Numer. Math. 21, 43—62 (1973)

On the Convergence of a Mixed Finite-Element Method for Plate Bending Problems

Claes Johnson

Chalmers Institute of Technology and the University of Göteborg

Received August 15, 1972

Summary. In this paper we justify a finite element method for biharmonic boundary value problems. The method is based on a stationary variational principle (the Reissner principle), and was introduced by Hellan, Herrmann and Visser. We prove error estimates and the existence of a finite element solution.

1. Introduction

Finite element methods of Rayleigh-Ritz type based on the Dirichlet minimum principle have been studied by several authors during the last few years (cf. [2], [3] and [11] for example). Using an alternative approach we analyze a finite-element method based on a stationary principle (referred to as the Reissner principle in engineering literature). This method has been introduced by Hellan [4], Herrmann [5] and Visser [13] and gives approximate solutions to those boundary-value problems which arise in connection with bending of elastic plates.

The term mixed refers to the fact that one utilizes two finite-dimensional spaces V_h and $\mathcal{M}_h$ which are of different types. Assuming that the domain Ω in the plane which is occupied by the plate has been triangulated, the space V_h consists of continuous functions (displacements) that are linear or quadratic in each triangle, and the space $\mathcal{M}_h$ consists of vector fields (moments) $\boldsymbol{M}=(M_{11}, M_{22}, M_{12})$, whose components are constant or linear in each triangle. It is possible to use polynomials of higher degree, but the number of unknowns then increases very rapidly.

If u is the solution of the biharmonic boundary value problem, then the mixed finite-element method produces an approximation in V_h to u and an approximation $\overline{\boldsymbol{M}}$ in $\mathcal{M}_h$ to $\nabla_2 u=(D^2_{x_1}u, D^2_{x_2}u, D_{x_1}D_{x_2}u)$. In engineering applications the second derivatives of u (the moments) are frequently more important than u itself.

If we start from the condition for a stationary point in the discrete (continuous) principle, and eliminate the displacement variables, then $\overline{\boldsymbol{M}}$ $(\nabla_2 u)$ is determined as the moment field which minimizes a positive definite expression, the complementary energy, over a finite dimensional hyperplane $\mathcal{H}_h$ (infinite dimensional hyperplane $\mathcal{H}$) in $L_2(\Omega)\times L_2(\Omega)\times L_2(\Omega)$. The hyperplane $\mathcal{H}_h$ has the following crucial property: The linear space determined by $\mathcal{H}_h$ is contained in the linear space determined by $\mathcal{H}$. In conformity with what is the case for the normal

Rayleigh-Ritz method, this reduces the problem of obtaining estimates for $\nabla_2 u - \overline{M}$ in the energy norm to a problem in approximation theory.

Since the mixed finite-element method is based on a stationary principle, the existence of a finite element solution requires special attention. However, the resulting indefinite system of equations can in fact be solved by merely inverting positive definite matrices.

The purpose of this paper is to prove some convergence results for the mixed finite-element method. As far as we know, methods of this type have not been discussed in the mathematical literature and our error estimates are believed to be new. In order to avoid complications at curved parts of the boundary we consider polygonal domains. Since the solution to the boundary value problem then has singularities at the vertices of the boundary, we have to use some weighted Sobolev spaces. For simplicity we limit our study to the case of a clamped plate, i.e., to the Dirichlet problem for the biharmonic operator. However, problems with other types of boundary conditions can be treated in a similar way.

The plan of the paper is the following:

In Section 2 we give some facts about the Dirichlet problem for the biharmonic operator on a plane polygonal domain.

In Section 3 the mixed finite-element method is presented, and we prove the existence of a finite-element solution.

In Section 4, finally, we prove error estimates using a technique introduced by Bramble and Hilbert [2].

2. The Dirichlet Problem for the Biharmonic Operator

Let Ω be a bounded, simply or multiply connected domain in the plane R^2 with a boundary $\partial\Omega$ consisting of a finite number of non-intersecting polygons Γ_j, $j=0, 1, \ldots, S$, such that $\Gamma_1, \ldots, \Gamma_S$ lie inside Γ_0. Let $W_2^{(n)}(\Omega)$ denote the space of all real-valued functions which, together with their generalized derivatives up the n^{th} order, belong to $L_2(\Omega)$. The norm in $W_2^{(n)}(\Omega)$ is given by

$$\|u\|_{n,\Omega} = \left[\sum_{|\alpha|\leq n} \int_\Omega |D^\alpha u|^2 \, dA\right]^{\frac{1}{2}}.$$

Here $\alpha = (\alpha_1, \alpha_2)$, α_1 and α_2 are integers,

$$|\alpha| = \alpha_1 + \alpha_2, \qquad \text{and} \qquad D^\alpha u = \frac{\partial^{\alpha_1+\alpha_2} u}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2}}.$$

Let $V = \overset{\circ}{W}{}_2^{(2)}(\Omega)$ denote the completion with respect to the norm $\|\cdot\|_{2,\Omega}$ of $C_0^\infty(\Omega)$, the set of real-valued infinitely differentiable functions with compact support in Ω. We introduce the bilinear form

$$E(\nabla_2 u, \nabla_2 w) = \int_\Omega D_{x_i} D_{x_j} u \, D_{x_i} D_{x_j} w \, dA$$

(using the summation convention), where $\nabla_2 u = (D_{x_1}^2 u, D_{x_2}^2 u, D_{x_1} D_{x_2} u)$.

Let now L be a continuous linear functional on V. It is well known that there exists exactly one function $u_L \in V$ such that

$$E(\nabla_2 u_L, \nabla_2 w) = L(w), \qquad \forall w \in V \tag{2.1}$$

(cf. [8, page 216]). If L is given by a function f which is Hölder continuous, then u_L is the classical solution of the following problem:

$$\Delta \Delta u = f \quad \text{in } \Omega,$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Here $\partial/\partial n$ denotes differentiation in the outward normal direction to $\partial\Omega$.

We now give two simple reformulations of (2.1). We introduce the real Hilbert space $\mathscr{M} = L_2(\Omega) \times L_2(\Omega) \times L_2(\Omega)$, equipped with the scalar product

$$E(\boldsymbol{M}^1, \boldsymbol{M}^2) = \int_\Omega M_{ij}^1 M_{ij}^2 \, dA,$$

where $\boldsymbol{M}^k = (M_{11}^k, M_{22}^k, M_{12}^k)$ and $M_{12}^k = M_{21}^k$, for $k = 1, 2$. We also introduce the (complementary energy) norm $\|\boldsymbol{M}\|_\Omega = (E(\boldsymbol{M}, \boldsymbol{M}))^{\frac{1}{2}}$ in $\mathscr{M}$. We can now state:

The Complementary Minimum Principle (cf. [10, 12]). Let $u_L \in V$ and assume that u_L satisfies (2.1). Then $\nabla_2 u_L$ is the unique minimum point for the functional

$$E(\boldsymbol{M}, \boldsymbol{M}), \qquad \boldsymbol{M} \in \mathscr{H}_L,$$

where $\mathscr{H}_L = \{\boldsymbol{M} \in \mathscr{M} : E(\boldsymbol{M}, \nabla_2 w) = L(w), \ \forall w \in V\}$.

Proof. It is obvious that $\nabla_2 u_L \in \mathscr{H}_L$. If also $\boldsymbol{M} \in \mathscr{H}_L$, we have since $u_L \in V$,

$$\|\nabla_2 u_L\|_\Omega^2 = E(\nabla_2 u_L, \nabla_2 u_L) = L(u_L) = E(\boldsymbol{M}, \nabla_2 u_L).$$

By Cauchy's inequality we therefore obtain

$$\|\nabla_2 u_L\|_\Omega \leqq \|\boldsymbol{M}\|_\Omega, \qquad \forall \boldsymbol{M} \in \mathscr{H}_L,$$

which completes the proof.

Remark. The space $\mathscr{H}_L$ has the following physical meaning: The moment field $\boldsymbol{M}$ is in equilibrium with the load L if $\boldsymbol{M} \in \mathscr{H}_L$.

Another way of reformulating (2.1) is:

The Stationary Principle. Let $u_L \in V$ and assume that u_L satisfies (2.1). Then the pair $(u_L, \nabla_2 u_L)$ is the unique stationary point for the functional

$$\tfrac{1}{2} E(\boldsymbol{M}, \boldsymbol{M}) - E(\boldsymbol{M}, \nabla_2 w) + L(w), \qquad (w, \boldsymbol{M}) \in V \times \mathscr{M}. \tag{2.2}$$

Proof. The conditions for $(\overline{w}, \overline{\boldsymbol{M}})$ to be a stationary point for the functional (2.2) are the following:

$$E(\boldsymbol{M}, \overline{\boldsymbol{M}}) = E(\boldsymbol{M}, \nabla_2 \overline{w}), \qquad \forall \boldsymbol{M} \in \mathscr{M},$$

$$E(\overline{\boldsymbol{M}}, \nabla_2 w) = L(w), \qquad \forall w \in V.$$

This obviously proves the stationary principle.

The mixed finite-element method in question will be based on the stationary principle.

