(平面应力问题－位移法) 1. Plane problem

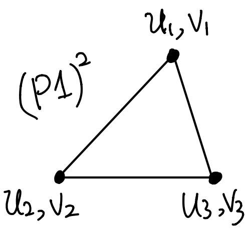
**1.1. Elasticity problem in 2D.** Given a plate $\Omega$ with thickness $e$, suppose $\boldsymbol{\sigma}(\boldsymbol{u}) = \frac{Ee}{1-\nu^2}((1-\nu)\epsilon(\boldsymbol{u}) + \nu \operatorname{div} \boldsymbol{u}\boldsymbol{\delta})$

$$
\begin{aligned}
-\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{f} \text{ in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{u}_0, \text{ on } \Gamma_1,\\
\boldsymbol{\sigma n} &= \boldsymbol{g} \text{ on } \Gamma_2.
\end{aligned}
$$

Given a triangulation $\mathcal{T}_h$ of $\Omega$, define the $P_1$ conforming element space as

$$V_h(\Omega) := \{v \in H^1(\Omega) \mid v|_K \in P_1(K) \text{ for all } K \in \mathcal{T}_h\}.$$

The discrete problem is as follows: Seek




试探 $\boldsymbol{u}_h \in \{\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in (V_h(\Omega))^2 | \boldsymbol{v}|_{\Gamma_1} = \Pi_1 \boldsymbol{u}_0\}$

$\Pi_1 \to$ 单元节点的一次插值
要求 $u_0 \in H^1(\Gamma_1)$
在边界1维情况下, $H^1(\Gamma) \hookrightarrow C^0(\Gamma)$
从而可定义节点值.否则可能需要平均.

such that

$$\boxed{(\boldsymbol{\sigma}(\boldsymbol{u}_h), \epsilon(\boldsymbol{v})) = (\boldsymbol{f}, \boldsymbol{v}) + \langle \boldsymbol{g}, \boldsymbol{v}\rangle_{\Gamma_2}}$$

for all

检验 $\boldsymbol{v} \in \{\boldsymbol{v} \in (V_h(\Omega))^2 | \boldsymbol{v}|_{\Gamma_1} = 0\}$

**1.2. Bending problem.** Suppose $\mathcal{K}(u) = -\nabla^2 u$ and $\mathcal{M}(u) = \frac{Ee^3}{12(1-\nu^2)}((1-\nu)\mathcal{K}(u) + \nu \operatorname{tr}(\mathcal{K}(u))\boldsymbol{\delta})$. The bending problem reads

(薄板弯曲问题－混合元)

$$
\begin{aligned}
-\operatorname{div}\operatorname{div} \mathcal{M}(u) &= f \text{ in } \Omega,\\
u = u_0, \frac{\partial u}{\partial n} &= g \text{ on } \Gamma_1,\\
u = u_0, \boldsymbol{n}^T\mathcal{M}(u)\boldsymbol{n} &= \mathcal{M}_0 \text{ on } \Gamma_2,\\
\operatorname{div} \mathcal{M}(u)\cdot \boldsymbol{n} + \frac{\partial(\boldsymbol{n}^T\mathcal{M}(u)\boldsymbol{t})}{\partial t} = q_0, \boldsymbol{n}^T\mathcal{M}(u)\boldsymbol{n} &= \mathcal{M}_0 \text{ on } \Gamma_3,\\
[\boldsymbol{n}^T\mathcal{M}(u)\boldsymbol{t}]|_{P_i} &= 0 \text{ at corner } P_i \text{ of } \Gamma_3.
\end{aligned}
$$

强制

强制

当使用HHJ方法求解弯矩时

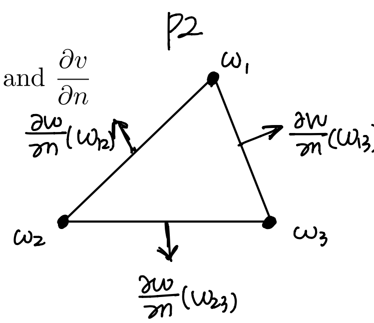
(不考虑集中力)

位移法

给出用Morley非协调元求解板问题的变分和空间

Define the Morley element space

$V_h^M(\Omega) = \{v \in L^2(\Omega) : v \in P_2(K)$ for all $K \in \mathcal{T}_h$, $v$ is continuous at vertices and $\frac{\partial v}{\partial n}$ is continuous at the midpoint of interior edges$\}$.




