THE DIRICHLET PROBLEM IN LIPSCHITZ DOMAINS FOR HIGHER ORDER ELLIPTIC SYSTEMS WITH ROUGH COEFFICIENTS *

By

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Abstract. We study the Dirichlet problem, in Lipschitz domains and with boundary data in Besov spaces, for divergence form strongly elliptic systems of arbitrary order with bounded, complex-valued coefficients. A sharp corollary of our main solvability result is that the operator of this problem performs an isomorphism between weighted Sobolev spaces when its coefficients and the unit normal of the boundary belong to the space VMO.

1 Introduction

1.1 Formulation of the main result. A fundamental theme in the theory of partial differential equations, which has profound and intriguing connections with many other subareas of analysis, is the well-posedness of various classes of boundary value problems under minimal smoothness assumptions on the boundary of the domain and on the coefficients of the corresponding differential operator. The main result of this paper is *the solution of the Dirichlet problem for higher order, strongly elliptic systems in divergence form, with complex-valued, bounded, measurable coefficients in Lipschitz domains, and for boundary data in Besov spaces, under sharp smoothness assumptions*. In order to be more specific, we need to introduce some notation.

Let $m, l \in \mathbb{N}$ be two fixed integers and, for a bounded Lipschitz domain Ω in $\mathbb{R}^n$, $n \geq 2$, with outward unit normal $\nu = (\nu_1, \ldots, \nu_n)$ consider the Dirichlet problem for the operator

$$(1.1) \qquad \mathcal{L}(X, D_X)\mathcal{U} := \sum_{|\alpha|=|\beta|=m} D^\alpha (A_{\alpha\beta}(X) D^\beta \mathcal{U}),$$

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i.e.,

$$\begin{cases} \sum_{|\alpha|=|\beta|=m} D^\alpha(A_{\alpha\beta}(X)\, D^\beta\, \mathcal{U}) = 0 & \text{for } X \in \Omega, \\ \frac{\partial^k \mathcal{U}}{\partial \nu^k} = g_k & \text{on } \partial\Omega,\ 0 \le k \le m-1. \end{cases} \tag{1.2}$$

Here and elsewhere, $D^\alpha = (-i\partial/\partial x_1)^{\alpha_1} \cdots (-i\partial/\partial x_n)^{\alpha_n}$ if $\alpha = (\alpha_1, \dots, \alpha_n)$. The coefficients $A_{\alpha\beta}$ are $l \times l$ matrix-valued functions with measurable, complex-valued entries, for which there exists some finite constant $\kappa > 0$ (referred to in the sequel as the ellipticity constant of $\mathcal{L}$) such that

$$\sum_{|\alpha|=|\beta|=m} \|A_{\alpha\beta}\|_{L_\infty(\Omega)} \le \kappa^{-1} \tag{1.3}$$

and such that the coercivity condition

$$\Re \int_\Omega \sum_{|\alpha|=|\beta|=m} \langle A_{\alpha\beta}(X) D^\beta\, \mathcal{V}(X),\, D^\alpha\, \mathcal{V}(X)\rangle\, dX \ge \kappa \sum_{|\alpha|=m} \|D^\alpha\, \mathcal{V}\|^2_{L_2(\Omega)} \tag{1.4}$$

holds for all $\mathbb{C}^l$-valued functions $\mathcal{V} \in C_0^\infty(\Omega)$. Throughout the paper, $\Re\, z$ denotes the real part of $z \in \mathbb{C}$ and $\langle \cdot, \cdot \rangle$ stands for the canonical inner product in $\mathbb{C}^l$. Since, generally speaking, ν is merely bounded and measurable, care should be exercised when defining iterated normal derivatives. For the setting we have in mind it is natural to take $\partial^k/\partial\nu^k := (\sum_{j=1}^n \xi_j \partial/\partial x_j)^k\, |_{\xi=\nu}$ or, more precisely,

$$\frac{\partial^k \mathcal{U}}{\partial \nu^k} := i^k \sum_{|\alpha|=k} \frac{k!}{\alpha!}\, \nu^\alpha\, \mathrm{Tr}\, [D^\alpha \mathcal{U}], \quad 0 \le k \le m-1, \tag{1.5}$$

where Tr is the boundary trace operator and $\nu^\alpha := \nu_1^{\alpha_1} \cdots \nu_n^{\alpha_n}$ if $\alpha = (\alpha_1, \dots, \alpha_n)$. Now, if $p \in (1, \infty)$, $a \in (-1/p, 1-1/p)$ are fixed and $\rho(X) := \mathrm{dist}\,(X, \partial\Omega)$, a solution for (1.2) is sought in $W_p^{m,a}(\Omega)$, defined as the space of vector-valued functions for which

$$\Big(\sum_{|\alpha| \le m} \int_\Omega |D^\alpha \mathcal{U}(X)|^p \rho(X)^{pa}\, dX \Big)^{1/p} < \infty. \tag{1.6}$$

In particular, as explained later on, the traces in (1.5) exist in the Besov space $B_p^s(\partial\Omega)$, where $s := 1 - a - 1/p \in (0,1)$, for any $\mathcal{U} \in W_p^{m,a}(\Omega)$. Recall that, with $d\sigma$ denoting the area element on $\partial\Omega$,

(1.7)

$$f \in B_p^s(\partial\Omega) \Leftrightarrow \|f\|_{B_p^s(\partial\Omega)} := \|f\|_{L_p(\partial\Omega)} + \Big(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(X) - f(Y)|^p}{|X-Y|^{n-1+sp}}\, d\sigma_X\, d\sigma_Y \Big)^{1/p} < \infty.$$

The above definition takes advantage of the Lipschitz manifold structure of $\partial\Omega$. On such manifolds, smoothness spaces of index $s \in (0,1)$ can be defined in an intrinsic, invariant fashion by lifting their Euclidean counterparts onto the manifold itself via local charts. Nonetheless, the very nature of the problem investigated in this paper requires the consideration of *higher order smoothness spaces on* $\partial\Omega$, in which case the above approach is no longer effective. An alternative point of view has been introduced by H. Whitney in [59] where he considered higher order Lipschitz spaces on arbitrary closed sets (see also C. Fefferman's article [18] for related issues). An extension of this circle of ideas pertaining to the full scale of Besov and Sobolev spaces on irregular subsets of $\mathbb{R}^n$ can be found in the book [30] by A. Jonsson and H. Wallin. Here we further refine this theory in the context of Lipschitz domains. For the purpose of this introduction we note that one possible description of these higher order Besov spaces on the boundary of a Lipschitz domain $\Omega \subset \mathbb{R}^n$, and for $m \in \mathbb{N}$, $p \in (1,\infty)$, $s \in (0,1)$, reads

(1.8)
$$\dot{B}_p^{m-1+s}(\partial\Omega) = \text{the closure of } \Big\{ (i^{|\alpha|} D^\alpha \mathcal{V}|_{\partial\Omega})_{|\alpha|\le m-1} : \mathcal{V} \in C_0^\infty(\mathbb{R}^n) \Big\} \text{ in } B_p^s(\partial\Omega)$$

(making no notational distinction between a Banach space $\mathfrak{X}$ and $\mathfrak{X}^N = \mathfrak{X} \oplus \cdots \oplus \mathfrak{X}$). A formal definition, which involves higher order Taylor remainders in place of $f(X) - f(Y)$ in (1.7), along with other equivalent characterizations of $\dot{B}_p^{m-1+s}(\partial\Omega)$ can be found in §7.1. Given (1.5)–(1.6) and (1.8), a necessary condition for the boundary data $\{g_k\}_{0\le k\le m-1}$ in (1.2) is that

$$(1.9)\quad g_k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \nu^\alpha f_\alpha, \quad 0 \le k \le m-1,$$
$$\text{for some } \dot{f} = \{f_\alpha\}_{|\alpha|\le m-1} \in \dot{B}_p^{m-1+s}(\partial\Omega).$$

Let BMO and VMO stand, respectively, for the John–Nirenberg space of functions of bounded mean oscillations and the Sarason space of functions of vanishing mean oscillations (considered either on Ω, or on $\partial\Omega$). The simplest version of our main result, pertaining to the well-posedness of the problem (1.2), then reads as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and assume that the operator $\mathcal{L}$ in* (1.1) *satisfies* (1.3)–(1.4). *Then the Dirichlet problem* (1.2) *with boundary data as in* (1.9) *has a unique solution $\mathcal{U}$ for which* (1.6) *holds, for each $p \in (1,\infty)$, $s \in (0,1)$ and with $a := 1-s-1/p$, provided the coefficient matrices $A_{\alpha\beta}$ belong to* $\mathrm{VMO}(\Omega)$ *and the exterior normal vector ν to $\partial\Omega$ belongs to* $\mathrm{VMO}(\partial\Omega)$.

The above result (along with its inhomogeneous version, presented in §8.1) is sharp. See §8.2 for a discussion.

1.2 Some consequences of the main result and of its proof. We shall, in fact, prove a quantitative version of Theorem 1.1 in which we identify the analytical and geometrical conditions under which the Dirichlet problem (1.2) formulated as above, in the setting of Sobolev–Besov spaces, is well-posed for each given $p\in(1,\infty)$ and $s\in(0,1)$. To state this formally, we need one final piece of terminology. By the *BMO mod VMO character* of a function $F\in L_1(\Omega)$ we shall understand the quantity

$$\{F\}_{*,\Omega} := \lim_{\varepsilon\to 0^+}\Big(\sup_{(B_\varepsilon)_\Omega} \fint_{B_\varepsilon\cap\Omega}\fint_{B_\varepsilon\cap\Omega}|F(X)-F(Y)|dXdY\Big), \tag{1.10}$$

where $(B_\varepsilon)_\Omega$ stands for the collection of arbitrary balls centered at points of Ω and of radii $\le\varepsilon$, and the barred integral is the mean value. In a similar fashion, if $(B_\varepsilon)_{\partial\Omega}$ is the collection of n-dimensional balls with centers on $\partial\Omega$ and of radii $\le\varepsilon$, and if $f\in L_1(\partial\Omega)$, we set

$$\{f\}_{*,\partial\Omega} := \lim_{\varepsilon\to 0^+}\Big(\sup_{(B_\varepsilon)_{\partial\Omega}} \fint_{B_\varepsilon\cap\partial\Omega}\fint_{B_\varepsilon\cap\partial\Omega}|f(X)-f(Y)|d\sigma_X d\sigma_Y\Big). \tag{1.11}$$

What we shall prove then is that *there exists* $c>0$*, depending only on the Lipschitz constant of* Ω *and the ellipticity constant of* $\mathcal{L}$*, with the following significance. Assume that* $p\in(1,\infty)$ *and* $s\in(0,1)$ *are given, and set* $a:=1-s-1/p$*. Then the Dirichlet problem* (1.2)*, formulated in the functional analytic context described in Theorem 1.1, has a unique solution* $\mathcal{U}$ *satisfying* (1.6) *granted that*

$$\{\nu\}_{*,\partial\Omega}+\sum_{|\alpha|=|\beta|=m}\{A_{\alpha\beta}\}_{*,\Omega}\le c\,s(1-s)\Big(p^2(p-1)^{-1}+s^{-1}(1-s)^{-1}\Big)^{-1}. \tag{1.12}$$

For an arbitrary function F we obviously have (with the dependence on the domain dropped) $\{F\}_*\le 2\,\mathrm{dist}\,(F,\mathrm{VMO})$, where the distance is taken in BMO. Moreover, as a consequence of a result due to D. Sarason (cf. Lemma 2 on p. 393 of [46]), there exists $C>0$ such that $\mathrm{dist}\,(F,\mathrm{VMO})\le C\{F\}_*$. Thus, all together, $\{F\}_*\sim\mathrm{dist}\,(F,\mathrm{VMO})$ so that condition (1.12) is satisfied provided

$$\mathrm{dist}\,(\nu,\mathrm{VMO})+\sum_{|\alpha|=|\beta|=m}\mathrm{dist}\,(A_{\alpha\beta},\mathrm{VMO})\le c\,s(1-s)\Big(p^2(p-1)^{-1}+s^{-1}(1-s)^{-1}\Big)^{-1}. \tag{1.13}$$

In particular, (1.13) is obviously valid when $\nu\in\mathrm{VMO}(\partial\Omega)$ and the $A_{\alpha\beta}$'s belong to $\mathrm{VMO}(\Omega)$, irrespective of p, s, the ellipticity constant of $\mathcal{L}$ and the Lipschitz constant of Ω. The latter is precisely the setting dealt with in Theorem 1.1.

As corollary we obtain that, under the same hypotheses, the problem

$$
\begin{cases} \sum_{|\alpha|=|\beta|=m} D^\alpha (A_{\alpha\beta}(X)\, D^\beta \mathcal{U}) = 0 & \text{in } \Omega \\ i^{|\gamma|}\, \mathrm{Tr}\, [D^\gamma \mathcal{U}] = f_\gamma & \text{on } \partial\Omega,\ |\gamma| \le m-1 \end{cases} \tag{1.14}
$$

has a unique solution $\mathcal{U}$ which satisfies (1.6) whenever

$$
\dot{f} := \{f_\gamma\}_{|\gamma|\le m-1} \in \dot{B}_p^{m-1+s}(\partial\Omega). \tag{1.15}
$$

However, an advantage of the classical formulation (1.2) over (1.14) is that, in the former case, the number of boundary conditions is *minimal*, i.e., m in place

$$
\sum_{k=0}^{m-1} \binom{n+k-1}{n-1}, \tag{1.16}
$$

which represents the cardinality of the set of multi-indices $\alpha \in \mathbb{N}_0^n$ of length $\le m-1$. For a domain $\Omega \subset \mathbb{R}^2$ with boundary of class C^{1+r}, $\frac{1}{2} < r < 1$, and for real, constant coefficient, scalar operators, the limiting case $p = \infty$ of the Dirichlet problem (1.14) has been considered by S. Agmon in [2]. Exploiting the special nature of the layer potentials associated with the equation in the two-dimensional setting, he has proved that there exists a unique solution $\mathcal{U} \in C^{m-1+s}(\overline{\Omega})$, $0 < s < r$, whenever $f_\gamma = i^{|\gamma|} D^\gamma \mathcal{V}|_{\partial\Omega}$, $|\gamma| \le m-1$, for some function $\mathcal{V} \in C^{m-1+s}(\overline{\Omega})$. See also [3] for a related problem.

The innovation that allows us to consider boundary data in Besov spaces as in (1.15) is the systematic use of *weighted Sobolev spaces* such as those associated with the norm in (1.6). In relation to the standard Besov scale in $\mathbb{R}^n$, we would like to point out that, thanks to Theorem 4.1 in [29] on the one hand, and Theorem 1.4.2.4 and Theorem 1.4.4.4 in [22] on the other, we have

$$
\begin{aligned} a = 1 - s - \tfrac{1}{p} \in (0, 1-1/p) &\Longrightarrow W_p^{m,a}(\Omega) \hookrightarrow B_p^{m-1+s+1/p}(\Omega), \\ a = 1 - s - \tfrac{1}{p} \in (-1/p, 0) &\Longrightarrow B_p^{m-1+s+1/p}(\Omega) \hookrightarrow W_p^{m,a}(\Omega). \end{aligned} \tag{1.17}
$$

Of course, $W_p^{m,a}(\Omega)$ is just the classical Sobolev space $W_p^m(\Omega)$ when $a = 0$.

Remarkably, the classical trace theory for ordinary Sobolev spaces in domains with smooth boundaries turns out to have a most satisfactory analogue in this weighted context and for Lipschitz domains. One of our main results in this regard is identifying the correct class of boundary data for higher order Dirichlet problems for functions in $W_p^{m,a}(\Omega)$. In the process, we establish that

$$
\begin{aligned} &\mathcal{U} \in W_p^{m,a}(\Omega) \text{ and } \tfrac{\partial^k \mathcal{U}}{\partial \nu^k} = 0 \text{ on } \partial\Omega \text{ for } 0 \le k \le m-1 \\ &\Longleftrightarrow \mathcal{U} \text{ belongs to the closure of } C_0^\infty(\Omega) \text{ in } W_p^{m,a}(\Omega), \end{aligned} \tag{1.18}
$$

which provides an answer to the question raised by J. Nečas in Problem 4.1 on page 91 of his 1967 book [43]. In the context of unweighted Sobolev spaces and for smoother domains, such a result has been known for a long time (cf., e.g., P. Grisvard, S. M. Nikol'skiĭ and H. Triebel's monographs [22], [44], [54] and the references therein).

As a consequence of the trace theory developed in §7 we have that, in the context of Theorem 1.1,

$$\sum_{|\alpha|\le m-1} \|\mathrm{Tr}\,[D^\alpha \mathcal{U}]\|_{B_p^s(\partial\Omega)} \sim \Big(\sum_{|\alpha|\le m} \int_\Omega \rho(X)^{p(1-s)-1}\,|D^\alpha \mathcal{U}(X)|^p\,dX\Big)^{1/p}, \tag{1.19}$$

uniformly in $\mathcal{U}$ satisfying $\mathcal{L}(X, D_X)\mathcal{U} = 0$ in Ω. The estimate (1.19) can be viewed as a significant generalization of a well-known characterization of the membership of a function to a Besov space in $\mathbb{R}^{n-1}$ in terms of weighted Sobolev norm estimates for its harmonic extension to $\mathbb{R}^n_+$ (see, e.g., Proposition $7'$ on p. 151 of E. Stein's book [53]).

1.3 A brief overview of related work. Broadly speaking, there are two types of questions pertaining to the well-posedness of the Dirichlet problem in a Lipschitz domain Ω for a divergence form, strongly elliptic system (1.1) of order $2m$ with boundary data in Besov spaces indexed by s and p.

Question I. Granted that the coefficients of $\mathcal{L}$ exhibit a certain amount of smoothness, identify the indices p, s for which this boundary value problem is well-posed.

Question II. Alternatively, having fixed the indices s and p, characterize the smoothness of $\partial\Omega$ and of the coefficients of $\mathcal{L}$ for which the aforementioned problem is well-posed.

These, as well as other related issues, have been a driving force behind many exciting, recent developments in partial differential equations and allied fields. An authoritative account of their impact is given by C. Kenig in [31] where he describes the state of the art in this field of research up to the mid 1990's. One generic problem which falls under the scope of Question I is to determine the optimal scale of spaces on which the Dirichlet problem for a strongly elliptic system of order $2m$ is solvable in an *arbitrary Lipschitz domain* Ω in $\mathbb{R}^n$. The most basic case, that of the constant coefficient Laplacian in arbitrary Lipschitz domains in $\mathbb{R}^n$, is now well-understood thanks to the work of B. Dahlberg and C. Kenig [14], in the case of L_p-data, and D. Jerison and C. Kenig [29], in the case of Besov data. The case of (8.15) for boundary data exhibiting higher regularity (i.e., $s > 1$) has been recently dealt with by V. Maz'ya and T. Shaposhnikova in [41] where optimal smoothness conditions for $\partial\Omega$ are found in terms of the properties of ν as a Sobolev space multiplier.

In spite of substantial progress in recent years, there remain many basic open questions, particularly for $l > 1$ and/or $m > 1$, even in the case of *constant coefficient* operators in Lipschitz domains. In this context, one significant problem is to determine the sharp range of p's for which the Dirichlet problem for strongly elliptic systems with L_p-boundary data is well-posed. In [45], J. Pipher and G. Verchota have developed a L_p-theory for real, constant coefficient, higher order systems $L = \sum_{|\alpha|=2m} A_\alpha D^\alpha$ when p is near 2, i.e., $2-\varepsilon < p < 2+\varepsilon$ with $\varepsilon > 0$ depending on the Lipschitz character of Ω. On p. 2 of [45] the authors ask whether the L_p-Dirichlet problem for these operators is solvable in a given Lipschitz domain for $p \in (2-\varepsilon, 2(n-1)/n-3+\varepsilon)$, and a positive answer has been recently given by Z. Shen in [47]. Let us also mention here the work [1] of V. Adolfsson and J. Pipher who have dealt with the Dirichlet problem for the biharmonic operator in arbitrary Lipschitz domains and with data in Besov spaces, [57] where G. Verchota formulates and solves the Neumann problem for the bi-Laplacian in Lipschitz domains and with boundary data in L_2, as well as the paper [32] by V. Kozlov and V. Maz'ya, which contains an explicit description of the asymptotic behavior of null-solutions of constant coefficient, higher order, elliptic operators near points on the boundary of a domain with a sufficiently small Lipschitz constant.

A successful strategy for dealing with Question II consists of formulating and solving the analogue of the original problem in a standard case, typically when $\Omega = \mathbb{R}^n_+$ and $\mathcal{L}$ has constant coefficients, and then deviating from this most standard setting by allowing perturbations of a certain magnitude. A paradigm result in this regard, going back to the work of S. Agmon, A. Douglis, L. Nirenberg and V. A. Solonnikov in the 50's and 60's, is that the Dirichlet problem is solvable in the context of Sobolev–Besov spaces if $\partial\Omega$ is sufficiently smooth and if $\mathcal{L}$ has continuous coefficients. The latter requirement is an artifact of the method of proof (based on Korn's trick of freezing the coefficients) which requires measuring the size of the oscillations of the coefficients in a *pointwise sense* (as opposed to integral sense, as in (1.10)). For a version of Question II, corresponding to boundary data selected from $\prod_{k=0}^{m-1} B_p^{m-1-k+s}(\partial\Omega)$, optimal results have been obtained by V. Maz'ya and T. Shaposhnikova in [39]. In this context, the natural language for describing the smoothness of the domain Ω is that of Sobolev space multipliers.

In the smooth context, problems such as (1.2) have been investigated by many authors, including L. Gårding [21], M. I. Višik [58], F. E. Browder [7], S. Agmon, A. Douglis and L. Nirenberg [2], [3], [4], V. Solonnikov [51], [52], L. Hörmander

[27], G. Grubb and N. J. Kokholm [23]. A related result is as follows. If $\Omega \subset \mathbb{R}^n$ is an *arbitrary bounded open set*, and $g \in W_2^m(\Omega)$, $m \in \mathbb{N}$, is given, then the problem

$$\Delta^m u = 0 \text{ in } \Omega, \quad u \in W_2^m(\Omega), \quad D^\alpha(u-g)|_{\partial\Omega} = 0 \text{ for } |\alpha| \leq m-1, \tag{1.20}$$

where the boundary traces are taken in a generalized sense, has a unique solution. Building on some earlier work of K. Friedrichs, S. L. Sobolev has considered this problem in [49] in the case when $\partial\Omega$ consists of a finite union of submanifolds of $\mathbb{R}^n$ of arbitrary codimension. This result also appears in Sobolev's 1950 monograph [50]. For arbitrary domains, the well-posedness of (1.20) has been established by L. I. Hedberg in [24], [25]. The issue of continuity of the variational solution for higher order equations at boundary points has been studied by V. Maz'ya in [38].

1.4 Comments on the proof of Theorem 1.1 and the layout of the paper While the study of boundary value problems for elliptic differential operators with rough coefficients goes a long way back (it suffices to point to the connections with Hilbert's 19-th problem and De Giorgi–Nash–Moser theory), a lot of attention has been devoted lately to the class of operators with coefficients in VMO. Part of the impetus for the recent surge of interest in this particular line of work stems from a key observation made by F. Chiarenza, M. Frasca and P. Longo in the early 1990's. More specifically, while investigating interior estimates for the solution of a scalar, second-order elliptic differential equation of the form $\mathcal{L}\mathcal{U} = F$, these authors noticed that $\mathcal{U}$ can be related to F via a potential theoretic representation formula in which the residual terms are commutators between operators of Mikhlin–Calderón–Zygmund type, on the one hand, and operators of multiplication by the coefficients of $\mathcal{L}$, on the other hand. This made it possible to control these terms by invoking the commutator estimate of Coifman–Rochberg–Weiss ([12]). An alternative method, based on maximal operators and good-λ inequalities, has been developed by L. Caffarelli and I. Peral in [10], whereas when $\Omega = \mathbb{R}^n$, an approach based on estimates for the Riesz transforms has been devised by T. Iwaniec and C. Sbordone in [28]. Further related results can be found in [5], [9], [11], [17], [35].

Compared to the aforementioned works, our approach is more akin to that of F. Chiarenza and collaborators [11] though there are fundamental differences between solving boundary problems for higher order and for second order operators. One difficulty inherently linked with the case $m > 1$ arises from the way the norm in (1.6) behaves under a change of variables $\varkappa : \Omega = \{(X', X_n) : X_n > \varphi(X')\} \to \mathbb{R}^n_+$ designed to flatten the Lipschitz surface $\partial\Omega$. When $m = 1$, a simple bi-Lipschitz changes of variables such as $\Omega \ni (X', X_n) \mapsto (X', X_n - \varphi(X')) \in \mathbb{R}^n_+$ will do, but matters are considerable more subtle in the case $m > 1$. In this latter situation, we employ a special global flattening map first introduced by J. Nečas (in a different context; cf. p. 188 in [43]) and then independently rediscovered and/or further adapted to new settings by several authors, including V. Maz'ya and T. Shaposhnikova in [39], B. Dahlberg, C. Kenig J. Pipher, E. Stein and G. Verchota (cf. [13] and the discussion in [15]), and S. Hofmann and J. Lewis in [26]. Our main novel contribution in this regard is adapting this circle of ideas to the context when one seeks pointwise estimates for higher order derivatives of $\varkappa$ and $\lambda := \varkappa^{-1}$ in terms of $[\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})}$.

Another ingredient of independent interest is deriving estimates for $D_x^\alpha D_y^\beta G(x,y)$ where G is the Green function associated with a constant (complex) coefficient system $L(D)$ of order $2m$ in the upper half space, which are sufficiently well-suited for deriving commutator estimates in the spirit of [12]. The methods employed in earlier works are largely based on explicit representation formulas for $G(x,y)$ and, hence, cannot be adapted easily to the case of general, non-symmetric, complex coefficient, higher order systems. By way of contrast, our approach consists of proving directly that the residual part $R(x,y) := G(x,y) - \Phi(x-y)$, where Φ is a fundamental solution for $L(D)$, has the property that $D_x^\alpha D_y^\beta R(x,y)$ is a Hardy-type kernel whenever $|\alpha| = |\beta| = m$. See also [5] for a discussion of the difficulties encountered when estimating the residual part $R(x,y)$ in the case when $\mathcal{L}$ is not necessarily symmetric and has complex coefficients.

The layout of the paper is as follows. Section 2 contains estimates for the Green function in the upper-half space. Section 3 deals with integral operators (of Mikhlin–Calderón–Zygmund and Hardy type) as well as commutator estimates on weighted Lebesgue spaces. In the last part of this section we also revisit Gagliardo's extension operator and establish estimates in the context of BMO. Section 4 contains a discussion of the Dirichlet problem for higher order, variable coefficient, strongly elliptic systems in the upper-half space. The adjustments necessary to treat the case of an unbounded domain lying above the graph of a Lipschitz function are presented in Section 5, whereas in Section 6 we explain how to handle the case of a bounded Lipschitz domain. In Section 7 we study traces and extension operators for higher order smoothness spaces on Lipschitz domains. Finally, in Section 8, we deal with the inhomogeneous version of (1.2); cf. Theorem 8.1 from which Theorem 1.1 follows.