We shall need some regularity properties of u_L which are expressed by the use of certain weighted Sobolev space. Let $\lambda_1, \ldots, \lambda_K$ be real numbers and set $\sigma(x) =$

$\min_k\{\operatorname{dist}(x, Q_k)^{\lambda_k}\}$, where the minimum is taken over all vertices Q_k, $k=1, \ldots, K$, of the boundary polygons $\Gamma_0, \ldots, \Gamma_S$. For n a natural number let $W_{2,\sigma}^{(n)}(\Omega)$ be the set of realvalued functions having generalized derivatives of order less than or equal to n such that the norm

$$\|u\|_{n,\sigma,\Omega} = \left[\sum_{|\alpha|\leq n} \int_\Omega |D^\alpha u|^2 \sigma \, dA\right]^{\frac{1}{2}}$$

is finite. We shall also use the homogeneous seminorm

$$|u|_{n,\sigma,\Omega} = \left[\sum_{|\alpha|=n} \int_\Omega |D^\alpha u|^2 \sigma \, dA\right]^{\frac{1}{2}}.$$

The following lemma (cf. [6, Theorem 3.3, p. 256]), is sufficient for our needs.

Lemma 2.1. Let $u_0 \in W_2^{(4)}(\Omega)$ and $f \in L_2(\Omega)$. Assume that $u - u_0 \in \mathring{W}_2^{(2)}(\Omega)$ and that

$$E(\nabla_2 u, \nabla_2 w) = \int_\Omega f w \, dA, \qquad \forall w \in \mathring{W}_2^{(2)}(\Omega).$$

Then there exist constants C and λ_k with $0 \leq \lambda_k < 4$ for $k=1, \ldots, K$, such that

$$\|u\|_{4,\sigma,\Omega} \leq C\{\|f\|_{0,\Omega} + \|u_0\|_{4,\Omega} + \|u\|_{2,\Omega}\},$$

where $\sigma(x) = \min_k\{\operatorname{dist}(x, Q_k)^{\lambda_k}\}$ and $Q_1, \ldots, Q_K$, are the vertices of $\partial\Omega$.

3. The Finite Element Method

Assume that Ω has been triangulated, i.e., Ω has been covered by a number of closed triangles $\bar{T}_1, \ldots, \bar{T}_J$ (T_j denotes the open triangle), such that any two of the triangles are either disjoint or have a common vertex or side. The vertices of the triangulation are denoted by $P_1, \ldots, P_I$. Let h be the maximum length of the diameters of the triangles T_j.

The mixed finite-element method can be regarded as an application of the stationary principle (cf. Section 2), using a finite dimensional space V_h in the place of $V = \mathring{W}_2^{(2)}(\Omega)$ and a finite dimensional subspace $\mathcal{M}_h$ of $\mathcal{M} = L_2(\Omega) \times L_2(\Omega) \times L_2(\Omega)$. Indeed, the method is to find a stationary point $(\bar{v}, \bar{\boldsymbol{M}})$ for the functional

$$\tfrac{1}{2} E(\boldsymbol{M}, \boldsymbol{M}) - E(\boldsymbol{M}, \nabla_2 v) + L(v), \qquad (v, \boldsymbol{M}) \in V_h \times \mathcal{M}_h.$$

Here $V_h = V_h^{(m)}$ denotes the set of continuous functions defined on Ω which vanish on the boundary of Ω and agree with a polynomial of degree at most m in each triangle T_j of the triangulation of Ω. We postpone the definition of $\mathcal{M}_h$ for a moment. The pair $(\bar{v}, \bar{\boldsymbol{M}})$ will be an approximation of the pair $(u_L, \nabla_2 u_L)$, where u_L is the exact solution. In the sequel we assume that L is a bounded measure on the set of continuous function on Ω so that $L(v)$ is defined if $v \in V_h^{(m)}$.

Since V_h is not a subspace of V, the meaning of $E(\boldsymbol{M}, \nabla_2 v)$, where $v \in V_h$ and $\boldsymbol{M} \in \mathcal{M}_h$, has to be clarified. For this purpose we give the following lemma

Lemma 3.1. Let $\{T_j\}_1^J$ be the triangulation of Ω. Let the vertices of the triangle T_j be P_j^i, $i=1, 2, 3$, and let the sides S_j^k of T_j, $k=1, 2, 3$, have outward normals $n_j^k = (\cos\alpha_j^k, \sin\alpha_j^k)$. Assume that u is a function defined on Ω which is either smooth on Ω, or continuous on Ω and equal to a polynomial on each T_j. Assume further that each component of the vector field $\boldsymbol{M}=(M_{11}, M_{22}, M_{12})$ is either smooth on Ω, or equal to a polynomial on each T_j. Then one has

$$\begin{aligned}\sum_{j=1}^{J} E(\boldsymbol{M}, \nabla_2 u)_{T_j} = \sum_{j=1}^{J} \int_{\partial T_j} M_n(\boldsymbol{M}) \frac{\partial u}{\partial n} ds - \sum_{j=1}^{J} \int_{\partial T_j} V_n(\boldsymbol{M}) u\, ds \\ + \sum_{i=1}^{I} H(\boldsymbol{M}, P_i) u(P_i) + \sum_{j=1}^{J} \int_{T_j} D_2(\boldsymbol{M}) u\, dA.\end{aligned} \tag{3.1}$$

Here $\partial/\partial n$ denotes differentiation in the outward normal direction to ∂T_j,

$$E(\boldsymbol{M}, \nabla_2 u)_{T_j} \quad \int_{T_j} M_{ij} D_{x_i} D_{x_j} u\, dA,$$

$$M_n(\boldsymbol{M}) = M_{n_j^k}(\boldsymbol{M}_{T_j}) = M_{11} \cos^2\alpha_j^k + M_{22} \sin^2\alpha_j^k + 2M_{12} \cos\alpha_j^k \sin\alpha_j^k \quad \text{on} \quad S_j^k,$$

where $\boldsymbol{M}_{T_j}$ denotes the restriction of $\boldsymbol{M}$ to the triangle T_j and also the continuous extension of $\boldsymbol{M}_{T_j}$ to $\bar{T}_j$. Further,

$$D_2(\boldsymbol{M}) = D_{x_1}^2 M_{11} + D_{x_2}^2 M_{22} + 2D_{x_1} D_{x_2} M_{12} \quad \text{in} \quad T_j$$

and

$$V_n(\boldsymbol{M}) = V_{n_j^k}(\boldsymbol{M}_{T_j}) \quad \text{on} \quad S_j^k,$$

where $V_{n_j^k}(\boldsymbol{M}_{T_j})$ is a certain linear combination of first derivatives of the components of $\boldsymbol{M}_{T_j}$. Finally, $H(\boldsymbol{M}, P_i)$ is a certain linear combination of the values at P_i of the components of $\boldsymbol{M}_{T_j}$ for the triangles T_j which have the common vertex P_i.

Proof. The lemma follows from application of the divergence theorem on each triangle T_j and summation over j. We omit the details.

Remark. The quantities $M_{n_j^k}(\boldsymbol{M}_{T_j})$ and $V_{n_j^k}(\boldsymbol{M}_{T_j})$ have the physical meanings of normal moment and Kirchhoff normal force acting on the boundary of S_j^k. The quantity $H(\boldsymbol{M}, P_i)$ represents a concentrated force at P_i.

We now define the space $\mathcal{M}_h = \mathcal{M}_h^{(m)}$ as

$\mathcal{M}_h^{(m)} = \{\boldsymbol{M} \in \mathcal{M}$: each component of $\boldsymbol{M}$ is a polynomial of degree m in each triangle T_j, and $M_n(\boldsymbol{M})$ takes equal values on abutting triangle sides$\}$.

Here and in the sequel m is either 0 or 1. Each member of $\mathcal{M}_h^{(m)}$ is determined by the values of certain parameters which we choose in the following way. Consider the restriction of $\boldsymbol{M}=(M_{11}, M_{22}, M_{12})$ to the triangle T_j. Put

$$M_{n_j^k} = M_{11} \cos^2 \alpha_j^k + M_{22} \sin^2\alpha_j^k + 2M_{12} \sin\alpha_j^k \cos\alpha_j^k \quad \text{for} \quad k=1, 2, 3,$$

(cf. Lemma 3.1), or in matrix notation

$$\begin{aligned}\boldsymbol{M}_n = \begin{bmatrix} M_{n_j^1} \\ M_{n_j^2} \\ M_{n_j^3} \end{bmatrix} = \begin{bmatrix} \cos^2\alpha_j^1 & \sin^2\alpha_j^1 & 2\sin\alpha_j^1 \cos\alpha_j^1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} \\ = B\boldsymbol{M}\end{aligned} \tag{3.2}$$

It is easily seen that the matrix B is nonsingular.

Let now $M \in \mathscr{M}_h^{(0)}$. Then M is uniquely determined in each triangle T_j by the three quantities $M_{n_j^k}$, $k=1, 2, 3$, which are supposed to be equal for abutting triangle sides. We can thus choose the values of $M_n(M)$ along the triangle sides to be the parameters defining the vector field $M \in \mathscr{M}_h^{(0)}$. We let $\{M^{i,0}\}_{i=1}^{I_0}$ be the basis associated with this choice of parameters, i.e., for each $M^{i,0}$ one parameter is 1 and the others are zero.

In order to treat the case when M is linear in each T_j, we let L_i be the linear function on T_j such that

$$L_i(P_j^k) = \begin{cases} 1 & \text{if } \ k=i \\ 0 & \text{if } \ k\neq i \end{cases} \quad \text{for} \quad i=1, 2, 3.$$

If M is linear in T_j we then have

$$M = \sum_{i=1}^{3} L_i M(P_j^i) = \sum_{i=1}^{3} L_i B^{-1} M_n(P_j^i) = B^{-1} \sum_{i=1}^{3} L_i M_n(P_j^i) \quad \text{in } T_j. \tag{3.3}$$

Hence, the nine values $M_{n_j^k}(P_j^i)$, $i, k=1, 2, 3$, uniquely determine M in T_j. It is then obvious that we may choose the parameters defining a vector field $M \in \mathscr{M}_h^{(1)}$ to be the values of $M_n(M)$ at the end points of each triangle side (equal for abutting sides), together with for each triangle T_j the three remaining values $M_{n_j^k}(P_j^i)$, $i=1, 2, 3$, where k is chosen in such a way that S_j^k is the side opposite P_j^i. We let $\{M^{i,1}\}_{j=1}^{I_1}$ be the basis associated with this choice of parameters.

