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Modeling and Justification of Eigenvalue Problems for Junctions between Elastic Structures

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We consider a problem in three-dimensional linearized elasticity, posed over a domain consisting of a plate with thickness 2ε inserted into a solid whose Lamé constants and density are independent of ε. If the Lamé constants of the material constituting the plate vary as ε^{-3} and its density as ε^{-1}, we show that the solutions of the three-dimensional eigenvalue problem converge, as ε approaches zero, to the solutions of a "coupled," "pluri-dimensional" eigenvalue problem of a new type, posed simultaneously over a three-dimensional open set with a slit and a two-dimensional open set. © 1989 Academic Press, Inc.

1. INTRODUCTION

The modeling of the junction between a three-dimensional linearly elastic structure and a linearly elastic plate has recently been analyzed by Ciarlet, Le Dret, and Nzengwa [19, 20], who showed that, once appropriately scaled, the solution of the three-dimensional *static problem* converges, as the thickness of the plate approaches zero, to the solution of a "coupled," "pluri-dimensional" problem of a new type, posed simultaneously over a three-dimensional open set with a slit and a two-dimensional open set. We consider here the associated *eigenvalue problem.*

Our approach relies on two basic tools: *First,* we use the *asymptotic method advocated and developed by Lions* [40] *for studying abstract variational problems that contain a "small" parameter.* This method has since

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then proved to be a powerful tool for justifying lower-dimensional elastic models: see Ciarlet and Destuynder [15, 16], Ciarlet and Kesavan [17], Ciarlet [9, 10, 13], Destuynder [25, 27], and Raoult [47, 48] for plates; Destuynder [26] for shells: and Bermudez and Viaño [4], Aganovič and Tutek [1], Cimetière, Geymonat, Le Dret, Raoult, and Tutek [22], and Trabucho and Viaño [53, 54, 55] for rods. In this respect, mention should be also made of the related pioneering contributions of Friedrichs and Dressler [28], Goldenveizer [32], John [33] (who was the first to mathematically justify the Kirchhoff–Love approximation), Rigolot [49, 50], Rigolot [51] (where the asymptotic expansion method is used to study the flexural vibrations of elastic rods), and Caillerie [8] (who gave an asymptotic analysis of plates that are "sandwiched" between three-dimensional structures).

Second, we combine this method with the approach developed in Ciarlet, Le Dret, and Nzengwa [20] for analyzing *junctions between three-dimensional elastic structures and elastic plates.* This approach is of wide applicability, since it can also be used for modeling *junctions between plates* (folded plates, possibly with corners; cf. Le Dret [36, 37, 38]), *junctions between plates and rods* (cf. Ciarlet [14]), *junctions between rods* (cf. Le Dret [39]), junctions connecting *rigid and elastic structures* (cf. Ciarlet and Le Dret [18]), and the corresponding *nonlinear problems* (cf. Aufranc [3]). See also Ciarlet [11] for an overview of this approach and Aufranc [2] for numerical results.

In each instance, one or several portions of the whole three-dimensional structure have a "small" thickness, or diameter, which is proportional to a *dimensionless parameter* ε. *If the various data* (Lamé constants, applied body or surface force densities) *behave as specific powers of* ε *as* $\varepsilon \to 0$, *one can establish the* H^1*-convergence of the* (appropriately scaled) *components of the displacement vector field towards the solution of a "limit" variational problem of a new type*, posed simultaneously over an open subset of $\mathbb{R}^m$ and an open subset of $\mathbb{R}^n$, with $1 \leqslant m, n \leqslant 3$.

In the present paper, we likewise establish the *convergence of the eigenvalues* and the H^1*-convergence of the associated eigenfunctions* (the components of the associated displacement vector fields) towards the solutions of a coupled, pluri-dimensional eigenvalue problem of a new type, which is precisely the eigenvalue problem associated with the "limit" variational problem obtained by Ciarlet, Le Dret and Nzengwa [20]. Accordingly, the techniques used here for proving the convergence rely on those used by Ciarlet and Kesavan [17] for the limit analysis of the eigenvalue problem for a "single" plate, and on those used by Ciarlet, Le Dret, and Nzengwa [20] for the limit analysis of the junctions between three-dimensional structures and plates.

The crucial idea for treating the junctions consists in *scaling the different* *parts of the full structure independently of each other* (in particular, the plate is scaled as is usually done in "single plate" theory), but *counting the junction twice*, once in each portion that it connects. The scaled components of the displacement, which are defined in this fashion on two separate domains, thus contain the information about the junction twice. That they correspond to the same displacement of the whole structure then yields the "*junction conditions*" that the solution of the limit problem must satisfy.

2. The Three-Dimensional Problem

Latin indices take their values in the set $\{1, 2, 3\}$ and Greek indices take their values in the set $\{1, 2\}$; the repeated index convention for summation is used systematically in conjunction with the above rules. Vector-valued functions and their associated function spaces are denoted by boldface letters.

We are given constants $a_1, b_1, a_2, a_3, b_3, \beta$ which are all >0 and we assume that $\beta < b_1$. For each $\varepsilon > 0$, we let (cf. Fig. 1)

$$\begin{aligned}
&\omega = \{(x_1, x_2) \in \mathbb{R}^2;\ 0 < x_1 < b_1,\ |x_2| < a_2\}, && \Omega^\varepsilon = \omega \times \left]-\varepsilon, \varepsilon\right[,\\
&\gamma_0 = \{(b_1, x_2) \in \mathbb{R}^2;\ |x_2| \leqslant a_2\}, && \Gamma_0^\varepsilon = \gamma_0 \times \left]-\varepsilon, \varepsilon\right[,\\
&\omega_\beta = \{(x_1, x_2) \in \mathbb{R}^2;\ 0 < x_1 < \beta,\ |x_2| < a_2\}, && \Omega_\beta^\varepsilon = \omega_\beta \times \left]-\varepsilon, \varepsilon\right[,\\
&0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3;\ -a_1 < x_1 < \beta,\ |x_2| < a_2,\ -a_3 < x_3 < b_3\}, &&\\
&0_\beta^\varepsilon = 0 - \bar{\Omega}_\beta^\varepsilon, && S^\varepsilon = 0 \cup \Omega^\varepsilon,
\end{aligned}$$

and we denote by $x^\varepsilon = (x_i^\varepsilon)$ a generic point in the set S^ε and by ∂_i^ε the partial derivative $\partial/\partial x_i^\varepsilon$.

Remark. Since ε is to be understood as a *dimensionless parameter*, the thickness of the thin structure should be written as $2\varepsilon h$ for some fixed *length* $h > 0$. We assume here that $h = 1$, thus saving another notation. ∎

The set $\bar{0}_\beta^\varepsilon$ is the reference configuration of a linearly elastic body whose Lamé constants $\tilde{\gamma} > 0$, $\tilde{\mu} > 0$ and density $\tilde{\rho} > 0$ are assumed to be *independent of* ε; the set $\bar{\Omega}^\varepsilon$ is the reference configuration of a linearly elastic body whose Lamé constants λ^ε, μ^ε and density ρ^ε are assumed to be of the form

$$\lambda^\varepsilon = \varepsilon^{-3}\lambda, \qquad \mu^\varepsilon = \varepsilon^{-3}\mu, \qquad \rho^\varepsilon = \varepsilon^{-1}\rho, \tag{2.1}$$

where $\lambda > 0$, $\mu > 0$, $\rho > 0$ are three constants *independent of* ε.

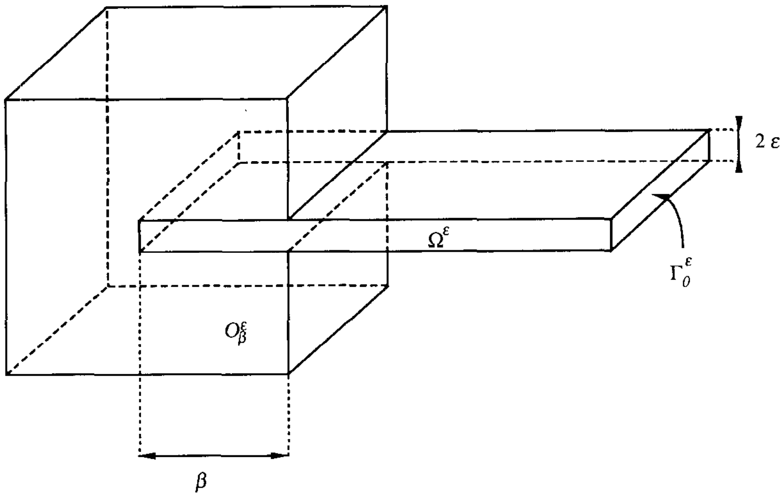

FIG. 1. The three-dimensional elastic structure.

In *linearized elastodynamics*, the *displacement vector field* $\mathbf{w}^\varepsilon = (w_i^\varepsilon)\colon \bar{S}^\varepsilon \times [0, +\infty[\to \mathbb{R}^3$ satisfies the equations

$$\tilde{\rho} \frac{\partial^2 w_i^\varepsilon}{\partial t^2} = \partial_j^\varepsilon \{\tilde{\lambda} e_{pp}(\mathbf{w}^\varepsilon)\, \delta_{ij} + 2\tilde{\mu} e_{ij}(\mathbf{w}^\varepsilon)\} \qquad \text{in } O_\beta^\varepsilon \text{ for all } t \geqslant 0,$$

$$\rho^\varepsilon \frac{\partial w_i^\varepsilon}{\partial t^2} = \partial_j^\varepsilon \{\lambda^\varepsilon e_{pp}(\mathbf{w}^\varepsilon)\, \delta_{ij} + 2\mu^\varepsilon e_{ij}(\mathbf{w}^\varepsilon)\} \qquad \text{in } \Omega^\varepsilon \text{ for all } t \geqslant 0,$$

if there are no applied body or surface forces, where $e_{ij}(\mathbf{w}^\varepsilon) = \frac{1}{2}(\partial_i^\varepsilon w_j^\varepsilon + \partial_j^\varepsilon w_i^\varepsilon)$ denote the components of the linearized strain tensor.

The problem of finding *stationary solutions* of these equations, i.e., solutions of the particular forms (cf., e.g., Courant and Hilbert [23, p. 308 ff.])

$$\mathbf{w}^\varepsilon(x^\varepsilon, t) = \mathbf{u}^\varepsilon(x^\varepsilon) \cos \sqrt{\Lambda^\varepsilon}\, t \qquad \text{or} \qquad \mathbf{u}(x^\varepsilon) \sin \sqrt{\Lambda^\varepsilon}\, t, \qquad x^\varepsilon \in S^\varepsilon,\ t \geqslant 0,$$

where Λ^ε is > 0, thus reduces to finding the associated *eigenvalues* Λ^ε and *eigenfunctions* $\mathbf{u}^\varepsilon$, which satisfy

$$-\tilde{\rho} \Lambda^\varepsilon u_i^\varepsilon = \partial_j^\varepsilon \{\tilde{\lambda} e_{pp}(\mathbf{u}^\varepsilon)\, \delta_{ij} + 2\tilde{\mu} e_{ij}(\mathbf{u}^\varepsilon)\} \qquad \text{in } 0_\beta^\varepsilon,$$

$$-\rho^\varepsilon \Lambda^\varepsilon u_i^\varepsilon = \partial_j^\varepsilon \{\lambda^\varepsilon e_{pp}(\mathbf{u}^\varepsilon)\, \delta_{ij} + 2\mu^\varepsilon e_{ij}(\mathbf{u}^\varepsilon)\} \qquad \text{in } \Omega^\varepsilon.$$

We shall further assume that the displacement vector field satisfies a *boundary condition of place* $\mathbf{w}^\varepsilon = \mathbf{0}$ on $\Gamma_0^\varepsilon \times [0, +\infty[$, so that we must also have

$\mathbf{u}^\varepsilon = \mathbf{0}$ on Γ_0. In this fashion, we find that each *eigensolution* $(\Lambda^\varepsilon, \mathbf{u}^\varepsilon)$ solves the variational equations

$$B^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) = \Lambda^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon)^\varepsilon, \qquad \text{for all} \quad \mathbf{v}^\varepsilon \in \mathbf{V}^\varepsilon, \tag{2.2}$$

where

$$B^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) = \int_{0^\varepsilon_\beta} \{\tilde{\lambda} e_{pp}(\mathbf{u}^\varepsilon)\, e_{qq}(\mathbf{v}^\varepsilon) + 2\tilde{\mu} e_{ij}(\mathbf{u}^\varepsilon)\, e_{ij}(\mathbf{v}^\varepsilon)\}\, dx^\varepsilon$$
$$+ \int_{\Omega^\varepsilon} \{\lambda^\varepsilon e_{pp}(\mathbf{u}^\varepsilon)\, e_{qq}(\mathbf{v}^\varepsilon) + 2\mu^\varepsilon e_{ij}(\mathbf{u}^\varepsilon)\, e_{ij}(\mathbf{v}^\varepsilon)\}\, dx^\varepsilon, \tag{2.3}$$

$$(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon)^\varepsilon = \int_{0^\varepsilon_\beta} \tilde{\rho} u_i^\varepsilon v_i^\varepsilon\, dx^\varepsilon + \int_{\Omega^\varepsilon} \rho^\varepsilon u_i^\varepsilon v_i^\varepsilon\, dx^\varepsilon, \tag{2.4}$$

and where the space $\mathbf{V}^\varepsilon$ is defined by

$$\mathbf{V}^\varepsilon = \{\mathbf{v}^\varepsilon = (v_i^\varepsilon) \in \mathbf{H}^1(S^\varepsilon);\ \mathbf{v}^\varepsilon = \mathbf{0} \text{ on } \Gamma_0^\varepsilon\}. \tag{2.5}$$

The positiveness of the Lamé constants $\tilde{\lambda}$, $\tilde{\mu}$, λ^ε, μ^ε, Korn's inequality, and the boundary condition of place together imply that the symmetric bilinear form $\mathbf{B}^\varepsilon(\cdot,\cdot)$ defined in (2.3), is coercive over the space $\mathbf{V}^\varepsilon$ of (2.5). This property and the compactness of the injection from $\mathbf{V}^\varepsilon$ into $\mathbf{L}^2(\Omega^\varepsilon)$ imply that the symmetric mapping

$$\mathbf{G}^\varepsilon\colon \mathbf{u}^\varepsilon \in \mathbf{V}^\varepsilon \to \mathbf{G}^\varepsilon\mathbf{u}^\varepsilon \in \mathbf{V}^\varepsilon$$

defined by

$$B^\varepsilon(\mathbf{G}^\varepsilon\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) = (\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon)^\varepsilon \qquad \text{for all} \quad \mathbf{v}^\varepsilon \in \mathbf{V}^\varepsilon$$

is compact and positive definite. By the spectral theory of such operators (see e.g. Taylor [52, Chap. 6] and Dautray and Lions [24, p. 51]), all the *eigenvalues* $\Lambda^{l,\varepsilon}$, $l \geqslant 1$, of this problem are > 0. They can be arranged to satisfy

$$0 < \Lambda^{1,\varepsilon} \leqslant \Lambda^{2,\varepsilon} \leqslant \cdots \leqslant \Lambda^{l,\varepsilon} \leqslant \Lambda^{l+1,\varepsilon} \leqslant \cdots, \qquad \text{with } \lim_{l\to\infty} \Lambda^{l,\varepsilon} = +\infty; \tag{2.6}$$

there exists an associated sequence of *eigenfunctions* $\mathbf{u}^{l,\varepsilon} \in \mathbf{V}^\varepsilon$, $l \geqslant 1$, i.e., which satisfy

$$B^\varepsilon(\mathbf{u}^{l,\varepsilon}, \mathbf{v}^\varepsilon) = \Lambda^{l,\varepsilon}(\mathbf{u}^{l,\varepsilon}, \mathbf{v}^\varepsilon)^\varepsilon \qquad \text{for all} \quad \mathbf{v}^\varepsilon \in \mathbf{V}^\varepsilon,\ l \geqslant 1, \tag{2.7}$$

and which form a complete orthogonal set in both Hilbert spaces $\mathbf{V}^\varepsilon$ and

$\mathbf{L}^2(\Omega^\varepsilon)$. We shall further assume here that the functions $\mathbf{u}^{l,\varepsilon}$ are orthonormalized in such a way that

$$B^\varepsilon(\mathbf{u}^{k,\varepsilon}, \mathbf{u}^{l,\varepsilon}) = \varepsilon^2 \Lambda^{k,\varepsilon}\, \delta_{kl}, \qquad \text{and thus} \qquad (\mathbf{u}^{k,\varepsilon}, \mathbf{u}^{l,\varepsilon})^\varepsilon = \varepsilon^2\, \delta_{kl}, \qquad 1 \leqslant k, l. \tag{2.8}$$

Note that the numbers $\Lambda^{l,\varepsilon}$ and the functions $\mathbf{u}^{l,\varepsilon}$ are respectively the inverses of the eigenvalues and of the eigenfunctions of the operator $\mathbf{G}^\varepsilon$.