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2 Green's matrix estimates in the half-space

2.1 Statement of the main result. Fix two nonnegative integers m, l and let $L(D_x)$ be a matrix-valued differential operator

$$L(D_x) = \sum_{|\alpha|=2m} A_\alpha D_x^\alpha, \tag{2.1}$$

where the A_α's are constant $l \times l$ matrices with complex entries. Throughout the paper, $D_x^\alpha := i^{-|\alpha|}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n}$ if $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n$. Here and elsewhere, $\mathbb{N}$ stands for the collection of all positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Assume that L is strongly elliptic, i.e., there exists $\kappa > 0$ such that $\sum_{|\alpha|=m} \|A_\alpha\|_{\mathbb{C}^{l\times l}} \le \kappa^{-1}$ and

$$\Re\,\langle L(\xi)\eta, \eta\rangle_{\mathbb{C}^l} \ge \kappa\,|\xi|^{2m}\,\|\eta\|_{\mathbb{C}^l}^2, \quad \forall\,\xi \in \mathbb{R}^n,\ \forall\,\eta \in \mathbb{C}^l. \tag{2.2}$$

In what follows, in order to simplify notations, we shall denote the norms in different finite-dimensional real Euclidean spaces by $|\cdot|$ irrespective of their dimensions. Also, quite frequently, we shall make no notational distinction between a space of scalar functions, call it $\mathfrak{X}$, and the space of vector-valued functions (of a fixed, finite dimension) whose components are in $\mathfrak{X}$. We denote by $F(x)$ a fundamental matrix of the operator $L(D_x)$, i.e., an $l \times l$ matrix solution of the system

$$L(D_x)F(x) = \delta(x)I_l \quad \text{in } \mathbb{R}^n, \tag{2.3}$$

where I_l is the $l \times l$ identity matrix and δ is the Dirac function. We consider the Dirichlet problem

$$\begin{cases} L(D_x)u = f & \text{in } \mathbb{R}^n_+, \\ \mathrm{Tr}\,[\partial^j u/\partial x_n^j] = f_j \quad j = 0, 1, \dots, m-1, & \text{on } \mathbb{R}^{n-1}, \end{cases} \tag{2.4}$$

where $\mathbb{R}^n_+ := \{x = (x', x_n) : x' \in \mathbb{R}^{n-1},\ x_n > 0\}$ and Tr is the boundary trace operator. Hereafter, we shall identify $\partial\mathbb{R}^n_+$ with $\mathbb{R}^{n-1}$ in a canonical fashion.

For each $y' \in \mathbb{R}^{n-1}$ we introduce the Poisson matrices $P_0, \dots, P_{m-1}$ for the problem (2.4), i.e., the solutions of the boundary-value problems

$$\begin{cases} L(D_x)P_j(x, y') = 0\, I_l & \text{in } \mathbb{R}^n_+, \\ \left(\frac{\partial^k}{\partial x_n^k} P_j\right)(\,(x', 0), y'\,) = \delta_{jk}\,\delta(x' - y')I_l & \text{for } x' \in \mathbb{R}^{n-1},\ 0 \le k \le m-1, \end{cases} \tag{2.5}$$

where δ_{jk} is the usual Kronecker symbol and $0 \le j \le m-1$. The matrix-valued function $P_j(x, 0')$ is positive homogeneous of degree $j + 1 - n$, i.e.,

$$P_j(x, 0') = |x|^{j+1-n}\,P_j(x/|x|, 0'), \quad x \in \mathbb{R}^n, \tag{2.6}$$

where $0'$ denotes the origin of $\mathbb{R}^{n-1}$. The restriction of $P_j(\cdot, 0')$ to the upper half-sphere S^{n-1}_+ is smooth and vanishes on the equator along with all of its derivatives up to order $m-1$ (see, for example, §10.3 in [34], and [4]). Hence,

$$\|P_j(x,0')\|_{\mathbb{C}^{l\times l}} \le C\,\frac{x_n^m}{|x|^{n+m-1-j}}, \quad x\in\mathbb{R}^n_+, \tag{2.7}$$

and, consequently,

$$\|P_j(x,y')\|_{\mathbb{C}^{l\times l}} \le C\,\frac{x_n^m}{|x-(y',0)|^{n+m-1-j}}, \quad x\in\mathbb{R}^n_+, \quad y'\in\mathbb{R}^{n-1}. \tag{2.8}$$

By $G(x,y)$ we shall denote the Green's matrix of the problem (2.4), i.e., the unique solution of the boundary-value problem

$$\begin{cases} L(D_x)G(x,y) = \delta(x-y)I_l & \text{for } x\in\mathbb{R}^n, \\ \left(\frac{\partial^j}{\partial x_n^j}G\right)((x',0),y) = 0\,I_l & \text{for } x'\in\mathbb{R}^{n-1},\ 0\le j\le m-1, \end{cases} \tag{2.9}$$

where $y\in\mathbb{R}^n_+$ is regarded as a parameter. We now introduce the matrix

$$R(x,y) := F(x-y) - G(x,y), \quad x,y\in\mathbb{R}^n_+, \tag{2.10}$$

so that, for each fixed $y\in\mathbb{R}^n_+$,

$$\begin{cases} L(D_x)\,R(x,y) = 0 & \text{for } x\in\mathbb{R}^n, \\ \left(\frac{\partial^j}{\partial x_n^j}R\right)((x',0),y) = \left(\frac{\partial^j}{\partial x_n^j}F\right)((x',0)-y) & \text{for } x'\in\mathbb{R}^{n-1},\ 0\le j\le m-1. \end{cases} \tag{2.11}$$

Our goal is to establish the following estimate (related results are proved in [4]).

Theorem 2.1. *For all multi-indices α,β of length m*

$$\|D_x^\alpha D_y^\beta R(x,y)\|_{\mathbb{C}^{l\times l}} \le C\,|x-\bar{y}|^{-n}, \tag{2.12}$$

for $x,y\in\mathbb{R}^n_+$, where $\bar{y} := (y',-y_n)$ is the reflection of the point $y\in\mathbb{R}^n_+$ with respect to $\partial\mathbb{R}^n_+$.

In the proof of Theorem 2.1 we distinguish two cases, $n>2m$ and $n\le 2m$, which we shall treat separately. Our argument pertaining to the situation when $n>2m$ is based on the following useful estimate for a parameter-dependent integral.

Lemma 2.2. *Let $a>0$ and $b\ge 0$ be two real numbers and assume that $\zeta\in\mathbb{R}^N$. Then for every $\varepsilon>0$ and $0<\delta<N$ there exists a constant $c(N,\varepsilon,\delta)>0$ such that*

$$\int_{\mathbb{R}^N}\frac{d\eta}{(|\eta|+a)^{N+\varepsilon}(|\eta-\zeta|+b)^{N-\delta}} \le \frac{c(N,\varepsilon,\delta)}{a^\varepsilon(|\zeta|+a+b)^{N-\delta}}. \tag{2.13}$$

The proof is postponed for §2.4, to prevent disrupting the flow of the presentation.

2.2 Proof of Theorem 2.1 for $n > 2m$. In the case when $n > 2m$ there exists a unique fundamental matrix $F(x)$ for the operator (2.1) which is positive homogeneous of degree $2m - n$. We shall use the integral representation formula

$$R(x,y) = R_0(x,y) + \cdots + R_{m-1}(x,y), \quad x,y \in \mathbb{R}^n_+, \tag{2.14}$$

where $R(x,y)$ has been introduced in (2.10) and, with P_j as in (2.5), we set

$$R_j(x,y) := \int_{\mathbb{R}^{n-1}} P_j(x,\xi') \left(\frac{\partial^j}{\partial x_n^j} F\right) ((\xi',0) - y)\, d\xi', \quad 0 \le j \le m-1. \tag{2.15}$$

Then, thanks to (2.7) we have

$$\|R_j(x,y)\|_{\mathbb{C}^{l\times l}} \le C \int_{\mathbb{R}^{n-1}} \frac{x_n^m}{|x-(\xi',0)|^{n+m-1-j}} \cdot \frac{d\xi'}{|(\xi',0)-y|^{n-2m+j}}. \tag{2.16}$$

Next, using Lemma 2.2 with

$$\begin{aligned} N &= n-1, \quad a = x_n, \quad \delta = 2m-j-1, \\ \varepsilon &= m-j, \quad b = y_n, \quad \zeta = y' - x', \end{aligned} \tag{2.17}$$

we obtain from (2.16)

$$\|R_j(x,y)\|_{\mathbb{C}^{l\times l}} \le \frac{C\, x_n^j}{(|y'-x'| + x_n + y_n)^{n-2m+j}}, \quad 0 \le j \le m-1. \tag{2.18}$$

Summing up over $j = 0, \dots, m-1$ gives, by virtue of (2.14), the estimate

$$\|R(x,y)\|_{\mathbb{C}^{l\times l}} \le C\, |x - \bar{y}|^{2m-n}, \quad x,y \in \mathbb{R}^n_+. \tag{2.19}$$

To obtain pointwise estimates for derivatives of $R(x,y)$, we make use of the following local estimate for a solution of problem (2.4) with $f = 0$. Recall that W^s_p stands for the classical L_p-based Sobolev space of order s. The particle *loc* is used to brand the local versions of these (and other) spaces.

Lemma 2.3 (see [4]). *Let ζ and ζ_0 be functions in $C_0^\infty(\mathbb{R}^n)$ such that $\zeta_0 = 1$ in a neighborhood of* $\mathrm{supp}\zeta$. *Then the solution $u \in W_2^m(\mathbb{R}^n_+, loc)$ of problem* (2.4) *with $f = 0$ and $f_j \in W_p^{k+1-j-1/p}(\mathbb{R}^{n-1}, loc)$, where $k \ge m$ and $p \in (1,\infty)$, belongs to $W_p^{k+1}(\mathbb{R}^n_+, loc)$ and satisfies the estimate*

$$\|\zeta u\|_{W_p^{k+1}(\mathbb{R}^n_+)} \le C \left(\sum_{j=0}^{m-1} \|\zeta_0 f_j\|_{W_p^{k+1-j-1/p}(\mathbb{R}^{n-1})} + \|\zeta_0 u\|_{L_p(\mathbb{R}^n_+)} \right), \tag{2.20}$$

where C is independent of u and f_j.

Let $B(x,r)$ denote the ball of radius $r > 0$ centered at x.

Corollary 2.4. *Assume that $u \in W_2^m(\mathbb{R}^n_+, loc)$ is a solution of problem* (2.4) *with $f = 0$ and $f_j \in C^{k+1-j}(\mathbb{R}^{n-1}, loc)$. Then for any $z \in \overline{\mathbb{R}^n_+}$ and $\rho > 0$,*

(2.21)

$$\sup_{\mathbb{R}^n_+ \cap B(z,\rho)} |\nabla_k u| \leq C \left(\rho^{-k} \sup_{\mathbb{R}^n_+ \cap B(z,2\rho)} |u| + \sum_{j=0}^{m-1} \sum_{s=0}^{k+1-j} \rho^{s+j-k} \sup_{\mathbb{R}^{n-1} \cap B(z,2\rho)} |\nabla'_s f_j| \right),$$

where ∇'_s is the gradient of order s in $\mathbb{R}^{n-1}$. Here C is a constant independent of ρ, z, u and f_j.

Proof. Given the dilation invariant nature of the estimate we seek, it suffices to assume that $\rho = 1$. Given $\phi \in C^{k+1-j}(\mathbb{R}^{n-1})$ supported in $\mathbb{R}^{n-1} \cap B(z, 2)$, we observe that, for a suitable $\theta \in (0, 1)$,

(2.22)

$$\|\phi\|_{W_p^{k+1-j-1/p}(\mathbb{R}^{n-1})} \leq C \|\phi\|^{\theta}_{L_p(\mathbb{R}^{n-1})} \|\phi\|^{1-\theta}_{W_p^{k+1-j}(\mathbb{R}^{n-1})} \leq C \sum_{s=0}^{k+1-j} \sup_{\mathbb{R}^{n-1} \cap B(z,2)} |\nabla'_s \phi|.$$

Also, if $p > n$,

$$\sup_{\mathbb{R}^n_+} |\nabla_k v| \leq C \, \|v\|_{W_p^{k+1}(\mathbb{R}^n_+)}, \tag{2.23}$$

by Sobolev's inequality. Now, (2.21) follows by combining (2.22), (2.23) with Lemma 2.3. $\square$

Given $x, y \in \mathbb{R}^n_+$, set $\rho := |x - \bar{y}|/5$ and pick $z \in \partial\mathbb{R}^n_+$ such that $|x - z| = \rho/2$. It follows that for any $w \in \mathbb{R}^n_+ \cap B(z, 2\rho)$ we have $|x - \bar{y}| \leq |x - z| + |z - w| + |w - \bar{y}| \leq \rho/2 + 2\rho + |w - \bar{y}| \leq |x - \bar{y}|/2 + |w - \bar{y}|$. Consequently, $|x - \bar{y}|/2 \leq |w - \bar{y}|$ for every $w \in \mathbb{R}^n_+ \cap B(z, 2\rho)$, so that, ultimately,

$$\rho^{\nu-k} \sup_{w \in \mathbb{R}^{n-1} \cap B(z,2\rho)} \|\nabla'_\nu F(w - y)\|_{\mathbb{C}^{l \times l}} \leq \frac{C}{|x - \bar{y}|^{n-2m+k}}, \tag{2.24}$$

for each $\nu \in \mathbb{N}_0$. Granted (2.19) and our choice of ρ, we altogether obtain that

$$\|D_x^\alpha R(x, y)\|_{\mathbb{C}^{l \times l}} \leq C_k \, |x - \bar{y}|^{2m-n-k}, \quad x, y \in \mathbb{R}^n_+, \quad \forall\, \alpha \in \mathbb{N}_0^n, \ |\alpha| \leq k. \tag{2.25}$$

In the following two formulas, it will be convenient to use the notation $R_{\mathcal{L}}$ for the matrix R associated with the operator $\mathcal{L}(D_x)$ as in (2.10). By Green's formula

$$R_{\mathcal{L}}(y, x) = \big[R_{\mathcal{L}^*}(x, y)\big]^*, \quad x, y \in \mathbb{R}^n_+, \tag{2.26}$$

where the superscript star indicates adjunction. In order to estimate *mixed* partial derivatives, we observe that (2.26) entails

$$(D_y^\beta R_{\mathcal{L}})(x, y) = \big[(D_x^\beta R_{\mathcal{L}^*})(y, x)\big]^* \tag{2.27}$$

and remark that $\mathcal{L}^*$ has properties similar to $\mathcal{L}$. This, in concert with (2.25) and the fact that $|x-\bar{y}| = |\bar{x}-y|$ for $x, y \in \mathbb{R}^n_+$, yields

$$\|D_y^\beta R(x,y)\|_{\mathbb{C}^{l\times l}} \leq C_\beta\, |x-\bar{y}|^{2m-n-|\beta|}. \tag{2.28}$$

Let us also point out that by formally differentiating (2.11) with respect to y we obtain

(2.29)

$$\begin{cases} \mathcal{L}(D_x)\,[D_y^\beta R_{\mathcal{L}}(x,y)] = 0 & \text{for } x \in \mathbb{R}^n, \\ \big(\frac{\partial^j}{\partial x_n^j} D_y^\beta R\big)((x',0),y) = \big(\frac{\partial^j}{\partial x_n^j}(-D)^\beta F\big)((x',0)-y), & x' \in \mathbb{R}^{n-1}, 0 \leq j \leq m-1. \end{cases}$$

With (2.28) and (2.29) in place of (2.19) and (2.11), respectively, we can now run the same program as above and obtain the estimate

$$\|D_x^\alpha D_y^\beta R(x,y)\|_{\mathbb{C}^{l\times l}} \leq C_{\alpha\beta}\, |x-\bar{y}|^{2m-n-|\alpha|-|\beta|}, \quad \forall\, \alpha, \beta \in \mathbb{N}_0. \tag{2.30}$$

2.3 Proof of Theorem 2.1 for $n \leq 2m$. When $n \leq 2m$ we shall use the method of descent. To get started, fix an integer N such that $N > 2m$ and let $(x,z) \mapsto \mathcal{G}(x,y,z-\zeta)$ denote the Green matrix with singularity at $(y,\zeta) \in \mathbb{R}^n \times \mathbb{R}^{N-n}$ of the Dirichlet problem for the operator $\mathcal{L}(D_x) + (-\Delta_z)^m$ in the N-dimensional half-space

$$\mathbb{R}^N_+ := \{(x,z) :\, z \in \mathbb{R}^{N-n},\, x = (x', x_n),\, x' \in \mathbb{R}^{n-1},\, x_n > 0\}. \tag{2.31}$$

Also, recall that $G(x,y)$ stands for the Green matrix of the problem (2.4).

Lemma 2.5. *For all multi-indices α and β of order m and for all x and y in $\mathbb{R}^n_+$,*

$$D_x^\alpha D_y^\beta G(x',y) = \int_{\mathbb{R}^{N-n}} D_x^\alpha D_y^\beta \mathcal{G}(x,y,-\zeta)\, d\zeta. \tag{2.32}$$

Proof. The strategy is to show that

$$\int_{\mathbb{R}^n_+} D_x^\alpha D_y^\beta G(x,y)\, f_\beta(y)\, dy = \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^{N-n}} D_x^\alpha D_y^\beta \mathcal{G}(x,y,-\zeta)\, d\zeta\, f_\beta(y)\, dy \tag{2.33}$$

for each $f_\beta \in C_0^\infty(\mathbb{R}^n_+)$, from which (2.32) clearly follows. To justify (2.33) for a fixed, arbitrary $f_\beta \in C_0^\infty(\mathbb{R}^n_+)$, we let u be the unique vector-valued function satisfying $D^\alpha u \in L^2(\mathbb{R}^n_+)$ for all α with $|\alpha| = m$, and such that

$$\begin{cases} \mathcal{L}(D_x)u = D_x^\beta f_\beta & \text{in } \mathbb{R}^n_+, \\ \left(\frac{\partial^j u}{\partial x_n^j}\right)(x',0) = 0 & \text{on } \mathbb{R}^{n-1},\ 0 \leq j \leq m-1. \end{cases} \tag{2.34}$$

It is well-known that for each $\gamma \in \mathbb{N}_0^n$,

$$|D^\gamma u(x)| \leq C_\gamma\, |x|^{m-n-|\gamma|} \quad for\ |x| > 1. \tag{2.35}$$

This follows, for instance, from Theorem 6.1.4 in [33] combined with Theorem 10.3.2 in [34]. Also, as a consequence of Green's formula, the solution of the problem (2.34) satisfies

$$D_x^\alpha u(x) = \int_{\mathbb{R}_+^n} D_x^\alpha (-D_y)^\beta G(x,y)\, f_\beta(y)\, dy. \tag{2.36}$$

We shall now derive yet another integral representation formula for $D_x^\alpha u$ in terms of (derivatives of) $\mathcal{G}$ which is similar in spirit to (2.36). Since $N > 2m$, (2.30) implies

$$\|D_x^\alpha D_y^\beta \mathcal{G}(x,y,-\zeta)\|_{\mathbb{C}^{l\times l}} \leq c\,(|x-y| + |\zeta|)^{-N}. \tag{2.37}$$

Let us now fix $x \in \mathbb{R}_+^n$, $\rho > 0$ and introduce a cut-off function $H \in C^\infty(\mathbb{R}^{N-n})$ which satisfies $H(z) = 1$ for $|z| \leq 1$ and $H(z) = 0$ for $|z| \geq 2$. We may then write

$$u(x) = \int_{\mathbb{R}^N} \mathcal{G}(x,y,-\zeta)\Big[H\big(\zeta/\rho\big) D^\beta f_\beta(y) + (-\Delta_\zeta)^m \big(H\big(\zeta/\rho\big)\, u(y)\big)\Big]\, dy\, d\zeta, \tag{2.38}$$

which further implies

$$\begin{aligned} &\left| D_x^\alpha u(x) - \int_{\mathbb{R}^N} D_x^\alpha (-D_y)^\beta\, \mathcal{G}(x,y,-\zeta)\, H\big(\zeta/\rho\big)\, f_\beta(y)\, dy\, d\zeta \right| \\ &\qquad \leq c \sum_{|\gamma|=m} \int_{\mathbb{R}_+^N} \|D_x^\alpha D_\zeta^\gamma\, \mathcal{G}(x,y,-\zeta)\|_{\mathbb{C}^{l\times l}}\, \big|u(y)\, D_\zeta^\gamma\big(H\big(\zeta/\rho\big)\big)\big|\, d\zeta. \end{aligned} \tag{2.39}$$

By (2.35) and (2.37), the expression in the right-hand side of (2.39) does not exceed

$$\begin{aligned} &c\,\rho^{-m} \int_{\rho<|\zeta|<2\rho} d\zeta \int_{\mathbb{R}^{n-1}} (|x-y| + |\zeta|)^{-N}\, |y|^{m-n}\, dy \\ &\qquad\qquad \leq c\,\rho^{N-n-m} \int_{\mathbb{R}^{n-1}} (|y| + \rho)^{-N} |y|^{m-n}\, dy = c\,\rho^{-n}. \end{aligned}$$

This estimate, in concert with 2.37, allows us to obtain, after making $\rho \to \infty$, that

$$D_x^\alpha u(x) = \int_{\mathbb{R}_+^n} \int_{\mathbb{R}^{N-n}} D_x^\alpha (-D_y)^\beta \mathcal{G}(x,y,-\zeta)\, d\zeta\, f_\beta(y)\, dy. \tag{2.40}$$

Now (2.33) follows readily from this and (2.36). $\square$

Having disposed of Lemma 2.5, we are ready to discuss the

End of Proof of Theorem 2.1. Assume that $2m \geq n$ and let N be again an integer such that $N > 2m$. Denote by $\mathcal{F}(x,z)$ the fundamental solution of the operator $\mathcal{L}(D_x) + (-\Delta_z)^m$, which is positive homogeneous of degree $2m - N$ and is singular at $(0,0) \in \mathbb{R}^n \times \mathbb{R}^{N-n}$. Then the identity

$$D_x^{\alpha+\beta} F(x) = \int_{\mathbb{R}^{N-n}} D_x^{\alpha+\beta} \mathcal{F}(x, -\zeta)\, d\zeta \tag{2.41}$$

can be established as in the proof of Lemma 2.5. Combining (2.41) with Lemma 2.5, we arrive at

$$D_x^\alpha D_y^\beta R(x', y) = \int_{\mathbb{R}^{N-n}} D_x^\alpha D_y^\beta \mathcal{R}(x, y, -\zeta)\, d\zeta, \tag{2.42}$$

where $\mathcal{R}(x,y,z) := \mathcal{G}(x,y,z) - \mathcal{F}(x-y,z)$. Consequently,

$$\|D_x^\alpha D_y^\beta \mathcal{R}(x,y,-\zeta)\|_{\mathbb{C}^{l\times l}} \leq C(|x - \bar{y}| + |\zeta|)^{-N} \tag{2.43}$$

by (2.25) with $k = 0$ and N in place of n. This estimate, together with (2.42), then yields (2.12). $\square$

2.4 Proof of Lemma 2.2. Via scaling, there is no loss of generality in assuming that $a = 1$. Assuming that this is the case, write $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$ where $\mathcal{J}$ stands for the integral in the left side of (2.13), whereas $\mathcal{J}_1$ and $\mathcal{J}_2$ denote the integrals obtained by splitting the domain of integration in $\mathcal{J}$ into the ball $B_1 = \{\eta \in \mathbb{R}^N : |\eta| < 1\}$ and $\mathbb{R}^n \setminus B_1$, respectively. If $|\zeta| < 2$, then

$$\mathcal{J}_1 \leq \int_{B_1} \frac{d\eta}{(|\eta - \zeta| + b)^{N-\delta}} \leq c \int_{B_4} \frac{d\xi}{(|\xi| + b)^{N-\delta}}. \tag{2.44}$$

Hence $\mathcal{J}_1 \leq c\,(\max\{1,b\})^{-N+\delta}$ so that, in particular,

$$|\zeta| < 2 \Longrightarrow \mathcal{J}_1 \leq c\,(|\zeta| + 1 + b)^{\delta - N}. \tag{2.45}$$

Let us now assume that $|\zeta| > 2$. Then

$$\mathcal{J}_1 \leq \int_{B_1} \frac{d\eta}{(|\eta| + 1)^{N+\varepsilon}} \frac{c}{(|\zeta| + b)^{N-\delta}} \leq c\,(|\zeta| + 1 + b)^{\delta - N}, \tag{2.46}$$

which is of the right order. As for $\mathcal{J}_2$, we write

$$\mathcal{J}_2 \leq \int_{\mathbb{R}^n \setminus B_1} \frac{d\eta}{|\eta|^{N+\varepsilon}(|\eta - \zeta| + b)^{N-\delta}} = \mathcal{J}_{2,1} + \mathcal{J}_{2,2} \tag{2.47}$$

where $\mathcal{J}_{2,1}$ and $\mathcal{J}_{2,2}$ are obtained by splitting the domain of integration in the above integral into the set $\{\eta : |\eta| > \max\{1, 2|\zeta|\}\}$ and its complement in $\mathbb{R}^n \backslash B_1$. We have

$$\begin{aligned}
\mathcal{J}_{2,1} \leq & \int_{|\eta|>\max\{1,b,2|\zeta|\}} \frac{d\eta}{|\eta|^{N+\varepsilon}(|\eta|+b)^{N-\delta}} + \int_{b>|\eta|>\max\{1,b,2|\zeta|\}} \frac{d\eta}{|\eta|^{N+\varepsilon}(|\eta|+b)^{N-\delta}} \\
\leq & c\Bigg(\int_{|\eta|>\max\{1,b,2|\zeta|\}} \frac{d\eta}{|\eta|^{2N+\varepsilon-\delta}} + \frac{1}{b^{N-\delta}} \int_{b>|\eta|>\max\{1,b,2|\zeta|\}} \frac{d\eta}{|\eta|^{N+\varepsilon}} \Bigg) \\
\leq & \frac{c}{(1+b+|\zeta|)^{N+\varepsilon-\delta}} + \frac{c}{(1+b+|\zeta|)^{N-\delta}} \frac{c}{(1+b+|\zeta|)^{N-\delta}}.
\end{aligned} \tag{2.48}$$

There remains to estimate the integral

$$\mathcal{J}_{2,2} = \int_{B_{2|\zeta|}\backslash B_1} \frac{d\eta}{|\eta|^{N+\varepsilon}(|\eta-\zeta|+b)^{N-\delta}} = \mathcal{J}_{2,2}^{(1)} + \mathcal{J}_{2,2}^{(2)}, \tag{2.49}$$

where $\mathcal{J}_{2,2}^{(1)}$ and $\mathcal{J}_{2,2}^{(2)}$ are obtained by splitting the domain of integration in $\mathcal{J}_{2,2}$ into $B_{|\zeta|/2} \backslash B_1$ and its complement (relative to $B_{2|\zeta|} \backslash B_1$). On the one hand,

$$\mathcal{J}_{2,2}^{(1)} \leq \frac{c}{(|\zeta|+b)^{N-\delta}} \int_{B_{|\zeta|/2}\backslash B_1} \frac{d\eta}{|\eta|^{N+\varepsilon}} \leq \frac{c}{(|\zeta|+1+b)^{N-\delta}}. \tag{2.50}$$

On the other hand, whenever $|\zeta| > 1/2$, the integral $\mathcal{J}_{2,2}^{(2)}$, which extends over all η's such that $|\eta| > 1$, $2|\zeta| > |\eta| > |\zeta|/2$, can be estimated as

$$\begin{aligned}
\mathcal{J}_{2,2}^{(2)} \leq & \frac{c}{|\zeta|^{N+\varepsilon}} \int_{B_{2|\zeta|}\backslash B_1} \frac{d\eta}{(|\eta-\zeta|+b)^{N-\delta}} \leq \frac{c}{|\zeta|^{N+\varepsilon}} \int_{B_{4|\zeta|}} \frac{d\xi}{(|\xi|+b)^{N-\delta}} \\
\leq & \frac{c}{|\zeta|^{N+\varepsilon}} \Bigg(\int_{\substack{|\xi|<4|\zeta| \\ |\xi|<b}} \frac{d\xi}{(|\xi|+b)^{N-\delta}} + \int_{\substack{|\xi|<4|\zeta| \\ |\xi|>b}} \frac{d\xi}{(|\xi|+b)^{N-\delta}} \Bigg).
\end{aligned} \tag{2.51}$$

Consequently,

$$\mathcal{J}_{2,2}^{(2)} \leq c \, \frac{\min\{|\zeta|, b\}^N}{|\zeta|^{N+\varepsilon} b^{N-\delta}}. \tag{2.52}$$

Using $|\zeta| > 1/2$ and the obvious inequality

$$\min\{|\zeta|, b\}^N \cdot \max\{|\zeta|, b\}^{N-\delta} \leq |\zeta|^N \, b^{N-\delta} \tag{2.53}$$

we arrive at

$$\mathcal{J}_{2,2}^{(2)} \leq c\,(|\zeta|+1+b)^{\delta-N}. \tag{2.54}$$

The estimate (2.54), along with (2.50) and (2.49), gives the upper bound $c\,(|\zeta|+1+b)^{\delta-N}$ for $\mathcal{J}_{2,2}$. Combining this with (2.48) we obtain the same majorant for $\mathcal{J}_2$ which, together with a similar result for $\mathcal{J}_1$ already obtained, leads to (2.13). The proof of the lemma is therefore complete.