We shall also need bases for the spaces $V_h^{(1)}$ and $V_h^{(2)}$ which were introduced in the beginning of this section. If $v \in V_h^{(1)}$ $(V_h^{(2)})$, then v is uniquely determined by the values of v at the vertices (the values of v at the vertices and at the midpoints of each side). These values are chosen to be the parameters defining $v \in V_h^{(m+1)}$, $m=0, 1$. We let $\{v^{j,m}\}_{j=1}^{J_m}$ for $m=0, 1$ be the bases associated with this choice of parameters.

In order to motivate our definition of $E(M, \nabla_2 v)$, where $M \in \mathscr{M}_h^{(m)}$ and $v \in V_h^{(m+1)}$, we let $M \in \mathscr{M}_h^{(m)}$ and $u \in C_0^\infty(\Omega)$. We then have by Lemma 3.1,

$$\begin{aligned} E(M, \nabla_2 u) = & \sum_j \int_{\partial T_j} M_n(M) \frac{\partial u}{\partial n} ds - \sum_j \int_{\partial T_j} V_n(M)\, u\, ds \\ & + \sum_i H(M, P_i)\, u(P_i) - \sum_j \int_{T_j} D_2(M)\, u\, dA. \end{aligned}$$

But $D_2(M)=0$ in each triangle since M is constant or linear in each triangle. Moreover,

$$\sum_j \int_{\partial T_j} M_n(M) \frac{\partial u}{\partial n} ds = 0,$$

since $M_n(M)$ takes equal values while $\partial u/\partial n$ takes values of opposite signs on abutting triangle sides and since $\partial u/\partial n = 0$ on $\partial\Omega$. Hence,

$$E(M, \nabla_2 u) = \sum_i H(M, P_i)\, u(P_i) - \sum_j \int_{\partial T_j} V_n(M)\, u\, ds, \tag{3.4}$$

if $u \in C_0^\infty(\Omega)$ and $M \in \mathscr{M}_h^{(m)}$. By the Sobolev imbedding theorem and a trace theorem it is obvious that (3.4) is valid even for $u \in V$.

With (3.4) in mind we now give the precise formulation of:

The Mixed Finite-Element Method. Find a stationary point $(\bar{v}, \overline{\boldsymbol{M}})$ for the functional

$$\tfrac{1}{2}E(\boldsymbol{M},\boldsymbol{M}) - \sum_i \mathrm{H}(\boldsymbol{M}, P_i)\,v(P_i) + \sum_j \int_{\partial T_j} V_n(\boldsymbol{M})\,v\,ds + L(v), \tag{3.5}$$
$$(v, \boldsymbol{M}) \in V_h^{(m+1)} \times \mathcal{M}_h^{(m)}.$$

Let us arrange the coordinates of $\bar{v}$ and $\overline{\boldsymbol{M}}$ in the bases given above in two column vectors $\overline{V}$ and $\overline{M}$. Then $(\bar{v}, \overline{\boldsymbol{M}})$ is a stationary point for the functional (3.5) if and only if $(\overline{V}, \overline{M})$ is a solution of the following system of equations

$$\begin{bmatrix} -A & D \\ 0 & A^t \end{bmatrix} \begin{bmatrix} V \\ M \end{bmatrix} = \begin{bmatrix} 0 \\ F \end{bmatrix} \tag{3.6}$$

where $F = \left(L(v^{1,m}), \ldots, L(v^{J_m,m})\right)$, D is an $I_m \times I_m$-matrix with elements $E(\boldsymbol{M}^{i,m}, \boldsymbol{M}^{k,m})$, $i, k = 1, \ldots, I_m$, and A is an $I_m \times J_m$-matrix with elements

$$\sum_i H(\boldsymbol{M}^{p,m}, P_i)\,v^{q,m}(P_i) - \sum_j \int_{\partial T_j} V_n(\boldsymbol{M}^{p,m})\,v^{q,m}\,ds, \quad p = 1, \ldots, I_m, \quad q = 1, \ldots, J_m.$$

The matrix A^t is the transpose of A. D and A are sparse matrices. D is positive definite, but the matrix of the system is obviously indefinite.

We shall now prove that there exists a unique stationary point for the functional (3.5), i.e., that the system of equations (3.6) has a unique solution. The proof is based on the following five lemmas. Three of these lemmas are needed only because we deal with a polygonal region. The essential part of the proof is Lemma 3.6 and its corollary, where we prove that the columns of the matrix A are lineary independent.

In the first four lemmas we concentrate our attention on the triangle T with sides S^k, $k = 1, 2, 3$. Let $\varrho(x) = \operatorname{dist}(x, P)$, where P is one of the vertices of T. Given a natural number n and a real number μ, we let $W_{2,\varrho^\mu}^{(n)}(T)$ be defined in the same way as $W_{2,\sigma}^{(n)}(\Omega)$, i.e., with the weight function $\varrho(x)^\mu$.

Lemma 3.2. For any $\mu \geqq 0$ and positive integer n there exists a constant C such that for $u \in W_{2,\varrho^\mu}^{(n)}(T)$,

$$\|u\|_{n-1,\varrho^{\mu'},T} \leqq C\,\|u\|_{n,\varrho^\mu,T},$$

where $\mu' = \mu - 2$ if $\mu > 1$ and $\mu' = 0$ if $0 \leqq \mu \leqq 1$.

Proof. See [7, Theorem 2.4, page 607].

Lemma 3.3. Let S be the side opposite the vertex P of the triangle T. For any $\mu > 1$ there exists a constant C, depending only on μ and the angles of T, such that for any $u \in W_{2,\varrho^\mu}^{(1)}(T)$,

$$\int_S |u|^2 \varrho^{\mu-1}\,ds \leqq C\left\{\int_T |u|^2 \varrho^{\mu-2}\,dA + \sum_{|\alpha|=1} \int_T |D^\alpha u|^2 \varrho^\mu\,dA\right\}.$$

Proof. We have by Theorem 1.5, p. 15 in [9],

$$\int_S |u|^2\,ds \leqq C\left\{\int_{T'} |u|^2\,dA + \sum_{|\alpha|=1} \int_{T'} |D^\alpha u|^2\,dA\right\},$$

where T' is the triangle with one side coincident with S and one vertex at the center of gravity of T. Introducing the various powers of ϱ and using a homothety, we obtain the statement of the lemma.

Lemma 3.4. For any $\mu < 2$ there exists a constant C such that for $u \in W^{(1)}_{2,\varrho^\mu}(T)$,

$$\int_{\partial T} |u| \, ds \leqq C \|u\|_{1,\varrho^\mu,T}. \tag{3.7}$$

Proof. Let $\alpha, \beta, b > 0$, $a < 0$ and $0 \leqq \lambda < 2$. Set

$$V = \{(\xi, \eta) \in R^2 : a \leqq \xi \leqq b,\ \alpha|\xi| \leqq \eta \leqq \alpha|\xi| + \beta\},$$

and let $u \in C^1(V)$. Then if $(\xi, \eta) \in V$ it follows that

$$|u(\xi, \alpha|\xi|)| = \left| u(\xi, \eta) - \int_{\alpha|\xi|}^{\eta} D_t u(\xi, t) \, dt \right|$$
$$\leqq |u(\xi, \eta)| + \int_{\alpha|\xi|}^{\alpha|\xi|+\beta} |D_t u(\xi, t)| \, dt.$$

Integrating this inequality over V, we get

$$\beta \int_a^b |u(\xi, \alpha|\xi|)| \, d\xi \leqq \int_V |u(\xi, \eta)| \, dA + \beta \int_V |D_t u(\xi, t)| \, dA$$
$$\leqq \Big(\int_V \varrho^{-\lambda} dA\Big)^{\frac{1}{2}} \Big(\int_V |u|^2 \varrho^\lambda dA\Big)^{\frac{1}{2}} + \beta \Big(\int_V \varrho^{-\lambda} dA\Big)^{\frac{1}{2}} \Big(\int_V |D_t u|^2 \varrho^\lambda dA\Big)^{\frac{1}{2}},$$

by Cauchys inequality. Recalling that $\varrho(x) = \operatorname{dist}(x, P)$ and $\lambda < 2$, it is obvious that $\int_V \varrho^{-\lambda} dA < \infty$.

We apply this inequality at each vertex of T, choosing the ξ, η-system so that $(0, 0)$ coincides with the vertex and $(a, \alpha|a|)$ and $(b, \alpha|b|)$ with the midpoints of the sides meeting at the vertex. At the vertex P we take $\lambda = \mu$ and at the other vertices we take $\lambda = 0$. This will prove (3.7) for u smooth enough. Since $C^1(\bar{T})$ is dense in $W^{(1)}_{2,\varrho^\mu}(T)$ (cf. [9]) we may extend (3.7) to hold for $u \in W^{(1)}_{2,\varrho^\mu}(T)$.