The primal form (Morley element): Seek

试探 $u_h \in \{v \in V_h^M(\Omega) \mid \Pi_1 v|_{\Gamma_1\cup\Gamma_2} = \Pi_1 u_0, \Pi_0(\frac{\partial v}{\partial n})|_{\Gamma_1} = \Pi_0 g\}$

such that

$$\boxed{(\mathcal{M}_h(u_h), \mathcal{K}_h(v)) = (f, v) - \langle \mathcal{M}_0, \frac{\partial v}{\partial n}\rangle_{\Gamma_2\cup\Gamma_3} + \langle q_0, v\rangle_{\Gamma_3} + [n^T M(u) t]|_{P_i} v(P_i)}$$

(若集中力不为0时)

for all

检验 $v \in \{v \in V_h^M(\Omega) \mid \Pi_1 v|_{\Gamma_1\cup\Gamma_2} = 0, \Pi_0(\frac{\partial v}{\partial n})|_{\Gamma_1} = 0\}.$

Define the space

$$\Sigma_h(\Omega) := \{\tau \in L^2(\Omega) \,|\, \cancel{\mathcal{M}_h}\,\tau \in P_0(K;\mathbb{S}) \text{ for all } K \in \mathcal{T}_h, \boldsymbol{n}^T\tau\boldsymbol{n} \text{ is continuous across edges}\}.$$

The HHJ method reads as follows. Let $\sigma = \mathcal{M}(u)$. Seek $P_0 M_0$ (其中$P_0$是$L^2$投影, $M_0$可以是函数)

$$\sigma_h \in \{\tau \in \Sigma_h(\Omega) \mid \boldsymbol{n}^T\tau\boldsymbol{n}|_{\Gamma_2\cup\Gamma_3} = \cancel{P_0 M_0}\}$$

$$u_h \in \{v \in V_h(\Omega) \mid v|_{\Gamma_1\cup\Gamma_2} = \Pi_1 u_0\}$$

such that

$$(\mathcal{C}\sigma_h, \tau) - \sum_K \big((\tau, -\nabla^2 u_h)_K + \langle \boldsymbol{n}^T\tau\boldsymbol{n}, \frac{\partial u_h}{\partial n}\rangle_{\partial K}\big) = -\langle g, \boldsymbol{n}^T\tau\boldsymbol{n}\rangle_{\Gamma_1}$$

$$-\sum_K \big((\sigma_h, -\nabla^2 v)_K + \langle \boldsymbol{n}^T\sigma_h\boldsymbol{n}, \frac{\partial v}{\partial n}\rangle_{\partial K}\big) = (f, v) + \langle q_0, v\rangle_{\Gamma_3}$$

for all

$$\tau \in \{\tau \in \Sigma_h(\Omega) \mid \boldsymbol{n}^T\tau\boldsymbol{n}|_{\Gamma_2\cup\Gamma_3} = 0\},$$

$$v \in \{v \in V_h(\Omega) \mid v|_{\Gamma_1\cup\Gamma_2} = 0\}.$$

$P_1-P_0$元求解

①先搞明白之前只有固支边界时的变分

②验证真解满足右侧的变分提法

③注意在内部边界处, $n^T\tau n$是连续的. $\frac{\partial u_h}{\partial n}$是抵消的. 只剩边界的部分要处理.

## 2. Two plates

Suppose two plates $S$ and $\utilde{S}$ with boundary $\Gamma$. Suppose $\partial S_0 \subset S$. (For simplicity, the present case only considers zero boundary conditions on $\partial S_0$ and the rigid hinge on $\Gamma$.)

