H(0)$ using singular value decomposition [Eq. (5.30)].

3) Determine the order of the system by examining the singular values of the Hankel matrix $H(0)$ [Eq. (5.30)].

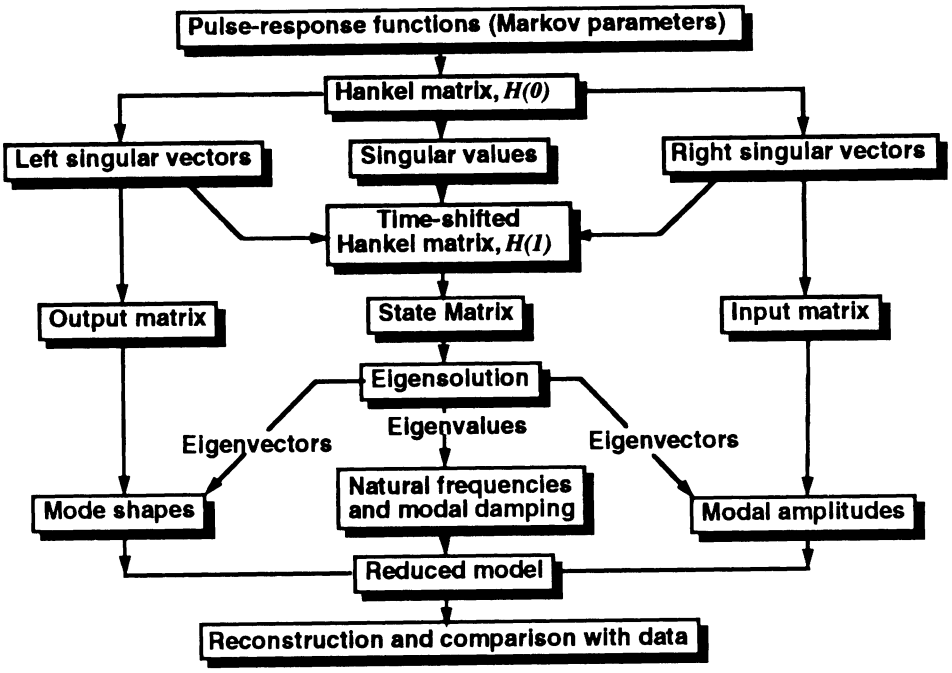

Figure 5.1: Flowchart for the ERA.

4) Construct a minimum order realization $[\hat{A},\ \hat{B},\ \hat{C}]$ using a shifted block Hankel matrix $H(1)$ [Eq. (5.34)].

5) Find the eigensolution of the realized state matrix and transform the realized model to modal coordinates [Eq. (5.37)] to calculate the system damping and frequencies.

6) Calculate the modal amplitude coherence [Eq. (5.49)] and mode singular values [Eq. (5.50)] to quantify the system and noise modes.

7) Determine the reduced system model based on the accuracy indicators computed in step 6, reconstruct Markov parameters Y_k [Eq. (5.23)], and compare with the measured Markov parameters.

Note that the optimum determination of α, β, s_i $(i = 1, 2, \ldots, \alpha)$ and t_i $(i = 1, 2, \ldots, \beta)$ in step 1 requires some engineering intuition. This determination is related to the choice of the measurement data to minimize the size of the Hankel matrix $H(0)$ with the rank unchanged, as will be discussed in a later chapter.

Example 5.3

Consider a mass-spring-dashpot system as shown in FIG 5.2. Let the system be excited by the control force $u(t)$ with random signal and measured by an accelerometer at the sampling rate 0.2 seconds. Both control force actuator and accelerometer are collocated at the end mass. The physical parameters for the system and the corresponding second-order differential equation are shown in the figure. Conventionally the force and acceleration signals are transformed to the sampled s-transform or z-transform domain by the discrete Fast Fourier Transform (FFT) procedures. If the control force is a zero-order hold signal, implying that the force remains constant in the sampling periods, then the FFT transforms the signal to the z-transform domain. However, if the input signal has been smoothed or interpolated in the sampling periods, then the FFT transforms the force signal to the sampled s-transform domain. From the FFT-transformed force and acceleration signals, the sampled impulse response or pulse response (Markov parameters) for the system can be obtained by the procedure shown in a later chapter. This conventional procedure is in the frequency domain. In the figure, the OKID means the observer/Kalman filter identification which identifies the Markov parameters in the time domain. The OKID method will be discussed in a later chapter.

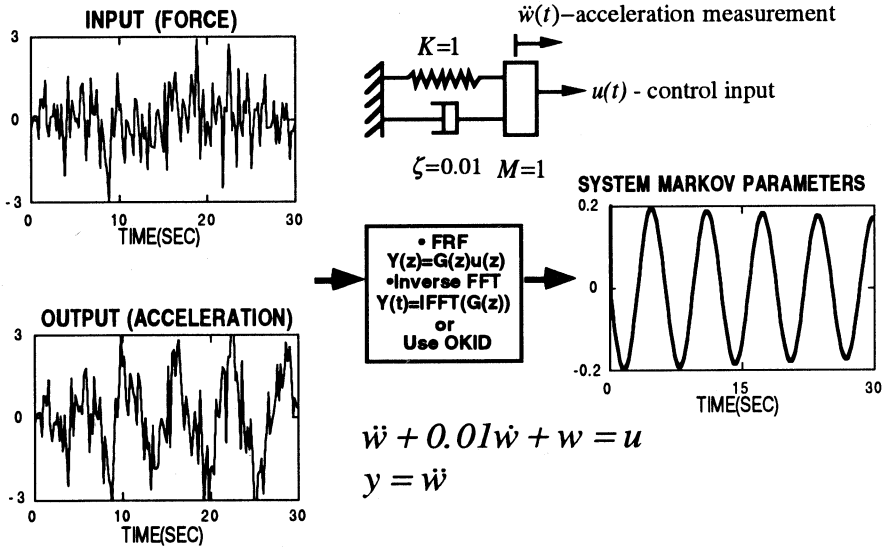

Figure 5.2: Typical test procedure to obtain Markov parameters.

For simplicity and illustration, assume that the system is noise-free and only "seven" Markov parameters are computed, i.e., $Y_0 = 1.0$, $Y_1 = -0.021905$, $Y_2 = -0.060816$, $Y_3 = -0.097227$, $Y_4 = -0.12969$, $Y_5 = -0.15693$, $Y_6 = -0.17786$. From the Markov parameters, first form the Hankel matrix $H(0)$ and the shifted Hankel matrix $H(1)$

$$H(0) = \begin{bmatrix} -0.021905 & -0.060816 & -0.097227 \\ -0.060816 & -0.097227 & -0.12969 \\ -0.097227 & -0.12969 & -0.15693 \end{bmatrix}$$

$$H(1) = \begin{bmatrix} -0.060816 & -0.097227 & -0.12969 \\ -0.097227 & -0.12969 & -0.15693 \\ -0.01.297 & -0.15693 & -0.17786 \end{bmatrix}$$

Take singular value decomposition of $H(0)$ to yield

$$H(0) = R\Sigma S^T = \begin{bmatrix} R_2 & \vdots & R_0 \end{bmatrix} \begin{bmatrix} \Sigma_2 & 0 \\ 0 & \Sigma_0 \end{bmatrix} \begin{bmatrix} S_2 & \vdots & S_0 \end{bmatrix}^T$$

$$= \begin{bmatrix} -0.3729 & -0.8308 & -0.4133 \\ -0.5657 & -0.1496 & 0.8.110 \\ -0.7355 & 0.5362 & -0.4141 \end{bmatrix} \begin{bmatrix} 0.3060 & 0 & 0 \\ 0 & 0.0299 & 0 \\ 0 & 0 & 0.31e-17 \end{bmatrix}$$

$$\begin{bmatrix} 0.3729 & -0.8308 & 0.4133 \\ 0.5657 & -0.1496 & -0.8110 \\ 0.7355 & 0.5362 & 0.4141 \end{bmatrix}^T$$

The singular value $0.31e-17$ is obviously too small to have any contribution to the system. Thus the system has only two significant singular values implying that the system order is two. The matrices R_2 and S_2 contain, respectively, the first two columns of the matrices R and S, corresponding to the first two singular values. The ERA identified model becomes

$$\hat{A} = \Sigma_2^{-1/2} R_2^T H(1) S_2 \Sigma_2^{-1/2} = \begin{bmatrix} 1.2710 & -0.3531 \\ 0.3531 & 0.6870 \end{bmatrix}$$

$$\hat{B} = \Sigma_2^{1/2} S_2^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.2062 \\ -0.1436 \end{bmatrix}$$

$$\hat{C} = [1 \quad 0 \quad 0] R_2 \Sigma_2^{1/2} = [-0.2062 \quad -0.1436]$$

$$\hat{D} = Y_0 = 1$$

Now compute the eigenvalues and eigenvectors of the estimated discrete-time state matrix such that $\hat{A} = \hat{\Psi}\hat{\Lambda}\hat{\Psi}^{-1}$ and transform the identified matrices into modal coordinates to yield

$$\hat{\Lambda} = \begin{bmatrix} \hat{\lambda}_1 & 0 \\ 0 & \hat{\lambda}_2 \end{bmatrix} = \begin{bmatrix} 0.9791 - 0.1985i & 0.0 \\ 0.0 & 0.9791 + 0.1985i \end{bmatrix}$$

$$\begin{aligned}\hat{B}_m &= \hat{\Psi}^{-1}\hat{B} = \begin{bmatrix}\hat{b}_1\\ \hat{b}_2\end{bmatrix} = \begin{bmatrix}-0.4089 - 0.1016i\\ -0.4089 + 0.1016i\end{bmatrix}\\ \hat{C}_m &= \hat{C}\hat{\Psi} = [\hat{c}_1 \quad \hat{c}_2] = [0.0820 + 0.2219i \quad 0.0820 - 0.2219i]\\ \hat{D} &= Y_0 = 1\end{aligned}$$