Lemma 3.5. Let $u \in W^{(4)}_{2,\varrho^\mu}(T)$, where $\mu < 4$. Then there exists a unique vector field $\boldsymbol{M} = (M_{11}, M_{22}, M_{12})$ of degree m which interpolates $V_2 u$ on T in the sense that

if $m = 0$ we have

(i) $$\int_{S^k} M_{nk}(\boldsymbol{M}) \, ds = \int_{S^k} \frac{\partial^2 u}{\partial n^2} \, ds \quad \text{for } k = 1, 2, 3,$$

and if $m = 1$ the following two conditions are fulfilled

(ii) $$\int_{S^k} M_{nk}(\boldsymbol{M}) \, v \, ds = \int_{S^k} \frac{\partial^2 u}{\partial n^2} \, v \, ds \quad \text{for any linear function } v,\ k = 1, 2, 3,$$

(ii′) $$\int_T \boldsymbol{M} \, dA = \int_T V_2 u \, dA.$$

Here $\int_T \boldsymbol{M} dA = \Big(\int_T M_{11} dA, \int_T M_{22} dA, \int_T M_{12} dA\Big)$ and $\int_T V_2 u \, dA$ is defined in a similar way. Further, $M_{nk}(\boldsymbol{M}) = M_{11} \cos^2\alpha^k + M_{22} \sin^2\alpha^k + 2M_{12} \cos\alpha^k \sin\alpha^k$ on S^k, where $(\cos\alpha^k, \sin\alpha^k)$ is the outward normal to the side S^k.

Proof. We first note that the right hand sides of (i), (ii) and (ii′) are well defined in view of Lemmas 3.2 and 3.4. From (3.2) we obtain that there exists a unique vector field which is constant in T and fulfills (i) in the lemma. We next consider the case $m=1$. Recalling (3.3), we have with $\boldsymbol{M}_n = \boldsymbol{M}_n(\boldsymbol{M})$,

$$\boldsymbol{M} = B^{-1} \sum_{i=1}^{3} L_i \boldsymbol{M}_n(P^i), \tag{3.8}$$

where the P^i are the vertices of T. Since M_{nk} varies linearly along S^k, condition (ii) determines the value of M_{nk} at the end points of S^k for $k=1, 2, 3$. Thus two of the three numbers $M_{nk}(P^i)$, $k=1, 2, 3$, are determined by (ii) at each vertex P^i. By (3.8) we can write condition (ii′) as

$$\int_T \nabla_2 u \, dA = B^{-1} \sum_{i=1}^{3} \int_T L_i \, dA \, \boldsymbol{M}_n(P^i) = B^{-1} \frac{\text{area}(T)}{3} \sum_{i=1}^{3} \boldsymbol{M}_n(P^i),$$

i.e.,

$$\sum_{i=1}^{3} \boldsymbol{M}_n(P^i) = \frac{3}{\text{area}(T)} B \int_T \nabla_2 u \, dA. \tag{3.9}$$

This equation determines the remaining number $M_{nk}(P^i)$ at each vertex P^i, and the existence of a linear vector field satisfying (ii) and (ii′) is proved.

Lemma 3.6. For any point $P \in \Omega$ there exists a vector field $\boldsymbol{M} \in \mathscr{M}_h^{(m)}$ such that

$$\sum_i H(\boldsymbol{M}, P_i) v(P_i) - \sum_j \int_{\partial T_j} V_n(\boldsymbol{M}) v \, ds = v(P), \qquad \forall v \in V_h^{(m+1)}. \tag{3.10}$$

Proof. Let L be the Dirac measure at P and let G, the Green's function with pole at P, be the corresponding solution of (2.1), i.e., let $G \in V$ and $E(\nabla_2 G, \nabla_2 w) = w(P)$, $\forall w \in V$. We shall prove that

$$\sum_j E(\nabla_2 G, \nabla_2 v)_{T_j} = \sum_j \int_{\partial T_j} \frac{\partial^2 G}{\partial n^2} \frac{\partial v}{\partial n} ds + v(P), \qquad \forall v \in V_h^{(m)}, \tag{3.11}$$

and use G to construct $\boldsymbol{M} \in \mathscr{M}_h^{(m)}$ satisfying (3.10).

Suppose without loss of generality that $P=0$, and substract from G its singular part $E_0 = \frac{1}{8\pi} r^2 \log 1/r$, where $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$. It is well known that $G_0 = G - E_0$ is smooth on Ω except at the vertices Q_k of $\partial\Omega$. We remove from Ω small polygonal domains D_{k_ε} of diameter ε at each vertex Q_k (cf. Fig. 1). We assume that D_{k_ε} is similar to D_{k_δ} for $0 < \varepsilon < \delta$.

Put $D_\varepsilon = \bigcup_k D_{k_\varepsilon}$ and $T_{j_\varepsilon} = T_j \setminus (T_j \cap D_\varepsilon)$. Using the divergence theorem on each polygon T_{j_ε}, we get by simple computations

$$\begin{aligned}\sum_j E(\nabla_2 G_0, \nabla_2 v)_{T_{j_\varepsilon}} &= \sum_j \int_{\partial T_{j_\varepsilon}} \frac{\partial^2 G_0}{\partial n^2} \frac{\partial v}{\partial n} ds + \sum_j \int_{\partial T_{j_\varepsilon}} \frac{\partial^2 G_0}{\partial n \, \partial s} \frac{\partial v}{\partial s} ds \\ &\quad + \sum_j \int_{\partial T_{j_\varepsilon}} v \frac{\partial}{\partial n} \Delta G_0 \, ds + \sum_j \int_{T_{j_\varepsilon}} v \Delta\Delta G_0 \, dA, \qquad \forall v \in V_h^{(m+1)}.\end{aligned} \tag{3.12}$$

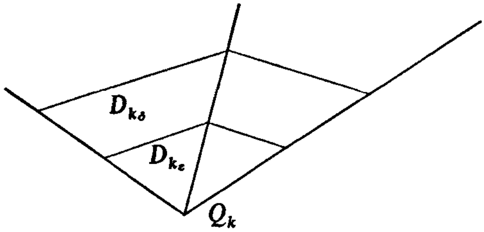

Fig. 1

But $\frac{\partial^2 G_0}{\partial s\,\partial n}$ takes equal values and $\frac{\partial v}{\partial s}$ takes values of opposite signs on abutting triangle sides. Further, $v=0$ on $\partial\Omega$. Using the notation $\partial D'_{k_\varepsilon}=\partial D_{k_\varepsilon}\cap\Omega$ we therefore get

$$\sum_j \int_{\partial T_{j_\varepsilon}} \frac{\partial^2 G_0}{\partial s\,\partial n}\,\frac{\partial v}{\partial s}\,ds=\sum_k \int_{\partial D'_{k_\varepsilon}} \frac{\partial^2 G_0}{\partial s\,\partial n}\,\frac{\partial v}{\partial s}\,ds. \tag{3.13}$$

By a similar argument it is seen that

$$\sum_j \int_{\partial T_{j_\varepsilon}} v\,\frac{\partial}{\partial n}\,\Delta G_0\,ds=\sum_k \int_{\partial D'_{k_\varepsilon}} v\,\frac{\partial}{\partial n}\,\Delta G_0\,ds. \tag{3.14}$$

To see that the right-hand sides of (3.13) and (3.14) tend to zero when $\varepsilon\to 0$, we need some information about the behavior of G_0 near the vertices Q_k. By Lemma 2.1 we know that there exist real numbers $\lambda_k<4$ such that $G_0\in W^{(4)}_{2,\sigma}(\Omega)$, where $\sigma(x)=\min_k\{\operatorname{dist}(x,Q_k)^{\lambda_k}\}$. Applying Lemma 3.3 to the triangles which make up D_{k_ε} and using the notation $\varrho_k(x)=\operatorname{dist}(x,Q_k)$, we get for $|\alpha|=3$,

$$\begin{aligned}\int_{\partial D'_{k_\varepsilon}} |D^\alpha G_0|^2\varrho_k^{\lambda_k-1}ds &\leqq C\Big\{\int_{D_{k_\varepsilon}} |D^\alpha G_0|^2\varrho_k^{\lambda_k-2}dA\\ &\quad+\sum_{|\beta|=4}\int_{D_{k_\varepsilon}}|D^\beta G_0|^2\varrho_k^{\lambda_k}dA\Big\}\leqq C_1\max_{|\beta|\leqq 4}\int_{D_{k_\varepsilon}}|D^\beta G_0|^2\varrho_k^{\lambda_k}dA\\ &\leqq C_1\|G_0\|^2_{4,\sigma,\Omega}=C_2\qquad\text{for } 0<\varepsilon<\delta,\end{aligned} \tag{3.15}$$

where the second inequality follows from Lemma 3.2.

Now $v=0$ on $\partial\Omega$ and v is either linear or quadratic in each triangle. We therefore have by Cauchy's inequality and (3.15),

$$\begin{aligned}\left|\int_{\partial D'_{k_\varepsilon}} v\,\frac{\partial}{\partial n}\,\Delta G_0\,ds\right| &\leqq C\int_{\partial D'_{k_\varepsilon}}\varrho_k\left|\frac{\partial}{\partial n}\,\Delta G_0\right|ds\\ &\leqq C\Big\{\int_{\partial D'_{k_\varepsilon}}\left|\frac{\partial}{\partial n}\,\Delta G_0\right|\varrho_k^{\lambda_k-1}ds\Big\}^{\frac12}\Big\{\int_{\partial D'_{k_\varepsilon}}\varrho_k^{3-\lambda_k}ds\Big\}^{\frac12}\\ &\leqq C_3\,\varepsilon^{4-\lambda_k}\to 0\qquad\text{as }\varepsilon\to 0,\end{aligned} \tag{3.16}$$

since $\lambda_k < 4$. In a similar way we obtain for $|\alpha| = 2$,

$$\int_{\partial D'_{k_\varepsilon}} \frac{\partial v}{\partial s} (D^\alpha G_0)\, ds \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.17}$$

Letting $\varepsilon \to 0$ in (3.12), we find from (3.13), ..., (3.17) that

$$\sum_j E(\nabla_2 G_0, \nabla_2 v)_{T_j} = \sum_j \int_{\partial T_j} \frac{\partial^2 G_0}{\partial n^2} \frac{\partial v}{\partial n}\, ds, \qquad \forall v \in V_h^{(m+1)}, \tag{3.18}$$

where we have used the fact that $\Delta\Delta G_0 = 0$ in Ω.

