ASYMPTOTIC METHODS IN THE OPTIMAL CONTROL OF DISTRIBUTED SYSTEMS

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1. INTRODUCTION

One of the main difficulties one meets in the Optimal Control of Distributed Systems lies obviously in the size of the problem, in particular for numerical computations.

A natural idea is therefore to use, among other things, *asymptotic methods*, in order to "simplify" the situation.

This idea has been used extensively for the optimal control of systems governed by ordinary differential equations (lumped systems) ; we refer to Kokotovic and Yackel [1], Kokotovic and Sannuti [2], Haddad and Kokotovic [3], R. O'Malley [4][5] and to the bibliography therein.

In this brief survey ([1]), we would like to report on the trends and problems in the asymptotic methods for the optimal control of distributed systems.

Very many different situations can arise and it is therefore useful to begin with a general picture of these situations. ∎

Let us define - in "abstract" form, for the time being - the state equation by

$$\Lambda y = f + Bv \; ; \tag{1.1}$$

in (1.1) Λ is an *unbounded* operator - linear or non linear - ; we look for y in $D(\Lambda)$ (the *domain* of Λ) ; in the right hand side of (1.1), f is given and $v \in \mathcal{U}$ = space of controls ; B is a linear operator from $\mathcal{U}$ to the range of Λ. We suppose that (1.1) admits a unique solution, which is denoted by $y(v)$ and which is the *state of the system*.

The *cost function* $J(v)$ is given by

$$J(v) = \Phi(y(v)) + \Psi(v), \tag{1.2}$$

where Φ and Ψ are functionals given on the range of Λ and on $\mathcal{U}$ respectively

The set of *admissible controls* $\mathcal{U}_{ad}$ is defined by :

(i) constraints on v - say $v \in \mathcal{U}^1_{ad} \subset \mathcal{U}$;

(ii) constraints on $y(v)$ - say $y(v) \in \mathcal{X} \subset$ range of Λ .

The problem of optimal control is to find

$$\inf J(v), \; v \in \mathcal{U}_{ad} \tag{1.3}$$

and to find one element $u \in \mathcal{U}_{ad}$, if it exists, which satisfies

$$J(u) = \inf J(v) \; ; \tag{1.4}$$

such an element u is said to be *an optimal control*. ∎

([1]) The complete proofs, which would be very long, are not given here.

Asymptotic Methods

One can think of using *asymptotic methods* when there are, in the data of the problems, coefficients with different orders of magnitude.

We shall denote by ε a small > 0 parameter (1).

One can distinguish three main cases - each case being subdivided into several cases !

Case I : Perturbations of the state equation

Let Λ^ε be a family of unbounded operators - partial differential operators in the examples we have in mind.
The state equation is now :

$$\Lambda^\varepsilon y_\varepsilon = f + Bv \tag{1.5}$$

which is supposed to admit a unique solution

$$y_\varepsilon(v).$$

The cost function is

$$J_\varepsilon(v) = \Phi(y_\varepsilon(v)) + \Psi(v). \tag{1.6}$$

Let u_ε be an optimal control of (1.6) - assumed to exist. We suppose that when $\varepsilon \to 0$, Λ^ε "converges" in some sense towards Λ^o, *a "simpler" operator than* Λ^ε for $\varepsilon > 0$. This means that, in some topology, $y_\varepsilon(v)$ converges to $y_o(v)$, where

$$\Lambda^o y_o(v) = f + Bv, \qquad y_o(v) \in D(\Lambda^o). \tag{1.7}$$

There are now two distinct cases :

Case (i): The cost function is continuous on $D(\Lambda^o)$
Then the "limit" problem is to minimize on $\mathcal{U}_{ad}$ the functional

$$J_o(v) = \Phi(y_o(v)) + \Psi(v). \tag{1.8}$$

Case (ii):The cost function u is not defined on $D(\Lambda^o)$
This case is much more complicated and it does not seem to have been considered before. Examples of this situation are presented in Section 3 below.
In all the cases the problem consists in :

1) solving the limit problem - which is a "simpler" problem than the initial one ;
2) finding in which sense the original problem is "approximated" by the limit problem - and for instance in finding, if possible, *asymptotic expansions* for u_ε. ∎

Case II: Perturbations of the cost function

Let the state equation be given by (1.1) and let Φ_o and Φ_1 be two given functionals on $D(\Lambda)$. We suppose that the cost function is given by

$$J_\varepsilon(v) = \Phi_o(y(v)) + \varepsilon\,\Phi_1(y(v)) + \Psi(v). \tag{1.9}$$

The "limit" problem is now to minimize

$$J_o(v) = \Phi_o(y(v)) + \Psi(v) \tag{1.10}$$

a problem which can be simpler than the original one (we give an example in Section 4 below). As in Case I, the next step is to see in which manner the limit problem "approximates" the original one. ∎

(1) The situations where there are *several* small parameters are not studied here.

Case III : Degeneracy of the cost function (cheap control)

Let the state equation be again given by (1.1) and let the cost function be given by

$$J_\varepsilon(v) = \Phi(y(v)) + \varepsilon\Psi(v) \tag{1.11}$$

Formally the "limit" problem is to minimize

$$J_o(v) = \Phi(y(v)) \tag{1.12}$$

which can be a singular problem.
The problems are then the same than above. ∎

Remark 1.1. One can of course consider situations where one has several of the above problems *at the same time* ; for instance the state equation can be given by (1.5) and the cost function by

$$J_\varepsilon(v) = \Phi(y_\varepsilon(v)) + \varepsilon^k\Psi(v) \; ; \tag{1.13}$$

this is a "combination" of Cases I and III. ∎

2. PERTURBATION OF THE STATE EQUATION. THE CASE OF A "CONTINUOUS" COST FUNCTION.

We give now *examples* of Case I (i) :

2.1. Singular perturbations

Let Ω be a bounded open set of $\mathbb{R}^n$, with boundary Γ.
Let A be a second order elliptic operator given by

$$\left.\begin{aligned} &A\varphi = -\sum \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial\varphi}{\partial x_j}\right), \\ &a_{ij} \in L^\infty(\Omega), \\ &\sum a_{ij}(x)\zeta_i\zeta_j \geq \alpha \sum \zeta_i^2, \quad \alpha>0, \text{ a.e. in } \Omega . \end{aligned}\right. \tag{2.1}$$

The state is given by

$$\left.\begin{aligned} &\varepsilon A\, y_\varepsilon(v) + y_\varepsilon(v) = f + v \text{ in } \Omega, \\ &\frac{\partial y_\varepsilon}{\partial \nu_A}(v) = 0 \quad \text{on } \Gamma \;{}^{(1)} \end{aligned}\right\} \tag{2.2}$$

In *variational form*, we introduce the *Sobolev space* :

$$H^1(\Omega) = \{ \varphi \mid \varphi, \frac{\partial\varphi}{\partial x_i} \in L^2(\Omega), \; i=1,\dots,n\}$$

provided with its usual Hilbertian structure ; for $\varphi,\psi \in H^1(\Omega)$ we set

$$\left.\begin{aligned} &a(\varphi,\psi) = \sum \int_\Omega a_{ij}(x)\frac{\partial\varphi}{\partial x_j}\frac{\partial\psi}{\partial x_i}\,dx, \\ &(f,\psi) = \int_\Omega f\,\psi\,dx\;; \end{aligned}\right\} \tag{2.3}$$

then (2.2) is equivalent to :

$$\left.\begin{aligned} &\varepsilon\, a(y_\varepsilon(v),\varphi) + (y_\varepsilon(v),\varphi) = (f+v,\varphi) \quad \forall\varphi \in H^1(\Omega), \\ &y_\varepsilon(v) \in H^1(\Omega). \end{aligned}\right\} \tag{2.4}$$

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(1) $\dfrac{\partial}{\partial \nu_A}$ denotes the conormal derivative associated to A.