From this result, one can compute the vector $\hat{q}_1$ shown in Eq. (5.42),

$$\begin{aligned}\hat{q}_1 &= [\hat{b}_1 \quad \hat{\lambda}_1\hat{b}_1 \quad \hat{\lambda}_1^2\hat{b}_1]\\ &= [-0.4089 + 0.1016i \quad -0.3802 + 0.1806i \quad -0.3364 + 0.2523i]\end{aligned}$$

which is composed of three time steps of the unit pulse response history associated with the first eigenvalue of the identified model. The three time steps rather than the length of data are chosen just for illustration. The vector $\hat{q}_2$ for the second eigenvalue is the complex conjugate of $\hat{q}_1$ because the second eigenvalue is the complex conjugate of the first eigenvalue. The corresponding vectors $\bar{q}_1$ and $\bar{q}_2$ shown in Eq. (5.47) are the first and second rows of the matrix $[\hat{\Psi}^{-1}\Sigma_2^{1/2}S_2^T]$, respectively. These vectors are directly obtained from the pulse response (Markov parameters) used in the identification. The vectors $\bar{q}_1$ and $\bar{q}_2$ are identical to the vectors $\hat{q}_1$ and $\hat{q}_2$ because the system in this example is assumed to be noise-free. Therefore, the Modal Amplitude Coherence (MAC) for the first and second eigenvalues is one.

The Mode Singular Value (MSV) defined in Eq. (5.50) with $\ell = 7$ (seven Markov parameters) for the first eigenvalue can be readily computed as

$$MSV_1 = \sqrt{|c_1|(1 + |\lambda_1| + |\lambda_1^2| + \cdots + |\lambda_1^5|)|b_1|} = 0.7724$$

If more Markov parameters are used in the identification, say $\ell = 2048$, one obtains $MSV_1 = 9.3812$ which approaches $\sqrt{|c_1||b_1|/(1 - |\lambda_1|)} = 9.9917$. The MSV_2 for the second eigenvalue should be identical to the MSV_1 for the first eigenvalue because again the second eigenvalue is the complex conjugate of the first eigenvalue.

The identified discrete-time model can be transformed to obtain a continuous-time model,

$$A_c = \begin{bmatrix}1.466 & -1.779\\ 1.779 & -1.476\end{bmatrix} \quad \text{and} \quad B_c = \begin{bmatrix}0.7487\\ -1.005\end{bmatrix}$$

Note that the C and D matrices for this continuous-time model are the same as those for the discrete-time model. The identified state matrix A_c has two eigenvalues, $-0.005 \pm 1.0i$, which are also the eigenvalues of the second order differential equation as shown in FIG 5.2.

5.4 The ERA with Data Correlations (ERA/DC)

Whereas the standard ERA method proceeds using the block data matrix $H(0)$ shown in Eq. (5.24), the ERA method with Data Correlations (ERA/DC) requires the definition of a square matrix of order $\gamma = m\alpha$ where m is the number of outputs,

$$
\begin{aligned}
\mathcal{R}_{hh}(k) &= H(k)H^T(0) \\
&= \begin{bmatrix} Y_{k+1} & Y_{k+2} & \cdots & Y_{k+\beta} \\ Y_{k+2} & Y_{k+3} & \cdots & Y_{k+\beta+1} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{k+\alpha} & Y_{k+\alpha+1} & \cdots & Y_{k+\alpha+\beta-1} \end{bmatrix} \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_\beta \\ Y_2 & Y_3 & \cdots & Y_{\beta+1} \\ \vdots & \vdots & \ddots & \vdots \\ Y_\alpha & Y_{\alpha+1} & \cdots & Y_{\alpha+\beta-1} \end{bmatrix}^T \\
&= \begin{bmatrix} \sum_{i=1}^{\beta} Y_{k+i}Y_i^T & \sum_{i=1}^{\beta} Y_{k+i}Y_{i+1}^T & \cdots & \sum_{i=1}^{\beta} Y_{k+i}Y_{\alpha+i-1}^T \\ \sum_{i=1}^{\beta} Y_{k+i+1}Y_i^T & \sum_{i=1}^{\beta} Y_{k+i+1}Y_{i+1}^T & \cdots & \sum_{i=1}^{\beta} Y_{k+i+1}Y_{\alpha+i-1}^T \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{\beta} Y_{k+\alpha+i-1}Y_i^T & \sum_{i=1}^{\beta} Y_{k+\alpha+i-1}Y_{i+1}^T & \cdots & \sum_{i=1}^{\beta} Y_{k+\alpha+i-1}Y_{\alpha+i-1}^T \end{bmatrix}
\end{aligned} \tag{5.51}
$$

Here $Y(k)$ is an $m \times r$ matrix whose columns are the Markov parameters (pulse response samples) corresponding to the m outputs. The size of $H(k)$ and $H(0)$ is $\alpha m \times \beta r$, whereas the size of $\mathcal{R}_{hh}(k)$ is $\alpha m \times \alpha m$. The data correlation matrix $\mathcal{R}_{hh}(k)$ is smaller in size than the Hankel matrix $H(k)$ in particular when the number of rows of the Hankel matrix is sufficiently large. For the case where $k = 0$, the correlation matrix $\mathcal{R}_{hh}(0)$ becomes

$$
\begin{aligned}
\mathcal{R}_{hh}(0) &= H(0)H^T(0) \\
&= \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_\beta \\ Y_2 & Y_3 & \cdots & Y_{\beta+1} \\ \vdots & \vdots & \ddots & \vdots \\ Y_\alpha & Y_{\alpha+1} & \cdots & Y_{\alpha+\beta-1} \end{bmatrix} \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_\beta \\ Y_2 & Y_3 & \cdots & Y_{\beta+1} \\ \vdots & \vdots & \ddots & \vdots \\ Y_\alpha & Y_{\alpha+1} & \cdots & Y_{\alpha+\beta-1} \end{bmatrix}^T \\
&= \begin{bmatrix} \sum_{i=1}^{\beta} Y_iY_i^T & \sum_{i=1}^{\beta} Y_iY_{i+1}^T & \cdots & \sum_{i=1}^{\beta} Y_iY_{\alpha+i-1}^T \\ \sum_{i=1}^{\beta} Y_{i+1}Y_i^T & \sum_{i=1}^{\beta} Y_{i+1}Y_{i+1}^T & \cdots & \sum_{i=1}^{\beta} Y_{i+1}Y_{\alpha+i-1}^T \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{\beta} Y_{\alpha+i-1}Y_i^T & \sum_{i=1}^{\beta} Y_{\alpha+i-1}Y_{i+1}^T & \cdots & \sum_{i=1}^{\beta} Y_{\alpha+i-1}Y_{\alpha+i-1}^T \end{bmatrix}
\end{aligned} \tag{5.52}
$$

The matrix $\mathcal{R}_{hh}(0)$ consists of auto-correlations of Markov parameters such as $\sum_{i=1}^{\beta} Y_i Y_i^T$ and cross-correlations between outputs such as $\sum_{i=1}^{\beta} Y_i Y_{i+1}^T$ at lag time values in the range $\pm\alpha$, summed over the r inputs. If noises in the Markov parameters are not correlated, the correlation matrix $\mathcal{R}_{hh}(0)$ will contain less noise than the Hankel matrix $H(0)$.

Application of Eq. (5.25) to Eq. (5.51) yields

$$\mathcal{R}_{hh}(k) = H(k)H^T(0) = \mathcal{P}_\alpha A^k \mathcal{Q}_\beta \mathcal{Q}_\beta^T \mathcal{P}_\alpha^T = \mathcal{P}_\alpha A^k \mathcal{Q}_c \tag{5.53}$$

where $\mathcal{Q}_c = \mathcal{Q}_\beta \mathcal{Q}_\beta^T \mathcal{P}_\alpha^T$. Now let us compare the correlation matrix $\mathcal{R}_{hh}(k) = \mathcal{P}_\alpha A^k \mathcal{Q}_c$ with the Markov parameter $Y(k) = CA^{k-1}B$. Both the correlation matrix $\mathcal{R}_{hh}(k)$ and the Markov parameters $Y(k)$ are the product of three matrices with the discrete-time state matrix in between. The ERA shown in the last section was developed to solve for A, B, and C based on the Hankel matrices $H(0)$ and $H(1)$. Similarly, a block correlation Hankel matrix can be formed and used to solve for A, $\mathcal{Q}_c$, and $\mathcal{P}_\alpha$.