Further, it is easy to show that for the fundamental solution $E_0 = \frac{1}{8\pi} r^2 \log 1/r$, we have

$$\sum_j E(\nabla_2 E_0, \nabla_2 v)_{T_j} = \sum_j \int_{\partial T_j} \frac{\partial^2 E_0}{\partial n^2} \frac{\partial v}{\partial n}\, ds + v(0), \qquad \forall v \in V_h^{(m+1)}. \tag{3.19}$$

Addition of (3.18) and (3.19) now proves (3.11).

With the aid of G and (3.11) we now construct a vector field as required in the lemma. Let $\boldsymbol{M} \in \mathscr{M}_h^{(m)}$ interpolate $\nabla_2 G$ on each T_j in the sense of Lemma 3.5, i.e., let $\boldsymbol{M}$ be the unique member of $\mathscr{M}_h^{(m)}$ such that

$$\int_{S_j^k} M_n(\boldsymbol{M})\, ds = \int_{S_j^k} \frac{\partial^2 G}{\partial n^2}\, ds \qquad \text{for } k = 1, 2, 3, \quad j = 1, \dots, J,$$

if $m = 0$, and

$$\int_{S_j^k} M_n(\boldsymbol{M})\, v\, ds = \int_{S_j^k} \frac{\partial^2 G}{\partial n^2}\, v\, ds \quad \text{for any linear function } v, \quad k = 1, 2, 3, \quad j = 1, \dots, J,$$

$$\int_{T_j} \boldsymbol{M}\, dA = \int_{T_j} \nabla_2 G\, dA \qquad \text{for } j = 1, \dots, J$$

if $m = 1$. Thus by (3.11),

$$\sum_j E(\boldsymbol{M}, \nabla_2 v)_{T_j} = \sum_j \int_{\partial T_j} M_n(\boldsymbol{M}) \frac{\partial v}{\partial n}\, ds + v(P), \qquad \forall v \in V_h^{(m+1)}, \tag{3.20}$$

where we have used the facts that

$$\nabla_2 v = 0 \text{ in each } T_j \text{ and } \frac{\partial v}{\partial n} \text{ is constant on each } S_j^k \text{ if } m = 0$$

and

$$\nabla_2 v \text{ is constant in } T_j \text{ and } \frac{\partial v}{\partial n} \text{ is linear on } S_j^k \text{ if } m = 1.$$

But by Lemma 3.1 we have

$$\begin{aligned} \sum_j E(\boldsymbol{M}, \nabla_2 v)_{T_j} = \sum_j \int_{\partial T_j} M_n(\boldsymbol{M}) \frac{\partial v}{\partial n}\, ds + \sum_j H(\boldsymbol{M}, P_i)\, v(P_i) \\ - \sum_j \int_{\partial T_j} V_n(\boldsymbol{M})\, v\, ds, \qquad \forall v \in V_h^{(m+1)}. \end{aligned} \tag{3.21}$$

Combination of (3.20) and (3.21) finishes the proof.

Corollary. The columns of the matrix A in (3.6) are linearly independent.

Proof. Let v be any member of $V_h^{(m+1)}$ and assume that

$$\sum_i H(\boldsymbol{M}, P_i)\, v(P_i) - \sum_j \int_{\partial T_j} V_n(\boldsymbol{M})\, v\, ds = 0, \qquad \forall \boldsymbol{M} \in \mathcal{M}_h^{(m)}.$$

It then follows from Lemma 3.6 that $v \equiv 0$. This proves the corollary.

We can now prove the existence of a finite element solution.

Proposition. There is a unique stationary point $(\bar{v}, \overline{\boldsymbol{M}})$ for the functional

$$\tfrac{1}{2} E(\boldsymbol{M}, \boldsymbol{M}) - \sum_i H(\boldsymbol{M}, P_i)\, v(P_i) + \sum_j \int_{\partial T_j} V_n(\boldsymbol{M})\, v\, ds + L(v), \qquad (v, \boldsymbol{M}) \in V_h^{(m+1)} \times \mathcal{M}_h^{(m)}.$$

Proof. We prove that the system of equations (cf. (3.6))

$$\begin{bmatrix} -A & D \\ 0 & A^t \end{bmatrix} \begin{bmatrix} V \\ M \end{bmatrix} = \begin{bmatrix} 0 \\ F \end{bmatrix}$$

has a unique solution. Since D is positive definite, we can write this system of equations in the following way,

$$\begin{aligned} M &= D^{-1} A V \\ A^t D^{-1} A V &= F \end{aligned} \tag{3.22}$$

Since D^{-1} is positive definite, we have that $V^t A^t D^{-1} A V = (A V)^t D^{-1} (A V) > 0$ if $A V \neq 0$. But $A V \neq 0$ if $V \neq 0$ by the corollary of Lemma 3.6. It follows that the matrix $A^t D^{-1} A$ is positive definite, thus proving the existence of a unique solution of the system of Eq. (3.22). This completes the existence proof.

We shall now prove that the present mixed finite-element method has a simple basic structure (this was first observed by Allmann [1] in the case $m = 0$). We introduce the finite dimensional hyperplane

$$\mathcal{H}_{L,h} = \{\boldsymbol{M} \in \mathcal{M}_h^{(m)} : \sum_i H(\boldsymbol{M}, P_i)\, v(P_i) - \sum_j \int_{\partial T_j} V_n(\boldsymbol{M})\, v\, ds = L(v), \quad \forall v \in V_h^{(m+1)}\}.$$

Since $(\bar{v}, \overline{\boldsymbol{M}})$ is a stationary point for the functional (3.5), we have $\overline{\boldsymbol{M}} \in \mathcal{H}_{L,h}$. The definition of $\mathcal{H}_{L,h}$ is analogous to the definition of $\mathcal{H}_L = \{\boldsymbol{M} \in \mathcal{M} : E(\boldsymbol{M}, \nabla_2 w) = L(w), \forall w \in V\}$, (cf. Section 2). The following property of $\mathcal{H}_{L,h}$ will imply that the vector field $\overline{\boldsymbol{M}}$ produced by the mixed finite-element method is the projection of $\nabla_2 u_L$ onto the hyperplane $\mathcal{H}_{L,h}$, where u_L is the exact solution.

Lemma 3.7. If $L = 0$, then $\mathcal{H}_{L,h} \subset \mathcal{H}_L$.

Proof. Let $\boldsymbol{M}$ be an arbitrary member of $\mathcal{H}_{L,h}$. By the definition of $\mathcal{H}_{L,h}$ for $L = 0$, we have

$$\sum_i H(\boldsymbol{M}, P_i)\, v(P_i) - \sum_j \int_{\partial T_j} V_n(\boldsymbol{M})\, v\, ds = 0, \qquad \forall v \in V_h^{(m+1)}. \tag{3.23}$$

If $m = 0$, then second sum in (3.23) vanishes. Letting v run through the basis $\{v^{j,0}\}$ of $V_h^{(1)}$, we then obtain that

$$H(\boldsymbol{M}, P_i) = 0 \quad \text{for} \quad i = 1, \ldots N, \tag{3.24}$$

where $P_1, \ldots, P_N$ are the vertices in Ω.

In the case $m=1$ we denote by $S_1, \ldots, S_p$, the triangle sides not contained in $\partial\Omega$. Each side S_q is common to two triangles T'_q and T''_q. Let $v^{q,1}$ be the basis function such that $v^{q,1}(P_i)=0$ for $i=1, \ldots, N$, and $v^{q,1}=0$ on S_p for $p \neq q$. Since $V_n(M_{T'_q})$ and $V_n(M_{T''_q})$ are constant on S_q, we have by (3.23),

$$0=\int_{S_q} V_n(M_{T'_q})v^{q,1}ds+\int_{S_q} V_n(M_{T''_q})v^{q,1}ds=[V_n(M_{T'_q})+V_n(M_{T''_q})]\int_{S_q} v^{q,1}ds.$$

But $\int_{S_q} v^{q,1}ds \neq 0$, so that

$$V_n(M_{T'_q})+V_n(M_{T''_q})=0 \qquad \text{for } q=1, \ldots, P. \tag{3.25}$$

The second sum in (3.23) therefore vanishes and by the same argument as in the case $m=0$, we obtain (3.24) also in the case $m=1$. In either case we obtain from (3.24) and (3.25) that

$$\sum_i H(M, P_i)u(P_i)-\sum_j \int_{\partial T_j} V_n(M)u\,ds=0, \qquad \forall u \in V.$$

It follows from (3.4) that $M \in \mathscr{H}_L=\mathscr{H}_0$, which completes the proof.