Consider the *Rayleigh quotient*

$$R^\varepsilon(\mathbf{v}^\varepsilon) = \frac{B^\varepsilon(\mathbf{v}^\varepsilon, \mathbf{v}^\varepsilon)}{(\mathbf{v}^\varepsilon, \mathbf{v}^\varepsilon)^\varepsilon}, \tag{2.9}$$

which is defined for all $\mathbf{v}^\varepsilon \in \mathbf{V}^\varepsilon - \{\mathbf{0}\}$. Then the eigenvalues $\Lambda^{l,\varepsilon}$ satisfy the *minimum principle* (cf. Courant and Hilbert [23, Chap. 6], Dautray and Lions [24, p. 123])

$$\begin{cases} \Lambda^{0,\varepsilon} = \min\{R^\varepsilon(\mathbf{v}^\varepsilon);\ \mathbf{v}^\varepsilon \in \mathbf{V}^\varepsilon - \{\mathbf{0}\}\}, \\ \Lambda^{l,\varepsilon} = \min\{R^\varepsilon(\mathbf{v}^\varepsilon);\ \mathbf{v}^\varepsilon \in \mathbf{V}^\varepsilon - \{\mathbf{0}\};\ (\mathbf{v}^\varepsilon, \mathbf{u}^{k,\varepsilon})^\varepsilon = 0,\ 1 \leqslant k \leqslant l-1\},\ l \geqslant 1, \end{cases} \tag{2.10}$$

and the *min-max principle*, (cf. Poincaré [46], Weinberger [56])

$$\Lambda^{l,\varepsilon} = \min_{U^\varepsilon \in \mathscr{V}^{l,\varepsilon}} \{\max R^\varepsilon(\mathbf{v}^\varepsilon);\ \mathbf{v}^\varepsilon \in U^\varepsilon\}, \tag{2.11}$$

respectively, where $\mathscr{V}^{l,\varepsilon}$ denotes for each integer $l \geqslant 1$ the family of all subspaces of dimension l of $\mathbf{V}^\varepsilon$.

If we let

$$\begin{aligned} \tilde{\mathbf{A}} &= (\tilde{\lambda}\, \delta_{ij}\, \delta_{kl} + \tilde{\mu}\{\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}\}), \\ \mathbf{A}^\varepsilon &= \varepsilon^{-3}\mathbf{A}, \\ \mathbf{A} &= (\lambda\, \delta_{ij}\, \delta_{kl} + \mu\{\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}\}), \\ (\mathbf{Be})_{ij} &= b_{ijkl} e_{kl} \qquad \text{if} \quad \mathbf{B} = (b_{ijkl}),\ \mathbf{e} = (e_{ij}), \end{aligned}$$

we find that each pair $(\Lambda^{l,\varepsilon}, \mathbf{u}^{l,\varepsilon})$ is also, at least formally, the solution of a classical *eigenvalue problem* of three-dimensional linearized elasticity, which takes here the form

$$-\mathbf{div}^\varepsilon\{\tilde{\mathbf{A}}\mathbf{e}(\mathbf{u}^\varepsilon)\} = \tilde{\rho}\Lambda^\varepsilon \mathbf{u}^\varepsilon \qquad \text{in } 0^\varepsilon_\beta, \tag{2.12}$$

$$-\mathbf{div}^\varepsilon\{\mathbf{A}^\varepsilon\mathbf{e}(\mathbf{u}^\varepsilon)\} = \rho^\varepsilon\Lambda^\varepsilon \mathbf{u}^\varepsilon \qquad \text{in } \Omega^\varepsilon, \tag{2.13}$$

$$\mathbf{u}^\varepsilon = \mathbf{0} \qquad \text{on } \Gamma^\varepsilon_0, \tag{2.14}$$

$$\tilde{\mathbf{A}}\mathbf{e}(\mathbf{u}^\varepsilon)\,\tilde{\mathbf{n}}^\varepsilon = \mathbf{0} \qquad \text{on } \partial O^\varepsilon_\beta - \partial\Omega^\varepsilon, \tag{2.15}$$

$$\mathbf{A}^\varepsilon\mathbf{e}(\mathbf{u}^\varepsilon)\,\mathbf{n}^\varepsilon = \mathbf{0} \qquad \text{on } \partial\Omega^\varepsilon - \partial O^\varepsilon_\beta, \tag{2.16}$$

$$(\mathbf{u}^\varepsilon_{|0^\varepsilon_\beta})_{|\partial 0^\varepsilon_\beta \cap \partial\Omega^\varepsilon} = (\mathbf{u}^\varepsilon_{|\Omega^\varepsilon})_{|\partial O^\varepsilon_\beta \cap \partial\Omega^\varepsilon}, \tag{2.17}$$

$$\tilde{\mathbf{A}}\mathbf{e}(\mathbf{u}^\varepsilon)\,\tilde{\mathbf{n}}^\varepsilon + \mathbf{A}^\varepsilon\mathbf{e}(\mathbf{u}^\varepsilon)\,\mathbf{n}^\varepsilon = \mathbf{0} \qquad \text{on } \partial 0^\varepsilon_\beta \cap \partial\Omega^\varepsilon, \tag{2.18}$$

where

$$(\mathbf{div}^\varepsilon\,\mathbf{a}^\varepsilon)_i = \partial^\varepsilon_j\,a^\varepsilon_{ij} \qquad \text{if} \quad \mathbf{a}^\varepsilon = (a^\varepsilon_{ij}),$$

$\tilde{\mathbf{n}}^\varepsilon$ and $\mathbf{n}^\varepsilon$ denote the unit outer normal vectors along the boundaries of the sets 0^ε_β and Ω^ε_β, respectively, and $w_{|B}$ denotes the restriction of a function w to a set B.

Relations (2.12) and (2.13) show that each solution $\mathbf{u}^{l,\varepsilon}$ is indeed an *eigenfunction* associated with the *eigenvalue* $\Lambda^{l,\varepsilon}$. Relations (2.17) and (2.18), which formally express the continuity of the displacement vectors and the stress vectors along the common portion of the two boundaries, are called *transmission conditions*; details about such transmission problems are found in Dautray and Lions [24, p. 1245]. Relation (2.17) shows that we are modeling a situation where *the inserted portion of the plate is glued to the three-dimensional structure*: we are thus excluding here situations where the inserted portion could slide along or part away from the three-dimensional structure.

3. Equivalent Formulation of the Three-Dimensional Problem over Two Open Sets Independent of ε

With the sets Ω^ε and 0, *which overlap over the "inserted" part Ω^ε_β of the "thin" part Ω^ε, we associate two disjoint sets Ω and $\tilde{\Omega}$*, as follows: First, as in the case of a single plate (cf. Ciarlet and Destuynder [15]), we let $\Omega = \omega \times]-1, 1[$; with each point $x^\varepsilon = (x_1, x_2, x^\varepsilon_3) \in \bar{\Omega}^\varepsilon$, we associate the point $x = (x_1, x_2, \varepsilon^{-1}x^\varepsilon_3) \in \bar{\Omega}$ (cf. Fig. 2); finally with the restriction (still denoted) $\mathbf{u}^\varepsilon = (u^\varepsilon_i)$: $\bar{\Omega}^\varepsilon \to \mathbb{R}^3$ of the unknown $\mathbf{u}^\varepsilon$ to the set $\bar{\Omega}^\varepsilon$, we associate the function $\mathbf{u}(\varepsilon) = (u_i(\varepsilon))$: $\bar{\Omega} \to \mathbb{R}^3$ defined by the *scalings*

$$u^\varepsilon_\alpha(x^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(x) \qquad \text{for all} \quad x^\varepsilon \in \bar{\Omega}^\varepsilon, \tag{3.1}$$

$$u^\varepsilon_3(x^\varepsilon) = \varepsilon u_3(\varepsilon)(x) \qquad \text{for all} \quad x^\varepsilon \in \bar{\Omega}^\varepsilon. \tag{3.2}$$

Second, we define the translated set $\tilde{\Omega} = 0 + \mathbf{t}$, the vector $\mathbf{t}$ being such that $\{\tilde{\Omega}\}^- \cap \bar{\Omega} = \varnothing$. Then, with each point $x^\varepsilon \in \bar{0}$, we associate the translated point $\tilde{x} = (x^\varepsilon + \mathbf{t}) \in \{\tilde{\Omega}\}^-$ (cf. Fig. 2) and with the restriction (still

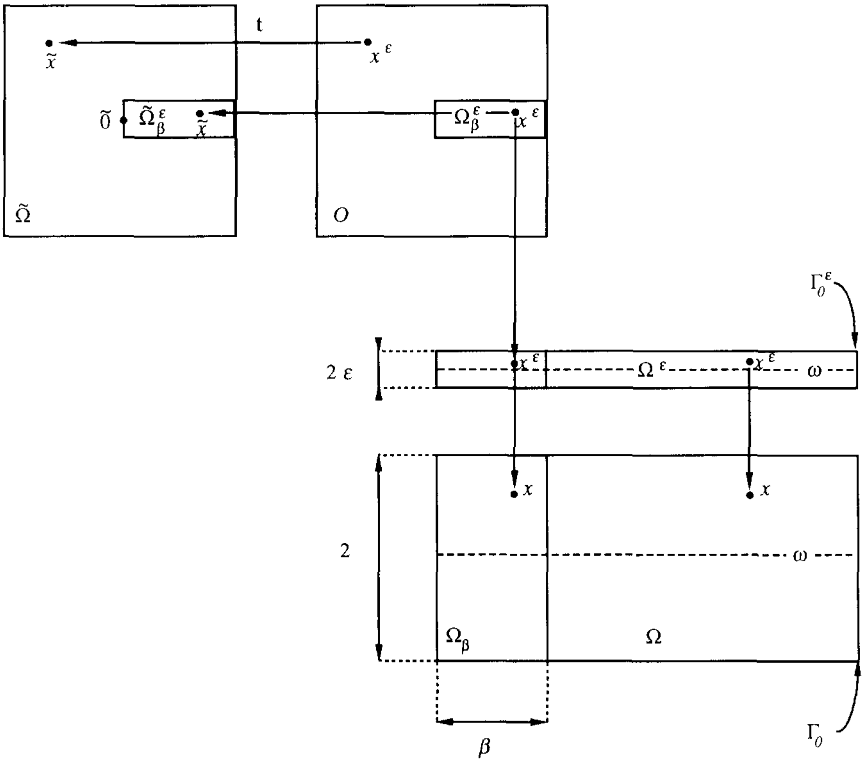

FIG. 2. The sets $\bar{\Omega}^\varepsilon$ and $\bar{0}$, which are respectively occupied by the "thin" part and the "three-dimensional" part of the elastic structure, are mapped into two disjoint sets $\bar{\Omega}$ and $\{\tilde{\Omega}\}^-$. The "inserted" part $\bar{\Omega}^\varepsilon_\beta$ of the thin part is thus mapped twice, once onto $\bar{\Omega}_\beta \subset \bar{\Omega}$ and once onto $\{\tilde{\Omega}^\varepsilon_\beta\}^- \subset \{\tilde{\Omega}\}^-$.

denoted) $\mathbf{u}^\varepsilon = (u^\varepsilon_i): \bar{0} \to \mathbb{R}^3$ of the unknown $\mathbf{u}^\varepsilon$ to the set $\bar{0}$, we associate the function $\tilde{\mathbf{u}}(\varepsilon) = (\tilde{u}_i(\varepsilon)): \{\tilde{\Omega}\}^- \to \mathbb{R}^3$ defined by the scalings

$$u^\varepsilon_i(x^\varepsilon) = \varepsilon \tilde{u}_i(\varepsilon)(\tilde{x}) \qquad \text{for all} \quad x^\varepsilon \in \bar{0}. \tag{3.3}$$

The function $\mathbf{u}^\varepsilon \in \mathbf{V}^\varepsilon$, where $\mathbf{V}^\varepsilon$ is the space defined in (2.3), is thus mapped through relations (3.1)–(3.3) into a function $(\mathbf{u}(\varepsilon), \mathbf{u}(\varepsilon))$ which belongs to the space $\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega)$, which satisfies the boundary condition $\mathbf{u}(\varepsilon) = \mathbf{0}$ on $\Gamma_0 = \gamma_0 \times]-1, 1[$, and which satisfies the *junction conditions for the three-dimensional problem*

$$\tilde{u}_\alpha(\varepsilon)(\tilde{x}) = \varepsilon u_\alpha(\varepsilon)(x), \tag{3.4}$$

$$\tilde{u}_3(\varepsilon)(\tilde{x}) = u_3(\varepsilon)(x), \tag{3.5}$$

at each corresponding point $\tilde{x} \in \tilde{\Omega}^{\varepsilon}_{\beta} = \Omega^{\varepsilon}_{\beta} + \mathbf{t}$ and $x \in \Omega_{\beta} = \omega_{\beta} \times]-1, 1[$, i.e., that correspond to the same point $x^{\varepsilon} \in \Omega^{\varepsilon}_{\beta}$ (Fig. 2).

Using the assumptions (2.1) on the data and the scalings (3.1)–(3.3), we can thus re-formulate the variational problem (2.2) in the following *equivalent* form: The function $(\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon))$ *constructed in* (3.1)–(3.3) *belongs to the space*

$$\begin{aligned} \mathbf{V}(\varepsilon) \overset{\text{def}}{=} \{ & (\tilde{\mathbf{v}}, \mathbf{v}) \in \mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega); \\ & \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0,\ \tilde{v}_\alpha(\tilde{x}) = \varepsilon v_\alpha(x),\ \tilde{v}_3(\tilde{x}) = v_3(x) \\ & \text{at all corresponding points } \tilde{x} \in \tilde{\Omega}^{\varepsilon}_{\beta} \text{ and } x \in \Omega_\beta \}, \end{aligned} \tag{3.6}$$

and $(\Lambda^{\varepsilon}, (\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)))$ *satisfies the variational equations*

$$\begin{aligned} & \int_{\tilde{\Omega}} \chi(\tilde{0}^{\varepsilon}_{\beta}) \{ \tilde{\lambda} e_{pp}(\tilde{\mathbf{u}}(\varepsilon))\, e_{qq}(\tilde{\mathbf{v}}) + 2\tilde{\mu} e_{ij}(\tilde{\mathbf{u}}(\varepsilon))\, e_{ij}(\tilde{\mathbf{v}}) \}\, d\tilde{x} \\ & \quad + \int_{\Omega} \{ \lambda e_{\alpha\alpha}(\mathbf{u}(\varepsilon))\, e_{\beta\beta}(\mathbf{v}) + 2\mu e_{\alpha\beta}(\mathbf{u}(\varepsilon))\, e_{\alpha\beta}(\mathbf{v}) \}\, \mathrm{dx} \\ & \quad + \frac{1}{\varepsilon^2} \int_{\Omega} \{ \lambda [e_{\alpha\alpha}(\mathbf{u}(\varepsilon))\, e_{33}(\mathbf{v}) + e_{33}(\mathbf{u}(\varepsilon))\, e_{\alpha\alpha}(\mathbf{v})] \\ & \qquad + 4\mu e_{\alpha 3}(\mathbf{u}(\varepsilon))\, e_{\alpha 3}(\mathbf{v}) \}\, dx \\ & \quad + \frac{1}{\varepsilon^4} \int_{\Omega} (\lambda + 2\mu)\, e_{33}(\mathbf{u}(\varepsilon))\, e_{33}(\mathbf{v})\, dx \\ & = \Lambda^{\varepsilon} \left\{ \int_{\tilde{\Omega}} \chi(\tilde{0}^{\varepsilon}_{\beta})\, \tilde{\rho} \tilde{u}_i(\varepsilon)\, \tilde{v}_i\, d\tilde{x} + \varepsilon^2 \int_{\Omega} \rho u_\alpha(\varepsilon)\, v_\alpha\, dx + \int_{\Omega} \rho u_3(\varepsilon)\, v_3\, dx \right\} \\ & \qquad \text{for all} \quad (\tilde{\mathbf{v}}, \mathbf{v}) \in \mathbf{V}(\varepsilon), \end{aligned} \tag{3.7}$$

where $\chi(A)$ denotes the characteristic function of a set A and $\tilde{0}^{\varepsilon}_{\beta} = 0^{\varepsilon}_{\beta} + \mathbf{t}$.

In this fashion, to each *eigensolution* $(\Lambda^{l,\varepsilon}, \mathbf{u}^{l,\varepsilon})$, $l \geqslant 1$, of (2.7) there corresponds an *eigensolution* $(\Lambda^l(\varepsilon), (\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon))$ of (3.7), where

$$\Lambda^l(\varepsilon) = \Lambda^{l,\varepsilon}, \tag{3.8}$$

and where the *eigenfunction* $(\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon)) \in \mathbf{V}(\varepsilon)$ associated with the *eigenvalue* $\Lambda^l(\varepsilon)$ is constructed from $\mathbf{u}^{l,\varepsilon} \in \mathbf{V}^{\varepsilon}$ as in (3.1)–(3.3). Note that, in view of (2.8), these eigenfunctions satisfy

$$\begin{aligned} & \int_{\tilde{\Omega}} \chi(\tilde{0}^{\varepsilon}_{\beta})\, \tilde{\rho} u^k_i(\varepsilon)\, u^l_i(\varepsilon)\, d\tilde{x} + \varepsilon^2 \int_{\Omega} \rho u^k_\alpha(\varepsilon)\, u^l_\alpha(\varepsilon)\, dx \\ & \quad + \int_{\Omega} \rho u^k_3(\varepsilon)\, u^l_3(\varepsilon)\, dx = \delta_{kl}, \qquad k, l \geqslant 1. \end{aligned} \tag{3.9}$$

The reason we introduce a new notation $\Lambda^l(\varepsilon)$ for the eigenvalues (cf. (3.8)) is a reminder that the eigenvalues may be also scaled in certain situations (see, e.g., Ciarlet and Kesavan, [17, Eq. (3.5)]).