Indeed, let a $\xi \times \zeta$ block correlation Hankel matrix be formed as

$$\begin{aligned}
\mathcal{H}(k) &= \begin{bmatrix} \mathcal{R}_{hh}(k) & \mathcal{R}_{hh}(k+\tau) & \cdots & \mathcal{R}_{hh}(k+\zeta\tau) \\ \mathcal{R}_{hh}(k+\tau) & \mathcal{R}_{hh}(k+2\tau) & \cdots & \mathcal{R}_{hh}(k+(\zeta+1)\tau) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{R}_{hh}(k+\xi\tau) & \mathcal{R}_{hh}(k+(\xi+1)\tau) & \cdots & \mathcal{R}_{hh}(k+(\xi+\zeta)\tau) \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{P}_\alpha \\ \mathcal{P}_\alpha A^\tau \\ \vdots \\ \mathcal{P}_\alpha A^{\xi\tau} \end{bmatrix} A^k \begin{bmatrix} \mathcal{Q}_c & A^\tau \mathcal{Q}_c & \cdots & A^{\zeta\tau}\mathcal{Q}_c \end{bmatrix} \\
&= \mathcal{P}_\xi A^k \mathcal{Q}_\zeta
\end{aligned} \tag{5.54}$$

or, for $k = 0$,

$$\begin{aligned}
\mathcal{H}(0) &= \begin{bmatrix} \mathcal{R}_{hh}(0) & \mathcal{R}_{hh}(\tau) & \cdots & \mathcal{R}_{hh}(\zeta\tau) \\ \mathcal{R}_{hh}(\tau) & \mathcal{R}_{hh}(2\tau) & \cdots & \mathcal{R}_{hh}((\zeta+1)\tau) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{R}_{hh}(\xi\tau) & \mathcal{R}_{hh}((\xi+1)\tau) & \cdots & \mathcal{R}_{hh}((\xi+\zeta)\tau) \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{P}_\alpha \\ \mathcal{P}_\alpha A^\tau \\ \vdots \\ \mathcal{P}_\alpha A^{\xi\tau} \end{bmatrix} \begin{bmatrix} \mathcal{Q}_c & A^\tau \mathcal{Q}_c & \cdots & A^{\zeta\tau}\mathcal{Q}_c \end{bmatrix} = \mathcal{P}_\xi \mathcal{Q}_\zeta
\end{aligned} \tag{5.55}$$

where k is an integer chosen to avoid correlation terms which give rise to bias when noise is present, and τ is an integer chosen to prevent significant overlap of adjacent R blocks. The integers ξ and ζ define how many correlation lags are included in the analysis. The matrices $\mathcal{P}_\xi$ and $\mathcal{Q}_\zeta$ can be called block correlation observability and controllability matrices of dimension $m\alpha(\xi+1) \times n$ and $n \times m\alpha(\zeta+1)$, respectively. To this end, we have shown the relationship between the system model matrices A, B, C and the correlation matrix $\mathcal{H}(0)$ formed from Markov parameters (i.e., pulse response). The basic formulation of the realization for the ERA/DC is shown in the following.

Similar to the ERA, the ERA/DC process continues with the factorization of the block correlation matrix $\mathcal{H}(0)$ (as opposed to $H(0)$ in the ERA) using singular value decomposition so that

$$\mathcal{H}(0) = R\Sigma S^T \tag{5.56}$$

where the columns of matrices R and S are orthonormal, i.e., $R^TR = I$, (I=identity), $S^TS = I$, and Σ is a rectangular matrix

$$\Sigma = \begin{bmatrix} \Sigma_n & 0 \\ 0 & 0 \end{bmatrix}$$

with

$$\Sigma_n = \text{diag}[\sigma_1,\ \sigma_2, \cdots, \sigma_i,\ \sigma_{i+1}, \cdots, \sigma_n]$$

and monotonically non-increasing σ_i $(i = 1,\ 2,\ \ldots,\ n)$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_i \geq \sigma_{i+1} \geq \cdots \sigma_n \geq 0$$

Next, let R_n and S_n be the matrices formed by the first n columns of R and S, respectively. The columns of $R_n(m\alpha(\xi+1)\times n)$ and $S_n(m\alpha(\zeta+1)\times n)$ are orthonormal and Σ is a diagonal matrix containing the n singular values that are considered significant, based on some truncation procedure discussed in the ERA process. Note that the above factorization is approximate if noise is present because the discarded singular values are nonzero.

Hence, the matrix $\mathcal{H}(0)$ and its pseudo-inverse become

$$\mathcal{H}(0) = R_n\Sigma_n S_n^T \quad \text{where} \quad R_n^TR_n = I_n = S_n^TS_n \tag{5.57}$$

and

$$\mathcal{H}^\dagger = S_n\Sigma_n^{-1}R_n^T. \tag{5.58}$$

Equation (5.58) can be readily proved by observing that

$$\mathcal{H}(0)\mathcal{H}^{\dagger}\mathcal{H}(0) = \mathcal{H}(0) \tag{5.59}$$

similar to that shown in Eq. (5.29). Using Eq. (5.54) and (5.59) thus yields

$$\mathcal{P}_{\xi}\mathcal{Q}_{\zeta}\mathcal{H}^{\dagger}\mathcal{P}_{\xi}\mathcal{Q}_{\zeta} = \mathcal{P}_{\xi}\mathcal{Q}_{\zeta} \quad \text{or} \quad \mathcal{P}_{\xi}\Big[\mathcal{Q}_{\zeta}\mathcal{H}^{\dagger}\mathcal{P}_{\xi}\Big]\mathcal{Q}_{\zeta} = \mathcal{P}_{\xi}\mathcal{Q}_{\zeta}$$

which implies that

$$\mathcal{Q}_{\zeta}\mathcal{H}^{\dagger}\mathcal{P}_{\xi} = I_n \tag{5.60}$$

with the condition that both $\mathcal{Q}_{\zeta}$ and $\mathcal{P}_{\xi}$ are of rank n. Define O_{γ} as a null matrix of order γ, I_{γ} an identity matrix of order γ, and $E_{\gamma}^{T} = [I_{\gamma}\ O_{\gamma}\cdots O_{\gamma}]$. Following an approach similar to that presented before for the ERA, a minimum (or reduced) order realization of dimension n can be obtained, with the aid of Eqs. (5.53)-(5.60), from

$$\begin{aligned}
\mathcal{R}_{hh}(k) &= E_{\gamma}^{T}\mathcal{H}(k)E_{\gamma} \quad \text{(from Eq. (5.54))} \\
&= E_{\gamma}^{T}\mathcal{P}_{\xi}\Big[\mathcal{Q}_{\zeta}\mathcal{H}^{\dagger}\mathcal{P}_{\xi}\Big]A^{k}\Big[\mathcal{Q}_{\zeta}\mathcal{H}^{\dagger}\mathcal{P}_{\xi}\Big]\mathcal{Q}_{\zeta}E_{\gamma} \\
&\quad \text{(use Eq.(5.54) and (5.60))} \\
&= E_{\gamma}^{T}[\mathcal{P}_{\xi}\mathcal{Q}_{\zeta}]\mathcal{H}^{\dagger}\Big[\mathcal{P}_{\xi}A^{k}\mathcal{Q}_{\zeta}\Big]\mathcal{H}^{\dagger}[\mathcal{P}_{\xi}\mathcal{Q}_{\zeta}]E_{\gamma} \quad \text{(regroup)} \\
&= E_{\gamma}^{T}\mathcal{H}(0)\mathcal{H}^{\dagger}\Big[\mathcal{P}_{\xi}A^{k}\mathcal{Q}_{\zeta}\Big]\mathcal{H}^{\dagger}\mathcal{H}(0)E_{\gamma} \quad \text{(use Eq. (5.54))} \\
&= E_{\gamma}^{T}\Big[R_n\Sigma_n S_n^{T}\Big]S_n\Sigma_n^{-1}R_n^{T}\Big[\mathcal{P}_{\xi}A^{k}\mathcal{Q}_{\zeta}\Big]S_n\Sigma_n^{-1}R_n^{T}\Big[R_n\Sigma_n S_n^{T}\Big]E_{\gamma} \\
&\quad \text{(use Eqs. (5.57) and (5.58))} \\
&= E_{\gamma}^{T}R_n\Sigma_n^{1/2}\Big[\Sigma_n^{-1/2}R_n^{T}\Big[\mathcal{P}_{\xi}A^{k}\mathcal{Q}_{\zeta}\Big]S_n\Sigma_n^{-1/2}\Big]\Sigma_n^{1/2}S_n^{T}E_{\gamma} \\
&\quad \text{(use } R_n^{T}R_n = I_n = S_n^{T}S_n\text{)} \\
&= E_{\gamma}^{T}R_n\Sigma_n^{1/2}\Big[\Sigma_n^{-1/2}R_n^{T}\mathcal{H}(1)S_n\Sigma_n^{-1/2}\Big]^{k}\Sigma_n^{1/2}S_n^{T}E_{\gamma}
\end{aligned} \tag{5.61}$$

This is the basic formulation of the realization for the ERA/DC. In comparison with Eq. (5.53), it follows that the triplet

$$[\,\Sigma_n^{-1/2}R_n^{T}\mathcal{H}(1)S_n\Sigma_n^{-1/2} \quad \Sigma_n^{1/2}S_n^{T}E_{\gamma} \quad E_{\gamma}^{T}R_n\Sigma_n^{1/2}\,] \tag{5.62}$$

is a minimum realization of $[A,\ \mathcal{Q}_c,\ \mathcal{P}_\alpha]$. Now, using Eq. (5.25) for $H(0)$,

$$H(0) = \mathcal{P}_\alpha \mathcal{Q}_\beta = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{\alpha-1} \end{bmatrix} [B \quad AB \quad A^2B \quad \cdots \quad A^{\beta-1}B] \tag{5.63}$$

an expression for $\mathcal{Q}_\beta$ can be found

$$\mathcal{Q}_\beta = \mathcal{P}_\alpha^\dagger H(0) = [E_\gamma^T R_n \Sigma_n^{1/2}]^\dagger H(0) \tag{5.64}$$