Lemma 3.8. Let $(\bar{v}, \overline{M})$ be the unique stationary point for the functional (3.5). Let u be an arbitrary member of V. Then

$$\|V_2 u-\overline{M}\|_\Omega \leq \|V_2 u-M\|_\Omega, \qquad \forall M \in \mathscr{H}_{L,h}.$$

Proof. The conditions for $(\bar{v}, \overline{M})$ to be a stationary point for the functional (3.5) are the following:

$$E(\overline{M}, M)=\sum_i H(M, P_i)\bar{v}(P_i)-\sum_j \int_{\partial T_j} V_n(M)\bar{v}\,ds, \qquad \forall M \in \mathscr{M}_h^{(m)}, \tag{3.26}$$

$$\sum_i H(\overline{M}, P_i)v(P_i)-\sum_j \int_{\partial T_j} V_n(\overline{M})v\,ds=L(v), \qquad \forall v \in V_h^{(m+1)}. \tag{3.27}$$

Let $M \in \mathscr{H}_{L,h}$. Then

$$\begin{aligned} E(\overline{M}, \overline{M}) &= \sum_i H(\overline{M}, P_i)\bar{v}(P_i)-\sum_j \int_{\partial T_j} V_n(\overline{M})\bar{v}\,ds=L(\bar{v}) \\ &= \sum_i H(M, P_i)\bar{v}(P_i)-\sum_j \int_{\partial T_j} V_n(M)\bar{v}\,ds=E(\overline{M}, M), \end{aligned}$$

so that

$$E(\overline{M}, \overline{M}-M)=0.$$

Since $\overline{M}-M \in \mathscr{H}_{0,h}$, we also have by Lemma 3.7,

$$E(V_2 u, \overline{M}-M)=0.$$

Hence,

$$\begin{aligned} E(\overline{M}-V_2 u, \overline{M}-V_2 u) &= E(\overline{M}-V_2 u, M-V_2 u+\overline{M}-M) \\ &= E(\overline{M}-V_2 u, M-V_2 u). \end{aligned}$$

An application of Cauchy's inequality concludes the proof.

Corollary. Let $(\bar{v}, \overline{M})$ be the stationary point for the functional (3.5). Let u_L be the unique member of V which satisfies (2.1). Then

$$\|V_2 u_L-\overline{M}\|_\Omega \leq \|V_2 u_L-M\|_\Omega, \qquad \forall M \in \mathscr{H}_{L,h}.$$

The corollary implies that the rate of convergence of the finite-element solution to the exact solution in the norm $\|\cdot\|_\Omega$ depends only on the approximation theoretic properties of $\mathscr{H}_{L,h}$ as $h\to 0$. Finite element methods of Rayleigh-Ritz type based on the Dirichlet minimum principle show a similar behavior.

Remark. Taking $u=0$ in Lemma 3.8, we obtain the following discrete analogue of the complementary minimum principle in Section 2: If $(\bar{v}, \overline{\boldsymbol{M}})$ is the stationary point for the functional (3.5), then $\overline{\boldsymbol{M}}$ is the minimum point for the functional

$$E(\boldsymbol{M},\boldsymbol{M}),\qquad \boldsymbol{M}\in\mathscr{H}_{L,h}.$$

4. Error Estimates

We know by Lemma 3.8 that the vector field $\overline{\boldsymbol{M}}$ produced by the mixed finite-element method is closest to the exact moment field $\nabla_2 u_L$ in the sense that

$$\|\nabla_2 u_L-\overline{\boldsymbol{M}}\|_\Omega \leqq \|\nabla_2 u_L-\boldsymbol{M}\|_\Omega,\qquad \forall \boldsymbol{M}\in\mathscr{H}_{L,h}.$$

To obtain an error estimate in the norm $\|\cdot\|_\Omega$ it is therefore sufficient to construct a vector field $\boldsymbol{M}\in\mathscr{H}_{L,h}$ which is close to $\nabla_2 u_L$. As $\boldsymbol{M}$ we shall choose the unique member of $\mathscr{M}_h^{(m)}$ which interpolates $\nabla_2 u_L$ in the sense of Lemma 3.5 on each triangle T_j. We shall use a technique of Bramble of Hilbert to estimate $\|\nabla_2 u_L-\boldsymbol{M}\|^2_{T_j}=E(\nabla_2 u_L-\boldsymbol{M}, \nabla_2 u_L-\boldsymbol{M})_{T_j}$ and then combine these local estimates to get a global estimate. The basic technical tool is the Poincaré inequality given in Lemma 4.2 below.

In the following three lemmas we shall concentrate our attention on the triangle T_1 with vertices at (0, 0), (0, 1), and (1, 0). Let $W^{(n)}_{2,\varrho^\mu}(T_1)$ be the space introduced in Section 3, the weight function being $\varrho^\mu(x)=(x_1^2+x_2^2)^{\mu/2}=|x|^\mu$.

Lemma 4.1. The identity map of $W^{(n)}_{2,\varrho^\mu}(T_1)$ into $W^{(n-1)}_{2,\varrho^\mu}(T_1)$ is completely continuous if $\mu\geqq 0$ and n is a positive integer.

Proof. The lemma follows from a simple modification of the proof of Theorem 2.5, page 287, in [9].

Lemma 4.2. For any $\mu\geqq 0$ and natural number n there exists a constant C_1 such that for $u\in W^{(n)}_{2,\varrho^\mu}(T_1)$,

$$\|u\|^2_{n,\varrho^\mu,T_1}\leqq C_1\Big\{|u|_{n,\varrho^\mu,T_1}+\sum_{|\alpha|<n}\Big|\int_{T_1} D^\alpha u\,dA\Big|^2\Big\}.$$

Here

$$|u|^2_{n,\varrho^\mu,T_1}=\sum_{|\alpha|=n}\int_{T_1}|D^\alpha u|^2\varrho^\mu\,dA.$$

Proof. The proof is based on Lemma 4.1 and is the same as the proof of Theorem 1.5, page 18, in [9].

The next lemma shows a simple but useful application of Lemma 4.2.

Lemma 4.3. (Bramble and Hilbert [2]). Let F be a bounded linear functional on $W^{(n)}_{2,\varrho^\mu}(T_1)$,

$$|F(u)|\leqq C_2\|u\|_{n,\varrho^\mu,T_1},\qquad \forall u\in W^{(n)}_{2,\varrho^\mu}(T_1),$$

and let $F(q)=0$ for every polynomial q of degree less than n. Then

$$|F(u)| \leqq C_1 C_2 |u|_{n,\varrho^\mu,T_1}, \qquad \forall u \in W_{2,\varrho^\mu}^{(n)}(T_1).$$

Here C_1 is the constant in Lemma 4.2.

Proof. Given $u \in W_{2,\varrho^\mu}^{(n)}(T_1)$ there is a unique polynomial q of degree less than n such that

$$\int_{T_1} D^\alpha(u+q)\,dA = 0 \quad \text{for} \quad |\alpha| < n.$$

Application of Lemma 4.2 now completes the proof.

In the following lemma we obtain a local error estimate.

Lemma 4.4. Let T be a triangle with diameter d and smallest angle $\Theta \geqq \Theta_0 > 0$, and let T have one vertex at $(0, 0)$. Let $u \in W_{2,\varrho_m}^{(3+m)}(T)$, where $\varrho_m = |x|^{\mu+2(m-1)}$ and $2(1-m) \leqq \mu < 4$. Further, let $\boldsymbol{M}^u$ be the constant (if $m=0$), or linear (if $m=1$), vector-field which interpolates $V_2 u$ on T in the sense of Lemma 3.5. Then there exists for any μ with $2(1-m) \leqq \mu < 4$, a constant C_m which depends only on Θ_0 and μ, such that

$$\|V_2 u - \boldsymbol{M}^u\|_T \leqq C_m d^{2-\mu/2} |u|_{3+m,\varrho_m,T}.$$

Proof. Let the vertices of T have the coordinates $(0,0)$, (a_1, a_2) and (b_1, b_2). Consider the mapping X of T_1 onto T given by

$$x = X(\xi) = (a_1\xi_1 + b_1\xi_2,\ a_2\xi_1 + b_2\xi_2).$$

We recall that one half times the Jacobian of X is equal to the area of T, which is not less than $\frac{1}{4}d^2 \sin\Theta$. It follows easily that

$$\left|\frac{\partial \xi_j}{\partial x_i}\right| \leqq \frac{2}{d\sin\Theta} \qquad \text{for } i, j = 1, 2. \tag{4.1}$$

Let n^k, $k=1, 2, 3$, denote the normal directions of the sides of T, and let the sides of T_1 be denoted by S_1^k, $k=1,2,3$. Set $\widehat{V_2 u}(\xi) = V_2 u(X(\xi))$ and $\widehat{\boldsymbol{M}^u}(\xi) = \boldsymbol{M}^u(X(\xi))$. After changing coordinates in the condition that $\boldsymbol{M}^u$ interpolates $V_2 u$ on T in the sense of Lemma 3.5, we then have

$$\int_{S_1^k} M_{nk}(\widehat{\boldsymbol{M}^u})\,ds = \int_{S_1^k} M_{nk}(\widehat{V_2 u})\,ds \qquad \text{for } k=1, 2, 3, \tag{4.2}$$

if $m=0$, and

$$\int_{S_1^k} M_{nk}(\widehat{\boldsymbol{M}^u})\,v\,ds = \int_{S_1^k} M_{nk}(\widehat{V_2 u})\,v\,ds \quad \text{for any linear function } v, \quad k=1, 2, 3, \tag{4.3}$$

$$\int_{T_1} \widehat{\boldsymbol{M}^u}\,dA = \int_{T_1} \widehat{V_2 u}\,dA, \tag{4.4}$$

if $m=1$.