4. Convergence of $(\Lambda^l(\varepsilon), (\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l\varepsilon))$, $l \geqslant 1$, as $\varepsilon \to 0$

We use the following notations: The norms of the space $L^2(A)$ and of the Sobolev spaces $H^m(A)$, $m \geqslant 1$, where A is an open subset in $\mathbb{R}^n$, are respectively denoted $|\cdot|_{0,A}$ and $\|\cdot\|_{m,A}$; the same notations are also used for the norms of the spaces $\mathbf{L}^2(A)$ and $\mathbf{H}^m(A)$ (whose elements are vector-valued functions). Strong and weak convergences are respectively denoted $\to$ and $\rightharpoonup$.

In (4.1) and subsequently, $\tilde{\omega}_\beta$ denotes the translated set $(\omega_\beta + \mathbf{t})$; $w_{|A}$ denotes the trace of a function w on the set A in the sense of Sobolev spaces (for instance, the trace $\tilde{v}_{3|\tilde{\omega}_\beta}$ is to be understood as a function in the space $H^{1/2}(\tilde{\omega}_\beta)$, etc.); the equality $\tilde{v}_{3|\tilde{\omega}_\beta} = \eta_{3|\omega_\beta}$ is to be understood as holding up to a translation by the vector $\mathbf{t}$; finally, ∂_ν denotes the outer normal derivative operator along $\partial\omega$.

We now show that, *for each integer* $l \geqslant 1$, *the family* $(\Lambda^l(\varepsilon), (\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon)))$, $\varepsilon > 0$ *of eigensolutions, orthonormalized as in* (3.9), *converges to a limit* $(\Lambda^l, (\tilde{\mathbf{u}}^l, \mathbf{u}^l))$ *in the space* $]0, +\infty[\times \mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega)$ *as* $\varepsilon \to 0$, *which is precisely the* lth *eigensolution of a "limit" variational problem.* Note that the next theorem contains as a special case the convergence proof established in Ciarlet and Kesavan [17] for a "single plate."

Theorem 1. (a) *Define the space*

$$[\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta = \{(\tilde{\mathbf{v}}, \eta_3) \in \mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega);\ \eta_3 = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0,$$
$$\tilde{v}_{3|\tilde{\omega}_\beta} = \eta_{3|\omega_\beta},\ \tilde{v}_{\alpha|\tilde{\omega}_\beta} = 0\}, \tag{4.1}$$

and consider the eigenvalue problem: Find all solutions $(\Lambda, (\tilde{\mathbf{u}}, \zeta_3)) \in\ ]0, +\infty[\times [\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta$ *of the variational equations*

$$\int_{\tilde{\Omega}} \{\tilde{\lambda} e_{pp}(\tilde{\mathbf{u}})\, e_{qq}(\tilde{\mathbf{v}}) + 2\tilde{\mu} e_{ij}(\tilde{\mathbf{u}})\, e_{ij}(\tilde{\mathbf{v}})\}\, d\tilde{x}$$
$$- \int_\omega m_{\alpha\beta}(\zeta_3)\, \partial_{\alpha\beta}\, \eta_3\, d\omega = \Lambda \left\{ \int_{\tilde{\Omega}} \tilde{\rho} \tilde{u}_i \tilde{v}_i\, d\tilde{x} + 2 \int_\omega \rho \zeta_3 \eta_3\, d\omega \right\},$$
$$\textit{for all} \quad (\tilde{\mathbf{v}}, \eta_3) \in [\mathbf{H}^1(\Omega) \times H^2(\omega)]_\beta, \tag{4.2}$$

where

$$m_{\alpha\beta}(\zeta_3) \stackrel{\text{def}}{=} -\frac{4\mu}{3} \left\{ \partial_{\alpha\beta}\, \zeta_3 + \frac{\lambda}{\lambda + 2\mu}\, \Delta\zeta_3\, \delta_{\alpha\beta} \right\}. \tag{4.3}$$

This problem has an infinite sequence of eigenvalues Λ^l, $l \geqslant 1$ which can be arranged to satisfy

$$0 < \Lambda^1 \leqslant \Lambda^2 \leqslant \cdots \leqslant \Lambda^l \leqslant \Lambda^{l+1} \leqslant \ldots, \qquad \textit{with} \quad \lim_{l \to \infty} \Lambda^l = +\infty. \tag{4.4}$$

(b) *For each integer $l \geqslant 1$,*

$$\lim \Lambda^l(\varepsilon) = \Lambda^l \qquad \textit{as} \quad \varepsilon \to 0. \tag{4.5}$$

(c) *If Λ^l is a simple eigenvalue of Problem* (4.2), *there exists $\varepsilon_0(l) > 0$ such that, for all $\varepsilon \leqslant \varepsilon_0(l)$, $\Lambda^l(\varepsilon)$ is also a simple eigenvalue of Problem* (3.7) *and there exists an eigenfunction $(\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon))$ associated with $\Lambda^l(\varepsilon)$, normalized as in* (3.9), *that converges in the space $\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega)$ to a limit $(\tilde{\mathbf{u}}^l, \mathbf{u}^l)$, possessing the following properties: There exists a function $\zeta_3^l \in H^2(\omega)$ satisfying $\zeta_3^l = \partial_\nu \zeta_3^l = 0$ on γ_0 such that*

$$u_\alpha^l(x_1, x_2, x_3) = -x_3\, \partial_\alpha \zeta_3^l(x_1, x_2) \qquad \textit{for all} \quad (x_1, x_2, x_3) \in \bar{\Omega}, \tag{4.6}$$

$$u_3^l(x_1, x_2, x_3) = \zeta_3^l(x_1, x_2) \qquad \textit{for all} \quad (x_1, x_2, x_3) \in \bar{\Omega}, \tag{4.7}$$

and the pair $(\mathbf{u}^l, \zeta_3^l)$ is an eigenfunction of Problem (4.2), *associated with the eigenvalue Λ^l.*

(d) *If Λ^l is not a simple eigenvalue of Problem* (4.2), *there exists a subsequence of eigenfunctions associated with the eigenvalues $\Lambda^l(\varepsilon)$ that satisfy the conclusions of Part* (c).

(e) *The eigenfunctions, obtained as in* (c) *or* (d), *form a complete set in both spaces $[\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta$ and $\mathbf{L}^2(\tilde{\Omega}) \times L^2(\omega)$, and they satisfy*

$$\int_{\tilde{\Omega}} \tilde{\rho} \tilde{u}_i^k \tilde{u}_i^l \, d\tilde{x} + 2 \int_\omega \rho \zeta_3^k \zeta_3^l \, d\omega = \delta_{kl}, \qquad k, l \geqslant 1. \tag{4.8}$$

∎

The proof of Theorem 1 is long and technical and for these reasons, is broken into a series of lemmas (Lemma 1 to Lemma 12). As is usually the case in asymptotic analysis, the first, and crucial, step consists in obtaining *a priori* bounds independent of the parameter. This is the object of Lemmas 1 and 2, where we show that for each l, the family $(\Lambda^l(\varepsilon), (\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon)))_{\varepsilon > 0}$ is bounded in the space $]0, +\infty[\times (\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega))$.

LEMMA 1. *For each integer $l \geqslant 1$, the family $(\Lambda^l(\varepsilon))_{\varepsilon > 0}$ is bounded.*

Proof. We first transform the Rayleigh quotient $R^\varepsilon(\mathbf{v}^\varepsilon)$ of (2.9) into a quotient $R(\varepsilon)(\tilde{\mathbf{v}}, \mathbf{v})$ expressed in terms of the functions $(\tilde{\mathbf{v}}, \mathbf{v}) \in \mathbf{V}(\varepsilon)$, where

$\mathbf{V}(\varepsilon)$ is the space defined in (3.6), and where the functions $(\tilde{\mathbf{v}}, \mathbf{v})$ are derived from the functions $\mathbf{v}^\varepsilon \in \mathbf{V}^\varepsilon$ by the same formulas as in (3.1)–(3.3), viz.,

$$v_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 v_\alpha(x) \quad \text{and} \quad v_3^\varepsilon(x^\varepsilon) = \varepsilon v_3(x) \quad \text{for all} \quad x^\varepsilon \in \bar{\Omega}^\varepsilon, \tag{4.9}$$

$$v_i^\varepsilon(x^\varepsilon) = \varepsilon^2 v_i(\tilde{x}) \quad \text{for all} \quad x^\varepsilon \in \bar{0}. \tag{4.10}$$

In so doing, we also take advantage of the fact that the points x^ε in the set $\bar{\Omega}_\beta^\varepsilon = \bar{\Omega}^\varepsilon \cap \bar{0}$ may be mapped either by (4.9) or by (4.10): We split the integral over the set Ω_β^ε appearing in the *denominator* (cf. (4.13) below) of the quotient $R(\varepsilon)(\tilde{\mathbf{v}}, \mathbf{v})$ into two equal (for definiteness) parts; one part is then mapped as an integral over the set $\tilde{\Omega}_\beta^\varepsilon$, and the other as an integral over the set Ω_β. In this fashion, we find that

$$R^\varepsilon(\mathbf{v}^\varepsilon) = R(\varepsilon)(\tilde{\mathbf{v}}, \mathbf{v}) \overset{\text{def}}{=} \frac{N(\varepsilon)(\tilde{\mathbf{v}}, \mathbf{v})}{D(\varepsilon)(\tilde{\mathbf{v}}, \mathbf{v})}, \tag{4.11}$$

where

$$N(\varepsilon)(\tilde{\mathbf{v}}, \mathbf{v}) = \int_{\tilde{\Omega}} \chi(\tilde{0}_\beta^\varepsilon)\, \tilde{\mathbf{A}}\mathbf{e}(\tilde{\mathbf{v}}) : e(\tilde{\mathbf{v}})\, d\tilde{x} + \int_\Omega \mathbf{A}\boldsymbol{\kappa}(\mathbf{v}) : \boldsymbol{\kappa}(\mathbf{v})\, dx, \tag{4.12}$$

with

$$\kappa_{\alpha\beta}(\mathbf{v}) = e_{\alpha\beta}(\mathbf{v}), \qquad \kappa_{\alpha 3}(\mathbf{v}) = \kappa_{3\alpha}(\mathbf{v}) = \frac{1}{\varepsilon} e_{\alpha 3}(\mathbf{v}), \qquad \kappa_{33}(\mathbf{v}) = \frac{1}{\varepsilon^2} e_{33}(\mathbf{v}),$$

$$D(\varepsilon)(\tilde{\mathbf{v}}, \mathbf{v}) = \int_{\tilde{\Omega}} \left\{ \chi(\tilde{0}_\beta^\varepsilon)\, \tilde{\rho} + \frac{1}{2\varepsilon} \chi(\tilde{\Omega}_\beta^\varepsilon) \rho \right\} \tilde{\mathbf{v}} \cdot \tilde{\mathbf{v}}\, d\tilde{x}$$
$$+ \int_\Omega \left\{ \frac{1}{2} \chi(\Omega_\beta) + \chi(\Omega - \Omega_\beta) \right\} \rho(\varepsilon^2 v_\alpha v_\alpha + v_3 v_3)\, dx, \tag{4.13}$$

and where we have let

$$\mathbf{a} : \mathbf{b} = a_{ij} b_{ij} \quad \text{if} \quad \mathbf{a} = (a_{ij}),\ b = (b_{ij}).$$
$$\mathbf{u} \cdot \mathbf{v} = u_i v_i \quad \text{if} \quad \mathbf{u} = (u_i),\ \mathbf{v} = (v_i).$$

Hence

$$R(\varepsilon)(\tilde{\mathbf{v}}, \mathbf{v}) \leqslant 2 \frac{\int_{\tilde{\Omega}} \tilde{\mathbf{A}}\mathbf{e}(\tilde{\mathbf{v}}) : \mathbf{e}(\tilde{\mathbf{v}})\, d\tilde{x}}{\int_{\tilde{\Omega}} \tilde{r}\tilde{\mathbf{v}} \cdot \tilde{\mathbf{v}}\, d\tilde{x}} + 2 \frac{\int_\Omega \mathbf{A}\boldsymbol{\kappa}(\mathbf{v}) : \boldsymbol{\kappa}(\mathbf{v})\, dx}{\int_\Omega \rho v_3 v_3\, dx},$$

where $\tilde{r} = \min\{\tilde{\rho}, \rho\}$.

Following Ciarlet and Kesavan [17, Lemma 1], we next use particular "test functions" $(\tilde{\mathbf{v}}, \mathbf{v})$, whose specific form over the set Ω is suggested by the study of the stationary problem. More specifically, we let

$$\begin{aligned}\mathbf{W}(\varepsilon)=\{(\tilde{\mathbf{v}}, \mathbf{v}) \in \mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega);\ &\mathbf{v}=\mathbf{0} \text{ on } \Gamma_0,\ \tilde{\mathbf{v}}=\mathbf{0} \text{ on } \tilde{\Omega}_\beta^\varepsilon,\\ &\mathbf{v}=\mathbf{0} \text{ on } \Omega_\beta,\ v_\alpha=-x_3\,\partial_\alpha \eta_3, \text{ and } v_3=\eta_3 \text{ in } \Omega-\Omega_\beta\\ &\text{with } \eta_3 \in H_0^2(\omega-\omega_\beta)\}.\end{aligned}$$

Clearly, $\mathbf{W}(\varepsilon) \subset \mathbf{V}(\varepsilon)$; hence, by the *min–max principle*,

$$\Lambda^l(\varepsilon) \leqslant \min_{X(\varepsilon) \in \mathscr{X}^l(\varepsilon)} \max\{2\tilde{R}(\tilde{\mathbf{v}})+2R(\eta_3);\ (\tilde{\mathbf{v}}, \eta_3) \in X(\varepsilon)\},$$

where

$$\tilde{R}(\tilde{\mathbf{v}}) \stackrel{\text{def}}{=} \frac{\int_{\tilde{\Omega}} \tilde{\mathbf{A}}\mathbf{e}(\tilde{\mathbf{v}}):\mathbf{e}(\tilde{\mathbf{v}})\,d\tilde{x}}{\int_{\tilde{\Omega}} \tilde{r}\tilde{\mathbf{v}} \cdot \tilde{\mathbf{v}}\,d\tilde{x}}, \qquad R(\eta_3) \stackrel{\text{def}}{=} \frac{\lambda+2\mu}{3} \frac{\int_{\omega-\omega_\beta} \Delta\eta_3\,\Delta\eta_3\,d\omega}{\int_{\omega-\omega_\beta} \rho\eta_3\eta_3\,d\omega},$$

and $\mathscr{X}^l(\varepsilon)$ denotes the family of all subspaces of dimension l of the space $\{\tilde{\mathbf{v}} \in \mathbf{H}^1(\tilde{\Omega});\ \tilde{\mathbf{v}}=\mathbf{0} \text{ on } \tilde{\Omega}_\beta^\varepsilon\} \times H_0^2(\omega-\omega_\beta)$.

Let $(\tilde{\lambda}^k(\varepsilon), \tilde{\mathbf{v}}^k(\varepsilon))_{k \geqslant 1}$ and $(\mu^k, \zeta_3^k)_{k \geqslant 1}$ be the eigensolutions of

$$\begin{aligned}-\widetilde{\mathbf{div}}\,\tilde{\mathbf{A}}\mathbf{e}(\tilde{\mathbf{v}})&=\tilde{\lambda}\tilde{r}\tilde{\mathbf{v}} && \text{in } \tilde{0}_\beta^\varepsilon,\\ \tilde{\mathbf{v}}&=\mathbf{0} && \text{on } \partial\tilde{0}_\beta^\varepsilon \cap \partial\Omega_\beta^\varepsilon,\end{aligned}$$

and

$$\begin{aligned}\frac{\lambda+2\mu}{3} \Delta^2 \zeta_3&=\mu\rho\zeta_3 && \text{in } \omega-\omega_\beta,\\ \zeta_3=\partial_\nu \zeta_3&=0 && \text{on } \partial(\omega-\omega_\beta).\end{aligned}$$

We shall assume (without loss of generality) that the eigenfunctions form complete orthogonal sets, and that

$$0<\tilde{\lambda}^1(\varepsilon) \leqslant \cdots \leqslant \tilde{\lambda}^k(\varepsilon) \leqslant \tilde{\lambda}^{k+1}(\varepsilon) \leqslant \cdots$$

and

$$0<\mu^1 \leqslant \cdots \leqslant \mu^k \leqslant \mu^{k+1} \leqslant \cdots.$$

Then the space $X^l(\varepsilon)$ spanned by the functions $(\tilde{\mathbf{v}}^k(\varepsilon), \zeta_3^k)$, $1 \leqslant k \leqslant l$, is of dimension l, and thus

$$\begin{aligned}\Lambda^l(\varepsilon) &\leqslant \max\{2R(\tilde{\mathbf{v}})+2R(\eta_3);\ (\tilde{\mathbf{v}}, \eta_3) \in X^l(\varepsilon)\}\\ &\leqslant 2 \max\{R(\tilde{\mathbf{v}};\ \mathbf{v} \in \operatorname{span}\{\tilde{\mathbf{v}}^k(\varepsilon)\}_{k=1}^l\}\\ &\quad+2 \max\{R(\eta_3);\ \eta_3 \in \operatorname{span}\{\zeta_3^k\}_{k=1}^l\}=2(\tilde{\lambda}^l(\varepsilon)+\zeta_3^l).\end{aligned}$$

Another application of the *min-max principle* shows that, for each $l \geq 1$, $\tilde{\lambda}^l(\varepsilon)$ is a decreasing function as $\varepsilon \to 0$ since

$$\{\tilde{\mathbf{v}} \in \mathbf{H}^1(\tilde{\Omega}); \tilde{\mathbf{v}} = \mathbf{0} \text{ on } \tilde{\Omega}^\varepsilon_\beta\} \subset \{\tilde{\mathbf{v}} \in \mathbf{H}^1(\tilde{\Omega}), \tilde{\mathbf{v}} = \mathbf{0} \text{ on } \tilde{\Omega}^{\varepsilon'}_\beta\} \qquad \text{if} \quad \varepsilon' \leqslant \varepsilon.$$

Hence, the family $(\Lambda^l(\varepsilon))_{\varepsilon > 0}$ is bounded. ∎

In what follows, any subsequence of a given family will be denoted for notational convenience by the same symbol as the family itself.