The output matrix C and the input matrix B from Eq. (5.63) can thus be identified from the first m rows of $\mathcal{P}_\alpha$ and the first r columns of $\mathcal{Q}_\beta$, respectively. Hence, a realization for $\left[\hat{A},\ \hat{B},\ \hat{C}\right]$ can be shown to be

$$\begin{aligned} \hat{A} &= \Sigma_n^{-1/2} R_n^T \mathcal{H}(1) S_n \Sigma_n^{-1/2} \\ \hat{B} &= [E_\gamma^T R_n \Sigma_n^{1/2}]^\dagger H(0) E_r \\ \hat{C} &= E_m^T R_n \Sigma_n^{1/2} \end{aligned} \tag{5.65}$$

The system frequencies and dampings may then be computed from the eigenvalues of the realized state transition matrix as for the ERA. The eigenvectors allow a transformation of the realization to modal space and hence the determination of the complex (or damped) mode shapes and the initial modal amplitudes (or modal participation factors). The modal amplitude coherence and mode singular values developed for the ERA can also be calculated for the ERA/DC to indicate any noise modes present due to imperfect singular value truncation.

It is worth pointing out that while the ERA is, in essence, a least-squares fit to the pulse response measurements, the ERA/DC involves a fit to the output auto-correlations and cross-correlations over a defined number of lag values. The bias terms affecting the ERA when "white" measurement noise is present can, in principle, be omitted in the ERA/DC by properly choosing the integer k shown in Eq. (5.54). The integer k may be determined by sensor characteristics such as covariance matrix. In order to avoid overlap of adjacent correlation terms in the block correlation matrix, it is required that $\tau \geq \alpha$ (see Eq. (5.54)). The structure of the $\mathcal{R}_{hh}(k)$ matrix, and hence

the block correlation Hankel matrix $\mathcal{H}(k)$, is significantly affected by the choice of the parameter ξ. When $\xi = 0$, the structure is simplest, but does not necessarily yield the best answer. However, it is the easiest way of implementing the ERA/DC method.

5.4.1 Computational Steps of ERA/DC

Similar to the computational steps for the ERA shown in FIG 5.1, the computational steps for the ERA/DC are shown in FIG 5.3 and summarized as follows:

1) Construct a block correlation matrix $\mathcal{H}(0)$ [Eq. (5.54)] with the elements $\mathcal{R}_{hh}(k+\tau)$, which is the product of a Hankel matrix $H(0)$ and a shifted Hankel matrix $H(k+\tau)$ [Eq. (5.53)].

2) Decompose $\mathcal{H}(0)$ using singular value decomposition [Eq. (5.56)].

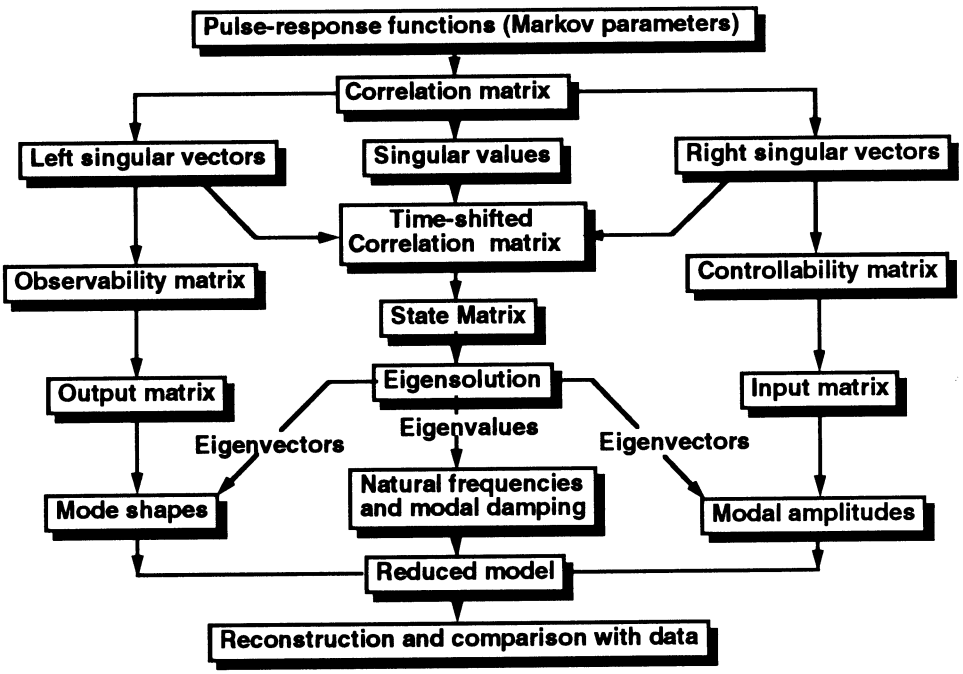

Figure 5.3: Flowchart for the ERA/DC.

3) Determine the order of the system by examining the singular values of the Hankel matrix $\mathcal{H}(0)$ [Eq. (5.56)].

4) Construct a minimum order realization $[A,\ \mathcal{Q}_c,\ \mathcal{P}_\alpha]$ using the shifted block Hankel matrix $\mathcal{H}(1)$ [Eq. (5.62)].

5) Compute the controllability matrix $\mathcal{Q}_\beta$ [Eq. (5.64)] and determine a minimum order realization $[\hat{A},\ \hat{B},\ \hat{C}]$ [Eq. (5.65)].

6) Find the eigensolution of the realized state matrix and transform the realized model into modal coordinates [Eq. (5.37)] to compute system damping and frequencies.

7) Calculate the modal amplitude coherence [Eq. (5.49)] and mode singular values [Eq. (5.50)] to quantify the system and noise modes.

8) Determine the reduced system model based on the accuracy indicators computed in step 7, reconstruct the Markov parameters Y_k [Eq. (5.23)], and compare with the measured Markov parameters.

The computational steps for the ERA/DC are almost identical to those for the ERA except that one additional step (i.e., step 5) is required for the ERA/DC. However, the ERA/DC in general uses a smaller matrix in step 2 for the singular value decomposition and thus requires less computation time than the ERA.

Example 5.4

Consider the mass-spring-dashpot system shown in FIG 5.2. Again, assume that the system is noise-free. Use the Hankel matrix $H(0)$ and the shifted Hankel matrix $H(1)$ shown in Example 5.3 to form the correlation matrices $\mathcal{R}_{hh}(0)$ and $\mathcal{R}_{hh}(1)$ as shown in Eqs. (5.51) and (5.52)

$$\mathcal{R}_{hh}(0) = H(0)H^T(0) = \begin{bmatrix} 0.01363 & 0.01986 & 0.02528 \\ 0.01986 & 0.02997 & 0.03888 \\ 0.02528 & 0.03888 & 0.05090 \end{bmatrix}$$

$$\mathcal{R}_{hh}(1) = H(1)H^T(0) = \begin{bmatrix} 0.01986 & 0.02997 & 0.03888 \\ 0.02528 & 0.03888 & 0.05090 \\ 0.02968 & 0.04621 & 0.06087 \end{bmatrix}$$

Let the correlation matrix $\mathcal{H}(k)$ shown in Eq. (5.54) contain only one element in it such that

$$\mathcal{H}(k) = \mathcal{R}_{hh}(k) \quad \text{i.e.,} \quad \mathcal{H}(0) = \mathcal{R}_{hh}(0) \quad \text{and} \quad \mathcal{H}(1) = \mathcal{R}_{hh}(1)$$

Take the singular value decomposition of $\mathcal{H}(0)$ to yield

$$\mathcal{H}(0) = R\Sigma S^T = \begin{bmatrix} R_2 & \vdots & R_0 \end{bmatrix} \begin{bmatrix} \Sigma_2 & 0 \\ 0 & \Sigma_0 \end{bmatrix} \begin{bmatrix} S_2 & \vdots & S_0 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0.3729 & -0.8308 & -0.4133 \\ 0.5657 & -0.1496 & 0.8.110 \\ 0.7355 & 0.5362 & -0.4141 \end{bmatrix} \begin{bmatrix} 0.09361 & 0 & 0 \\ 0 & 0.08938e{-}2 & 0 \\ 0 & 0 & 0.01408e{-}16 \end{bmatrix}$$

$$\begin{bmatrix} 0.3729 & -0.8308 & -0.4133 \\ 0.5657 & -0.1496 & 0.8.110 \\ 0.7355 & 0.5362 & -0.4141 \end{bmatrix}^T$$

