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The Cahn–Hilliard gradient theory for phase separation with non-smooth free energy Part I: Mathematical analysis

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A mathematical analysis is carried out for the Cahn–Hilliard equation where the free energy takes the form of a double well potential function with infinite walls. Existence and uniqueness are proved for a weak formulation of the problem which possesses a Lyapunov functional. Regularity results are presented for the weak formulation, and consideration is given to the asymptotic behaviour as the time becomes infinite. An investigation of the associated stationary problem is undertaken proving the existence of a nontrivial stationary solution and further regularity results for any stationary solution. Stationary solutions are constructed in one and two dimensions; a formula for the number of stationary solutions in one dimension is derived. It is then natural to study the asymptotic behaviour as the phenomenological parameter $\gamma \to 0$, the main result being that the interface between the two phases has minimal area.

1 Introduction

When a binary alloy (or mixture), comprising of species **A** and **B**, is prepared at a uniform temperature T_I, greater than the critical temperature T_c, the system is stable with mean composition u_m. Suppose now that the temperature is quenched (rapid reduction of temperature) to a temperature T_m less than T_c. Then experimentally one observes that the concentration $u(x,t)$, i.e., the difference between the mass fractions of each of the two species, of the alloy changes from the uniform mixed state to that of a spatially separated

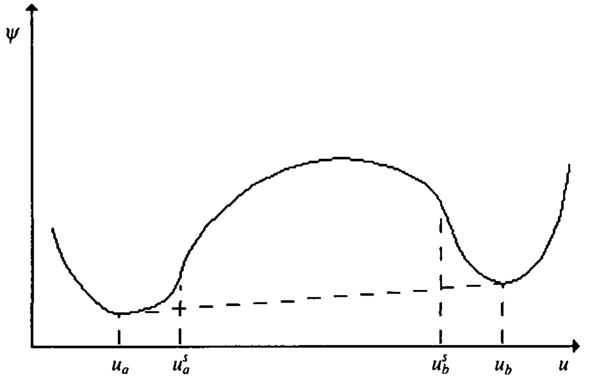

FIGURE 1. Free energy of the system below the critical temperature.

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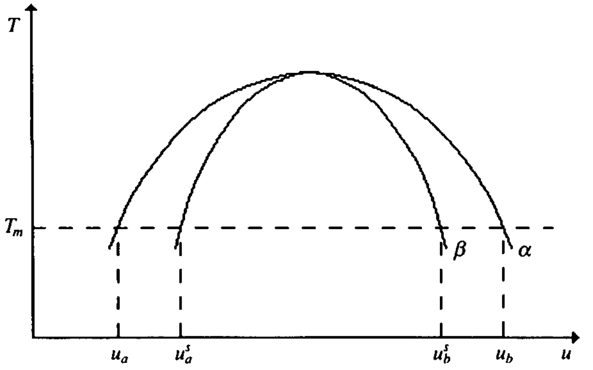

FIGURE 2. Phase diagram for a binary alloy.

two-phase structure, each phase being characterized by a different concentration value which is either u_a or u_b. The kinetics of this decomposition is called phase separation. We refer to Cahn & Hilliard (1958, 1971) and Cahn (1961). (See also the reviews by Gunton *et al.* 1983 and Skripov & Skripov 1979.)

A phenomenological theory describing the above is provided by consideration of a free energy $\psi(u, T)$, where for $T > T_c$, $\psi_{uu}(u, T) > 0$ and for $T < T_c$, $\psi_{uu}(u, T) < 0$ in just one interval $[u_a^s, u_b^s]$, called the spinodal interval (see figure 1). Connected with this description is the phase diagram depicted in figure 2. The spinodal curve β is the locus of points where $\psi_{uu}(u, T) = 0$. Above the coexistence curve α, any uniform concentration is stable. Below the spinodal curve, the state (u_m, T_m) is unstable, and the alloy separates into two values characterized by the values u_a and u_b, where the line $T = T_m$ crosses the coexistence curve.

In order to model surface energy of the interface separating the phases Cahn & Hilliard (1958) modify the free energy by adding the gradient term $\gamma|\nabla u|^2/2$ where $\gamma > 0$ so that the free energy becomes

$$\Psi = \psi(u) + \tfrac{1}{2}\gamma|\nabla u|^2, \tag{1.1}$$

and $\psi(\cdot)$ is called the homogeneous free energy. Van der Waals (1893) had previously used gradients to model the surface energy of the interfaces separating phases. The Cahn–Hilliard–van der Waals model for the equilibrium description of phase separation is thus to find

$$\begin{aligned} &\min \quad \mathscr{E}_\gamma(u) \\ &\text{subject to} \quad \int_\Omega u(x)\,\mathrm{d}x = u_m|\Omega|, \end{aligned} \tag{1.2}$$

where $\mathscr{E}_\gamma(\cdot)$ is the Ginzburg–Landau energy functional

$$\mathscr{E}_\gamma(u) = \tfrac{1}{2}\gamma\int_\Omega |\nabla u|^2\,\mathrm{d}x + \int_\Omega \psi(u)\,\mathrm{d}x. \tag{1.3}$$

It is noteworthy that the generalized chemical potential w is the functional derivative of $\mathscr{E}_\gamma$

$$w = \psi'(u) - \gamma\Delta u. \tag{1.4a}$$

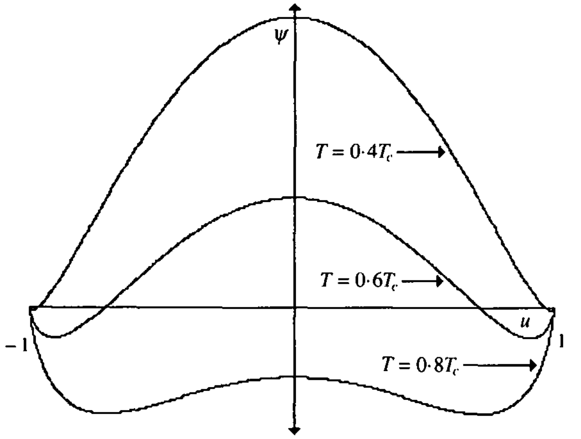

FIGURE 3. Non-differentiable homogeneous free energy for three different values of T.

The mass flux is given by $-M\nabla w$, where M is the mobility, so that the generalized diffusion equation for this non-equilibrium gradient theory of phase separation is (Cahn 1961)

$$\partial u/\partial t = \nabla \cdot (M\nabla w). \tag{1.4b}$$

(1.4a, b) can be written equivalently as

$$\partial u/\partial t = \nabla \cdot (M\nabla(\psi'(u) - \gamma\Delta u)), \quad x \in \Omega, t > 0. \tag{1.5}$$

This fourth-order in space, nonlinear time-dependent partial differential equation is called the Cahn–Hilliard equation. For a closed system there is no mass flux so that

$$M(\nabla w)\cdot \underline{n} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.6a}$$

and for the other boundary condition we take the natural boundary condition associated with the variational problem (1.2),

$$\gamma(\nabla u)\cdot \underline{n} = 0 \quad \text{on} \quad \partial\Omega. \tag{1.6b}$$

The initial boundary-value problem for a closed system is then to solve (1.5) subject to the boundary conditions (1.6*a*, *b*), and the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{1.7}$$

The form of $u_0(x)$ that is of interest in modelling the quenching process described earlier is

$$u_0(x) = u_m + \xi(x), \quad \int_\Omega \xi(x)\,dx = 0, \quad |\xi(x)| \ll 1. \tag{1.8}$$

The solution to the initial value problem satisfies

$$\frac{d\mathscr{E}_\gamma(u)}{dt} = \int_\Omega [\psi'(u)\,u_t + \gamma\nabla u\,\nabla u_t]\,dx = \int_\Omega w u_t\,dx = -\int_\Omega M|\nabla w|^2\,dx, \tag{1.9a}$$

$$\frac{d}{dt}\int_\Omega u\,dx = 0, \quad \int_\Omega u(x,t)\,dx = \int_\Omega u_0\,dx. \tag{1.9b}$$

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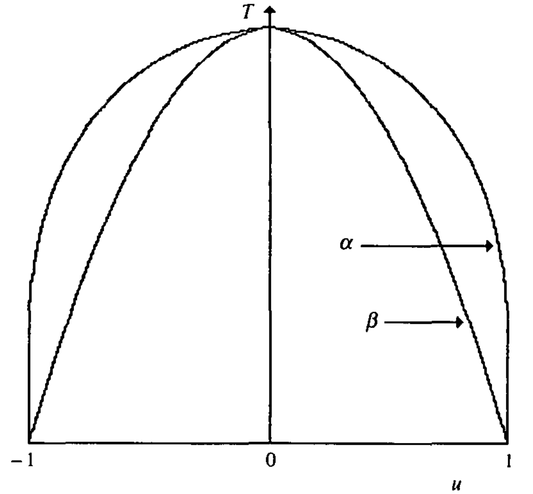

FIGURE 4. Phase diagram for non-differentiable homogeneous free energy.

This is in accordance with the requirement for this model of kinetics of phase separation that the evolution of a non-equilibrium composition is to a composition of lower energy whilst conserving the mass. Indeed, it is natural to ask the questions: does the time-dependent solution to the initial value problem converge to a minimizer of the energy as $t \to \infty$, and if we were to let $\gamma \to 0$ for a sequence of minimizers, would the measure of the interface be minimized?

A fuller description of the above, and some further considerations, are given in a review of the mathematical and numerical analysis which may be found in Elliott (1989).

Up to this point we have assumed that the homogeneous free energy ψ is differentiable in u. We now consider ψ to be of the form

$$\psi(u) = \psi(0, T) - \tfrac{1}{2}kT_c u^2 + \tfrac{1}{2}kT[(1-u)\log_e(1-u) + (1+u)\log_e(1+u)] \qquad (1.10)$$

(see figure 3), where k is Boltzmann's constant, T is considered as a parameter, $\psi(0, T)$ is continuous in T and the concentration u varies between the values ± 1, which correspond to either atoms of type **A**, say, or type **B**. This form of the free energy was suggested by Cahn & Hilliard (1958), where

$$\psi(0, T) = \tfrac{1}{2}kT_c - kT\log_e 2.$$

Since

$$\psi'(u) = -kT_c u + \tfrac{1}{2}kT\log_e\left[\frac{1+u}{1-u}\right], \quad \psi''(u) = -kT_c + \frac{kT}{1-u^2}, \qquad (1.11)$$

it follows that for $T > T_c$, ψ is a convex function on $(-1, 1)$, and for $T < T_c$, ψ has the required double-well form. The values u_a and u_b defining the minima of $\psi(\cdot)$ are $u_b = -u_a = \beta$, where β is the positive root of

$$2T_c/T = \log_e[(1+\beta)/(1-\beta)]/\beta. \qquad (1.12)$$

Furthermore, the spinodal interval in which $\psi''(\cdot)$ is negative is $(-(1-T/T_c)^{\frac{1}{2}}, (1-T/T_c)^{\frac{1}{2}})$ (see figure 4 for the phase diagram associated with ψ). So, in summary, this free energy has the desired properties described above.

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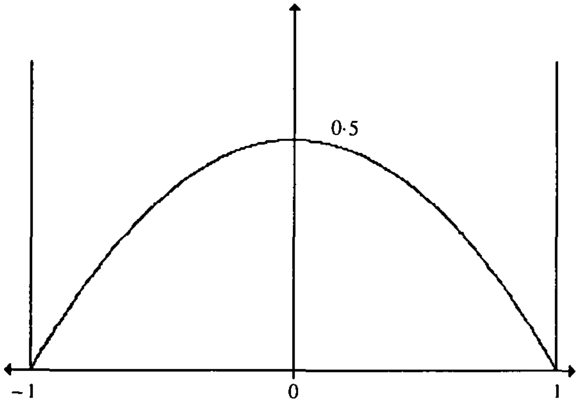

FIGURE 5. Homogeneous free energy under consideration.

For T close to T_c we may approximate $\psi(\cdot)$ by the simple quartic polynomial retaining the double well form, namely

$$\psi(u) = \tfrac{1}{4}\alpha(u^2-\beta^2)^2. \tag{1.13}$$

It is in this form of ψ that the Cahn–Hilliard equation has been widely studied from the analytical and numerical point of view (cf. Elliott & Zheng 1986; Novick-Cohen & Segel 1984; Copetti & Elliott 1990). A mathematical analysis for arbitrary polynomial ψ has been carried out by von Wahl (1985), Témam (1988) and Nicaelenko *et al.* (1989).

We consider a limit corresponding to a deep quench for which $T/T_c \ll 1$. it is clear that as $T/T_c \to 0$, the spinodal interval expands to $(-1, 1)$, and ψ'' tends to a constant. For this reason, we study the homogeneous free energy depicted in figure 5 given by

$$\psi(u) = \begin{cases} \tfrac{1}{2}(1-u^2) & \text{if } |u| \leqslant 1, \\ +\infty & \text{if } |u| > 1. \end{cases} \tag{1.14}$$

This form of the free energy has been proposed by Oono & Puri (1988), who performed a numerical study of a discrete cell dynamical system.

The Ginzburg–Landau energy functional $\mathscr{E}_\gamma(\cdot)$ is still defined by (1.3), but since $\mathscr{E}_\gamma(\cdot)$ is now non-differentiable, the definition of the generalized chemical potential formally becomes

$$w + \gamma\Delta u + u \in \partial I(u), \tag{1.15}$$

where $\partial I(\cdot)$ is the subdifferential of the indicator function $I(\cdot)$ of the set $[-1, 1]$.

Rescaling x and t, and taking the mobility to be a constant, we are led to the following free-boundary problem:

Given $\gamma > 0$ find $\{u(x,t), w(x,t)\}$ such that on $\Omega_T = \Omega \times (0, T)$,

$$\partial u/\partial t = \Delta w, \tag{1.16}$$

$$\begin{gathered} (|u|-1)(-\gamma\Delta u - u - w) = 0, \\ (\gamma\Delta u + u + w)\operatorname{sign} u \geqslant 0, \\ |u| \leqslant 1, \end{gathered} \tag{1.17}$$

and $|u| = 1$, $\nabla u = 0$ on the free boundary, subject to the initial and boundary conditions

$$u(x,0) = u_0(x) \quad \text{for} \quad x \in \Omega, \quad \partial u/\partial \nu = \partial w/\partial \nu = 0 \quad \text{on} \quad \partial\Omega, \tag{1.18}$$

where $\partial\Omega$ is smooth. This corresponds to the chemical potential being

$$w = -\gamma\Delta u - u, \quad |u| < 1. \tag{1.19}$$

This problem can be studied with periodic boundary conditions.

We now give a brief description of the contents of this paper. In §2 a global existence theorem for a weak formulation possessing a Lyapunov functional is proven. Regularity results are presented for the weak formulation, and consideration is given to the asymptotic behaviour of $u(x, t)$ in time.

In §3 an investigation of the associated stationary problem is undertaken, proving the existence of a non-trivial stationary solution for γ small enough, and further regularity results for any stationary solution. Stationary solutions are constructed in one and two space dimensions; a formula for the number of stationary solutions in one dimension is derived. It is then natural to study the asymptotic behaviour as $\gamma \to 0$ of a sequence of minimizers of $\mathscr{E}_\gamma(\cdot)$ over the set where the mass is fixed and modulus is less than or equal to one almost everywhere. In particular, the existence of a sequence of minimizers which converges to a function in $L^1(\Omega)$ is proven. The limit is piecewise ± 1 *a.e.* with the interface between the two sets having minimal area. Moreover, the associated sequence of Lagrange multipliers for the sequence of minimizers converges to 0 at a prescribed rate. These results are an application of the theorems of Modica (1987) and Luckhaus & Modica (1989).

Remarks This problem also arises in the study of the Stefan problem with surface tension. It corresponds to a model proposed by Visintin (1984, 1989) where u denotes the phase parameter, w the temperature, and the specific heat is zero. An existence proof for a weak solution of a related initial boundary value problem is given by Visintin (1984).

After this work was completed, Elliott & Luckhaus (1990) then showed that (1.5), (1.6a, b) and (1.7) have a unique solution with ψ given by (1.10), and that as $T/T_c \to 0$ this solution converges to a solution of (1.16)–(1.18).

2 Evolutionary problem

2.1 Existence and uniqueness

Throughout, Ω denotes a bounded domain in $\mathbb{R}^d$, $(d = 1, 2, 3)$, we denote the norm of $H^p(\Omega)$ $(p \geqslant 0)$ by $\|\cdot\|_p$, the semi-norm $\|D^p\eta\|_0$ by $|\eta|_p$ and the $L^2(\Omega)$ inner-product by $(.,.)$. For $d = 2, 3$ we assume that $\partial\Omega$ is Lipschitz continuous.

We introduce the Green's operator $\mathscr{G}_N$ for the inverse of the Laplacian with zero Neumann boundary data: given $f \in \mathscr{F} \equiv \{f \in (H^1(\Omega))' : \langle f, 1\rangle = 0\}$ we define $\mathscr{G}_N f \in H^1(\Omega)$ to be the unique solution of

$$(\nabla \mathscr{G}_N f, \nabla \eta) = \langle f, \eta\rangle \quad \forall \eta \in H^1(\Omega), \tag{2.1 a}$$

$$(\mathscr{G}_N f, 1) = 0, \tag{2.1 b}$$

where $\langle .,. \rangle$ denotes the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$ such that

$$\langle f, \eta\rangle \equiv (f, \eta) \quad \forall f \in L^2(\Omega). \tag{2.2}$$

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The existence and uniqueness of $\mathscr{G}_N f$ follows from the Poincaré inequality

$$|\eta|_0 \leqslant C_P\{|(\eta, 1)| + |\eta|_1\} \quad \forall \eta \in H^1(\Omega), \tag{2.3}$$

and the Lax–Milgram theorem.

For $f \in \mathscr{F}$ we define

$$\|f\|_{-1} \equiv |\mathscr{G}_N f|_1, \tag{2.4}$$

and note that if $f \in \mathscr{F} \cap L^2(\Omega)$ then

$$\|f\|_{-1} \equiv (\mathscr{G}_N f, f)^{\frac{1}{2}}. \tag{2.5}$$

For $f \in \mathscr{F} \cap L^2(\Omega)$, the Poincaré inequality (2.3) and (2.5) yields

$$\|f\|_{-1}^2 \leqslant |f|_0 |\mathscr{G}_N f|_0 \leqslant C_P |f|_0 |\mathscr{G}_N f|_1 = C_P |f|_0 \|f\|_{-1},$$

so that

$$\|f\|_{-1} \leqslant C_P |f|_0. \tag{2.6}$$

Given $\gamma > 0$ and $u_0 \in K = \{\eta \in H^1(\Omega): -1 \leqslant \eta \leqslant 1\}$ with $m \equiv (u_0, 1) \in (-|\Omega|, |\Omega|)$, we consider the problems **(P)** and **(Q)** defined as follows:

(P) Find $\{u, w\}$ such that $u \in H^1(0, T; (H^1(\Omega))') \cap L^\infty(0, T; H^1(\Omega))$, $u \in K$ for *a.e.* $t \in (0, T)$ and $w \in L^2(0, T; H^1(\Omega))$

$$\langle \partial u/\partial t, \eta \rangle + (\nabla w, \nabla \eta) = 0 \quad \forall \eta \in H^1(\Omega), \quad a.e.\ t \in (0, T), \tag{2.7a}$$

$$\gamma(\nabla u, \nabla \eta - \nabla u) - (u, \eta - u) \geqslant (w, \eta - u) \quad \forall \eta \in K, \quad a.e.\ t \in (0, T), \tag{2.7b}$$

and

$$u(0) = u_0. \tag{2.7c}$$

(Q) Find $u \in H^1(0, T; (H^1(\Omega))') \cap L^\infty(0, T; H^1(\Omega))$, $u \in K_m$ for *a.e.* $t \in (0, T)$ such that

$$\gamma(\nabla u, \nabla \eta - \nabla u) + (\mathscr{G}_N \partial u/\partial t, \eta - u) - (u, \eta - u) \geqslant 0 \quad \forall \eta \in K_m, \quad a.e.\ t \in (0, T), \tag{2.8a}$$

and

$$u(0) = u_0, \tag{2.8b}$$

where $K_m = \{\eta \in K: (\eta, 1) = (u_0, 1) = m\}$.