We assume that

$$v \in \mathcal{U}_{ad} = \text{closed convex subset of } L^2(\Omega). \tag{2.5}$$

Equation (2.4) admits a unique solution.
The cost function is given by

$$J_\varepsilon(v) = \int_\Omega |y_\varepsilon(v)-z_d|^2 \, dx + N \int_\Omega v^2 dx, \tag{2.6}$$

where z_d is given in $L^2(\Omega)$ and where $N > 0$. ∎

It is well known (cf. for instance Lions [6] and the bibliography therein) th when $\varepsilon \to 0$, $y_\varepsilon(v) \to y(v)$ in $L^2(\Omega)$ where

$$y(v) = f + v \tag{2.7}$$

The limit problem is straightforward

$$J(v) = \int_\Omega [|f + v - z_d|^2 + Nv^2] \, dx. \tag{2.8}$$

If u denotes the solution of

$$\inf J(v), \quad v \in \mathcal{U}_{ad} \tag{2.9}$$

then $u_\varepsilon \to u$ *in* $L^2(\Omega)$ *as* $\varepsilon \to 0$. ∎

The next step is to obtain an expansion for u_ε.
Let us write the *optimality system* for problem (2.6) (cf. Lions [7]). We intr duce the *adjoint state* p_ε defined by

$$\left.\begin{aligned} &\varepsilon A^* p_\varepsilon + p_\varepsilon = y_\varepsilon - z_d \text{ in } \Omega, \\ &\frac{\partial p_\varepsilon}{\partial \nu_{A^*}} = 0 \text{ on } \Gamma \end{aligned}\right\} \tag{2.10}$$

where $A^*\varphi = -\sum \frac{\partial}{\partial x_i}(a^*_{ij}(x) \frac{\partial}{\partial x_j})$, $a^*_{ij}(x) = a_{ji}(x)$,

and where

$$y_\varepsilon(u_\varepsilon) = y_\varepsilon. \tag{2.11}$$

Then u_ε is optimal iff

$$\left.\begin{aligned} &\int_\Omega (p_\varepsilon + Nu_\varepsilon)(v-u_\varepsilon)\,dx \geq 0 \quad \forall v \in \mathcal{U}_{ad}, \\ &u_\varepsilon \in \mathcal{U}_{ad}. \end{aligned}\right\} \tag{2.12}$$

The optimality system is given by

$$\left.\begin{aligned} &\varepsilon A y_\varepsilon + y_\varepsilon = f + u_\varepsilon \text{ in } \Omega, \\ &\frac{\partial y_\varepsilon}{\partial \nu_A} = 0 \text{ on } \Gamma \end{aligned}\right\} \tag{2.13}$$

together with (2.10) and (2.12). ∎

In the particular case where there are *no constraints*, i.e.

$$\mathcal{U}_{ad} = L^2(\Omega), \tag{2.14}$$

then (2.12) reduces to

$$p_\varepsilon + Nu_\varepsilon = 0 \tag{2.15}$$

and the optimality system becomes

$$\left.\begin{aligned}&\varepsilon A y_\varepsilon + y_\varepsilon + \tfrac{1}{N} p_\varepsilon = f \ ,\\ &\varepsilon A^* p_\varepsilon + p_\varepsilon - y_\varepsilon = - z_d \quad \text{in } \Omega \ ,\\ &\frac{\partial y_\varepsilon}{\partial \nu_A} = 0 \ , \quad \frac{\partial p_\varepsilon}{\partial \nu_{A^*}} = 0 \quad \text{on } \Gamma.\end{aligned}\right\} \tag{2.16}$$

■

If we look for an expansion in the form

$$\left.\begin{aligned}y_\varepsilon &= y + \varepsilon y^1 + \dots\\ p_\varepsilon &= p + \varepsilon p^1 + \dots\end{aligned}\right\} \tag{2.17}$$

we obtain

$$\left.\begin{aligned}&y + \tfrac{1}{N} p = f \ ,\\ &p - y = - z_d\end{aligned}\right\} \tag{2.18}$$

(which is of course the optimality system for the limit problem (2.9) when $\mathcal{U}_{ad} = L^2(\Omega)$), and

$$\left.\begin{aligned}&A y + y^1 + \tfrac{1}{N} p^1 = 0,\\ &A^* p + p^1 - y^1 = 0 .\end{aligned}\right\} \tag{2.19}$$

System (2.19) will give y^1, p^1 in $L^2(\Omega)$ if we *assume* that

$$Ay, \quad A^* p \in L^2(\Omega)$$

which is satisfied if f and $z_d \in H^2(\Omega)$.

But, as it is classical in singular perturbations, we need *correctors* in (2.17) in order to take care of *boundary conditions*. That is we look for

$$\left.\begin{aligned}y_\varepsilon &= y + \varepsilon y^1 + \eta_\varepsilon + \dots\\ p_\varepsilon &= p + \varepsilon p^1 + \pi_\varepsilon + \dots\end{aligned}\right\} \tag{2.20}$$

where the η_ε and π_ε are *correctors "concentrated" in the neighborhood of* Γ : these are *the boundary layers*.

By using local maps one reduces the problem to the case where A is with constant coefficients and where $\Omega = \{x \mid x_n > 0\}$. (Cf. Visik-Liousternik [8] [9], W. Eckhaus and E.M. de Jager [10]).
We keep only the *normal* derivatives so that we *define* η_ε and π_ε by

$$\left.\begin{aligned}&- \varepsilon \frac{d^2 \eta_\varepsilon}{d x_n^2} + \eta_\varepsilon + \frac{1}{N} \pi_\varepsilon = 0,\\ &- \varepsilon \frac{d^2 \pi_\varepsilon}{d x_n^2} + \pi_\varepsilon - \eta_\varepsilon = 0 \quad , \quad x_n > 0 \ ,\end{aligned}\right\} \tag{2.21}$$

$$\left.\begin{aligned}&\frac{d\eta_\varepsilon}{dx_n}(o) + \frac{\partial y}{\partial x_n}(x',o) = 0 \quad x' = \{x_1, \dots, x_{n-1}\},\\ &\frac{d\pi_\varepsilon}{dx_n}(o) + \frac{\partial p}{\partial x_n}(x',o) = 0,\end{aligned}\right\} \tag{2.22}$$

and η_ε, π_ε being with *exponential decrease* as $x_n \to \infty$.