3 Properties of integral operators in a half-space

In §3.1 and §3.2 we prove estimates for commutators (and certain commutator-like operators) between integral operators in $\mathbb{R}^n_+$ and multiplication operators with functions of bounded mean oscillations, in weighted Lebesgue spaces on $\mathbb{R}^n_+$. Subsection 3.3 contains BMO and pointwise estimates for extension operators from $\mathbb{R}^{n-1}$ onto $\mathbb{R}^n_+$. Throughout, given two Banach spaces E, F, we let $\mathfrak{L}(E,F)$ stand for the space of bounded linear operators from E into F, and abbreviate $\mathfrak{L}(E) := \mathfrak{L}(E,E)$. Also, given $p \in [1,\infty]$, an open set $\mathcal{O} \subset \mathbb{R}^n$ and a measurable nonnegative function w on $\mathcal{O}$, we let $L_p(\mathcal{O}, w(x)\,dx)$ denote the Lebesgue space of (classes of) functions which are p-th power integrable with respect to the weighted measure $w(x)\,dx$ on $\mathcal{O}$. Finally, following a well-established custom, $A(r) \sim B(r)$ will mean that each quantity is bounded by a fixed multiple of the other, uniformly in the parameter r.

3.1 Kernels with singularities along $\partial\mathbb{R}^n_+$. Recall that $L_p(\mathbb{R}^n_+, x_n^{ap}\,dx)$ stands for the weighted Lebesgue space of p-th power integrable functions in $\mathbb{R}^n_+$ corresponding to the weight $w(x) := x_n^{ap}$, $x = (x', x_n) \in \mathbb{R}^n_+$.

Proposition 3.1. *Let $a \in \mathbb{R}$, $1 < p < \infty$, and assume that $\mathcal{Q}$ is a nonnegative measurable function on $\{\zeta = (\zeta', \zeta_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \zeta_n > -1\}$, which also satisfies*

$$\int_{\mathbb{R}^n_+} \mathcal{Q}(\zeta', \zeta_n - 1)\, \zeta_n^{-a-1/p}\, d\zeta < \infty. \tag{3.1}$$

Then the operator

$$Qf(x) := x_n^{-n} \int_{\mathbb{R}^n_+} \mathcal{Q}\Big(\frac{y-x}{x_n}\Big) f(y)\, dy, \quad x = (x', x_n) \in \mathbb{R}^n_+, \tag{3.2}$$

initially defined on functions $f \in L_p(\mathbb{R}^n_+)$ with compact support in $\mathbb{R}^n_+$, can be extended by continuity to an operator acting from $L_p(\mathbb{R}^n_+, x_n^{ap}\,dx)$ into itself, with the norm satisfying

$$\|Q\|_{\mathfrak{L}(L_p(\mathbb{R}^n_+,\, x_n^{ap}dx))} \le \int_{\mathbb{R}^n_+} \mathcal{Q}(\zeta', \zeta_n - 1)\, \zeta_n^{-a-1/p}\, d\zeta. \tag{3.3}$$

Proof. Introducing the new variable $\zeta := (x_n^{-1}(y' - x'), x_n^{-1} y_n) \in \mathbb{R}^n_+$, we may write

$$|Qf(x)| \le \int_{\mathbb{R}^n_+} \mathcal{Q}(\zeta', \zeta_n - 1) |f(x' + x_n\zeta', x_n\zeta_n)| d\zeta, \quad \forall\, x \in \mathbb{R}^n_+. \tag{3.4}$$

Then, by Minkowski's inequality,

$$\|Qf\|_{L_p(\mathbb{R}^n_+,x_n^{ap}\,dx)} \leq \int_{\mathbb{R}^n_+} \mathcal{Q}(\zeta',\zeta_n-1)\Big(\int_{\mathbb{R}^n_+} x_n^{ap}|f(x'+x_n\zeta',x_n\zeta_n)|^p\,dx\Big)^{1/p} d\zeta$$
$$=\Big(\int_{\mathbb{R}^n_+} \mathcal{Q}(\zeta',\zeta_n-1)\,\zeta_n^{-a-1/p}\,d\zeta\Big)\|f\|_{L_p(\mathbb{R}^n_+,x_n^{ap}\,dx)}, \tag{3.5}$$

as desired. □

Recall that $\bar{y} := (y',-y_n)$ if $y=(y',y_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$.

Corollary 3.2. *Consider*

$$Rf(x) := \int_{\mathbb{R}^n_+} \frac{\log\big(\frac{|x-y|}{x_n}+2\big)}{|x-\bar{y}|^n} f(y)\,dy, \qquad x=(x',x_n)\in\mathbb{R}^n_+. \tag{3.6}$$

Then for each $1<p<\infty$ *and each* $a\in(-1/p,1-1/p)$ *the operator* R *is bounded from* $L_p(\mathbb{R}^n_+,\,x_n^{ap}\,dx)$ *into itself. Moreover, there exists* $c(n)$*, independent of* a*,* p *and* $s:=1-a-1/p$*, for which*

$$\|R\|_{\mathfrak{L}(L_p(\mathbb{R}^n_+,\,x_n^{ap}\,dx))} \leq \frac{c(n)\,p^2}{(pa+1)(p(1-a)-1)} = \frac{c(n)}{s(1-s)}. \tag{3.7}$$

Proof. The result follows from Proposition 3.1 with

$$\mathcal{Q}(\zeta) := \frac{\log\,(|\zeta|+2)}{(|\zeta|^2+1)^{n/2}}, \tag{3.8}$$

and from the obvious inequality $2|x-\bar{y}|^2 \geq |x-y|^2+x_n^2$. □

Let us note here that Corollary 3.2 immediately yields the following.

Corollary 3.3. *Consider*

$$Kf(x) := \int_{\mathbb{R}^n_+} \frac{f(y)}{|x-\bar{y}|^n}\,dy, \qquad x\in\mathbb{R}^n_+. \tag{3.9}$$

Then for each $1<p<\infty$ *and* $a\in(-1/p,1-1/p)$ *the operator* K *is bounded from* $L_p(\mathbb{R}^n_+,\,x_n^{ap}\,dx)$ *into itself. In addition, there exists* $c(n)$*, independent of* a*,* p *and* $s:=1-a-1/p$*, such that*

$$\|K\|_{\mathfrak{L}(L_p(\mathbb{R}^n_+,\,x_n^{ap}\,dx))} \leq \frac{c(n)\,p^2}{(pa+1)(p(1-a)-1)} = \frac{c(n)}{s(1-s)}. \tag{3.10}$$

Recall that the barred integral stands for the mean value (taken in the integral sense).

Lemma 3.4. *Assume that* $1 < p < \infty$, $a \in (-1/p, 1-1/p)$, *and recall the operator* K *introduced in* (3.9). *Further, consider a nonnegative, measurable function* w *defined on* $\mathbb{R}^n_+$ *and fix a family of balls* $\mathcal{F}$ *which form a Whitney covering of* $\mathbb{R}^n_+$. *Then the norm of* wK *as an operator from* $L_p(\mathbb{R}^n_+, x_n^{ap}\,dx)$ *into itself is equivalent to*

$$\sup_{B\in\mathcal{F}} \fint_B w(y)^p\,dy. \tag{3.11}$$

Furthermore, there exists $c(n)$, *independent of* w, p, a *and* $s := 1-a-1/p$, *such that*

$$\|w\,K\|_{\mathfrak{L}(L_p(\mathbb{R}^n_+,\,x_n^{ap}\,dx))} \le \frac{c(n)}{s(1-s)} \sup_{B\in\mathcal{F}} \left(\fint_B w(y)^p\,dy\right)^{1/p}. \tag{3.12}$$

Proof. Fix $f \ge 0$ and denote by $|B|$ the Euclidean volume of B. Sobolev's embedding theorem allows us to write

$$\|Kf\|^p_{L_\infty(B)} \le c(n)\,|B|^{-1}\sum_{j=0}^{n} |B|^{jp/n}\|\nabla_j Kf\|^p_{L_p(B)}, \quad \forall\, B\in\mathcal{F}. \tag{3.13}$$

Hence,

$$\int_{\mathbb{R}^n_+} |x_n^a w(x)(Kf)(x)|^p\,dx \le c(n)\ \sup_{B\in\mathcal{F}} \fint_B w(y)^p\,dy \int_{\mathbb{R}^n_+} x_n^{pa} \sum_{0\le j\le l} x_n^{jp}|\nabla_j Kf|^p\,dx. \tag{3.14}$$

Observing that $x_n^j|\nabla_j\,Kf| \le c(n)\,Kf$ and referring to Corollary 3.3, we arrive at the required upper estimate for the norm of wK. The lower estimate is obvious. $\Box$

We momentarily pause in order to collect some definitions and set up basic notation pertaining to functions with bounded mean oscillations. Let f be a locally integrable function defined on $\mathbb{R}^n$ and define the seminorm

$$[f]_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_B \fint_B \left| f(x) - \fint_B f(y)\,dy\right| dx, \tag{3.15}$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. If f is a locally integrable function on $\mathbb{R}^n_+$, set

$$[f]_{\mathrm{BMO}(\mathbb{R}^n_+)} := \sup_{(B)} \fint_{B\cap\mathbb{R}^n_+} \left| f(x) - \fint_{B\cap\mathbb{R}^n_+} f(y)\,dy\right| dx, \tag{3.16}$$

where, this time, the supremum is taken over the collection (B) of all balls B with centers in $\overline{\mathbb{R}^n_+}$. Then the following inequalities are straightforward:

$$[f]_{\mathrm{BMO}(\mathbb{R}^n_+)} \le \sup_{(B)} \fint_{B\cap\mathbb{R}^n_+} \fint_{B\cap\mathbb{R}^n_+} |f(x)-f(y)|\,dxdy \le 2\,[f]_{\mathrm{BMO}(\mathbb{R}^n_+)}. \tag{3.17}$$

We also record here the equivalence relation

$$[f]_{\mathrm{BMO}(\mathbb{R}^n_+)} \sim [\mathrm{Ext}\, f]_{\mathrm{BMO}(\mathbb{R}^n)}, \tag{3.18}$$

where $\mathrm{Ext}\, f$ is the extension of f onto $\mathbb{R}^n$ as an even function in x_n. Finally, by $\mathrm{BMO}(\mathbb{R}^n_+)$ we denote the collection of equivalence classes, mod constants, of functions f on $\mathbb{R}^n_+$ for which $[f]_{\mathrm{BMO}(\mathbb{R}^n_+)} < \infty$.

Proposition 3.5. *Let $b \in \mathrm{BMO}(\mathbb{R}^n_+)$ and consider the operator*

$$Tf(x) := \int_{\mathbb{R}^n_+} \frac{|b(x) - b(y)|}{|x - \bar{y}|^n} f(y)\, dy, \quad x \in \mathbb{R}^n_+. \tag{3.19}$$

Then for each $p \in (1, \infty)$ and $a \in (-1/p, 1 - 1/p)$,

$$T : L_p(\mathbb{R}^n_+,\, x_n^{ap}\, dx) \longrightarrow L_p(\mathbb{R}^n_+,\, x_n^{ap}\, dx) \tag{3.20}$$

is a well-defined, bounded operator, such that if $s := 1 - a - 1/p$ then

$$\|T\|_{\mathfrak{L}(L_p(\mathbb{R}^n_+,\, x_n^{ap}\, dx))} \leq \frac{c(n)}{s(1-s)}\, [b]_{\mathrm{BMO}(\mathbb{R}^n_+)}. \tag{3.21}$$

Proof. Given $x \in \mathbb{R}^n_+$ and $r > 0$, we shall use the abbreviations

$$\bar{b}_r(x) := \fint_{B(x,r)\cap\mathbb{R}^n_+} b(y)\, dy, \quad D_r(x) := |b(x) - \bar{b}_r(x)|, \tag{3.22}$$

and make use of the integral operator

$$Sf(x) := \int_{\mathbb{R}^n_+} \frac{D_{|x-\bar{y}|}(x)}{|x - \bar{y}|^n}\, f(y)\, dy, \quad x \in \mathbb{R}^n_+, \tag{3.23}$$

as well as its adjoint S^*. For each nonnegative, measurable function f on $\mathbb{R}^n_+$ and each $x \in \mathbb{R}^n_+$,

$$\begin{aligned} Tf(x) \leq & Sf(x) + S^*f(x) + \int_{\mathbb{R}^n_+} \frac{|\bar{b}_{|x-\bar{y}|}(x) - \bar{b}_{|x-\bar{y}|}(y)|}{|x - \bar{y}|^n}\, f(y)\, dy \\ \leq & Sf(x) + S^*f(x) + c(n)\, [b]_{\mathrm{BMO}(\mathbb{R}^n_+)} Kf(x), \end{aligned} \tag{3.24}$$

where K has been introduced in (3.9). Making use of Corollary 3.3, we need to estimate only the norm of S. Obviously,

$$Sf(x) \leq D_{x_n}(x) Kf(x) + \int_{\mathbb{R}^n_+} \frac{|\bar{b}_{x_n}(x) - \bar{b}_{|x-\bar{y}|}(x)|}{|x - \bar{y}|^n}\, f(y)\, dy. \tag{3.25}$$

Setting $r = |x - \bar{y}|$ and $\rho = x_n$ in the standard inequality

$$|\bar{b}_\rho(x) - \bar{b}_r(x)| \leq c(n) \log\Big(\frac{r}{\rho} + 1\Big)[b]_{\mathrm{BMO}(\mathbb{R}^n_+)}, \tag{3.26}$$

where $r > \rho$, we arrive at

$$Sf(x) \leq D_{x_n}(x)Kf(x) + c(n)\,[b]_{BMO(\mathbb{R}^n_+)}\,Rf(x), \tag{3.27}$$

where R is defined in (3.6). Let $\mathcal{F}$ be a Whitney covering of $\mathbb{R}^n_+$ with open balls. For an arbitrary $B \in \mathcal{F}$, denote by δ the radius of B. By Lemma 3.4 with $w(x) := D_{x_n}(x)$, the norm of the operator $D_{x_n}(x)K$ does not exceed

$$\begin{aligned}\sup_{B\in\mathcal{F}}\Big(\fint_B |D_{x_n}(x)|^p\,dx\Big)^{1/p} &\leq c(n)\,\sup_{B\in\mathcal{F}}\Big(\fint_B |b(x) - \bar{b}_\delta(x)|^p\,dx\Big)^{1/p} + c(n)\,[b]_{\mathrm{BMO}(\mathbb{R}^n_+)}\\ &\leq c(n)\,[b]_{\mathrm{BMO}(\mathbb{R}^n_+)},\end{aligned} \tag{3.28}$$

by the John–Nirenberg inequality. Here we have also used the triangle inequality and the estimate (3.26) in order to replace $\bar{b}_{x_n}(x)$ in the definition of $D_{x_n}(x)$ by $\bar{b}_\delta(x)$. The intervening logarithmic factor is bounded independently of x since x_n is comparable with δ, uniformly for $x \in B$. With this estimate in hand, a reference to Corollary 3.2 gives that

$$\begin{aligned}&S : L_p(\mathbb{R}^n_+,\, x_n^{ap}\,dx) \to L_p(\mathbb{R}^n_+,\, x_n^{ap}\,dx) \text{ boundedly}\\ &\text{for each } p \in (1,\infty) \text{ and each } a \in (-1/p, 1-1/p).\end{aligned} \tag{3.29}$$

The corresponding estimate for the norm of S is implicit in the above argument. By duality, it follows that S^* enjoys the same property and, hence, the operator T is bounded on $L_p(\mathbb{R}^n_+,\, x_n^{ap}\,dx)$ for each $p \in (1,\infty)$ and $a \in (-1/p, 1-1/p)$, thanks to (3.24) and Corollary 3.3. The fact that the operator norm of T can be estimated in the desired fashion is implicit in the above reasoning. □

3.2 Preliminary estimates for singular integrals on weighted Lebesgue spaces. We need the analogue of Proposition 3.5 for the class of Mikhlin–Calderón–Zygmund singular integral operators. Let S^{n-1} stand for the unit sphere in $\mathbb{R}^n$ and recall that

$$\mathcal{S}f(x) = p.v.\int_{\mathbb{R}^n} k(x, x-y)f(y)\,dy, \quad x \in \mathbb{R}^n \tag{3.30}$$

(where $p.v.$ indicates that the integral is taken in the principal value sense) is called a Mikhlin–Calderón–Zygmund operator provided the function $k : \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}) \to \mathbb{R}$ satisfies:

(i) $k(x,\cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and, for almost each $x \in \mathbb{R}^n$,

$$\max_{|\alpha| \le 2n} \|D_z^\alpha k(x,z)\|_{L_\infty(\mathbb{R}^n \times S^{n-1})} < \infty. \tag{3.31}$$

(ii) $k(x,\lambda z) = \lambda^{-n} k(x,z)$ for each $z \in \mathbb{R}^n$ and each $\lambda \in \mathbb{R}$, $\lambda > 0$;

(iii) $\int_{S^{n-1}} k(x,\omega)\, d\omega = 0$, where $d\omega$ indicates integration with respect to $\omega \in S^{n-1}$.

It is well-known that the Mikhlin–Calderón–Zygmund operator $\mathcal{S}$ and its commutator $[\mathcal{S}, b]$ with the operator of multiplication by a function $b \in \mathrm{BMO}(\mathbb{R}^n_+)$ are bounded operators in $L_p(\mathbb{R}^n_+)$ for each $1 < p < \infty$. Then

$$\|\mathcal{S}\|_{\mathfrak{L}(L_p(\mathbb{R}^n_+))} \le c(n)\, p\, p', \quad \|\, [\mathcal{S}, b]\, \|_{\mathfrak{L}(L_p(\mathbb{R}^n_+))} \le c(n)\, p\, p'\, [b]_{\mathrm{BMO}(\mathbb{R}^n_+)}, \tag{3.32}$$

where $1/p + 1/p' = 1$ and $c(n)$ depends only on n and the quantity in (3.31). The first estimate in (3.32) goes back to the work of A. Calderón and A. Zygmund (see also the comment on p. 22 of [53] regarding the dependence on the parameter p of the constants involved). The second estimate in (3.32) was originally proved for convolution-type operators by R. Coifman, R. Rochberg and G. Weiss in [12] and a standard expansion in spherical harmonics allows one to extend this result to the case of operators with variable-kernels of the type considered above.

We wish to extend (3.32) to the case when the Lebesgue measure is replaced by $x_n^{ap}\, dx$, with $1 < p < \infty$ and $a \in (-1/p, 1 - 1/p)$. Incidentally, $a \in (-1/p, 1 - 1/p)$ corresponds precisely to the range of a's for which $w(x) := x_n^{ap}$ is a weight in Muckenhoupt's A_p class, although here we prefer to give a direct, elementary proof.

Proposition 3.6. *Retain the above conventions and hypotheses. Then the operator $\mathcal{S}$ and its commutator $[\mathcal{S}, b]$ with a function $b \in \mathrm{BMO}(\mathbb{R}^n_+)$ are bounded when acting from $L_p(\mathbb{R}^n_+, x_n^{ap}\, dx)$ into itself for each $p \in (1,\infty)$ and $a \in (-1/p, 1 - 1/p)$. Then, with $s := 1 - a - 1/p$ and $1/p + 1/p' = 1$,*

$$\|\mathcal{S}\|_{\mathfrak{L}(L_p(\mathbb{R}^n_+,\, x_n^{ap}\, dx))} \le c(n)\Big(p\, p' + \frac{1}{s(1-s)}\Big), \tag{3.33}$$

$$\|\, [\mathcal{S}, b]\, \|_{\mathfrak{L}(L_p(\mathbb{R}^n_+,\, x_n^{ap}\, dx))} \le c(n)\Big(p\, p' + \frac{1}{s(1-s)}\Big)\, [b]_{\mathrm{BMO}(\mathbb{R}^n_+)}. \tag{3.34}$$

Proof. Let χ_j be the characteristic function of the layer $2^{j/2} < x_n \le 2^{1+j/2}$, $j = 0, \pm 1, \ldots$, so that $\sum_{j \in \mathbb{Z}} \chi_j = 2$. We then write $\mathcal{S}$ as the sum $\mathcal{S}_1 + \mathcal{S}_2$, where

$$\mathcal{S}_1 := \frac{1}{4} \sum_{|j-k| \le 3} \chi_j \mathcal{S} \chi_k. \tag{3.35}$$

The following chain of inequalities is evident:

$$\begin{aligned}\|\mathcal{S}_1 f\|_{L_p(\mathbb{R}^n_+,\, x_n^{ap}\, dx)} &\leq \Big(\sum_j \int_{\mathbb{R}^n_+} \chi_j(x) \Big|\mathcal{S}\Big(\sum_{|k-j|\leq 3} \chi_k f\Big)(x)\Big|^p x_n^{ap}\, dx\Big)^{1/p} \\ &\leq c(n)\Big(\sum_j \int_{\mathbb{R}^n_+} \Big|\mathcal{S}\Big(\sum_{|k-j|\leq 3} \chi_k 2^{ja/2} f\Big)(x)\Big|^p dx\Big)^{1/p}. \end{aligned} \tag{3.36}$$

In concert with the first estimate in (3.32), this entails

$$\begin{aligned}\|\mathcal{S}_1 f\|_{L_p(\mathbb{R}^n_+,\, x_n^{ap}\, dx)} &\leq c(n)\, p\, p'\Big(\sum_j \int_{\mathbb{R}^n_+} \Big(\sum_{|k-j|\leq 3} \chi_k 2^{ja/2}|f|\Big)^p dx\Big)^{1/p} \\ &\leq c(n)\, p\, p'\Big(\int_{\mathbb{R}^n_+} |f(x)|^p\, x_n^{ap}\, dx\Big)^{1/p}, \end{aligned} \tag{3.37}$$

which is further equivalent to

$$\|\mathcal{S}_1\|_{\mathfrak{L}(L_p(\mathbb{R}^n_+,\, x_n^{ap}\, dx))} \leq c(n)\, p\, p'. \tag{3.38}$$

Applying the same argument to $[\mathcal{S}_1, b]$ and referring to (3.32), we arrive at

$$\|\, [\mathcal{S}_1, b]\, \|_{\mathfrak{L}(L_p(\mathbb{R}^n_+,\, x_n^{ap}\, dx))} \leq c(n)\, p\, p'\, [b]_{\mathrm{BMO}(\mathbb{R}^n_+)}. \tag{3.39}$$

It remains to obtain the analogues of (3.38) and (3.39) with $\mathcal{S}_2$ in place of $\mathcal{S}_1$. One can check directly that the modulus of the kernel of $\mathcal{S}_2$ does not exceed $c(n)\,|x-\bar{y}|^{-n}$ and that the modulus of the kernel of $[\mathcal{S}_2, b]$ is majorized by $c(n)\,|b(x)-b(y)|\,|x-\bar{y}|^{-n}$. Then the desired conclusions follow from Corollary 3.3 and Proposition 3.5. $\square$

3.3 BMO-type estimates for Gagliardo's extension operator. Here we shall revisit a certain operator T, extending functions defined on $\mathbb{R}^{n-1}$ into functions defined on $\mathbb{R}^n_+$, first introduced by E. Gagliardo in [20]. Fix a smooth, radial, decreasing, even, nonnegative function ζ in $\mathbb{R}^{n-1}$ such that $\zeta(t) = 0$ for $|t| \geq 1$ and

$$\int\limits_{\mathbb{R}^{n-1}} \zeta(t)\, dt = 1. \tag{3.40}$$

(A standard choice is $\zeta(t) := c\exp\left(-1/(1-|t|^2)_+\right)$ for a suitable c.) Following [20] we then define

$$(T\varphi)(x', x_n) := \int\limits_{\mathbb{R}^{n-1}} \zeta(t)\varphi(x' + x_n t)\, dt, \quad (x', x_n) \in \mathbb{R}^n_+, \tag{3.41}$$

acting on functions φ from $L_1(\mathbb{R}^{n-1}, loc)$. To get started, we note that

$$\nabla_{x'}(T\varphi)(x', x_n) = \int_{\mathbb{R}^{n-1}} \zeta(t)\nabla\varphi(x' + tx_n)\, dt, \tag{3.42}$$

$$\frac{\partial}{\partial x_n}(T\varphi)(x', x_n) = \int_{\mathbb{R}^{n-1}} \zeta(t)\, t\, \nabla\varphi(x' + tx_n)\, dt, \tag{3.43}$$

and, hence, we have the estimate

$$\|\nabla_x\, (T\varphi)\|_{L_\infty(\mathbb{R}^n_+)} \le c\, \|\nabla_{x'}\, \varphi\|_{L_\infty(\mathbb{R}^{n-1})}. \tag{3.44}$$

Refinements of (3.44) are contained in Lemmas 3.7 and 3.8 below.

Lemma 3.7. (i) *For all multi-indices α with $|\alpha| > 1$ there exists $c > 0$ such that*

$$\Big|D^\alpha_x(T\varphi)(x)\Big| \le c\, x_n^{1-|\alpha|}[\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})}, \quad \forall\, x = (x', x_n) \in \mathbb{R}^n_+. \tag{3.45}$$

(ii) *There exists $c > 0$ such that*

$$\Big|(T\varphi)(x) - \varphi(x')\Big| \le c\, x_n [\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})}, \quad \forall\, x = (x', x_n) \in \mathbb{R}^n_+. \tag{3.46}$$

Proof. Rewriting (3.43) as

$$\frac{\partial}{\partial x_n}(T\varphi)(x', x_n) = x_n^{1-n} \int_{\mathbb{R}^{n-1}} \zeta\Big(\frac{\xi - x'}{x_n}\Big)\frac{\xi - x'}{x_n}\Big(\nabla\varphi(\xi) - \fint_{|z-x'|<x_n} \nabla\varphi(z)dz\Big)d\xi \tag{3.47}$$

we obtain

$$\Big|D^\gamma_x \frac{\partial}{\partial x_n}(T\varphi)(x)\Big| \le c\, x_n^{-|\gamma|}[\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})} \tag{3.48}$$

for every non-zero multi-index γ. Furthermore, for $i = 1, \dots, n-1$, by (3.42)

$$\frac{\partial}{\partial x_i}\nabla_{x'}(T\varphi)(x) = x_n^{1-n} \int_{\mathbb{R}^{n-1}} \partial_i\zeta\Big(\frac{\xi - x'}{x_n}\Big)\Big(\nabla\varphi(\xi) - \fint_{|z-x'|<x_n} \nabla\varphi(z)dz\Big)d\xi. \tag{3.49}$$

Hence, once again

$$\Big|D^\gamma_x \frac{\partial}{\partial x_i}\nabla_{x'}(T\varphi)(x)\Big| \le c\, x_n^{-|\gamma|-1}[\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})}, \tag{3.50}$$

and the estimate claimed in (i) follows. Finally, (ii) is a simple consequence of (i) and the fact that $(T\varphi)|_{\mathbb{R}^{n-1}} = \varphi$. □

Remark 3.1. In concert with Theorem 2 on pp. 62–63 in [53], formula (3.42) yields the pointwise estimate

$$|\nabla\,(T\varphi)(x)| \leq c\,\mathcal{M}(\nabla\varphi)(x'), \quad x=(x',x_n)\in\mathbb{R}^n_+, \tag{3.51}$$

where $\mathcal{M}$ is the classical Hardy–Littlewood maximal function (cf., e.g., Chapter I in [53]). As for higher order derivatives, an inspection of the above proof reveals that

$$\Big|D^\alpha_x(T\varphi)(x)\Big| \leq c\,x_n^{1-|\alpha|}(\nabla\varphi)^{\#}(x'), \quad (x',x_n)\in\mathbb{R}^n, \tag{3.52}$$

holds for each multi-index α with $|\alpha|>1$, where $(\cdot)^{\#}$ is the Fefferman–Stein sharp maximal function (cf. [19]).