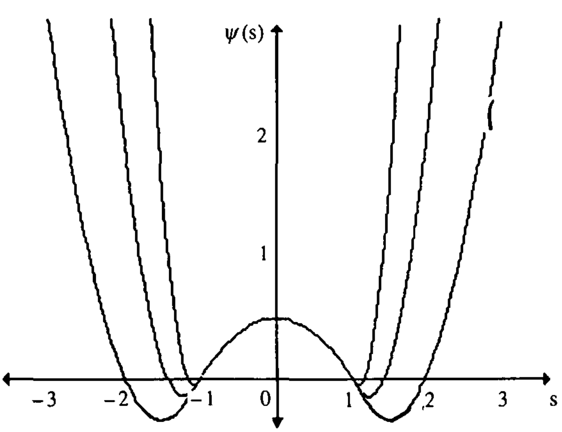

FIGURE 6. Penalized homogeneous free energy ψ_ϵ for three values of ϵ.

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Remarks

(1) We note that if $|m| = |\Omega|$ then the problems have the unique trivial solution $u(x,t) = \operatorname{sign} m$, $\forall (x,t) \in \Omega \times (0,T)$, and if $m > |\Omega|$ then there cannot be a solution.

(2) Plainly, if $\{u, w\}$ solves (**P**), then we can see that u solves (**Q**) simply by observing that

$$w = -\mathcal{G}_N \partial u/\partial t + \lambda,$$

where $\lambda = (w, 1)/|\Omega|$.

Given $0 < \epsilon < 1$ we introduce the homogeneous free energy $\psi_\epsilon \in C^2(\mathbb{R})$ defined as follows:

$$\psi_\epsilon(r) := \begin{cases} \dfrac{1}{2\epsilon}\left(r - \left(1 + \dfrac{\epsilon}{2}\right)\right)^2 + \tfrac{1}{2}(1-r^2) + \dfrac{\epsilon}{24} & \text{for} \quad r \geqslant 1+\epsilon, \\ \dfrac{1}{6\epsilon^2}(r-1)^3 + \tfrac{1}{2}(1-r^2) & \text{for} \quad 1 < r < 1+\epsilon, \\ \tfrac{1}{2}(1-r^2) & \text{for} \quad |r| \leqslant 1, \\ -\dfrac{1}{6\epsilon^2}(r+1)^3 + \tfrac{1}{2}(1-r^2) & \text{for} \quad -1-\epsilon < r < -1, \\ \dfrac{1}{2\epsilon}\left(r + \left(1 + \dfrac{\epsilon}{2}\right)\right)^2 + \tfrac{1}{2}(1-r^2) + \dfrac{\epsilon}{24} & \text{for} \quad r \leqslant -1-\epsilon, \end{cases} \tag{2.9}$$

(see figure 6). It is easy to show that for $\epsilon < \frac{1}{4}$

$$\psi_\epsilon(r) \geqslant -C_0\,\epsilon, \tag{2.10}$$

where C_0 is a positive constant bounded independently of ϵ. We define $\beta_\epsilon \in C^1(\mathbb{R})$ as follows:

$$\beta_\epsilon(r) = \epsilon(r + \psi_\epsilon'(r)) := \begin{cases} r - \left(1 + \dfrac{\epsilon}{2}\right) & \text{for} \quad r \geqslant 1+\epsilon, \\ \dfrac{1}{2\epsilon}(r-1)^2 & \text{for} \quad 1 < r < 1+\epsilon, \\ 0 & \text{for} \quad |r| \leqslant 1, \\ -\dfrac{1}{2\epsilon}(r+1)^2 & \text{for} \quad -1-\epsilon < r < -1, \\ r + \left(1 + \dfrac{\epsilon}{2}\right) & \text{for} \quad r \leqslant -1-\epsilon. \end{cases} \tag{2.11}$$

We note that β_ϵ is a Lipschitz continuous function where

$$0 \leqslant \beta_\epsilon' \leqslant 1. \tag{2.12}$$

It will also be useful to introduce the convex function

$$\hat{\psi}_\epsilon(r) := \psi_\epsilon(r) - \tfrac{1}{2}(1-r^2), \tag{2.13}$$

which obviously satisfies

$$\hat{\psi}_\epsilon' = \tfrac{1}{\epsilon}\beta_\epsilon \quad \text{and} \quad 0 \leqslant \hat{\psi}_\epsilon'' \leqslant 1/\epsilon. \tag{2.14}$$

Also, it follows from the definition of $\hat{\psi}_\epsilon$ that

$$\hat{\psi}_\epsilon(r) \geqslant \frac{1}{2\epsilon}\beta_\epsilon(r)^2, \tag{2.15}$$

and from the convexity of $\hat{\psi}_\epsilon$ that

$$\hat{\psi}_\epsilon(r) \geqslant \hat{\psi}_\epsilon(s) + \frac{1}{\epsilon}\beta_\epsilon(s)(r-s). \tag{2.16}$$

It is convenient to introduce the following penalized problem:

$(\mathbf{P}_\epsilon)$ For $\epsilon > 0$, find $\{u_\epsilon, w_\epsilon\}$ such that $u_\epsilon \in H^1(0,T;(H^1(\Omega))') \cap L^\infty(0,T;H^1(\Omega))$ and $w_\epsilon \in L^2(0,T;H^1(\Omega))$,

$$\left\langle \frac{\partial u_\epsilon}{\partial t}, \eta \right\rangle + (\nabla w_\epsilon, \nabla \eta) = 0 \quad \forall \eta \in H^1(\Omega), \quad a.e.\ t \in (0,T), \tag{2.17a}$$

$$\gamma(\nabla u_\epsilon, \nabla \eta) + (\psi'_\epsilon(u_\epsilon), \eta) = (w_\epsilon, \eta) \quad \forall \eta \in H^1(\Omega), \quad a.e.\ t \in (0,T), \tag{2.17b}$$

and

$$u_\epsilon(0) = u_0. \tag{2.17c}$$

$(\mathbf{P}_\epsilon)$ is a weak formulation of the Cahn–Hilliard equation (1.5).

Theorem 2.1 *For $0 < \epsilon < \frac{1}{4}$ there exists a unique solution to $(\mathbf{P}_\epsilon)$ such that:*

$$\|u_\epsilon\|_{H^1(0,T;(H^1(\Omega))')} \leqslant C, \tag{2.18a}$$

$$\|u_\epsilon\|_{L^\infty(0,T;H^1(\Omega))} \leqslant C, \tag{2.18b}$$

$$\|w_\epsilon\|_{L^2(0,T;H^1(\Omega))} \leqslant C(1+T^{\frac{1}{2}}), \tag{2.18c}$$

for constants C independent of ϵ and T.

Proof We will prove existence using the classical Galerkin method of Lions (1969). Let $\{z_j\}_{j=1}^\infty$ be the orthogonal basis for $H^1(\Omega)$ consisting of the eigenfunctions for

$$-\Delta z + z = \mu z; \quad \partial z/\partial \nu = 0, \tag{2.19}$$

and normalized so that

$$(z_i, z_j) = \delta_{ij}. \tag{2.20}$$

Note that $\{z_j\}$ is an orthonormal basis for $L^2(\Omega)$. Let V^k denote the finite dimensional subspace of $H^1(\Omega)$ spanned by $\{z_j\}_{j=1}^k$. A Galerkin approximation to $(\mathbf{P}_\epsilon)$ is the following:

$$u^k(t) = \sum_{j=1}^{k} c_j(t)\, z_j, \quad w^k(t) = \sum_{j=1}^{k} \mathrm{d}_j(t)\, z_j, \tag{2.21a}$$

$$(du^k/\mathrm{d}t, \eta^k) + (\nabla w^k, \nabla \eta^k) = 0 \quad \forall \eta^k \in V^k, \tag{2.21b}$$

$$\gamma(\nabla u^k, \nabla \eta^k) + \frac{1}{\epsilon}(\beta_\epsilon(u^k), \eta^k) - (u^k, \eta^k) = (w^k, \eta^k) \quad \forall \eta^k \in V^k, \tag{2.21c}$$

$$u^k(0) = P^k(u_0), \tag{2.21d}$$

where $P^k: H^1(\Omega) \to V^k$ is the projection defined by:

$$P^k v = \sum_{j=1}^{k} (v, z_j) z_j \quad \forall \eta^k \in V^k \quad \text{and} \quad (P^k v - v, \eta^k) = 0 \quad \forall \eta^k \in V^k. \tag{2.22}$$

It is easy to prove global existence and uniqueness for u^k and w^k by a standard result in systems of ordinary differential equations.

Let us consider the Ginzburg–Landau free energy functional

$$\mathscr{E}^\epsilon(v) := \tfrac{1}{2}\gamma |v|_1^2 + (\psi_\epsilon(v), 1) \; v \in H^1(\Omega). \tag{2.23}$$

Since $\mathrm{d}u^k/\mathrm{d}t \in V^k$ for each t, differentiating $\mathscr{E}^\epsilon(u^k)$ with respect to t we obtain

$$\begin{aligned} \frac{\mathrm{d}\mathscr{E}^\epsilon(u^k)}{\mathrm{d}t} &= \gamma\left(\frac{\nabla u^k, \nabla \mathrm{d}u^k}{\mathrm{d}t}\right) + \left(\frac{\psi_\epsilon'(u^k), \mathrm{d}u^k}{\mathrm{d}t}\right) = \left(\frac{\omega^k, \mathrm{d}u^k}{\mathrm{d}t}\right), \\ &= -|w^k|_1^2. \end{aligned} \tag{2.24}$$

Integrating over $(0, t)$, we obtain

$$\mathscr{E}^\epsilon(u^k) + \int_0^t |w^k(s)|_1^2 \, \mathrm{d}s \leqslant \mathscr{E}^\epsilon(P^k(u_0)). \tag{2.25}$$

As $|P^k(u_0)|_1 \leqslant |u_0|_1$, (2.13), setting $s = P^k u_0$ and $r = u_0$ in (2.16), $-1 \leqslant u_0 \leqslant 1$, from Lipschitz continuity of β_ϵ and the strong convergence of $P^k u_0$ to u_0 in $L^2(\Omega)$ it follows that

$$\begin{aligned} \mathscr{E}^\epsilon(P^k u_0) &= \tfrac{1}{2}\gamma |P^k u_0|_1^2 + (\psi_\epsilon(P^k u_0), 1), \\ &\leqslant \tfrac{1}{2}\gamma |u_0|_1^2 + \tfrac{1}{2}(1 - (P^k u_0)^2, 1) + \frac{1}{\epsilon}(\beta_\epsilon(P^k u_0), P^k u_0 - u_0), \end{aligned}$$

and hence
$$\limsup_{k\to\infty} \mathscr{E}^\epsilon(P^k u_0) \leqslant \mathscr{E}^\epsilon(u_0) = \mathscr{E}_\gamma(u_0); \tag{2.26}$$

in particular, since $u_0 \in H^1(\Omega)$

$$\mathscr{E}^\epsilon(P^k(u_0)) \leqslant C,$$

where C is independent of ϵ and k.

Now from (2.10) for $\epsilon < \frac{1}{4}$, we have

$$(\psi_\epsilon(u^k), 1) \geqslant -C_0 |\Omega| \, \epsilon.$$

So (2.23) and (2.25) yield

$$\tfrac{1}{2}\gamma |u^k|_1^2 + \int_0^t |w^k(s)|_1^2 \, \mathrm{d}s \leqslant C, \tag{2.27}$$

where C is independent of T, ϵ and k. From (2.22) we have

$$\begin{aligned} \left\| \frac{\mathrm{d}u^k}{\mathrm{d}t} \right\|_{-1}^2 &= \left(\frac{\mathrm{d}u^k}{\mathrm{d}t}, \mathscr{G}_N \frac{\mathrm{d}u^k}{\mathrm{d}t}\right) = \left(\frac{\mathrm{d}u^k}{\mathrm{d}t}, P^k \mathscr{G}_N \frac{\mathrm{d}u^k}{\mathrm{d}t}\right) = -\left(\nabla w^k, \nabla P^k \mathscr{G}_N \frac{\mathrm{d}u^k}{\mathrm{d}t}\right), \\ &= -\left(\nabla w^k, \nabla \mathscr{G}_N \frac{\mathrm{d}u^k}{\mathrm{d}t}\right) = -\left(w^k, \frac{\mathrm{d}u^k}{\mathrm{d}t}\right) = |w^k|_1^2, \end{aligned} \tag{2.28}$$

so (2.25) implies the bound

$$\left\| \frac{\mathrm{d}u^k}{\mathrm{d}t} \right\|_{L^2(0,T;(H^1(\Omega))')} \leqslant C, \tag{2.29}$$

where C is independent of T, ϵ and k.

Now substituting $\eta^k = 1/|\Omega|^{\frac{1}{2}}$ in (2.21 b, c) yields

$$\left(\frac{\mathrm{d}u^k}{\mathrm{d}t}, 1\right) = 0 = (u^k, 1) + (w^k, 1) - \frac{1}{\epsilon}(\beta_\epsilon(u^k), 1), \tag{2.30}$$

$$(u^k(t), 1) = (u(0), 1) = (P^k(u_0), 1). \tag{2.31}$$

From (2.31), (2.27) and the Poincaré inequality, (2.3), we obtain the bound

$$\|u^k\|_{L^\infty(0,T;H^1(\Omega))} \leqslant C. \tag{2.32}$$

In order to obtain a bound on w^k in $H^1(\Omega)$ we must estimate $|(w^k, 1)|$. As $\beta_\epsilon \equiv 0$ on $[-1, 1]$ it follows from the definition of β_ϵ that $|\beta_\epsilon(r)| \leqslant r\beta_\epsilon(r)$, hence

$$\frac{1}{\epsilon}(|\beta_\epsilon(u_k)|, 1) \leqslant \frac{1}{\epsilon}(\beta_\epsilon(u^k), u^k). \tag{2.33}$$

Using (2.30) it follows that

$$|(w^k, 1)| \leqslant |(u^k, 1)| + \frac{1}{\epsilon}(|\beta_\epsilon(u^k)|, 1),$$

$$\leqslant |(u_0, 1)| + \frac{1}{\epsilon}(\beta_\epsilon(u^k), u^k).$$

So, setting $\eta^k = u^k$ in (2.21 c) we obtain

$$|(w^k, 1)| \leqslant |(u_0, 1)| + |u^k|_0^2 - \gamma|u^k|_1^2 + (w^k, u^k). \tag{2.34}$$

Now by using the definition of $\mathscr{G}_N$, (2.1 a b), and noting that $(u^k, 1) = m$ we obtain

$$\begin{aligned}(w^k, u^k) &= (-u^k, 1/|\Omega|(u^k, 1)) + (u^k, 1)(w^k, 1)/|\Omega|,\\ &= (\nabla w^k, \nabla \mathscr{G}_N(u^k - m/|\Omega|)) + m(w^k, 1)/|\Omega|,\\ &\leqslant |w^k|_1 |\mathscr{G}_N(u^k - m/|\Omega|)|_1 + m(w^k, 1)/|\Omega|,\end{aligned} \tag{2.35}$$

because $|(u_0, 1)| = |m| < |\Omega|$. Upon rearranging (2.34) using (2.35) and (2.6) we obtain

$$|(w^k, 1)| \leqslant \frac{|m| + |u^k|_0^2 - \gamma|u^k|_1^2 + |w^k|_1 \| u^k - m/|\Omega| \|_{-1}}{1 - |m|/|\Omega|},$$

$$\leqslant \frac{|m| + |u^k|_0^2 - \gamma|u^k|_1^2 + C_P|w^k|_1 \| u^k - m/|\Omega| \|_0}{1 - |m|/|\Omega|}. \tag{2.36}$$

Hence using the Poincaré inequality (2.3)

$$\|w^k\|_1 \leqslant C + C|w^k|_1, \tag{2.37}$$

so $w^k(t)$ is bounded in $L^2(0, T; H^1(\Omega))$ independently of k and ϵ.

From compactness arguments we deduce the existence of subsequences $\{u^k, w^k\}$ having the following properties:

$$u^k \rightharpoonup u_\epsilon \quad \text{in} \quad H^1(0,T;(H^1(\Omega))') \cap L^2(0,T;H^1(\Omega)), \tag{2.38a}$$

$$u^k \rightharpoonup^* u_\epsilon \quad \text{in} \quad L^\infty(0,T;H^1(\Omega)), \tag{2.38b}$$

$$w^k \rightharpoonup w_\epsilon \quad \text{in} \quad L^2(0,T;H^1(\Omega)), \tag{2.38c}$$

$$u^k \to u_\epsilon \quad \text{in} \quad L^2(\Omega_T), \tag{2.38d}$$

(2.38d) being a consequence of a compactness theorem (cf. Lions 1969). Furthermore, $H^1(0,T;(H^1(\Omega))') \cap L^2(0,T;H^1(\Omega)) \hookrightarrow C([0,T];L^2(\Omega))$ (cf. Témam 1977), which together with (2.38a) and the strong convergence in $L^2(\Omega)$ of $P^k(u_0)$ to u_0 implies that $u_\epsilon(0) = u_0$.

For any $\eta \in H^1(\Omega)$ set $\eta^k = P^k\eta$ in (2.21b, c). We can immediately pass to the limit *a.e.* in (2.21b) to obtain (2.7a). To yield the result it remains to prove that

$$(\beta_\epsilon(u^k), \eta^k) \to (\beta_\epsilon(u_\epsilon), \eta) \quad \text{as} \quad k \to \infty.$$

This is proved quite simply using the Lipschitz continuity of β_ϵ, properties of P^k and the strong convergence of u^k to u_ϵ in $L^2(\Omega)$:

$$\begin{aligned} |(\beta_\epsilon(u^k), \eta^k) - (\beta_\epsilon(u_\epsilon), \eta)| &\leqslant |(\beta_\epsilon(u^k) - \beta_\epsilon(u_\epsilon), \eta^k) - (\beta_\epsilon(u_\epsilon), \eta - \eta^k)|, \\ &\leqslant |u^k - u_\epsilon|_0 |\eta|_0 + |\beta_\epsilon(u_\epsilon)|_0 |\eta - \eta^k|_0 \to 0 \quad \text{as} \quad k \to \infty. \end{aligned}$$

Hence we obtain

$$\gamma(\nabla u_\epsilon, \nabla \eta) + (\psi_\epsilon'(u_\epsilon), \eta) = (w_\epsilon, \eta).$$

Finally, we prove uniqueness. Let $\{u_1, w_1\}$ and $\{u_2, w_2\}$ be two solutions to $(\mathbf{P}_\epsilon)$, define $\theta^u = u_1 - u_2$ and $\theta^w = w_1 - w_2$. Subtract (2.17a), when u_1 is the solution, from (2.17a), when u_2 is the solution, and subtract (2.17b), when u_1 is the solution and $\eta = \theta^u$, from (2.17b), when u_2 is the solution and $\eta = \theta^u$. Using the monotonicity of β_ϵ yields

$$\langle \partial\theta^u/\partial t, \eta \rangle + (\nabla\theta^w, \nabla\eta) = 0 \quad \forall \eta \in H^1(\Omega), \tag{2.39a}$$

$$\gamma|\theta^u|_1^2 - |\theta^u|_0^2 \leqslant (\theta^w, \theta^u). \tag{2.39b}$$

Setting $\eta = \mathscr{G}_N \theta^u$ in (2.39a) and substituting into (2.39b) yields

$$\langle \partial\theta^u/\partial t, \mathscr{G}_N \theta^u \rangle + \gamma|\theta^u|_1^2 \leqslant |\theta^u|_0^2. \tag{2.40}$$

Now using the definition of $\mathscr{G}_N$, (2.1a, b), the identity

$$\langle \partial\theta^u/\partial t, \mathscr{G}_N \theta^u \rangle = \frac{\mathrm{d}}{\mathrm{d}t} \|\theta^u\|_{-1}^2,$$

and the Cauchy–Schwarz inequality, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\theta^u\|_{-1}^2 + \gamma|\theta^u|_1^2 \leqslant |\theta^u|_0^2 = (\nabla\theta^u \, \nabla\mathscr{G}_N \theta^u) \leqslant \tfrac{1}{2}\gamma|\theta^u|_1^2 + \frac{1}{2\gamma} \|\theta^u\|_{-1}^2,$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\theta^u\|_{-1}^2 - \gamma|\theta^u|_1^2 \leqslant \frac{1}{\gamma} \|\theta^u\|_{-1}^2.$$

Multiplying through by $\exp^{-t/\gamma}$ and integrating over $(0, t)$ yields

$$\exp^{-t/\gamma}\|\theta^u(t)\|_{-1}^2+\gamma\int_0^t \exp^{-s/\gamma}|\theta^u(s)|_1^2\,\mathrm{d}s \leqslant \|\theta^u(0)\|_{-1}^2 = 0. \tag{2.41}$$