One finds that

$$\eta_\varepsilon = \frac{\sqrt{\varepsilon}}{\sqrt{1+N}}\, e^{-\lambda x_n/\sqrt{\varepsilon}} \frac{\partial p}{\partial x_n}(x',o)\left[\lambda \sin\frac{\mu x_n}{\sqrt{\varepsilon}} + \mu \cos\frac{\mu x_n}{\sqrt{\varepsilon}}\right] + $$
$$+ \frac{\sqrt{\varepsilon}\sqrt{N}}{\sqrt{1+N}}\, e^{-\lambda x_n/\sqrt{\varepsilon}} \frac{\partial y}{\partial x_n}(x',o)\left[\lambda \cos\frac{\mu x_n}{\sqrt{\varepsilon}} - \mu \sin\frac{\mu x_n}{\sqrt{\varepsilon}}\right] , \tag{2.23}$$

$$\pi_\varepsilon = -\frac{\sqrt{\varepsilon}\sqrt{N}}{\sqrt{1+N}}\, e^{-\lambda x_n/\sqrt{\varepsilon}} \frac{\partial p}{\partial x_n}(x',o)\left[-\lambda \cos\frac{\mu x_n}{\sqrt{\varepsilon}} + \mu \sin\frac{\mu x_n}{\sqrt{\varepsilon}}\right] -$$
$$- \frac{\sqrt{\varepsilon}}{\sqrt{1+N}}\, e^{-\lambda x_n/\sqrt{\varepsilon}} \frac{\partial y}{\partial x_n}(x',o)\left[\lambda \sin\frac{\mu x_n}{\sqrt{\varepsilon}} + \mu \cos\frac{\mu x_n}{\sqrt{\varepsilon}}\right] , \tag{2.24}$$

where $\lambda, \mu > 0$, $\lambda^2 = \frac{1}{2}\left(1+\sqrt{1+\frac{1}{N}}\right)$, $\mu^2 = \frac{1}{2}\left(\sqrt{1+\frac{1}{N}} - 1\right)$.

For the general case, the boundary layer is given by (2.23)(2.24) where x_n is replaced by $d(x,\Gamma)$ = distance from x to Γ , and where

$\frac{\partial p}{\partial x_n}(x',o)$, $\frac{\partial y}{\partial x_n}(x',o)$ are respectively replaced by

$$-\frac{1}{1+\frac{1}{N}}\frac{\partial}{\partial \nu_{A^*}}(f-z_d), \quad -\frac{1}{1+\frac{1}{N}}\frac{\partial}{\partial \nu_A}(f+\frac{1}{N}z_d) \text{ on } \Gamma .$$ ■

With these corrections, one proves (Lions [6], chapter 7)

$$\left\| u_\varepsilon - \left(-\frac{1}{N}(p+\varepsilon p^1+\pi_\varepsilon)\right)\right\|_{L^2(\Omega)} \le C\varepsilon^{3/2}. \tag{2.25}$$

For other results along these lines, cf. Lions [6], Chapter 7. ■
For non singular perturbations for non linear systems, we refer to C.M.Brauner [11] , C.M. Brauner and P.Penel [12] .

2.2. Homogenization

In the study of *composite materials* one considers

$$A^\varepsilon y_\varepsilon = f + v \text{ in } \Omega , \tag{2.26}$$

$$y_\varepsilon = 0 \text{ on } \Gamma \tag{2.27}$$

where

$$A^\varepsilon \varphi = -\sum \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial \varphi}{\partial x_j}\right) ; \tag{2.28}$$

in (2.28) the functions $a_{ij}(y)$ are *periodic* (with period 1 to fix ideas) *in all variables* and they satisfy

$$\sum_{i,j} a_{ij}(y)\, \zeta_i \zeta_j \ge \alpha \sum \zeta_i^2 , \quad \alpha > 0, \text{ for almost every } y \text{ in } \mathbb{R}^n. \tag{2.29}$$

Let us assume that the *cost function* is given by

$$J_\varepsilon(v) = \int_\Omega |y_\varepsilon(v)-z_d|^2 dx + N\int_\Omega v^2 dx \tag{2.30}$$

where $\left| \begin{array}{l} y_\varepsilon(v) = \text{solution of } (2.26)(2.27), \\ z_d \in L^2(\Omega), \quad N > o. \end{array} \right.$

It is known (cf. de Giorgi and Spagnolo [13] and A. Bensoussan, J.L. Lions and G Papanicolaou [14] and the bibliography therein) that there *exists an operator $\mathcal{A}$ elliptic with constant coefficients* such that the *limit problem* of

$$\inf \; J_\varepsilon(v) \quad , \quad v \in \mathcal{U}_{ad} \tag{2.31}$$

is given by

$$\left.\begin{array}{l} \inf J(v), \quad v \in \mathcal{U}_{ad}, \\ J(v) = \int_\Omega |y(v)-z_d|^2 \, dx + N \int_\Omega v^2 dx, \end{array}\right\} \tag{2.32}$$

$$\left.\begin{array}{l} \mathcal{A} y(v) = f + v \,, \\ y(v) = 0 \quad \text{on } \Gamma . \end{array}\right\} \tag{2.33}$$

The operator $\mathcal{A}$ is the so called *homogenized operator* associated to A^ε.

Remark 2.1. We refer to the book by A. Bensoussan, G. Papanicolaou and the Author for the formulas giving $\mathcal{A}$.

Remark 2.2. A systematic study of the optimal control and homogenization is given in Kesavan and Vanninathan [15].

3. PERTURBATION OF THE STATE EQUATION. THE CASE WHERE THE COST FUNCTION IS NOT DEFINED ON THE LIMIT SPACE.

3.1. Orientation

In all cases studied in Section 2, J(v) is defined (and actually it is continuous) on $\mathcal{U}_{ad}$. We now consider cases when $J_\varepsilon(v)$ is continuous on $\mathcal{U}_{ad}$ but the limit cost function J(v) *is not defined on* $\mathcal{U}_{ad}$. (This corresponds to case I (ii) in the classification of the Introduction).

3.2. A stationary problem.

The state equation is given by (2.2), under hypothesis (2.1), i.e.

$$\left.\begin{array}{l} \varepsilon A \, y_\varepsilon(v) + y_\varepsilon(v) = f + v \text{ in } \Omega \,, \\ \dfrac{\partial y_\varepsilon}{\partial \nu_A}(v) = 0 \text{ on } \Gamma. \end{array}\right\} \tag{3.1}$$

We assume that *the cost function* is given by

$$J_\varepsilon(v) = \int_\Gamma |y_\varepsilon(v)-z_d|^2 \, d\Gamma + N \int_\Omega v^2 dx, \tag{3.2}$$

and we want to minimize $J_\varepsilon(v)$ on $\mathcal{U}_{ad}$ = closed convex subset of $L^2(\Omega)$. Let u_ε be the unique solution of this problem and let us set

$$y_\varepsilon(u_\varepsilon) = y_\varepsilon \,. \tag{3.3}$$

The optimality system is given by

$$\varepsilon A y_\varepsilon + y_\varepsilon = f + u_\varepsilon \,, \quad \varepsilon A^* p_\varepsilon + p_\varepsilon = 0 \text{ in } \Omega \,, \tag{3.4}$$

$$\frac{\partial y_\varepsilon}{\partial \nu_A} = 0 \ , \ \varepsilon \frac{\partial p_\varepsilon}{\partial \nu_{A^*}} = y_\varepsilon - z_d \text{ on } \Gamma \tag{3.5}$$

and

$$\int_\Omega (p_\varepsilon + N u_\varepsilon)(v - u_\varepsilon)\,dx \geq 0 \quad \forall v \in \mathcal{U}_{ad}, \quad u_\varepsilon \in \mathcal{U}_{ad}. \tag{3.6}$$ ■

Formally the limit problem is given as follows : it is known that $y_\varepsilon(v) \to y(v)$ in $L^2(\Omega)$ where

$$y(v) = f + v \tag{3.7}$$

and

$$J(v) = \int_\Gamma |y(v) - z_d|^2 d\Gamma + N \int_\Omega v^2 dx. \tag{3.8}$$

But *this problem does not make sense*, since (3.8) involves taking the trace on Γ of a function $(f+v)$ which belongs to $L^2(\Omega)$!