LEMMA 2. *For each integer $l \geqslant 1$, the family $(\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon))_{\varepsilon>0}$, orthonormalized as in* (3.9), *is bounded in the space $\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega)$.*

Thus there exists a subsequence which can be chosen to be the same for all integers $l \geqslant 1$, and there exist for each l a number $\Lambda^l \geqslant 0$ and a function $(\tilde{\mathbf{u}}^l, \mathbf{u}^l) \in \mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega)$ such that

$$\Lambda^l(\varepsilon) \to \Lambda^l \qquad \textit{as} \quad \varepsilon \to 0, \tag{4.14}$$

$$\tilde{\mathbf{u}}^l(\varepsilon) \rightharpoonup \tilde{\mathbf{u}}^l \textit{ in } \mathbf{H}^1(\tilde{\Omega}) \qquad \textit{as} \quad \varepsilon \to 0, \tag{4.15}$$

$$\mathbf{u}^l(\varepsilon) \rightharpoonup \mathbf{u}^l \textit{ in } \mathbf{H}^1(\Omega) \textit{ as } \varepsilon \rightharpoonup 0 \qquad \textit{and} \qquad \mathbf{u}^l = \mathbf{0} \textit{ on } \Gamma_0. \tag{4.16}$$

Proof. The trick now consists in splitting the integral over the set Ω^ε_β appearing in the *numerator* of the quotient $R(\varepsilon)(\tilde{\mathbf{v}}, \mathbf{v})$ of (4.11) into two equal (for definiteness) parts; one part is then mapped as an integral over the set $\tilde{\Omega}^\varepsilon_\beta$, and the other is mapped as an integral over the set Ω_β. In this fashion, we find that

$$\begin{aligned}
&\int_{\tilde{\Omega}} \{\chi(\tilde{0}^\varepsilon_\beta)\, \tilde{\mathbf{A}}\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)) : \mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)) \\
&\quad + \frac{1}{2\varepsilon^3} \chi(\tilde{\Omega}^\varepsilon_\beta)\, \mathbf{A}\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)) : \mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon))\}\, d\tilde{x} \\
&\quad + \int_\Omega \{\tfrac{1}{2}\chi(\Omega_\beta) + \chi(\Omega - \Omega_\beta)\}\, \mathbf{A}\boldsymbol{\kappa}(\mathbf{u}^l(\varepsilon)) : \boldsymbol{\kappa}(\mathbf{u}^l(\varepsilon))\, dx = \Lambda^l(\varepsilon),
\end{aligned} \tag{4.17}$$

where we have let

$$\begin{aligned}
&\kappa_{\alpha\beta}(\mathbf{u}^l(\varepsilon)) = e_{\alpha\beta}(\mathbf{u}^l(\varepsilon)), \\
&\kappa_{\alpha 3}(\mathbf{u}^l(\varepsilon)) = \kappa_{3\alpha}(\mathbf{u}^l(\varepsilon)) = \frac{1}{\varepsilon} e_{\alpha 3}(\mathbf{u}^l(\varepsilon)), \\
&\kappa_{33}(\mathbf{u}^l(\varepsilon)) = \frac{1}{\varepsilon^2} e_{33}(\mathbf{u}^l(\varepsilon)).
\end{aligned} \tag{4.18}$$

Since there exists a constant $c=c(\lambda, \mu, \tilde{\lambda}, \tilde{\mu})$ such that

$$c \geqslant 0 \qquad \text{and} \qquad \tilde{\mathbf{A}}\mathbf{e} : \mathbf{e} \geqslant c\mathbf{e} : \mathbf{e}, \ \mathbf{A}\mathbf{e} : \mathbf{e} \geqslant c\mathbf{e} : \mathbf{e} \tag{4.19}$$

for all symmetric tensors $\mathbf{e}=(e_{ij})$, and since without loss of generality, we may restrict ourselves to values of ε that are $\leqslant 1$, we infer from (4.17) and (4.19) that

$$\begin{aligned} |\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon))|^2_{0,\tilde{\Omega}} + |\mathbf{e}(\mathbf{u}^l(\varepsilon))|^2_{0,\Omega} &\leqslant |\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon))|^2_{0,\tilde{\Omega}} + |\boldsymbol{\kappa}(\mathbf{u}^l(\varepsilon))|^2_{0,\Omega} \\ &\leqslant 2c^{-1}A^l(\varepsilon). \end{aligned} \tag{4.20}$$

By Lemma 1 of Ciarlet, Le Dret, and Nzengwa [20], there exists a constant C independent of ε such that

$$\|\tilde{\mathbf{v}}\|^2_{1,\tilde{\Omega}} + \|\mathbf{v}\|^2_{1,\Omega} \leqslant C(|\mathbf{e}(\tilde{\mathbf{v}})|^2_{0,\tilde{\Omega}} + |\mathbf{e}(\mathbf{v})|^2_{0,\Omega}) \quad \text{for all} \quad (\tilde{\mathbf{v}}, \mathbf{v}) \in \mathbf{V}(\varepsilon). \tag{4.21}$$

Hence we conclude from Lemma 1 and inequalities (4.20)–(4.21) that the family $(\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon))_{\varepsilon>0}$ is bounded independently of ε in the space $\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega)$. The other conclusions of Lemma 2 follow from this property (that the subsequence may be chosen to be the same of all integers $l \geqslant 1$ follows from the diagonal procedure). ▮

The next lemmas, 3 to 10, closely follow Lemmas 3 to 10 of Ciarlet, Le Dret, and Nzengwa [20] and for this reason, only the significantly different parts of their proofs will be given.

As in the case of "single" plates (see Destuynder, [25, 27] or Ciarlet and Kesavan [17]), we next show that the weak limit $\mathbf{u}^l \in \mathbf{H}^1(\Omega)$ found in (4.16) is a *Kirchhoff–Love vector field* over the set Ω, in the sense that it belongs to the closed vector space $\mathbf{V}_{KL}(\Omega)$ defined in the next lemma. Note that $\mathbf{V}_{KL}(\Omega)$ is *strictly* contained in the space $\{\mathbf{v} \in \mathbf{H}^1(\Omega);\ \mathbf{v}=\mathbf{0} \text{ on } \Gamma_0\}$.

LEMMA 3. *For each integer $l \geqslant 1$, the function $\mathbf{u}^l$ belongs to the space*

$$\mathbf{V}_{KL}(\Omega) \stackrel{\text{def}}{=} \{\mathbf{v} \in \mathbf{H}^1(\Omega);\ e_{i3}(\mathbf{v})=0 \ \textit{in}\ \Omega,\ \mathbf{v}=\mathbf{0}\ \textit{on}\ \Gamma_0\}, \tag{4.22}$$

which can also be defined as

$$\begin{aligned} \mathbf{V}_{KL}(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega);\ & v_\alpha = \eta_\alpha - x_3\,\partial_\alpha \eta_3,\ v_3=\eta_3, \\ & \text{with } \eta_\alpha \in H^1(\omega),\ \eta_3 \in H^2(\omega),\ \eta_i = \partial_\nu \eta_3 = 0 \ \textit{on}\ \gamma_0\}. \end{aligned} \tag{4.23}$$

Proof. The second inequality in (4.20) shows that the sequence $(\kappa(\mathbf{u}^l(\varepsilon)))_{\varepsilon>0}$ is bounded in the space $\mathbf{L}^2(\Omega)$. Hence (cf. (4.18)) there exists in particular a constant C independent of ε such that

$$|e_{\alpha 3}(\mathbf{u}^l(\varepsilon))|_{0,\Omega} \leqslant C\varepsilon, \qquad |e_{33}(\mathbf{u}^l(\varepsilon))|_{0,\Omega} \leqslant C\varepsilon^2. \tag{4.24}$$

The weak lower semicontinuity of the norm implies that

$$|e_{i3}(\mathbf{u}^l)|_{0,\Omega} \leqslant \liminf_{\varepsilon \to 0} |e_{i3}(\mathbf{u}^l(\varepsilon))|_{0,\Omega} = 0;$$

consequently, $\mathbf{u}^l$ belongs to the space $\mathbf{V}_{KL}(\Omega)$ defined in (4.22). The equivalence between Definitions (4.22) and (4.23) is established as in Ciarlet and Destuynder [15]. ∎

We next identify (cf. (4.25)) the *junction conditions* that the pair $(\tilde{\mathbf{u}}^l, \mathbf{u}^l)$ must satisfy. Note that, in (4.25), the second equality is to be understood *up to a translation by the vector* **t**.

LEMMA 4. *For each integer* $l \geqslant 1$, *the weak limit* $(\tilde{\mathbf{u}}^l, \mathbf{u}^l)$ *satisfies*

$$\tilde{u}^l_{\alpha|\tilde{\omega}_\beta} = 0 \qquad \textit{and} \qquad \tilde{u}^l_{3|\tilde{\omega}_\beta} = u^l_{3|\omega_\beta}. \tag{4.25}$$

Proof. By definition of the space $\mathbf{V}(\varepsilon)$ (cf. (3.6)),

$$\tilde{u}^l_\alpha(\varepsilon)(\tilde{x}) = \varepsilon u^l_\alpha(\varepsilon)(x) \qquad \text{and} \qquad \tilde{u}^l_3(\varepsilon)(\tilde{x}) = u^l_3(\varepsilon)(x)$$

at all corresponding points $\tilde{x} \in \tilde{\Omega}^\varepsilon_\beta$ and $x \in \Omega_\beta$. Hence

$$\tilde{u}^l_\alpha(\varepsilon)_{|\tilde{\omega}_\beta} = \varepsilon u^l_\alpha(\varepsilon)_{|\omega_\beta} \quad \text{and} \quad \tilde{u}^l_3(\varepsilon)_{|\tilde{\omega}_\beta} = u^l_3(\varepsilon)_{|\omega_\beta} \quad \text{for each} \quad \varepsilon > 0. \tag{4.26}$$

Since

$$\tilde{\mathbf{u}}^l(\varepsilon)_{|\tilde{\omega}_\beta} \rightharpoonup \tilde{\mathbf{u}}^l_{|\tilde{\omega}_\beta} \text{ in } \mathbf{H}^{1/2}(\tilde{\omega}_\beta) \qquad \text{and} \qquad \mathbf{u}^l(\varepsilon)_{|\omega_\beta} \rightharpoonup \mathbf{u}^l_{|\omega_\beta} \text{ in } \mathbf{H}^{1/2}(\omega_\beta) \tag{4.27}$$

(the trace operators from $H^1(\tilde{\Omega})$ onto $H^{1/2}(\tilde{\omega}_\beta)$ and from $H^1(\Omega)$ onto $H^{1/2}(\omega_\beta)$ are strongly continuous, and a linear mapping that is strongly continuous is also continuous with respect to the weak topology; cf., e.g., Brezis [7, p. 39]), the second equality in (4.25) follows from the second equality in (4.26) and from (4.27).

Since $(u^l_\alpha(\varepsilon)_{|\omega_\beta})_{\varepsilon > 0}$ is a weakly convergent sequence, it is bounded; therefore the sequence $(\varepsilon u^l_\alpha(\varepsilon)_{|\omega_\beta})_{\varepsilon > 0}$ converges strongly to 0 in the space $H^{1/2}(\tilde{\omega}_\beta)$. This fact, combined with the first equality of (4.26) and with (4.27), implies that the first equality of (4.25) holds. ∎

In the next two lemmas, we state two technical results, which play a key role in the identification of the "limit" variational problem solved by the pair $(\tilde{\mathbf{u}}, \mathbf{u})$.

LEMMA 5. *There exists a subsequence, which can be chosen to be the same for all integers* $l \geqslant 1$, *such that*

$$\kappa_{\alpha 3}(\mathbf{u}^l(\varepsilon)) = \frac{1}{\varepsilon}\, e_{\alpha 3}(\mathbf{u}^l(\varepsilon)) \rightharpoonup 0 \qquad \textit{in } L^2(\Omega), \tag{4.28}$$

$$\kappa_{33}(\mathbf{u}^l(\varepsilon)) = \frac{1}{\varepsilon^2}\, e_{33}(\mathbf{u}^l(\varepsilon)) \rightharpoonup -\frac{\lambda}{(\lambda+2\mu)}\, e_{\gamma\gamma}(\mathbf{u}^l) \qquad \textit{in } L^2(\Omega). \tag{4.29}$$

Proof. The proof is *verbatim* that of Lemma 5 of Ciarlet, Le Dret, and Nzengwa [20], once the right-hand sides f_i of Eq. (4.44) in [20] are replaced by

$$f_\alpha = -\varepsilon^2 \rho \Lambda^l(\varepsilon)\, u^l_\alpha(\varepsilon) \qquad \text{and} \qquad f_3 = -\rho \Lambda^l(\varepsilon)\, u^l_3(\varepsilon),$$

since all that was needed in that proof was the boundedness of these right-hand sides in the space $L^2(\Omega)$. ∎

It follows from Lemmas 2, 3, and 4 that, for each integer $l \geqslant 1$, the weak limit $(\tilde{\mathbf{u}}^l, \mathbf{u}^l)$ belongs to the space

$$[\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{V}_{KL}(\Omega)]_\beta \overset{\text{def}}{=} \{(\tilde{\mathbf{v}}, \mathbf{v}) \in \mathbf{H}^1(\tilde{\Omega}) \times \mathbf{V}_{KL}(\Omega);$$
$$\tilde{v}_{\alpha|\tilde{\omega}_\beta} = 0,\ \tilde{v}_{3|\tilde{\omega}_\beta} = v_{3|\omega_\beta}\}. \tag{4.30}$$

The next lemma shows how any function lying in *two particular subspaces* of the space $[\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{V}_{KL}(\Omega)]_\beta$ can be approximated as well as we please by functions $(\tilde{\mathbf{v}}(\varepsilon), \mathbf{v}(\varepsilon))$ in the space $\mathbf{V}(\varepsilon)$, whose components $\mathbf{v}(\varepsilon)$ lie in addition in the space $\mathbf{V}_{KL}(\Omega)$. If we take limits as $\varepsilon \to 0$, this density property will later enable us to find variational equations satisfied by $(\tilde{\mathbf{u}}^l, \mathbf{u}^l)$. In what follows, $\tilde{\omega}$ designates the intersection of the set $\tilde{\Omega}$ by the plane that contains the set $\tilde{\omega}_\beta$, and *we are assuming that the origin* $\tilde{0}$ *for the points* $\tilde{x} \in \tilde{\Omega}$ *belongs to the left edge of the set* $\tilde{\omega}_\beta$ (cf. Fig. 2); the spaces $\mathbf{V}(\varepsilon)$ and $\mathbf{V}_{KL}(\Omega)$ have been defined in (3.6) and (4.22).

LEMMA 6. *Let* $(\tilde{\mathbf{v}}, \mathbf{v})$ *be a function in the space* $[\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{V}_{KL}(\Omega)]_\beta$ *of* (4.30) *such that either* $\operatorname{supp} \tilde{\mathbf{v}}$ *is contained in the set* $\{\tilde{x} = (\tilde{x}_i) \in \tilde{\Omega};\ \tilde{x}_1 \leqslant 0\}$ *and* $\mathbf{v} = \mathbf{0}$, *or* $\tilde{\mathbf{v}}_{|\tilde{\omega}} \in \mathbf{H}^1(\tilde{\omega})$. *Then there exists a sequence* $(\tilde{\mathbf{v}}(\varepsilon), \mathbf{v}(\varepsilon))$ *such that*

$$(\tilde{\mathbf{v}}(\varepsilon), \mathbf{v}(\varepsilon)) \in \mathbf{V}(\varepsilon) \qquad \textit{for all} \quad \varepsilon > 0, \tag{4.31}$$

$$\mathbf{v}(\varepsilon) \in \mathbf{V}_{KL}(\Omega) \qquad \textit{for all} \quad \varepsilon > 0, \tag{4.32}$$

$$\|\mathbf{v}(\varepsilon) - \mathbf{v}\|_{1,\Omega} \xrightarrow[\varepsilon \to 0]{} 0, \tag{4.33}$$

$$\|\tilde{\mathbf{v}}(\varepsilon) - \tilde{\mathbf{v}}\|_{1,\tilde{\Omega}} \xrightarrow[\varepsilon \to 0]{} 0. \tag{4.34}$$

Proof. See the proof of Lemma 6 of Ciarlet, Le Dret, and Nzengwa [20] which itself makes an essential use of an idea of Caillerie [8]. ∎

As a first step towards identifying the "limit" eigenvalue problem solved by the weak limit $(\tilde{\mathbf{u}}^l, \mathbf{u}^l)$, we obtain the variational equations that this weak limit should satisfy when the test-functions $(\tilde{\mathbf{v}}, \mathbf{v})$ are subjected to the same restrictions as in Lemma 6.