Also noting that from (2.17a), $(\theta^u, 1) = 0$ and the Poincaré inequality (2.3) we obtain the uniqueness of u. Now using (2.17b) with $\eta = 1$ and the uniqueness of u we obtain $(\theta^w, 1) = 0$. Also

$$|\theta^w|_1^2 = -\langle \partial\theta^u/\partial t, \theta^w\rangle = 0,$$

and so again from the Poincaré inequality (2.3) we obtain uniqueness of w_ϵ, thus proving existence and uniqueness to the problem $(\mathbf{P}_\epsilon)$. □

Remark It will be of use to note that, setting $s = u_\epsilon$ and $r = u^k$ in (2.16), noting (2.13), and the convergence of u^k to u_ϵ in $H^1(\Omega)$, yields

$$\begin{aligned}\mathscr{E}^\epsilon(u^k) &= \tfrac{1}{2}\gamma|u^k|_1^2+(\psi_\epsilon(u^k), 1),\\ &\geqslant \mathscr{E}_\gamma(u^k)+(\hat{\psi}_\epsilon(u_\epsilon), 1)+\frac{1}{\epsilon}(\beta_\epsilon(u_\epsilon), u^k-u_\epsilon);\end{aligned}$$

hence
$$\liminf_{k\to\infty}\mathscr{E}^\epsilon(u^k) \geqslant \mathscr{E}_\gamma(u_\epsilon)+(\hat{\psi}_\epsilon(u_\epsilon), 1) = \mathscr{E}^\epsilon(u_\epsilon),$$

which together with (2.26) and (2.25) yield that

$$\mathscr{E}^\epsilon(u_\epsilon)+\int_0^t |w_\epsilon(s)|_1^2\,\mathrm{d}s \leqslant \mathscr{E}^\epsilon(u_0). \tag{2.42}$$

We now take the limit of $(\mathbf{P}_\epsilon)$. It is convenient to define $\beta: \mathbb{R}\to\mathbb{R}$ as follows:

$$\beta(r) = \lim_{\epsilon\to 0}\beta_\epsilon(r) = \begin{cases} r-1 & \text{for} \quad r\geqslant 1,\\ 0 & \text{for} \quad |r|\leqslant 1,\\ r+1 & \text{for} \quad r\leqslant -1.\end{cases} \tag{2.43}$$

We note that β is a Lipschitz continuous function, and that

$$|\beta(r)-\beta_\epsilon(r)| \leqslant \tfrac{1}{2}\epsilon \quad \forall\, r\in\mathbb{R}, \quad \text{and} \quad |\beta(r)-\beta(s)| \leqslant |r-s| \quad \forall\, r, s\in\mathbb{R}. \tag{2.44}$$

Theorem 2.2 *Problems* (**P**) *and* (**Q**) *have unique solutions and are equivalent. Also given initial data u_0 and v_0 and denoting the solutions to* (**Q**) *by $u(t)$ and $v(t)$, respectively, then*

$$\|u(t)-v(t)\|_{-1} \leqslant C(t)\,\|u_0-v_0\|_{-1}. \tag{2.45}$$

Furthermore, $\forall\,\delta > 0$ $u\in C([\delta, T]; H^1(\Omega))$ and $\mathscr{E}_\gamma$ defined by (1.3) *is a Lyapunov functional for* (**P**) *and* (**Q**), *namely*

$$\forall\, 0 < t' < t, \quad \mathscr{E}_\gamma(u(t))+\int_{t'}^t |w(s)|_1^2\,\mathrm{d}s \leqslant \mathscr{E}_\gamma(u(t')), \tag{2.46a}$$

$$\forall\, 0 \leqslant t, \quad \mathscr{E}_\gamma(u(t))+\int_0^t |w(s)|_1^2\,\mathrm{d}s \leqslant \mathscr{E}_\gamma(u_0). \tag{2.46b}$$

Proof We observe that $u_\epsilon \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))')$ and $w_\epsilon \in L^2(0,T;H^1(\Omega))$ are bounded independently of ϵ; thus by compactness there exists a subsequence of $\{u_\epsilon\}$ and $\{w_\epsilon\}$ such that

$$u_\epsilon \rightharpoonup u \quad \text{in} \quad H^1(0,T;(H^1(\Omega))') \cap L^2(0,T;H^1(\Omega)), \tag{2.47a}$$

$$u_\epsilon \rightharpoonup^* u \quad \text{in} \quad L^\infty(0,T;H^1(\Omega)), \tag{2.47b}$$

$$w_\epsilon \rightharpoonup w \quad \text{in} \quad L^2(0,T;H^1(\Omega)). \tag{2.47c}$$

From the estimates obtained above we can pass to the limit in (2.17a) to obtain (2.7b). Now let us set $\eta = \beta_\epsilon(u_\epsilon) \in H^1(\Omega)$ in (2.17b); so, using the Cauchy–Schwarz inequality,

$$\begin{aligned}\gamma(\nabla u_\epsilon, \nabla\beta_\epsilon(u_\epsilon)) + \frac{1}{\epsilon}|\beta_\epsilon(u_\epsilon)|_0^2 &= (u_\epsilon + w_\epsilon, \beta_\epsilon(u_\epsilon)) \leqslant (|u_\epsilon|_0 + |w_\epsilon|_0)|\beta_\epsilon(u_\epsilon)|_0, \\ &\leqslant \epsilon(|u_\epsilon|_0^2 + |w_\epsilon|_0^2) + \frac{1}{2\epsilon}|\beta_\epsilon(u_\epsilon)|_0^2. \end{aligned} \tag{2.48}$$

Since $0 \leqslant \beta'_\epsilon(\cdot) \leqslant 1$

$$\begin{aligned}(\nabla u_\epsilon, \nabla\beta_\epsilon(u_\epsilon)) = \int_\Omega \beta'_\epsilon(u_\epsilon)\nabla u_\epsilon \cdot \nabla u_\epsilon \,\mathrm{d}x &\geqslant \int_\Omega (\beta'_\epsilon(u_\epsilon))^2 \nabla u_\epsilon \cdot \nabla u_\epsilon \,\mathrm{d}x, \\ &= |\beta_\epsilon(u_\epsilon)|_1^2 \geqslant 0, \end{aligned} \tag{2.49}$$

it follows from (2.48) and the stability bounds in Theorem 2.1 that

$$\|\beta_\epsilon(u_\epsilon)\|_{L^2(\Omega_T)} \leqslant C\epsilon. \tag{2.50}$$

From (2.49), (2.48) and the stability bounds in Theorem 2.1,

$$\|\beta_\epsilon(u_\epsilon)\|_{L^2(0,T;H^1(\Omega))} \leqslant C\epsilon^{\frac{1}{2}}, \tag{2.51}$$

so if we let $\epsilon \to 0$ then from (2.50) we conclude that for *a.e.* $t \in (0,T)$

$$\lim_{\epsilon\to 0} |\beta_\epsilon(u_\epsilon)|_0 = 0.$$

Using (2.44), the Lipschitz continuity of β, and (2.50)

$$\begin{aligned}\int_0^T |(\beta(u),\eta)| &\leqslant \int_0^T (|\beta(u)-\beta(u_\epsilon)|_0 + |\beta(u_\epsilon)-\beta_\epsilon(u_\epsilon)|_0 + |\beta_\epsilon(u_\epsilon)|_0)|\eta|_0, \\ &\leqslant C(|u_\epsilon - u|_{L^2(\Omega_T)} + \epsilon)|\eta|_{L^2(\Omega_T)},\end{aligned}$$

so that from the compactness result of Lions, u_ϵ converges strongly to u in $L^2(\Omega_T)$, $\beta(u) = 0$ *a.e.*; that is, $u \in K$.

Let $v \in K$; thus $\beta_\epsilon(v) = 0$ and

$$\gamma(\nabla u_\epsilon, \nabla v - \nabla u_\epsilon) - (w_\epsilon + u_\epsilon, v - u_\epsilon) = \frac{1}{\epsilon}(\beta_\epsilon(v) - \beta_\epsilon(u_\epsilon), v - u_\epsilon) \geqslant 0. \tag{2.52}$$

From all of the convergence properties of u_ϵ and w_ϵ we deduce the existence of a solution to (**P**).

To prove existence to (**Q**) we note that

$$w = -\mathscr{G}_N(\partial u/\partial t) + \lambda, \tag{2.53}$$

where $\lambda = (w, 1)/|\Omega|$. It follows that u solves (**Q**) by substituting (2.53) into (2.7b) and restricting those $\eta \in K$ to have a fixed mass, i.e. $(\eta, 1) = (u_0, 1)$.

It only remains to prove uniqueness.

Suppose that we have two solutions to (**P**), $\{u_1, w_1\}$ and $\{u_2, w_2\}$; then set $\eta = u_2$ in (2.7b) when $\{u_1, w_1\}$ is the solution and similarly when $\{u_2, w_2\}$ is the solution, add the two inequalities together, and obtain

$$-(\theta^u, \theta^w) + \gamma|\theta^u|_1^2 \leqslant |\theta^u|_0^2,$$

so setting $\eta = \mathscr{G}_N \theta^u$ in (2.7a) and subtracting gives

$$-(\theta^u, \theta^w) = \langle \partial\theta^u/\partial t, \mathscr{G}_N \theta^u \rangle.$$

We now repeat the arguments used in proving uniqueness for u_ϵ in ($\mathbf{P}_\epsilon$) to prove uniqueness for u. If we note that

$$|\theta^w|_1^2 = -\langle \partial\theta^u/\partial t, \theta^w \rangle = 0,$$

then we see that w is unique up to addition of a constant. As $u \in L^\infty(0, T; H^1(\Omega))$ we may define, in the H^1 sense, the open set

$$\Omega_0(t) := \{x \in \Omega : |u(x)| < 1\} \quad a.e.\ t. \tag{2.54}$$

Since $(u, 1) = m \in (-|\Omega|, |\Omega|)$, $\Omega_0(t)$ is non-empty. Take $\eta = u \pm \delta\phi$ in (2.7b), where $\phi \in C_0^\infty(\Omega_0(t))$, and δ is chosen so that $\eta \in K$. Then

$$\gamma(\nabla u, \nabla \phi) = (u + w, \phi) \quad \forall \phi \in C_0^\infty(\Omega_0(t)), \quad a.e.\ t,$$

and we conclude that

$$(\theta^w(t), \phi) = 0 \quad \forall \phi \in C_0^\infty(\Omega_0(t)), \quad a.e.\ t,$$

from which uniqueness for w follows.

Let us now take initial data u_0 and v_0; then as in the uniqueness proof for ($\mathbf{P}_\epsilon$), (compare with (2.41)), it is possible to show that for all $t > 0$

$$\exp^{-t/\gamma}\|u(t) - v(t)\|_{-1}^2 + \gamma \int_0^t \exp^{-s/\gamma}|u(s) - v(s)|_1^2 \, ds \leqslant \|u(0) - v(0)\|_{-1}^2, \tag{2.55}$$

from which it is easy to deduce (2.45).

If we differentiate (2.21c) with respect to t and set $\eta^k = du^k/dt \in V^k$, then

$$\gamma|du^k/dt|_1^2 + \frac{1}{\epsilon}(\beta_\epsilon'(u^k), (du^k/dt)^2) - \left|\frac{du^k}{dt}\right|_0^2 = (dw^k/dt, du^k/dt);$$

noting that $\beta_\epsilon'(\cdot) \geqslant 0$ and setting $\eta^k = (dw^k/dt)$ in (2.21b) we obtain

$$\gamma|du^k/dt|_1^2 + (\nabla\, dw^k/dt, \nabla w^k) \leqslant |du^k/dt|_0^2,$$

$$\leqslant \tfrac{1}{2}\gamma\left|\frac{du^k}{dt}\right|_1^2 + \frac{1}{2\gamma}\|du^k/dt\|_{-1}^2,$$

so that
$$\gamma|\mathrm{d}u^k/\mathrm{d}t|_1^2+\mathrm{d}/\mathrm{d}t|w^k|_1^2 \leqslant \frac{1}{\gamma}\left\|\frac{\mathrm{d}u^k}{\mathrm{d}t}\right\|_{-1}^2. \tag{2.56}$$

Multiplication by $t>0$, and noting (2.28), yields

$$\mathrm{d}/\mathrm{d}t[t|w^k|_1^2]+t\gamma|\mathrm{d}u^k/\mathrm{d}t|_1^2 \leqslant |w^k|_1^2+\frac{t}{\gamma}\|\mathrm{d}u^k/\mathrm{d}t\|_{-1}^2,$$
$$= (1+t/\gamma)|w^k|_1^2,$$

so that from integration over $(0,t)$ and (2.27)

$$t|w^k|_1^2+\gamma\int_0^t s|\mathrm{d}u^k/\mathrm{d}t|_1^2\,\mathrm{d}s \leqslant \int_0^t (1+s/\gamma)|w^k(s)|_1^2\,\mathrm{d}s,$$
$$\leqslant (1+t)\,C(u_0). \tag{2.57}$$

Hence, for $\delta>0$, from the Poincaré inequality it follows that $u^k\in H^1(\delta,T;H^1(\Omega))$ and passing to the limit in k and ϵ we obtain $u\in C^0([\delta,T];H^1(\Omega))$.

By the weak convergence of u_ϵ to u in $H^1(\Omega)$, (2.15) and (2.50),

$$\liminf_{\epsilon\to 0}\mathscr{E}^\epsilon(u_\epsilon) = \liminf_{\epsilon\to 0}\mathscr{E}_\gamma(u_\epsilon)+\liminf_{\epsilon\to 0}(\hat{\psi}_\epsilon(u_\epsilon),1),$$
$$\geqslant \liminf_{\epsilon\to 0}\mathscr{E}_\gamma(u_\epsilon)+\lim_{\epsilon\to 0}1/2\epsilon|\beta_\epsilon(u_\epsilon)|_0^2 \geqslant \mathscr{E}_\gamma(u).$$

Hence by the weak convergence of w_ϵ to w in $H^1(\Omega)$, (2.46 b) follows from (2.42). To prove that (2.46 a) holds we use a 'stop-start' argument. As $u\in C([\delta,T];H^1(\Omega))$, $\forall\,\delta>0$, we may set for $t'>0$ $U_0=u(t')$ and then solve **(P)** with the initial data U_0 to obtain $U(t)\,\forall\,t\geqslant t'$, which satisfies

$$\mathscr{E}_\gamma(U(t))+\int_{t'}^t\|U_t(s)\|_{-1}^2\,\mathrm{d}s \leqslant \mathscr{E}_\gamma(U_0)) = \mathscr{E}_\gamma(u(t')).$$

By uniqueness we have that $u(t)\equiv U(t)$ and so (2.46 a) holds. □

2.2 Regularity

We suppose $\partial\Omega$ to be sufficiently smooth, so that if z is the weak solution to

$$-\Delta z=f,\quad \partial z/\partial\nu=0, \tag{2.58}$$

where $f\in\mathscr{F}\cap L^2(\Omega)$ then

$$\|z\|_{H^2(\Omega)} \leqslant C|\Delta z|_0, \tag{2.59}$$

(see Grisvard 1985 for sufficient conditions).

Proposition 2.3 *For Ω sufficiently smooth we have the regularity results*

$$u\in L^2(0,T;H^2(\Omega)) \quad \textit{and} \quad \partial u/\partial\nu=0 \ \textit{on}\ \partial\Omega\ \textit{for a.e.}\ t, \tag{2.60 a}$$

and in addition

$$\forall\, t>0 \quad \min\{t^{\frac{1}{2}},1\}\,\|u(t)\|_2 \leqslant C(u_0), \quad \min\{t^{\frac{1}{2}},1\}\|w(t)\|_1 \leqslant C(u_0). \tag{2.60 b}$$

Proof Setting $\eta^k = -\Delta u^k \in V^k$ in (2.21 c),

$$\gamma|\Delta u^k|_0^2 - |u^k|_1^2 + \frac{1}{\epsilon}(\nabla\beta_\epsilon(u^k), \nabla u^k) = (w^k, -\Delta u^k) = (\nabla w^k, \nabla u^k),$$

$$= -(\mathrm{d}u^k/\mathrm{d}t, u^k) = -\frac{\mathrm{d}}{\mathrm{d}t}|u^k|_0^2, \tag{2.61}$$

and rearranging yields

$$\gamma|\Delta u^k|_0^2 + \frac{1}{\epsilon}(\nabla\beta_\epsilon(u^k), \nabla u^k) + \frac{\mathrm{d}}{\mathrm{d}t}|u^k|_0^2 = |u^k|_1^2.$$

It follows from $\beta'_\epsilon \geqslant 0$, $(\nabla\beta_\epsilon(u^k), \nabla u^k) \geqslant 0$, that

$$\gamma|\Delta u^k|_0^2 + \frac{\mathrm{d}}{\mathrm{d}t}|u^k|_0^2 \leqslant |u^k|_1^2,$$

so integrating over $(0,t)$ for $t\in[0,T]$ and noting that $u^k \in L^\infty(0,T;H^1(\Omega))$ yields

$$|u^k(t)|_0^2 + 2\gamma\int_0^t |\Delta u^k|_0^2 \leqslant 2\int_0^T |u^k|_1^2 + |u^k(0)|_0^2,$$

$$\leqslant C \quad \forall\, t\in[0,T]. \tag{2.62}$$

It follows from (2.59) that u^k is bounded in $L^2(0,T;H^2(\Omega))$ independently of ϵ and k, so by the usual compactness arguments we conclude that $u \in L^2(0,T;H^2(\Omega))$. Furthermore, since $\partial u^k/\partial\nu = 0$ on $\partial\Omega$ it follows by the weak convergence of u^k to u in $H^2(\Omega)$ that $\partial u/\partial\nu = 0$ in $L^2(\partial\Omega)$.