Note that matrices R and S are identical in this case because the correlation matrix $\mathcal{H}(0)$ is symmetric. The singular value $0.01408e - 16$ is obviously too small to have any contribution to the system. Thus the system has only two significant singular values implying that the system order is two. The matrix R_2 contains the first two columns of the matrix R, corresponding to the first two singular values. A minimum order realization $[A,\ \mathcal{Q}_c,\ \mathcal{P}_\alpha]$ using the shift block Hankel matrix $\mathcal{H}(1)$ shown above can then be obtained as (see Eq. (5.62))

$$\begin{aligned} \hat{A} &= \Sigma_2^{-1/2} R_2^T \mathcal{H}(1) R_2 \Sigma_2^{-1/2} = \begin{bmatrix} 1.2710 & 0.1104 \\ -1.130 & 0.6870 \end{bmatrix} \\ \mathcal{Q}_c &= \Sigma_2^{1/2} R_2^T \\ \mathcal{P}_\alpha &= R_2 \Sigma_2^{1/2} \end{aligned}$$

The pseudo-inverse of $\mathcal{P}_\alpha$ is simply

$$\begin{aligned} \mathcal{P}_\alpha^\dagger &= [\mathcal{P}_\alpha^T \mathcal{P}_\alpha]^{-1} \mathcal{P}_\alpha^T = [\Sigma_2^{1/2} R_2^T R_2 \Sigma_2^{1/2}]^{-1} \Sigma_2^{1/2} R_2^T \\ &= \Sigma_2^{-1} \Sigma_2^{1/2} R_2^T = \Sigma_2^{-1/2} R_2^T \end{aligned}$$

where $R_2^T R_2 = I_2$ (an identity matrix of order 2) has been used in the derivation. The ERA/DC-identified model thus becomes

$$\begin{aligned} \hat{A} &= \Sigma_2^{-1/2} R_2^T \mathcal{H}(1) S_2 \Sigma_2^{-1/2} = \begin{bmatrix} 1.2710 & 0.1104 \\ -1.1296 & 0.6870 \end{bmatrix} \\ \hat{B} &= \mathcal{P}_\alpha^\dagger H(0) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \Sigma_2^{-1/2} R_2^T H(0) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.3729 \\ -0.8308 \end{bmatrix} \\ \hat{C} &= [1 \quad 0 \quad 0] \mathcal{P}_\alpha = [1 \quad 0 \quad 0] R_2 \Sigma_2^{1/2} = [0.1141 \quad -0.0248] \\ \hat{D} &= Y_0 = 1 \end{aligned}$$

Now compute the eigenvalues and eigenvectors of the identified state matrix such that $\hat{A} = \hat{\Psi}\hat{\Lambda}\hat{\Psi}^{-1}$ and transform the identified matrices into modal coordinates to yield

$$\begin{aligned}
\hat{\Lambda} &= \begin{bmatrix} \hat{\lambda}_1 & 0 \\ 0 & \hat{\lambda}_2 \end{bmatrix} = \begin{bmatrix} 0.9791 - 0.1985i & 0.0 \\ 0.0 & 0.9791 + 0.1985i \end{bmatrix} \\
\hat{B}_m &= \hat{\Psi}^{-1}\hat{B} = \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \end{bmatrix} = \begin{bmatrix} -1.7521 - 0.4352i \\ -1.7521 + 0.4352i \end{bmatrix} \\
\hat{C}_m &= \hat{C}\hat{\Psi} = [\,\hat{c}_1 \quad \hat{c}_2\,] = [\,0.0191 + 0.0519i \quad 0.0191 - 0.0519i\,] \\
\hat{D} &= Y_0 = 1
\end{aligned}$$

Instead of computing the vector $\hat{q}_1$ shown in Eq. (5.42), let us use the observability matrix in modal coordinates to yield

$$\hat{q}_1 = \begin{bmatrix} \hat{c}_1 \\ \hat{c}_1\hat{\lambda}_1 \\ \hat{c}_1\hat{\lambda}_1^2 \end{bmatrix} = \begin{bmatrix} 0.0191 + 0.0519i \\ 0.0290 + 0.0470i \\ 0.0377 + 0.0462i \end{bmatrix}$$

which is composed of three time steps of the unit pulse response history associated with the first eigenvalue of the identified model. The vector $\hat{q}_2$ for the second eigenvalue is the complex conjugate of $\hat{q}_1$ because the second eigenvalue is the complex conjugate of the first eigenvalue. The corresponding vectors $\bar{q}_1$ and $\bar{q}_2$ from pulse response data (Markov parameters) are the first and second columns of the matrix $[R_2\Sigma_2^{1/2}\hat{\Psi}]$, respectively. These vectors are directly obtained from the pulse response data used in the identification. The vectors $\bar{q}_1$ and $\bar{q}_2$ are identical to the vectors $\hat{q}_1$ and $\hat{q}_2$ because the system in this example is assumed noise-free. Therefore the Modal Amplitude Coherence (MAC) for the first eigenvalue is one and also for the second eigenvalue. In this example, we have shown another way of computing the MAC, namely from the observability matrix in modal coordinates. It is possible only because controllability and observability are dual in nature.

The Mode Singular Value (MSV) defined in Eq. (5.50) with $\ell = 7$ (seven Markov parameters) for each identified eigenvalue can also be computed by the same way as shown in Example 5.3. The MSV computed from ERA/DC should be the same as that from ERA, because the identified eigenvalue λ_1 and the absolute value $|c_1||b_1| = |c_1||b_1| = |c_1 b_1| = |Y_1| = 0.021905$ from both ERA and ERA/DC should be identical for noise-free systems. In general, they are not the same because of noise and truncation of nonzero singular values.

The A_c and B_c matrices for the corresponding continuous-time model are

$$\hat{A}_c = \begin{bmatrix} 1.466 & 0.5561 \\ -5.691 & -1.476 \end{bmatrix} \quad \text{and} \quad \hat{B}_c = \begin{bmatrix} -1.354 \\ -5.815 \end{bmatrix}$$

The C and D matrices for the continuous-time model are the same as those for the discrete-time model. The ERA/DC-identified matrices are different from the corresponding ERA-identified matrices. However the identified eigenvalues from both ERA and ERA/DC should be identical. Indeed, the identified eigenvalues from both methods in the continuous-time domain are $-0.005 \pm 1.0i$ which are exactly the eigenvalues of the system.

5.5 Canonical-Form Realization

During the past two decades, several algorithms for the construction of a canonical-form representation of linear systems have appeared in the controls literature [6, 7]. Researchers in the field of controls are mainly concerned with determining a passive or an active network that has a prescribed impedance or transfer function. Among several available methods for canonical-form realization, one based on Hankel matrices will be addressed in this section.

In the field of structures, a method which is similar, if not identical, to a canonical-form realization was developed in [85, 86] using frequency response functions for identification of modal parameters from multi-input and multi-output measurement data. The method is referred to as the Polyreference technique. The mathematical background of the Polyreference technique will be presented in this section using a new approach which provides insights to the relationship between the Polyreference technique and the canonical-form realization.

Form the $\alpha \times \beta$ block Hankel matrix as

$$H(0) = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_\beta \\ Y_2 & Y_3 & \cdots & Y_{\beta+1} \\ \vdots & \vdots & \ddots & \vdots \\ Y_\alpha & Y_{\alpha+1} & \cdots & Y_{\alpha+\beta-1} \end{bmatrix}. \tag{5.66}$$

where α and β are integers which are chosen to be larger than the order of the system. Using the singular value decomposition, find the nonsingular matrices R and S such that

$$H(0) = R\Sigma S^T = [R_n \;\; R_o] \begin{bmatrix} \Sigma_n & 0 \\ 0 & 0 \end{bmatrix} [S_n \;\; S_o]^T \tag{5.67}$$

where Σ_n is a diagonal matrix containing monotonically non-increasing nonzero singular values. The integer n is determined by the characteristics of the system noise as discussed in the previous sections. All singular

values numbered after n are considered as zero values. The matrices R_n and R_o denote, respectively, the first n and the last remaining columns of the orthonormal matrix R. Similarly, S_n and S_o denote, respectively, the first n and the remaining columns of the orthonormal matrix S.

Now from the definition of Markov parameters, we have

$$H(0) = \mathcal{P}_\alpha \mathcal{Q}_\beta \tag{5.68}$$

where

$$\mathcal{P}_\alpha = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{\alpha-1} \end{bmatrix} \quad \text{and} \quad \mathcal{Q}_\beta = [B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{\beta-1}B]$$

Note that $\mathcal{P}_\alpha$ and $\mathcal{Q}_\beta$ are observability and controllability matrices, respectively. Since the matrices R and S are orthonormal, i.e., $R^TR = S^TS = I$, Eqs. (5.67) and (5.68) lead to

$$\begin{aligned} R^TH(0)S &= R^T\mathcal{P}_\alpha\mathcal{Q}_\beta S \\ &= \begin{bmatrix} R_n^T\mathcal{P}_\alpha\mathcal{Q}_\beta S_n & R_n^T\mathcal{P}_\alpha\mathcal{Q}_\beta S_o \\ R_o^T\mathcal{P}_\alpha\mathcal{Q}_\beta S_n & R_o^T\mathcal{P}_\alpha\mathcal{Q}_\beta S_o \end{bmatrix} = \begin{bmatrix} \Sigma_n & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{5.69}$$

which yields

$$R_n^T\mathcal{P}_\alpha\mathcal{Q}_\beta S_n = \Sigma_n \tag{5.70}$$

$$R_n^T\mathcal{P}_\alpha\mathcal{Q}_\beta S_o = 0, \quad R_o^T\mathcal{P}_\alpha\mathcal{Q}_\beta S_n = 0, \tag{5.71}$$

and

$$R_o\mathcal{P}_\alpha\mathcal{Q}_\beta S_o = 0 \tag{5.72}$$

Recall that the ERA algorithm is developed using the matrices R_n, S_n and Σ_n as shown in Eq. (5.70). If the system is assumed controllable and observable, each of the five matrices R_n, $\mathcal{P}_\alpha$, $\mathcal{Q}_\beta$, S_n and Σ_n are of rank n. By Eq. (5.70), this means that the ranks of matrices $R_n^T\mathcal{P}_\alpha$ and $\mathcal{Q}_\beta S_n$ are n. Thus, Eqs. (5.71) and (5.72) imply

$$R_o^T\mathcal{P}_\alpha = 0 \quad \text{and} \quad \mathcal{Q}_\beta S_o = 0 \tag{5.73}$$