LEMMA 7. *Let* $(\tilde{\mathbf{v}}, \mathbf{v})$ *be a function in the space* $[\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{V}_{KL}(\Omega)]_\beta$ *of* (4.30) *such that either* $\operatorname{supp} \tilde{\mathbf{v}}$ *is contained in the set* $\{\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_i) \in \tilde{\Omega};\ \tilde{x}_1 \leqslant 0\}$ *and* $\mathbf{v} = \mathbf{0}$, *or* $\tilde{\mathbf{v}}_{|\tilde{\omega}} \in \mathbf{H}^1(\tilde{\omega})$. *Then, for each integer* $l \geqslant 1$, *the weak limit* $(\tilde{\mathbf{u}}^l, \mathbf{u}^l) \in [\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{V}_{KL}(\Omega)]_\beta$ *satisfies*

$$\int_{\tilde{\Omega}} \tilde{\mathbf{A}}\mathbf{e}(\tilde{\mathbf{u}}^l) : \mathbf{e}(\tilde{\mathbf{v}})\, d\tilde{x} + \int_\omega \frac{4}{3} \left\{ \mu \partial_{\alpha\beta} \zeta_3^l + \frac{\lambda\mu}{\lambda + 2\mu} \Delta \zeta_3^l \delta_{\alpha\beta} \right\} \partial_{\alpha\beta} \eta_3 \, d\omega$$

$$+ \int_\omega 4 \left\{ \mu \left(\frac{\partial_\alpha \zeta_\beta^l + \partial_\beta \zeta_\alpha^l}{2} \right) + \frac{\lambda\mu}{\lambda + 2\mu} \partial_\gamma \zeta_\gamma^l \delta_{\alpha\beta} \right\} \partial_\alpha \eta_\beta \, d\omega$$

$$= \Lambda^l \left\{ \int_{\tilde{\Omega}} \tilde{\rho} \tilde{\mathbf{u}}^l \cdot \tilde{\mathbf{v}} \, d\tilde{x} + 2 \int_\omega \rho \zeta_3^l \eta_3 \, d\omega \right\}, \tag{4.35}$$

with (*cf. Lemma* 3)

$$u_\alpha^l = \zeta_\alpha^l - x_3 \partial_\alpha \zeta_3^l, \qquad u_3^l = \zeta_3^l, \qquad \zeta_\alpha^l \in H^1(\omega),$$
$$\zeta_3^l \in H^2(\omega), \qquad \zeta_i^l = \partial_\nu \zeta_3^l = 0 \qquad \text{on } \gamma_0, \tag{4.36}$$

and

$$v_\alpha = \eta_\alpha - x_3 \partial_\alpha \eta_3, \qquad v_3 = \eta_3, \qquad \eta_\alpha \in H^1(\omega),$$
$$\eta_3 \in H^2(\omega), \qquad \eta_i = \partial_\nu \eta_3 = 0 \qquad \text{on } \gamma_0. \tag{4.37}$$

Proof. We use the functions $(\tilde{\mathbf{v}}(\varepsilon), \mathbf{v}(\varepsilon))$ constructed in Lemma 6 to approximate the function $(\tilde{\mathbf{v}}, \mathbf{v})$, as test-functions in the variational equations (3.7) satisfied by the triple $(\Lambda^l(\varepsilon), (\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon))$. Since $e_{i3}(\mathbf{v}(\varepsilon)) = 0$ in Ω by construction, these equations reduce to

$$\int_{\tilde{\Omega}} \chi(\tilde{0}_\beta^\varepsilon)\, \tilde{A}\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)) : \mathbf{e}(\tilde{\mathbf{v}}(\varepsilon))\, d\tilde{x}$$

$$+ \int_\Omega \{ 2\mu e_{\alpha\beta}(\mathbf{u}^l(\varepsilon))\, e_{\alpha\beta}(\mathbf{v}(\varepsilon))$$

$$+ \lambda [e_{\alpha\alpha}(\mathbf{u}^l(\varepsilon)) + \kappa_{33}(\mathbf{u}^l(\varepsilon))]\, e_{\beta\beta}(\mathbf{v}(\varepsilon)) \}\, dx$$

$$= \Lambda^l(\varepsilon) \left\{ \int_{\tilde{\Omega}} \chi(\tilde{0}_\beta^\varepsilon)\, \tilde{\rho} \tilde{u}_i^l(\varepsilon)\, \tilde{v}_i(\varepsilon)\, d\tilde{x} \right.$$

$$\left. + \varepsilon^2 \int_\Omega \rho u_\alpha^l(\varepsilon)\, v_\alpha(\varepsilon)\, dx + \int_\Omega \rho u_3^l(\varepsilon)\, v_3(\varepsilon)\, dx \right\}, \tag{4.38}$$

where $\kappa_{33}(\mathbf{u}^l(\varepsilon)) = \varepsilon^{-2} e_{33}(\mathbf{u}^l(\varepsilon))$ (cf. (4.29)). Let then ε approach 0 in Eqs. (4.38). Since (cf. Lemmas 2, 5, 6)

$$\tilde{\mathbf{u}}^l(\varepsilon) \rightharpoonup \tilde{\mathbf{u}}^l \text{ in } \mathbf{H}^1(\tilde{\Omega}), \qquad \mathbf{u}^l(\varepsilon) \rightharpoonup \mathbf{u}^l \text{ in } \mathbf{H}^1(\Omega),$$

$$\kappa_{33}(\mathbf{u}^l(\varepsilon)) \rightharpoonup -\frac{\lambda}{(\lambda+2\mu)} e_{\gamma\gamma}(\mathbf{u}^l(\varepsilon)) \text{ in } L^2(\Omega),$$

$$\tilde{\mathbf{v}}(\varepsilon) \to \tilde{\mathbf{v}} \text{ in } \mathbf{H}^1(\tilde{\Omega}), \qquad \mathbf{v}(\varepsilon) \to \mathbf{v} \text{ in } \mathbf{H}^1(\Omega),$$

we can pass to the limit in Eq. (4.38) (whenever B is a strongly continuous bilinear form, $u_n \rightharpoonup u$ and $v_n \to v$ implies $B(u_n, v_n) \to B(u, v)$). We obtain in this fashion Eq. (4.35) after replacing the components of $\mathbf{u}^l$ and $\mathbf{v}^l$ by their expressions (4.36)–(4.37). ∎

Remark. Only the weak convergence of the sequence $(\kappa_{33}(\mathbf{u}^l(\varepsilon))$ is thus needed here; the weak convergence of the sequence $\kappa_{\alpha 3}(\mathbf{u}^l(\varepsilon))$ will not be used until Lemma 10. ∎

It turns out that the limit problem consists of *two independent problems*, the first one being a "genuine" eigenvalue problem with $(\Lambda^l, (\tilde{\mathbf{u}}^l, \zeta_3^l))$ as the unknown, the second one being a "degenerate" problem with $(\zeta_1^l, \zeta_2^l) = 0$ as its unique solution. Accordingly, our identification of the limit problem comprises two stages (Lemmas 8 and 9).

LEMMA 8. *For each integer $l \geq 1$, the function $(\tilde{\mathbf{u}}^l, \zeta_3^l)$ belongs to the space*

$$[\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta \stackrel{\text{def}}{=} \{(\tilde{\mathbf{v}}, \eta_3) \in \mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega);\ \eta_3 = \partial_\nu \eta_3 = 0 \ on\ \gamma_0,$$
$$\tilde{v}_{3|\tilde{\omega}_\beta} = \eta_{3|\omega_\beta},\ \tilde{v}_{\alpha|\tilde{\omega}_\beta} = 0\}, \tag{4.39}$$

and it satisfies

$$\int_{\tilde{\Omega}} \tilde{\mathbf{A}}\mathbf{e}(\tilde{\mathbf{u}}^l) : \mathbf{e}(\tilde{\mathbf{v}})\, d\tilde{x} + \int_\omega \frac{4}{3}\left\{\mu \partial_{\alpha\beta}\zeta_3^l + \frac{\lambda\mu}{\lambda+2\mu}\Delta\zeta_3^l\,\delta_{\alpha\beta}\right\}\partial_{\alpha\beta}\eta_3\, d\omega$$
$$= \Lambda^l\left\{\int_{\tilde{\Omega}} \tilde{\rho}\tilde{\mathbf{u}}^l \cdot \tilde{\mathbf{v}}\, d\tilde{x} + 2\int_\omega \rho\zeta_3^l \eta_3\, d\omega\right\}$$
$$for\ all \quad (\tilde{\mathbf{v}}, \eta_3) \in [\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta. \tag{4.40}$$

The bilinear form appearing in the left-hand side of Eq. (4.40) *is symmetric and coercive over the space $[\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta$; hence, each eigenvalue Λ^l is necessarily > 0. The eigenfunctions $(\tilde{\mathbf{u}}^l, \zeta_3^l)$, $l \geq 1$, found in this fashion satisfy*

$$\int_{\tilde{\Omega}} \tilde{\rho}\tilde{\mathbf{u}}^k \cdot \mathbf{u}^l\, d\tilde{x} + 2\int_\omega \rho\zeta_3^k\zeta_3^l\, d\omega = \delta_{kl}, \qquad k, l \geq 1. \tag{4.41}$$

Proof. By Lemma 7, the variational equation (4.35) is satisfied in particular by any function of the form $(\tilde{\mathbf{v}}, (-\partial_1 \eta_3, -\partial_2 \eta_3, \eta_3))$, such that $(\tilde{\mathbf{v}}, \eta_3) \in [\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta$ and either $\operatorname{supp} \tilde{\mathbf{v}}$ is contained in the set $\{\tilde{x} = (\tilde{x}_i) \in \tilde{\Omega};\ \tilde{x}_1 \leqslant 0\}$ and $\eta_3 = 0$, or $\tilde{\mathbf{v}}_{|\tilde{\omega}} \in \mathbf{H}^1(\tilde{\omega})$, and in both cases they reduce to Eq. (4.40). Given an arbitrary function $(\tilde{\mathbf{v}}, \eta_3) \in [\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta$, let $\tilde{\eta}_3 \in H^2(\tilde{\omega})$ denote an extension of $\tilde{\eta}_{3|\tilde{\omega}_\beta}$, and let $\tilde{w}_3(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \tilde{\eta}_3(\tilde{x}_1, \tilde{x}_2)$ for all $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in \tilde{\Omega}$. Since the function $(\tilde{\mathbf{w}}^*, \eta_3) \overset{\text{def}}{=} ((0, 0, \tilde{w}_3), \eta_3)$ belongs to the space $[\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta$ and satisfies $\tilde{\mathbf{w}}^*_{|\tilde{\omega}} \in \mathbf{H}^2(\tilde{\omega}) \subsetneq \mathbf{H}^1(\tilde{\omega})$, it satisfies the variational equation (4.40). Since Eq. (4.40) is linear with respect to $(\tilde{\mathbf{v}}, \eta_3)$, it thus suffices to show that it is satisfied for all pairs of the form $(\tilde{\mathbf{v}}, 0) \in [\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta$, with functions $\tilde{\mathbf{v}} = (\tilde{v}_i) \in \mathbf{H}^1(\tilde{\Omega})$ satisfying $\tilde{v}_{3|\tilde{\omega}_\beta} = 0$.

To this end, we use the following result, proved in Ciarlet, Le Dret, and Nzengwa [20]: Given any function $\tilde{v} \in H^1(\tilde{\Omega})$ that satisfies

$$\tilde{v}_{|\tilde{\omega}_\beta} = 0,$$

there exist functions $\tilde{r}^n$ and $\tilde{s}^n$, $n \geqslant 1$, with the following properties:

$$\begin{aligned} \tilde{r}^n &\in H^1(\tilde{\Omega}) && \text{and} && \tilde{r}^n_{|\tilde{\omega}} \in H^1(\tilde{\omega}), \\ \tilde{s}^n &\in H^1(\tilde{\Omega}) && \text{and} && \operatorname{supp} \tilde{s}^n \subset \{\tilde{x} = (\tilde{x}_i) \in \tilde{\Omega};\ \tilde{x}_1 \leqslant 0\}, \\ (\tilde{r}^n + \tilde{s}^n) &\xrightarrow[n \to +\infty]{} \tilde{v} && \text{in } H^1(\tilde{\Omega}). \end{aligned}$$

Since the variational equation (4.40) is separately satisfied by the functions $((\tilde{r}^n_i), 0)$ and $((\tilde{s}^n_i), 0)$, and since they are linear and continuous with respect to $\tilde{\mathbf{v}} \in \mathbf{H}^1(\tilde{\Omega})$, the conclusion then follows.

In order to show that the bilinear form appearing in the left-hand side of the variational equation (4.40) is coercive over the space $[\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta$, it suffices to observe that the mapping

$$(\tilde{\mathbf{v}}, \eta_3) \to \{ |\mathbf{e}(\tilde{\mathbf{v}})|^2_{0,\tilde{\Omega}} + |\partial_{\alpha\beta} \eta_3\, \partial_{\alpha\beta} \eta_3|^2_{0,\omega} \}^{1/2}$$

is a norm over the space $[\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta$, equivalent to the norm

$$(\tilde{\mathbf{v}}, \eta_3) \to \{ \|\tilde{\mathbf{v}}\|^2_{1,\tilde{\Omega}} + \|\eta_3\|^2_{2,\omega} \}^{1/2};$$

the proof of this last result is similar to that of Lemma 1 of Ciarlet, Le Dret, and Nzengwa [20] and, for this reason, is omitted.

The orthonormalization conditions (4.41) follow from the orthonormalization conditions (3.9) and from the compact imbedding from $\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega)$ into $\mathbf{L}^2(\tilde{\Omega}) \times \mathbf{L}^2(\Omega)$ (each sequence $(\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon))_{\varepsilon > 0}$, $l \geqslant 1$, converges strongly to $(\tilde{\mathbf{u}}^l, \mathbf{u}^l)$ in the latter space). ∎

LEMMA 9. *For each integer $l \geqslant 1$, the pair (ζ_1^l, ζ_2^l) belongs to the space*

$$\mathbf{H}(\omega) = \{(\eta_1, \eta_2) \in H^1(\omega) \times H^1(\omega);\ \eta_\alpha = 0 \text{ on } \gamma_0\}, \tag{4.42}$$

and it satisfies

$$\int_\omega 4\left\{\mu\left(\frac{\partial_\alpha \zeta_\beta^l + \partial_\beta \zeta_\alpha^l}{2}\right) + \frac{\lambda\mu}{\lambda + 2\mu}\,\partial_\gamma \zeta_\gamma^l\,\delta_{\alpha\beta}\right\}\partial_\alpha \eta_\beta\, d\omega = 0 \tag{4.43}$$

for all $(\eta_1, \eta_2) \in \mathbf{H}(\omega)$.

The bilinear form appearing in the left-hand side of Eq. (4.43) *is symmetric and coercive over the space $\mathbf{H}(\omega)$, and thus*

$$\zeta_1^l = 0 \qquad \text{and} \qquad \zeta_2^l = 0. \tag{4.44}$$

is its unique solution.

Proof. By Lemma 7, the variational equation (4.35) is satisfies in particular by any function of the form $(\tilde{\mathbf{0}}, (\eta_1, \eta_2, 0))$ such that $(\eta_1, \eta_2) \in \mathbf{H}(\omega)$ (since $\tilde{\mathbf{0}}_{|\tilde\omega} \in \mathbf{H}^1(\tilde\omega)$), in which case it reduces to Eq. (4.43). The coerciveness of the associated variational problem is a simple consequence of the two-dimensional Korn inequality. ∎

We now establish the *strong convergence* in the space $\mathbf{H}^1(\tilde\Omega) \times \mathbf{H}^1(\Omega)$ of the subsequences $(\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon))_{\varepsilon > 0}$, which so far are only known to weakly converge in this space. We likewise establish the strong convergence in $\mathbf{L}^2(\Omega)$ of the subsequence $(\boldsymbol{\kappa}(\mathbf{u}^l(\varepsilon)))$ (cf. (4.18)).

LEMMA 10. *By Lemmas* 2 *and* 5, *there exists at least one subsequence with the properties*

$$A^l(\varepsilon) \to A^l \qquad \text{as} \quad \varepsilon \to 0, \tag{4.45}$$

$$(\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon)) \rightharpoonup (\tilde{\mathbf{u}}^l, \mathbf{u}^l) \qquad \text{in } \mathbf{H}^1(\tilde\Omega) \times \mathbf{H}^1(\Omega) \text{ as } \varepsilon \to 0, \tag{4.46}$$

$$\kappa_{\alpha 3}(\mathbf{u}^l(\varepsilon)) = \frac{1}{\varepsilon}\, e_{\alpha 3}(\mathbf{u}^l(\varepsilon)) \rightharpoonup 0 \qquad \text{in } L^2(\Omega) \text{ as } \varepsilon \to 0, \tag{4.47}$$

$$\kappa_{33}(\mathbf{u}^l(\varepsilon)) = \frac{1}{\varepsilon^2}\, e_{33}(\mathbf{u}^l(\varepsilon)) \rightharpoonup -\frac{\lambda}{(\lambda + 2\mu)}\, e_{\gamma\gamma}(\mathbf{u}^l) \quad \text{in } L^2(\Omega) \text{ as } \varepsilon \to 0. \tag{4.48}$$

for all integers $l \geqslant 1$. Then all convergences ((4.46), (4.47), (4.48)) *are strong.*

Proof. Let $l \geqslant 1$ be a given integer throughout the proof. By the Rellich–Kondrasov theorem, the sequence $(\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon))$ strongly converges to $(\tilde{\mathbf{u}}^l, \mathbf{u}^l)$ in $\mathbf{L}^2(\tilde\Omega) \times \mathbf{L}^2(\Omega)$. Hence it suffices to show that the family

$(\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)), \mathbf{e}(\mathbf{u}^l(\varepsilon)))_{\varepsilon>0}$ strongly converges in the space $\mathbf{L}^2(\tilde{\Omega}) \times \mathbf{L}^2(\Omega)$, as the conclusion will then follow from Korn's inequality applied in the spaces $\mathbf{H}^1(\tilde{\Omega})$ and $\mathbf{H}^1(\Omega)$.