Now noting (2.61) we deduce that

$$\gamma|\Delta u^k|_0^2 \leqslant |u^k|_1^2 + |w^k|_1|u^k|_1,$$

and from (2.57), (2.37) and (2.27) we obtain

$$\|w^k\|_1^2 \leqslant C(u_0)(1+1/t),$$

$$|\Delta u^k|_0^2 \leqslant C(u_0)(1+1/t),$$

which we can rewrite as

$$\surd(t/(1+t))\,\|w^k\|_1 \leqslant C(u_0),$$

$$\surd(t/(1+t))\,|\Delta u^k|_0 \leqslant C(u_0).$$

Using (2.59) and the simple inequality

$$\surd(t/(1+t)) \geqslant \min\{\surd t, 1\}/\surd 2 \quad \forall\, t\geqslant 0,$$

(2.60 b) easily follows. □

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Corollary 2.4 *For a.e. $t \in (0, T)$ u satisfies the following complementarity problem*:

$$-1 \leqslant u \leqslant 1 \quad a.e.\ in\ \Omega, \tag{2.63a}$$

$$(-\gamma\Delta u - u - w)(|u| - 1) = 0 \quad a.e.\ \text{in}\ \Omega, \tag{2.63b}$$

$$\text{sign}\,(u)(\gamma\Delta u + u + w) \geqslant 0 \quad a.e.\ \text{in}\ \Omega. \tag{2.63c}$$

Proof By Proposition 2.3 for *a.e.* $t \in (0, T)$, integrating by parts in (2.7b) yields

$$-\int_\Omega (\gamma\Delta u + u + w)(v - u) \geqslant 0 \quad \forall\, v \in K. \tag{2.64}$$

Let us define the sets Ω_0, Ω_+, Ω_- to be

$$\Omega_0 = \{x \in \Omega : -1 < u(x) < 1\},$$
$$\Omega_+ = \{x \in \bar{\Omega} : u(x) = 1\},$$
$$\Omega_- = \{x \in \bar{\Omega} : u(x) = -1\}.$$

By the Sobolev imbedding theorem, u is continuous, and hence Ω_0 is open, so by choosing $\zeta \in C_0^\infty(\Omega_0)$, setting $v = u \pm \epsilon\zeta \in K$ and assuming that $|\epsilon|$ is small enough yields

$$-\gamma\Delta u - u = w \quad \text{in}\ \Omega_0. \tag{2.65}$$

In a similar manner, by letting $\zeta \in C_0^\infty(\Omega_\pm)$ where $-2 \leqslant \pm\zeta \leqslant 0$ so that $v = u + \zeta \in K$ we arrive at (2.63c). □

We now suppose further regularity on $\partial\Omega$, so that if z is the weak solution to

$$-\Delta z = f, \quad \partial z/\partial \nu = 0, \tag{2.66}$$

where $f \in \mathscr{F} \cap H^1(\Omega)$ then

$$\|z\|_{H^3(\Omega)} \leqslant C\|\Delta z\|_1. \tag{2.67}$$

Proposition 2.5 *If Ω is sufficiently smooth, and we assume further regularity on the initial data $\Delta u_0 \in H^1(\Omega)$ and $\partial u_0/\partial\nu = 0$ on $\partial\Omega$, then*

$$u \in H^1(0, T; H^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega)),$$

and
$$w \in L^2(0, T; H^3(\Omega)) \cap L^\infty(0, T; H^1(\Omega)).$$

Proof Integrating (2.56) over $(0, t)$ we obtain the estimate that, $\forall\, t \in [0, T]$,

$$|w^k(t)|_1^2 + \gamma\int_0^t |\mathrm{d}u^k/\mathrm{d}t|_1^2 \leqslant |w^k(0)|_1^2 + 1/\gamma\int_0^T \|\mathrm{d}u^k/\mathrm{d}t\|_{-1}^2. \tag{2.68}$$

We must now estimate $|w^k(0)|_1$. Let $\eta^k \in V^k$ so that

$$(P^k\Delta u_0, \eta^k) = (\Delta u_0, \eta^k) = -(\nabla u_0, \nabla\eta^k) = -(\nabla P^k u_0, \nabla\eta^k) = (\Delta P^k u_0, \eta^k);$$

thus in $H^1(\Omega)$,

$$\lim_{k\to\infty} \Delta u^k(0) \equiv \lim_{k\to\infty} \Delta P^k u_0 = \lim_{k\to\infty} P^k\Delta u_0 = \Delta u_0.$$

So by the Sobolev imbedding theorem $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, $u^k(0) \to u_0$ in $L^\infty(\Omega)$. Now

$$w^k(0) = -\gamma \Delta u^k(0) - u^k(0) + 1/\epsilon\, P^k \beta_\epsilon(u^k(0)),$$

so that

$$\begin{aligned} |w^k(0)|_1 &\leqslant \gamma |\Delta u^k(0)|_1 + |u^k(0)|_1 + 1/\epsilon |\beta_\epsilon(u^k(0))|_1, \\ &\leqslant \gamma |\Delta u^k(0)|_1 + |u^k(0)|_1 + 1/\epsilon |\beta'_\epsilon(u^k(0))|_{L^\infty(\Omega)} |u^k(0)|_1. \end{aligned} \tag{2.69}$$

Since β'_ϵ is continuous and $-1 \leqslant u_0 \leqslant 1$, as $k \to \infty$, $|\beta'_\epsilon(u^k(0))|_{L^\infty(\Omega)} \to 0$, so that from (2.69) the regularity of the initial data yields

$$|w^k(0)|_1 \leqslant C,$$

where C is independent of k and ϵ. Hence it follows from (2.68) and (2.29) that

$$|w^k(t)|_1^2 + \gamma \int_0^t \left| \frac{\mathrm{d}u^k}{\mathrm{d}t} \right|_1^2 \leqslant C, \tag{2.70}$$

where C is independent k, ϵ and T, yielding $\mathrm{d}u^k/\mathrm{d}t \in L^2(0, T; H^1(\Omega))$. Noting (2.37) and (2.70), we may conclude that $w^k \in L^\infty(0, T; H^1(\Omega))$. Also, from the smoothness of $\partial\Omega$, $-\Delta w^k = \mathrm{d}u^k/\mathrm{d}t$ and (2.67) yield that $w^k \in L^2(0, T; H^3(\Omega))$, where the bound is independent of k, ϵ and T. Finally, if we note that

$$w^k \in L^\infty(0, T; H^1(\Omega)), \quad (\nabla w^k, \nabla u^k) \leqslant |w^k|_1 |u^k|_1, \quad \text{(2.61) and (2.59)},$$

then we obtain the bound $u^k \in L^\infty(0, T; H^2(\Omega))$.

Note that each bound in this proof is independent of k, ϵ and T; the usual compactness results yield the result. □

2.3 Asymptotic behaviour in t

For $t \geqslant 0$ let $S(t)$ be the solution operator for the initial-value problem (2.7 a–c), so that $S(t)\, u_0 \equiv u(t)$. Define X to be the metric space consisting of the set K_m endowed with the metric

$$\mathrm{d}(\phi, \xi) \equiv \|\phi - \xi\|_{-1} \quad \text{for } \phi, \xi \in K_m.$$

By definition, $\{S(t)\}_{t \geqslant 0}$ is a semi-group of operators mapping X into itself. Furthermore, by (2.45), for $v_j, v \in X$,

$$\|S(t)\, v_j - S(t)\, v\|_{-1} \leqslant C(t)\, \|v_j - v\|_{-1};$$

it follows that $S(t): X \to X$ is continuous for each $t \geqslant 0$. For $u_0 \in X$, let $\omega(u_0)$ denote the ω-limit set

$$\omega(u_0) := \{v \in X : \exists t_n \to \infty \text{ such that } \lim_{n \to \infty} S(t_n)\, u_0 = v \text{ in } X\}. \tag{2.71}$$

We assume that $\partial\Omega$ is sufficiently regular that Proposition 2.3 holds. Since

$$\min\{1, \sqrt{t}\}\, \|u(t)\|_2 \leqslant C(u_0)$$

for all t, it follows that for each $t_0 > 0$, $\bigcup_{t \geqslant t_0} S(t)\, u_0$ is relatively compact in X. Hence, by standard arguments in topological dynamical systems (see Témam 1988), it follows that

(i) $\omega(u_0)$ is compact and connected;
(ii) $S(t)\,\omega(u_0) \subset \omega(u_0) \quad \forall\, t \geqslant 0$.

We note that $\mathscr{E}_\gamma(\cdot)$ is continuous on $H^1(\Omega)$ and, from (2.46a), that for $0 < t' < t$

$$\mathscr{E}_\gamma(S(t)\,u_0) + \int_{t'}^{t} \|\mathrm{d}u(s)/\mathrm{d}t\|_{-1}^2\,\mathrm{d}s \leqslant \mathscr{E}_\gamma(S(t')\,u_0) \quad \forall\, u_0 \in X. \tag{2.72}$$

Since, by Proposition 2.3 for $T > 0$, $u \in C((0, T];\, H^1(\Omega))$ and, since $\mathscr{E}_\gamma$ is bounded below on X, it follows from (2.72) that

$$\mathscr{E}_\gamma^\infty = \lim_{t\to\infty} \mathscr{E}_\gamma(S(t)\,u_0),$$

is well-defined. Furthermore, since $\|u(t)\|_2 \leqslant C(u_0, t_0)$ for $t > t_0$ it follows that for $v \in \omega(u_0)$ there exists $\{t_n\}_{n\geqslant 0}$ such that $v = \lim_{n\to\infty} S(t_n)\,u_0$ in $H^1(\Omega)$. Hence, by the continuity of $\mathscr{E}_\gamma(\cdot)$ on $H^1(\Omega)$,

$$\mathscr{E}_\gamma(v) = \lim_{n\to\infty} \mathscr{E}_\gamma(S(t_n)\,u_0) = \mathscr{E}_\gamma^\infty,$$

and we have that $\mathscr{E}_\gamma(\cdot)$ is constant on $\omega(u_0)$. Inequality (2.72) yields that

$$\mathrm{d}/\mathrm{d}t\,(S(t)\,v) \equiv 0 \quad \text{for} \quad v \in \omega(u_0),$$

so that v is a fixed point of $S(t)$. Hence if $v \in \omega(u_0)$, then

$$\gamma(\nabla v, \nabla\chi - \nabla v) - (v, \chi - v) \geqslant 0 \quad \forall\, \eta \in K_m, \tag{2.73}$$

and there exists $\lambda \in \mathbb{R}$ such that

$$\gamma(\nabla v, \nabla\chi - \nabla v) - (v, \chi - v) \geqslant \lambda(1, \chi - v) \quad \forall\, \eta \in K. \tag{2.74}$$

2.4 The linear equation

When $|u_0| < 1$ the solution for small time is given by

$$u(x, t) = \frac{m}{|\Omega|} + \sum_{j=2}^{\infty} \alpha_j(0) \exp^{(-\gamma(\mu_j-1)^2 + (\mu_j - 1))t}\, z_j(x), \tag{2.75}$$

where $\alpha_j(0) = (z_j, u_0)$. The maximum growth rate is for the wave number $\mu_c \approx 1 + 1/(2\gamma)$. Neglecting all other terms, the critical time t_c, at which $|u(x, t)| = 1$, is given by

$$t_c = 4\gamma \log\left(\frac{\min\{|1 - m/|\Omega||, |-1 - m/|\Omega||\}}{|(z_c, u_0)|\, \|z_c\|_\infty}\right).$$

3 The stationary problem

We now focus our attention upon the stationary problem:
(S_γ) Find $\{u, \lambda\} \in K_m \times \mathbb{R}$ such that $\forall \eta \in K$,

$$\gamma(\nabla u, \nabla\eta - \nabla u) - (u, \eta - u) - \lambda(1, \eta - u) \geqslant 0, \tag{3.1}$$

where $K = \{\eta \in H^1(\Omega) : -1 \leqslant \eta \leqslant 1\}$.

We also consider the related minimization problem:

($\mathbf{M}_\gamma$) Find $u \in K_m$ such that

$$\mathscr{E}_\gamma(u) = \min_{\eta \in K_m} \mathscr{E}_\gamma(\eta) := \tfrac{1}{2}\gamma(\nabla\eta, \nabla\eta) + \tfrac{1}{2}(1-\eta^2, 1), \tag{3.2}$$

and the critical point problem:

($\mathbf{C}_\gamma$) Find $u \in K_m$ such that

$$\gamma(\nabla u, \nabla\eta - \nabla u) - (u, \eta - u) \geqslant 0 \quad \forall\, \eta \in K_m. \tag{3.3}$$

Remarks

1. In (3.1) λ is the constant steady state value of the chemical potential.
2. It is obvious that if $\{u, \lambda\}$ solves ($\mathbf{S}_\gamma$) then u solves ($\mathbf{C}_\gamma$). Furthermore, ($\mathbf{S}_\gamma$) and ($\mathbf{C}_\gamma$) are the stationary versions of (**P**) and (**Q**), respectively.
3. As remarked previously in §2, we may assume that $|m| < |\Omega|$, otherwise the problem ($\mathbf{S}_\gamma$) is trivial.

3.1 Existence and regularity

Proposition 3.1 *There exists a minimizer to the minimization problem* ($\mathbf{M}_\gamma$).

Proof Since $\mathscr{E}_\gamma(\eta) \geqslant \gamma/2|\eta|_1^2$, for $\eta \in K_m$, existence follows by a standard minimization argument. □

Remark Observe that the trivial solution $u = m/|\Omega|$, $\lambda = -m/|\Omega|$ and $u = m/|\Omega|$ always solves ($\mathbf{S}_\gamma$) and ($\mathbf{C}_\gamma$), respectively. Later we will prove that for $\gamma \leqslant C_P^2$ there exists a non-trivial solution to ($\mathbf{S}_\gamma$) and ($\mathbf{C}_\gamma$).

Lemma 3.2 *Let $\{u, \lambda\}$ be a solution of* ($\mathbf{S}_\gamma$). *Then for Ω sufficiently regular, $u \in W^{2,p}(\Omega)$, $1 \leqslant p < \infty$, $u \in C^{1,\alpha}(\bar{\Omega})$ for $0 < \alpha < 1$ and*

$$\partial u/\partial \nu = 0 \quad \textit{a.e.} \quad \textit{on } \partial\Omega, \tag{3.4a}$$

$$-1 \leqslant u \leqslant 1 \quad \textit{a.e.} \quad \textit{in } \Omega, \tag{3.4b}$$

$$(-\gamma\Delta u - u - \lambda)(|u| - 1) = 0 \quad \textit{a.e.} \quad \textit{in } \Omega, \tag{3.4c}$$

$$\operatorname{sign}(u)(\gamma\Delta u + u + \lambda) \geqslant 0 \quad \textit{a.e.} \quad \textit{in } \Omega, \tag{3.4d}$$

$$\lambda = -1/|\Omega_0| \int_{\Omega_0} u(x)\,\mathrm{d}x, \quad -1 < \lambda < 1, \tag{3.4e}$$

$$2\mathscr{E}_\gamma(u) = |\Omega_0|(1-\lambda^2), \tag{3.4f}$$

where $\Omega_0 := \{x \in \Omega : |u(x)| < 1\}$.

Proof Given $\{u, \lambda\}$ we set $f = 2u + \lambda$ and rewrite (3.1) as

$$\gamma(\nabla u, \nabla\eta - \nabla u) + (u, \eta - u) \geqslant (f, \eta - u) \quad \forall\, \eta \in K. \tag{3.5}$$

We can approximate u via the following penalized problem:

$(\mathrm{S}_{\gamma,\epsilon})$ Find $u_\epsilon \in H^1(\Omega)$ such that $\forall \eta \in H^1(\Omega)$

$$\gamma(\nabla u_\epsilon, \nabla \eta) + 1/\epsilon(\beta_\epsilon(u_\epsilon), \eta) + (u_\epsilon, \eta) = (f, \eta), \tag{3.6}$$

where $\beta_\epsilon(\cdot)$ is given in §2 by (2.11) and $f \in H^1(\Omega)$. Existence and uniqueness are proved via a Galerkin approximation as in §2. By setting $\eta = (\beta_\epsilon(u_\epsilon))^{p-1}$ in (3.6), where $p \geqslant 2$ is even, and noting that

$$(\nabla u_\epsilon, \nabla(\beta_\epsilon(u_\epsilon))^{p-1}) = \int_\Omega (p-1)(\beta_\epsilon(u_\epsilon))^{p-2}\beta_\epsilon'(u_\epsilon)\nabla u_\epsilon \cdot \nabla u_\epsilon \, \mathrm{d}x \geqslant 0,$$

we obtain

$$\begin{aligned}\frac{1}{\epsilon}|\beta_\epsilon(u_\epsilon)|^p_{L^p(\Omega)} &\leqslant (f-u_\epsilon, (\beta_\epsilon(u_\epsilon))^{p-1}),\\ &\leqslant (|f|_{L^p(\Omega)} + |u_\epsilon|_{L^p(\Omega)})\left(\int_\Omega (\beta_\epsilon(u_\epsilon))^{(p-1)q}\right)^{1/q},\\ &= (|f|_{L^p(\Omega)} + |u_\epsilon|_{L^p(\Omega)})|\beta_\epsilon(u_\epsilon)|^{p-1/p}_{L^p(\Omega)},\end{aligned}$$

so that

$$1/\epsilon|\beta_\epsilon(u_\epsilon)|_{L^p(\Omega)} \leqslant |f|_{L^p(\Omega)} + |u_\epsilon|_{L^p(\Omega)}.$$

In particular, for $p = 2$, $|1/\epsilon\, \beta_\epsilon(u_\epsilon)|_0 \leqslant C$, so we conclude that

$$\gamma|\Delta u_\epsilon|_0 \leqslant |f|_0 + 1/\epsilon|\beta_\epsilon(u_\epsilon)|_0 + |u_\epsilon|_0 \leqslant C,$$

where C is independent of ϵ. Passing to the limit as $\epsilon \to 0$, standard arguments yield weak convergence in $H^2(\Omega)$ of u_ϵ to u for Ω sufficiently smooth. Since $\partial u_\epsilon/\partial \nu = 0$ in the sense of traces on $\partial\Omega$, we have that $\partial u/\partial \nu = 0$. Now, since $H^2(\Omega) \hookrightarrow L^p(\Omega)$ for $1 \leqslant p < \infty$, where $\mathrm{d} \leqslant 3$, it follows that $f \in L^p(\Omega)$, and, from the above calculations,

$$\gamma|\Delta u_\epsilon|_{L^p(\Omega)} \leqslant |f|_{L^p(\Omega)} + 1/\epsilon|\beta_\epsilon(u_\epsilon)|_{L^p(\Omega)} + |u_\epsilon|_{L^p(\Omega)} \leqslant C.$$

Hence for $2 \leqslant p$ and p even, from the regularity of elliptic equations for Ω smooth enough, $u \in W^{2,p}(\Omega)$; however, since Ω is bounded we immediately deduce that for all $1 \leqslant p < \infty$, $u \in W^{2,p}(\Omega)$ and, from the Sobolev imbedding theorem, $u \in C^{1,\alpha}(\bar{\Omega})$.

The regularity of u allows the use of Green's theorem to deduce from (3.1) that

$$-\int_\Omega (\gamma\Delta u + u + \lambda)(\eta - u) \geqslant 0 \quad \forall \eta \in K. \tag{3.7}$$

Let us define the sets Ω_0, Ω_+, Ω_- to be the following:

$$\begin{aligned}\Omega_0 &= \{x \in \Omega : -1 < u(x) < 1\},\\ \Omega_+ &= \{x \in \bar{\Omega} : u(x) = 1\},\\ \Omega_- &= \{x \in \bar{\Omega} : u(x) = -1\}.\end{aligned}$$

Since u is continuous, Ω_0 is open, so by choosing $\zeta \in C_0^\infty(\Omega_0)$ and setting $\eta = u \pm \delta\zeta \in K$ in (3.7), for $|\delta|$ small enough, we obtain

$$\gamma\int_{\Omega_0} \nabla u \cdot \nabla\zeta \, \mathrm{d}x = \int_{\Omega_0} (\lambda + u)\zeta \, \mathrm{d}x \Rightarrow -\gamma\Delta u - u = \lambda \quad \text{in } \Omega_0. \tag{3.8}$$

In a similar manner, let $\zeta \in C_0^\infty(\Omega_\pm)$, where $-2 \leqslant \pm\zeta \leqslant 0$, so that setting $\eta = u + \zeta \in K$ in (3.7) yields (3.4d). If we note that

$$0 = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \mathrm{d}s = \int_\Omega \Delta u \,\mathrm{d}x = \int_{\Omega_+ \cup \Omega_0 \cup \Omega_-} \Delta u \,\mathrm{d}x = \int_{\Omega_0} \Delta u \,\mathrm{d}x,$$

then integrating (3.8) over Ω_0 yields

$$\lambda |\Omega_0| = -\int_{\Omega_0} (\lambda \Delta u(x) + u(x)) \,\mathrm{d}x = -\int_{\Omega_0} u(x) \,\mathrm{d}x, \tag{3.9}$$

from which we obtain (3.4e), since

$$|\lambda| \leqslant \frac{1}{|\Omega_0|} \int_{\Omega_0} |u(x)| \,\mathrm{d}x < 1.$$

A direct calculation yields

$$\begin{aligned} 2\mathscr{E}_\gamma(u) &= -\gamma \int_\Omega u \Delta u \,\mathrm{d}x + |\Omega| - \int_\Omega u^2 \,\mathrm{d}x, \\ &= \int_{\Omega_0} [u^2 + u\lambda] \,\mathrm{d}x + |\Omega| - \int_\Omega u^2 \,\mathrm{d}x, \\ &= (1 - \lambda^2) |\Omega_0|. \qquad \square \end{aligned}$$

Proposition 3.3 *The problems* $(\mathbf{S}_\gamma)$ *and* $(\mathbf{C}_\gamma)$ *are equivalent.*

Proof It has already been remarked that if $\{u, \lambda\}$ solves $(\mathbf{S}_\gamma)$ then u solves $(\mathbf{C}_\gamma)$. Let $u \in K_m$ solve $(\mathbf{C}_\gamma)$. We now consider the problem:

$(\hat{\mathbf{S}}_\mu)$ Given $\mu \in [-3, 3]$ find $u_\mu \in k$ satisfying for all $\eta \in k$

$$\gamma(\nabla u_\mu, \nabla \eta - \nabla u_\mu) + (u_\mu, \eta - u_\mu) \geqslant (f, \eta - u_\mu) + \mu(1, \eta - u_\mu), \tag{3.10}$$

where $f = 2u$. We note that there exists a unique solution to $(\hat{\mathbf{S}}_\mu)$. Define the mapping $\mathscr{M}: [-3, 3] \to \mathbb{R}$ by

$$\mathscr{M}(\mu) = (u_\mu, 1).$$

Let $\mu_1, \mu_2 \in [-3, 3]$, then set $\mu = \mu_1$, $\eta = u_{\mu_2}$ and $\mu = \mu_2$, $\eta = u_{\mu_1}$ in (3.10), add the resulting inequalities, and we obtain

$$0 \leqslant \gamma |u_{\mu_1} - u_{\mu_2}|_1^2 + |u_{\mu_1} - u_{\mu_2}|_0^2 \leqslant (\mathscr{M}(\mu_1) - \mathscr{M}(\mu_2))(\mu_1 - \mu_2); \tag{3.11}$$

also from the Cauchy–Schwarz inequality and (3.11),

$$|\mathscr{M}(\mu_1) - \mathscr{M}(\mu_2)|^2 / |\Omega| \leqslant |u_{\mu_1} - u_{\mu_2}|_0^2 \leqslant (\mathscr{M}(\mu_1) - \mathscr{M}(\mu_2))(\mu_1 - \mu_2),$$

so that $\mathscr{M}$ is monotone and continuous. Now if we note the trivial inequalities

$$(1 - 3 - 2u)(\eta - 1) \geqslant 0 \quad \text{and} \quad (-1 + 3 - 2u)(\eta + 1) \geqslant 0 \quad \forall -1 \leqslant \eta \leqslant 1,$$

then it is clear that $u_3 = 1$ and $u_{-3} = -1$, so that $\mathscr{M}(-3) = -|\Omega|$ and $\mathscr{M}(3) = |\Omega|$. It follows from the intermediate value theorem that there exists $\lambda \in [-3, 3]$ such that

$\mathcal{M}(\lambda) = (u_\lambda, 1) = m$. Now setting $\eta = u_\lambda$ in (3.3) and $\eta = u$ in (3.10) and adding the resulting inequalities, yields

$$\gamma|u-u_\lambda|_1^2+|u-u_\lambda|_0^2 \leqslant 0, \tag{3.12}$$

i.e. $u_\lambda = u$ and from (3.4e) the uniqueness of λ follows. □

Proposition 3.4 *For $\gamma > C_P^2$ there exists a unique solution to* (3.1). *Let μ_1 be the smallest positive eigenvalue and e_1 the associated eigenfunction for the eigenvalue problem*

$$-\Delta e = \mu e; \quad \partial e/\partial \nu = 0 \quad on \quad \partial\Omega. \tag{3.13}$$

Then, for $\gamma \leqslant 1/\mu_1$, there exists a nontrivial solution to ($\mathbf{S}_\gamma$).