Lemma 3.8. *If* $\nabla_{x'}\varphi\in\mathrm{BMO}(\mathbb{R}^{n-1})$ *then* $\nabla(T\varphi)\in\mathrm{BMO}(\mathbb{R}^n_+)$ *and*

$$[\nabla(T\varphi)]_{\mathrm{BMO}(\mathbb{R}^n_+)} \leq c\,[\nabla_{x'}\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})}. \tag{3.53}$$

Proof. Since $(T\varphi)(x',x_n)$ is even with respect to x_n, it suffices to estimate $[\nabla_x(T\varphi)]_{\mathrm{BMO}(\mathbb{R}^n)}$. Let Q_r denote a cube with side-length r centered at the point $\eta=(\eta',\eta_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$. Also, let Q'_r be the projection of Q_r on $\mathbb{R}^{n-1}$. Clearly,

$$\nabla_{x'}(T\varphi)(x',x_n)-\nabla_{x'}\varphi(x') = x_n^{1-n}\int\limits_{\mathbb{R}^{n-1}}\zeta\Big(\frac{\xi-x'}{x_n}\Big)(\nabla\varphi(\xi)-\nabla\varphi(x'))\,d\xi. \tag{3.54}$$

Suppose that $|\eta_n|<2r$ and write

$$\begin{aligned}\int_{Q_r}\Big|\nabla_{x'}(T\varphi)(x',x_n)-\nabla_{x'}\varphi(x')\Big|\,dx &\leq c\,r^{2-n}\int_{Q'_{4r}}\int_{Q'_{4r}}|\nabla\varphi(\xi)-\nabla\varphi(z)|\,dz\,d\xi.\\ &\leq c\,r^n[\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})}.\end{aligned} \tag{3.55}$$

Therefore, for $|\eta_n|<2r$,

$$\begin{aligned}&\fint_{Q_r}\fint_{Q_r}|\nabla_{x'}T\varphi(x)-\nabla_{y'}T\varphi(y)|\,dxdy\\ &\qquad\leq 2\fint_{Q_r}|\nabla_{x'}T\varphi(x)-\nabla\varphi(x')|\,dx+\fint_{Q'_r}\fint_{Q'_r}|\nabla\varphi(x')-\nabla\varphi(y')|\,dx'dy'\\ &\qquad\leq c\,[\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})}.\end{aligned} \tag{3.56}$$

Next, consider the case when $|\eta_n|\geq 2r$ and let x and y be arbitrary points in $Q_r(\eta)$.

Then, using the generic abbreviation $\bar{f}_E := \fint_E f$, we may write

$$
\begin{aligned}
&|\nabla_{x'} T\varphi(x) - \nabla_{y'} T\varphi(y)| \\
&\qquad \le \int\limits_{\mathbb{R}^{n-1}} \Big| x_n^{1-n} \zeta\Big(\frac{\xi - x'}{x_n}\Big) - y_n^{1-n} \zeta\Big(\frac{\xi - y'}{y_n}\Big)\Big| \Big| \nabla\varphi(\xi) - \overline{\nabla\varphi}_{Q'_{2|\eta_n|}} \Big| \, d\xi \\
&\qquad \le \frac{c\, r}{|\eta_n|^n} \int_{Q'_{2|\eta_n|}} \Big| \nabla\varphi(\xi) - \overline{\nabla\varphi}_{Q'_{2|\eta_n|}} \Big| \, d\xi \\
&\qquad \le c\, [\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})}. \qquad (3.57)
\end{aligned}
$$

Consequently, for $|\eta_n| \ge 2r$,

$$
\fint_{Q_r} \fint_{Q_r} |\nabla_{x'} T\varphi(x) - \nabla_{y'} T\varphi(y)| \, dx dy \le c\, [\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})} \qquad (3.58)
$$

which, together with (3.56), gives

$$
[\nabla_{x'} T\varphi]_{\mathrm{BMO}(\mathbb{R}^n)} \le c\, [\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})}. \qquad (3.59)
$$

This inequality and (3.48) with $|\gamma| = 0$ imply (3.53). □

4 The Dirichlet problem in $\mathbb{R}^n_+$ for variable coefficient systems

4.1 The setup.

For

$$
1 < p < \infty, \quad -\frac{1}{p} < a < 1 - \frac{1}{p} \quad \text{and} \quad m \in \mathbb{N}, \qquad (4.1)
$$

we let $V_p^{m,a}(\mathbb{R}^n_+)$ denote the weighted Sobolev space associated with the norm

$$
\|u\|_{V_p^{m,a}(\mathbb{R}^n_+)} := \Big(\sum_{|\beta| \le m} \int_{\mathbb{R}^n_+} |x_n^{|\beta| - m} D^\beta u(x)|^p \, x_n^{pa} \, dx \Big)^{1/p}. \qquad (4.2)
$$

It is easily proved that $C_0^\infty(\mathbb{R}^n_+)$ is dense in $V_p^{m,a}(\mathbb{R}^n_+)$. Moreover, by the one-dimensional Hardy's inequality (see, for instance, [37], formula (1.3/1)), we have (with $s := 1 - a - 1/p$, as usual)

$$
\|u\|_{V_p^{m,a}(\mathbb{R}^n_+)} \le C\, s^{-1} \Big(\sum_{|\beta| = m} \int_{\mathbb{R}^n_+} |D^\beta u(x)|^p \, x_n^{pa} \, dx \Big)^{1/p} \quad \text{for } u \in C_0^\infty(\mathbb{R}^n_+). \qquad (4.3)
$$

The dual of $V_{p'}^{m,-a}(\mathbb{R}^n_+)$ will be denoted by $V_p^{m,a}(\mathbb{R}^n_+)$, where $1/p + 1/p' = 1$.

Consider now the operator

$$L(x, D_x)u := \sum_{|\alpha|,|\beta| \le m} D_x^\alpha (A_{\alpha\beta}(x)\, x_n^{|\alpha|+|\beta|-2m} D_x^\beta\, u) \tag{4.4}$$

where $A_{\alpha\beta}$ are $\mathbb{C}^{l\times l}$-valued functions in $L_\infty(\mathbb{R}^n_+)$. We shall use the notation $\mathring{L}(x, D_x)$ for the principal part of $L(x, D_x)$, i.e.,

$$\mathring{L}(x, D_x)u := \sum_{|\alpha|=|\beta|=m} D_x^\alpha (A_{\alpha\beta}(x)\, D_x^\beta\, u). \tag{4.5}$$

4.2 Solvability and regularity result

Lemma 4.1. *Assume that there exists $\kappa = const > 0$ such that the coercivity condition*

$$\Re \int_{\mathbb{R}^n_+} \sum_{|\alpha|=|\beta|=m} \langle A_{\alpha\beta}(x)\, D^\beta u(x),\, D^\alpha u(x)\rangle_{\mathbb{C}^l}\, dx \ge \kappa \sum_{|\gamma|=m} \|D^\gamma\, u\|^2_{L_2(\mathbb{R}^n_+)} \tag{4.6}$$

holds for all $u \in C_0^\infty(\mathbb{R}^n_+)$, and that

$$\sum_{|\alpha|=|\beta|=m} \|A_{\alpha\beta}\|_{L_\infty(\mathbb{R}^n_+)} \le \kappa^{-1}. \tag{4.7}$$

(i) *Let $p \in (1, \infty)$, $a \in (-1/p, 1 - 1/p)$, and suppose that*

$$\frac{1}{s(1-s)} \sum_{\substack{|\alpha|+|\beta|<2m \\ 0\le|\alpha|,|\beta|\le m}} \|A_{\alpha\beta}\|_{L_\infty(\mathbb{R}^n_+)} + \sum_{|\alpha|=|\beta|=m} [A_{\alpha\beta}]_{\mathrm{BMO}(\mathbb{R}^n_+)} \le \delta, \tag{4.8}$$

where $s := 1 - a - 1/p$, $1/p + 1/p' = 1$, and δ satisfies

$$\Big(p\,p' + \frac{1}{s(1-s)}\Big)\, \delta < c(n, m, \kappa) \tag{4.9}$$

with a sufficiently small constant $c(n, m, \kappa) > 0$. Then

$$L = L(x, D_x) : V_p^{m,a}(\mathbb{R}^n_+) \longrightarrow V_p^{-m,a}(\mathbb{R}^n_+) \quad \textit{isomorphically.} \tag{4.10}$$

(ii) *Let $p_i \in (1, \infty)$ and $-1/p_i < a_i < 1 - 1/p_i$, where $i = 1, 2$. Suppose that* (4.9) *holds with p_i and $s_i := 1 - a_i - 1/p_i$ in place of p and s. Then, if the function $u \in V_{p_1}^{m,a_1}(\mathbb{R}^n_+)$ is such that $Lu \in V_{p_1}^{-m,a_1}(\mathbb{R}^n_+) \cap V_{p_2}^{-m,a_2}(\mathbb{R}^n_+)$, it follows that $u \in V_{p_2}^{m,a_2}(\mathbb{R}^n_+)$.*

Proof. The fact that the operator in (4.10) is continuous is obvious. Also, the existence of a bounded inverse L^{-1} for $p = 2$ and $a = 0$ follows from (4.6) and (4.8)–(4.9) with $p = 2$, $a = 0$, which allow us to implement the Lax–Milgram lemma. We shall use the notation $\mathring{L}_y$ for the operator $\mathring{L}(y, D_x)$, corresponding to (4.5) in which the coefficients have been frozen at $y \in \mathbb{R}^n_+$, and the notation G_y for the solution operator for the Dirichlet problem for $\mathring{L}_y$ in $\mathbb{R}^n_+$ with homogeneous boundary conditions. Next, given $u \in V^{m,a}_p(\mathbb{R}^n_+)$, set $f := Lu \in V^{-m,a}_p(\mathbb{R}^n_+)$ so that

$$\begin{cases} L(x, D)u = f & \text{in } \mathbb{R}^n_+, \\ \frac{\partial^j u}{\partial x_n^j}(x', 0) = 0 & \text{on } \mathbb{R}^{n-1},\ 0 \le j \le m-1. \end{cases} \tag{4.11}$$

We may now write

$$u(x) = (G_y f)(x) - (G_y(\mathring{L} - \mathring{L}_y)u)(x) - (G_y(L - \mathring{L})u)(x), \quad x \in \mathbb{R}^n_+, \tag{4.12}$$

and aim to use (4.12) in order to express u in terms of f (cf. (4.26)–(4.27) below) via integral operators whose norms we can control. First, we claim that whenever $|\gamma| = m$, the norm of the operator

$$V^{m,a}_p(\mathbb{R}^n_+) \ni u \mapsto D^\gamma_x(G_y(\mathring{L} - \mathring{L}_y)u)(x)\Big|_{x=y} \in L_p(\mathbb{R}^n_+, y_n^{ap}\, dy) \tag{4.13}$$

does not exceed

$$C\left(pp' + \frac{1}{s(1-s)}\right) \sum_{|\alpha|=|\beta|=m} [A_{\alpha\beta}]_{\mathrm{BMO}(\mathbb{R}^n_+)}. \tag{4.14}$$

Given the hypotheses under which we operate, the expression (4.14) is therefore small if δ is small.

In what follows, we denote by $G_y(x, z)$ the integral kernel of G_y and integrate by parts in order to move derivatives of the form D^α_z with $|\alpha| = m$ from $(\mathring{L} - \mathring{L}_y)u$ onto $G_y(x, z)$ (the absence of boundary terms is due to the fact that $G_y(x, \cdot)$ satisfies homogeneous Dirichlet boundary conditions). That (4.14) bounds the norm of (4.13) can now be seen by combining Theorem 2.1 with (3.21) and Proposition 3.6. Let γ and α be multi-indices with $|\gamma| = m$, $|\alpha| \le m$ and consider the assignment

$$C^\infty_0(\mathbb{R}^n_+) \ni \Psi \mapsto \left(D^\gamma_x \int_{\mathbb{R}^n_+} G_y(x, z) D^\alpha_z \frac{\Psi(z)}{z_n^{m-|\alpha|}}\, dz\right)\Big|_{x=y}. \tag{4.15}$$

After integrating by parts, the action of this operator can be rewritten in the form

$$\left(D^\gamma_x \int_{\mathbb{R}^n_+} \left[\left(\frac{-1}{i}\frac{\partial}{\partial z_n}\right)^{m-|\alpha|} (-D_z)^\alpha G_y(x, z)\right] \Gamma_\alpha(z)\, dz\right)\Big|_{x=y}, \tag{4.16}$$

where

$$\Gamma_\alpha(z) := \begin{cases} \Psi(z), & \text{if } |\alpha| = m, \\ \frac{(-1)^{m-|\alpha|}}{(m-|\alpha|-1)!} \int_{z_n}^\infty (t - z_n)^{m-|\alpha|-1} \frac{\Psi(z', t)}{t^{m-|\alpha|}}\, dt, & \text{if } |\alpha| < m. \end{cases} \tag{4.17}$$

Using Theorem 2.1 along with (3.21) and Proposition 3.6, we may therefore conclude that

$$\Big\| \Big(D_x^\gamma \int_{\mathbb{R}^n_+} \Big[\Big(\frac{-1}{i} \frac{\partial}{\partial z_n} \Big)^{m-|\alpha|} (-D_z)^\alpha G_y(x,z) \Big] \Gamma_\alpha(z)\, dz \Big) \Big|_{x=y} \Big\|_{L_p(\mathbb{R}^n_+,\, y_n^{ap}\, dy)} \le C \Big(p\,p' + \frac{1}{s(1-s)} \Big) \|\Gamma_\alpha\|_{L_p(\mathbb{R}^n_+,\, x_n^{ap}\, dx)}. \tag{4.18}$$

On the other hand, Hardy's inequality gives

$$\|\Gamma_\alpha\|_{L_p(\mathbb{R}^n_+,\, x_n^{ap}\, dx)} \le \frac{C}{1-s} \|\Psi\|_{L_p(\mathbb{R}^n_+,\, x_n^{ap}\, dx)} \tag{4.19}$$

and, hence, the operator (4.15) can be extended to a linear mapping from $C_0^\infty(\mathbb{R}^n_+)$ to $L_p(\mathbb{R}^n_+,\, x_n^{ap}\, dx)$ with norm $\le \frac{C}{1-s}\Big(p\,p' + \frac{1}{s(1-s)} \Big)$. Next, given $u \in V_p^{m,a}(\mathbb{R}^n_+)$, we let $\Psi = \Psi_{\alpha\beta}$ in (4.15) with

$$\Psi_{\alpha\beta}(z) := z_n^{|\beta|-m} A_{\alpha\beta}\, D^\beta u(z), \quad |\alpha| + |\beta| < 2m, \tag{4.20}$$

and conclude that the norm of the operator

$$V_p^{m,a}(\mathbb{R}^n_+) \ni u \mapsto D_x^\gamma \,(G_y\,(L - \mathring{L})u)(x)\Big|_{x=y} \in L_p(\mathbb{R}^n_+, y_n^{ap}\, dy) \tag{4.21}$$

does not exceed

$$\frac{C}{1-s} \Big(p\,p' + \frac{1}{s(1-s)} \Big) \sum_{\substack{|\alpha|+|\beta|<2m \\ |\alpha|,|\beta|\le m}} \|A_{\alpha\beta}\|_{L_\infty(\mathbb{R}^n_+)}. \tag{4.22}$$

It is well-known (cf. (1.1.10/6) on p. 22 of [37]) that any $u \in V_p^{m,a}(\mathbb{R}^n_+)$ can be represented as

$$u = K\{D^\sigma u\}_{|\sigma|=m}, \tag{4.23}$$

where K is a linear operator with the property that

$$D^\alpha K : L_p(\mathbb{R}^n_+, x_n^{ap}\, dx) \longrightarrow L_p(\mathbb{R}^n_+, x_n^{ap}\, dx) \tag{4.24}$$

is bounded for every multi-index α with $|\alpha| = m$. In particular, by (4.3),

$$\|K\{D^\sigma u\}_{|\sigma|=m}\|_{V_p^{m,a}(\mathbb{R}^n_+)} \le C\, s^{-1} \|\{D^\sigma u\}_{|\sigma|=m}\|_{L_p(\mathbb{R}^n_+, x_n^{ap}\, dx)}. \tag{4.25}$$

At this stage, we transform the identity (4.12) as follows. First, we express the two u's occurring inside the Green operator G_y in the left-hand side of (4.12) as in (4.23). Second, for each $\gamma \in \mathbb{N}_0$ with $|\gamma| = m$, we apply D^γ to both sides of (4.12) and, finally, set $x = y$. The resulting identity reads

$$\{D^\gamma u\}_{|\gamma|=m} + S\{D^\sigma u\}_{|\sigma|=m} = Q\, f, \tag{4.26}$$

where Q is a bounded operator from $V_p^{-m,a}(\mathbb{R}^n_+)$ into $L_p(\mathbb{R}^n_+,\, x_n^{ap}\, dx)$ and S is a linear operator mapping $L_p(\mathbb{R}^n_+,\, x_n^{ap}\, dx)$ into itself. Furthermore, on account of (4.13)–(4.14), (4.21)–(4.22) and (4.25), we can bound $\|S\|_{\mathfrak{L}(L_p(\mathbb{R}^n_+,\, x_n^{ap}\, dx))}$ by

(4.27)
$$C\left(p\,p' + \frac{1}{s(1-s)}\right)\left(\sum_{|\alpha|=|\beta|=m}[A_{\alpha\beta}]_{\mathrm{BMO}(\mathbb{R}^n_+)} + \frac{1}{s(1-s)}\sum_{\substack{|\alpha|+|\beta|<2m\\ 0\leq|\alpha|,|\beta|\leq m}}\|A_{\alpha\beta}\|_{L^\infty(\mathbb{R}^n_+)}\right).$$

Owing to (4.8)–(4.9) and with the integral representation formula (4.26) and the bound (4.27) in hand, a Neumann series argument and standard functional analysis allow us to simultaneously settle the claims (i) and (ii) in the statement of the lemma. □

5 The Dirichlet problem in a special Lipschitz domain

In this section as well as in subsequent ones, we shall work with an unbounded domain of the form

$$G = \{X = (X', X_n) \in \mathbb{R}^n : X' \in \mathbb{R}^{n-1},\ X_n > \varphi(X')\}, \tag{5.1}$$

where $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function.

5.1 The space $\mathrm{BMO}(G)$**.** The space of functions of bounded mean oscillations in G can be introduced in a similar fashion to the case $G = \mathbb{R}^n_+$. Specifically, a locally integrable function on G belongs to the space $\mathrm{BMO}(G)$ if

$$[f]_{\mathrm{BMO}(G)} := \sup_{(B)} ⨍_{B\cap G}\Big|f(X) - ⨍_{B\cap G} f(Y)\,dY\Big|\,dX < \infty, \tag{5.2}$$

where the supremum is taken over all balls B centered at points in $\bar{G}$. Much as before,

$$[f]_{\mathrm{BMO}(G)} \sim \sup_{(B)} ⨍_{B\cap G}⨍_{B\cap G}\Big|f(X) - f(Y)\Big|\,dX\,dY. \tag{5.3}$$

This implies the equivalence relation

$$[f]_{\mathrm{BMO}(G)} \sim [f\circ\lambda]_{\mathrm{BMO}(\mathbb{R}^n_+)} \tag{5.4}$$

for each bi-Lipschitz diffeomorphism λ of $\mathbb{R}^n_+$ onto G. As consequences of definitions, we also have

$$\Big[\prod_{1\leq j\leq N} f_j\Big]_{\mathrm{BMO}(G)} \leq c\,\|f\|^{N-1}_{L_\infty(G)}[f]_{\mathrm{BMO}(G)}, \quad \text{where } f = (f_1,\dots,f_N), \tag{5.5}$$

$$[f^{-1}]_{\mathrm{BMO}(G)} \leq c\,\|f^{-1}\|^2_{L_\infty(G)}[f]_{\mathrm{BMO}(G)}. \tag{5.6}$$

5.2 A bi-Lipschitz map $\lambda : \mathbb{R}^n_+ \to G$ and its inverse. Let $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ be the Lipschitz function whose graph is ∂G and set $M := \|\nabla\varphi\|_{L_\infty(\mathbb{R}^{n-1})}$. Next, let T be the extension operator defined as in (3.41) and, for a fixed, sufficiently large constant $C > 0$, consider the Lipschitz mapping

$$\lambda : \mathbb{R}^n_+ \ni (x', x_n) \mapsto (X', X_n) \in G \tag{5.7}$$

defined by the equalities

$$X' := x', \quad X_n := C\, M\, x_n + (T\varphi)(x', x_n) \tag{5.8}$$

(see [39], §6.5.1 and an earlier, less accessible, reference [40]). The Jacobi matrix of λ is given by

$$\lambda' = \begin{pmatrix} I & 0 \\ \nabla_{x'}(T\varphi) & CM + \partial(T\varphi)/\partial x_n \end{pmatrix}, \tag{5.9}$$

where I is the identity $(n-1)\times(n-1)$ matrix. Since $|\partial(T\varphi)/\partial x_n| \le c\, M$ by (3.43), it follows that $\det \lambda' > (C - c)\, M > 0$. Thanks to (3.46) and (5.7)–(5.8) we have

$$X_n - \varphi(X') \sim x_n. \tag{5.10}$$

Also, based on (3.53) we may write

$$[\lambda']_{\mathrm{BMO}(\mathbb{R}^n_+)} \le c\, [\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})} \tag{5.11}$$

and further, by (3.44) and (3.45),

$$\|D^\alpha \lambda'(x)\|_{\mathbb{R}^{n\times n}} \le c(M)\, x_n^{-|\alpha|}[\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})}, \quad \forall\, \alpha\, :\, |\alpha| \ge 1. \tag{5.12}$$

Next, by closely mimicking the proof of Proposition 2.6 from [41] it is possible to show the existence of the inverse Lipschitz mapping $\varkappa := \lambda^{-1} : G \to \mathbb{R}^n_+$. Owing to (5.4), the inequality (5.11) implies

$$[\lambda' \circ \varkappa]_{\mathrm{BMO}(G)} \le c\, [\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})}. \tag{5.13}$$

Furthermore, (5.12) is equivalent to

$$\|(D^\alpha \lambda')(\varkappa(X))\|_{\mathbb{R}^{n\times n}} \le c(M,\alpha)(X_n - \varphi(X'))^{-|\alpha|}[\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})}, \tag{5.14}$$

whenever $|\alpha| > 0$. Since $\varkappa' = (\lambda' \circ \varkappa)^{-1}$ we obtain from (5.6) and (5.13)

$$[\varkappa']_{\mathrm{BMO}(G)} \le c\, [\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})}. \tag{5.15}$$

On the other hand, using $\varkappa' = (\lambda' \circ \varkappa)^{-1}$ and (5.14) one can prove by induction on the order of differentiation that, for all $X \in G$ and $\alpha \in \mathbb{N}_0$ with $|\alpha| > 0$,

$$\|D^\alpha \varkappa'(X)\|_{\mathbb{R}^{n\times n}} \le c(M,\alpha)\, (X_n - \varphi(X'))^{-|\alpha|}[\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})}. \tag{5.16}$$

5.3 The space $V_p^{m,a}(G)$. Analogously to $V_p^{m,a}(\mathbb{R}^n_+)$, we define the weighted Sobolev space $V_p^{m,a}(G)$ naturally associated with the norm

(5.17)
$$\|\mathcal{U}\|_{V_p^{m,a}(G)} := \Big(\sum_{|\gamma|\le m}\int_G |(X_n-\varphi(X'))^{|\gamma|-m} D^\gamma \mathcal{U}(X)|^p\,(X_n-\varphi(X'))^{pa}\,dX\Big)^{1/p}.$$

Replacing the function $X_n-\varphi(X')$ by either $\rho(X):=\operatorname{dist}(X,\partial G)$, or by the so-called regularized distance function $\rho_{\rm reg}(X)$ (defined as on pp. 170–171 of [53]), yields equivalent norms on $V_p^{m,a}(G)$. A standard localization argument involving a cut-off function vanishing near ∂G (for example, take $\eta(\rho_{\rm reg}/\varepsilon)$ where $\eta\in C_0^\infty(\mathbb{R})$ satisfies $\eta(t)=0$ for $|t|<1$ and $\eta(t)=1$ for $|t|>2$) shows that

$$C_0^\infty(G)\hookrightarrow V_p^{m,a}(G)\quad\text{densely.} \tag{5.18}$$

Next, we observe that for each $\mathcal{U}\in C_0^\infty(G)$,

(5.19)
$$C\,s\,\|\mathcal{U}\|_{V_p^{m,a}(G)} \le \Big(\sum_{|\gamma|=m}\int_G |D^\gamma\mathcal{U}(X)|^p\,(X_n-\varphi(X'))^{pa}\,dX\Big)^{1/p} \le \|\mathcal{U}\|_{V_p^{m,a}(G)},$$

where, as before, $s=1-a-1/p$. Indeed, for each multi-index γ with $|\gamma|\le m$, the one-dimensional Hardy's inequality gives

$$\begin{aligned}\int_G |(X_n-\varphi(X'))^{|\gamma|-m}D^\gamma\mathcal{U}(X)|^p\,(X_n-\varphi(X'))^{pa}\,dX\\ \le \big(C/s\big)^p\sum_{|\alpha|=m}\int_G |D^\alpha\mathcal{U}(X)|^p\,(X_n-\varphi(X'))^{pa}\,dX,\end{aligned} \tag{5.20}$$

and the first inequality in (5.19) follows readily from it. Also, the second inequality in (5.19) is a trivial consequence of (5.17). Going further, we aim to establish that

$$c_1\,\|u\|_{V_p^{m,a}(\mathbb{R}^n_+)} \le \|u\circ\varkappa\|_{V_p^{m,a}(G)} \le c_2\,\|u\|_{V_p^{m,a}(\mathbb{R}^n_+)}, \tag{5.21}$$

where c_1 and c_2 do not depend on p and s, and $\varkappa: G\to\mathbb{R}^n_+$ is the map introduced in §5.2. Clearly, it suffices to prove the upper estimate for $\|u\circ\varkappa\|_{V_p^{m,a}(G)}$ in (5.21). As a preliminary matter, we note that

$$\begin{aligned}D^\gamma\big(u(\varkappa(X))\big) &= \big((\varkappa'^*(X)\xi)^\gamma_{\xi=D}\,u\big)(\varkappa(X))\\ &+\sum_{1\le|\tau|<|\gamma|}(D^\tau u)(\varkappa(X))\sum_\sigma c_\sigma\prod_{i=1}^n\prod_j D^{\sigma_{ij}}\varkappa_i(X),\end{aligned} \tag{5.22}$$

where

$$\sigma=(\sigma_{ij}),\quad \sum_{i,j}\sigma_{ij}=\gamma,\quad |\sigma_{ij}|\ge 1,\quad \sum_{i,j}(|\sigma_{ij}|-1)=|\gamma|-|\tau|. \tag{5.23}$$