Let $\boldsymbol{\kappa}^l = (\kappa_{ij}^l)$, with

$$\kappa_{\alpha\beta}^l = e_{\alpha\beta}(\mathbf{u}^l), \qquad \kappa_{\alpha 3}^l = \kappa_{3\alpha}^l = 0, \qquad \kappa_{33}^l = -\frac{\lambda}{(\lambda+2\mu)} e_{\gamma\gamma}(\mathbf{u}^l), \quad (4.49)$$

denote the weak limit in $\mathbf{L}^2(\Omega)$ of the sequence $(\boldsymbol{\kappa}^l(\mathbf{u}^l(\varepsilon)))_{\varepsilon>0}$. Expressing that the variational equation (4.40) is satisfied in particular by $(\tilde{\mathbf{v}}, \eta_3) = (\tilde{\mathbf{u}}^l, \zeta_3^l)$, and taking into account Equation (4.41), we easily obtain

$$\int_{\tilde{\Omega}} \mathbf{A}\mathbf{e}(\tilde{\mathbf{u}}^l) : \mathbf{e}(\tilde{\mathbf{u}}^l)\, d\tilde{x} + \int_{\Omega} \mathbf{A}\boldsymbol{\kappa}^l : \boldsymbol{\kappa}^l\, dx = \Lambda^l. \quad (4.50)$$

By Inequalities (4.19), there exists a constant $c > 0$ such that

$$\begin{aligned} & c(|\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)) - \mathbf{e}(\tilde{\mathbf{u}}^l)|_{0,\tilde{\Omega}}^2 + |\boldsymbol{\kappa}(\mathbf{u}^l(\varepsilon)) - \boldsymbol{\kappa}^l|_{0,\Omega}^2) \\ & \leqslant \int_{\tilde{\Omega}} \chi(\tilde{\Omega}_\beta^\varepsilon)\, \mathbf{A}(\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)) - \mathbf{e}(\tilde{\mathbf{u}}^l)) : (\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)) - \mathbf{e}(\tilde{\mathbf{u}}^l))\, d\tilde{x} \\ & \quad + \int_{\tilde{\Omega}} \chi(\tilde{\mathrm{O}}_\beta^\varepsilon)\, \tilde{\mathbf{A}}(\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)) - \mathbf{e}(\tilde{\mathbf{u}}^l)) : (\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)) - \mathbf{e}(\tilde{\mathbf{u}}^l))\, d\tilde{x} \\ & \quad + \int_{\Omega} \mathbf{A}(\boldsymbol{\kappa}(\mathbf{u}^l(\varepsilon)) - \boldsymbol{\kappa}^l) : (\boldsymbol{\kappa}(\mathbf{u}^l(\varepsilon)) - \boldsymbol{\kappa}^l)\, dx. \end{aligned} \quad (4.51)$$

We shall now show that the right-hand side of (4.51) approaches 0 as $\varepsilon \to 0$.

First, a simple computation based on the junction conditions (3.4)–(3.5) for the three-dimensional problem shows that

$$\int_{\tilde{\Omega}} \chi(\tilde{\Omega}_\beta^\varepsilon)\, \mathbf{A}\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)) : \mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon))\, d\tilde{x} = \varepsilon^3 \int_{\Omega_\beta} \mathbf{A}\boldsymbol{\kappa}(\mathbf{u}^l(\varepsilon)) : \boldsymbol{\kappa}(\mathbf{u}^l(\varepsilon))\, dx \xrightarrow[\varepsilon \to 0]{} 0,$$

since the weakly convergent family $(\boldsymbol{\kappa}(\mathbf{u}^l(\varepsilon)))_{\varepsilon>0}$ is bounded in the space $\mathbf{L}^2(\Omega)$. Further,

$$\int_{\tilde{\Omega}} \chi(\tilde{\Omega}_\beta^\varepsilon)\, \mathbf{A}\tilde{\mathbf{e}}(\tilde{\mathbf{u}}^l) : (\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)) - \mathbf{e}(\tilde{\mathbf{u}}^l))\, d\tilde{x} \xrightarrow[\varepsilon \to 0]{} 0,$$

since the family $(\chi(\tilde{\Omega}_\beta^\varepsilon)\, \mathbf{A}\mathbf{e}(\tilde{\mathbf{u}}^l))_{\varepsilon>0}$ converges strongly to $\mathbf{0}$ in $\mathbf{L}^2(\tilde{\Omega})$, the family $(\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)) - \mathbf{e}(\tilde{\mathbf{u}}^l))_{\varepsilon>0}$ converges weakly to $\mathbf{0}$ in $\mathbf{L}^2(\tilde{\Omega})$, and the inner product in the space $\mathbf{L}^2(\tilde{\Omega})$ is a continuous bilinear form (this argument will be used at several other places in what follows). Finally,

$$\int_{\tilde{\Omega}} \chi(\tilde{\Omega}^{\varepsilon}_{\beta})\, \mathbf{A}\mathbf{e}(\tilde{\mathbf{u}}^l) : \mathbf{e}(\tilde{\mathbf{u}}^l)\, d\tilde{x} \xrightarrow[\varepsilon \to 0]{} 0,$$

since the $d\tilde{x}$-measure of the set $\tilde{\Omega}^{\varepsilon}_{\beta}$ approaches 0 as $\varepsilon \to 0$. Hence

$$\int_{\tilde{\Omega}} \chi(\tilde{\Omega}^{\varepsilon}_{\beta})\, \mathbf{A}(\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)) - \mathbf{e}(\tilde{\mathbf{u}}^l)) : (\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)) - \mathbf{e}(\tilde{\mathbf{u}}^l))\, d\tilde{x} \xrightarrow[\varepsilon \to 0]{} 0.$$

The remaining terms in the right-hand side of (4.51) can be rewritten as

$$\begin{aligned}
&\int_{\tilde{\Omega}} \chi(\tilde{0}^{\varepsilon}_{\beta})\, \mathbf{A}(\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)) - \mathbf{e}(\tilde{\mathbf{u}}^l)) : (\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)) - \mathbf{e}(\tilde{\mathbf{u}}^l))\, d\tilde{x} \\
&\quad + \int_{\Omega} \mathbf{A}(\boldsymbol{\kappa}(\mathbf{u}^l(\varepsilon)) - \boldsymbol{\kappa}^l) : (\boldsymbol{\kappa}(\mathbf{u}^l(\varepsilon) - \boldsymbol{\kappa}^l)\, dx \\
&= \int_{\tilde{\Omega}} \chi(\tilde{0}^{\varepsilon}_{\beta})\, \tilde{\mathbf{A}}\mathbf{e}(\tilde{\mathbf{u}}^l) : (\mathbf{e}(\tilde{\mathbf{u}}^l) - 2\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)))\, d\tilde{x} \\
&\quad + \int_{\Omega} \mathbf{A}\boldsymbol{\kappa}^l : (\boldsymbol{\kappa}^l - 2\boldsymbol{\kappa}(\mathbf{u}^l(\varepsilon))\, dx \\
&\quad + \int_{\tilde{\Omega}} \chi(\tilde{0}^{\varepsilon}_{\beta}) \tilde{\mathbf{A}}\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)) : \mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon))\, d\tilde{x} \\
&\quad + \int_{\Omega} \mathbf{A}\boldsymbol{\kappa}(\mathbf{u}^l(\varepsilon)) : \boldsymbol{\kappa}(\mathbf{u}^l(\varepsilon))\, dx.
\end{aligned}$$

First,

$$\begin{aligned}
&\int_{\tilde{\Omega}} \chi(\tilde{0}^{\varepsilon}_{\beta})\, \tilde{\mathbf{A}}\mathbf{e}(\tilde{\mathbf{u}}^l) : (\mathbf{e}(\tilde{\mathbf{u}}^l) - 2\mathbf{e}(\tilde{\mathbf{u}}^l(\varepsilon)))\, dx + \int_{\Omega} \mathbf{A}\boldsymbol{\kappa}^l : (\boldsymbol{\kappa}^l - 2\boldsymbol{\kappa}(\mathbf{u}^l(\varepsilon)))\, dx \\
&\xrightarrow[\varepsilon \to 0]{} \left\{ \int_{\tilde{\Omega}} \tilde{\mathbf{A}}\tilde{\mathbf{e}}(\tilde{\mathbf{u}}^l) : \mathbf{e}(\tilde{\mathbf{u}}^l)\, d\tilde{x} + \int_{\Omega} \mathbf{A}\boldsymbol{\kappa}^l : \boldsymbol{\kappa}^l)\, dx \right\} = -\Lambda^l
\end{aligned}$$

by (4.50). Second,

$$\int_{\tilde{\Omega}} \chi(\tilde{0}^{\varepsilon}_{\beta})\, \tilde{\mathbf{A}}\mathbf{e}(\mathbf{u}^l(\varepsilon)) : \mathbf{e}(\tilde{\mathbf{u}}^l(\mathrm{e}))\, d\tilde{x} + \int_{\Omega} \mathbf{A}\boldsymbol{\kappa}^l(\varepsilon) : \boldsymbol{\kappa}^l(\varepsilon)\, dx = \Lambda^l(\varepsilon)$$

by (3.7), written with $(\tilde{\mathbf{v}}, \mathbf{v}) = (\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon))$, and (3.9). Since $\Lambda^l(\varepsilon) \to \Lambda^l$ as $\varepsilon \to 0$, we thus conclude that the right-hand side of (4.51) converges to 0 as $\varepsilon \to 0$. ∎

Though we have proved that each subsequence $(\Lambda^l(\varepsilon), (\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon)))_{\varepsilon>0}$, $l \geqslant 1$, strongly converges in $]0, +\infty[\times \mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega)$ to a solution $(\Lambda^l, (\tilde{\mathbf{u}}^l, \mathbf{u}^l))$ of the "limit" eigenvalue problem (4.40) (cf. Lemmas 8, 9, 10), nothing tells us so far whether Λ^l *is precisely the* lth *eigenvalue* (counting multiplicities) of (4.40), nor whether the set $(\tilde{\mathbf{u}}^l, \zeta_3^l)_{l=1}^{\infty}$ forms a *complete set* in the space $[\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta$ of (4.39). We shall answer these questions by the affirmative in the next lemma. There, as in Lemma 12, we shall make an essential use of ideas first developed by Kesavan [34] in an abstract setting, then applied to eigenvalue problems for a "single" plate by Ciarlet and Kesavan [17].

LEMMA 11. *Let* $(\Lambda^l, (\mathbf{u}^l, \zeta_3^l))$, $l \geqslant 1$, *be the eigensolutions of Problem* (4.40) *found as limits of the subsequence* $(\Lambda^l(\varepsilon), (\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}(\varepsilon)))_{\varepsilon>0}$, $l \geqslant 1$ *of eigensolutions, orthonormalized as in* (3.9), *of Problem* (3.7).

Then the sequence $(\Lambda^l)_{l=1}^{\infty}$ *comprises all eigenvalues, counting multiplicities, of Problem* (4.40), *and the associated sequence* $((\mathbf{u}^l, \zeta_3^l))_{l=1}^{\infty}$ *of eigenfunctions, orthonormalized as in* (4.41), *forms a complete set in the space* $[\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta$.

Proof. We first show that

$$0 < \Lambda^1 \leqslant \Lambda^2 \leqslant \cdots \leqslant \Lambda^l \leqslant \Lambda^{l+1} \leqslant \ldots, \qquad \text{with} \quad \lim_{l \to +\infty} \Lambda^l = +\infty. \tag{4.52}$$

Since $0 < \Lambda^1(\varepsilon) \leqslant \Lambda^2(\varepsilon) \leqslant \ldots$ for each $\varepsilon > 0$, it follows that $0 \leqslant \Lambda^1 \leqslant \Lambda^2 \leqslant \ldots$; since the bilinear form associated with the left-hand side of Eq. (4.40) is coercive over the space $[\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta$, it follows that $\Lambda^1 > 0$. If the sequence $(\Lambda^l)_{l=1}^{\infty}$ were bounded, the eigenvalue problem (4.40) could have only a finite number of linearly independent eigensolutions, since its associated operator is compact over the space $[\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta$; but this would contradict the orthonormalization condition (4.41). Hence all relations (4.52) hold.

We next show that, if Λ is any eigenvalue of Problem (4.40), there exists an integer $l \geqslant 1$ such that $\Lambda = \Lambda^l$. Suppose that the contrary holds, i.e., that $\Lambda \neq \Lambda^l$ for all $l \geqslant 1$, and let $(\tilde{\mathbf{u}}, \zeta_3) \in [\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta$ denote an associated eigenfunction, which thus satisfies

$$\begin{aligned}
&\int_{\tilde{\Omega}} \tilde{\mathbf{A}}\tilde{\mathbf{e}}(\tilde{\mathbf{u}}) : \mathbf{e}(\tilde{\mathbf{v}})\, d\tilde{x} + \int_\omega \frac{4}{3}\left\{\mu\, \partial_{\alpha\beta}\zeta_3 + \frac{\lambda\mu}{\lambda + 2\mu} \Delta\zeta_3\, \partial_{\alpha\beta}\right\} \partial_{\alpha\beta}\eta_3\, d\omega \\
&\quad = \Lambda \left\{\int_{\tilde{\Omega}} \tilde{\rho}\tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}}\, dx + 2\int_\omega \rho\zeta_3\eta_3\, d\omega\right\} \\
&\qquad \text{for all} \quad (\tilde{\mathbf{v}}, \eta_3) \in [\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta,
\end{aligned} \tag{4.53}$$

$$\int_{\tilde{\Omega}} \tilde{\rho}\tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}}^l \, dx + 2 \int_{\omega} \rho \zeta_3 \zeta_3^l \, d\omega = 0, \qquad l \geqslant 1, \tag{4.54}$$

$$\int_{\tilde{\Omega}} \tilde{\rho}\tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \, dx + 2 \int_{\omega} \rho \zeta_3 \zeta_3 \, d\omega = 1. \tag{4.55}$$

If $(\tilde{\mathbf{w}}, \mathbf{w})$ and $(\tilde{\mathbf{v}}, \mathbf{v})$ denote arbitrary elements in the space $\mathbf{V}(\varepsilon)$ of (3.6), we let

$$\begin{aligned} &N(\varepsilon)((\tilde{\mathbf{w}}, \mathbf{w}), (\tilde{\mathbf{v}}, \mathbf{v})) \\ &\quad = \int_{\tilde{\Omega}} \chi(\tilde{0}^{\varepsilon}_{\beta}) \, \tilde{\mathbf{A}}\mathbf{e}(\tilde{\mathbf{w}}) : \mathbf{e}(\tilde{\mathbf{v}}) \, d\tilde{x} + \int_{\Omega} \mathbf{A}\boldsymbol{\kappa}(\mathbf{w}) : \boldsymbol{\kappa}(\mathbf{v}) \, dx, \end{aligned} \tag{4.56}$$

$$\begin{aligned} &D(\varepsilon)((\tilde{\mathbf{w}}, \mathbf{w}), (\tilde{\mathbf{v}}, \mathbf{v})) \\ &\quad = \int_{\tilde{\Omega}} \chi(\tilde{0}^{\varepsilon}_{\beta}) \, \tilde{\rho}\tilde{\mathbf{w}} \cdot \tilde{\mathbf{v}} \, d\tilde{x} + \int_{\Omega} \rho(\varepsilon^2 u_\alpha v_\alpha + u_3 v_3) \, dx. \end{aligned} \tag{4.57}$$

If $(\tilde{\mathbf{w}}, \mathbf{w}) = (\tilde{\mathbf{v}}, \mathbf{v})$, we use the shorter notations $N(\varepsilon)(\tilde{\mathbf{v}}, \mathbf{v})$ and $D(\varepsilon)(\tilde{\mathbf{v}}, \mathbf{v})$, respectively, already defined in (4.12)–(4.13). For each $\varepsilon > 0$, let $(\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)) \in \mathbf{V}(\varepsilon)$ be the unique solution of

$$\begin{aligned} &N(\varepsilon)((\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)), (\tilde{\mathbf{v}}, \mathbf{v})) \\ &\quad = \Lambda \left\{ \int_{\tilde{\Omega}} \chi(\tilde{0}^{\varepsilon}_{\beta}) \, \tilde{\rho}\tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}} \, d\tilde{x} + \int_{\Omega} \rho \zeta_3 v_3 \, dx \right\} \qquad \text{for all} \quad (\tilde{\mathbf{v}}, \mathbf{v}) \in \mathbf{V}(\varepsilon)), \end{aligned} \tag{4.58}$$

where $(\tilde{\mathbf{u}}, \zeta_3)$ is the function that satisfies (4.53)–(4.55) and ζ_3 is identified with a function independent of x_3 in the space $H^1(\Omega)$ in the last integral found in (4.58).