Proof Suppose that $\{u_1, \lambda_1\}$ and $\{u_2, \lambda_2\} \in K_m \times \mathbb{R}$ solve (3.1). Setting $\eta = u_2$ when $\{u_1, \lambda_1\}$ is the solution, and vice-versa, then adding the two inequalities leads to

$$\gamma|u_1-u_2|_1^2-|u_1-u_2|_0^2-(\lambda_1-\lambda_2)(u_1-u_2, 1) \leqslant 0.$$

Thus, noting that $(u_1-u_2, 1) = 0$, and the Poincaré inequality, we obtain

$$(\gamma/C_P^2-1)|u_1-u_2|_0^2 \leqslant 0,$$

so that u is unique; however, noting that we always have the trivial solution we conclude that λ is unique.

In order to prove the existence of a nontrivial solution it is enough to note that the trivial solution is not the solution to ($\mathbf{M}_\gamma$).

When $\gamma = 1/\mu_1$ and $\eta = \delta e_1+m/|\Omega|$, where $|\delta| \leqslant (1-|m|/|\Omega|)/\|e_1\|_\infty$, so that $|\eta| \leqslant 1$ and $(\eta, 1) = m$, then η solves ($\mathbf{C}_\gamma$) and hence ($\mathbf{S}_\gamma$). Now let $\gamma < 1/\mu_1$, and calculate the energy for η:

$$\begin{aligned} 2\mathscr{E}_\gamma(\eta) &= \gamma\delta^2|e_1|_1^2-(\delta^2|e_1|_0^2+m^2/|\Omega|)+|\Omega|, \\ &= (\gamma-1/\mu_1)\,\delta^2|e_1|_1^2+\delta^2(|e_1|_1^2/\mu_1-|e_1|_0^2)-m^2/|\Omega|+|\Omega|, \\ &< |\Omega|-m^2/|\Omega| = 2\mathscr{E}_\gamma(m/|\Omega|). \qquad \square \end{aligned}$$

Remark It is clear that if $\{1/\gamma, e\}$ is an eigenvalue and eigenfunction of (3.13) then $\delta e+m/|\Omega|$, where $|\delta| \leqslant (1-|m|/|\Omega|)/\|e\|_\infty$ so that $|\eta| \leqslant 1$ and $(\eta, 1) = m$, solves ($\mathbf{C}_\gamma$), and hence we have a continuum of solutions. It will become clear in the next subsection that, when we construct solutions solving ($\mathbf{S}_\gamma$) in one dimension, there may exist a continuum of solutions to the free-boundary value problem.

3.2 The one-dimensional problem

In this subsection we construct all the solutions for the one-dimensional problem ($\mathbf{S}_\gamma$) with $\Omega = (0, l)$ and consider the minimization problem ($\mathbf{M}_\gamma$).

From Lemma 3.2 it is clear that the problem we must solve is to find $u \in C^{1,1}(\bar{\Omega})$ and $\lambda \in \mathbb{R}$ such that

$$\gamma\, \mathrm{d}^2u/\mathrm{d}x^2+u(x)+\lambda = 0 \quad \text{for} \quad x \in \Omega_0 := \{x \in \Omega : |u(x)| < 1\}, \tag{3.14a}$$

$$u'(x) = 0 \quad \text{for} \quad x \in \partial\Omega_0 \cup \partial\Omega, \tag{3.14b}$$

where $$-\lambda = \frac{1}{|\Omega_0|}\int_{\Omega_0} u(x)\,\mathrm{d}x \quad \text{and} \quad |u(x)| \leqslant 1 \quad \text{for} \quad x \in \Omega. \tag{3.14c}$$

We may partition $\Omega_0 = \bigcup_{i=1}^{J} \Omega^i$ where $\Omega^i = (x_L^i, x_R^i)$ and Ω^i is defined to be the maximally-connected set such that $|u(x)| < 1$ on Ω^i and $\Omega^i \cap \Omega^j = \varnothing$ if $i \neq j$, ordered so that

$$0 \leqslant x_L^1 < x_R^1 \leqslant \ldots \leqslant x_L^i < x_R^i \leqslant \ldots \leqslant x_L^J < x_R^J \leqslant l.$$

Hence we have

$$(0, l) = (0, x_L^1] \cup \bigcup_{i=1}^{J} (x_L^i, x_R^i) \cup \bigcup_{i=1}^{J-1} [x_R^i, x_L^{i+1}] \cup [x_R^J, l), \tag{3.15}$$

where $$|u(x)| = 1 \quad \text{for} \quad x \in \Omega - \Omega_0 = (0, x_L^1] \cup \bigcup_{i=1}^{J-1} [x_R^i, x_L^{i+1}] \cup [x_R^J, l).$$

On each Ω^i $(i = 1, \ldots, J)$

$$\gamma\,\mathrm{d}^2u/\mathrm{d}x^2 + u + \lambda = 0, \tag{3.16a}$$

and $$u'(x_L^i) = u'(x_R^i) = 0, \tag{3.16b}$$

for which the general solution is

$$u(x) = a_i \cos\left(\frac{x - x_L^i}{\sqrt{\gamma}}\right) - \lambda, \quad \text{where} \quad x_R^i - x_L^i = k^i \sqrt{\gamma}\,\pi, \quad k^i \in \mathbb{N}. \tag{3.17}$$

It is easy to see that for each i we have the following exhaustive and mutually exclusive possibilities:

Type 1: $i \in [1, J]$, $u(x_L^i)\,u(x_R^i) < 0$ and $|u(x_L^i)| = 1$, $|u(x_R^i)| = 1$;
Type 2(i): $i \in [1, J]$, $u(x_L^i)\,u(x_R^i) > 0$ and $|u(x_L^i)| = 1$, $|u(x_R^i)| = 1$;
Type 2(ii): $i = 1$, $\Omega^1 = (0, x_R^1)$ and $|u(0)| < 1$, $|u(x_R^1)| = 1$;
Type 2(iii): $i = J$, $\Omega^J = (x_L^J, l)$ and $|u(l)| < 1$, $|u(x_L^J)| = 1$;
Type 3: $J = 1$, $\Omega^1 = (0, l)$ and $|u(0)| < 1$, $|u(l)| < 1$.

We now consider each case separately.

Type 1 Without loss of generality, assume that $u(x_L^i) = 1$ and $u(x_R^i) = -1$. Simple calculations reveal that the solution is given by

$$\lambda = 0 \quad \text{and} \quad u(x) = \cos(x - x_L^i)/\sqrt{\gamma} \quad \text{on} \quad \Omega^i \quad \text{where} \quad x_R^i - x_L^i = \sqrt{\gamma}\,\pi. \tag{3.18}$$

Type 2 (i) Without loss of generality, let $u(x_L^i) = 1$ and $u(x_R^i) = 1$. Again, simple calculations reveal that on Ω^i

$$-1 < \lambda < 0 \quad \text{and} \quad u(x) = (1 + \lambda)\cos(x - x_L^i)/\sqrt{\gamma} - \lambda \quad \text{where} \quad x_R^i - x_L^i = 2\sqrt{\gamma}\,\pi.$$

(ii) Without loss of generality, let $u(x_R^1) = 1$. Then the solution on Ω^1 is given by

$$-1 < \lambda < 0 \quad \text{and} \quad u(x) = -(1 + \lambda)\cos x/\sqrt{\gamma} - \lambda \quad \text{where} \quad x_R^1 = \sqrt{\gamma}\,\pi.$$

(iii) Without loss of generality, let $u(x_L^J) = 1$. Then the solution on Ω^J is given by

$$-1 < \lambda < 0 \quad \text{and} \quad u(x) = (1 + \lambda)\cos(x - x_L^J)/\sqrt{\gamma} - \lambda \quad \text{where} \quad l - x_L^J = \sqrt{\gamma}\,\pi.$$

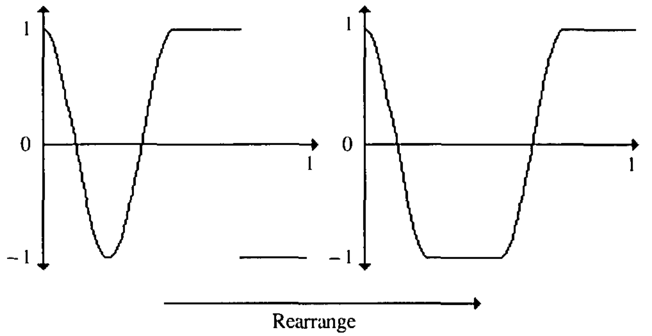

FIGURE 7. Rearranging a constructed stationary solution when $m = 0$.

It follows that since the existence of an interval of *Type* 2 implies $\lambda \neq 0$.

A solution of (S_γ) cannot have an interval of *Type* 1 and an interval of *Type* 2.

If there exists an interval of *Type* 2, then $\lambda \neq 0$ and (3.21) implies $m \neq 0$.

Type 3 Since $\Omega^1 = (0, l)$ and $\int_0^l u(x)\,dx = m$, it follows that

$$\lambda = -m/l \quad \text{and} \quad u(x) = a\cos x/\sqrt{\gamma} + m/l,$$

where either $\gamma = k^2\pi^2/l^2$ and $0 < |a| < 1 - |m|/l$, where $k \in \mathbb{N}$ or $a = 0$.

We have found u on the J intervals Ω^i, $i = 1, 2, \ldots, J$. From (3.15) we see that we must piece the curves back together so that u is defined on $(0, l)$. Hence we consider the following three mutually exclusive cases:

Case A (Piecing together J curves of *Type* 1). For the moment assume that $m = 0$. We observe that

$$\int_{x_L^i}^{x_R^i} u(x)\,dx = 0, \quad \text{so} \quad 0 = \int_\Omega u(x)\,dx = |\Omega_+| - |\Omega_-|,$$

i.e. $|\Omega_+| = |\Omega_-|$. Note that

$$|\Omega_0| = \sum_{i=1}^{J} |\Omega^i| = J\sqrt{\gamma}\,\pi \leqslant l, \tag{3.19}$$

so $|\Omega_+| = (l - J\sqrt{\gamma}\,\pi)/2$. Now let $u(x) = 1$ on an interval of width length $|\Omega_+|$, and likewise let $u(x) = -1$ on an interval of width length $|\Omega_-|$. Then by a simple rearrangement we have constructed our solution (see figure 7).

From (3.19) it is clear that there are a finite number of possible Js, and in particular for

$$\frac{l^2}{(N+1)^2\pi^2} < \gamma \leqslant \frac{l^2}{N^2\pi^2} \Rightarrow 1 \leqslant J \leqslant N. \tag{3.20}$$

Now suppose that $m \neq 0$, set $u(x) = \operatorname{sign} m$ on an interval of length $|m|$. If we then redefine the length of our interval to be $l - |m|$ then we reduce our problem with $m = 0$. So if we construct our solution on the interval of length $l - |m|$ as discussed previously, then, upon rearrangement (see figure 8), we may ensure that all the conditions are satisfied.

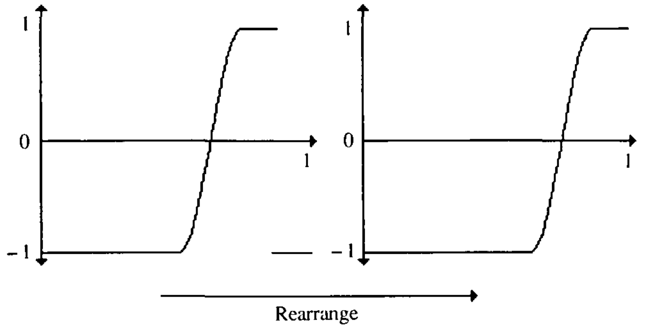

FIGURE 8. Rearranging a constructed stationary solution when $m \neq 0$.

Case B (Piecing together J curves of *Type* 2(i)–(iii)). Without loss of generality consider $u(x) > -1$ on Ω and $\lambda < 0$. Noting

$$m = \int_0^l u(x)\,\mathrm{d}x = |\Omega_+| + \int_{\Omega_0} u(x)\,\mathrm{d}x = |\Omega_+| - |\Omega_0|\,\lambda,$$

$$= l - (1+\lambda)|\Omega_0|, \tag{3.21}$$

and rearranging yields

$$\lambda = (l-m)/|\Omega_0| - 1, \tag{3.22}$$

and since $|\Omega_0| \leqslant l$ it follows that

$$-m/l \leqslant \lambda < 0, \tag{3.23}$$

as a consequence, $m > 0$. Substituting (3.22) into (2.23) and rearranging we obtain

$$0 < l - m < |\Omega_0| \leqslant l. \tag{3.24}$$

Furthermore, since

$$|\Omega_0| = \sum_{i=1}^{J} |\Omega^i|, \tag{3.25}$$

it follows that

$$|\Omega_0| = J'\sqrt{\gamma}\,\pi, \tag{3.26}$$

where

$$J' = \begin{cases} 2J, & \text{if we have pieced together curves of } Type\ 2\text{(i) only,} \\ 2J-1, & \text{if we have pieced together curves of } Type\ 2\text{(i) and (ii)} \\ & \text{or 2(i) and (iii),} \\ 2J-2, & \text{if we have pieced together curves of } Type\ 2\text{(i), (ii) and (iii).} \end{cases}$$

Substituting (3.26) into (3.24) it follows that

$$\frac{l-m}{\sqrt{\gamma}\,\pi} < J' \leqslant \frac{l}{\sqrt{\gamma}\,\pi}. \tag{3.27}$$

Case C. This is simply the solution of *Type* 3.

In the theorem that follows an equation for the number of stationary solutions is given. However, this should be understood in the following sense: as the position of the free boundary is unknown *a priori*, if there are enough points on the free boundary, at least two or three, then the free-boundary points are unrestricted in their position. We count two stationary solutions as the same if the constraint $(u, 1) = m$ is satisfied, and it is possible to obtain one stationary solution from the other by moving the free-boundary points, which should not cross in the process. This also means that in certain cases we have a continuum of solutions.

Theorem 3.5 *Given $\gamma > 0$ and m where $|m| < l$, let*

$$N := \left[\frac{l}{\pi\sqrt{\gamma}}\right].$$

Then there are exactly $2N+1$ solutions to $(\mathbf{S}_\gamma)$.

Proof Let us define

$$M = \left[\frac{l-|m|}{\pi\sqrt{\gamma}}\right],$$

and $\underline{J}_e$, $\overline{J}_e$, $\underline{J}_0$, $\overline{J}_0$ to be the smallest/largest, even/odd J' satisfying

$$\frac{l-|m|}{\sqrt{\gamma}\pi} < J' \leqslant N, \tag{3.28}$$

i.e. $M+1 \leqslant J' \leqslant N$. We consider cases A–C separately.

Case A. Without loss of generality we assume that $m = 0$ (see the previous reference for case A). It is enough to consider

$$u(x) = \begin{cases} 1 & \text{on } (0, x_L^1], \\ \cos\dfrac{(x-x_L^1)}{\sqrt{\gamma}} & \text{on } (x_L^1, x_R^1), \\ (-1)^J & \text{on } [x_R^1, l), \end{cases} \tag{3.29}$$

where $x_L^1 = (l - J\sqrt{\gamma}\pi)/2$, $x_R^1 = (l + J\sqrt{\gamma}\pi)/2$ and $1 \leqslant J \leqslant M$, because piecing together curves from Case 1 is the same as unpicking and rearranging (3.29). There are M solutions of this type, and reflection about $u = 0$ yields a further M solutions. Hence we have $2M$ solutions from Case A.

Case B. We note that, if $m = 0$, no solutions arise from Case B. So, for $|m| > 0$, we separate B into the four possible cases:

1. Curves of *Type* 2(i) alone. Let $J' = 2J$ satisfy (3.28). As in Case A, consideration of the curve

$$u(x) = \begin{cases} -\lambda + (1+\lambda)\cos x/\sqrt{\gamma} & \text{on } (0, 2J\sqrt{\gamma}\pi), \\ 1 & \text{on } [2J\sqrt{\gamma}\pi, l), \end{cases} \tag{3.30}$$

Table 1 *The number of stationary solutions to case B*

$M+1$	even	even	odd	odd
N	even	odd	even	odd
Case B				
1.	$\frac{N-(M+1)}{2}+1$	$\frac{(N-1)-(M+1)}{2}+1$	$\frac{N-(M+2)}{2}+1$	$\frac{(N-1)-(M+2)}{2}+1$
2.	$\frac{(N-1)-(M+2)}{2}+1$	$\frac{N-(M+2)}{2}+1$	$\frac{(N-1)-(M+1)}{2}+1$	$\frac{N-(M+1)}{2}+1$
3.	$\frac{(N-1)-(M+2)}{2}+1$	$\frac{N-(M+2)}{2}+1$	$\frac{(N-1)-(M+1)}{2}+1$	$\frac{N-(M+1)}{2}+1$
4.	$\frac{N-(M+1)}{2}+1$	$\frac{(N-1)-(M+1)}{2}+1$	$\frac{N-(M+2)}{2}+1$	$\frac{(N\text{-}1)-(M+2)}{2}+1$
Total	$2N-2M$	$2N-2M$	$2N-2M$	$2N-2M$

yields that we have precisely $(\bar{J}_e-\underline{J}_e)/2+1$ solutions.

2. Curves of *Type* 2(i) and (ii). Let $J'=2J-1$ satisfy (3.28). As in Case A, consideration of the curve

$$u(x)=\begin{cases} -\lambda-(1+\lambda)\cos x/\sqrt{\gamma} & \text{on } (0,\sqrt{\gamma}\,\pi], \\ -\lambda+(1+\lambda)\cos(x-x_L)/\sqrt{\gamma} & \text{on } (x_L, x_R), \\ 1 & \text{on } [x_R, l), \end{cases} \tag{3.31}$$

where $x_L=\sqrt{\gamma}\,\pi$ and $x_R=(2J-1)\sqrt{\gamma}\,\pi$, yields that we have precisely $(\bar{J}_0-\underline{J}_0)/2+1$ solutions.

3. Curves of *Type* 2(i) and (iii). Let $J'=2J-1$ satisfy (3.28). Then simply by reflecting $u(x)$, as given in (3.31), about $x=l/2$ yields that we have precisely $(\bar{J}_0-\underline{J}_0)/2+1$ solutions.