But of course it does make sense to ask for the limit (if it exists) *of* $J_\varepsilon(u_\varepsilon)$ and also possibly for the limit of u_ε , and this is the question we want to consider.

3.3. A priori estimates

We make *the hypothesis*

$$\left.\begin{array}{l} \text{there exists } v_o \in \mathcal{U}_{ad} \text{ such that} \\ f + v_o \in H^1(\Omega). \end{array}\right\} \tag{3.9}$$

Example 3.1.

If $\mathcal{U}_{ad} = L^2(\Omega)$, (3.9) is always satisfied ; one can take $v_o = -f$.

Example 3.2.

If $\mathcal{U}_{ad} = \{v \mid v \geq 0 \text{ a.e. in } \Omega\}$, and $f = f_o + f_1$, $f_o \in H^1(\Omega)$, $f_1 \in L^2(\Omega)$, $f_1 \leq 0$, one has (3.9) ; one can take $v = -f_1$.

Remark 3.1. It would be enough for the validity of the estimates which follow to assume that there exists $v_o \in \mathcal{U}_{ad}$ such that $f + v_o \in H^{\frac{1}{2}}(\Omega)$. ■

We now verify : under the hypothesis (3.9) one has

$$|u_\varepsilon| \leq C, \qquad |y_\varepsilon| \leq C, \qquad \|y_\varepsilon(u_\varepsilon)\|_{L^2(\Gamma)} \leq C \tag{3.10}$$

where $|f|$ = norm of f in $L^2(\Omega)$ and where the C's denote various constants (independent of ε).

Proof :

We have

$$\varepsilon A y_\varepsilon(v_o) + y_\varepsilon(v_o) = f + v_o, \qquad \frac{\partial y_\varepsilon(v_o)}{\partial \nu_A} = 0 \text{ on } \Gamma . \tag{3.11}$$

Multiplying (3.11) by $y_\varepsilon(v_o)$ and writing $a(\varphi) = a(\varphi,\varphi)$ (cf.(2.3)) we obtain

$$\varepsilon\, a(y_\varepsilon(v_o)) + |y_\varepsilon(v_o)|^2 = (f + v_o, y_\varepsilon(v_o)) \tag{3.12}$$

hence it follows that

$$|y_\varepsilon(v_o)| \leq C. \tag{3.13}$$

We now multiply (3.11) by $Ay_\varepsilon(v_o)$. Since $f+v_o \in H^1(\Omega)$, it follows that

$$\varepsilon|Ay_\varepsilon(v_o)|^2 + a(y_\varepsilon(v_o)) = a(y_\varepsilon(v_o), f+v_o) \tag{3.14}$$

hence it follows that

$$\left|\frac{\partial y_\varepsilon(v_o)}{\partial x_i}\right| \leq C \qquad \forall i \tag{3.15}$$

It follows from (3.13)(3.15) that $y_\varepsilon(v_o)$ is bounded in $H^1(\Omega)$ and therefore $y_\varepsilon(v_o)|_\Gamma$ *is bounded* (in particular) *in* $L^2(\Gamma)$.

Therefore

$$J_\varepsilon(v_o) \leq C$$

and since $J_\varepsilon(u_\varepsilon) \leq J_\varepsilon(v_o)$, it follows that

$$N|u_\varepsilon|^2 + \int_\Gamma |y_\varepsilon - z_d|^2 \, d\Gamma \leq C. \tag{3.16}$$

Consequently $|u_\varepsilon| \leq C$ and since $\varepsilon A y_\varepsilon + y_\varepsilon = f + u_\varepsilon$, $\dfrac{\partial y_\varepsilon}{\partial \nu_A} = 0$, we have

$$|y_\varepsilon| \leq C.$$

Using (3.16) we see that $y_\varepsilon(u_\varepsilon) = y_\varepsilon$ is bounded in $L^2(\Gamma)$ and (3.10) is proven ∎

We also remark that

$$J_\varepsilon(u_\varepsilon) \leq C. \tag{3.17}$$

∎

The problem is now to study the behaviour of u_ε, y_ε, $J_\varepsilon(u_\varepsilon)$ as $\varepsilon \to 0$. We consider the case *without constraints*.

3.4. The case without constraints $\mathcal{U}_{ad} = L^2(\Omega)$.

In that case the optimality system becomes

$$\left.\begin{aligned} &\varepsilon A y_\varepsilon + y_\varepsilon + \frac{1}{N} p_\varepsilon = f \\ &\varepsilon A^* p_\varepsilon + p_\varepsilon = 0, \end{aligned}\right\} \tag{3.18}$$

$$\frac{\partial y_\varepsilon}{\partial \nu_A} = 0 \quad , \; \varepsilon \frac{\partial p_\varepsilon}{\partial \nu_{A^*}} = y_\varepsilon - z_d \text{ on } \Gamma \tag{3.19}$$

and

$$p_\varepsilon + N u_\varepsilon = 0. \tag{3.20}$$

It follows from (3.10) and (3.20) that

$$|p_\varepsilon| \leq C.$$

We can extract a subsequence, still denoted by u_ε, y_ε, p_ε such that

$$u_\varepsilon \to u, \quad y_\varepsilon \to y, \quad p_\varepsilon \to p \quad \text{in } L^2(\Omega) \text{ weakly.} \tag{3.21}$$

It follows from the second equation (3.18) that $p = 0$ and therefore $u = 0$. The first equation (3.18) gives $y = f$. The limit being unique, we do not have to extract subsequences and we have

$$u_\varepsilon \to 0, \quad p_\varepsilon \to 0, \quad y_\varepsilon \to f \quad \text{in } L^2(\Omega) \text{ weakly.} \tag{3.22}$$

∎

We compute now an *asymptotic expansion* for (3.18)(3.19). We consider the particular case when

$$\Omega = \{x \mid x_n > 0\}, \quad A = -\Delta. \tag{3.23}$$

The "interior" expansion of y_ε, p_ε gives

$$y_\varepsilon = f - \varepsilon Af + \dots \tag{3.24}$$

where we assume from now on that

$$f \in H^2(\Omega) ; \tag{3.25}$$

interior terms are zero for p_ε.

We look for *boundary layers* η_ε, π_ε.

$$\left.\begin{aligned} y_\varepsilon &= f - \varepsilon Af + \eta_\varepsilon + \dots \\ p_\varepsilon &= \pi_\varepsilon + \dots \quad ; \end{aligned}\right\} \tag{3.26}$$

we use (3.26) in (3.18) and we only keep normal derivatives (and we neglect higher order terms) ; it comes

$$\left.\begin{aligned} -\varepsilon \frac{d^2\eta_\varepsilon}{dx_n^2} + \eta_\varepsilon + \frac{1}{N}\pi_\varepsilon = 0 \ , \\ -\varepsilon \frac{d^2\pi_\varepsilon}{dx_n^2} + \pi_\varepsilon = 0, \end{aligned}\right\} \tag{3.27}$$

$$\left.\begin{aligned} \frac{d\eta_\varepsilon}{dx_n}(o) + \frac{\partial f}{\partial x_n}(x',o) = 0, \\ -\varepsilon \frac{d\pi_\varepsilon}{dx_n}(x',o) = f(x',o) - z_d(x') + \eta_\varepsilon(o) \ ; \end{aligned}\right\} \tag{3.28}$$

and we compute the solution which is with exponential decrease as $x_n \to 0$.
We obtain