Proceeding as in Ciarlet, Le Dret, and Nzengwa [20] (or, for that matter, as in Lemmas 2 to 9 of the present paper; there is no need here fore the analog of Lemma 1), we find that the sequence $((\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)))_{\varepsilon > 0}$ converges strongly in $\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega)$ to an element $(\tilde{\mathbf{w}}, \mathbf{w})$ such that $w_\alpha = -x_3 \, \partial_\alpha \theta_3$, $w_3 = \theta_3$, where $(\tilde{\mathbf{w}}, \theta_3) \in [\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta$ is the unique solution of

$$\begin{aligned} &\int_{\tilde{\Omega}} \mathbf{A}\mathbf{e}(\tilde{\mathbf{w}}) : \mathbf{e}(\tilde{\mathbf{v}}) \, d\tilde{x} + \int_{\omega} \frac{4}{3} \left\{ \mu \partial_{\alpha\beta} \theta_3 + \frac{\lambda\mu}{\lambda + 2\mu} \Delta\theta_3 \, \delta_{\alpha\beta} \right\} \partial_{\alpha\beta} \eta_3 \, d\omega \\ &\quad = \Lambda \left\{ \int_{\tilde{\Omega}} \tilde{\rho}\tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}} \, d\tilde{x} + 2 \int_{\omega} \rho \zeta_3 \eta_3 \, d\omega \right\} \\ &\qquad \text{for all} \quad (\tilde{\mathbf{v}}, \eta_3) \in [\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta. \end{aligned} \tag{4.59}$$

By comparing (4.53) and (4.59), we conclude that $(\mathbf{w}, \theta_3) = (\mathbf{u}, \zeta_3)$. In other words, we have shown that

$$(\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)) \to (\tilde{\mathbf{u}}, (-x_3 \partial_1 \zeta_3, -x_3 \partial_2 \zeta_3)) \quad \text{in} \quad \mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega) \text{ as } \to 0. \tag{4.60}$$

By virtue of (4.52), there exists an integer m such that

$$\Lambda < \Lambda^{m+1}. \tag{4.61}$$

For each $\varepsilon > 0$, let

$$\begin{aligned}
(\tilde{\mathbf{v}}(\varepsilon), \mathbf{v}(\varepsilon)) &= (\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)) && \text{if} \quad m = 0,\\
(\tilde{\mathbf{v}}(\varepsilon), \mathbf{v}(\varepsilon)) &= (\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)) \\
&\quad - \sum_{k=1}^{m} D(\varepsilon)((\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)), (\tilde{\mathbf{u}}^k(\varepsilon), \mathbf{u}^k(\varepsilon)))\, (\tilde{\mathbf{u}}^k(\varepsilon), \mathbf{u}^k(\varepsilon)) \\
& && \text{if} \quad m \geqslant 1,
\end{aligned} \tag{4.62}$$

so that, by construction, if $m \geqslant 1$

$$D(\varepsilon)((\tilde{\mathbf{v}}(\varepsilon), \mathbf{v}(\varepsilon)), (\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon))) = 0, \qquad 1 \leqslant l \leqslant m.$$

Hence the *minimum principle* (2.10) gives us

$$R(\varepsilon)(\tilde{\mathbf{v}}(\varepsilon), \mathbf{v}(\varepsilon)) \overset{\text{def}}{=} \frac{N(\varepsilon)(\tilde{\mathbf{v}}(\varepsilon), \mathbf{v}(\varepsilon))}{D(\varepsilon)(\tilde{\mathbf{v}}(\varepsilon), \mathbf{v}(\varepsilon))} \geqslant \Lambda^{m+1}(\varepsilon) \qquad \text{for all} \quad \varepsilon > 0. \tag{4.63}$$

Let us study the behavior of the quotient $R(\varepsilon)(\tilde{\mathbf{v}}(\varepsilon), \mathbf{v}(\varepsilon))$ as $\varepsilon \to 0$. Using Definition (4.62), we find that

$$\begin{aligned}
N(\varepsilon)(\tilde{\mathbf{v}}(\varepsilon), \mathbf{v}(\varepsilon)) = N(\varepsilon)(\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon))& \\
- 2 \sum_{k=1}^{m} D(\varepsilon)((\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)), (\tilde{\mathbf{u}}^k(\varepsilon), \mathbf{u}^k(\varepsilon)))& \\
\times N(\varepsilon)((\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)), (\tilde{\mathbf{u}}^k(\varepsilon), \mathbf{u}^k(\varepsilon)))& \\
+ \sum_{k,l=1}^{m} D(\varepsilon)((\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)), (\tilde{\mathbf{u}}^k(\varepsilon), \mathbf{u}^k(\varepsilon)))& \\
\times D(\varepsilon)((\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)), \tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon)))& \\
\times N(\varepsilon)((\tilde{\mathbf{u}}^k(\varepsilon), \mathbf{u}^k(\varepsilon)), (\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon)).&
\end{aligned} \tag{4.64}$$

The definition of $D(\varepsilon)$ (cf. (4.57)), the convergence (4.60), the convergence of each sequence $(\tilde{\mathbf{u}}^k(\varepsilon), \mathbf{u}^k(\varepsilon))_{\varepsilon>0}$, $1 \leqslant k \leqslant m$, and the orthonormalization condition (4.54) then yield

$$\lim_{\varepsilon \to 0} D(\varepsilon)((\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)), (\tilde{\mathbf{u}}^k(\varepsilon), \mathbf{u}^k(\varepsilon)))$$
$$= \int_{\tilde{\Omega}} \tilde{\rho} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}}^k \, d\tilde{x} + 2 \int_{\omega} \rho \zeta_3 \zeta_3^k \, d\omega = 0. \tag{4.65}$$

From the definition of $N(\varepsilon)$ (cf. (4.56)), we deduce that there exists a constant C independent of ε such that, for arbitrary elements $(\tilde{\mathbf{w}}, \mathbf{w})$ and $(\tilde{\mathbf{v}}, \mathbf{v})$ in the space $\mathbf{V}(\varepsilon)$,

$$|N(\varepsilon)((\tilde{\mathbf{w}}, \mathbf{w}), (\tilde{\mathbf{v}}, \mathbf{v}))|$$
$$\leqslant C\{|\mathbf{e}(\tilde{\mathbf{w}})|_{0,\tilde{\Omega}}^2 + |\boldsymbol{\kappa}(\mathbf{w})|_{0,\Omega}^2\}^{1/2} \{|\mathbf{e}(\tilde{\mathbf{v}})|_{0,\tilde{\Omega}}^2 + |\boldsymbol{\kappa}(\mathbf{v})|_{0,\Omega}^2\}^{1/2}. \tag{4.66}$$

Hence the sequences $(N(\varepsilon)((\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)), (\tilde{\mathbf{u}}^k(\varepsilon), \mathbf{u}^k(\varepsilon))))_{\varepsilon>0}$ and $(N(\varepsilon)((\tilde{\mathbf{u}}^k(\varepsilon), \mathbf{u}^k(\varepsilon)), (\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon))))_{\varepsilon>0}$, $1 \leqslant k \leqslant m$ (which appear in (4.64)) are bounded independently of ε since all sequences $(\tilde{\mathbf{u}}(\varepsilon), \boldsymbol{\kappa}(\mathbf{u}(\varepsilon)))_{\varepsilon>0}$ and $(\tilde{\mathbf{u}}^k(\varepsilon), \boldsymbol{\kappa}(\mathbf{u}^k(\varepsilon)))_{\varepsilon>0}$, $1 \leqslant k \leqslant m$, converge in $\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{L}^2(\Omega)$.

Letting $(\tilde{\mathbf{v}}, \mathbf{v}) = (\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon))$ in Eq. (4.58) gives

$$N(\varepsilon)(\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)) = \Lambda \left\{ \int_{\tilde{\Omega}} \chi(\tilde{0}_\beta^\varepsilon) \, \tilde{\rho} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}}(\varepsilon) \, d\tilde{x} + \int_{\Omega} \rho \zeta_3 u_3(\varepsilon) \, dx \right\},$$

and thus, by (4.55), (4.60), (4.64), (4.65), and (4.66),

$$\lim_{\varepsilon \to 0} N(\varepsilon)(\tilde{\mathbf{v}}(\varepsilon), \mathbf{v}(\varepsilon)) = \lim_{\varepsilon \to 0} N(\varepsilon)((\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon))$$
$$= \Lambda \left\{ \int_{\tilde{\Omega}} \tilde{\rho} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \, d\tilde{x} + 2 \int_{\omega} \rho \zeta_3 \zeta_3 \, d\omega \right\} = \Lambda. \tag{4.67}$$

From the definition of $D(\varepsilon)$ (cf. (4.57)), we deduce that there exists a constant C independent of ε such that, for arbitrary elements $(\tilde{\mathbf{w}}, \mathbf{w})$ and $(\tilde{\mathbf{v}}, \mathbf{v})$ in the space $\mathbf{V}(\varepsilon)$,

$$|D(\varepsilon)((\tilde{\mathbf{w}}, \mathbf{w}), (\tilde{\mathbf{v}}, \mathbf{v}))| \leqslant C\{|\tilde{\mathbf{w}}|_{0,\tilde{\Omega}}^2 + |\mathbf{w}|_{0,\Omega}^2\}^{1/2} \{|\tilde{\mathbf{v}}|_{0,\tilde{\Omega}}^2 + |\mathbf{v}|_{0,\Omega}^2\}^{1/2}. \tag{4.68}$$

Since, from (4.62), (4.65), and Lemma 2,

$$\{(\tilde{\mathbf{v}}(\varepsilon), \mathbf{v}(\varepsilon)) - (\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon))\} \to 0 \quad \text{in} \quad \mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega) \text{ as } \varepsilon \to 0, \tag{4.69}$$

whence *a fortiori* in $\mathbf{L}^2(\tilde{\Omega}) \times \mathbf{L}^2(\Omega)$, we conclude from (4.68), (4.69), and (4.55) that

$$\lim_{\varepsilon \to 0} D(\varepsilon)((\tilde{\mathbf{v}}(\varepsilon), \mathbf{v}(\varepsilon))$$

$$= \lim_{\varepsilon \to 0} \left\{ \int_{\tilde{\Omega}} \chi(\tilde{0}_\beta^\varepsilon)\, \tilde{\rho}\tilde{\mathbf{u}}(\varepsilon) \cdot \tilde{\mathbf{u}}(\varepsilon)\, d\tilde{x} + \int_\Omega \rho(\varepsilon^2 u_\alpha(\varepsilon)\, u_\alpha(\varepsilon) + u_3(\varepsilon)\, u_3(\varepsilon))\, dx \right\}$$

$$= \int_{\tilde{\Omega}} \tilde{\rho}\tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}}\, d\tilde{x} + 2 \int_\Omega \rho \zeta_3 \zeta_3\, d\omega = 1. \tag{4.70}$$

Therefore, Relations (4.63), (4.67), (4.70) together imply that

$$\Lambda^{m+1} = \lim_{\varepsilon \to 0} \Lambda^{m+1}(\varepsilon) \leqslant \lim_{\varepsilon \to 0} R(\varepsilon)(\tilde{\mathbf{v}}(\varepsilon), \mathbf{v}(\varepsilon)) = \Lambda,$$

but this last inequality contradicts Inequality (4.61). Hence *each eigenvalue Λ of Problem* (4.40) *is equal to at least one of the limits Λ^l, $l \geqslant 1$.*

Let us finally show that *the sequence* $(\tilde{\mathbf{u}}^l, \mathbf{u}^l)$ *forms a complete set in the space* $[\mathbf{H}^1(\tilde{\Omega}) \times H^2(\omega)]_\beta$. Otherwise, by the spectral theory of compact operators, there would exist an eigensolution $(\Lambda, (\tilde{\mathbf{u}}, \zeta_3))$ of Problem (4.40) with the property that the eigenfunction $(\tilde{\mathbf{u}}, \zeta_3)$ is orthogonal in the sense of (4.54) to all the eigenfunctions $(\tilde{\mathbf{u}}^l, \zeta_3^l)$, $l \geqslant 1$, found as limits. By virtue of (4.52), there exists an integer $m \geqslant 0$ such that (4.61) holds, and thus the same argument as above would again lead to a contradiction. Hence the assertion follows. ∎

In our last lemma, we show that any one of the convergences so far established only for a subsequence holds in fact for the whole family, except those of eigenfunctions corresponding to a multiple eigenvalue of the limit problem.

LEMMA 12. *For each integer $l \geqslant 1$, the whole family $(\Lambda^l(\varepsilon))_{\varepsilon > 0}$ converges as $\varepsilon \to 0$.*

If, for a given integer $l \geqslant 1$, the eigenvalue Λ^l is simple, there exists $\varepsilon_0(l) > 0$ such that, for all $\varepsilon < \varepsilon_0(l)$, the eigenvalue $\Lambda^l(\varepsilon)$ is simple and further there exists, for all $\varepsilon \leqslant \varepsilon_0(l)$, an eigenfunction $(\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon))$, orthonormalized as in (3.9), *such that the whole family $((\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon)))_{\varepsilon > 0}$ converges in the space $\mathbf{H}^1(\tilde{\Omega}) \times \mathbf{H}^1(\Omega)$ as $\varepsilon \to 0$.*

Proof. If instead of the whole family, i.e., indexed by all $\varepsilon > 0$, we had started out with one of its subsequences, the conclusions of Lemmas 1 to 11 would hold *verbatim* for a further subsequence. In this fashion, we see in particular that any subsequence of the whole family $(\Lambda^l(\varepsilon))_{\varepsilon > 0}$ has a further subsequence that always converges to the *same* limit, viz., the lth eigenvalue Λ^l of the limit problem (cf. Lemma 11); hence the entire family $(\Lambda^l(\varepsilon))_{\varepsilon>0}$ converges to Λ^l for each $l \geqslant 1$.

Since $\lim_{\varepsilon\to 0} \Lambda^l(\varepsilon) = \Lambda^l$, there exists for each integer $l \geqslant 1$ a number $\varepsilon_0(l) > 0$ such that the multiplicity of $\Lambda^l(\varepsilon)$ is less than or equal to that of Λ^l for $\varepsilon \leqslant \varepsilon_0(l)$. In particular, $\Lambda^l(\varepsilon)$ is simple for $\varepsilon \leqslant \varepsilon_0(l)$ if Λ^l is simple.

If Λ^l is simple, let $(\tilde{\mathbf{u}}^l, \mathbf{u}^l)$ be one of its two corresponding orthonormalized eigenfunctions (the other one being $(-\tilde{\mathbf{u}}^l, -\mathbf{u}^l)$). Then for each $\varepsilon \leqslant \varepsilon_0(l)$, $\Lambda^l(\varepsilon)$ is also simple, and of its two distinct eigenfunctions, there exists at least one of them, denoted $(\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon))$, that satisfies

$$\int_{\tilde{\Omega}} \tilde{\rho}\tilde{u}_i^l \tilde{u}_i^l(\varepsilon)\, d\tilde{x} + \int_{\Omega} \rho \zeta_3^l u_3^l(\varepsilon)\, dx \geqslant 0. \tag{4.71}$$

Then again the previous arguments can be repeated *verbatim*: Given any subsequence of the whole family $(\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon))_{0<\varepsilon\leqslant\varepsilon_0(l)}$ whose elements satisfy (4.71), there exists a further subsequence that converges to an eigenfunction of the limit problem (4.40). Since this convergence can be only to $(\tilde{\mathbf{u}}^l, \mathbf{u}^l)$ in view of (4.71), we conclude that the whole family $(\tilde{\mathbf{u}}^l(\varepsilon), \mathbf{u}^l(\varepsilon))_{0\leqslant\varepsilon\leqslant\varepsilon_0(l)}$ whose elements satisfy (4.71) converges. ∎

5. Interpretation of the Limit Problem as a Boundary Value Problem

It remains to describe the eigenvalue problem that is, at least formally, associated with the variational equations (4.2). To begin with, we define the open set

$$\tilde{\Omega}_\beta = \tilde{\Omega} - \{\tilde{\omega}_\beta\}^-, \tag{5.1}$$

which is thus a *three-dimensional open set with a two-dimensional slit*, and we let $\tilde{\omega}_\beta^+$ and $\tilde{\omega}_\beta^-$ denote the upper and lower faces of the slit. When viewed as sets, these faces are fictitiously distinguished, since they coincide with the set $\tilde{\omega}_\beta$; on the other hand, the introduction of different notations allows for a convenient distinction between the trace from above and the trace from below of a function defined over the set $\tilde{\Omega}_\beta$, as in Eq. (5.5). Finally, we denote by $\tilde{\mathbf{n}} = (\tilde{n}_i)$ the unit outer normal vector along the set $\partial\tilde{\Omega}_\beta - \{\omega_\beta^+ \cup \omega_\beta^-\}$; by (ν_α) and (τ_α) the unit outer normal and unit tangential vectors along $\partial\omega$; and by ∂_τ the tangential derivative operator along $\partial\omega$.