4. Curves of type 2(i), (ii) and (iii). Let $2J-2$ satisfy (3.28). As in Case A, consideration of the curve

$$u(x)=\begin{cases} -\lambda-(1+\lambda)\cos x/\sqrt{\gamma} & \text{on } (0,\sqrt{\gamma}\,\pi], \\ -\lambda+(1+\lambda)\cos(x-x_L)/\sqrt{\gamma} & \text{on } (x_L, x_R), \\ 1 & \text{on } [x_R, l-\sqrt{\gamma}\,\pi), \\ -\lambda+(1+\lambda)\cos(l-x-\sqrt{\gamma}\,\pi)/\sqrt{\gamma} & \text{on } [l-\sqrt{\gamma}\,\pi, l), \end{cases} \tag{3.32}$$

where $x_L=\sqrt{\gamma}\,\pi$ and $x_R=(2J-1)\sqrt{\gamma}\,\pi$, yields that we have precisely $(\bar{J}_e-\underline{J}_e)/2+1$ solutions.

We combine the total number of solutions to 1–4 in Table 1, where we assume $M+1\leqslant N$; otherwise, $M+1>N$; i.e. $M=N$, and there are no solutions from Case B.

Case C. From previous calculations we see that we have only one possibility.

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Table 2 *Splitting the number of stationary solutions for various m and γ*

$\gamma =$	0.005			0.001		
$m =$	0	-0.6	-0.95	0	-0.6	-0.95
A	2×4	2×1	0	2×10	2×4	0
B	0	6	8	0	12	20
C	1	1	1	1	1	1

Thus, combining the number of solutions from cases A, B and C:

if $M+1 \leqslant N$ we have $2M+(2N-2M)+1$ solutions in total, and

if $M+1 > N$, i.e. $M = N$, then we have $2N+1$ solutions in total. □

Table 2 splits the different types of stationary solutions for various values of m and γ when $l = 1$.

Theorem 3.6 *For $\gamma < l^2/\pi^2$, the minimizers to* $(\mathbf{M}_\gamma)$ *are the steady state solutions with smallest* $|\Omega_0|$.

Proof We first compute the minimum energies for steady state solutions of cases A, B and C denoted $\mathscr{E}_\gamma(u_A)$, $\mathscr{E}_\gamma(u_B)$ and $\mathscr{E}_\gamma(u_C)$, respectively.

For a solution of Case A, $\lambda = 0$ so that by (3.4f) we have

$$2\mathscr{E}_\gamma(u) = |\Omega_0|;$$

thus $\mathscr{E}_\gamma(\cdot)$ is smallest when $|\Omega_0|$ is minimal, that is u_A has one interface, and hence $2\mathscr{E}_\gamma(u_A) = \pi\sqrt{\gamma}$.

Without loss of generality we consider $m > 0$, so that for a solution to Case B, from (3.4f) and (3.22), it follows that

$$2\mathscr{E}_\gamma(u) = 2(l-m)-(l-m)^2/|\Omega_0|,$$

so the energy is smallest when $|\Omega_0|$ is minimal and $|\Omega_0| \geqslant 2\sqrt{\gamma}\pi$. From (3.24) we deduce

$$l-m < 2\mathscr{E}_\gamma(u_B) \leqslant l-m^2/l. \tag{3.33}$$

For a solution to Case C, setting $\lambda = -m/l$ and $|\Omega_0| = l$ in (3.4f) yields

$$2\mathscr{E}_\gamma(u_C) = l-m^2/l.$$

Since
$$\gamma \leqslant \left(\frac{l-|m|}{\pi}\right)^2 \Leftrightarrow \pi\sqrt{\gamma} \leqslant l-|m| < l-\frac{m^2}{l},$$

it follows that $\mathscr{E}_\gamma(u_A) < \min\{\mathscr{E}_\gamma(u_B), \mathscr{E}_\gamma(u_C)\}$ for $\gamma \leqslant ((l-|m|)/\pi)^2$.

We now turn to the situation $((l-|m|)/\pi)^2 < \gamma < (l/\pi)^2$. In this case, no solution to Case A exists; however, from (3.33) $\mathscr{E}_\gamma(u_B) \leqslant \mathscr{E}_\gamma(u_C)$ with equality only when $|\Omega_0| = l$. □

Remarks

1. Noting the Poincaré inequality $C_P = l/\pi$, it follows from Proposition 3.4 for $\gamma > (l/\pi)^2$ that we have uniqueness to (S_γ).

2. For $\gamma < ((l-|m|)/\pi)^2$ the minimum is unique, up to reversal, and

$$\Omega_+ = \left(0, \frac{l+m-\pi\sqrt{\gamma}}{2}\right], \quad \Omega_0 = \left(\frac{l+m-\pi\sqrt{\gamma}}{2}, \frac{l+m+\pi\sqrt{\gamma}}{2}\right), \quad \Omega_- = \left[\frac{l+m+\pi\sqrt{\gamma}}{2}, l\right).$$

This compares with Carr *et al.* (1984), who proved that in the Cahn–Hilliard equation for γ small enough, the associated Lyapunov functional has a unique minimizer with one interface (note that again the reversal has the same energy).

3. Zheng (1986) has shown that, in one space dimension, if $|m|/|\Omega|$ lies in the spinodal interval, then there are a finite number of solutions; in particular, when the free-energy is given by $\psi(u) = (u^2-1)^2/4$ and $m = 0$ then there are $2N_0+1$ solutions where

$$N_0 \leqslant l/(\pi\sqrt{\gamma}) < N_0+1.$$

3.3 The two-dimensional problem on a square

It is the purpose of this subsection to construct stationary solutions on the square $\Omega = (0, l) \times (0, l)$.

Using the results from the previous subsection, a two-dimensional solution may be obtained from a one-dimensional solution with many interfaces; for example, if $x_L, x_L + \pi\sqrt{\gamma} \in [0, l]$ then

$$u(x, y) = \begin{cases} 1 & \text{if } x \in [0, x_L), \\ \cos(x - x_L)/\sqrt{\gamma} & \text{if } x \in [x_L, x_L + \pi\sqrt{\gamma}], \\ -1 & \text{if } x \in (x_L + \pi\sqrt{\gamma}, l], \end{cases}$$

where x_L is chosen so that $(u, 1) = m$, is a stationary solution.

Radially symmetric solutions can also be used to construct solutions on the square. In order for there to be a radial solution satisfying the zero Neumann boundary data condition, the solution should be radial on a quarter or half annulus intersecting with the boundary or on a full annulus contained within the interior of the square. Now some radially symmetric solutions are constructed.

We look for $u(x, y) = u(r)$ where $r = (x^2+y^2)^{\frac{1}{2}}$, on

$$A := \{(x, y) \in \mathbb{R}^2 : r_{\text{in}} < (x^2+y^2)^{\frac{1}{2}} < r_{\text{out}}\}, \tag{3.34}$$

and $0 \leqslant r_{\text{in}} < r_{\text{out}} \leqslant l$. From the regularity of u it follows without loss of generality that

$$u'(r_{\text{in}}) = u'(r_{\text{out}}) = 0, \tag{3.35a}$$

$$u(r_{\text{in}}) = -u(r_{\text{out}}) = 1, \tag{3.35b}$$

otherwise we multiply u by -1. And from (3.4c) we obtain

$$-r(ru')'/\gamma - u = \lambda\, r_{\text{in}} < r < r_{\text{out}}. \tag{3.36}$$

Table 3 *Some zeros of J_0' and the scaling values*

k	R^k	R^{k+1}	c_0	λ
0	0	3.832	1.426	0.4258
1	3.832	7.016	-2.846	0.1460
2	7.016	10.173	3.638	9.686×10^{-2}
3	10.173	13.324	-4.279	6.697×10^{-2}
4	13.324	16.470	4.821	5.278×10^{-2}
5	16.470	19.616	-5.312	4.356×10^{-2}
6	19.616	22.760	5.760	3.709×10^{-2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Multiplying (3.36) by $u'(r)$ and integration over $(r_{\text{in}}, r_{\text{out}})$ yields, upon noting (3.35a, b), that

$$-2\lambda = \lambda \int_{r_{\text{in}}}^{r_{\text{out}}} u'(r)\,\mathrm{d}r = -\int_{r_{\text{in}}}^{r_{\text{out}}} \left(\gamma u''(r)\,u'(r) + \gamma \frac{(u'(r))^2}{r} + u'(r)\,u(r)\right)\mathrm{d}r,$$

$$= -\gamma \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{(u'(r))^2}{r}\,\mathrm{d}r < 0,$$

hence $\lambda > 0$.

It is easy to show that for some constants c_0 and c_1 the general solution of (3.36) is

$$u(r) = -\lambda + c_0 J_0(r/\sqrt{\gamma}) + c_1 Y_0(r/\sqrt{\gamma}), \tag{3.37}$$

where J_0 and Y_0 are the Bessel functions of order zero of the first and second kind, respectively.

For simplicity we choose only to seek solutions of the form

$$u(r) = -\lambda + c_0 J_0(r/\sqrt{\gamma}). \tag{3.38}$$

Applying (3.35a) to (3.38) yields

$$J_0'(r_{\text{in}}/\sqrt{\gamma}) = J_0'(r_{\text{out}}/\sqrt{\gamma}) = 0,$$

so $r_{\text{in}}/\sqrt{\gamma}$ and $r_{\text{out}}/\sqrt{\gamma}$ are taken to be successive zeros, R^k and R^{k+1}, respectively, of J_0', where $|J_0'(r)| > 0$ on (R^k, R^{k+1}). Applying (3.35b) to (3.38) yields

$$c_0 = \frac{2}{J_0(r_{\text{in}}/\sqrt{\gamma}) - J_0(r_{\text{out}}/\sqrt{\gamma})} \quad \text{and} \quad \lambda = \frac{J_0(r_{\text{in}}/\sqrt{\gamma}) + J_0(r_{\text{out}}/\sqrt{\gamma})}{J_0(r_{\text{in}}/\sqrt{\gamma}) - J_0(r_{\text{out}}/\sqrt{\gamma})}. \tag{3.39}$$

Since J_0 is monotone on our chosen interval, it follows that $|\lambda| < 1$. (See Table 3 for some of the important values.) Note that J_0 has the following properties (Birkhoff & Rota 1959):

(i) $J_0'(r)$ has an infinite number of zeros.
(ii) The difference between consecutive zeros of $J_0'(r)$ converges to π as $r \to \infty$.

(iii) $J_0(r)$ has an asymptotic expansion

$$J_0(r) = \frac{\sqrt{2}}{\sqrt{(\pi r)}}\cos(r+\tfrac{1}{4}\pi)+O(r^{-\frac{3}{2}}) \quad \text{as} \quad r\to\infty. \tag{3.40}$$

We now consider the mass constraint upon u. From (3.4e) if $\Omega_0 = A$, a single quarter annulus, then

$$\iint_A u(x,y)\,\mathrm{d}x\,\mathrm{d}y = \tfrac{1}{4}\pi\lambda(r_{\text{in}}^2 - r_{\text{out}}^2). \tag{3.41}$$

Let Ω_0 consist of n quarter annuli of the same radii; then

$$\begin{aligned} m &= \int_\Omega u(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega_+\cup\Omega_0\cup\Omega_-} u(x,y)\,\mathrm{d}x\,\mathrm{d}y,\\ &= \frac{n\pi}{4}r_{\text{in}}^2 + \lambda\frac{n\pi}{4}(r_{\text{in}}^2 - r_{\text{out}}^2) - \left(l^2 - \frac{n\pi}{4}r_{\text{out}}^2\right),\\ &= \frac{n\pi}{4}((1+\lambda)\,r_{\text{in}}^2 + (1-\lambda)\,r_{\text{out}}^2) - l^2. \end{aligned} \tag{3.42}$$

Since $r_{\text{in}} = R^k\sqrt{\gamma}$ and $r_{\text{out}} = R^{k+1}\sqrt{\gamma}$, (3.42) may be rewritten as

$$m = \frac{n\pi\gamma}{4}((1+\lambda)(R^k)^2 + (1-\lambda)(R^{k+1})^2) - l^2. \tag{3.43}$$

As an example, using the values in Table 3.3 with $\Omega = (0,5)\times(0,5)$, and $\gamma = 0.08$, we obtain Table 4.

It is useful to introduce

$$R := \left(\frac{2(m+l^2)}{\pi}\right)^{\frac{1}{2}},$$

so that

$$\frac{\pi R^2}{4} - \left(l^2 - \frac{\pi R^2}{4}\right) = m.$$

We now construct a sequence of radial solutions $\{u^k\}$, where Ω_0 consists of a single quarter annulus and

$$-l^2 < m < (\tfrac{1}{2}\pi - 1)\,l^2, \tag{3.44}$$

so that $0 < R < l$.

Define

$$c_0^k = \frac{2}{J_0(R^k) - J_0(R^{k+1})}, \tag{3.45a}$$

$$\lambda^k = \frac{J_0(R^k) + J_0(R^{k+1})}{J_0(R^k) - J_0(R^{k+1})}, \tag{3.45b}$$

$$\gamma^k = \frac{4(l^2+m)}{\pi((1+\lambda^k)(R^k)^2 + (1-\lambda^k)(R^{k+1})^2)}. \tag{3.45c}$$

Table 4 *Mass of a radially symmetric stationary solution*

			m			
k	r_{in}	r_{out}	$n=1$	$n=2$	$n=3$	$n=4$
0	0.00	1.08	−24.5	−23.9	−23.4	−22.9
1	1.08	1.98	−21.3	−17.6	−13.9	−10.2
2	1.98	2.88	−15.8	−6.5	2.8	12.1
3	2.88	3.77	−7.7	9.7		
4	3.77	4.66	2.9			

Since the sequence $\{R^k\}$ is unbounded and $|\lambda^k| < 1$, it follows that $\lim_{k\to\infty} \gamma^k = 0$. If we rewrite (3.43) when $n = 1$ and note that $R^k < R^{k+1}$, then it follows that

$$\gamma^k(R^k)^2 < R^2 = \frac{\gamma^k[(1+\lambda^k)(R^k)^2+(1-\lambda^k)(R^{k+1})^2]}{2} < \gamma^k(R^{k+1})^2, \tag{3.46}$$

so that from (3.40) $$\lim_{k\to\infty} R^{k+1}-R^k = \pi;$$

since $\gamma^k \to 0$ as $k \to \infty$, it follows that

$$\lim_{k\to\infty} R^k\sqrt{\gamma^k} = \left(\frac{2(m+l^2)}{\pi}\right)^{\frac{1}{2}} = R. \tag{3.47}$$

Now define k^* to be the smallest integer so that

$$\sqrt{\gamma^{k^*}}\, R^{k^*+1} \leqslant l. \tag{3.48}$$

Then it follows that for $k \geqslant k^*$ $\{u^k, \lambda^k\}$ solves (S_{γ^k}).

We now show that the sequence of Lagrange multipliers λ^k converges to 0 at a prescribed rate.

From (3.45b) and (3.47), and the asymptotic expansion (3.40),

$$\lim_{k\to\infty} \frac{\lambda^k}{\sqrt{\gamma^k}} = \lim_{k\to\infty} \frac{R^k}{R} \times \frac{J_0(R^k)+J_0(R^{k+1})}{J_0(R^k)-J_0(R^{k+1})},$$

$$= \lim_{k\to\infty} \frac{R^k}{R} \times \frac{1/\sqrt{R^k}-1/\sqrt{R^{k+1}}}{1/\sqrt{R^k}+1/\sqrt{R^{k+1}}};$$

since $\lim_{k\to\infty} R^{k+1}-R^k = \pi$, it follows that

$$\lim_{k\to\infty} \frac{\lambda^k}{\sqrt{\gamma^k}} = \frac{\pi}{4R}. \tag{3.49}$$

We consider more general results of this nature in the next subsection.

3.4 The asymptotic behaviour as $\gamma \to 0$

In this subsection we consider the limit $\gamma \to 0$ in the problem $(\mathbf{M}_\gamma)$ for $\Omega \subset \mathbb{R}^d$, $(1 \leqslant d \leqslant 3)$, being a bounded domain with Lipschitz continuous boundary. Following Modica (1987), we establish that, as $\gamma \to 0$, there is a sequence of minimizers $\{u^\gamma\}$, of $(\mathbf{M}_\gamma)$ which converges to a function u^* that only takes on the values $+1$ and -1 with the interface between the two sets being minimal. This result is interesting, as the phenomenological parameter γ was introduced to model interfacial energy. A different approach has been studied by Gurtin (1985). Some results of Luckhaus & Modica (1989), $2 \leqslant d \leqslant 3$, are then duplicated which verify the Gibbs–Thomson relation for surface tension.

Define $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(s) = \tfrac{1}{2}(1-s^2);$$

for $\gamma > 0$, $u^\gamma \in K$, we note that

$$\mathscr{E}_\gamma(u^\gamma) = \tfrac{1}{2}\gamma |u^\gamma|_1^2 + (\psi(u^\gamma), 1), \tag{3.50}$$

and

$$\int_{-1}^{1} \psi^{\frac{1}{2}}(s)\,\mathrm{d}s = \frac{\pi}{2\sqrt{2}}. \tag{3.51}$$

We remark that Modica (1987) and Luckhaus & Modica (1989) study the free energy functional (3.50) with $\psi(\cdot)$ being a C^0 and C^2 function, respectively, having a double well form; the purpose of this subsection is to apply their results to this situation.

For any open subset of $E \subset \mathbb{R}^d$, $1 \leqslant d \leqslant 3$, and $v \in L^1_{loc}(E)$, we define

$$\int_E |Dv| := \sup\left\{ \int_E v \nabla \cdot g(x)\,\mathrm{d}x : g \in C_0^\infty(E; \mathbb{R}^d), |g| \leqslant 1 \right\}. \tag{3.52}$$

Also,

$$BV(E) := \left\{ v \in L^1(\Omega) : \int_E |Dv| < \infty \right\}.$$

The Sobolev space $W^{1,1}(E)$ is contained in $BV(E)$, and for $v \in W^{1,1}(E)$, $\int_E |Dv|$ equals the ordinary Lebesgue integral $\int_E |\nabla v|\,\mathrm{d}x$.

If $\{v_k\}$ is a sequence in $L^1(\Omega)$ which converges in $L^1(\Omega)$ to v_∞, then

$$\int_\Omega |Dv_\infty| \leqslant \liminf_{k \to \infty} \int_\Omega |Dv_k|. \tag{3.53}$$

When m and c are real constants the set

$$\left\{ v \in L^1(\Omega) : (v, 1) = m, \int_\Omega |Dv| \leqslant c \right\} \text{ is compact in } L^1(\Omega). \tag{3.54}$$

If E is any measurable subset of $\mathbb{R}^d$, we denote by I_E the characteristic function of E, and define

$$P_\Omega(E) := \int_\Omega |DI_E|.$$

It can be proved that $P_\Omega(E) \leqslant \mathscr{H}_{d-1}(\partial E \cap \Omega)$, where $\mathscr{H}_{d-1}$ is the $(d-1)$-dimensional Hausdorff measure, with equality if $\partial E \cap \Omega$ is a Lipschitz continuous hypersurface.

Let $v \in BV(\mathbb{R}^d)$, then the function $t \to P_\Omega(\{x \in \mathbb{R}^d : v(x) > t\})$ is Lebesgue measurable on $\mathbb{R}$, and the Fleming–Rishel formula (1960) holds:

$$\int_\Omega |Dv| = \int_{-\infty}^{\infty} P_\Omega(\{x \in \mathbb{R}^d : v(x) > t\})\, \mathrm{d}t. \tag{3.55}$$

All of the results above can be found in Giusti (1984).

Theorem 3.7 *For $\gamma > 0$ and $m \in (-|\Omega|, |\Omega|)$, suppose that u^γ is a solution to*

$$\mathscr{E}_\gamma(u^\gamma) = \min\{\mathscr{E}_\gamma(v) : v \in K, (v, 1) = m\}. \tag{3.56}$$

Then there exists a sequence $\{u^{\gamma_k}\}$ such that:

(i) *As $\gamma_k \to 0$, u^{γ_k} converges to u^* in $L^1(\Omega)$ where $|u^*(x)| = 1$ for a.e. $x \in \Omega$.*

(ii) *The set $\Omega^*_- = \{x \in \Omega : u^*(x) = -1\}$ is a solution of the variational problem*

$$P_\Omega(\Omega^*_-) = \min\left\{P_\Omega(F) : F \subseteq \Omega, |F| = \frac{|\Omega| - m}{2}\right\}.$$

(iii)

$$\lim_{k\to\infty} \gamma_k^{-\frac{1}{2}} \mathscr{E}_{\gamma_k}(u^{\gamma_k}) = \tfrac{1}{2}\pi P_\Omega(\Omega^*_-).$$

Note that $P_\Omega(\Omega^*_-) = P_\Omega(\Omega^*_+)$, where $\Omega^*_+ = \{x \in \Omega : u^*(x) = 1\}$.