Given $\hat{u} \in W_{2,\varrho_m}^{(3+m)}(T_1)$ put $u(x) = \hat{u}(X^{-1}(x))$ and let $\boldsymbol{M}^u$ interpolate $V_2 u$ on T. Consider now the linear functional F on $W_{2,\varrho_m}^{(3+m)}(T_1)$ given by

$$F(\hat{u}) = E(\widehat{V_2 u} - \widehat{\boldsymbol{M}^u}, \boldsymbol{M})_{T_1},$$

where $\boldsymbol{M}$ is an arbitrary vector field belonging to $[L_2(T_1)]^3$.

Suppose we can prove that

$$\|\widehat{\boldsymbol{M}^u}\|_{T_1} \leqq C \max_{|\alpha|=2} \|\widehat{D^\alpha u}\|_{1+m,\varrho_m,T_1}, \tag{4.5}$$

where $\widehat{D^\alpha u}(\xi) = D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2} u(X(\xi))$. Then we get by Cauchy's inequality,

$$\begin{aligned} |F(\hat{u})| &\leqq \|\widehat{V_2 u} - \widehat{\boldsymbol{M}^u}\|_{T_1} \|\boldsymbol{M}\|_{T_1} \leqq (\|\widehat{V_2 u}\|_{T_1} + \|\widehat{\boldsymbol{M}^u}\|_{T_1}) \|\boldsymbol{M}\|_{T_1} \\ &\leqq C \|\boldsymbol{M}\|_{T_1} \max_{|\alpha|=2} \|\widehat{D^\alpha u}\|_{1+m,\varrho_m,T_1}, \end{aligned} \tag{4.6}$$

where the last inequality follows from (4.5) and Lemma 3.2.

In view of (4.1) we easily find from (4.6) that

$$|F(\hat{u})| \leqq C \|\boldsymbol{M}\|_{T_1} d^{-2} (\sin\Theta)^{-2} \|\hat{u}\|_{3+m,\varrho_m,T_1}.$$

If $\hat{u}$ is a polynomial of degree less than $m+3$, then so is u. Since $\boldsymbol{M}^u$ is unique, it follows that $V_2 u = \boldsymbol{M}^u$ and therefore $F(\hat{u}) = 0$. By Lemma 4.3 we thus have

$$|F(\hat{u})| \leqq C \|\boldsymbol{M}\|_{T_1} d^{-2} (\sin\Theta)^{-2} |\hat{u}|_{3+m,\varrho_m,T_1}.$$

Choosing $\boldsymbol{M} = \widehat{V_2 u} - \widehat{\boldsymbol{M}^u}$ we obtain the estimate

$$\|\widehat{V_2 u} - \widehat{\boldsymbol{M}^u}\|_{T_1} \leqq C d^{-2} (\sin\Theta)^{-2} |\hat{u}|_{3+m,\varrho_m,T_1}.$$

The conclusion in the lemma now follows by a change of coordinates.

It remains to give a proof of (4.5). From Lemma 3.4 and (4.2) we conclude in the case $m=0$ that

$$|\boldsymbol{M}_{nk}(\widehat{\boldsymbol{M}^u})| \leqq C \max_{|\alpha|=2} \|\widehat{D^\alpha u}\|_{1,\varrho_0,T_1}. \tag{4.7}$$

But $\widehat{\boldsymbol{M}^u} = A^{-1} \widehat{\boldsymbol{M}_n}$, where $\widehat{\boldsymbol{M}_n} = (M_{n^1}(\widehat{\boldsymbol{M}^u}), M_{n^2}(\widehat{\boldsymbol{M}^u}), M_{n^3}(\widehat{\boldsymbol{M}^u}))$. The determinant of the matrix B is bounded away from zero for triangles T with smallest angle $\Theta \geqq \Theta_0 > 0$. This proves (4.5) in the case $m=0$.

In the case $m=1$ the condition (4.3) determines two of the numbers $M_{nk}(\widehat{\boldsymbol{M}^u})(P_1^i)$, $k=1, 2, 3$, at each vertex P_1^i of T_1. For these two numbers at each vertex the estimate (4.7) holds. But we know that (cf. (3.9)),

$$\sum_{i=1}^{3} \widehat{\boldsymbol{M}_n}(P_1^i) = \frac{3}{\text{area}(T_1)} B \int_{T_1} \widehat{V_2 u}\, dA,$$

thus determining the third number $M_{nk}(\widehat{\boldsymbol{M}^u})(P_1^i)$. By Lemma 3.2 we therefore obtain

$$\max_{i,k=1,2,3} |M_{nk}(\widehat{\boldsymbol{M}^u})(P_1^i)| \leqq C \max_{|\alpha|=2} \|\widehat{D^\alpha u}\|_{2,\varrho_1,T_1}.$$

This proves (4.5) even in the case $m=1$ and the proof is complete.

We are now able to state and prove the main result of this paper. We first introduce three numbers associated with the triangulation $\{T_j\}_1^J$ of Ω. Let $Q_1, \ldots, Q_K$

be the vertices of $\partial\Omega$. We assume that if $Q_k \in \overline{T}_j$, then Q_k is actually a vertex of T_j. Let

$$\delta_j = \min_k \{\operatorname{dist}(T_j, Q_k)/\operatorname{diam}(T_j)\}$$

and set

Θ = smallest angle in any triangle,

h = largest diameter of any triangle,

$\delta = \min \delta_j$, taken over all δ_j such that $\delta_j > 0$.

Theorem. For a given $f \in L_2(\Omega)$ let u_L be the unique function in $\mathring{W}_2^{(2)}(\Omega)$ such that

$$E(\nabla_2 u_L, \nabla_2 w) = \int_\Omega f w \, dA = L(w), \qquad \forall w \in \mathring{W}_2^{(2)}(\Omega).$$

Assume that $u_L \in W_{2,\sigma}^{(4)}(\Omega)$, where $\sigma(x) = \min_k \{\operatorname{dist}(x, Q_k)^{\lambda_k}\}$, $2(1-m) \leqq \lambda_k < 4$, $k = 1, \ldots, K$, and m is either 0 or 1. Let the parameters associated with the triangulation $\{T_j\}$ be Θ, δ and h. Suppose that there exist positive numbers Θ_0 and δ_0 independent of h such that $\Theta \geqq \Theta_0$ and $\delta > \delta_0$ for h sufficiently small. Further, let $(\bar{v}, \overline{\boldsymbol{M}})$ be the stationary point for the functional

$$\tfrac{1}{2}E(\boldsymbol{M}, \boldsymbol{M}) - \sum_i H(\boldsymbol{M}, P_i)\, v(P_i) + \sum_j \int_{\partial T_j} V_n(\boldsymbol{M})\, v\, ds + \int_\Omega f v\, ds, \quad (v, \boldsymbol{M}) \in V_h^{(m+1)} \times \mathscr{M}_h^{(m)}.$$

Then it follows that:

1) There exist a constant C depending only on Θ_0, δ_0 and the λ_k, and a constant h_0 such that for $h \leqq h_0$,

$$\|\nabla_2 u_L - \overline{\boldsymbol{M}}\|_\Omega \leqq C h^{2 - \frac{\lambda_{\max}}{2}} |u_L|_{3+m, \sigma_m, \Omega}.$$

Here $\lambda_{\max} = \max_k \lambda_k$ and $\sigma_m(x) = \min_k \{(\operatorname{dist}(x, Q_k))^{\lambda_k + 2(m-1)}\}$.

2) for any compact set $A \subset \Omega$ there exists a constant C_A, depending only on A, such that

$$\max_{P_i \in A} |u_L(P_i) - \bar{v}(P_i)| \leqq C_A \|\nabla_2 u_L - \overline{\boldsymbol{M}}\|_\Omega \quad \text{for} \quad 0 < h \leqq h_0.$$

Here $\{P_i\}$ are the vertices of $\{T_j\}$.

Remark. If u_L is smooth enough, then the rate of convergence of $\overline{\boldsymbol{M}}$ to $\nabla_2 u_L$ is optimal, i.e., $\|\nabla_2 u_L - \overline{\boldsymbol{M}}\|_\Omega = \mathcal{O}(h)$ if $m=0$ and $\|\nabla_2 u_L - \overline{\boldsymbol{M}}\|_\Omega = \mathcal{O}(h^2)$ if $m=1$ as $h \to 0$.

Proof of the Theorem. Let $\boldsymbol{M}$ be the unique element in $\mathscr{M}_h^{(m)}$ which interpolates $\nabla_2 u_L$ on each triangle T_j in the sense of Lemma 3.5. Since $u_L \in W_{2,\sigma}^{(4)}(\Omega)$, we can use the argument leading to (3.11) to show that

$$E(\nabla_2 u_L, \nabla_2 v)_{T_j} = \sum_j \int_{\partial T_j} \frac{\partial^2 u_L}{\partial n^2} \frac{\partial v}{\partial n}\, ds + \int_\Omega f v\, dA, \qquad \forall v \in V_h^{(m+1)}.$$

By (3.21) and the argument preceding (3.21) it then follows that $\boldsymbol{M} \in \mathscr{H}_{L,h}$, where L is given by the function f. Recalling Lemma 3.8 we therefore have

$\|\nabla_2 u - \overline{\boldsymbol{M}}\|_\Omega \leqq \|\nabla_2 u - \boldsymbol{M}\|_\Omega$. We now estimate $\|\nabla_2 u - \boldsymbol{M}\|_\Omega$ by using Lemma 4.4 on each triangle T_j.