The discrete problem reads: Seek

$$(\boldsymbol{u}_h, \utilde{\boldsymbol{u}_h}) \in \{(\boldsymbol{v}, \utilde{\boldsymbol{v}}) \in (V_h(S))^3 \times (V_h(\utilde{S}))^3 \mid \boldsymbol{v}|_\Gamma = \utilde{\boldsymbol{v}}|_\Gamma, \boldsymbol{v}|_{\partial S_0} = 0\},$$

$$(\sigma_h, \utilde{\sigma_h}) \in \{(\tau, \utilde{\tau}) \in \Sigma_h(S) \times \Sigma_h(\utilde{S}) \mid \boldsymbol{n}^T\tau\boldsymbol{n}|_\Gamma = \utilde{\boldsymbol{n}}^T\utilde{\tau}\utilde{\boldsymbol{n}}|_\Gamma\}$$

such that

$$(\boldsymbol{\sigma}(\boldsymbol{u}_h), \epsilon(\boldsymbol{v}))_S + (\boldsymbol{\sigma}(\utilde{\boldsymbol{u}_h}), \epsilon(\utilde{\boldsymbol{v}}))_{\utilde{S}} + (\mathcal{C}\sigma_h, \tau)_S - \sum_K \big((\tau, -\nabla^2 \boldsymbol{u}_h^3)_K + \langle \boldsymbol{n}^T\tau\boldsymbol{n}, \frac{\partial \boldsymbol{u}_h^3}{\partial n}\rangle_{\partial K}\big)$$

$$+ (\mathcal{C}\utilde{\sigma_h}, \utilde{\tau})_{\utilde{S}} - \sum_{\utilde{K}} \big((\utilde{\tau}, -\utilde{\nabla}^2 \utilde{\boldsymbol{u}_h}^3)_{\utilde{K}} + \langle \utilde{\boldsymbol{n}}^T\utilde{\tau}\utilde{\boldsymbol{n}}, \frac{\partial \utilde{\boldsymbol{u}_h}^3}{\partial \utilde{n}}\rangle_{\partial \utilde{K}}\big)$$

$$-\sum_K \big((\sigma_h, -\nabla^2 \boldsymbol{v}^3)_K + \langle \boldsymbol{n}^T\sigma_h\boldsymbol{n}, \frac{\partial \boldsymbol{v}^3}{\partial n}\rangle_{\partial K}\big) - \sum_{\utilde{K}} \big((\utilde{\sigma_h}, -\utilde{\nabla}^2 \utilde{\boldsymbol{v}}^3)_{\utilde{K}} + \langle \utilde{\boldsymbol{n}}^T\utilde{\sigma_h}\utilde{\boldsymbol{n}}, \frac{\partial \utilde{\boldsymbol{v}}^3}{\partial \utilde{n}}\rangle_{\partial \utilde{K}}\big)$$

$$= (\boldsymbol{f}, \boldsymbol{v})_S + (\utilde{\boldsymbol{f}}, \utilde{\boldsymbol{v}})_{\utilde{S}}$$

for all

$$(\boldsymbol{v}, \utilde{\boldsymbol{v}}) \in \{(\boldsymbol{v}, \utilde{\boldsymbol{v}}) \in (V_h(S))^3 \times (V_h(\utilde{S}))^3 \mid \boldsymbol{v}|_\Gamma = \utilde{\boldsymbol{v}}|_\Gamma, \boldsymbol{v}|_{\partial S_0} = 0\},$$

$$(\tau, \utilde{\tau}) \in \{(\tau, \utilde{\tau}) \in \Sigma_h(S) \times \Sigma_h(\utilde{S}) \mid \boldsymbol{n}^T\tau\boldsymbol{n}|_\Gamma = \utilde{\boldsymbol{n}}^T\utilde{\tau}\utilde{\boldsymbol{n}}|_\Gamma\}$$

Here $\boldsymbol{v}^3$ denotes the third component of $\boldsymbol{v}$ in the corresponding coordinate. $\boldsymbol{v}|_\Gamma = \utilde{\boldsymbol{v}}|_\Gamma$ is related to the angle between two plates.