THEOREM 2. *A smooth enough solution* $(\Lambda, (\tilde{\mathbf{u}}, \zeta_3))$ *of the variational equations* (4.2) *solves the following equations*:

(a) *in the set* $\tilde{\Omega}_\beta$,

$$-\tilde{\partial}_j \tilde{\sigma}_{ij}(\tilde{\mathbf{u}}) = \Lambda \tilde{\rho} \tilde{u}_i \qquad \text{in } \tilde{\Omega}_\beta, \tag{5.2}$$

$$\tilde{\sigma}_{ij}(\tilde{\mathbf{u}})\, \tilde{n}_j = 0 \qquad \text{on } \partial\tilde{\Omega}_\beta - \{\omega_\beta^+ \cup \omega_\beta^-\}, \tag{5.3}$$

where $\tilde{\partial}_i = \partial/\partial \tilde{x}_i$,

$$\tilde{\sigma}_{ij}(\tilde{\mathbf{u}}) = \tilde{\lambda} e_{pp}(\tilde{\mathbf{u}}) + 2\tilde{\mu} e_{ij}(\tilde{\mathbf{u}}); \tag{5.4}$$

(b) *in the set* ω,

$$\frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)} \Delta^2 \zeta_3 = 2\Lambda \rho \zeta_3 + \chi(\omega_\beta)\{\tilde{\sigma}_{33}(\tilde{\mathbf{u}})_{|\omega_\beta^+} - \tilde{\sigma}_{33}(\tilde{\mathbf{u}})_{|\omega_\beta^-}\} \qquad \text{in } \omega, \tag{5.5}$$

$$\zeta_3 = \partial_\nu \zeta_3 = 0 \qquad \text{on } \gamma_0, \tag{5.6}$$

$$m_{\alpha\beta}(\zeta_3)\, \nu_\alpha \nu_\beta = 0 \qquad \text{on } (\partial\omega - \gamma_0), \tag{5.7}$$

$$\partial_\tau(m_{\alpha\beta}(\zeta_3)\, \nu_\alpha \tau_\beta) + \{\partial_\alpha m_{\alpha\beta}(\zeta_3)\}\, \nu_\beta = 0 \qquad \text{on } (\partial\omega - \gamma_0), \tag{5.8}$$

where $m_{\alpha\beta}(\zeta_3)$ is defined as in (4.3);

(c) *at the "junction" between the sets* $\tilde{\Omega}_\beta$ *and* ω,

$$\tilde{u}_{3|\tilde{\omega}_\beta^+} = \tilde{u}_{3|\tilde{\omega}_\beta^-} = \zeta_{3|\omega_\beta}, \tag{5.9}$$

$$\tilde{u}_{\alpha|\tilde{\omega}_\beta^+} = \tilde{u}_{\alpha|\tilde{\omega}_\beta^-} = 0. \tag{5.10}$$

Proof. In the variational equation (4.2), let $\eta_3 = 0$ and let $\tilde{\mathbf{v}}$ vary in the space $\{\tilde{\mathbf{v}} \in \mathscr{C}^\infty(\{\tilde{\Omega}\}^-);\ \tilde{\mathbf{v}} = \mathbf{0}$ in a neighborhood of $\tilde{\omega}_\beta\}$; this shows that Eqs. (5.2) and (5.3) are satisfied. If $\tilde{\mathbf{v}} \in \mathbf{H}^1(\tilde{\Omega})$ satisfies $\tilde{v}_{\alpha|\tilde{\omega}_\beta^+} = \tilde{v}_{\alpha|\tilde{\omega}_\beta^-} = 0$, and if $\tilde{\mathbf{u}}$ is smooth enough (as is customarily assumed when variational equations are interpreted as boundary value problems), we thus have

$$\int_{\tilde{\Omega}} \{\tilde{\lambda} e_{pp}(\tilde{\mathbf{u}})\, e_{qq}(\tilde{\mathbf{v}}) + 2\tilde{\mu} e_{ij}(\tilde{\mathbf{u}})\, e_{ij}(\tilde{\mathbf{v}})\}\, d\tilde{x} - \Lambda \int_{\tilde{\Omega}} \tilde{\rho} \tilde{u}_i \tilde{v}_i\, d\tilde{x}$$

$$= -\int_{\tilde{\omega}_\beta} \{\tilde{\sigma}_{33}(\tilde{\mathbf{u}})_{|\omega_\beta^+} - \tilde{\sigma}_{33}(\tilde{\mathbf{u}})_{|\omega_\beta^-}\}\, \tilde{v}_3\, d\tilde{\omega}, \tag{5.11}$$

where the quantity $\tilde{\sigma}_{33}(\tilde{\mathbf{u}})$ is defined as in (5.4). Since $\tilde{v}_{3|\tilde{\omega}_\beta} = \eta_{3|\omega_\beta}$, the variational equation (4.2) thus reduces to

$$-\int_\omega m_{\alpha\beta}(\zeta_3)\, \partial_{\alpha\beta} \eta_3\, d\omega$$

$$= \int_\omega (2\Lambda\rho\zeta_3 + \chi(\omega_\beta)\{\tilde{\sigma}_{33}(\tilde{\mathbf{u}})_{|\omega_\beta^+} - \tilde{\sigma}_{33}(\tilde{\mathbf{u}})_{|\omega_\beta^-}\})\, \eta_3\, d\omega \tag{5.12}$$

for all $\eta_3 \in H^2(\omega)$ that satisfy $\eta_{3|\gamma_0} = \partial\eta_{3|\gamma_0} = 0$. That the variational problem (5.12) is formally equivalent to the boundary value problem (5.5)–(5.8) is classical (see, e.g., Germain [30, p. 83 ff.]). ∎

6. Comments on Theorems 1 and 2

(i) To begin with, we must "de-scale" Eqs. (5.2)–(5.10) in order that the "de-scaled" unknowns $\tilde{\mathbf{u}}^\varepsilon$ and ζ_3^ε, as defined in (6.1)–(6.2) below, may be attached to the actual structure. Let (cf. Fig. 3)

$$0_\beta = \tilde{\Omega}_\beta - \mathbf{t}, \qquad \omega_\beta^+ = \tilde{\omega}_\beta^+ - \mathbf{t}, \qquad \omega_\beta^- = \tilde{\omega}_\beta^- - \mathbf{t}.$$

Then with the "limit" vector field $\tilde{\mathbf{u}} = (\tilde{u}_i) : \tilde{\Omega}_\beta \to \mathbb{R}^3$, we associate the "limit" vector field $\tilde{\mathbf{u}}^\varepsilon = (\tilde{u}_i^\varepsilon) : \bar{0}_\beta \to \mathbb{R}^3$ by letting, in view of (3.3),

$$\tilde{u}_i^\varepsilon(x^\varepsilon) = \varepsilon \tilde{u}_i(\tilde{x}) \tag{6.1}$$

at all corresponding points $x^\varepsilon \in O_\beta$ and $\tilde{x} \in \tilde{\Omega}_\beta$, and with the "limit" vector field $\zeta_3 : \omega \to \mathbb{R}^3$ we associate the "limit" vector field $\zeta_3^\varepsilon : \omega \to \mathbb{R}^3$ by letting, in view of (3.2),

$$\zeta_3^\varepsilon(x_1, x_2) = \varepsilon \zeta_3(x_1, x_2) \qquad \text{for all} \quad (x_1, x_2) \in \omega. \tag{6.2}$$

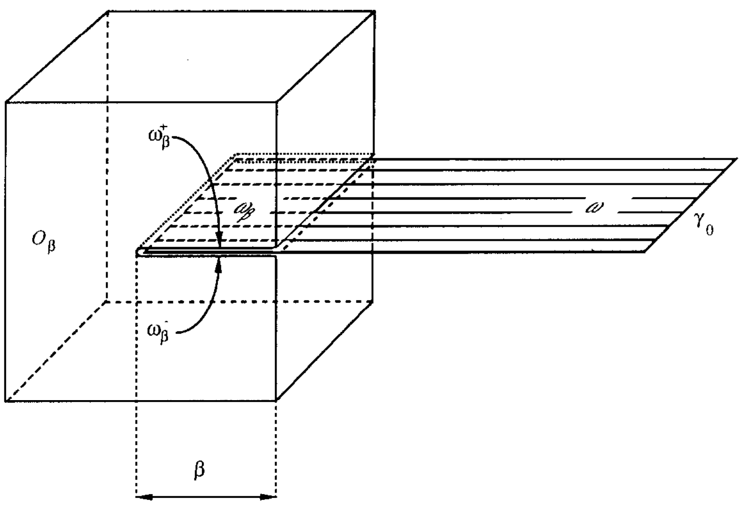

FIG. 3. The limit problem is a coupled, pluri-dimensional eigenvalue problem posed over the three-dimensional open set 0_β and the two-dimensional open set ω. The three-dimensional set has a two-dimensional slit into which the two-dimensional set is inserted.

In this fashion, we find that $(\Lambda, (\tilde{\mathbf{u}}^\varepsilon, \zeta_3^\varepsilon))$ solves the following equations:

(a) in the set 0_β and on its boundary,

$$-\tilde{\partial}_j^\varepsilon \tilde{\sigma}_{ij}(\tilde{\mathbf{u}}^\varepsilon) = \Lambda \tilde{\rho} \tilde{u}_i^\varepsilon \qquad \text{in } 0_\beta, \tag{6.3}$$

$$\tilde{\sigma}_{ij}(\tilde{\mathbf{u}}^\varepsilon)\, n_j = 0 \qquad \text{on } \partial 0_\beta - \{\omega_\beta^+ \cup \omega_\beta^-\}, \tag{6.4}$$

where $\tilde{\partial}_i^\varepsilon = \partial/\partial \tilde{x}_i^\varepsilon$,

$$\tilde{\sigma}_{ij}(\tilde{\mathbf{u}}^\varepsilon) = \tilde{\lambda} e_{pp}(\tilde{\mathbf{u}}^\varepsilon) + 2\tilde{\mu} e_{ij}(\tilde{\mathbf{u}}^\varepsilon), \tag{6.5}$$

and $\tilde{\lambda}$, $\tilde{\mu}$ are the "true" Lamé constants of the "three-dimensional" portion of the structure;

(b) in the set ω and on its boundary,

$$\frac{8\mu^\varepsilon(\lambda^\varepsilon + \mu^\varepsilon)}{3(\lambda^\varepsilon + 2\mu^\varepsilon)} \varepsilon^3 \Delta^2 \zeta_3^\varepsilon$$

$$= 2\Lambda \varepsilon \rho^\varepsilon \zeta_3^\varepsilon + \chi(\omega_\beta)\{\tilde{\sigma}_{33}(\tilde{\mathbf{u}}^\varepsilon)_{|\omega_\beta^+} - \tilde{\sigma}_{33}(\tilde{\mathbf{u}}^\varepsilon)_{|\omega_\beta^-}\} \qquad \text{in } \omega, \tag{6.6}$$

$$\zeta_3^\varepsilon = \partial_\nu \zeta_3^\varepsilon = 0 \qquad \text{on } \gamma_0, \tag{6.7}$$

$$m_{\alpha\beta}^\varepsilon(\zeta_3^\varepsilon)\, \nu_\alpha \nu_\beta = 0 \qquad \text{on } (\partial\omega - \gamma_0), \tag{6.8}$$

$$\partial_\tau (m_{\alpha\beta}^\varepsilon(\zeta_3^\varepsilon)\, \nu_\alpha \tau_\beta + \{\partial_\alpha m_{\alpha\beta}^\varepsilon(\zeta_3^\varepsilon)\}\, \nu_\beta = 0 \qquad \text{on } (\partial\omega - \gamma_0), \tag{6.9}$$

where

$$m_{\alpha\beta}^\varepsilon(\zeta_3) = -\frac{4\mu^\varepsilon \varepsilon^3}{3}\left\{\partial_{\alpha\beta}\zeta_3^\varepsilon + \frac{\lambda^\varepsilon}{\lambda^\varepsilon + 2\mu^\varepsilon} \Delta\zeta_3^\varepsilon \delta_{\alpha\beta}\right\}; \tag{6.10}$$

(c) at the "junction" between the sets 0_β and ω,

$$\tilde{u}^\varepsilon_{3\,|\,\omega_\beta^+} = \tilde{u}^\varepsilon_{3\,|\,\omega_\beta^-} = \zeta^\varepsilon_{3\,|\,\omega_\beta}, \tag{6.11}$$

$$\tilde{u}^\varepsilon_{\alpha\,|\,\omega_\beta^+} = \tilde{u}^\varepsilon_{\alpha\,|\,\omega_\beta^-} = 0. \tag{6.12}$$

(ii) A major conclusion is thus that $(\Lambda, (\tilde{\mathbf{u}}^\varepsilon, \zeta_3^\varepsilon))$ *solves a coupled, pluri-dimensional, eigenvalue problem of a new type, posed over a subspace of* $\mathbf{H}^1(0_\beta) \times H^2(\omega)$, *whose elements satisfy the junction conditions* (6.11–6.12). *Further, this problem is precisely the eigenvalue problem associated with the "static" problem found in Ciarlet, Le Dret & Nzengwa* [20].

(iii) Problem (6.3)–(6.12) may be equivalently formulated as a *variational problem*: *find all solutions* $(\Lambda, (\tilde{\mathbf{u}}^\varepsilon, \zeta_3^\varepsilon)) \in \mathbb{R} \times [\mathbf{H}^1(0) \times H^2(\omega)]_\beta$, where

$$[\mathbf{H}^1(0) \times H^2(\omega)]_\beta = \{(\tilde{\mathbf{v}}, \eta_3) \in \mathbf{H}^1(O) \times H^2(\omega), \qquad \eta_3 = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0,$$

$$\tilde{v}_{3\,|\,\omega_\beta} = \eta_{3\,|\,\omega_\beta},\ \tilde{v}_{\alpha\,|\,\omega_\beta} = 0\}, \tag{6.13}$$

such that

$$\int_O \tilde{\mathbf{A}}\mathbf{e}(\tilde{\mathbf{u}}^\varepsilon) : \mathbf{e}(\mathbf{v})\, d\tilde{x} + \varepsilon^3 \int_\omega \frac{4\mu^\varepsilon}{3} \left\{ \partial_{\alpha\beta}\, \zeta_3^\varepsilon + \frac{\lambda^\varepsilon}{\lambda^\varepsilon + 2\mu^\varepsilon} \Delta\zeta_3^\varepsilon\, \delta_{\alpha\beta} \right\} d\omega$$

$$= \Lambda \left\{ \int_0 \tilde{\rho}\tilde{u}_i^\varepsilon \tilde{v}_i\, d\tilde{x} + 2\varepsilon \int_\omega \rho^\varepsilon \zeta_3^\varepsilon \eta_3\, d\omega \right\}$$

$$\text{for all} \quad (\mathbf{v}, \eta_3) \in [\mathbf{H}^1(0) \times H^2(\omega)]_\beta. \tag{6.14}$$

This de-scaled limit problem provides an example of a *"stiff" spectral problem*, in the sense that different powers of ε (respectively, 0 and 3) appear in front of the two bilinear forms found in (6.14). For a *given* $\varepsilon > 0$ (in which case the data λ^ε, μ^ε, ρ^ε should be viewed as constant), this problem thus becomes amenable to the techniques developed by Lions [41, p. 184 ff.] for expanding the solution of (6.14) as a power series with respect to ε (see also Panasenko [45]).

In the same spirit, it would be worthwhile to derive the associated "limit" *time-dependent problem* (whose identification and justification should rely on the techniques developed by Raoult [47] for a "single" plate), which surely falls in the category of *"stiff" evolution problems*, as studied by Lions [40, 41].

(iv) While the junction conditions $\tilde{u}^\varepsilon_{3|\omega_\beta^+} = \tilde{u}^\varepsilon_{3|\omega_\beta^-} = \zeta^\varepsilon_{3|\omega_\beta}$ express the *continuity of the vertical displacement along the inserted portion of the plate*, the other junction condition $\tilde{u}^\varepsilon_{\alpha|\omega_\beta^+} = \tilde{u}^\varepsilon_{\alpha|\omega_\beta^-} = 0$ looks as though it involves only the three-dimensional part of the structure. This is an apparent paradox, for the convergence result obtained in Theorem 1 implies that the limit vector field $\mathbf{u} = (u_i)$ satisfies

$$\zeta_\alpha(x_1, x_2) \stackrel{\text{def}}{=} u_\alpha(x_1, x_2, 0) = 0. \tag{6.15}$$

Thus the de-scaled unknowns $\zeta_\alpha^\varepsilon : \omega \to \mathbb{R}$ defined by

$$\zeta_\alpha^\varepsilon(x_1, x_2) = \varepsilon^2 \zeta_\alpha(x_1, x_2), \tag{6.16}$$

in accordance with (3.1), satisfy

$$\zeta_\alpha^\varepsilon = 0 \qquad \text{in } \omega \tag{6.17}$$

to within the second order with respect to ε. Therefore, to within the first order with respect to ε, Conditions (6.12) may also be viewed as "true" junction conditions since $\tilde{u}_\alpha^\varepsilon = 0$ on ω_β to within the first order with respect to ε, by (5.10) and (6.1).

Note in passing that conditions (6.15) or their de-scaled counterparts (6.17) are in agreement with the conclusions reached by Ciarlet, Le Dret and Nzengwa [20] in the static case; there, it was found that applied forces with horizontal components of order $1/\varepsilon$ were needed in the plate in order to produce non-zero "limit" functions ζ_α (here the corresponding right hand sides $-\rho^\varepsilon \Lambda^\varepsilon u_\alpha^\varepsilon$ are of order ε).

(v) Relations (2.1) express that the *rigidity of the material constituting the thin portion of the structure should increase as* ε^{-3} *when* $\varepsilon \to 0$. *That such asymptotic orders are inevitable assumptions in order that a limit problem exists*, has already been observed by Caillerie [8] and Ciarlet [9] in the case of a "single" plate. The reader is referred to Ciarlet [9, 10, 13] for more detailed discussions of the meaning of such asymptotic orders.

(vi) The *controllability* of structures with junctions is a problem of outstanding *practical* interest, particularly in aerospace engineering, where the stabilization of large multi-structures, such as space stations, is a crucial problem. The controllability of the limit problem can then be approached by the *Hilbert Uniqueness Method* (HUM) of Lions [42, 43] (see also Lagnese and Lions [35] and Glowinski, Li and Lions [31]).

(vii) The numerical analysis of the limit eigenvalue problem found in Sect. 5 should be performed by methods adapted to its pluri-dimensional character, such as *model synthesis by substructuting methods* (see Bourquin [6]). There remains however the challenging assessment of the range of practical validity of the limit problem (in this direction, see Miara [44] for a single plate).

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