In turn, (5.22)–(5.23) and (5.16) allow us to conclude that

$$|D^\gamma\big(u(\varkappa(X))\big)| \leq c \sum_{1\leq|\tau|\leq|\gamma|} x_n^{|\tau|-|\gamma|}\,|D^\tau u(x)|, \tag{5.24}$$

which, in view of (5.10), yields the desired conclusion. Finally, if, as usual, $p' = p/(p-1)$, we set

$$V_p^{-m,a}(G) := \Big(V_{p'}^{m,-a}(G)\Big)^*. \tag{5.25}$$

5.4 Solvability and regularity result for the Dirichlet problem in the domain G. Let us consider the differential operator

$$\mathcal{L}\mathcal{U} = \mathcal{L}(X, D_X)\mathcal{U} = \sum_{|\alpha|=|\beta|=m} D^\alpha(\mathfrak{A}_{\alpha\beta}(X)\,D^\beta\mathcal{U}), \quad X \in G, \tag{5.26}$$

whose matrix-valued coefficients satisfy

$$\sum_{|\alpha|=|\beta|=m} \|\mathfrak{A}_{\alpha\beta}\|_{L_\infty(G)} \leq \kappa^{-1}. \tag{5.27}$$

This operator generates the sesquilinear form $\mathcal{L}(\cdot,\cdot) : V_p^{m,a}(G) \times V_{p'}^{m,-a}(G) \to \mathbb{C}$ where p' is the conjugate exponent of p, defined by

$$\mathcal{L}(\mathcal{U},\mathcal{V}) := \sum_{|\alpha|=|\beta|=m} \int_G \langle \mathfrak{A}_{\alpha\beta}(X)\,D^\beta\mathcal{U}(X),\, D^\alpha\mathcal{V}(X)\rangle\,dX. \tag{5.28}$$

We assume that

$$\Re\,\mathcal{L}(\mathcal{U},\mathcal{U}) \geq \kappa \sum_{|\gamma|=m} \|D^\gamma\,\mathcal{U}\|^2_{L_2(G)} \quad \text{holds for all } \mathcal{U} \in V_2^{m,0}(G). \tag{5.29}$$

Lemma 5.1. (i) *Let $p \in (1,\infty)$, $1/p + 1/p' = 1$, $-1/p < a < 1 - 1/p$, and set $s := 1 - a - 1/p$. Also, suppose that*

$$[\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})} + \sum_{|\alpha|=|\beta|=m} [\mathfrak{A}_{\alpha\beta}]_{\mathrm{BMO}(G)} \leq \delta, \tag{5.30}$$

where δ satisfies

$$\Big(p\,p' + \frac{1}{s(1-s)}\Big)\,\frac{\delta}{s(1-s)} < C(n, m, \kappa, \|\nabla\varphi\|_{L_\infty(\mathbb{R}^{n-1})}) \tag{5.31}$$

with a sufficiently small constant C, independent of p and s. Then

$$\mathcal{L}(X, D_X) : V_p^{m,a}(G) \longrightarrow V_p^{-m,a}(G) \quad \textit{isomorphically}. \tag{5.32}$$

(ii) *Suppose that $p_i \in (1,\infty)$, $-1/p_i < a_i < 1 - 1/p_i$, $i = 1, 2$. and that* (4.9) *holds with p_i and $s_i := 1 - a_i - 1/p_i$ in place of p and s. If $\mathcal{U} \in V_{p_1}^{m,a_1}(G)$ and $\mathcal{L}\mathcal{U} \in V_{p_1}^{-m,a_1}(G) \cap V_{p_2}^{-m,a_2}(G)$, then $\mathcal{U} \in V_{p_2}^{m,a_2}(G)$.*

Proof. We shall use extensively the flattening mapping λ and its inverse studied in §5.2. The assertions (i) and (ii) will follow directly from Lemma 4.1 as soon as we show that the operator

$$L(\mathcal{U}\circ\lambda) := (\mathcal{L}\mathcal{U})\circ\lambda \tag{5.33}$$

satisfies all the hypotheses in that lemma. The sesquilinear form corresponding to the operator L will be denoted by $L(u,v)$. Set $u(x) := \mathcal{U}(\lambda(x))$, $v(x) := \mathcal{V}(\lambda(x))$ and note that (5.22) implies

$$D^\beta\mathcal{U}(X) = \big((\varkappa'^*(\lambda(x))\xi)^\beta_{\xi=D}\,u\big)(x) + \sum_{1\le|\tau|<|\beta|} K_{\beta\tau}(x)\,x_n^{|\tau|-|\beta|}D^\tau u(x), \tag{5.34}$$

$$D^\alpha\mathcal{V}(X) = \big((\varkappa'^*(\lambda(x))\xi)^\alpha_{\xi=D}\,v\big)(x) + \sum_{1\le|\tau|<|\alpha|} K_{\alpha\tau}(x)\,x_n^{|\tau|-|\alpha|}D^\tau v(x), \tag{5.35}$$

where, thanks to (5.16), the coefficients $K_{\gamma\tau}$ satisfy

$$\|K_{\gamma\tau}\|_{L_\infty(\mathbb{R}^n_+)} \le c\,[\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})}. \tag{5.36}$$

Plugging (5.34) and (5.35) into the definition of $\mathcal{L}(\mathcal{U},\mathcal{V})$, we arrive at

$$\mathcal{L}(\mathcal{U},\mathcal{V}) = L_0(u,v) + \sum_{\substack{1\le|\alpha|,|\beta|\le m\\ |\alpha|+|\beta|<2m}} \int_{\mathbb{R}^n_+} \langle A_{\alpha\beta}(x)\,x_n^{|\alpha|+|\beta|-2m}D^\beta\,u(x),\ D^\alpha v(x)\rangle\,dx, \tag{5.37}$$

where

$$L_0(u,v) = \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n_+} \langle(\mathfrak{A}_{\alpha\beta}\circ\lambda)((\varkappa'^*\circ\lambda)\xi)^\beta_{\xi=D}\,u,\ ((\varkappa'^*\circ\lambda)\xi)^\alpha_{\xi=D}\,v\rangle\det\lambda'\,dx. \tag{5.38}$$

It follows from (5.34)–(5.36) that the coefficient matrices $A_{\alpha\beta}$ obey

$$\sum_{\substack{1\le|\alpha|,|\beta|\le m\\ |\alpha|+|\beta|<2m}} \|A_{\alpha\beta}\|_{L_\infty(\mathbb{R}^n_+)} \le c\kappa^{-1}\,[\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})}, \tag{5.39}$$

where c depends on m, n, and $\|\nabla\varphi\|_{L_\infty(\mathbb{R}^{n-1})}$. We can write the form $L_0(u,v)$ as

$$\sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n_+} \langle A_{\alpha\beta}(x)\,D^\beta u(x),\ D^\alpha v(x)\rangle\,dx, \tag{5.40}$$

where the coefficient matrices $A_{\alpha\beta}$ are given by

$$A_{\alpha\beta} = \det\lambda' \sum_{|\gamma|=|\tau|=m} P^{\gamma\tau}_{\alpha\beta}(\varkappa'\circ\lambda)(\mathfrak{A}_{\gamma\tau}\circ\lambda), \tag{5.41}$$

for some scalar homogeneous polynomials $P^{\gamma\tau}_{\alpha\beta}$ of the elements of the matrix $\varkappa'(\lambda(x))$ of degree $2m$. In view of (5.5)–(5.15),

$$\sum_{|\alpha|=|\beta|=m}[A_{\alpha\beta}]_{\mathrm{BMO}(\mathbb{R}^n_+)} \le c\Big(\kappa^{-1}[\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})} + \sum_{|\alpha|=|\beta|=m}[\mathfrak{A}_{\alpha\beta}]_{\mathrm{BMO}(G)}\Big), \tag{5.42}$$

where c depends on n, m, and $\|\nabla\varphi\|_{L_\infty(\mathbb{R}^{n-1})}$. By (5.39),

$$|L(u,u) - L_0(u,u)| \le c\,\delta\|u\|^2_{V_2^{m,0}(\mathbb{R}^n_+)} \tag{5.43}$$

and, therefore,

$$\Re\, L_0(u,u) \ge \Re\,\mathcal{L}(\mathcal{U},\mathcal{U}) - c\,\delta\|u\|^2_{V_2^{m,0}(\mathbb{R}^n_+)}. \tag{5.44}$$

Using (5.29) and the equivalence (cf. the discussion in §5.3)

$$\|\mathcal{U}\|_{V_2^{m,0}(G)} \sim \|u\|_{V_2^{m,0}(\mathbb{R}^n_+)}, \tag{5.45}$$

we arrive at (4.6). Thus, all conditions of Lemma 4.1 hold and the result follows. □

6 The Dirichlet problem in a bounded Lipschitz domain

6.1 Background. Let Ω be a *bounded Lipschitz domain* in $\mathbb{R}^n$, which means (cf. [53], p. 189) that there exists a finite open covering $\{\mathcal{O}_j\}_{1\le j\le N}$ of $\partial\Omega$ with the property that, for every $j\in\{1,\dots,N\}$, $\mathcal{O}_j\cap\Omega$ coincides with the portion of $\mathcal{O}_j$ lying in the over-graph of a Lipschitz function $\varphi_j:\mathbb{R}^{n-1}\to\mathbb{R}$ (where $\mathbb{R}^{n-1}\times\mathbb{R}$ is a new system of coordinates obtained from the original one via a rigid motion). We then define the *Lipschitz constant* of a bounded Lipschitz domain $\Omega\subset\mathbb{R}^n$ as

$$\inf\Big(\max\{\|\nabla\varphi_j\|_{L_\infty(\mathbb{R}^{n-1})}:\,1\le j\le N\}\Big), \tag{6.1}$$

where the infimum is taken over all possible families $\{\varphi_j\}_{1\le j\le N}$ as above. Such domains are called *minimally smooth* in E. Stein's book [53]. It is a classical result that, for a Lipschitz domain Ω, the surface measure $d\sigma$ is well-defined on $\partial\Omega$ and an outward-pointing normal vector ν exists a.e. on $\partial\Omega$.

We denote by $\rho(X)$ the distance from $X\in\mathbb{R}^n$ to $\partial\Omega$ and, for p, a and m as in (4.1), introduce the weighted Sobolev space $V_p^{m,a}(\Omega)$ naturally associated with the norm

$$\|\mathcal{U}\|_{V_p^{m,a}(\Omega)} := \Big(\sum_{0\le|\beta|\le m}\int_\Omega |\rho(X)^{|\beta|-m}D^\beta\mathcal{U}(X)|^p\,\rho(X)^{pa}\,dX\Big)^{1/p}. \tag{6.2}$$

One can check the equivalence of the norms

$$\|\mathcal{U}\|_{V_p^{m,a}(\Omega)} \sim \|\rho_{\mathrm{reg}}^a \mathcal{U}\|_{V_p^{m,0}(\Omega)}, \tag{6.3}$$

where $\rho_{\mathrm{reg}}(X)$ stands for the regularized distance from X to $\partial\Omega$ (in the sense of Theorem 2, p. 171 in [53]). Much as with (5.18), it is also easily proved that $C_0^\infty(\Omega)$ is dense in $V_p^{m,a}(\Omega)$ and that

$$\|\mathcal{U}\|_{V_p^{m,a}(\Omega)} \sim \Big(\sum_{|\beta|=m} \int_\Omega |D^\beta \mathcal{U}(X)|^p \,\rho(X)^{pa}\, dX\Big)^{1/p} \tag{6.4}$$

uniformly for $\mathcal{U} \in C_0^\infty(\Omega)$. As in (5.25), we set

$$V_p^{-m,a}(\Omega) := \Big(V_{p'}^{m,-a}(\Omega)\Big)^*. \tag{6.5}$$

Let us fix a Cartesian coordinates system and consider the differential operator

$$\mathcal{A}\mathcal{U} = \mathcal{A}(X, D_X)\mathcal{U} := \sum_{|\alpha|=|\beta|=m} D^\alpha(\mathcal{A}_{\alpha\beta}(X)\, D^\beta \mathcal{U}), \qquad X \in \Omega, \tag{6.6}$$

with measurable $l \times l$ matrix-valued coefficients. The corresponding sesquilinear form will be denoted by $\mathcal{A}(\mathcal{U}, \mathcal{V})$. Similarly to (5.27) and (5.29), we impose the conditions

$$\sum_{|\alpha|=|\beta|=m} \|\mathcal{A}_{\alpha\beta}\|_{L_\infty(\Omega)} \le \kappa^{-1} \tag{6.7}$$

and

$$\Re\, \mathcal{A}(\mathcal{U},\mathcal{U}) \ge \kappa \sum_{|\gamma|=m} \|D^\gamma \mathcal{U}\|^2_{L_2(\Omega)} \quad \text{for all } \mathcal{U} \in V_2^{m,0}(\Omega). \tag{6.8}$$

6.2 Interior regularity of solutions.

Lemma 6.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Pick two functions $\mathcal{H}, \mathcal{Z} \in C_0^\infty(\Omega)$ such that $\mathcal{H}\,\mathcal{Z} = \mathcal{H}$, and assume that, for a sufficiently small constant $c(m, n, \kappa) > 0$,*

$$\sum_{|\alpha|=|\beta|=m} [\mathcal{A}_{\alpha\beta}]_{\mathrm{BMO}(\Omega)} \le \delta, \tag{6.9}$$

where

$$\delta \le \frac{c(m,n,\kappa)}{p\,p'}. \tag{6.10}$$

If $\mathcal{U} \in W_q^m(\Omega, loc)$ for a certain $q < p$ and $\mathcal{A}\mathcal{U} \in W_p^{-m}(\Omega, loc)$, then $\mathcal{U} \in W_p^m(\Omega, loc)$ and

$$\|\mathcal{H}\mathcal{U}\|_{W_p^m(\Omega)} \le C\,(\|\mathcal{H}\,\mathcal{A}(\cdot, D)\,\mathcal{U}\|_{W_p^{-m}(\Omega)} + \|\mathcal{Z}\mathcal{U}\|_{W_q^m(\Omega)}). \tag{6.11}$$

Proof. We shall use the notation $\mathcal{A}_Y$ for the operator $\mathcal{A}(Y, D_X)$, where $Y \in \Omega$, and the notation Φ_Y for a fundamental solution of $\mathcal{A}_Y$ in $\mathbb{R}^n$. Then, with the star denoting the convolution product,

$$\mathcal{H}\mathcal{U} + \Phi_Y * (\mathcal{A} - \mathcal{A}_Y)(\mathcal{H}\mathcal{U}) = \Phi_Y * (\mathcal{H}\,\mathcal{A}\mathcal{U}) + \Phi_Y * ([\mathcal{A}, \mathcal{H}](\mathcal{Z}\,\mathcal{U})) \tag{6.12}$$

and, consequently, for each multi-index γ, $|\gamma| = m$,

$$\begin{aligned} D^\gamma(\mathcal{H}\mathcal{U}) + \sum_{|\alpha|=|\beta|=m} & D^{\alpha+\gamma}\Phi_Y * \big((\mathcal{A}_{\alpha\beta} - \mathcal{A}_{\alpha\beta}(Y))D^\beta(\mathcal{H}\mathcal{U})\big) \\ & = D^\gamma\Phi_Y * (\mathcal{H}\,\mathcal{A}\mathcal{U}) + D^\gamma\Phi_Y * ([\mathcal{A}, \mathcal{H}](\mathcal{Z}\,\mathcal{U})). \end{aligned} \tag{6.13}$$

Writing this equation at the point Y and using (3.32), we obtain

$$\begin{aligned} (1 - C\,p\,p'\delta) \sum_{|\gamma|=m} & \|D^\gamma(\mathcal{H}\mathcal{U})\|_{L_p(\Omega)} \\ & \leq C(p,\kappa)(\|\mathcal{H}\,\mathcal{A}\mathcal{U}\|_{W_p^{-m}(\Omega)} + \|[\mathcal{A}, \mathcal{H}](\mathcal{Z}\,\mathcal{U})\|_{W_p^{-m}(\Omega)}). \end{aligned} \tag{6.14}$$

Let $p' < n$. We have, for every $\mathcal{V} \in \mathring{W}_p^m(\Omega)$, the closure of $C_0^\infty(\Omega)$ in $W_p^m(\Omega)$,

$$\begin{aligned} \Big|\int_\Omega \langle [\mathcal{A}, \mathcal{H}](\mathcal{Z}\,\mathcal{U}), \mathcal{V}\rangle\, dX\Big| = & |\mathcal{A}(\mathcal{H}\mathcal{Z}\,\mathcal{U}, \mathcal{V}) - \mathcal{A}(\mathcal{Z}\,\mathcal{U}, \mathcal{H}\mathcal{V})| \\ \leq & c(\|\mathcal{Z}\,\mathcal{U}\|_{W_p^{m-1}(\Omega)}\|\mathcal{V}\|_{W_{p'}^m(\Omega)} \\ & + \|\mathcal{Z}\,\mathcal{U}\|_{W^m_{\frac{pn}{n+p}}(\Omega)}\|\mathcal{V}\|_{W^{m-1}_{\frac{p'n}{n-p'}}(\Omega)}). \end{aligned} \tag{6.15}$$

By Sobolev's theorem,

$$\|\mathcal{Z}\,\mathcal{U}\|_{W_p^{m-1}(\Omega)} \leq c\,\|\mathcal{Z}\,\mathcal{U}\|_{W^m_{\frac{pn}{n+p}}(\Omega)} \tag{6.16}$$

and

$$\|\mathcal{V}\|_{W^{m-1}_{\frac{p'n}{n-p'}}(\Omega)} \leq c\,\|\mathcal{V}\|_{W^m_{p'}(\Omega)}. \tag{6.17}$$

Therefore,

$$\Big|\int_\Omega \langle [\mathcal{A}, \mathcal{H}](\mathcal{Z}\,\mathcal{U}), \mathcal{V}\rangle\, dX\Big| \leq c\,\|\mathcal{Z}\,\mathcal{U}\|_{W^m_{\frac{pn}{n+p}}(\Omega)}\|\mathcal{V}\|_{W^m_{p'}(\Omega)}, \tag{6.18}$$

which is equivalent to the inequality

$$\|[\mathcal{A}, \mathcal{H}](\mathcal{Z}\,\mathcal{U})\|_{W_p^{-m}(\Omega)} \leq c\,\|\mathcal{Z}\,\mathcal{U}\|_{W^m_{\frac{pn}{n+p}}(\Omega)}. \tag{6.19}$$

In the case $p' \geq n$, the same argument leads to a similar inequality, where $pn/(n+p)$ is replaced by $1+\varepsilon$ with an arbitrary $\varepsilon > 0$ for $p' > n$ and $\varepsilon = 0$ for $p' = n$. Now, (6.11) follows from (6.14) if $p' \geq n$ and $p' < n$, $q \geq pn/(n+p)$. In the remaining case the goal is achieved by iterating this argument finitely many times. $\square$

Corollary 6.2. *Let $p \geq 2$ and suppose that* (6.9) *and* (6.10) *hold. If $\mathcal{U} \in W_2^m(\Omega, loc)$ and $\mathcal{A}\mathcal{U} \in W_p^{-m}(\Omega, loc)$, then $\mathcal{U} \in W_p^m(\Omega, loc)$ and*

$$\|\mathcal{H}\,\mathcal{U}\|_{W_p^m(\Omega)} \leq C\,(\|\mathcal{Z}\,\mathcal{A}(\cdot, D)\,\mathcal{U}\|_{W_p^{-m}(\Omega)} + \|\mathcal{Z}\,\mathcal{U}\|_{W_2^{m-1}(\Omega)}). \tag{6.20}$$

Proof. Let $\mathcal{Z}_0$ denote a real-valued function in $C_0^\infty(\Omega)$ such that $\mathcal{H}\mathcal{Z}_0 = \mathcal{H}$ and $\mathcal{Z}_0\mathcal{Z} = \mathcal{Z}_0$. By (6.11),

$$\|\mathcal{H}\,\mathcal{U}\|_{W_p^m(\Omega)} \leq C\,(\|\mathcal{H}\,\mathcal{A}(\cdot, D)\,\mathcal{U}\|_{W_p^{-m}(\Omega)} + \|\mathcal{Z}_0\,\mathcal{U}\|_{W_2^m(\Omega)}) \tag{6.21}$$

and it follows from (6.8) that

$$\|\mathcal{Z}_0\,\mathcal{U}\|^2_{W_2^m(\Omega)} \leq c\,\kappa^{-1}\Re\,\mathcal{A}(\mathcal{Z}_0\,\mathcal{U}, \mathcal{Z}_0\,\mathcal{U}). \tag{6.22}$$

Furthermore,

$$|\mathcal{A}(\mathcal{Z}_0\,\mathcal{U}, \mathcal{Z}_0\,\mathcal{U}) - \mathcal{A}(\mathcal{U}, \mathcal{Z}_0^2\,\mathcal{U})| \leq c\,\kappa^{-1}\|\mathcal{Z}\,\mathcal{U}\|_{W_2^{m-1}(\Omega)}\,\|\mathcal{Z}_0\,\mathcal{U}\|_{W_2^m(\Omega)}. \tag{6.23}$$

Hence

$$\begin{aligned}&\|\mathcal{Z}_0\,\mathcal{U}\|^2_{W_2^m(\Omega)} \\ &\quad\leq c\,\kappa^{-1}(\|\mathcal{Z}\,\mathcal{A}\mathcal{U}\|_{W_2^{-m}(\Omega)}\,\|\mathcal{Z}_0^2\,\mathcal{U}\|_{W_2^m(\Omega)} + \kappa^{-1}\|\mathcal{Z}\,\mathcal{U}\|_{W_2^{m-1}(\Omega)}\,\|\mathcal{Z}_0\,\mathcal{U}\|_{W_2^m(\Omega)})\end{aligned} \tag{6.24}$$

and, therefore,

$$\|\mathcal{Z}_0\,\mathcal{U}\|_{W_2^m(\Omega)} \leq c\,\kappa^{-1}(\|\mathcal{Z}\,\mathcal{A}\mathcal{U}\|_{W_2^{-m}(\Omega)} + \kappa^{-1}\|\mathcal{Z}\,\mathcal{U}\|_{W_2^{m-1}(\Omega)}). \tag{6.25}$$

Combining this inequality with (6.21) we arrive at (6.20). □

6.3 Invertibility of $\mathcal{A} : V_p^{m,a}(\Omega) \longrightarrow V_p^{-m,a}(\Omega)$. Recall the quantities defined in (1.10)–(1.11).

Theorem 6.3. *Let $1 < p < \infty$, $0 < s < 1$, and set $a = 1-s-1/p$. Furthermore, let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Suppose that the differential operator $\mathcal{A}$ is as in §6.1 and that, in addition,*

$$\sum_{|\alpha|=|\beta|=m}\{\mathcal{A}_{\alpha\beta}\}_{*,\Omega} + \{\nu\}_{*,\partial\Omega} \leq \delta, \tag{6.26}$$

where

$$\Big(p\,p' + \frac{1}{s(1-s)}\Big)\frac{\delta}{s(1-s)} \leq c \tag{6.27}$$

for a sufficiently small constant $c > 0$ independent of p and s. Then the operator

$$\mathcal{A} : V_p^{m,a}(\Omega) \longrightarrow V_p^{-m,a}(\Omega) \quad \textit{isomorphically}. \tag{6.28}$$

Proof. We shall proceed in a series of steps starting with

(i) *The construction of the auxiliary domain G and operator $\mathcal{L}$.* Let ε be small enough so that

$$\sum_{|\alpha|=|\beta|=m} \fint_{B_r\cap\Omega}\fint_{B_r\cap\Omega} |\mathcal{A}_{\alpha\beta}(X)-\mathcal{A}_{\alpha\beta}(Y)|\,dX\,dY \le 2\delta \tag{6.29}$$

for all balls in $\{B_r\}_\Omega$ with radii $r<\varepsilon$ and

$$\fint_{B_r\cap\partial\Omega}\fint_{B_r\cap\partial\Omega} \Big|\nu(X)-\nu(Y)\Big|\,d\sigma_X\,d\sigma_Y \le 2\delta \tag{6.30}$$

for all balls in $\{B_r\}_{\partial\Omega}$ with radii $r<\varepsilon$. We fix a ball B_ε in $(B_\varepsilon)_{\partial\Omega}$ and assume without loss of generality that, in a suitable system of Cartesian coordinates,

$$\Omega\cap B_\varepsilon=\{X=(X',X_n)\in B_\varepsilon : X_n>\varphi(X')\} \tag{6.31}$$

for some Lipschitz function $\varphi:\mathbb{R}^{n-1}\to\mathbb{R}$. Consider now the unique cube $Q(\varepsilon)$ (relative to this system of coordinates) which is inscribed in B_ε and denote its projection onto $\mathbb{R}^{n-1}$ by $Q'(\varepsilon)$. Since $\nabla\varphi=-\nu'/\nu_n$, it follows from (6.30) that

$$\fint_{B'_r}\fint_{B'_r} \Big|\nabla\varphi(X')-\nabla\varphi(Y')\Big|\,dX'dY' \le c(n)\,\delta, \tag{6.32}$$

where $B'_r=B_r\cap\mathbb{R}^{n-1}$, $r<\varepsilon$. Let us retain the notation φ for the mirror extension of the function φ from $Q'(\varepsilon)$ onto $\mathbb{R}^{n-1}$. We extend $\mathcal{A}_{\alpha\beta}$ from $Q(\varepsilon)\cap\Omega$ onto $Q(\varepsilon)\backslash\Omega$ by setting

$$\mathcal{A}_{\alpha\beta}(X):=\mathcal{A}_{\alpha\beta}(X',-X_n+2\varphi(X')),\qquad X\in Q(\varepsilon)\backslash\Omega, \tag{6.33}$$

and we shall use the notation $\mathfrak{A}_{\alpha\beta}$ for the periodic extension of $\mathcal{A}_{\alpha\beta}$ from $Q(\varepsilon)$ onto $\mathbb{R}^n$.

Consistent with the earlier discussion in Section 5, we shall denote the special Lipschitz domain $\{X=(X',X_n):\ X'\in\mathbb{R}^{n-1},\ X_n>\varphi(X')\}$ by G. One can easily see that, owing to the $2\varepsilon n^{-1/2}$-periodicity of φ and $\mathcal{A}_{\alpha\beta}$,

$$\sum_{|\alpha|=|\beta|=m} [\mathcal{A}_{\alpha\beta}]_{\mathrm{BMO}(G)}+[\nabla\varphi]_{\mathrm{BMO}(\mathbb{R}^{n-1})}\le c(n)\,\delta. \tag{6.34}$$

With the operator $\mathcal{A}(X,D_X)$ in Ω, we associate the auxiliary operator $\mathcal{L}(X,D_X)$ in G given by (5.26).