This theorem is a modification from Modica (1987), for which we require the following propositions and lemmas before proving the theorem. For completeness we include the proofs of Propositions 3.8 and 3.11 and theorem 3.7, which involve calculations similar to those of Modica.

Proposition 3.8 *Let $(v^\gamma)_{\gamma > 0}$ be a family of functions in K such that $v^\gamma \to v^*$ in $L^1(\Omega)$ as $\gamma \to 0^+$.*

(a) *If*

$$\liminf_{\gamma \to 0^+} \gamma^{-\frac{1}{2}} \mathscr{E}_\gamma(v^\gamma) < +\infty,$$

then $|v^*(x)| = 1$ for a.e. $x \in \Omega$.

(b) *If $|v^*(x)| = 1$ for a.e. $x \in \Omega$ then*

$$P_\Omega(E) \leqslant 2/\pi \liminf_{\gamma \to 0^+} \gamma^{-\frac{1}{2}} \mathscr{E}_\gamma(v^\gamma), \tag{3.57}$$

where $E := \{x \in \Omega : v^*(x) = -1\}$.

Proof Let $\{\gamma_k\}$ be a positive sequence converging to zero as $k \to \infty$ such that $\{v^{\gamma_k}\}$ converges pointwise to v^* *a.e.* in Ω, and

$$\lim_{k\to\infty} \mathscr{E}_{\gamma_k}(v^{\gamma_k}) = 0.$$

It follows that $|v^*(x)| \leqslant 1$ *a.e.* in Ω, and by Fatou's Lemma,

$$\int_\Omega \psi(v^*(x))\,\mathrm{d}x \leqslant \liminf_{k\to\infty} \int_\Omega \psi(v^{\gamma_k}(x))\,\mathrm{d}x \leqslant \liminf_{k\to\infty} \mathscr{E}_{\gamma_k}(v^{\gamma_k}) = 0,$$

since $\psi(s) \geqslant 0$ for $-1 \leqslant s \leqslant 1$, it follows that (a) holds.

For $t \in [-1, 1]$ define

$$\phi(t) := \int_{-1}^{t} \psi^{\frac{1}{2}}(s)\,\mathrm{d}s, \tag{3.58}$$

note that $\phi(-1) = 0$ and $\phi(1) = \pi/(2\sqrt{2})$. Set

$$w^*(x) = \phi(v^*(x)) \quad \text{and for} \quad \gamma > 0 \quad w^\gamma(x) = \phi(v^\gamma(x)), \quad x \in \Omega.$$

Since v^γ is equibounded and $\phi \in C^1$, w^γ converges to w^* in $L^1(\Omega)$ as $\gamma \to 0^+$, so by the lower semi-continuity (3.53)

$$\int_\Omega |Dw^*| \leqslant \liminf_{\gamma\to 0^+} \int_\Omega |Dw^\gamma|. \tag{3.59}$$

Since $P_\Omega(\Omega) = 0$ and $P_\Omega(A) = 0$ if $A \cap \Omega = \varnothing$, then by the Fleming–Rishel formula (3.55) and the hypothesis $|v^*(x)| = 1$ for *a.e.* $x \in \Omega$,

$$\begin{aligned}\int_\Omega |Dw^*| &= \int_{-\infty}^{\infty} P_\Omega(\{x \in \Omega : \phi(v^*(x)) > t\})\,\mathrm{d}t,\\ &= \int_{\phi(-1)}^{\phi(1)} P_\Omega(\Omega - \bar{E})\,\mathrm{d}t,\\ &= (\phi(1) - \phi(-1))\,P_\Omega(E) = \frac{\pi}{2\sqrt{2}} P_\Omega(E). \end{aligned}\tag{3.60}$$

On the other hand, $v^\gamma \in H^1(\Omega)$ implies that $\nabla w^\gamma(x) = \phi'(v^\gamma(x))\,\nabla v^\gamma(x)$ (cf. Marcus & Mizel 1973); hence

$$\int_\Omega |\nabla w^\gamma|\,\mathrm{d}x = \sqrt{2} \int_\Omega |\psi^{\frac{1}{2}}(v^\gamma(x))| \frac{1}{\sqrt{2}} |\nabla v^\gamma(x)|\,\mathrm{d}x, \tag{3.61}$$

$$\leqslant \frac{1}{\sqrt{2}} \int_\Omega \left\{ \frac{\gamma^{\frac{1}{2}}}{2} |\nabla v^\gamma|^2 + \gamma^{-\frac{1}{2}} \psi(v^\gamma) \right\} \mathrm{d}x,$$

$$= \frac{1}{\sqrt{2}} \gamma^{-\frac{1}{2}} \mathscr{E}_\gamma(v^\gamma). \tag{3.62}$$

By inserting (3.60) and (3.62) in (3.59) upon rearrangement we obtain (b). □

Lemma 3.9 *Let E be a measurable subset of Ω with $0 < |E| < |\Omega|$. If, for a fixed $\delta \geqslant 0$, we have $\delta \leqslant P_\Omega(A)$ for every open bounded subset A of $\mathbb{R}^d$ which has a smooth boundary and satisfies $\mathscr{H}_{d-1}(\partial A \cap \partial\Omega) = 0$, $|A \cap \Omega| = |E|$, then*

$$\delta \leqslant \min\{P_\Omega(F) : F \text{ a measurable subset of } \Omega, |F| = |E|\}.$$

If, in particular, $\delta = P_\Omega(E)$, then equality holds.

Proof See Modica (1987).

Lemma 3.10 *Let $A \subset \mathbb{R}^d$ be an open set with smooth, non-empty compact boundary. Define $h: \mathbb{R}^d \to \mathbb{R}$ by*

$$h(x) = \begin{cases} -\operatorname{dist}(x, \partial A), & \text{if } x \in A, \\ \operatorname{dist}(x, \partial A), & \text{if } x \notin A. \end{cases} \tag{3.63}$$

Then h is Lipschitz continuous, $|Dh(x)| = 1$ for a.e. $x \in \mathbb{R}^d$ and

$$\lim_{t \to 0} \mathscr{H}_{d-1}(S_t \cap \Omega) = \mathscr{H}_{d-1}(\partial A \cap \Omega),$$

where $S_t := \{x \in \mathbb{R}^d : h(x) = t\}$.

Proof See Modica (1987).

Proposition 3.11 *Let $A \subset \mathbb{R}^d$ be an open set with ∂A a non-empty, compact, smooth hypersurface and $\mathscr{H}_{d-1}(\partial A \cap \partial \Omega) = 0$. Define $v^* : \Omega \to \mathbb{R}$ by*

$$v^*(x) := \begin{cases} -1 & \text{if } x \in A \cap \Omega, \\ 1 & \text{if } x \in (\mathbb{R}^d - A) \cap \Omega. \end{cases} \tag{3.64}$$

Then there is a family $\{v^\gamma\}_{\gamma > 0}$ of Lipschitz continuous functions on Ω such that $v^\gamma \to v^*$ in $L^1(\Omega)$ as $\gamma \to 0^+$, $|v^\gamma| \leqslant 1$ for every $\gamma > 0$ and

(a) $(v^\gamma, 1) = (v^*, 1) = |\Omega - A| - |A \cap \Omega| \; \forall \gamma > 0$.

(b) $\dfrac{2}{\pi} \limsup_{\gamma < 0^+} \gamma^{-\frac{1}{2}} \mathscr{E}_\gamma(v^\gamma) \leqslant P_\Omega(A)$.

Proof Define $\chi^* : \mathbb{R} \to \mathbb{R}$ by

$$\chi^*(t) := \begin{cases} -1, & t < 0, \\ 1, & t \geqslant 0, \end{cases} \tag{3.65}$$

so that
$$v^*(x) = \chi^*(h(x)) \quad \text{for } x \in \Omega,$$

and define $\zeta^\gamma : \mathbb{R} \to \mathbb{R}$ by

$$\zeta^\gamma(t) := \begin{cases} -1, & t < 0, \\ -\cos(t/\sqrt{\gamma}), & 0 \leqslant t \leqslant \eta^\gamma = \pi\sqrt{\gamma}, \\ 1 & \eta^\gamma < t. \end{cases} \tag{3.66}$$

Plainly, $\zeta^\gamma \in C^{1,1}(\mathbb{R})$ and for $t \in [0, \eta^\gamma]$

$$\gamma^{\frac{1}{2}}(\zeta^\gamma)'(t) = \sqrt{2}\psi^{\frac{1}{2}}(\zeta^\gamma(t)). \tag{3.67}$$

Note that for all $t \in \mathbb{R}$

$$\zeta^\gamma(t) \leqslant \chi^*(t) \leqslant \zeta^\gamma(t + \eta^\gamma);$$

thus there exists $\delta^\gamma \in [0, \eta^\gamma]$ such that

$$\int_\Omega \zeta^\gamma(h(x)+\delta^\gamma)\,\mathrm{d}x = \int_\Omega \chi^*(h(x))\,\mathrm{d}x = \int_\Omega v^*(x)\,\mathrm{d}x. \tag{3.68}$$

Finally, for $t \in \mathbb{R}$, we define $\chi^\gamma(t) = \zeta^\gamma(t+\delta^\gamma)$ and for $x \in \Omega$, $v^\gamma(x) = \chi^\gamma(h(x))$.

Each v^γ is a Lipschitz continuous function, and $-1 \leqslant v^\gamma \leqslant 1$; by Lemma 3.10

$$\int_\Omega |v^\gamma - v^*|\,\mathrm{d}x = \int_\Omega |\chi^\gamma(h(x)) - \chi^*(h(x))|\,|Dh(x)|\,\mathrm{d}x,$$

so that the coarea formula (cf. Federer 1968)

$$\int_\Omega f(g(x))\,|Dg(x)|\,\mathrm{d}x = \int_{-\infty}^{\infty} f(t)\,\mathscr{H}_{d-1}(\{x \in \Omega : g(x) = t\})\,\mathrm{d}t,$$

which holds for any Lebesgue measurable function f and any Lipschitz continuous function g, implies that

$$\begin{aligned}\int_\Omega |v^\gamma - v^*|\,\mathrm{d}x &= \int_{-\delta^\gamma}^{\eta^\gamma-\delta^\gamma} |\chi^\gamma(t) - \chi^*(t)|\,\mathscr{H}_{d-1}(S_t \cap \Omega)\,\mathrm{d}t,\\ &\leqslant 2\eta^\gamma \sup_{|t| \leqslant \eta^\gamma} \mathscr{H}_{d-1}(S_t \cap \Omega),\end{aligned}$$

where $S_t := \{x \in \mathbb{R}^d : h(x) = t\}$. As $\eta^\gamma = \pi\sqrt{\gamma}$, applying Lemma 3.10 again, we conclude that v^γ converges to v^* in $L^1(\Omega)$ as $\gamma \to 0^+$.

Since (a) is a direct consequence of (3.68), it remains to prove (b).

Let $\xi^\gamma = \sup_{|t| \leqslant \eta^\gamma} \mathscr{H}_{d-1}(S_t \cap \Omega)$; from the coarea formula we obtain

$$\begin{aligned}\gamma^{-\frac{1}{2}}\mathscr{E}_\gamma(v^\gamma) &= \int_\Omega \left\{\frac{\gamma^{\frac{1}{2}}}{2}(v^\gamma)^2 + \gamma^{-\frac{1}{2}}\psi(v^\gamma)\right\}\mathrm{d}x,\\ &= \int_{-\infty}^{\infty} \left\{\frac{\gamma^{\frac{1}{2}}}{2}(\chi^\gamma)'(t)^2 + \gamma^{-\frac{1}{2}}\psi(\chi^\gamma(t))\right\}\mathscr{H}_{d-1}(S_t \cap \Omega)\,\mathrm{d}t,\\ &\leqslant \xi^\gamma \int_{-\delta^\gamma}^{\eta^\gamma-\delta^\gamma} \left\{\frac{\gamma^{\frac{1}{2}}}{2}((\zeta^\gamma)'(t+\delta^\gamma))^2 + \gamma^{-\frac{1}{2}}\psi(\zeta^\gamma(t+\delta^\gamma))\right\}\mathrm{d}t,\\ &= \xi^\gamma \int_0^{\eta^\gamma} \left\{\frac{\gamma^{\frac{1}{2}}}{2}((\zeta^\gamma)'(t))^2 + \gamma^{-\frac{1}{2}}\psi(\zeta^\gamma(t))\right\}\mathrm{d}t.\end{aligned}$$

Now, recalling (3.67),

$$\begin{aligned}\gamma^{-\frac{1}{2}}\mathscr{E}_\gamma(v^\gamma) &\leqslant \sqrt{2}\xi^\gamma \int_0^{\eta^\gamma} (\zeta^\gamma)'(t)\,\psi^{\frac{1}{2}}(\zeta^\gamma(t))\,\mathrm{d}t,\\ &= \sqrt{2}\xi^\gamma \int_{-1}^{1} \psi^{\frac{1}{2}}(s)\,\mathrm{d}s.\end{aligned}$$

Since Lemma 3.10 implies

$$\lim_{\gamma\to 0^+} \xi^\gamma = \mathscr{H}_{d-1}(\partial A \cap \Omega) = P_\Omega(A),$$

we conclude that

$$\limsup_{\gamma\to 0^+} \gamma^{-\frac{1}{2}}\mathscr{E}_\gamma(v^\gamma) \leqslant P_\Omega(A)\tfrac{1}{2}\pi. \qquad \square$$

Proof of Theorem 3.7 We select, as comparison functions for u^γ, the following piecewise affine functions w^γ, depending on the first variable x_1:

$$w^\gamma := \begin{cases} -1 & \text{if } x_1 \leqslant z-\sqrt{\gamma}, \\ \dfrac{x_1-z}{\sqrt{\gamma}} & \text{if } z-\sqrt{\gamma} < x_1 < z+\sqrt{\gamma}, \\ 1 & \text{if } x_1 \geqslant z+\sqrt{\gamma}, \end{cases}$$

with z chosen so that $\qquad (w^\gamma, 1) = m.$

If we let $T^\gamma = \{x\in\Omega : z-\sqrt{\gamma} \leqslant x_1 \leqslant z+\sqrt{\gamma}\}$, then by the boundedness of Ω, it follows that $|T^\gamma| \leqslant C\sqrt{\gamma}$ for every $\gamma > 0$ and some suitable constant C; hence by the minimizing property of u^γ,

$$\gamma^{-\frac{1}{2}}\mathscr{E}_\gamma(u_\gamma) \leqslant \gamma^{-\frac{1}{2}}\mathscr{E}_\gamma(w_\gamma) = \int_{T_\gamma}\left\{\frac{\gamma^{-\frac{1}{2}}}{2} + \gamma^{-\frac{1}{2}}\psi(w_\gamma(x))\right\}\mathrm{d}x,$$

$$\leqslant C\{1 + \max_{-1\leqslant s\leqslant 1} \psi(s)\} \leqslant C. \tag{3.69}$$

We now prove the existence of a sequence $u^{\gamma_k} \to u^*$ as $\gamma_k \to 0$. For $\gamma > 0$ define $v^\gamma = \phi(u^\gamma)$. As $0 \leqslant \sqrt{2}\psi^{\frac{1}{2}}(s) \leqslant 1$ for $s\in[-1,1]$, v^γ is bounded in $L^1(\Omega)$. However, recalling (3.62) and (3.69),

$$\int_\Omega |Dv^\gamma|\,\mathrm{d}x \leqslant \frac{\gamma^{-\frac{1}{2}}}{\sqrt{2}}\mathscr{E}_\gamma(u^\gamma) \leqslant C \quad \forall\, \gamma > 0.$$

Now from compactness (3.54) there is a sequence $\{\gamma_k\}$ of positive numbers converging to zero such that $v^{\gamma_k} \to v^*$ in $L^1(\Omega)$. We now return to the functions u^γ. As ϕ is strictly monotone increasing and continuous on $[-1,1]$, ϕ^{-1} is well defined, bounded and uniformly continuous on $[0, \pi/(2\sqrt{2})]$. Define $u^*(x) = \phi^{-1}(v^*(x))$; then by the uniform continuity we conclude that $u^{\gamma_k} = \phi^{-1}(v^{\gamma_k})$ converges in $L^1(\Omega)$ to u^* as $k\to\infty$, thus proving existence. Hence Proposition 3.8 applies and (i) is proved.

We now turn to the proof of (ii). Since

$$|\Omega^*_-| = \int_\Omega\left(\frac{1-u^*}{2}\right)\mathrm{d}x = \frac{|\Omega|-m}{2},$$

by Lemma 3.9 it suffices to verify that $P_\Omega(\Omega^*_-) \leqslant P_\Omega(A)$ for every open, bounded subset A of $\mathbb{R}^d$, with smooth boundary, such that $\mathscr{H}_{d-1}(\partial A \cap \partial\Omega) = 0$ and $|A\cap\Omega| = |\Omega^*_-|$. Fix such a subset A and note that $\partial A \cap \Omega \neq \varnothing$, because $0 < |A \cap \Omega| < |\Omega|$; using Proposition 3.11 we construct a family $\{w^\gamma\}_{\gamma>0}$ in $H^1(\Omega)$, different from the sequence of affine functions at the start of the proof, such that

$$(w^\gamma, 1) = |\Omega - A| - |A \cap \Omega| = (|\Omega| - |\Omega_-^*|) - |\Omega_-^*| = m,$$

and
$$2/\pi \limsup_{\gamma\to 0^+} \gamma^{-\frac{1}{2}} \mathscr{E}_\gamma(w^\gamma) \leqslant P_\Omega(A). \tag{3.70}$$

On the other hand, by (i) and Proposition 3.8,

$$P_\Omega(\Omega_-^*) \leqslant 2/\pi \liminf_{\gamma_k \to 0^+} \gamma_k^{-\frac{1}{2}} \mathscr{E}_{\gamma_k}(u^{\gamma_k}), \tag{3.71}$$

and by the minimizing property of u^{γ_k}

$$\mathscr{E}_{\gamma_k}(u^{\gamma_k}) \leqslant \mathscr{E}_{\gamma_k}(w^{\gamma_k}) \quad \forall k \in \mathbb{N}. \tag{3.72}$$

It is now obvious that (3.71), (3.72) and (3.70) yield $P_\Omega(\Omega_-^*) \leqslant P_\Omega(A)$. This completes the proof of (ii).

Finally, let us prove (iii). By taking into account (3.71), (3.72) and (3.70), we find that

$$\tfrac{1}{2}\pi P_\Omega(\Omega_-^*) \leqslant \liminf_{k\to\infty} \gamma_k^{\frac{1}{2}} \mathscr{E}_{\gamma_k}(u^{\gamma_k}) \leqslant \limsup_{k\to\infty} \gamma_k^{\frac{1}{2}} \mathscr{E}_{\gamma_k}(u^{\gamma_k}) \leqslant \tfrac{1}{2}\pi P_\Omega(A),$$

for any open, bounded set $A \subset \mathbb{R}^d$, with smooth boundary, such that $\mathscr{H}_{d-1}(\partial A \cap \partial\Omega) = 0$ and $|A \cap \Omega| = |\Omega_-^*|$. Applying Lemma 3.9 with $\delta = \limsup_{k\to\infty} \gamma_k^{\frac{1}{2}} \mathscr{E}_{\gamma_k}(u^{\gamma_k})$ in conjunction with (ii), we immediately obtain (iii). $\square$

Remark Let us consider the one-dimensional problem when $\Omega = (0, l)$. Let F be a subset of $(0, l)$ so that $|F| = (l-m)/2$; then

$$F = \bigcup_{i\in I} J^i,$$

where I is some index used to count the intervals, $J^i \cap J^j = \varnothing$ for $i \neq j, i, j \in I$ and $J^i = (x_L^i, x_R^i)$, $i \in I$. now

$$P_\Omega(F) := \sup\left\{\int_F g'(x)\,dx : g \in C_0^\infty((0,l);\mathbb{R}), |g| \leqslant 1\right\},$$

so that

$$\int_F g'(x)\,dx = \sum_{i\in I} \int_{x_L^i}^{x_R^i} g'(x)\,dx = \sum_{i\in I} g(x_R^i) - g(x_L^i).$$

If we assume that $|I|$ is finite, then $P_\Omega(F) = 2|I| - \{$the number of endpoints of Ω in $F\}$; hence considering all the possibilities, the minimum value that $P_\Omega(F)$ can take is 1. Hence, $\Omega_-^* = (0, (l-m)/2)$ or $\Omega_-^* = ((l+m)/2, l)$ so that $P_\Omega(\Omega_-^*) = 1$ and

$$\lim_{k\to\infty} \gamma_k^{-\frac{1}{2}} \mathscr{E}_{\gamma_k}(u^{\gamma_k}) = \tfrac{1}{2}\pi.$$

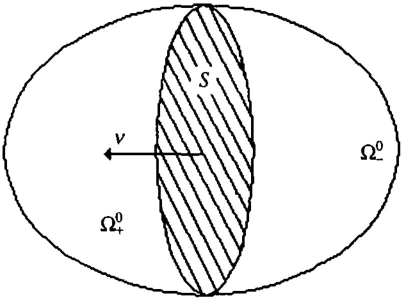

FIGURE 9. Domain decomposition of Ω into Ω_+ and Ω_-.