$$\left.\begin{aligned} \eta_\varepsilon &= (c_2 + c_3 x_n)\, e^{-x_n/\sqrt{\varepsilon}} \\ \pi_\varepsilon &= c_1\, e^{-x_n/\sqrt{\varepsilon}} \ , \end{aligned}\right\} \tag{3.29}$$

where

$$c_1 = \frac{\sqrt{\varepsilon}}{1 + \frac{\sqrt{\varepsilon}}{2N}} \left(f(x',o) - z_d + \sqrt{\varepsilon}\frac{\partial f}{\partial x_n}(x',o)\right) \ ,$$

$$c_2 = \frac{\sqrt{\varepsilon}}{1 + \frac{\sqrt{\varepsilon}}{2N}}\, g \quad , \quad g = \frac{\partial f}{\partial x_n}(x',o) - \frac{1}{2N}(f(x',o) - z_d),$$

$$c_3 = -\frac{c_1}{2N\sqrt{\varepsilon}}$$

Therefore

$$u_\varepsilon = -\frac{\sqrt{\varepsilon}}{N}(f(x',o) - z_d(x'))\, e^{-x_n/\sqrt{\varepsilon}} + \dots \tag{3.30}$$

In the general case, we find that

$$u_\varepsilon \sim -\frac{\sqrt{\varepsilon}}{N}(f(x') - z_d(x))\, e^{-d(x,\Gamma)/\sqrt{\varepsilon}} \tag{3.31}$$

where x' = nearest point of x on Γ.
We can compute the expansion of $J_\varepsilon(u_\varepsilon)$.
One finds

$$J_\varepsilon(u_\varepsilon) = \int_\Gamma (f(x',o)-z_d)^2 d\Gamma + 2\sqrt{\varepsilon} \int_\Gamma (f(x',o)-z_d) g d\Gamma -$$
$$- 2\varepsilon \int_\Gamma (f(x',o)-z_d)(Af(x',o)+\frac{g}{2N}) d\Gamma + \tag{3.32}$$
$$+ \frac{\varepsilon^{3/2}}{2N^2} \int_\Gamma (f(x',o)-z_d) g d\Gamma + \ldots$$

(where we assume that Af admits a trace on Γ).

3.5. Various remarks

Remark 3.2.

We conjecture that, under hypothesis (3.9), the expansion of $J_\varepsilon(u_\varepsilon)$ is of the form :

$$J_\varepsilon(u_\varepsilon) = J_o + \sqrt{\varepsilon}\, J_1 + \ldots \tag{3.33}$$

but this is an open question. We do not know the behaviour of u_ε if (3.9) is not satisfied, or at least if the hypothesis of Remark 3.1 is not satisfied. ∎

Remark 3.3. Expansion of the type (3.32) is somewhat reminiscent of the expansion of the following problem arising in visco-plasticity (cf. Mosolov and Miasnikov [16]). Let Ω be the complementary set in $\mathbb{R}^n$ of a bounded simply connected set ω with boundary Γ ; we consider

$$J_\varepsilon(v) = \frac{\varepsilon}{2} \int_\Omega |\operatorname{grad} v|^2 dx + \int_\Omega |\operatorname{grad} v| dx \tag{3.34}$$

and we consider

$$\left.\begin{aligned} &\inf J_\varepsilon(v), \quad v=1 \text{ on } \Gamma, \quad \operatorname{grad} v \in (L^1 \cap L^2(\Omega))^n, \\ &v \text{ is "small" at infinity.} \end{aligned}\right\} \tag{3.35}$$

Then

$$\inf_v J_\varepsilon(v) = J_o + \sqrt{\varepsilon}\, J_1 + \ldots \tag{3.36}$$

where J_0 and J_1 can be explicitely computed (cf. an introduction to the work of Mosolov and Miasnikov in Lions [17]). ∎

Remark 3.4. Let us make use of the duality in convex analysis, following R.T. Rockafellar [18], I. Ekeland and R. Temam [19]. We consider

$$F(v) = \frac{N}{2} \int_\Omega v^2 dx, \quad G(q) = \frac{1}{2} \int_\Gamma |q+\varphi_\varepsilon - z_d|^2 dx,$$

$$\varphi_\varepsilon = y_\varepsilon(o),$$

$$P_\varepsilon v = y_\varepsilon(v) - y_\varepsilon(o).$$

Then (we consider the case without constraints) :

$$\left.\begin{aligned} \inf \frac{1}{2} J_\varepsilon(v) &= \inf_v [F(v) + G(P_\varepsilon v)] = \\ &= - \inf_q [F^*(P_\varepsilon^* q) + G^*(-q)]. \end{aligned}\right\} \tag{3.37}$$

In (3.37) the dual functions F^*, G^* are given by

$$\left.\begin{aligned} F^*(v) &= \frac{1}{2N} \int_\Omega v^2 dx, \\ G^*(q) &= \frac{1}{2} \int_\Gamma q^2 d\Gamma - \int_\Gamma q(\varphi_\varepsilon - z_d) d\Gamma. \end{aligned}\right\} \tag{3.38}$$

The operator P_ε^* is given by

$$\left.\begin{aligned} &\varepsilon A^* z_\varepsilon + z_\varepsilon = 0, \\ &\varepsilon \frac{\partial z_\varepsilon}{\partial \nu_{A^*}} = q \text{ on } \Gamma. \end{aligned}\right\} \quad (3.39)$$

Then

$$z_\varepsilon = P_\varepsilon^* q. \quad (3.40)$$

Therefore if we define

$$\mathcal{J}_\varepsilon(q) = \frac{1}{N}\int_\Omega (z_\varepsilon(q))^2 dx + \int_\Gamma q^2 d\Gamma - 2\int_\Gamma q(z_d - \varphi_\varepsilon) d\Gamma \quad (3.41)$$

we have

$$\inf_v J_\varepsilon(v) = -\inf_q \mathcal{J}_\varepsilon(q). \quad (3.42)$$

This method can be useful in particular when there are *constraints on the state* $y_\varepsilon(v)$ (cf. in this respect, in situations without singularities, J. Mossino [20]). ∎

<u>Remark 3.5</u> Another example where the "limit" cost function is not defined on the "limit" space is the following. Let the state be given by

$$\varepsilon \frac{\partial y_\varepsilon}{\partial t} + A y_\varepsilon = f \quad \text{in } \Omega \times]0,T[\,, \quad (3.43)$$

where A is given by (2.1) with the initial condition

$$y_\varepsilon(x,o) = 0 \text{ in } \Omega \quad (3.44)$$

and the boundary condition

$$\left.\begin{aligned} &\frac{\partial y_\varepsilon}{\partial \nu_A} = v \text{ on } \Sigma = \Gamma \times]0,T[\,, \\ &v \in L^2(\Sigma) \,, \ (v = \text{control function}). \end{aligned}\right\} \quad (3.45)$$

The problem (3.43)(3.44)(3.45) admits a unique solution, denoted by

$$y_\varepsilon(x,t;v) = y_\varepsilon(v).$$

The cost function is defined by

$$J_\varepsilon(v) = \int_\Omega (y_\varepsilon(x,T;v) - (z_d(x))^2 dx + N\int_\Sigma v^2 d\Sigma \,, \quad (3.46)$$

where $T > 0$ given, z_d given in $L^2(\Omega)$, $N > 0$.

The problem of optimal control consists in minimizing $J_\varepsilon(v)$ on $\mathcal{U}_{ad}$ = closed convex subset of $L^2(\Sigma)$.