(ii) *Uniqueness.* Assuming that $\mathcal{U}\in V_p^{m,a}(\Omega)$ satisfies $\mathcal{L}\mathcal{U}=0$ in Ω, we shall show that $\mathcal{U}\in V_2^{m,0}(\Omega)$. This will imply that $\mathcal{U}=0$, which proves the injectivity of the operator in (6.28). To this end, pick a function $\mathcal{H}\in C_0^\infty(Q(\varepsilon))$ and write

$\mathcal{L}(\mathcal{H}\mathcal{U}) = [\mathcal{L}, \mathcal{H}]\mathcal{U}$. Also, fix a small $\theta > 0$ and select a smooth function Λ on $\mathbb{R}^1_+$, which is identically 1 on $[0, 1]$ and which vanishes identically on $(2, \infty)$. Then by (ii) in Lemma 5.1,

$$\mathcal{L}(\mathcal{H}\mathcal{U}) - [\mathcal{L}, \mathcal{H}]\,(\Lambda(\rho_{\rm reg}/\theta)\mathcal{U}) \in V_2^{-m,0}(G) \cap V_p^{-m,a}(G). \tag{6.35}$$

Note that the operator

$$[\mathcal{L}, \mathcal{H}]\rho_{\rm reg}^{-1} : V_p^{m,a}(G) \longrightarrow V_p^{-m,a}(G) \tag{6.36}$$

is bounded and that the norm of the multiplier $\rho_{\rm reg}\,\Lambda(\rho_{\rm reg}/\theta)$ in $V_p^{m,a}(G)$ is $O(\theta)$. Moreover, the same is true for $p = 2$ and $a = 0$. The inclusion (6.35) can be written in the form

$$\mathcal{L}(\mathcal{H}\mathcal{U}) + \mathcal{M}(\mathcal{Z}\mathcal{U}) \in V_p^{-m,a}(G) \cap V_2^{-m,0}(G), \tag{6.37}$$

where $\mathcal{Z} \in C_0^\infty(\mathbb{R}^n)$, $\mathcal{Z}\mathcal{H} = \mathcal{H}$ and $\mathcal{M}$ is a linear operator mapping

$$V_p^{m,a}(G) \to V_p^{-m,a}(G) \quad \text{and} \quad V_2^{m,0}(G) \to V_2^{-m,0}(G) \tag{6.38}$$

with both norms of order $O(\theta)$.

Select a finite covering of $\overline{\Omega}$ by cubes $Q_j(\varepsilon)$ and let $\{\mathcal{H}_j\}$ be a smooth partition of unity subordinate to $\{Q_j(\varepsilon)\}$. Also, let $\mathcal{Z}_j \in C_0^\infty(Q_j(\varepsilon))$ be such that $\mathcal{H}_j\mathcal{Z}_j = \mathcal{H}_j$. By G_j we denote the special Lipschitz domain generated by the cube $Q_j(\varepsilon)$ as in part (i) of the present proof. The corresponding operators $\mathcal{L}$ and $\mathcal{M}$ will be denoted by $\mathcal{L}_j$ and $\mathcal{M}_j$, respectively. It follows from (6.37) that

$$\mathcal{H}_j\mathcal{U} + \sum_k (\mathcal{L}_j^{-1}\,\mathcal{M}_j\,\mathcal{Z}_j\,\mathcal{Z}_k)(\mathcal{H}_k\mathcal{U}) \in V_p^{m,a}(\Omega) \cap V_2^{m,0}(\Omega). \tag{6.39}$$

Taking into account that the norms of the matrix operator $\mathcal{L}_j\,\mathcal{M}_j\,\mathcal{Z}_j\,\mathcal{Z}_k$ in the spaces $V_p^{m,a}(\Omega)$ and $V_2^{m,0}(\Omega)$ are $O(\theta)$, we may take $\theta > 0$ small enough and obtain $\mathcal{H}_j\mathcal{U} \in V_2^{m,0}(\Omega)$, i.e., $\mathcal{U} \in V_2^{m,0}(\Omega)$. Therefore, $\mathcal{L} : V_p^{m,a}(\Omega) \to V_p^{-m,a}(\Omega)$ is injective.

(iii) *A priori estimate*. Let $p \geq 2$ and assume that $\mathcal{U} \in V_p^{m,a}(\Omega)$. Referring to Corollary 6.2 and arguing as in part (ii) of the present proof, we arrive at the equation

$$\mathcal{H}_j\mathcal{U} + \sum_k (\mathcal{L}_j^{-1}\,\mathcal{M}_j\,\mathcal{Z}_j\,\mathcal{Z}_k)(\mathcal{H}_k\mathcal{U}) = \mathcal{F}, \tag{6.40}$$

whose right-hand side satisfies

$$\|\mathcal{F}\|_{V_p^{m,a}(\Omega)} \leq c(\|\mathcal{A}\mathcal{U}\|_{V_p^{-m,a}(\Omega)} + \|\mathcal{U}\|_{W_2^{m-1}(\omega)}), \tag{6.41}$$

for some domain ω with $\overline{\omega} \subset \Omega$. Since the $V_p^{m,a}(\Omega)$-norm of the sum in (6.40) does not exceed $C\theta\|\mathcal{U}\|_{V_p^{m,a}(\Omega)}$, we obtain the estimate

$$\|\mathcal{U}\|_{V_p^{m,a}(\Omega)} \le c\,(\|\mathcal{A}\mathcal{U}\|_{V_p^{-m,a}(\Omega)} + \|\mathcal{U}\|_{W_2^{m-1}(\omega)}). \tag{6.42}$$

(iv) *End of proof.* Let $p \ge 2$. The range of the operator $\mathcal{A} : V_p^{m,a}(\Omega) \to V_p^{-m,a}(\Omega)$ is closed by (6.40) and the compactness of the restriction operator:

$$V_p^{m,a}(\Omega) \to W_2^{m-1}(\omega).$$

Since the coefficients of the adjoint operator $\mathcal{L}^*$ satisfy the same conditions as those of $\mathcal{L}$, we may conclude that $\mathcal{L}^* : V_{p'}^{m,a}(\Omega) \to V_{p'}^{-m,-a}(\Omega)$ is injective. Therefore, $\mathcal{L} : V_p^{m,a}(\Omega) \to V_p^{-m,-a}(\Omega)$ is surjective. Being also injective, $\mathcal{L}$ is isomorphic if $p \ge 2$. Thus $\mathcal{L}^*$ is isomorphic for $p' \le 2$ so $\mathcal{L}$ is isomorphic for $p \ge 2$. □

7 Traces and extensions

7.1 Higher order Besov spaces on Lipschitz surfaces. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and, for $m \in \mathbb{N}$, $1 < p < \infty$ and $-1/p < a < 1 - 1/p$, consider a new space, $W_p^{m,a}(\Omega)$, consisting of functions $\mathcal{U} \in L_p(\Omega, loc)$ with the property that $\rho^a D^\alpha \mathcal{U} \in L_p(\Omega)$ for all multi-indices α with $|\alpha| = m$. We equip $W_p^{m,a}(\Omega)$ with the norm

$$\|\mathcal{U}\|_{W_p^{m,a}(\Omega)} := \sum_{|\alpha|=m} \|D^\alpha \mathcal{U}\|_{L_p(\Omega,\,\rho(X)^{ap}\,dX)} + \|\mathcal{U}\|_{L_p(\omega)}, \tag{7.1}$$

where ω is an open non-empty domain, $\overline{\omega} \subset \Omega$. An equivalent norm is given by the expression in (1.6). We omit the standard proof of the fact that

$$C^\infty(\overline{\Omega}) \hookrightarrow W_p^{m,a}(\Omega) \quad \text{densely.} \tag{7.2}$$

Recall that for $p \in (1,\infty)$ and $s \in (0,1)$ the Besov space $B_p^s(\partial\Omega)$ is then defined via the requirement (1.7). The nature of our problem requires that we work with Besov spaces (defined on Lipschitz boundaries) which exhibit a higher order of smoothness. In accordance with [30], [53], [59], we now make the following definition.

Definition 7.1. For $p \in (1,\infty)$, $m \in \mathbb{N}$ and $s \in (0,1)$, define the (higher order) Besov space $\dot{B}_p^{m-1+s}(\partial\Omega)$ as the collection of all families $\dot{f} = \{f_\alpha\}_{|\alpha|\le m-1}$ of measurable functions defined on $\partial\Omega$, such that if

$$R_\alpha(X,Y) := f_\alpha(X) - \sum_{|\beta|\le m-1-|\alpha|} \frac{1}{\beta!}\, f_{\alpha+\beta}(Y)\,(X-Y)^\beta, \quad X,Y \in \partial\Omega, \tag{7.3}$$

for each multi-index α of length $\leq m-1$, then

$$\begin{aligned}\|\dot{f}\|_{\dot{B}_p^{m-1+s}(\partial\Omega)} &:= \sum_{|\alpha|\leq m-1} \|f_\alpha\|_{L_p(\partial\Omega)} \\ &+ \sum_{|\alpha|\leq m-1} \Big(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|R_\alpha(X,Y)|^p}{|X-Y|^{p(m-1+s-|\alpha|)+n-1}}\,d\sigma_X d\sigma_Y\Big)^{1/p} < \infty. \end{aligned} \tag{7.4}$$

It is standard to prove that $\dot{B}_p^{m-1+s}(\partial\Omega)$ is a Banach space. Also, trivially, for any constant $\kappa > 0$,

(7.5)
$$ggg \sum_{|\alpha|\leq m-1} \|f_\alpha\|_{L_p(\partial\Omega)} + \sum_{|\alpha|\leq m-1} \Bigg(\iint\limits_{\substack{X,Y\in\partial\Omega\\ |X-Y|<\kappa}} \frac{|R_\alpha(X,Y)|^p}{|X-Y|^{p(m-1+s-|\alpha|)+n-1}}\,d\sigma_X d\sigma_Y\Bigg)^{1/p}$$

is an equivalent norm on $\dot{B}_p^{m-1+s}(\partial\Omega)$.

A few notational conventions which will occasionally simplify the presentation are as follows. Given a family of functions $\dot{f} = \{f_\alpha\}_{|\alpha|\leq m-1}$ on $\partial\Omega$ and $X \in \mathbb{R}^n$, $Y, Z \in \partial\Omega$, set

$$P_\alpha(X,Y) := \sum_{|\beta|\leq m-1-|\alpha|} \frac{1}{\beta!} f_{\alpha+\beta}(Y)\,(X-Y)^\beta, \quad \forall\,\alpha\,:\,|\alpha|\leq m-1, \tag{7.6}$$

$$P(X,Y) := P_{(0,\ldots,0)}(X,Y). \tag{7.7}$$

Then

$$R_\alpha(Y,Z) = f_\alpha(Y) - P_\alpha(Y,Z), \quad \forall\,\alpha\,:\,|\alpha|\leq m-1, \tag{7.8}$$

and the following elementary identities hold for each multi-index α of length $\leq m-1$:

$$i^{|\beta|} D_X^\beta P_\alpha(X,Y) = P_{\alpha+\beta}(X,Y), \quad |\beta|\leq m-1-|\alpha|, \tag{7.9}$$

$$P_\alpha(X,Y) - P_\alpha(X,Z) = \sum_{|\beta|\leq m-1-|\alpha|} \frac{1}{\beta!} R_{\alpha+\beta}(Y,Z)(X-Y)^\beta. \tag{7.10}$$

See, e.g., p. 177 in [53] for the last formula.

We now discuss how $\dot{B}_p^{m-1+s}(\partial\Omega)$ behaves under multiplication by a smooth function with compact support.

Lemma 7.1. *For each* $p \in (1,\infty)$*,* $m \in \mathbb{N}$ *and* $s \in (0,1)$*, the Besov space* $\dot{B}_p^{m-1+s}(\partial\Omega)$ *is a module over* $C_0^\infty(\mathbb{R}^n)$*, granted that for each* $\dot{f} = \{f_\alpha\}_{|\alpha|\leq m-1} \in \dot{B}_p^{m-1+s}(\partial\Omega)$ *and* $\psi \in C_0^\infty(\mathbb{R}^n)$ *we set*

$$\psi\,\dot{f} := \Big\{ \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} i^{|\beta|}\,\mathrm{Tr}\,[D^\beta\psi] f_\gamma \Big\}_{|\alpha|\leq m-1}. \tag{7.11}$$

Proof. Let $\dot{f} = \{f_\alpha\}_{|\alpha|\leq m-1} \in \dot{B}_p^{m-1+s}(\partial\Omega)$ and $\psi \in C_0^\infty(\mathbb{R}^n)$ be arbitrary and set

$$\dot{\psi} := \Big\{ i^{|\alpha|}\,\mathrm{Tr}\,[D^\alpha\psi] \Big\}_{|\alpha|\leq m-1}. \tag{7.12}$$

Denote by $\tilde{R}_\alpha(X,Y)$ the remainder (7.3) written for $\psi\,\dot{f}$ in place of $\dot{f} = \{f_\alpha\}_{|\alpha|\leq m-1}$. Also, let $\tilde{P}_\alpha(X,Y)$ and $\tilde{P}(X,Y)$ be the polynomials defined in (7.6) and (7.7) with the components of $\dot{f}$ replaced by those of $\dot{\psi}$.

Next, fix some $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m-1$ and, for each $X, Y \in \partial\Omega$, write

$$\begin{aligned} \tilde{R}_\alpha(X,Y) = & \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} i^{|\beta|}\, D^\beta\psi(X) f_\gamma(X) \\ & - \sum_{|\delta|\leq m-1-|\alpha|} \frac{1}{\delta!}\Big[\sum_{\sigma+\tau=\alpha+\delta} \frac{(\alpha+\delta)!}{\sigma!\tau!} i^{|\sigma|}\, D^\sigma\psi(Y) f_\tau(Y)\Big](X-Y)^\delta. \end{aligned} \tag{7.13}$$

The crux of the matter is establishing that for a fixed $\kappa > 0$ there exists $C > 0$ independent of $\dot{f}$ such that

$$\iint_{\substack{X,Y\in\partial\Omega \\ |X-Y|<\kappa}} \frac{|\tilde{R}_\alpha(X,Y)|^p}{|X-Y|^{p(m-1+s-|\alpha|)+n-1}}\, d\sigma_X d\sigma_Y \leq C\|\dot{f}\|^p_{\dot{B}_p^{m-1+s}(\partial\Omega)}. \tag{7.14}$$

To get started, we note that the first sum in (7.13) can be further expanded as

$$\begin{aligned} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} i^{|\beta|}\, D^\beta\psi(X) f_\gamma(X) = & \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!}\Big[i^{|\beta|}\, D^\beta\psi(X) - \tilde{P}_\beta(X,Y)\Big] f_\gamma(X) \\ & + \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \tilde{P}_\beta(X,Y)\Big[f_\gamma(X) - P_\gamma(X,Y)\Big] \\ & + \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \tilde{P}_\beta(X,Y) P_\gamma(X,Y). \end{aligned} \tag{7.15}$$

Now, if $|X-Y| \leq \kappa$, the first sum in the right-hand side of (7.15) is pointwise dominated by $C|X-Y|^{m-|\alpha|}\sum_{|\gamma|\leq m-1}|f_\gamma(X)|$, hence, when raised to the p-th power and then multiplied by $|X-Y|^{-p(m-1+s-|\alpha|)+n-1}$, it is dominated by

$$C \sum_{|\gamma|\leq m-1} \iint_{\substack{X,Y\in\partial\Omega \\ |X-Y|<\kappa}} \frac{|f_\gamma(X)|^p}{|X-Y|^{n-1-p(1-s)}}\, d\sigma_X d\sigma_Y \leq C \sum_{|\gamma|\leq m-1} \|f_\gamma\|^p_{L_p(\partial\Omega)}, \tag{7.16}$$

which suits our goal. Similarly, for $|X-Y|\leq\kappa$, the second sum in the right-hand side of (7.15) is $\leq C\sum_{|\gamma|\leq|\alpha|}|R_\gamma(X,Y)|$, thus, as before, its contribution in the context of estimating the left-hand side of (7.14) does not exceed

$$C\sum_{|\gamma|\leq|\alpha|}\iint_{\substack{X,Y\in\partial\Omega\\|X-Y|<\kappa}}\frac{|R_\gamma(X,Y)|^p}{|X-Y|^{p(m-1+s-|\gamma|)+n-1}}\,d\sigma_X d\sigma_Y\leq C\|\dot f\|^p_{\dot B_p^{m-1+s}(\partial\Omega)}. \tag{7.17}$$

As for the last sum in the right-hand side of (7.15), we employ (7.9), (7.7), Leibniz's rule and the definitions of $P(X,Y)$, $\tilde P(X,Y)$, in order to successively transform this sum into

$$\begin{aligned}\sum_{\beta+\gamma=\alpha}&\frac{\alpha!}{\beta!\gamma!}i^{|\beta|}D_X^\beta\tilde P(X,Y)i^{|\gamma|}D^\gamma P(X,Y)\\ &=i^{|\alpha|}D_X^\alpha[\tilde P(X,Y)P(X,Y)]\\ &=i^{|\alpha|}D_X^\alpha\Bigg(\sum_{|\sigma|,|\tau|\leq m-1}\frac{1}{\sigma!\tau!}i^{|\sigma|}D^\sigma\psi(Y)f_\tau(Y)(X-Y)^{\sigma+\tau}\Bigg).\end{aligned} \tag{7.18}$$

Note that $D_X^\alpha[(X-Y)^{\sigma+\tau}]=0$ unless it is possible to select $\delta\in\mathbb{N}_0^n$ such that $\sigma+\tau=\alpha+\delta$. In the latter situation, we use

$$i^{|\alpha|}D_X^\alpha[(X-Y)^{\alpha+\delta}]=\frac{(\alpha+\delta)!}{\delta!}(X-Y)^\delta$$

and re-write the last expression in (7.18) as

$$\sum_{|\delta|\leq 2(m-1)-|\alpha|}\frac{1}{\delta!}\Bigg[\sum_{\sigma+\tau=\alpha+\delta}\frac{(\alpha+\delta)!}{\sigma!\tau!}i^{|\sigma|}D^\sigma\psi(Y)f_\tau(Y)\Bigg](X-Y)^\delta. \tag{7.19}$$

Now, the second sum in (7.13) cancels the portion from (7.19) corresponding to the case when $|\delta|\leq m-1-|\alpha|$, and the remaining terms in this sum are $\leq C|X-Y|^{m-|\alpha|}\sum_{|\gamma|\leq m-1}|f_\gamma(X)|$. Consequently, in the context of (7.14), their contribution is estimated as we did in (7.16).

This analysis establishes (7.14). Since, trivially,

$$\|(\psi\dot f)_\alpha\|_{L_p(\partial\Omega)}\leq C\sum_{|\gamma|\leq m-1}\|f_\gamma\|_{L_p(\partial\Omega)},$$

the proof of the lemma is finished. □

Typically, the previous lemma is used to localize functions $\dot f\in\dot B_p^{m-1+s}(\partial\Omega)$ in such a way that the supports of their components are contained in suitably small open subsets of $\partial\Omega$, where the boundary can be described as a graph of a real-valued Lipschitz function defined in $\mathbb{R}^{n-1}$. Such an argument, involving a smooth partition of unity, is standard and will often be used tacitly hereafter.

We next discuss a special case of the general trace result we have in mind.

Lemma 7.2. *For each* $1 < p < \infty$, $-1/p < a < 1 - 1/p$ *and* $s := 1 - a - 1/p$, *the trace operator*

$$\mathrm{Tr} : W_p^{1,a}(\Omega) \longrightarrow B_p^s(\partial\Omega) \tag{7.20}$$

is well-defined, linear, bounded, onto and has $V_p^{1,a}(\Omega)$ *as its null-space. Furthermore, there exists a linear, continuous mapping*

$$\mathcal{E} : B_p^s(\partial\Omega) \longrightarrow W_p^{1,a}(\Omega), \tag{7.21}$$

called an extension operator, such that $\mathrm{Tr} \circ \mathcal{E} = I$ *(i.e., a bounded, linear right-inverse of trace).*

Proof. By a standard argument involving a smooth partition of unity it suffices to deal with the case when Ω is the domain lying above the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. Composing with the bi-Lipschitz homeomorphism $\mathbb{R}^n_+ \ni (X', X_n) \mapsto (X', \varphi(X') + X_n) \in \Omega$ further reduces matters to the case when $\Omega = \mathbb{R}^n_+$, in which situation the claims in the lemma have been proved (in greater generality) by S. V. Uspenskiĭ in [55] (a paper preceded by the significant work of E. Gagliardo in [20] in the unweighted case). $\square$

We need to establish an analogue of Lemma 7.2 for higher smoothness spaces. While for $\Omega = \mathbb{R}^n_+$ this has been done by S. V. Uspenskiĭ in [55], the flattening argument used in Lemma 7.2 is no longer effective in this context. Let us also mention here that a result similar in spirit, valid for any Lipschitz domain Ω but with $B^{m-1+s+1/p}(\Omega)$ in place of $W_p^{m,a}(\Omega)$, has been proved by A. Jonsson and H. Wallin in [30] (in fact, in this latter context, these authors have dealt with much more general sets than Lipschitz domains). The result which serves our purposes is as follows.

Proposition 7.3. *For* $1 < p < \infty$, $-1/p < a < 1 - 1/p$, $s := 1 - a - 1/p \in (0, 1)$ *and* $m \in \mathbb{N}$, *define the* **higher order** *trace operator*

$$\mathrm{tr}_{m-1} : W_p^{m,a}(\Omega) \longrightarrow \dot{B}_p^{m-1+s}(\partial\Omega) \tag{7.22}$$

by setting

$$\mathrm{tr}_{m-1}\, \mathcal{U} := \left\{ i^{|\alpha|}\, \mathrm{Tr}\, [D^\alpha \mathcal{U}] \right\}_{|\alpha| \le m-1}, \tag{7.23}$$

where the traces in the right-hand side are taken in the sense of Lemma 7.2. Then (7.22)–(7.23) *is a well-defined, linear, bounded operator, which is onto and has*

$V_p^{m,a}(\Omega)$ *as its null-space. Moreover, it has a bounded, linear right-inverse, i.e., there exists a linear, continuous operator*

$$\mathcal{E} : \dot{B}_p^{m-1+s}(\partial\Omega) \longrightarrow W_p^{m,a}(\Omega) \tag{7.24}$$

such that

$$\dot{f} = \{f_\alpha\}_{|\alpha|\leq m-1} \in \dot{B}_p^{m-1+s}(\partial\Omega) \Longrightarrow i^{|\alpha|}\,\mathrm{Tr}\,[D^\alpha(\mathcal{E}\,\dot{f})] = f_\alpha, \quad \forall\,\alpha\,:\,|\alpha|\leq m-1. \tag{7.25}$$

In order to facilitate the exposition, we isolate a couple of preliminary results prior to the proof of Proposition 7.3. The first is analogous to a Taylor remainder formula proved by H. Whitney in [60] using a different set of compatibility conditions than (7.26) below.

Lemma 7.4. *Assume that $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function and define $\Phi : \mathbb{R}^{n-1} \to \partial\Omega \hookrightarrow \mathbb{R}^n$ by setting $\Phi(X') := (X', \varphi(X'))$ at each $X' \in \mathbb{R}^{n-1}$. Next, consider the special Lipschitz domain $\Omega := \{X = (X', X_n) \in \mathbb{R}^n : X_n > \varphi(X')\}$ and, for some fixed $m \in \mathbb{N}$, a system of sufficiently nice functions $\{f_\alpha\}_{|\alpha|\leq m-1}$ with the property that*

$$\frac{\partial}{\partial X_k}[f_\alpha(\Phi(X'))] = \sum_{j=1}^n f_{\alpha+e_j}(\Phi(X'))\partial_k\Phi_j(X'), \quad 1\leq k\leq n-1, \quad |\alpha|\leq m-2, \tag{7.26}$$

where $\{e_j\}_j$ is the canonical orthonormal basis in $\mathbb{R}^n$. Then, with $R_\alpha(X,Y)$ defined as in (7.3)*, the following identity holds for each multi-index α of length $\leq m-2$ and a.e. $X', Y' \in \mathbb{R}^{n-1}$:*

$$\begin{aligned} &R_\alpha(\Phi(X'),\Phi(Y')) \\ &\quad = \sum_{j=1}^n \sum_{|\gamma|=m-2-|\alpha|} \frac{1}{\gamma!}\int_0^1 \big[f_{\alpha+\gamma+e_j}(\Phi(Y'+t(X'-Y'))) - f_{\alpha+\gamma+e_j}(\Phi(Y'))\big] \\ &\qquad \times (\Phi(X') - \Phi(Y'+t(X'-Y')))^\gamma \nabla\Phi_j(Y'+t(X'-Y'))\cdot(X'-Y')\,dt. \end{aligned} \tag{7.27}$$

Proof. We shall prove that for any system of functions $\{f_\alpha\}_{|\alpha|\leq m-1}$ which satisfies (7.26), any multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m-2$ and any $l \in \mathbb{N}$ with $l \leq m-1-|\alpha|$, there holds

$$\begin{aligned} &f_\alpha(\Phi(X')) - \sum_{|\beta|\leq l}\frac{1}{\beta!}f_{\alpha+\beta}(\Phi(Y'))(\Phi(X')-\Phi(Y'))^\beta \\ &\quad = \sum_{j=1}^n \sum_{|\gamma|=l-1} \frac{1}{\gamma!}\int_0^1 \Big[f_{\alpha+\gamma+e_j}(\Phi(Y'+t(X'-Y'))) - f_{\alpha+\gamma+e_j}(\Phi(Y'))\Big] \\ &\qquad \times (\Phi(X') - \Phi(Y'+t(X'-Y')))^\gamma \nabla\Phi_j(Y'+t(X'-Y'))\cdot(X'-Y')\,dt. \end{aligned} \tag{7.28}$$

Clearly, (7.27) follows from (7.3) and (7.28) by taking $l := m - 1 - |\alpha|$.

In order to justify (7.28) we proceed by induction on l. Concretely, when $l = 1$ we may write, based on (7.26) and the Fundamental Theorem of Calculus,

$$\begin{aligned} f_\alpha(\Phi(X')) - f_\alpha(\Phi(Y')) &- \sum_{j=1}^n f_{\alpha+e_j}(\Phi(Y'))(\Phi_j(X') - \Phi_j(Y')) \\ &= \int_0^1 \frac{d}{dt}\Big[f_\alpha(\Phi(Y' + t(X'-Y')))\Big]\,dt \\ &\quad - \sum_{j=1}^n f_{\alpha+e_j}(\Phi(Y'))\int_0^1 \frac{d}{dt}\Big[\Phi_j(Y' + t(X'-Y'))\Big]\,dt \\ &= \sum_{j=1}^n \Bigg\{\int_0^1 \Big[f_{\alpha+e_j}(\Phi(Y' + t(X'-Y'))) - f_{\alpha+e_j}(\Phi(Y'))\Big] \\ &\qquad\qquad \times \nabla\Phi_j(Y' + t(X'-Y'))\cdot(X'-Y')\,dt\Bigg\}, \end{aligned} \tag{7.29}$$

as wanted.