If $\min_k \operatorname{dist}(T_j, Q_k) \geqq \delta_0 \operatorname{diam}(T_j) = \delta_0 d_j$ we get with $\mu = 2(1-m)$ in Lemma 4.4,

$$\|\nabla_2 u_L - \boldsymbol{M}\|_{T_j} \leqq C_m d_j^{1+m} |u_L|_{3+m, T_j}.$$

But recalling that $\sigma_m(x) = \min_k \{(\operatorname{dist}(x, Q_k))^{\lambda_k + 2(m-1)}\}$ we have for $d_j \leqq 1$,

$$\begin{aligned} |u_L|^2_{3+m, T_j} &\leqq \{\max_{x \in T_j} (1/\sigma_m(x))\} \sum_{|\alpha|=3+m} \int_{T_j} |D^\alpha u_L|^2 \sigma_m \, dA \\ &\leqq C d_j^{-\lambda_{\max} - 2(m-1)} |u_L|^2_{3+m, \sigma_m, T_j}. \end{aligned}$$

Thus

$$\|\nabla_2 u_L - \boldsymbol{M}\|^2_{T_j} \leqq C d_j^{4-\lambda_{\max}} |u_L|^2_{3+m, \sigma_m, T_j}.$$

On the other hand, if $\min_k \operatorname{dist}(T_j, Q_k) < \delta_0 d_j$, then actually T_j has a vertex Q_{k_j} in common with $\partial\Omega$. From Lemma 4.4 with $\mu = \lambda_{k_j}$ we find that for $d_j \leqq h_0$ we have

$$\begin{aligned} \|\nabla_2 u_L - \boldsymbol{M}\|^2_{T_j} &\leqq C_m^2 d_j^{4-\lambda_{k_j}} |u_L|^2_{3+m, \sigma_m, T_j} \\ &\leqq C d_j^{4-\lambda_{\max}} |u_L|^2_{3+m, \sigma_m, T_j}. \end{aligned}$$

Summation over j then proves the first part of the theorem.

We now prove the second part of the theorem. Recalling (3.4) and (3.26) we have

$$\begin{aligned} E(\nabla_2 u_L - \overline{\boldsymbol{M}}, \boldsymbol{M}) &= \sum_i H(\boldsymbol{M}, P_i)\big(u_L(P_i) - \bar{v}(P_i)\big) \\ &\quad - \sum_j \int_{\partial T_j} V_n(\boldsymbol{M})(u_L - \bar{v})\, ds, \qquad \forall \boldsymbol{M} \in \mathcal{M}_h^{(m)}. \end{aligned} \tag{4.8}$$

Now, let $P_k \in \Omega$ be a vertex of the triangulation and let G_{P_k} be the Green's function with pole at P_k. Let $\boldsymbol{M}^{P_k} \in \mathcal{M}_h^{(m)}$ interpolate G_{P_k} on each triangle T_j. As usual we have $\boldsymbol{M}^{P_k} \in \mathcal{H}_{L_k, h}$, where L_k denotes the Dirac measure at P_k. Arguing as in the proof of Lemma 3.7 it follows that $V_n(\boldsymbol{M}^{P_k})$ takes values of opposite signs on abutting triangle sides and that

$$H(\boldsymbol{M}^{P_k}, P_i) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases}$$

where the P_i are the vertices in Ω. Consequently, we have by (4.8),

$$|u_L(P_k) - \bar{v}(P_k)| = |E(\nabla_2 u_L) - \overline{\boldsymbol{M}}, \boldsymbol{M}^{P_k})| \leqq \|\boldsymbol{M}^{P_k}\|_\Omega \|\nabla_2 u_L - \overline{\boldsymbol{M}}\|_\Omega.$$

But using Lemma 2.1 and the fact that $|x|^2 \log|x| \in W^{(4)}_{2, \varrho^\mu}(T)$ for any $\mu > 2$, where the triangle T has a vertex at (0, 0) and the weight function is $\varrho^\mu = |x|^\mu$, it is easy to see that there exists a constant C_A such that

$$\|\boldsymbol{M}^{P_k}\|_\Omega \leqq C_A \quad \text{for } P_k \in A, \quad h \leqq h_0.$$

This concludes the proof of the theorem.

Corollary. Let A be a compact subset of Ω and let $\varepsilon > 0$. Let G_P be the Green's function with pole at $P \in A$, and let $(\bar{v}, \overline{\boldsymbol{M}})$ be the corresponding finite element solution. Then there exists constants C and h_0 such that for $h \leqq h_0$,

$$\|V_2 G_{P_i} - \overline{\boldsymbol{M}}\|_\Omega \leqq C\left\{h^{1-\varepsilon} + h^{2-\frac{\lambda_{\max}}{2}}\right\}.$$

Proof. A simple modification of the proof of the theorem.

Example 1. Let Ω be convex, $f \in L_2(\Omega)$ and let u_L be the unique function in V such that

$$E(V_2 u_L, V_2 w) = \int_\Omega f w \, dA, \qquad \forall w \in V.$$

Then $u_L \in W_{2,\sigma}^{(4)}(\Omega)$ for any $\sigma(x) = \min_k \{\operatorname{dist}(x, Q_k)^{\lambda_k}\}$ with $\lambda_k > 2$, (cf. [6, p. 308]). Let $(\bar{v}, \overline{\boldsymbol{M}})$ be the corresponding finite element solution. By the theorem it follows that for $\varepsilon > 0$ and any compact set $A \subset \Omega$ there exist constants C, C_A and h_0 such that for $h \leqq h_0$,

$$\|V_2 u_L - \overline{\boldsymbol{M}}\|_\Omega \leqq C h^{1-\varepsilon},$$

$$\max_{P_i \in A} |u_L(P_i) - \bar{v}(P_i)| \leqq C_A C h^{1-\varepsilon}.$$

Example 2. Let Ω be convex, $P \in \Omega$ be a vertex of the triangulation and let G_P be the unique function in V such that

$$E(V_2 G_P, V_2 w) = L(w) = w(P), \qquad \forall w \in V. \tag{4.9}$$

Let $(\bar{v}, \overline{\boldsymbol{M}})$ be the corresponding finite element solution. By the corrollary of the theorem and Example 1 we know that for $\varepsilon > 0$ there are constants C and h_0 such that for $h \leqq h_0$,

$$\|V_2 G_P - \overline{\boldsymbol{M}}\|_\Omega \leqq C h^{1-\varepsilon}. \tag{4.10}$$

Further, as already noted in the proof of the Theorem, if L is given by the Dirac measure at P, then $\boldsymbol{M} \in \mathscr{H}_{L,h}$ is equivalent to requiring that $V_n(\boldsymbol{M})$ takes values of opposite signs on abutting triangle sides and that

$$H(\boldsymbol{M}, P_i) = 1 \quad \text{if} \quad P_i = P \quad \text{and} \quad H(\boldsymbol{M}, P_i) = 0 \quad \text{if} \quad P_i \neq P.$$

It follows by (3.26) and (3.4), respectively, that

$$E(\overline{\boldsymbol{M}}, \overline{\boldsymbol{M}}) = \bar{v}(P), \tag{4.11}$$

$$E(\overline{\boldsymbol{M}}, V_2 w) = w(P), \qquad \forall w \in V. \tag{4.12}$$

Combining (4.9) and (4.12), we get

$$E(V_2 G_P - \overline{\boldsymbol{M}}, V_2 G_P) = 0.$$

By (4.9), (4.10), and (4.11) we therefore have

$$\begin{aligned}|G_P(P) - \bar{v}(P)| &= |E(V_2 G_P, V_2 G_P) - E(\overline{\boldsymbol{M}}, \overline{\boldsymbol{M}})| = |E(V_2 G_P - \overline{\boldsymbol{M}}, V_2 G_P - \overline{\boldsymbol{M}})| \\ &\leqq C^2 h^{2-2\varepsilon}.\end{aligned}$$

Thus, we have a sharper estimate at the special point P.

References

1. Allman, D. J.: Triangular finite elements for plate bending with constant and linearly varying bending moments. Colloqium of the International Union of Theoretical and Applied Mechanics (IUTAM) on High Speed Computing of Elastic Structures, University of Liege, Belgium, August 23–28, 1970.
2. Bramble, J., Hilbert, S. R.: Estimation of linear functionals on Sobolev spaces with application to Fourier transforms and Spline interpolation. Siam. J. Numer. Anal. **7**, 112–124 (1970).
3. Bramble, J., Zlamal, M.: Triangular elements in the finite element method. Math. Comp., **24**, 809–820 (1970).
4. Hellan, K.: Analysis of elastic plates in flexure by a simplified finite element method. Acta Polytechnica Scandinavica, Ci 46, Trondheim, 1967.
5. Herrmann, L.: Finite element bending analysis for plates. J. of Mech., Div. ASCE, a3, EM5, 1967.
6. Kondratev, V. A.: Boundary value problems for elliptic equations with conical or angular points. Trans. Moscow Math. Soc., 1967, pp. 227–313.
7. Kufner, A.: Einige Eigenschaften der Sobolevschen Räume mit Belegungsfunktion. Czech. Math. J. **15**, 597–620 (1965).
8. Lions, J. L., Magenes, E.: Problemes aux limites non homogenes et applications. Vol. 1, Travaux Recherches Math., no. 17. Paris: Dunod 1968.
9. Necas, J.: Les methodes directes en theorie des equations elliptiques. Paris: Masson 1967.
10. Mikhlin, S. G.: Variational methods in mathematical physics. Berlin: Akademie Verlag 1962.
11. Strang, G., Fix, G.: An analysis of the finite element method. Prentice-Hall, Inc. (to appear).
12. Synge, J. L.: The hypercircle in mathematical physics. Cambridge at the University Press 1957.
13. Visser, W.: A refined mixed type plate bending element. A.I.A.A. Journal **7**, 1969.

Claes Johnson
Chalmers Institute of Technology
and the University of Göteborg
Department of Mathematics Fack
S-40220 Göteborg 5
Sweden