The matrix R_o provides the left orthonormal basis for the null subspace which is orthogonal to the observability matrix, whereas the matrix S_o gives

the right orthonormal basis for the null subspace which is orthonormal to the controllability matrix. Now partition the matrices R_o and S_o as

$$R_o^T = [R_{o1}^T \;\; R_{o2}^T \;\; \cdots \;\; R_{o\alpha}^T] \text{ and } S_o^T = [S_{o1}^T \;\; S_{o2}^T \;\; \cdots \;\; S_{o\beta}^T] \tag{5.74}$$

Substitution of Eqs. (5.68) and (5.74) into Eq. (5.73) yields

$$\sum_{i=1}^{\alpha} R_{oi}^T C A^{i-1} = R_{o1}^T C + R_{o2}^T CA + \cdots + R_{o\alpha}^T CA^{\alpha-1} = 0 \tag{5.75}$$

and

$$\sum_{i=1}^{\beta} A^{i-1} B S_{oi} = BS_{o1} + ABS_{o2} + \cdots + A^{\beta-1} BS_{o\beta} = 0 \tag{5.76}$$

Equations (5.75) and (5.76) are the basic formulation for the canonical-form realization (Polyreference technique). In fact, Equation (5.75) is the basis for an observable canonical-form realization whereas Eq. (5.76) is the basis for a controllable canonical-form realization (see [7], p. 321, Problem 6-19 and 6-21). Both Eqs. (5.75) and (5.76) should produce the same results. The question arises as to whether Eq. (5.75) is more favorable than Eq. (5.76) or vice versa. The answer is given in the following.

Observe that each submatrix R_{oi} $(i = 1, \ldots, \alpha)$ must have more columns than rows (the number of outputs m). Similarly, each submatrix S_{oi} $(i = 1, \cdots, \beta)$ has more columns than rows (the number of inputs r). Choose square matrices $\bar{R}_{oi}$ of order m and $\bar{S}_{oi}$ of order r respectively from matrices R_{oi} and S_{oi}, and rewrite Eqs. (5.75) and (5.76) such that

$$CA^{\alpha-1} = -\sum_{i=1}^{\alpha-1} [\bar{R}_{o\alpha}^T]^{-1} \bar{R}_{oi}^T CA^{i-1} \equiv -\sum_{i=1}^{\alpha-1} \tilde{R}_{oi}^T CA^{i-1} \tag{5.77}$$

and

$$A^{\beta-1} B = -\sum_{i=1}^{\beta-1} A^{i-1} B \bar{S}_{oi} [\bar{S}_{o\beta}]^{-1} \equiv -\sum_{i=1}^{\beta-1} A^{i-1} B \tilde{S}_{oi} \tag{5.78}$$

with

$$\tilde{R}_{oi}^T = [\bar{R}_{o\alpha}^T]^{-1} \bar{R}_{oi}^T \quad \text{and} \quad \tilde{S}_{oi} = \bar{S}_{oi} [\bar{S}_{o\beta}]^{-1}$$

Equations (5.77) and (5.78) can be rearranged into block companion matrix form as

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha-3} \\ CA^{\alpha-2} \end{bmatrix} A = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_m \\ -\tilde{R}_{o1}^T & -\tilde{R}_{o2}^T & -\tilde{R}_{o3}^T & \cdots & -\tilde{R}_{o(\alpha-1)}^T \end{bmatrix} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{\alpha-2} \end{bmatrix} \tag{5.79}$$

and

$$\begin{aligned} & A[B \;\; AB \;\; \cdots \;\; A^{\beta-3}B \;\; A^{\beta-2}B] \\ & = [B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{\beta-2}B] \begin{bmatrix} 0 & 0 & \cdots & 0 & \tilde{S}_{o1} \\ I_r & 0 & \cdots & 0 & \tilde{S}_{o2} \\ 0 & I_r & \cdots & 0 & \tilde{S}_{o3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_r & \tilde{S}_{o(\beta-1)} \end{bmatrix} \end{aligned} \tag{5.80}$$

The matrix I_m is an identity matrix of order m and the matrix I_r is an identity matrix of order r. Now it is claimed from Eq. (5.79) that the triple

$$\begin{aligned} \hat{A} &= \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_m \\ -\tilde{R}_{o1}^T & -\tilde{R}_{o2}^T & -\tilde{R}_{o3}^T & \cdots & -\tilde{R}_{o(\alpha-1)}^T \end{bmatrix} \\ \hat{B} &= \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_{\alpha-1} \end{bmatrix} \\ \hat{C} &= [I_m \;\; 0 \;\; \ldots \;\; 0 \;\; 0] \end{aligned} \tag{5.81}$$

is an $m(\alpha-1)$-dimensional realization of the system. Indeed, it can be readily verified that

$$Y_1 = \hat{C}\hat{B}, \;\; Y_2 = \hat{C}\hat{A}\hat{B}, \;\; \cdots, \;\; Y_\alpha = \hat{C}\hat{A}^{\alpha-1}\hat{B}$$

Because of the structure of $\hat{A}$ and $\hat{C}$, it is easy to show that the realization is observable, which will be discussed more in Chapter 7. However, it is not in

general controllable. This realization is called an observable canonical-form realization. It is not a minimum realization because it is not both observable and controllable.

Similarly, from Eq. (5.80), it can be verified that the triple

$$\begin{aligned}
\hat{A} &= \begin{bmatrix} 0 & 0 & \cdots & 0 & \tilde{S}_{o1} \\ I_r & 0 & \cdots & 0 & \tilde{S}_{o2} \\ 0 & I_r & \cdots & 0 & \tilde{S}_{o3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_r & \tilde{S}_{o(\beta-1)} \end{bmatrix} \\
\hat{B} &= \begin{bmatrix} I_r \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \\
\hat{C} &= [Y_1 \;\; Y_2 \;\; Y_3 \;\; \cdots \;\; Y_{\beta-1}]
\end{aligned} \tag{5.82}$$

is an $r(\beta-1)$-dimensional realization of the system. This is a controllable canonical-form realization (see Chapter 7 for more discussion) which is not in general observable. Again this realization is not of minimum order.

Equation (5.82) can be reduced to a minimum order realization by applying the reduction procedure shown in [7], Chapter 5. However, the canonical form will be destroyed after the application of the reduction procedure.

The order of either observable or controllable canonical-form realization, i.e., $m(\alpha-1)$ or $r(\beta-1)$, is required to be equal to or larger than the order of the system. From a computational point of view, one should choose the one with smaller dimension to work for modal parameter identification. The numerical problem for the eigensolution of the canonical-form realization can be solved in various ways. A technique suitable and efficient for mini-computer systems has been implemented and described in [86]. It should be remarked that only a subset of eigenvalues in the realized state matrix $\hat{A}$ (see Eq. (5.81)) belongs to the actual state matrix A, since the matrix $\hat{A}$ is generally oversized for multi-input and multi-output measurements.

The method used here to obtain the canonical-form realization is different from that shown in [85]. Orthonormal matrices R_o and S_o are computed through the application of the singular value decomposition to realize a block companion state matrix. Since the orthonormal matrices are very close to identity matrices, this generates a computationally well-behaved canonical-form realization.

5.6 Principal Hankel Component Algorithm

A minimum order canonical-form realization is generally impossible for multi-input, multi-output systems due to the constraint that the realized state matrix is in companion form. If the constraint is released, a minimum order realization can be obtained from Eq. (5.79).