To prove the version of (7.28) when l is replaced by $l + 1$ we split the sum in the left-hand side of (7.28), written for $l + 1$ in place of l, according to whether $|\beta| \le l$ or $|\beta| = l + 1$ and denote the expressions created in this fashion by S_1 and S_2, respectively. Next, based on (7.26) and the Fundamental Theorem of Calculus, we write

$$\begin{aligned} &f_{\alpha+\gamma+e_j}(\Phi(Y' + t(X'-Y'))) - f_{\alpha+\gamma+e_j}(\Phi(Y')) \\ &= \sum_{k=1}^n \int_0^t f_{\alpha+\gamma+e_j+e_k}(\Phi(Y' + \tau(X'-Y')))\nabla\Phi_k(Y' + \tau(X'-Y'))\cdot(X'-Y')\,d\tau \end{aligned} \tag{7.30}$$

and use the induction hypothesis to conclude that

$$\begin{aligned} S_1 = \sum_{j,k=1}^n \sum_{|\gamma|=l-1} \frac{1}{\gamma!}\int_0^1\int_0^t & f_{\alpha+\gamma+e_j+e_k}(\Phi(Y' + \tau(X'-Y'))) \\ &\times(\Phi(X') - \Phi(Y' + t(X'-Y')))^\gamma \nabla\Phi_j(Y' + t(X'-Y')) \\ &\times (X'-Y')\nabla\Phi_k(Y' + \tau(X'-Y'))\cdot(X'-Y')\,d\tau\,dt. \end{aligned} \tag{7.31}$$

Thus, if we set

$$F_j(t) := \Phi_j(X') - \Phi_j(Y' + t(X'-Y')), \quad 1 \le j \le n, \tag{7.32}$$

and use an elementary identity to the effect that for any $\mathbb{R}^n$-valued function $F = (F_1, \dots, F_n)$,

$$\frac{d}{dt}\Big[\frac{1}{\beta!}F(t)^\beta\Big] = \sum_{\gamma+e_j=\beta} \frac{1}{\gamma!}F(t)^\gamma F_j'(t), \quad \forall\,\beta \in \mathbb{N}_0^n, \tag{7.33}$$

we may express S_1 in the form

$$\begin{aligned} S_1 = -\sum_{k=1}^n \sum_{|\beta|=l} \int_0^1 \int_0^t & f_{\alpha+\beta+e_k}(\Phi(Y' + \tau(X'-Y'))) - f_{\alpha+\beta+e_k}(\Phi(Y'))] \\ & \times \frac{d}{dt}\Big[\frac{1}{\beta!}F(t)^\beta\Big] \nabla\Phi_k(Y' + \tau(X'-Y')) \cdot (X'-Y')\, d\tau\, dt \\ + \sum_{k=1}^n \sum_{|\beta|=l} & f_{\alpha+\beta+e_k}(\Phi(Y')) \int_0^1 \int_0^t \frac{d}{dt}\Big[\frac{1}{\beta!}F(t)^\beta\Big] F_k'(\tau)\, d\tau\, dt. \end{aligned} \tag{7.34}$$

Note that after changing the order of integration and using the Fundamental Theorem of Calculus, the first double sum above corresponds precisely to the expression in the right-hand side of (7.28) written for $l+1$ in place of l. By once again changing the order of integration and relying on (7.33), it becomes apparent that the second double sum in (7.34) is $-S_2$. Thus, $S_1 + S_2$ matches the right-hand side of (7.28) with l replaced by $l+1$, proving (7.28). □

Corollary 7.5. *Under the assumptions of Lemma 7.4, for each multi-index α of length $\le m-2$ the following estimate holds:*

$$\Big(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|R_\alpha(X,Y)|^p}{|X-Y|^{p(m-1+s-|\alpha|)+n-1}}\, d\sigma_X\, d\sigma_Y\Big)^{1/p} \le C \sum_{|\gamma|=m-1} \|f_\gamma\|_{B_p^s(\partial\Omega)}, \tag{7.35}$$

where the constant C depends only on n, p, s and $\|\nabla\varphi\|_{L_\infty(\mathbb{R}^{n-1})}$.

Proof. The identity (7.27) gives

$$\begin{aligned} & |R_\alpha(\Phi(X'), \Phi(Y'))| \\ & \quad \le C|X'-Y'|^{m-1-|\alpha|} \sum_{|\gamma|=m-1} \int_0^1 |f_\gamma(\Phi(Y' + t(X'-Y'))) - f_\gamma(\Phi(Y'))|\, dt \end{aligned} \tag{7.36}$$

for each $X', Y' \in \mathbb{R}^{n-1}$, where the constant C depends only on n and $\|\nabla\Phi\|_{L_\infty(\mathbb{R}^{n-1})}$ which, in turn, is controlled in terms of $\|\nabla\varphi\|_{L_\infty(\mathbb{R}^{n-1})}$. If we now integrate the p-th power of both sides in (7.36) for $X', Y' \in \mathbb{R}^{n-1}$, use Fubini's Theorem and make the change of variables $Z' := Y' + t(X' - Y')$, we obtain

$$
\begin{aligned}
&\int_{\partial\Omega}\int_{\partial\Omega} \frac{|R_\alpha(X,Y)|^p}{|X-Y|^{p(m-1-|\alpha|+s)+n-1}}\,d\sigma_X\,d\sigma_Y \\
&\qquad \leq C\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|R_\alpha(\Phi(X'),\Phi(Y'))|^p}{|X'-Y'|^{p(m-1-|\alpha|+s)+n-1}}\,dX'dY' \\
&\qquad \leq C\sum_{|\gamma|=m-1}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\int_0^1 \frac{|f_\gamma(\Phi(Y'+t(X'-Y'))) - f_\gamma(\Phi(Y'))|^p}{|X'-Y'|^{ps+n-1}}\,dt\,dX'dY' \\
&\qquad \leq C\sum_{|\gamma|=m-1}\int_0^1\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} t^{ps}\,\frac{|f_\gamma(\Phi(Z')) - f_\gamma(\Phi(Y'))|^p}{|Z'-Y'|^{ps+n-1}}\,dZ'dY'dt \\
&\qquad \leq C\sum_{|\gamma|=m-1}\|f_\gamma\|^p_{B^s_p(\partial\Omega)}, \qquad (7.37)
\end{aligned}
$$

since $|\nabla\Phi(X')| \sim 1$ and $|\Phi(X') - \Phi(Y')| \sim |X' - Y'|$, uniformly for $X', Y' \in \mathbb{R}^{n-1}$. □

After this preamble, we are in a position to present the

Proof of Proposition 7.3. We divide the proof into a series of steps, starting with

STEP I: *The well-definiteness of trace.* Thanks to (7.2), it suffices to study the action of the trace operator of a function $\mathcal{U} \in C^\infty(\overline{\Omega}) \hookrightarrow W^{m,a}_p(\Omega)$. If we now set

$$f_\alpha := i^{|\alpha|}\,\mathrm{Tr}\,[D^\alpha\,\mathcal{U}], \quad \forall\,\alpha\,:\,|\alpha| \leq m-1, \tag{7.38}$$

it follows from Lemma 7.2 that these trace functions are well-defined and, in fact,

$$\sum_{|\alpha|\leq m-1}\|f_\alpha\|_{B^s_p(\partial\Omega)} \leq C\|\,\mathcal{U}\|_{W^{m,a}_p(\Omega)}. \tag{7.39}$$

In order to prove that $\dot{f} := \{f_\alpha\}_{|\alpha|\leq m-1}$ belongs to $\dot{B}^{m-1+s}_p(\partial\Omega)$, let $R_\alpha(X,Y)$ be as in (7.3). Our goal is to show that for every multi-index α with $|\alpha| \leq m-1$,

$$\left(\int_{\partial\Omega}\int_{\partial\Omega}\frac{|R_\alpha(X,Y)|^p}{|X-Y|^{p(m-1-|\alpha|+s)+n-1}}\,d\sigma_X\,d\sigma_Y\right)^{1/p} \leq C\|\,\mathcal{U}\|_{W^{m,a}_p(\Omega)}. \tag{7.40}$$

To this end, we first observe that if $|\alpha| = m-1$, then the expression in the left-hand side of (7.40) is majorized by

$$C\left(\int_{\partial\Omega}\int_{\partial\Omega}|f_\alpha(X) - f_\alpha(Y)|^p|X-Y|^{-(ps+n-1)}\,d\sigma_X\,d\sigma_Y\right)^{1/p}$$

which, by (7.39), is indeed $\leq C\|\,\mathcal{U}\|_{W^{m,a}_p(\Omega)}$. To treat the case when $|\alpha| \leq m-2$ we assume that Ω is locally represented as $\{X\,:\,X_n > \varphi(X')\}$ for some Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ and, as before, set $\Phi(X') := (X', \varphi(X'))$, $X' \in \mathbb{R}^{n-1}$. Then (7.26) holds, thanks to (7.38), for every multi-index α of length $\leq m-2$. Consequently, Corollary 7.5 applies and, in concert with (7.39), yields (7.40). This proves that the operator (7.22)–(7.23) is well-defined and bounded.

STEP II: *The extension operator.* We introduce a co-boundary operator $\mathcal{E}$ which acts on $\dot{f} = \{f_\alpha\}_{|\alpha|\leq m-1} \in \dot{B}_p^{m-1+s}(\partial\Omega)$ according to

$$(\mathcal{E}\dot{f})(X) = \int_{\partial\Omega} \mathcal{K}(X,Y)\, P(X,Y)\, d\sigma_Y, \quad X \in \Omega, \tag{7.41}$$

where $P(X,Y)$ is the polynomial associated with $\dot{f}$ as in (7.7). The integral kernel $\mathcal{K}$ is assumed to satisfy

$$\int_{\partial\Omega} \mathcal{K}(X,Y)\, d\sigma_Y = 1 \quad \text{for all } X \in \Omega, \tag{7.42}$$

$$|D_X^\alpha \mathcal{K}(X,Y)| \leq c_\alpha\, \rho(X)^{1-n-|\alpha|}, \quad \forall\, X \in \Omega,\ \forall\, Y \in \partial\Omega, \tag{7.43}$$

where α is an arbitrary multi-index, and

$$\mathcal{K}(X,Y) = 0 \quad \text{if } |X-Y| \geq 2\rho(X). \tag{7.44}$$

One can take, for instance, the kernel

$$\mathcal{K}(X,Y) := \eta\left(\frac{X-Y}{\varkappa\rho_{\rm reg}(X)}\right)\left(\int_{\partial\Omega} \eta\left(\frac{X-Z}{\varkappa\rho_{\rm reg}(X)}\right) d\sigma_Z\right)^{-1}, \tag{7.45}$$

where $\eta \in C_0^\infty(B_2)$, $\eta = 1$ on B_1, $\eta \geq 0$ and $\varkappa$ is a positive constant depending on the Lipschitz constant of $\partial\Omega$. Here, as before, $\rho_{\rm reg}(X)$ stands for the regularized distance from X to $\partial\Omega$.

For each $X \in \Omega$ and $Z \in \partial\Omega$ and for every multi-index γ with $|\gamma| = m$, we then obtain

$$D^\gamma \mathcal{E}\dot{f}(X) = \sum_{\substack{\alpha+\beta=\gamma \\ |\alpha|\geq 1}} \frac{\gamma!}{\alpha!\beta!} \int_{\partial\Omega} D_X^\alpha \mathcal{K}(X,Y)\, (P_\beta(X,Y) - P_\beta(X,Z))\, d\sigma_Y. \tag{7.46}$$

Fix $\mu > 1$ and denote by $B(X,R)$ the ball of radius R centered at X. We may then estimate

$$\begin{aligned}
|D^\gamma \mathcal{E}\dot{f}(X)|^p \leq & C \sum_{\substack{\alpha+\beta=\gamma \\ |\alpha|\geq 1}} \rho(X)^{-p|\alpha|} ⨍_{B(X,\mu\rho(X))\cap\partial\Omega} |P_\beta(X,Y)-P_\beta(X,Z)|^p \, d\sigma_Y \\
\leq & C \sum_{\substack{\alpha+\beta=\gamma \\ |\alpha|\geq 1}} \sum_{|\beta|+|\delta|\leq m-1} \rho(X)^{-p|\alpha|} \\
& \times ⨍_{B(X,\mu\rho(X))\cap\partial\Omega} |R_{\delta+\beta}(Y,Z)|^p |X-Y|^{p|\delta|} d\sigma_Y, \\
\leq & C \sum_{|\tau|\leq m-1} \rho(X)^{p(|\tau|-m)} ⨍_{B(X,\mu\rho(X))\cap\partial\Omega} |R_\tau(Y,Z)|^p \, d\sigma_Y,
\end{aligned} \tag{7.47}$$

where we have used Hölder's inequality and (7.10). Averaging the extreme terms in (7.47) for Z in $B(X,\mu\rho(X))\cap\partial\Omega$, we arrive at

$$|D^\gamma \mathcal{E}\dot{f}(X)|^p \leq C \sum_{|\tau|\leq m-1} \rho(X)^{p(|\tau|-m)-2(n-1)} \iint\limits_{\substack{Y,Z\in\partial\Omega \\ |X-Y|,|X-Z|<\mu\rho(X)}} |R_\tau(Y,Z)|^p \, d\sigma_Y d\sigma_Z. \tag{7.48}$$

For each multi-index γ of length m we may then estimate

$$\begin{aligned}
& \int_\Omega |D^\gamma \mathcal{E}\dot{f}(X)|^p \rho(X)^{p(1-s)-1} \, dX \\
& \leq C \sum_{|\tau|\leq m-1} \int_{\partial\Omega}\int_{\partial\Omega} |R_\tau(Y,Z)|^p \\
& \qquad \times \Bigg(\int\limits_{\substack{X\in\Omega \\ |X-Y|,|X-Z|<\mu\rho(X)}} \rho(X)^{p(-m+1-s+|\tau|)-2n+1} \, dX \Bigg) d\sigma_Y d\sigma_Z \\
& \leq C \sum_{|\tau|\leq m-1} \int_{\partial\Omega}\int_{\partial\Omega} \frac{|R_\tau(Y,Z)|^p}{|Y-Z|^{p(m-1-|\tau|+s)+n-1}} \, d\sigma_Y d\sigma_Z \\
& \leq C \|\dot{f}\|^p_{\dot{B}_p^{m-1+s}(\partial\Omega)},
\end{aligned} \tag{7.49}$$

by (7.4), where we have used the readily verified fact that there exists $C>0$ such that

$$\int\limits_{\substack{X\in\Omega \\ |X-Y|,|X-Z|<\mu\rho(X)}} \rho(X)^{p(-m+1-s+|\tau|)-2n+1} \, dX \leq C|Y-Z|^{p(-m+1+|\tau|-s)-n+1}, \tag{7.50}$$

for any $Y,Z\in\partial\Omega$ and $\tau\in\mathbb{N}_0^n$ with $|\tau|\leq m-1$. This proves that the operator (7.41) is well-defined and bounded in the context of (7.24).

STEP III: *The right-invertibility property.* We shall now show that the operator (7.41) is a right-inverse for the trace operator (7.22), i.e., whenever $\dot{f} = \{f_\gamma\}_{|\gamma|\leq m-1} \in \dot{B}_p^{m-1+s}(\partial\Omega)$, there holds

$$f_\gamma = i^{|\gamma|}\,\mathrm{Tr}[D^\gamma \mathcal{E}\dot{f}] \tag{7.51}$$

for every multi-index γ of length $\leq m-1$. To this end, for $|\gamma|\leq m-1$ we write

$$D^\gamma\mathcal{E}\dot{f}(X) - \mathcal{E}_\gamma\dot{f}(X) = \sum_{\substack{\alpha+\beta=\gamma\\ |\alpha|\geq 1}} \frac{\gamma!}{\alpha!\beta!}\int_{\partial\Omega} D_X^\alpha\mathcal{K}(X,Y)(P_\beta(X,Y)-P_\beta(X,Z))\,d\sigma_Y, \tag{7.52}$$

where

$$\mathcal{E}_\gamma\dot{f}(X) := \int_{\partial\Omega}\mathcal{K}(X,Y)\,P_\gamma(X,Y)\,d\sigma_Y,\qquad X\in\Omega. \tag{7.53}$$

Estimating the right-hand side in (7.52) in the same way as we did with the right-hand side of (7.46), we obtain, using the boundedness of $\partial\Omega$,

$$\begin{aligned}\int_{\partial\Omega}&|D^\gamma\mathcal{E}\dot{f}(X)-\mathcal{E}_\gamma\dot{f}(X)|^p\rho(X)^{-ps-1}\,dX\\ &\leq C\sum_{|\tau|\leq m-1}\int_{\partial\Omega}\int_{\partial\Omega}\frac{|R_\tau(Y,Z)|^p}{|Y-Z|^{p(|\gamma|+s-|\tau|)+n-1}}\,d\sigma_Y d\sigma_Z\\ &\leq C\,\|\dot{f}\|^p_{\dot{B}_p^{m-1+s}(\partial\Omega)}.\end{aligned} \tag{7.54}$$

In a similar fashion, we check that

$$\begin{aligned}\int_{\partial\Omega}&|\nabla(D^\gamma\mathcal{E}\dot{f}(X)-\mathcal{E}_\gamma\dot{f}(X))|^p\rho(X)^{p-ps-1}\,dX\\ &\leq C\sum_{|\tau|\leq m-1}\int_{\partial\Omega}\int_{\partial\Omega}\frac{|R_\tau(Y,Z)|^p}{|Y-Z|^{p(|\gamma|+s-|\tau|)+n-1}}\,d\sigma_Y d\sigma_Z\\ &\leq C\,\|\dot{f}\|^p_{\dot{B}_p^{m-1+s}(\partial\Omega)}.\end{aligned} \tag{7.55}$$

The two last inequalities imply $D^\gamma\mathcal{E}\dot{f} - \mathcal{E}\dot{f}\in V_p^{1,a}(\Omega)$ and, therefore,

$$\mathrm{Tr}\,(D^\gamma\mathcal{E}\dot{f} - \mathcal{E}_\gamma\dot{f}) = 0. \tag{7.56}$$

Going further, let us set

$$Eg(X) := \int_{\partial\Omega}\mathcal{K}(X,Y)\,g(Y)\,d\sigma_Y,\qquad X\in\Omega. \tag{7.57}$$

A simpler version of the reasoning in Step II yields that E maps $B_p^s(\partial\Omega)$ boundedly into $W_p^{1,a}(\Omega)$. Also, a standard argument based on the Poisson kernel-like behavior of $\mathcal{K}(X,Y)$ shows that $\mathrm{Tr}\,Eg = g$ for each $g \in B_p^s(\partial\Omega)$. Based on (7.6)–(7.7) and (7.53) we have

$$
\begin{aligned}
&|\mathcal{E}_\gamma \dot{f}(X) - Ef_\gamma(X)|^p + \rho(X)^p |\nabla(\mathcal{E}_\gamma \dot{f}(X) - Ef_\gamma(X))|^p \\
&\qquad \le C \sum_{\substack{|\beta|\le m-1-|\gamma| \\ |\beta|\ge 1}} \rho(X)^{p|\beta|} \fint_{B(X,\mu\rho(X))\cap\partial\Omega} |f_{\gamma+\beta}(Y)|^p \, d\sigma_Y .
\end{aligned} \tag{7.58}
$$

Consequently, for an arbitrary Whitney cube $Q \subset \Omega$ of side-length l, we have

$$
\begin{aligned}
&\int_Q |\mathcal{E}_\gamma \dot{f}(X) - Ef_\gamma(X)|^p \rho(X)^{-ps-1} \, dX \\
&\qquad + \int_Q |\nabla(\mathcal{E}_\gamma \dot{f}(X) - Ef_\gamma(X))|^p \rho(X)^{p-ps-1} \, dX \\
&\qquad\qquad \le C \sum_{\substack{|\beta|\le m-1-|\gamma| \\ |\beta|\ge 1}} l^{p(|\beta|-s)} \int_{\partial\Omega\cap\varkappa Q} |f_{\gamma+\beta}(Y)|^p \, d\sigma_Y ,
\end{aligned} \tag{7.59}
$$

where $\varkappa Q$ is the concentric dilate of Q by some fixed factor $\varkappa > 1$. Summing over all cubes Q of a Whitney decomposition of Ω we find

$$
\|\mathcal{E}_\gamma \dot{f} - Ef_\gamma\|_{V_p^{1,a}(\Omega)} \le C \sum_{|\alpha|\le m-1} \|f_\alpha\|_{L_p(\partial\Omega)}, \tag{7.60}
$$

which implies

$$
\mathrm{Tr}\,(\mathcal{E}_\gamma \dot{f} - Ef_\gamma) = 0. \tag{7.61}
$$

Finally, combining (7.61), (7.56), and $\mathrm{Tr}\,Ef_\gamma = f_\gamma$, we arrive at (7.51).

STEP IV: *The kernel of the trace.* We now turn to the task of identifying the null-space of the trace operator (7.22)–(7.23). For each $k \in \mathbb{N}_0$ we denote by $\mathcal{P}_k$ the collection of all vector-valued, complex coefficient polynomials of degree $\le k$ (and agree that $\mathcal{P}_k = 0$ whenever k is a negative integer). The claim we make at this stage is that the null-space of the operator

$$
W_p^{m,a}(\Omega) \ni \mathcal{W} \mapsto \Big\{ i^{|\gamma|}\, \mathrm{Tr}\,[D^\gamma \mathcal{W}] \Big\}_{|\gamma|=m-1} \in B_p^s(\partial\Omega) \tag{7.62}
$$

is given by

$$
\mathcal{P}_{m-2} + V_p^{m,a}(\Omega). \tag{7.63}
$$

The fact that the null-space of the trace operator (7.22)–(7.23) is $V_p^{m,a}(\Omega)$ follows readily from this.

That (7.63) is included in the null-space of the operator (7.62) is obvious. The opposite inclusion amounts to showing that if $\mathcal{W} \in W_p^{m,a}(\Omega)$ is such that $\mathrm{Tr}\,[D^\gamma \mathcal{W}] = 0$ for all $\gamma \in \mathbb{N}_0$ with $|\gamma| = m-1$, then there exists $P_{m-2} \in \mathcal{P}_{m-2}$ with the property that $\mathcal{W} - P_{m-2} \in V_p^{m,a}(\Omega)$. To this end, we note that the case $m = 1$ is a consequence of (5.20) and consider next the case $m = 2$, i.e., when

$$\mathcal{W} \in W_p^{2,a}(\Omega), \quad \mathrm{Tr}\,[\nabla \mathcal{W}] = 0 \quad \text{on } \partial\Omega. \tag{7.64}$$

Assume that $\{\mathcal{W}_j\}_{j\geq 1}$ is a sequence of smooth (even polynomial) vector-valued functions in $\overline{\Omega}$, approximating $\mathcal{W}$ in $W_p^{2,a}(\Omega)$. In particular,

$$\mathrm{Tr}\,[\nabla \mathcal{W}_j] \to 0 \quad \text{in } L_p(\partial\Omega) \quad \text{as } j \to \infty. \tag{7.65}$$

If in a neighborhood of a point on $\partial\Omega$ the domain Ω is given by $\{X : X_n > \varphi(X')\}$ for some Lipschitz function φ, the following chain rule holds for the gradient of the function $w_j : B' \ni X' \mapsto \mathcal{W}_j(X', \varphi(X'))$, where B' is a $(n-1)$-dimensional ball:

(7.66)
$$\nabla w_j(X') = \Big(\nabla_{Y'} \mathcal{W}_j(Y', \varphi(X'))\Big)\Big|_{Y'=X'} + \Big(\frac{\partial}{\partial Y_n} \mathcal{W}_j(X', Y_n)\Big)\Big|_{Y_n = \varphi(X')} \nabla\varphi(X').$$

Since the sequence $\{w_j\}_{j\geq 1}$ is bounded in $L_p(B')$ and $\nabla w_j \to 0$ in $L_p(B')$, it follows that there exists a subsequence $\{j_i\}_i$ such that $w_{j_i} \to \mathit{const}$ in $L_p(B')$ (see Theorem 1.1.12/2 in [37]). Hence, $\mathrm{Tr}\,\mathcal{W} = P_0 = \mathit{const}$ on $\partial\Omega$. In view of $\mathrm{Tr}\,[\mathcal{W} - P_0] = 0$ and $\mathrm{Tr}\,[\nabla \mathcal{W}] = 0$, we may conclude that $\mathcal{W} - P_0 \in V_p^{2,a}(\Omega)$ by Hardy's inequality. The general case follows in an inductive fashion, by reasoning as before with $D^\alpha \mathcal{W}$ with $|\alpha| = m-2$ in place of $\mathcal{W}$. $\square$

We now present a short proof of (1.8), based on Proposition 7.3.

Proposition 7.6. *Assume that* $1 < p < \infty$, $s \in (0,1)$ *and* $m \in \mathbb{N}$. *Then*

$$\Big\{\{i^{|\alpha|} D^\alpha \mathcal{V}|_{\partial\Omega}\}_{|\alpha|\leq m-1} : \mathcal{V} \in C_0^\infty(\mathbb{R}^n)\Big\} \quad \textit{is dense in } \dot{B}_p^{m-1+s}(\partial\Omega) \tag{7.67}$$

and

$$\|\dot{f}\|_{\dot{B}_p^{m-1+s}(\partial\Omega)} \sim \sum_{|\alpha|\leq m-1} \|f_\alpha\|_{B_p^s(\partial\Omega)}, \tag{7.68}$$

uniformly for $\dot{f} = \{f_\alpha\}_{|\alpha|\leq m-1} \in \dot{B}_p^{m-1+s}(\partial\Omega)$. *As a consequence,* (1.8) *holds.*

Proof. That (7.67) holds is a direct consequence of (7.2) and Proposition 7.3. Next, this density result and (7.35) yield the left-pointing inequality in (7.68). As for the opposite inequality, let $\dot{f} = \{f_\alpha\}_{|\alpha|\leq m-1} \in \dot{B}_p^{m-1+s}(\partial\Omega)$ and, with

$a := 1 - s - 1/p$, consider $\mathcal{U} := \mathcal{E}(\dot{f}) \in W^{m,a}_p(\Omega)$. Then Lemma 7.2 implies that, for each multi-index α of length $\leq m-1$, the function $f_\alpha = i^{|\alpha|}\,\mathrm{Tr}\,[D^\alpha \mathcal{U}]$ belongs to $B^s_p(\partial\Omega)$, plus a naturally accompanying norm estimate. $\square$

We include one more equivalent characterization of the space $\dot{B}^{m-1+s}_p(\partial\Omega)$, in the spirit of work in [1], [45], [56]. To state it, recall that $\{e_j\}_j$ is the canonical orthonormal basis in $\mathbb{R}^n$.

Proposition 7.7. *Assume that* $1 < p < \infty$, $s \in (0,1)$ *and* $m \in \mathbb{N}$. *Then*
(7.69)

$$\{f_\alpha\}_{|\alpha|\leq m-1} \in \dot{B}^{m-1+s}_p(\partial\Omega) \Longleftrightarrow \begin{cases} f_\alpha \in B^s_p(\partial\Omega), \quad \forall\,\alpha\,:\,|\alpha| \leq m-1 \\ \qquad\qquad \textit{and} \\ (\nu_j\partial_k - \nu_k\partial_j)f_\alpha = \nu_j f_{\alpha+e_k} - \nu_k f_{\alpha+e_j} \\ \forall\,\alpha\,:\,|\alpha|\leq m-2, \quad \forall\, j,k \in \{1,\dots,n\}. \end{cases}$$

Proof. The left-to-right implication is a consequence of (7.68) and of the fact that (7.38) holds for some $\mathcal{U} \in W^{m,a}_p(\Omega)$ (cf. Proposition 7.3). As for the opposite implication, we multiply $\{f_\alpha\}_\alpha$ (as in (7.11)), with a function $\psi \in C^\infty_0(\mathbb{R}^n)$ which can be assumed to satisfy $\mathrm{supp}\,\psi\cap\partial\Omega = \mathrm{supp}\,\psi\cap\{(X',X_n):\, X_n > \varphi(X')\}$ for some Lipschitz function $\varphi:\mathbb{R}^{n-1}\to\mathbb{R}$. Furthermore, in this latter case, the compatibility conditions in (7.69) become equivalent to (7.26). Thus, given $\dot{f} := \{f_\alpha\}_{|\alpha|\leq m-1}$ whose components are as in the right-hand side of (7.69), we may proceed as in the proof of Corollary 7.5 and use the fact that $f_\alpha \in B^s_p(\partial\Omega)$ for each α to conclude that $\dot{f} \in \dot{B}^{m-1+s}_p(\partial\Omega)$. $\square$

7.2 The space of Dirichlet data and the main trace theorem. In this subsection, we study the mapping properties of the assignment

$$\mathcal{U} \mapsto \left\{\partial^k \mathcal{U}/\partial\nu^k\right\}_{0\leq k\leq m-1},$$

for $\mathcal{U} \in W^{m,a}_p(\Omega)$. In order to facilitate the subsequent discussion, for each $\ell \in \mathbb{N}$ we consider polynomial functions $P^{\alpha\beta}_{\gamma jk}$ such that
(7.70)

$$\nu^\beta D^\alpha - \nu^\alpha D^\beta = \sum_{|\gamma|=\ell-1}\sum_{j,k=1}^n P^{\alpha\beta}_{\gamma jk}(\nu)\frac{\partial}{\partial\tau_{jk}}D^\gamma, \quad \forall\,\alpha,\beta\in\mathbb{N}^n_0\,:\,|\alpha|=|\beta|=\ell,$$

where $\partial/\partial\tau_{jk}$ is the tangential derivative given by

$$\frac{\partial}{\partial\tau_{jk}} := \nu_j\frac{\partial}{\partial x_k} - \nu_k\frac{\partial}{\partial x_j}, \quad 1 \leq j,k \leq n. \tag{7.71}$$

We shall also need Sobolev spaces of order one on $\partial\Omega$. Concretely, let $\nabla_{\tan} := (\sum_j \nu_j \partial/\partial\tau_{jk})_{1\le k\le n}$ stand for the tangential gradient on the surface $\partial\Omega$ and, for $1 < p < \infty$, introduce the space

$$L_p^1(\partial\Omega) := \{f : \|f\|_{L_p^1(\partial\Omega)} := \|f\|_{L_p(\partial\Omega)} + \|\nabla_{\tan} f\|_{L_p(\partial\Omega)} < \infty\}. \tag{7.72}$$

Our main trace/extension result then reads as follows.