However, from the construction of stationary solutions and Theorem 3.6, for $0 < \gamma < (l-|m|)/\pi$, a minimizer u^γ of $\mathscr{E}_\gamma$ satisfies

$$\mathscr{E}_\gamma(u^\gamma) = \frac{1}{2}\left(\gamma \int_0^l \left(\frac{\mathrm{d}u^\gamma}{\mathrm{d}x}\right)^2 + l - \int_0^l (u^\gamma)^2\right) = \frac{\pi\sqrt{\gamma}}{2},$$

and, rewriting
$$\gamma^{-\frac{1}{2}}\mathscr{E}_\gamma(u^\gamma) = \pi/2 \quad \forall\, 0 < \gamma < \left(\frac{l-|m|}{\pi}\right)^2.$$

Throughout the remainder of this subsection, $\{u^\gamma\}$ is a subsequence of $\{u^\gamma\}$ converging to u^* in $L^1(\Omega)$ which was found in Theorem 3.7; likewise, $\{v^\gamma\}$ is the bounded sequence in $BV(\Omega)$, introduced in the proof of Theorem 3.7, defined by

$$v^\gamma = \phi(u^\gamma) \quad \gamma > 0 \quad \text{where} \quad v^\gamma \to v^* \text{ in } L^1(\Omega).$$

Since $u^* := \phi^{-1}(v^*)$ and $|u^*(x)| = 1$ for *a.e.* $x \in \Omega$, it follows that

$$v^*(x) = \begin{cases} \phi(1) & \forall\, x \in \Omega_+^*, \\ \phi(-1) & \forall\, x \in \Omega_-^* \end{cases}. \tag{3.73}$$

We now modify a Theorem of Luckhaus & Modica (1989). For simplicity we assume that $\Omega \subset \mathbb{R}^d$ $(d \geqslant 2)$ has a sufficiently smooth boundary so that Lemma 3.2 holds. For completeness we include the proof which again involves calculations similar to those of Luckhaus & Modica.

Let $\lambda^* \in \mathbb{R}$ be the Lagrange multiplier associated with the minimization problem $(\mathbf{M}_0)$. The constant λ^* has the following geometrical significance from the theory of minimal surfaces: the surface $S = \Omega_-^* \cap \Omega$ is a smooth hypersurface (Giusti 1984) whose mean curvature κ is constant, and is equal to $\lambda^*/(d-1)$.

Theorem 3.12 *Let the sequence $\{\lambda^\gamma\}$ be the associated sequence of Lagrange multipliers for the sequence $\{u^\gamma\}$; then*

$$\lim_{\gamma \to 0^+} \gamma^{-\frac{1}{2}}\lambda^\gamma = -\tfrac{1}{4}\pi(d-1)\,\kappa, \tag{3.74}$$

where κ is the (constant) mean curvature of the hypersurface $S = \partial\Omega_-^* \cap \Omega$.

Proof From (3.4c)

$$\gamma\Delta u^\gamma + u^\gamma + \lambda^\gamma = 0 \quad \text{on} \quad \Omega_0,$$
$$\nabla u^\gamma = \underline{0} \quad \text{on} \quad \Omega_- \cup \Omega_+.$$

Hence for a fixed vector field $\underline{\xi} \in C_0^\infty(\Omega; \mathbb{R}^d)$; since $u \in H^2(\Omega)$ the following equality holds

$$\int_\Omega (\gamma\Delta u^\gamma + u^\gamma + \lambda^\gamma)\,\xi_i D_i u^\gamma \,\mathrm{d}x = 0, \tag{3.75}$$

where $D_i = \nabla$ and summation is taken over repeated indices. Integration by parts yields

$$0 = \int_\Omega (-\gamma D_i u^\gamma D_i(\xi_j D_j u^\gamma) + u^\gamma \xi_i D_i u^\gamma + \lambda^\gamma \xi_i D_i u^\gamma)\,\mathrm{d}x. \tag{3.76}$$

Noting that

$$D_i u^\gamma D_i(\xi_j D_j u^\gamma) = D_i u^\gamma D_i \xi_j D_j u^\gamma + D_i u^\gamma \xi_j D_{ij}^2 u^\gamma, \tag{3.77}$$

$$\begin{aligned}\int_\Omega D_i u^\gamma \xi_j D_{ij}^2 u^\gamma \,\mathrm{d}x &= \frac{1}{2}\int_\Omega \xi_j D_j(D_i u^\gamma D_i u^\gamma)\,\mathrm{d}x,\\ &= -\frac{1}{2}\int_\Omega |\nabla u^\gamma|^2 \operatorname{div}\underline{\xi}\,\mathrm{d}x,\end{aligned} \tag{3.78}$$

$$\begin{aligned}\int_\Omega u^\gamma \xi_i D_i u^\gamma \,\mathrm{d}x &= -\frac{1}{2}\int_\Omega \xi_i D_i(1-(u^\gamma)^2)\,\mathrm{d}x,\\ &= \int_\Omega \psi(u^\gamma)\operatorname{div}\underline{\xi}\,\mathrm{d}x,\end{aligned} \tag{3.79}$$

$$-\int_\Omega \lambda^\gamma \xi_i D_i u^\gamma \,\mathrm{d}x = \lambda^\gamma \int_\Omega u^\gamma \operatorname{div}\underline{\xi}\,\mathrm{d}x. \tag{3.80}$$

Hence substituting (3.77)–(3.80) into (3.76), dividing by $\sqrt{\gamma}$, and rearranging yields

$$\begin{aligned}-\frac{\lambda^\gamma}{\sqrt{\gamma}}\int_\Omega u^\gamma \operatorname{div}\underline{\xi}\,\mathrm{d}x &= \int_\Omega \left(\frac{\sqrt{\gamma}}{2}|\nabla u^\gamma|^2 - \gamma^{-\frac{1}{2}}\psi(u^\gamma)\right)\operatorname{div}\underline{\xi}\,\mathrm{d}x\\ &\quad + \sqrt{\gamma}\int_\Omega (D_i u^\gamma D_i \xi_j D_j u^\gamma) - |\nabla u^\gamma|^2 \operatorname{div}\underline{\xi})\,\mathrm{d}x.\end{aligned} \tag{3.81}$$

It is obvious that we may pass to the limit on the left-hand side of (3.81); since S is smooth we obtain from the divergence theorem

$$\lim_{\gamma\to 0^+}\int_\Omega u^\gamma \operatorname{div}\underline{\xi}\,\mathrm{d}x = \int_\Omega u^* \operatorname{div}\underline{\xi}\,\mathrm{d}x = -2\int_S \underline{\xi}\cdot\underline{\nu}\,\mathrm{d}\mathscr{H}_{d-1}, \tag{3.82}$$

where $\underline{\nu}$ is the outer, unit normal vector on $S = \partial\Omega_-^* \cap \Omega$ (see figure 9). We postpone the remainder of the proof to prove two lemmas which estimate the two terms on the right-hand side of (3.81).

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Lemma 3.13 *We have the following three limits*:

$$\lim_{\gamma\to 0^+}\int_\Omega\left(\frac{\gamma^{\frac{1}{4}}}{\sqrt{2}}|\nabla u^\gamma|-\gamma^{-\frac{1}{4}}\psi^{\frac{1}{2}}(u^\gamma)\right)^2\mathrm{d}x,$$
$$=\lim_{\gamma\to 0^+}\int_\Omega\left|\frac{\gamma^{\frac{1}{2}}}{\sqrt{2}}|\nabla u^\gamma|^2-|\nabla u^\gamma|\,\psi^{\frac{1}{2}}(u^\gamma)\right|\mathrm{d}x,$$
$$=\lim_{\gamma\to 0^+}\int_\Omega\left|\frac{\gamma^{\frac{1}{2}}}{2}|\nabla u^\gamma|^2-\gamma^{-\frac{1}{2}}\psi(u^\gamma)\right|\mathrm{d}x=0. \tag{3.83}$$

Proof From theorem 3.7 part (iii)

$$\lim_{\gamma\to 0^+}\int_\Omega\left(\frac{\gamma^{\frac{1}{2}}}{2}|\nabla u^\gamma|^2+\gamma^{-\frac{1}{2}}\psi(u^\gamma)\right)\mathrm{d}x=\lim_{\gamma\to 0^+}\gamma^{-\frac{1}{2}}\mathscr{E}^\gamma(u^\gamma)=\tfrac{1}{2}\pi P_\Omega(\Omega_-^*); \tag{3.84}$$

for γ small enough this shows that the sequences $\{\gamma^{\frac{1}{4}}|\nabla u^\gamma|\}$ and $\{\gamma^{-\frac{1}{4}}\psi^{\frac{1}{2}}(u^\gamma)\}$ are bounded in $L^2(\Omega)$. From (3.61), (3.53) and (3.60)

$$\liminf_{\gamma\to 0^+}\int_\Omega\sqrt{2}|\nabla u^\gamma|\,\psi^{\frac{1}{2}}(u^\gamma)\,\mathrm{d}x=\sqrt{2}\liminf_{\gamma\to 0^+}\int_\Omega|\nabla v^\gamma|\,\mathrm{d}x,$$
$$\geqslant\sqrt{2}\int_\Omega|\nabla v^*|=\tfrac{1}{2}\pi P_\Omega(\Omega_-^*); \tag{3.85}$$

hence we conclude that

$$0\leqslant\lim_{\gamma\to 0^+}\int_\Omega\left(\frac{\gamma^{\frac{1}{4}}}{\sqrt{2}}|\nabla u^\gamma|-\gamma^{-\frac{1}{4}}\psi^{\frac{1}{2}}(u^\gamma)\right)^2\mathrm{d}x\leqslant\tfrac{1}{2}\pi P_\Omega(\Omega_-^*)-\tfrac{1}{2}\pi P_\Omega(\Omega_-^*)=0. \tag{3.86}$$

Noting the inequalities

$$|a^2-ab|\leqslant(a-b)^2+|(a-b)\,b|\quad\text{and}\quad|a^2-b^2|\leqslant(a-b)^2+2|(a-b)\,b|,$$

using the Cauchy–Schwarz inequality, (3.86) and the boundedness of the sequences $\{\gamma^{\frac{1}{4}}|\nabla u^\gamma|\}$ and $\{\gamma^{-\frac{1}{4}}\psi^{\frac{1}{2}}(u^\gamma)\}$ in $L^2(\Omega)$ easily yields the result. □

Lemma 3.14 *If the surface S is smooth inside the support of $\underline{\xi}$, then*

$$\lim_{\gamma\to 0^+}\gamma^{\frac{1}{2}}\int_\Omega((D_i u^\gamma D_i\xi_j D_j u^\gamma)-|\nabla u^\gamma|^2\operatorname{div}\underline{\xi})\,\mathrm{d}x=+\tfrac{1}{2}\pi\lambda^*\int_S\underline{\xi}\cdot\underline{\nu}\,\mathrm{d}\mathscr{H}_{d-1}. \tag{3.87}$$

Proof For $\gamma>0$ define

$$\Omega^\gamma=\{x\in\Omega:|\nabla v^\gamma(x)|>0\}. \tag{3.88}$$

Note that
$$\frac{\nabla u^\gamma(x)}{|\nabla u^\gamma(x)|}=\frac{\nabla v^\gamma(x)}{|\nabla v^\gamma(x)|}\quad\forall\,x\in\Omega^\gamma, \tag{3.89}$$

and for $x\in\Omega$, $\underline{\eta}\in\mathbb{R}^d$ ($\underline{\eta}\neq\underline{0}$) define

$$f(x,\underline{\eta})=\frac{\eta_i D_i\xi_j\eta_j}{\eta_i\eta_i}-\operatorname{div}\underline{\xi}(x). \tag{3.90}$$

It follows from Lemma 3.13 and the boundedness of f that

$$\lim_{\gamma\to 0^+} \gamma^{\frac{1}{2}} \int_\Omega ((D_i u^\gamma D_i \xi_j D_j u^\gamma) - |\nabla u^\gamma|^2 \operatorname{div} \underline{\xi})\,\mathrm{d}x$$

$$= \lim_{\gamma\to 0^+} \gamma^{\frac{1}{2}} \int_{\Omega^\gamma} f(x, \nabla v^\gamma)|\nabla u^\gamma|^2\,\mathrm{d}x = \sqrt{2}\lim_{\gamma\to 0^+}\int_{\Omega^\gamma} f(x,\nabla v^\gamma)|\nabla v^\gamma|\,\mathrm{d}x,$$

$$= \sqrt{2}\lim_{\gamma\to 0^+}\int_\Omega F(x,\nabla v^\gamma)\,\mathrm{d}x, \tag{3.91}$$

where

$$F(x,\underline{\eta}) = \begin{matrix} f(x,\underline{\eta})|\underline{\eta}| & \text{for } |\underline{\eta}|>0, \\ 0 & \text{otherwise.} \end{matrix} \tag{3.92}$$

We remark that $\{\nabla v^\gamma\}$ converges weakly in $L^1(\Omega)$, or weakly $*$ in the sense of vector measures to the measure

$$Dv^* = \frac{\pi}{2\sqrt{2}}\underline{\nu}\mathscr{H}_{d-1}|_S, \tag{3.93}$$

where $\mathscr{H}_{d-1}|_S$ denotes the $(d-1)$-dimension Hausdorff measure restricted to the surface S; in fact, from (3.73), for every vector field $\underline{\chi}\in C_0^1(\Omega)$,

$$\lim_{\gamma\to 0^+}\int_\Omega \chi_i D_i v^\gamma\,\mathrm{d}x = -\lim_{\gamma\to 0^+}\int_\Omega v^\gamma \operatorname{div}\underline{\chi}\,\mathrm{d}x = -\int_\Omega v^*\operatorname{div}\underline{\chi}\,\mathrm{d}x,$$

$$= -\int_{\Omega_-^*}\phi(-1)\operatorname{div}\underline{\chi}\,\mathrm{d}x - \int_{\Omega_+^*}\phi(1)\operatorname{div}\underline{\chi}\,\mathrm{d}x,$$

$$= +\frac{\pi}{2\sqrt{2}}\int_S \underline{\nu}\cdot\underline{\chi}\,\mathrm{d}\mathscr{H}_{d-1}. \tag{3.94}$$

Now $\int_\Omega|\nabla v^\gamma|\,\mathrm{d}x$ is bounded, and from (3.62) and the proof of Theorem 3.7,

$$\limsup_{\gamma\to 0^+}\int_\Omega|\nabla v^\gamma|\,\mathrm{d}x \leqslant \frac{\pi}{2\sqrt{2}}P_\Omega(\Omega_-^*), \tag{3.95}$$

and from (3.85)

$$\liminf_{\gamma\to 0^+}\int_\Omega|\nabla v^\gamma|\,\mathrm{d}x \geqslant \frac{\pi}{2\sqrt{2}}P_\Omega(\Omega_-^*), \tag{3.96}$$

which together with (3.70) imply that

$$\lim_{\gamma\to 0^+}\int_\Omega|\nabla v^\gamma|\,\mathrm{d}x = \frac{\pi}{2\sqrt{2}}P_\Omega(\Omega_-^*) = \int_\Omega |Dv^*|\,\mathrm{d}x. \tag{3.97}$$

We can now apply a result of Reshetnyak (see Luckhaus & Modica 1989) about the weak convergence of homogeneous function of measures, which yields

$$\lim_{\gamma\to 0^+}\int_\Omega F(x,\nabla v^\gamma(x))\,\mathrm{d}x = \frac{\pi}{2\sqrt{2}}\int_S F(x,\underline{\nu}(x))\,\mathrm{d}\mathscr{H}_{d-1}. \tag{3.98}$$

Let us introduce on S the tangential derivative operator

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$$\partial_i = D_i - (\nu_j D_j)\nu_i, \tag{3.99}$$

which is merely the component of ∇ in the tangential direction, and the corresponding divergence operator

$$\mathrm{div}_S = \sum_i \partial_i. \tag{3.100}$$

Elementary differential geometry yields that

$$F(x, \underline{\nu}(x)) = f(x, \underline{\nu}(x)) = -\mathrm{div}_S\, \underline{\xi}(x) \quad \text{and} \quad \mathrm{div}_S\, \underline{\nu} = \lambda^*.$$

Finally, the divergence theorem on curved hypersurfaces (see Massari & Miranda 1984) leads to

$$\begin{aligned}\int_S F(x, \underline{\nu}(x))\, \mathrm{d}\mathscr{H}_{d-1} &= -\int_S \mathrm{div}_S\, \underline{\xi}(x)\, \mathrm{d}\mathscr{H}_{d-1} = -\int_S \underline{\xi}\cdot\underline{\nu}\, \mathrm{div}_S\, \underline{\nu}\, \mathrm{d}\mathscr{H}_{d-1}, \\ &= -\lambda^* \int_S \underline{\xi}\cdot\underline{\nu}\, \mathrm{d}\mathscr{H}_{d-1},\end{aligned} \tag{3.101}$$

and Lemma 3.14 is proved. □

We now conclude the proof of Theorem 3.12.

Proof If we apply Lemma 3.13 to the first term on the right-hand side of (3.81), then we obtain

$$\lim_{\gamma\to 0^+} \int_\Omega \left(\frac{\gamma^{\frac{1}{2}}}{2}|\nabla u^\gamma|^2 - \gamma^{-\frac{1}{2}}\psi(u^\gamma)\right) \mathrm{div}\, \underline{\xi}\, \mathrm{d}x = 0. \tag{3.102}$$

As Ω^*_- is a minimal set in Ω and $2 \leqslant d \leqslant 3$ then it follows from the theory of minimal surfaces that $\partial\Omega^*_- \cap \Omega$ is an analytic hypersurface (see Giusti 1984); hence Lemma 3.14 implies that

$$\lim_{\gamma\to 0^+} \gamma^{\frac{1}{2}} \int_\Omega ((D_i u^\gamma D_i \xi_j D_j u^\gamma) - \mathrm{div}\, \underline{\xi} |\nabla u^\gamma|^2)\, \mathrm{d}x = -\tfrac{1}{2}\pi\lambda^* \int_S \underline{\xi}\cdot\underline{\nu}\, \mathrm{d}\mathscr{H}_{d-1}. \tag{3.103}$$

Any choice in (3.81) and (3.82) of $\underline{\xi} \in C_0^\infty(\Omega)$ such that $S \cap (\text{support } \underline{\xi})$ is smooth and

$$\int_S \underline{\xi}\cdot\underline{\nu}\, \mathrm{d}\mathscr{H}_{d-1} \neq 0,$$

yields the result. □

Remark If the sequence of radial solutions constructed in the previous subsection is a sequence of minimizers of $(\mathbf{M}_\gamma)$, then it is easily seen that (3.49) is in agreement theorem 3.12.

4 Conclusion

We have presented a mathematical analysis of a parabolic variational inequality and its steady state that arises from the deep quench limit of a model of phase separation in a binary mixture due to Cahn and Hilliard. This form of the free energy with 'infinite walls' had previously been suggested by Oono and Puri. The numerical analysis of this model is studied in Part II of this work. It is expected that variants of (1.16) and (1.17) will provide an alternative phase field model for the Stefan problem with interfacial energy effects. The advantage of this approach to approximate phase transformations with infinitesimal

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interfaces is that the order parameter is identically $+1$ or -1, except in a narrow interfacial transition layer. The implication of these remarks will be pursued in a future work.

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