Let us suppose that

$$\left.\begin{aligned} &0 \in \mathcal{U}_{ad}, \ ({}^1) \\ &f, \frac{\partial f}{\partial t} \in L^2(\Omega \times]0,T[) \,, \quad f(x,o) = 0. \end{aligned}\right\} \quad (3.47)$$

Then one can verify that, in particular,

$$|y_\varepsilon(x,T;0)|_{L^2(\Omega)} \leq C$$

so that

$$J_\varepsilon(o) \leq C$$

and therefore one obtains a priori estimates similar to those of Section 3.3. We do not explicit the asymptotic expansions calculations, in the "no constraint" case. ∎

(1) One can more generally assume that $\mathcal{U}_{ad}$ contains a function v_o such that $v_o, \frac{\partial v_o}{\partial t} \in L^2(\Sigma)$, $v_o(x,o) = 0$.

4. PEPTURBATION OF THE COST FUNCTION

We consider now an example of the Case II of the Intrōduction.

4.1. Setting of the problem

We consider A given by (2.1) and we assume that *the state equation* is given by

$$\left.\begin{aligned} &\frac{\partial y}{\partial t} + Ay = f \quad \text{in } \Omega\times]0,T[\ , \\ &\frac{\partial y}{\partial \nu_A} = v \quad \text{on } \Sigma\ , \\ &y(x,o) = y_o \text{ in } \Omega\ . \end{aligned}\right\} \tag{4.1}$$

where y_o is given in $L^2(\Omega)$.

We define

$$C\varphi = \int_\Omega \varphi dx \tag{4.2}$$

and we consider the *cost function*

$$J_\varepsilon(v) = \int_0^T (Cy(v) - z_1)^2 dt + \varepsilon\int_\Sigma (y(v) - z_2)^2 \, d\Sigma + N\int_\Sigma v^2 d\Sigma. \tag{4.3}$$

In (4.3), z_1 is given in $L^2(0,T)$ and z_2 is given in $L^2(\Sigma)$.

If $\mathcal{U}_{ad}$ is a closed convex subset of $L^2(\Sigma)$, we consider

$$\inf J_\varepsilon(v) \ , \quad v \in \mathcal{U}_{ad}\ ; \tag{4.4}$$

let u_ε be the unique solution of (4.4) and let us set $y(u_\varepsilon) = y_\varepsilon$. ∎

The *limit problem* (as $\varepsilon \to 0$) is here very simple. We define

$$J_o(v) = \int_0^T (Cy(v)-z_1)^2 dt + N\int_\Sigma v^2 d\Sigma \tag{4.5}$$

and we denote by u the solution of

$$J_o(u) = \inf J_o(v) \ , \quad v \in \mathcal{U}_{ad} \ , \ u \in \mathcal{U}_{ad}. \tag{4.6}$$

It is a simple matter to verify that

$$u_\varepsilon \to u \quad \text{in } L^2(\Sigma) \quad \text{as } \varepsilon \to 0. \tag{4.7}$$

But (4.6) *is a very simple problem.* Indeed it follows from (4.1) that

$$\left.\begin{aligned} &\frac{d}{dt}(Cy) - \int_\Gamma v d\Gamma = Cf, \\ &Cy(o) = Cy_o \end{aligned}\right\} \tag{4.8}$$

so that

$$Cy(v) = Cy_o + \int_o^t Cf(x,s)ds + \int_o^t ds \int_\Gamma v \, d\Gamma. \tag{4.9}$$

Therefore if we set

$$\tilde{z}_1 = z_1 - Cy_o - \int_o^t Cf(x,s)ds \tag{4.10}$$

we have

$$J_o(v) = \int_0^T \left(\int_o^t ds \int_\Gamma v\,d\Gamma - \tilde{z}_1(t)\right)^2 dt + N\int_\Sigma v^2 d\Sigma \qquad (4.11)$$

so that (4.6) is an elementary problem. ■
The next step is to look for *an asymptotic expansion.*

4.2. The case without constraints

In general, the optimality system is given as follows :

$$\left.\begin{aligned} &\frac{\partial y_\varepsilon}{\partial t} + Ay_\varepsilon = f\ , \\ &-\frac{\partial p_\varepsilon}{\partial t} + A^* p = Cy_\varepsilon - z_1 \end{aligned}\right\} \qquad (4.12)$$

$$\left.\begin{aligned} &\frac{\partial y_\varepsilon}{\partial \nu_A} = u_\varepsilon\ , \\ &\frac{\partial p_\varepsilon}{\partial \nu_{A^*}} = \varepsilon(y_\varepsilon - z_2) \text{ on } \Sigma \end{aligned}\right\} \qquad (4.13)$$

$$y_\varepsilon(x,o) = y_o(x), \quad p_\varepsilon(x,T) = 0, \qquad (4.14)$$

$$\int_\Sigma (p + Nu_\varepsilon)(v - u_\varepsilon)d\Sigma \geq 0 \qquad \forall v \in \mathcal{U}_{ad}, \quad u_\varepsilon \in \mathcal{U}_{ad}. \qquad (4.15)$$

In the case without constraints $p_\varepsilon + Nu_\varepsilon = 0$ and if we look for an expansion

$$y_\varepsilon = y^o + \varepsilon y^1 + \ldots\ , \ p_\varepsilon = p^o + \varepsilon p^1 \ \ldots \qquad (4.16)$$

we obtain for y^o, p^o the optimality system for (4.6) (but this is useless, by using (4.11) and for y^1, p^1 the system

$$\left.\begin{aligned} &\frac{\partial y^1}{\partial t} + Ay^1 = 0, \\ &-\frac{\partial p^1}{\partial t} + A^* p^1 = Cy^1, \end{aligned}\right\} \qquad (4.17)$$

$$y^1(x,o) = 0\ , \quad p^1(x,T) = 0\ , \qquad (4.18)$$

$$\frac{\partial y^1}{\partial \nu_A} + \frac{1}{N} p^1 = 0\ , \quad \frac{\partial p^1}{\partial \nu_{A^*}} = y^o - z_2 \text{ on } \Sigma. \qquad (4.19)$$

This system can be uncoupled.
Indeed it follows from (4.17)(4.18) and (4.19) that

$$\frac{d}{dt} Cy^1 = \frac{1}{N}\int_\Gamma p^1 d\Gamma\ , \ (Cy^1)(o) = 0,$$

so that

$$Cy^1(t) = \frac{1}{N} \int_o^t ds \int_\Gamma p^1 d\Gamma \qquad (4.20)$$

and therefore

$$- \frac{\partial p^1}{\partial t} + A^* p^1 = \frac{1}{N} \int_o^t ds \int p^1 d\Gamma ,$$

$$p^1(x,T) = 0, \tag{4.21}$$

$$\frac{\partial p^1}{\partial \nu_{A^*}} = y^o - z_2 \text{ on } \Sigma ;$$

in (4.21) $y^o_1 = y(u)$ where u is the solution of (4.6)(4.11).
Therefore p^1 can be computed *independently of* y^1 and

$$u_\varepsilon = u - \frac{\varepsilon}{N} p^1 + \dots \tag{4.22}$$

The convergence of the expansion (4.22) can be proven without difficulty.
One has

$$\| u_\varepsilon - (u - \frac{\varepsilon}{N} p^1) \|_{L^2(\Sigma)} \leq C\varepsilon^2 . \tag{4.23}$$