In view of Eqs. (5.67) and (5.68), the controllability and observability matrices can be expressed by the following equations:

$$\mathcal{P}_\alpha = R_n \Sigma_n^{1/2} \quad \text{and} \quad \mathcal{Q}_\beta = \Sigma_n^{1/2} S_n^T \tag{5.83}$$

Define $\mathcal{P}_\alpha^{\uparrow m}$ as the matrix formed by deleting the first m rows of $\mathcal{P}_\alpha$ and $\mathcal{P}_\alpha^{\downarrow m}$ as the matrix formed by deleting the last m rows of $\mathcal{P}_\alpha$, i.e.,

$$\mathcal{P}_\alpha = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha-2} \\ CA^{\alpha-1} \end{bmatrix} \Rightarrow \mathcal{P}_\alpha^{\uparrow m} = \begin{bmatrix} CA \\ \vdots \\ CA^{\alpha-2} \\ CA^{\alpha-1} \end{bmatrix} \text{ and } \mathcal{P}_\alpha^{\downarrow m} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha-2} \end{bmatrix} \tag{5.84}$$

Now note from Eq. (5.83) that

$$\mathcal{P}_\alpha^{\downarrow m} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha-2} \end{bmatrix} = R_n^{\downarrow m} \Sigma_n^{1/2} \tag{5.85}$$

where $R_n^{\downarrow m}$ simply means the matrix obtained by deleting the last m rows of the matrix R_n. Hence, Eq. (5.79) can be written with the aid of Eq. (5.81) as

$$R_n^{\downarrow m} \Sigma_n^{1/2} A = \hat{A} R_n^{\downarrow m} \Sigma_n^{1/2} \tag{5.86}$$

which yields

$$A = \Sigma_n^{-1/2} [R_n^{\downarrow m}]^\dagger \hat{A} R_n^{\downarrow m} \Sigma_n^{1/2} \tag{5.87}$$

where $\dagger$ means the pseudo-inverse. This is a minimum realization of order n. To compute Eq. (5.87), a simple procedure can be developed as follows. In view of the definition of $\mathcal{P}_\alpha^{\uparrow m}$ shown in Eq. (5.84), Eq. (5.79) can also be written as

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha-2} \end{bmatrix} A = \mathcal{P}_\alpha^{\uparrow m} = R_n^{\uparrow m} \Sigma_n^{1/2} = \hat{A} R_n^{\downarrow m} \Sigma_n^{1/2} \tag{5.88}$$

and thus Eq. (5.87) becomes

$$\hat{A} = \Sigma_n^{-1/2}[R_n^{\downarrow m}]^{\dagger} R_n^{\uparrow m}\Sigma_n^{1/2} \tag{5.89}$$

Here, $R_n^{\downarrow m}$ simply means the matrix obtained by deleting the last m rows of the matrix R_n, and $R_n^{\uparrow m}$ represents the matrix obtained by deleting the first m rows of the matrix R_n. This equation was first presented in [47] and named the Principal Hankel Component Algorithm. Since R_n is an orthonormal matrix, a special and efficient procedure can be developed to compute the pseudo-inverse of the matrix $R_n^{\downarrow m}$ using the matrix inversion lemma.

Indeed, let R_n be partitioned such that

$$R_n = \begin{bmatrix} R_n^{\downarrow m} \\ R_n^{m} \end{bmatrix} \tag{5.90}$$

where $R_n^{\downarrow m}$ is a $(\alpha - 1) \times n$ matrix and R_n^m is an $m \times n$ matrix. Application of the orthonormality property, $R_n^T R_n = I_n$, yields

$$R_n^T R_n = [R_n^{\downarrow m}]^T R_n^{\downarrow m} + [R_n^m]^T R_n^m = I_n$$

or

$$[R_n^{\downarrow m}]^T R_n^{\downarrow m} = I_n - [R_n^m]^T R_n^m \tag{5.91}$$

Use the matrix inversion lemma shown in the footnote of Section 3.2.3 in Chapter 3 to compute the inverse of the matrix $[R_n^{\downarrow m}]^T R_n^{\downarrow m}$, i.e.,

$$\begin{aligned} \left[[R_n^{\downarrow m}]^T R_n^{\downarrow m}\right]^{-1} &= \left[I_n - [R_n^m]^T R_n^m\right]^{-1} \\ &= I_n + [R_n^m]^T \left[I_m - R_n^m [R_n^m]^T\right]^{-1} R_n^m \end{aligned} \tag{5.92}$$

Note that I_m and I_n are identity matrices of order m and n, respectively. Therefore the pseudo-inverse of the matrix $R_n^{\downarrow m}$ becomes

$$\begin{aligned} [R_n^{\downarrow m}]^{\dagger} &= \left[[R_n^{\downarrow m}]^T R_n^{\downarrow m}\right]^{-1} [R_n^{\downarrow m}]^T \\ &= \left[I_n + [R_n^m]^T \left[I_m - R_n^m [R_n^m]^T\right]^{-1} R_n^m\right] [R_n^{\downarrow m}]^T \end{aligned} \tag{5.93}$$

When should this computation be used? When the number of sensors, m, is less than the order of the system, n, the matrix $\left[I_m - R_n^m [R_n^m]^T\right]$ of order m

to be inverted is smaller in size than the matrix $\left[[R_n^{\downarrow m}]^T R_n^{\downarrow m}\right]$ of order n. In particular, when m is considerably smaller than n, Eq. (5.93) not only saves computational time but also provides more accurate result.

Similarly, an equation for the determination of the state transition matrix A can be derived from Eq. (5.80) as

$$\hat{A} = \Sigma_n^{1/2}[S_n^{\uparrow r}]^T[S_n^{\downarrow r}]^{T\dagger}\Sigma_n^{-1/2} \tag{5.94}$$

where $S_n^{\downarrow r}$ means the matrix obtained by deleting the last r rows of matrix S_n, and $S_n^{\uparrow r}$ represents the matrix obtained by deleting the first r rows of the matrix S_n. The reader is encouraged to prove this result.

Equations (5.89) and (5.94) can also be derived by the ERA procedure. Let an oversized Hankel matrix $\hat{H}$ be formed such that

$$H(0) = \hat{H}^{\downarrow m} \quad \text{and} \quad H(1) = \hat{H}^{\uparrow m} \tag{5.95}$$

The Hankel matrix $H(0)$ is formed by deleting the last m rows of the Hankel matrix $\hat{H}$ whereas the Hankel matrix $H(1)$ is obtained by deleting the first m rows of matrix $\hat{H}$.

Find the orthonormal matrices R_n and S_n, and a diagonal matrix Σ_n such that

$$\hat{H} = R_n\Sigma_n S_n^T \tag{5.96}$$

Equation (5.95) thus becomes

$$H(0) = R_n^{\downarrow m}\Sigma_n S_n^T \quad \text{and} \quad H(1) = R_n^{\uparrow m}\Sigma_n S_n^T \tag{5.97}$$

Substituting this equation into the ERA basic formulation (5.34) and noting that $S_n^T S_n = I_n$ yields the triple

$$\begin{aligned} \hat{A} &= \Sigma_n^{-1/2}[R_n^{\downarrow m}]^\dagger R_n^{\uparrow m}\Sigma_n^{1/2} \\ \hat{B} &= \Sigma_n^{1/2} S_n^T E_r \\ \hat{C} &= E_m^T R_n \Sigma_n^{1/2} \end{aligned} \tag{5.98}$$

Note that the state matrix $\hat{A}$ is indeed identical to that in Eq. (5.89).

Similarly, it can be written that

$$H(0) = \hat{H}^{\rightarrow r} = R_n\Sigma_n[S_n^{\downarrow r}]^T \quad \text{and} \quad H(1) = \hat{H}^{\leftarrow r} = R_n\Sigma_n[S_n^{\uparrow r}]^T \tag{5.99}$$

where the Hankel matrix $H(0)$ is obtained by deleting the last r columns of the oversized Hankel matrix $\hat{H}$, and the Hankel matrix $H(1)$ is obtained by deleting the first r columns of the same matrix $\hat{H}$. Substitution of Eq. (5.99) into the ERA basic formulation, with the aid of $R_n^T R_n = I_n$, produces the triple

$$\begin{aligned} \hat{A} &= \Sigma_n^{1/2}[S_n^{\uparrow r}]^T[S_n^{\downarrow r}]^{T\dagger}\Sigma_n^{-1/2} \\ \hat{B} &= \Sigma_n^{1/2} S_n^T E_r \\ \hat{C} &= E_m^T R_n \Sigma_n^{1/2} \end{aligned} \tag{5.100}$$

The state matrix $\hat{A}$ is again identical to that in Eq. (5.94).

Realizations (5.98) and (5.100) preserve the same features as for the ERA, including a good numerical performance, internal balance, and flexibility in determining order-error tradeoff. Based on formulations (5.98) and (5.100), a close link between the ERA and the canonical-form realization (Polyreference technique) is established through the singular value decomposition and the generalized Hankel matrix.

5.7 Least-Squares Regression Technique

The least-squares regression technique for a discrete-time model has been derived and used for system identification for more than two decades (see Ref. [24], pp. 97-99). The same technique was rederived and further developed for the use of modal parameter identification in the field of structures as shown in Ref. [30, 31]. Here, the least-squares regression technique will be formulated using system realization theory, which provides a good basis for comparison with other methods.

In view of Eqs. (5.24), (5.27), and (5.28), the measurement function $Y(k)$ can be obtained through either of two other algorithms as follows:

$$\begin{aligned} Y_k &= E_m^T H(k-1) E_r \quad \text{(from Eq. (5.24))} \\ &= E_m^T \mathcal{P}_\alpha A^{k-1} \mathcal{Q}_\beta E_r \quad \text{(use Eqs. (5.24) and (5.27))} \\ &= E_m^T \mathcal{P}_\alpha A^{k-1} [\mathcal{Q}_\beta H^\dagger \mathcal{P}_\alpha] \mathcal{Q}_\beta E_r \quad \text{(use Eq. (5.28))} \\ &= E_m^T [\mathcal{P}_\alpha A \mathcal{Q}_\beta H^\dagger]^{k-1} \mathcal{P}_\alpha \mathcal{Q}_\beta E_r \quad \text{(use Eq. (5.28))} \\ &= E_m^T [H(1) H^\dagger]^{k-1} H(0) E_r \end{aligned} \tag{5.101}$$

or, with a slight change in the third equality,

$$\begin{aligned} Y_k &= E_m^T H(k-1) E_r \quad \text{(from Eq. (5.24))} \\ &= E_m^T \mathcal{P}_\alpha A^{k-1} \mathcal{Q}_\beta E_r \quad \text{(use Eqs. (5.24) and (5.27))} \\ &= E_m^T \mathcal{P}_\alpha [\mathcal{Q}_\beta H^\dagger \mathcal{P}_\alpha] A^{k-1} \mathcal{Q}_\beta E_r \quad \text{(use Eq. (5.28))} \\ &= E_m^T \mathcal{P}_\alpha \mathcal{Q}_\beta [H^\dagger \mathcal{P}_\alpha A \mathcal{Q}_\beta]^{k-1} E_r \quad \text{(use Eq. (5.28))} \\ &= E_m^T H(0) [H^\dagger H(1)]^{k-1} E_r \end{aligned} \tag{5.102}$$

Hence, the triplet $[H(1)H^\dagger,\ H(0)E_r,\ E_m^T]$ or $[H^\dagger H(1),\ E_r,\ E_m^T H(0)]$ is a realization. The matrix $H(1)H^\dagger$ or $H^\dagger H(1)$ constitutes the basis for the least-squares regression technique (see Ref. [24], pp. 97-99).