5. DEGENERACY OF THE COST FUNCTION (CHEAP CONTROL)

We consider now, very briefly ([1]), an example of Case III of Section 1.
Let the *state* $y(v)$ be given as the solution of (4.1) and let the *cost function* be given by

$$J_\varepsilon(v) = \int_\Sigma |y(v) - z_d|^2 d\Sigma + \varepsilon \int_\Sigma v^2 d\Sigma . \tag{5.1}$$

The *formal* limit problem is given by

$$J_o(v) = \int_\Sigma |y(v) - z_d|^2 d\Sigma . \tag{5.2}$$

In the case *without constraints* it is a simple matter to verify that

$$\inf_{v \in L^2(\Sigma)} J_o(v) = 0. \tag{5.3}$$

Indeed, if we consider a sequence of *smooth* functions g_n on Σ such that

$$g_n \to z_d \text{ in } L^2(\Sigma), \tag{5.4}$$

we define z_n as the solution of

$$\left.\begin{aligned} &\frac{\partial z_n}{\partial t} + A z_n = f , \\ &z_n = g_n \text{ on } \Sigma , \quad z_n(x,o) = y_o(x) ; \end{aligned}\right\} \tag{5.5}$$

we can assume that z_n is smooth (by taking g_n appropriately so that if we *define*

$$v_n = \frac{\partial z_n}{\partial \nu_A} \tag{5.6}$$

then $v_n \in L^2(\Sigma)$ and, of course, $y(v_n) = z_n$, so that, according to (5.4) :

$$J_o(v_n) \to 0.$$

But *in general there is no* $u \in L^2(\Sigma)$ *such that* $J_o(u) = 0$. Indeed, if we define z by

$$\left.\begin{aligned} &\frac{\partial z}{\partial t} + A z = f , \\ &z = z_d \text{ on } \Sigma , \quad z(x,o) = y_o(x) \end{aligned}\right\} \tag{5.7}$$

([1]) For more details and other examples, we refer to Lions [6][21][22][23].

then necessarily

$$u = \frac{\partial z}{\partial \nu_A} \quad ; \tag{5.8}$$

all this makes sense (cf. Lions-Magenes [24]) but (5.8) will in general define an element of $H^{-1}(\Sigma)$ *and not of* $L^2(\Sigma)$.

Therefore if u_ε is the solution of

$$J_\varepsilon(u_\varepsilon) = \inf J_\varepsilon(v) \tag{5.9}$$

then u_ε will not, in general, converge in $L^2(\Sigma)$ *but in a larger space*. This is typical of *singular perturbations*, and one can indeed see that this problem is closely connected to questions of singular perturbations by considering the optimality system.

6. PERTURBATIONS OF THE DOMAIN.

We consider now, as a last example of perturbations techniques in optimal control, a system described by a "perturbed domain".

6.1. Setting of the problem

Let Ω_o be a bounded open set of $\mathbb{R}^n$, with a smooth boundary Γ_o. If $x \in \Gamma_o$, we denote by $\nu(x)$ the unitary normal to Γ_o at x, directed towards the exterior of Ω_o.

Let $\alpha(x)$ be a scalar continuous function given on Γ_o.

For ε *small enough* (in order to prevent any topological difficulty) we define

$$\Gamma_\varepsilon = \{ x + \varepsilon\, \alpha(x)\nu(x) \quad | \; x \in \Gamma_o \} \tag{6.1}$$

and we denote by Ω_ε the open set "interior" to Γ_ε.

Let E and F be given sets contained in all the Ω_ε for ε small enough, E and F being measurable of >0 measure.

For $v \in L^2(E)$, we define the state $y_\varepsilon(v) = y_\varepsilon$ of the system by

$$\left.\begin{aligned} &Ay_\varepsilon = f + v\chi_E \text{ in } \Omega_\varepsilon, \\ &y_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon \quad ; \end{aligned}\right\} \tag{6.2}$$

in (6.2) A is a second order elliptic operator as in (2.1) where the a_{ij}'s are defined in a neighborhood of $\bigcup_\varepsilon \Omega_\varepsilon$; f is also given in such a neighborhood and belongs to L^2 ; χ_E is the characteristic function of E.

Let the *cost function* be given by

$$J_\varepsilon(v) = \int_F |y_\varepsilon(v) - z_d|^2 \, dx + N \int_E v^2 dx, \tag{6.3}$$

where z_d is given in $L^2(F)$. We look for $\inf J_\varepsilon(v)$, $v \in \mathcal{U}_{ad}$ = closed convex subset of $L^2(E)$.

This problem admits a unique solution u_ε, $y_\varepsilon(u_\varepsilon) = y_\varepsilon$. ∎

The *limit problem* is, formally, the following. One defines $y_o(v)$ as the solution of

$$\left.\begin{aligned} &Ay_o(v) = f + v\,\chi_E \text{ in } \Omega_o, \\ &y_o(v) = 0 \quad \text{on} \quad \Gamma_o \end{aligned}\right\} \tag{6.4}$$

and the limit problem is

$$\left.\begin{array}{l} \inf J_o(v), \quad v \in \mathcal{U}_{ad}, \\ J_o(v) = \int_F |y_o(v)-z_d|^2 dx + N\int_E v^2 dx. \end{array}\right\} \quad (6.5)$$

It is clear that if Γ_o is a "simple" boundary and if, on the other hand, Γ_ε is a "complicated" boundary (corresponding for instance to a rapidly oscillating function α), then (6.5) is much "simpler" than the original problem.

It is therefore a natural idea to try to expand u_ε and y_ε in terms of functions *computed on* Ω_o.

6.2. <u>Case without constraints</u>

In general the optimality system is given as follows :

$$\left.\begin{array}{l} Ay_\varepsilon = f + u_\varepsilon \chi_E, \\ A^* p_\varepsilon = (y_\varepsilon - z_d)\chi_F \end{array}\right\} \quad (6.6)$$

$$y_\varepsilon = p_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon, \quad (6.7)$$

$$\int_E (p_\varepsilon + Nu_\varepsilon)(v-u_\varepsilon)dx \geq 0 \qquad \forall\, v \in \mathcal{U}_{ad}, \quad u_\varepsilon \in \mathcal{U}_{ad}. \quad (6.8)$$

If $\mathcal{U}_{ad} = L^2(E)$, then $p_\varepsilon + Nu_\varepsilon = 0$ and (6.5) becomes

$$\left.\begin{array}{l} Ay_\varepsilon + \frac{1}{N} p_\varepsilon \chi_E = f, \\ A^* p_\varepsilon - y_\varepsilon \chi_F = -z_d\chi_F. \end{array}\right\} \quad (6.9)$$

If we look - in a formal fashion, which can be justified - for an expansion in the form

$$y_\varepsilon = y^o + \varepsilon y^1 + \ldots, \quad p_\varepsilon = p^0 + \varepsilon p^1 + \ldots \quad (6.10)$$

(this is formal since we are looking for $y^o, y^1, p^o, p^1, \ldots$ *defined in* Ω_o and that y_ε and p_ε are defined in Ω_ε), we obtain

$$\left.\begin{array}{l} Ay^o + \frac{1}{N}p^o \chi_E = f \\ A^*p^o - y^o\chi_F = -z_d\chi_F \quad \text{in } \Omega_o, \end{array}\right\} \quad (6.11)$$

$$\left.\begin{array}{l} Ay^1 + \frac{1}{N}p^1 \chi_E = 0, \\ A^* p^1 - y^1 \chi_F = 0 \quad \text{in } \Omega_o \end{array}\right\} \quad (6.12)$$

etc. The *boundary conditions* are obtained by writing that

$$y_\varepsilon(x+\varepsilon\alpha(x)\nu(x)) = 0 \quad p_\varepsilon(x+\varepsilon\,\alpha(x)\,\nu(x)) = 0, \quad x \in \Gamma_o. \quad (6.13)$$

Using (6.10) into (6.13) we obtain - always formally -

$$y^o(x+\varepsilon\alpha(x)\,\nu(x)) + \varepsilon y^1(x+\varepsilon\,\alpha(x)\,\nu(x)) + \ldots = 0, \quad (6.14)$$

(and similar equation for $p^o + \varepsilon p^1 + \ldots$) . We expand each term separately in (6.14). It follows that

$$y^o(x) = 0 \quad , \quad x \in \Gamma_o \tag{6.15}$$

$$y^1(x) + \alpha(x)\frac{\partial y^o}{\partial \nu} = 0 \ , \quad x \in \Gamma_o \ , \tag{6.16}$$

and similar relations for p^o, p^1 :

$$p^o(x) = 0 \quad \text{on} \quad \Gamma_o \ , \tag{6.17}$$

$$p^1(x) + \alpha(x)\frac{\partial p^o}{\partial \nu} = 0 \ \text{on} \ \Gamma_o. \tag{6.18}$$

We remark that (6.11)(6.15)(6.17) is the optimality system for the limit problem and therefore admits a unique solution. ■

It is interesting to see that the system (6.12)(6.16)(6.18) is the *optimality* system of a new problem of optimal control.

Given $v \in L^2(E)$, we define the state $y^1(v)$ by

$$\left.\begin{aligned} Ay^1(v) &= v\chi_E \ , \ \text{in} \ \Omega_o \\ y^1(v) &= -\alpha(x)\frac{\partial y^o}{\partial \nu}(x) \ \text{on} \ \Gamma_o. \end{aligned}\right\} \tag{6.19}$$

We assume that α, y^o, p^o are smooth so that

$$\alpha\frac{\partial y^o}{\partial \nu} \in H^{3/2}(\Gamma_o) \ , \quad \alpha\frac{\partial p^o}{\partial \nu} \in L^2(\Gamma_o) \tag{6.20}$$

The *cost function* is defined by

$$J_1(v) = \int_F |y^1(v)|^2 \,dx + N\int_E v^2 dx - 2\int_{\Gamma_o} \alpha \ \frac{\partial p^o}{\partial \nu}\ \frac{\partial y^1}{\partial \nu_A}(v)\ d\Gamma_o. \tag{6.21}$$

We observe that - assuming Γ_o smooth enough - the solution of (6.19) belongs to $H^2(\Omega_o)$ so that

$$\frac{\partial y^1}{\partial \nu_A} \in H^{1/2}(\Gamma_o)$$

and (6.21) makes sense. Moreover the mapping

$$v \longrightarrow \int_{\Gamma_o} \alpha\ \frac{\partial p^o}{\partial \nu}\ \frac{\partial y^1}{\partial \nu_A}(v)\ d\Gamma_o$$

is affine continuous on $L^2(E)$ so that

$$\inf_{v \in L^2(E)} J_1(v) = J_1(u^1) \ , \qquad u^1 \in L^2(E). \tag{6.22}$$

The adjoint state for problem (6.22) is defined by

$$\left.\begin{aligned} A^*p^1 &= y^1_o\chi_F \quad \text{in} \ \Omega_o, \\ p^1 &= \alpha\frac{\partial p^o}{\partial \nu} \quad \text{on} \ \Gamma_o. \end{aligned}\right\} \tag{6.23}$$

The optimality condition is

$$p^1 + Nu^1 = 0 \quad \text{in} \ E \ , \tag{6.24}$$

so that the optimality system for (6.22) is indeed (6.12)(6.16) and (6.18) (which among other things, implies the existence and uniqueness of the solution of this system). ■

Expansion of $J_\varepsilon(u_\varepsilon)$.

If we use (6.10) and $u_\varepsilon = -\frac{1}{N}p_\varepsilon$ in $J_\varepsilon(u_\varepsilon)$, we obtain

$$J_\varepsilon(u_\varepsilon)\ \int_F |y^o+\varepsilon y^1+\ \dots\ - z_d|^2 dx + \frac{1}{N}\int_E (p^o+\varepsilon p^1+\ \dots)^2$$

hence

$$\left.\begin{aligned}
&J_\varepsilon(u_\varepsilon) = J_o(u^o) + \varepsilon M + \varepsilon^2 N + \dots ,\\
&J_o(u^o) = \inf J_o(v),\\
&M = 2\int_F (y^o-z_d)y^1 dx + \frac{2}{N}\int_E p^o p^1 dx,\\
&N = \int_F (y^1)^2\, dx + \frac{1}{N}\int_E (p^1)^2 dx.
\end{aligned}\right\} \quad (6.25)$$

If we multiply the equations (6.11) by p^1 and y^1 respectively, we find that

$$\int_F (y^o-z_d)y^1 dx + \frac{1}{N}\int_E p^o p^1 dx = \int_{\Omega_o} (A^* p^o)y^1 dx - \int_{\Omega_o} (Ay^o-f)p^1 dx =$$

$$= -\int_{\Gamma_o} \frac{\partial p^o}{\partial \nu_{A^*}} y^1\, d\Gamma_o + \int_{\Omega_o} p^o(Ay^1)dx + \int_{\Gamma_o} \frac{\partial y^o}{\partial \nu_A} p^1 d\Gamma_o - \int_{\Omega_o} y^o(A^* p^1)dx + \int_{\Omega_o} fp^1 dx.$$

Using (6.12)(6.16)(6.18), the right hand side equals

$$\int_{\Gamma_o}\left(\alpha\frac{\partial p^o}{\partial \nu_{A^*}}\frac{\partial y^o}{\partial \nu} - \alpha\,\frac{\partial p^o}{\partial \nu}\,\frac{\partial y^o}{\partial \nu_A}\right)d\Gamma_o - \frac{1}{N}\int_E p^o p^1 dx - \int_F y^o y^1 dx + \int_{\Omega_o} fp^1 dx$$

so that

$$M = \int_{\Omega_o} f\ p^1 dx + \int_{\Gamma_o}\left(\alpha\frac{\partial p^o}{\partial \nu_{A^*}}\ \frac{\partial y^o}{\partial \nu} - \alpha\frac{\partial p^o}{\partial \nu}\ \frac{\partial y^o}{\partial \nu_A}\right)d\Gamma_o . \qquad (6.26)$$

If $A = A^* = -\Delta$ then $\frac{\partial}{\partial \nu_A} = \frac{\partial}{\partial \nu_{A^*}} = \frac{\partial}{\partial \nu}$ and (6.26) reduces to

$$M = \int_\Omega f\ p^1 dx. \qquad (6.27)$$

In the case when $A \neq A^*$ (but not necessarily with constant coefficients) one can again make the surface integral zero in (6.26) by using another representation of Γ_ε. One introduces ν_A = conormal vector to Γ_o, unitary, directed toward the exterior of Ω_o and one *defines*

$$\Gamma_\varepsilon = \{ x + \varepsilon\alpha(x)\ \nu_A(x)\ \ |x \in \Gamma_o \}. \qquad (6.28)$$

In this manner one has to replace $\frac{\partial}{\partial \nu}$ by $\frac{\partial}{\partial \nu_A}$ in the above formulas and if $A = A^*$ the surface integral in (6.26) drops out.

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