The matrix $H^\dagger$ is the pseudo-inverse of the matrix $H(0)$. For a single input, there exists a case where the rank of $H(0)$ equals the column number of $H(0)$, then

$$H^\dagger = [H(0)^T\ H(0)]^{-1} H(0)^T \tag{5.103}$$

On the other hand, there exists a case for a single output where the rank equals the row number, then

$$H^\dagger = H(0)^T [H(0) H(0)^T]^{-1} \tag{5.104}$$

The matrix $H(1)H^\dagger$ has been used in the structural dynamics field to identify system modes and frequencies (see Ref. [30, 31]). This is a special case representing a single input which cannot realize a system that has repeated eigenvalues or uses sufficiently low noise data unless the system order is known *a priori*.

These realizations given by Eqs. (5.101) and (5.102) are not of minimum order, since their order is the number of either columns or rows of the matrix $H(0)$ which is larger than the order of the state matrix A for multi-input and multi-output cases. Examination of (5.101) and (5.102) reveals that these two equations are special cases of ERA. Equation (5.101) is formulated by inserting the identity matrix $\mathcal{Q}_\beta H^\dagger \mathcal{P}_\alpha = I_n$ on the right-hand side of the state matrix A. On the other hand, Eq. (5.102) is obtained by inserting the identity matrix $\mathcal{Q}_\beta H^\dagger \mathcal{P}_\alpha = I_n$ on the left-hand side of the state matrix A. However, the ERA is formed by inserting the identity matrix $\mathcal{Q}_\beta H^\dagger \mathcal{P}_\alpha = I_n$ on both sides of the state matrix A as shown in Eq. (5.33). Because of the different insertion, the least-squares regression method does not minimize the order of the system. Mathematically, if the singular value decomposition technique or another rank detection technique is not included in the

computational procedures, the realized triplet obtained from Eq. (5.101) or (5.102) cannot be numerically implemented, unless a certain degree of artificial noise and/or system noise is present. Noise tends to make up the rank deficiency of the data matrix $H(0)$. Since the degree of noise contamination is generally unknown, the least-squares regression technique is not recommended for modal parameter identification in the structures field.

5.8 Summary

A summary of various minimum realization and canonical-form realization is given in TABS 5.1 and 5.2, respectively. The first column gives the algorithm name and the second column gives the formulation. The bottom defines the notation used in the equations. All formulations are based on the system Markov parameters.

In this chapter, several methods for system identification have been presented and derived using system realization theory. The relations between different techniques are reasonably well understood and the choice of methods can be made largely on the basis of the final purpose of the identification, for example, control of flexible structures. Most methods work well on simulated and test data. In spite of a large literature on identification, there are few technical papers which compare different techniques using experimental data. For a person engaged in application, it would be highly desirable to have comparisons available. This chapter illustrates the mathematical relations among different methods via system realization theory, which provides a basis and insight for comparison and evaluation.

Through the interaction of control and structure fields, the field of modal parameter identification will move towards unification and there will be more comparisons of different methods. One of the purposes of this chapter is to contribute to the goal of unification. The contribution may be fairly small but it should serve as a starting point to stimulate more research toward this goal. It is believed that interaction with other fields such as controls, artificial intelligence, etc. is essential for progress in developing identification algorithms for engineering systems.

There exist many other techniques in the field of modal parameter identification, which could not be addressed in a single chapter. The reader is directed to the Bibliography for further information.

Table 5.1: Minimum Realization Algorithms

Eigensystem Realization Algorithm (ERA)	$H(0) = R_n \Sigma_n S_n^T;\ R_n^T R_n = I_n = S_n^T S_n$ $\hat{A} = \Sigma_n^{-1/2} R_n^T H(1) S_n \Sigma_n^{-1/2}$ $\hat{B} = \Sigma_n^{1/2} S_n^T E_r$ $\hat{C} = E_m^T R_n \Sigma_n^{1/2}$
Eigensystem Realization Algorithm Using Data Correlation (ERA/DC)	$\mathcal{H}(0) = R_n \Sigma_n S_n^T;\ R_n^T R_n = I_n = S_n^T S_n$ $\hat{A} = \Sigma_n^{-1/2} R_n^T \mathcal{H}(1) S_n \Sigma_n^{-1/2}$ $\hat{B} = [E_\gamma^T R_n \Sigma_n^{1/2}]^\dagger H(0) E_r$ $\hat{C} = E_m^T R_n \Sigma_n^{1/2}$
Principal Hankel Component Algorithm (PHCA)	$\hat{H} = R_n \Sigma_n S_n^T;\ R_n^T R_n = I_n = S_n^T S_n$ $H(0) = R_n^{\downarrow m} \Sigma_n S_n^T;\ H(1) = R_n^{\uparrow m} \Sigma_n S_n^T$ $\hat{A} = \Sigma_n^{-1/2} [R_n^{\downarrow m}]^\dagger R_n^{\uparrow m} \Sigma_n^{1/2}$ $= \Sigma_n^{1/2} [S_n^{\uparrow r}]^T [S_n^{\downarrow r}]^{T\dagger} \Sigma_n^{-1/2}$ $\hat{B} = \Sigma_n^{1/2} S_n^T E_r$ $\hat{C} = E_m^T R_n \Sigma_n^{1/2}$

System Markov Parameters : $Y_1,\ Y_2,\ Y_3,\ \ldots,\ Y_\ell$

Hankel Matrix and Data Correlation Matrix

$$H(k-1) = \begin{bmatrix} Y_k & Y_{k+1} & \cdots & Y_{k+\beta-1} \\ Y_{k+1} & Y_{k+2} & \cdots & Y_{k+\beta} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{k+\alpha-1} & Y_{k+\alpha} & \cdots & Y_{k+\alpha+\beta-2} \end{bmatrix}, \quad \mathcal{R}_{hh}(k) = H(k) H^T(0)$$

Data Correlation Hankel Matrix

$$\mathcal{H}(k) = \begin{bmatrix} \mathcal{R}_{hh}(k) & \mathcal{R}_{hh}(k+\tau) & \cdots & \mathcal{R}_{hh}(k+\zeta\tau) \\ \mathcal{R}_{hh}(k+\tau) & \mathcal{R}_{hh}(k+2\tau) & \cdots & \mathcal{R}_{hh}(k+(\zeta+1)\tau) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{R}_{hh}(k+\xi\tau) & \mathcal{R}_{hh}(k+(\xi+1)\tau) & \cdots & \mathcal{R}_{hh}(k+(\xi+\zeta)\tau) \end{bmatrix}$$

Table 5.2: Canonical-Form Realization

Observable Canonical-form

$$\hat{A} = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_m \\ -\tilde{R}_{o1}^T & -\tilde{R}_{o2}^T & -\tilde{R}_{o3}^T & \cdots & -\tilde{R}_{o(\alpha-1)}^T \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_{\alpha-1} \end{bmatrix}$$

$$\hat{C} = [I_m \ 0 \ \ldots \ 0 \ 0]$$

Controllable Canonical-form

$$\hat{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \tilde{S}_{o1} \\ I_r & 0 & \cdots & 0 & \tilde{S}_{o2} \\ 0 & I_r & \cdots & 0 & \tilde{S}_{o3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_r & \tilde{S}_{o(\beta-1)} \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} I_r \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{C} = [Y_1 \ Y_2 \ Y_3 \ \cdots \ Y_{\beta-1}]$$

System Markov Parameters : $Y_1, Y_2, Y_3, \ldots, Y_\ell$

Hankel Matrix; $H(0) = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_\beta \\ Y_2 & Y_3 & \cdots & Y_{\beta+1} \\ \vdots & \vdots & \ddots & \vdots \\ Y_\alpha & Y_{\alpha+1} & \ldots & Y_{\alpha+\beta-1} \end{bmatrix}$

Singular Value Decomposition and Matrix Partition

$$H(0) = R\Sigma S^T = [R_n \ R_o] \begin{bmatrix} \Sigma_n & 0 \\ 0 & 0 \end{bmatrix} [S_n \ S_o]^T$$

$$R_o^T = [R_{o1}^T \ R_{o2}^T \ \cdots \ R_{o\alpha}^T]$$

$$S_o^T = [S_{o1}^T \ S_{o2}^T \ \cdots \ S_{o\beta}^T]$$