Theorem 7.8. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and, for $1 < p < \infty$, $-1/p < a < 1 - 1/p$, $s := 1 - a - 1/p \in (0, 1)$ and $m \in \mathbb{N}$, consider the mapping*

$$\operatorname{Tr}_{m-1} : W_p^{m,a}(\Omega) \ni \mathcal{U} \mapsto \left\{\frac{\partial^k \mathcal{U}}{\partial \nu^k}\right\}_{0\le k\le m-1} \in L_p(\partial\Omega). \tag{7.73}$$

Then its null-space is precisely $V_p^{m,a}(\Omega)$, and its image can be characterized as follows.

Given $(g_0, g_1, \dots, g_{m-1}) \in L_p(\partial\Omega)$, set

$$f_{(0,\dots,0)} := g_0 \tag{7.74}$$

and, inductively, having defined $\{f_\gamma\}_{|\gamma|\le \ell-1}$ for some $\ell \in \{1, \dots, m-1\}$, consider

$$f_\alpha := \nu^\alpha g_\ell + i \sum_{|\beta|=\ell}\ \sum_{|\gamma|=\ell-1}\ \sum_{j,k=1}^{n} \frac{\ell!}{\beta!}\, \nu^\beta\, P_{\gamma jk}^{\alpha\beta}(\nu)\, \frac{\partial f_\gamma}{\partial \tau_{jk}}, \quad \forall\, \alpha \in \mathbb{N}_0^n : |\alpha| = \ell, \tag{7.75}$$

where $P_{\gamma jk}^{\alpha\beta}$ is any fixed family of polynomials satisfying (7.70). *Then there exists $\mathcal{U} \in W_p^{m,a}(\Omega)$ such that $\operatorname{Tr}_{m-1}\mathcal{U} = \{g_k\}_{0\le k\le m-1}$ if and only if*

$$f_\alpha \in L_p^1(\partial\Omega) \quad \text{if } |\alpha| \le m-2, \quad \text{and } f_\alpha \in B_p^s(\partial\Omega) \quad \text{if } |\alpha| = m-1. \tag{7.76}$$

Furthermore, for Tr_{m-1} as in (7.73), *set*

$$\dot{W}_p^{m-1+s}(\partial\Omega) := \text{the image of the operator } \operatorname{Tr}_{m-1} \text{ in } (7.73), \tag{7.77}$$

$$\|g\|_{\dot{W}_p^{m-1+s}(\partial\Omega)} := \sum_{|\alpha|\le m-1} \|f_\alpha\|_{B_p^s(\partial\Omega)} \tag{7.78}$$

if the families $g := \{g_k\}_{0\le k\le m-1}$ and $\{f_\alpha\}_{|\alpha|\le m-1}$ are related to one another as in (7.74)–(7.75). *Then this space is independent of the particular choice of polynomials $P_{\gamma jk}^{\alpha\beta}$ satisfying* (7.70) *and the operator*

$$\operatorname{Tr}_{m-1} : W_p^{m,a}(\Omega) \longrightarrow \dot{W}_p^{m-1+s}(\partial\Omega) \tag{7.79}$$

is well-defined, bounded, and has a right-inverse. That is, there exists a bounded, linear operator

$$\operatorname{Ext} : \dot{W}_p^{m-1+s}(\partial\Omega) \longrightarrow W_p^{m,a}(\Omega) \tag{7.80}$$

such that $\operatorname{Tr}_{m-1} \circ \operatorname{Ext} = I$, the identity.

Prior to the proof of Theorem 7.8 we establish a useful approximation result, extending work done in [1] for the case $m = 2$.

Lemma 7.9. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with outward unit normal ν, and fix $1 < p < \infty$, $m \in \mathbb{N}$. Also, assume that $\{f_\alpha\}_{|\alpha|\leq m-1}$ is a family of functions satisfying*

$$f_\alpha \in L^1_p(\partial\Omega), \quad \forall\, \alpha \in \mathbb{N}_0^n \,:\, |\alpha| \leq m-1, \tag{7.81}$$

and

$$\nu_j f_{\alpha+e_k} - \nu_k f_{\alpha+e_j} = \frac{\partial f_\alpha}{\partial \tau_{jk}}, \quad \forall\, \alpha \,:\, |\alpha| \leq m-2, \quad \forall\, j,k \in \{1,\dots,n\}. \tag{7.82}$$

Then there exists a sequence of functions $F^\varepsilon \in C_0^\infty(\mathbb{R}^n)$, $\varepsilon > 0$, such that

$$i^{|\alpha|}\,\mathrm{Tr}\,[D^\alpha F^\varepsilon] \longrightarrow f_\alpha \quad \textit{in } L^1_p(\partial\Omega) \quad \textit{as } \varepsilon \to 0^+, \quad \forall\, \alpha \in \mathbb{N}_0^n \,:\, |\alpha| \leq m-1. \tag{7.83}$$

Proof. Since both the hypotheses (7.81)–(7.82) and the conclusion (7.83) are stable under multiplication by a smooth function with compact support as in (7.11), there is no loss of generality in assuming that $\Omega = \{X \,:\, X_n > \varphi(X')\}$ for some Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, and that the functions f_α have compact support. In this setting,

$$\|f_\alpha\|_{L^1_p(\partial\Omega)} \sim \|f_\alpha(\cdot\,,\varphi(\cdot))\|_{L^1_p(\mathbb{R}^{n-1})}, \quad \forall\, \alpha \in \mathbb{N}_0^n \,:\, |\alpha| \leq m-1, \tag{7.84}$$

and the compatibility conditions (7.82) can be written in the form

$$\begin{gathered} \tfrac{\partial}{\partial X_j}\Big[f_\alpha(X',\varphi(X'))\Big] = f_{\alpha+e_j}(X',\varphi(X')) + \partial_j\varphi(X') f_{\alpha+e_n}(X',\varphi(X')) \\ \text{for a.e. } X' \in \mathbb{R}^{n-1}, \quad \forall\, \alpha \in \mathbb{N}_0^n \,:\, |\alpha| \leq m-2, \quad 1 \leq j \leq n-1. \end{gathered} \tag{7.85}$$

For further reference, let us also fix $R > 0$ with the property that for each α, $\mathrm{supp}\, f_\alpha(\cdot,\varphi(\cdot)) \subseteq \{X' \in \mathbb{R}^{n-1} \,:\, |X'| \leq R\}$. Next, fix a nonnegative function $\eta \in C_0^\infty(\mathbb{R}^{n-1})$ which integrates to one and, for each $\varepsilon > 0$, set $\eta_\varepsilon(X') := \varepsilon^{1-n}\eta(X'/\varepsilon)$. Then, for each $|\alpha| \leq m-1$, $\varepsilon > 0$, and $X = (X', X_n) \in \mathbb{R}^n$, consider

$$F^\varepsilon_\alpha(X) := \Bigg(\sum_{\ell=0}^{m-1-|\alpha|} \frac{1}{\ell!}\Big[(x_n - \varphi(\cdot))^\ell f_{\alpha+\ell\, e_n}(\cdot\,,\varphi(\cdot))\Big] * \eta_\varepsilon\Bigg)(X'). \tag{7.86}$$

Clearly, for each multi-index α, the function $F^\varepsilon_\alpha(X)$ is C^∞-smooth for $X = (X', X_n)$ in $\mathbb{R}^n$, has compact support in the variable $X' \in \mathbb{R}^{n-1}$ (more precisely, $F^\varepsilon_\alpha(X', X_n) = 0$ if $|X'| > R+1$), and depends polynomially on X_n.

Based on (7.85), a straightforward calculation gives that whenever $r := m-1-|\alpha| \geq 1$ and $1 \leq j \leq n-1$,

$$\partial_j F^\varepsilon_\alpha(X) = F^\varepsilon_{\alpha+e_j}(X) + \frac{1}{(r-1)!}\Big[\Big((X_n-\varphi(\cdot))^{r-1}\partial_j\varphi(\cdot) f_{\alpha+r\,e_n}(\cdot\,,\varphi(\cdot))\Big)*\eta_\varepsilon\Big](X')$$
$$+\frac{1}{r!}\Big[\Big((X_n-\varphi(\cdot))^r f_{\alpha+r\,e_n}(\cdot\,,\varphi(\cdot))\Big)*\partial_j(\eta_\varepsilon)\Big](X'). \tag{7.87}$$

Thus, if $r := m-1-|\alpha| \geq 1$, after moving the derivative off of η_ε in the last term above we arrive at the recurrence formula

(7.88)
$$\partial_j F^\varepsilon_\alpha(X) = \begin{cases} F^\varepsilon_{\alpha+e_j}(X) + \frac{1}{r!}\Big[\Big((X_n-\varphi(\cdot))^r\partial_j(f_{\alpha+r\,e_n}(\cdot\,,\varphi(\cdot)))\Big)*\eta_\varepsilon\Big](X') & \text{if } j<n,\\ F^\varepsilon_{\alpha+e_n}(X) & \text{if } j=n.\end{cases}$$

Let us now pick $\psi \in C^\infty_0(\mathbb{R})$ such that $\psi(t)=1$ if $|t| < \sup\{|\varphi(X')| : |X'| \leq R+1\}$, and define $F^\varepsilon(X) := \psi(X_n)F^\varepsilon_{(0,\dots,0)}(X',X_n)$ for $X=(X',X_n)\in\mathbb{R}^n$. Then, obviously,

$$F_\varepsilon \in C^\infty_0(\mathbb{R}^n) \quad \text{and } F_\varepsilon(X',X_n) = F^\varepsilon_{(0,\dots,0)}(X',X_n)$$
$$\text{if } |X'| \leq R+1 \quad \text{and } X_n \text{ is near } \varphi(X'). \tag{7.89}$$

Hence, an inductive argument based on (7.88) shows that, for any multi-index $\alpha=(\alpha',\alpha_n)\in\mathbb{N}^{n-1}_0\times\mathbb{N}$ of length $|\alpha'|+\alpha_n \leq m-1$, the difference between $i^{|\alpha|}D^\alpha F^\varepsilon(X)$ and $F^\varepsilon_\alpha(X)$ can be expressed, when $X=(X',X_n)$ with $|X'|\leq R$ and X_n is near $\varphi(X')$, as a finite, constant coefficient linear combination of terms of the type

$$\varepsilon^{-|\gamma|}\Big((X_n-\varphi(\cdot))^{m-1-|\beta|-\alpha_n}\partial_j(f_\delta(\cdot\,,\varphi(\cdot)))\Big)*(\partial^\gamma\eta)_\varepsilon(X'), \tag{7.90}$$

where

(7.91)
$$1\leq j\leq n-1,\;\; \beta,\gamma\in\mathbb{N}^{n-1}_0 \quad \text{are such that } e_j+\beta+\gamma=\alpha',\text{ and } \delta\in\mathbb{N}^n_0,\ |\delta|=m-1.$$

Consequently, (7.83) will follow easily once we establish that for every $\alpha\in\mathbb{N}^n_0$ of length $\leq m-1$,

$$F^\varepsilon_\alpha(\cdot,\varphi(\cdot)) \to f_\alpha(\cdot,\varphi(\cdot)) \quad \text{in } L^1_p(\mathbb{R}^{n-1}) \quad \text{as } \varepsilon\to 0^+, \tag{7.92}$$

and that whenever the indices are as in (7.91) and $X_n=\varphi(X')$, the expression in (7.90) converges to zero in $L^1_p(\mathbb{R}^{n-1})$ as $\varepsilon\to 0^+$. As regards (7.92), we begin by noting that

(7.93)
$$F^\varepsilon_\alpha(X',\varphi(X')) = \sum_{\ell=0}^{m-1-|\alpha|}\frac{1}{\ell!}\int_{\mathbb{R}^{n-1}}\big(\varphi(X')-\varphi(Y')\big)^\ell f_{\alpha+\ell e_n}(Y',\varphi(Y'))\eta_\varepsilon(X'-Y')dY'.$$

Thus, $F_\alpha^\varepsilon(\cdot\,,\varphi(\cdot)) \to f_\alpha(\cdot\,,\varphi(\cdot))$ in $L_p(\mathbb{R}^{n-1})$ as $\varepsilon \to 0^+$ (indeed, based on the Lipschitzianity of φ and the fact that $\operatorname{supp}\eta_\varepsilon \subseteq \{X' \in \mathbb{R}^{n-1} : |X'| \leq \varepsilon\}$, one can easily show that any term in the right-hand side of (7.93) with $\ell > 0$ converges to zero in $L_p(\mathbb{R}^{n-1})$ as $\varepsilon \to 0^+$). Consequently, (7.92) is proved as soon as we show that $\nabla_{X'}[F_\alpha^\varepsilon(\cdot\,,\varphi(\cdot))] \to \nabla_{X'}[f_\alpha(\cdot\,,\varphi(\cdot))]$ in $L_p(\mathbb{R}^{n-1})$ as $\varepsilon \to 0^+$. We remark that this is obviously true if $|\alpha| = m-1$ (cf. (7.93)), so we consider the case when $|\alpha| \leq m-2$. In this situation, we use (7.93) (in which we treat the cases $\ell = 0$, $\ell = 1$ and $\ell \geq 2$ separately) to compute, for each $k \in \{1, \dots, n-1\}$,

$$
\begin{aligned}
\frac{\partial}{\partial X_k}&\Big[F_\alpha^\varepsilon(X',\varphi(X'))\Big] \\
&= \int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial Y_k}\Big[f_\alpha(Y',\varphi(Y'))\Big]\,\eta_\varepsilon(X'-Y')\,dY' \\
&\quad + \int_{\mathbb{R}^{n-1}} (\partial_k\varphi(X') - \partial_k\varphi(Y'))f_{\alpha+e_n}(Y',\varphi(Y'))\,\eta_\varepsilon(X'-Y')\,dY' \\
&\quad + \mathcal{R}(X'),
\end{aligned} \tag{7.94}
$$

where (based on the Lipschitzianity of φ, the fact that $\operatorname{supp}\eta_\varepsilon \subseteq \{X' \in \mathbb{R}^{n-1} : |X'| \leq \varepsilon\}$ and $\partial_k(\eta_\varepsilon) = \varepsilon^{-1}(\partial_k\eta)_\varepsilon$) it can be shown that the remainder satisfies the pointwise estimate

$$
|\mathcal{R}(X')| \leq C\varepsilon \sum_{|\gamma|\leq m-1} \Big(|f_\gamma(\cdot\,,\varphi(\cdot))| * (|\eta| + |\nabla\eta|)_\varepsilon\Big)(X'), \qquad \forall\, X' \in \mathbb{R}^{n-1}. \tag{7.95}
$$

Hence,

$$
\frac{\partial}{\partial X_k}\Big[F_\alpha^\varepsilon(X',\varphi(X'))\Big] \to \frac{\partial}{\partial X_k}\Big[f_\alpha(X',\varphi(X'))\Big] \text{ in } L_p(\mathbb{R}^{n-1}) \text{ as } \varepsilon \to 0^+,
$$

since $\|\mathcal{R}\|_{L_p(\mathbb{R}^{n-1})} \to 0$ as $\varepsilon \to 0^+$. This proves (7.92).

Let us now consider the expression (7.90) when $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m-1$ is fixed and the conditions in (7.91) hold. A direct estimate (based on familiar, by now, support considerations, etc.) shows that, when $X_n = \varphi(X')$, the L_p-norm of this quantity (as a function of the variable $X' \in \mathbb{R}^{n-1}$) is $\leq C\varepsilon^{m-|\alpha|} \to 0$ as $\varepsilon \to 0^+$. To finish the proof, take $X_n = \varphi(X')$ in (7.90) and, for an arbitrary $j \in \{1, \dots, n-1\}$, apply $\partial/\partial X_j$. Much as we just did, the L_p-norm of this quantity is $\leq C\varepsilon^{m-|\alpha|-1}$. Thus, provided that $|\alpha| \leq m-2$, this converges to zero as $\varepsilon \to 0^+$. The most delicate case is when $|\alpha| = m-1$. In this situation, we write out the terms obtained as a result of making $X_n = \varphi(X')$ in (7.90) and then applying $\partial/\partial X_k$ for some fixed $k \in \{1, \dots, n-1\}$. They are

$$
\varepsilon^{-r} \int_{\mathbb{R}^{n-1}} \Big(\varphi(X') - \varphi(Y')\Big)^r g(Y')(\partial^{\gamma+e_k}\eta)_\varepsilon(X'-Y')\,dY', \tag{7.96}
$$

and

$$(7.97)\qquad r\,\varepsilon^{1-r}\int_{\mathbb{R}^{n-1}}\Big(\varphi(X')-\varphi(Y')\Big)^{r-1}\partial_k\varphi(X')g(Y')(\partial^\gamma\eta)_\varepsilon(X'-Y')\,dY',$$

where we have set $r:=m-1-|\beta|-\alpha_n$ and $g:=\partial_j(f_\delta(\cdot\,,\varphi(\cdot)))$. Above, we have used the fact that $|\alpha|=m-1$ forces $|\gamma|=r-1$. Our goal is to prove that the L_p-norm of the sum between (7.97) and (7.96), viewed as functions in $X\in\mathbb{R}^{n-1}$, converges to zero as $\varepsilon\to 0^+$. To this end, write $\varphi(X')-\varphi(Y')=\Delta(X',Y')\,|X'-Y'|+\nabla\varphi(X')\cdot(X'-Y')$, where

$$(7.98)\qquad \Delta(X',Y'):=\frac{\varphi(X')-\varphi(Y')-(\nabla\varphi)(X')\cdot(Y'-X')}{|X'-Y'|},$$

then expand

(7.99)

$$\begin{aligned}\big(\varphi(X')-\varphi(Y')\big)^r&=\sum_{a+b=r}\frac{r!}{a!b!}\,\Delta(X',Y')^a\,|X'-Y'|^a\,(\nabla\varphi(X')\cdot(X'-Y'))^b\\&=\sum_{a+b=r}\sum_{\substack{|\sigma|=b\\ \sigma\in\mathbb{N}_0^{n-1}}}\frac{r!}{a!\sigma!}\Delta(X',Y')^a\,|X'-Y'|^a(\nabla\varphi(X'))^\sigma\,(X'-Y')^\sigma.\end{aligned}$$

Plugging this back into (7.96) finally yields

(7.100)

$$\sum_{a+b=r}\sum_{\substack{|\sigma|=b\\ \sigma\in\mathbb{N}_0^{n-1}}}\frac{r!}{a!\sigma!}\,(\nabla\varphi(X'))^\sigma\int_{\mathbb{R}^{n-1}}\Delta(X',Y')^a g(Y')(\Theta^{\sigma,a}_{\gamma+e_k})_\varepsilon(X'-Y')\,dY',$$

where we have used the notation

$$(7.101)\quad \Theta^{\sigma,a}_\tau(X'):=(X')^\sigma\,|X'|^a\,(\partial^\tau\eta)(X'),\quad X'\in\mathbb{R}^{n-1},\ \sigma,\,\tau\in\mathbb{N}_0^{n-1},\ a\in\mathbb{N}_0.$$

Each integral above is pointwise dominated by $C(\|\nabla\varphi\|_{L_\infty})\mathcal{M}g(X')$ uniformly with respect to $\varepsilon>0$ (recall that $\mathcal{M}$ is the Hardy–Littlewood maximal operator), and converges to zero as $\varepsilon\to 0^+$ whenever $a>0$ and X' is a differentiability point for the function φ. Thus, since φ is almost everywhere differentiable, by a well-known theorem of H. Rademacher, and since $\mathcal{M}$ is bounded on L_p if $1<p<\infty$, Lebesgue's Dominated Convergence Theorem gives that all integrals in (7.100) corresponding to $a>0$ converge to zero in $L_p(\mathbb{R}^{n-1})$ as $\varepsilon\to 0^+$.

On the other hand, in the context of (7.100), $a=0$ forces $|\sigma|=r=|\gamma+e_k|$. Note that in general, if $a=0$ and $|\sigma|=|\tau|$, definition (7.101) and repeated integrations by parts yield

$$(7.102)\qquad \int_{\mathbb{R}^{n-1}}\Theta^{\sigma,0}_\tau(X')\,dX'=\int_{\mathbb{R}^{n-1}}(X')^\sigma(\partial^\tau\eta)(X')\,dX'=(-1)^{|\tau|}\sigma!\,\delta_{\sigma\tau},$$

where $\delta_{\sigma\tau}$ is the Kronecker symbol. Consequently, as $\varepsilon \to 0^+$, the portion of (7.100) corresponding to $a = 0$ (and, hence, the *entire* expression in (7.100)) converges in $L_p(\mathbb{R}^{n-1})$ to

$$(-1)^r\, r!\, (\nabla\varphi)^{\gamma+e_k} g. \tag{7.103}$$

The analysis of (7.97) closely parallels that of (7.96). In fact, given the close analogy between (7.97) and (7.96), in order to compute the limit of the former in L_p as $\varepsilon \to 0^+$, we only need to make the following changes in (7.103): replace $\gamma+e_k$ by γ, r by $r-1$ and then multiply the result by $r\,\partial_k\varphi$. The resulting expression is precisely the opposite of (7.103), and this finishes the proof. □

After this preamble, we are now ready to present the

Proof of Theorem 7.8. The fact that $V_p^{m,a}(\Omega)$ is the null-space of Tr_{m-1} follows from Proposition 7.3 once we notice that (7.73) is the composition between (7.22) and

$$\dot{B}_p^{m-1+s}(\partial\Omega) \ni \dot{f} = \{f_\alpha\}_{|\alpha|\leq m-1} \mapsto \Big\{ \sum_{|\alpha|=k} \frac{k!}{\alpha!}\,\nu^\alpha\, f_\alpha \Big\}_{0\leq k\leq m-1} \in L_p(\partial\Omega), \tag{7.104}$$

and that the assignment (7.104) is one-to-one. The latter claim can be justified with the help of the identity

$$D^\alpha = i^{-|\alpha|}\,\nu^\alpha\,\frac{\partial^{|\alpha|}}{\partial\nu^{|\alpha|}} + \sum_{|\beta|=|\alpha|-1}\,\sum_{j,k=1}^n p_{jk}^{\alpha\beta}(\nu)\frac{\partial}{\partial\tau_{jk}}D^\beta, \tag{7.105}$$

where $p_{jk}^{\alpha\beta}$ are polynomial functions. Indeed, let $\dot{f} \in \dot{B}_p^{m-1+s}(\partial\Omega)$ be mapped to zero by the assignment (7.104) and consider $\mathcal{U} := \mathcal{E}(\dot{f}) \in W_p^{m,a}(\Omega)$. Then $f_\alpha = i^{|\alpha|}\,\mathrm{Tr}\,[D^\alpha\mathcal{U}]$ on $\partial\Omega$ for each α with $|\alpha| \leq m-1$ and, granted the current hypotheses, $\partial^k\mathcal{U}/\partial\nu^k = 0$ for $k = 0, 1, \dots, m-1$. Consequently, (7.105) and induction on $|\alpha|$ yield that $\mathrm{Tr}\,[D^\alpha\mathcal{U}] = 0$ on $\partial\Omega$ whenever $|\alpha| \leq m-1$. Thus, ultimately, $f_\alpha = 0$ for each α with $|\alpha| \leq m-1$, as desired. In turn, the identity (7.105) can be proved by writing

(7.106)

$$\begin{aligned} i^{|\alpha|}\,D^\alpha &= \prod_{j=1}^n \Big(\frac{\partial}{\partial x_j}\Big)^{\alpha_j} = \prod_{j=1}^n \Big[\sum_{k=1}^n \xi_k\Big(\xi_k\frac{\partial}{\partial x_j} - \xi_j\frac{\partial}{\partial x_k}\Big) + \sum_{k=1}^n \xi_j\xi_k\frac{\partial}{\partial x_k}\Big]^{\alpha_j}\Big|_{\xi=\nu} \\ &= \prod_{j=1}^n \Big[\sum_{l=0}^{\alpha_j} \frac{\alpha_j!}{l!(\alpha_j-l)!}\Big(\sum_{k=1}^n \xi_k\Big(\xi_k\frac{\partial}{\partial x_j} - \xi_j\frac{\partial}{\partial x_k}\Big)\Big)^{\alpha_j-l}\nu_j^l\frac{\partial^l}{\partial\nu^l}\Big]\Big|_{\xi=\nu} \\ &= \prod_{j=1}^n \Big[\nu_j^{\alpha_j}\frac{\partial^{\alpha_j}}{\partial\nu^{\alpha_j}} + \sum_{l=0}^{\alpha_j-1} \frac{\alpha_j!}{l!(\alpha_j-l)!}\Big(\sum_{k=1}^n \xi_k\Big(\xi_k\frac{\partial}{\partial x_j} - \xi_j\frac{\partial}{\partial x_k}\Big)\Big)^{\alpha_j-l}\nu_j^l\frac{\partial^l}{\partial\nu^l}\Big]\Big|_{\xi=\nu} \end{aligned}$$

and noticing that

$$\prod_{j=1}^{n} \nu_j^{\alpha_j} \partial^{\alpha_j}/\partial\nu^{\alpha_j} = \nu^\alpha \partial^{|\alpha|}/\partial\nu^{|\alpha|},$$

whereas

$$(\xi_k\partial/\partial x_j - \xi_j\partial/\partial x_k)|_{\xi=\nu} = -\partial/\partial\tau_{jk}.$$

Parenthetically, let us point out here that the identity (7.106) readily proves the existence of some polynomial function $P^{\alpha\beta}_{\gamma jk}$ such that (7.70) holds.

Turning to the characterization of the image of the operator (7.73), assume that $g_k \in L_p(\partial\Omega)$, $0 \leq k \leq m-1$, are such that the functions f_α defined as in (7.74)–(7.75) belong to $B^s_p(\partial\Omega)$. The claim that we make is that $\dot{f} := \{f_\alpha\}_{|\alpha|\leq m-1} \in \dot{B}^{m-1+s}_p(\partial\Omega)$ and

$$g_k = \sum_{|\alpha|=k} \frac{k!}{\alpha!}\, \nu^\alpha\, f_\alpha, \quad 0 \leq k \leq m-1. \tag{7.107}$$

Regarding the first part of the claim, by (7.76) and Proposition 7.7 it suffices to show that (7.82) holds. We shall prove by induction on $\ell := |\alpha| \in \{0,\dots,m-2\}$. Based on (7.74)–(7.75), we compute

$$f_\alpha = \nu^\alpha g_1 + \sum_{|\beta|=1} \nu^\beta \frac{\partial g_0}{\partial\tau_{\beta\alpha}}, \quad \forall\, \alpha \,:\, |\alpha| = 1, \tag{7.108}$$

from which the version of (7.82) with $\alpha = (0,\dots,0)$ is immediate. To prove the induction step, assume that (7.82) holds whenever $|\alpha| \leq \ell - 1$. By Lemma 7.9, there exists $F_\varepsilon \in C^\infty_0(\mathbb{R}^n)$, $\varepsilon > 0$, such that

$$|\gamma| \leq \ell \Longrightarrow i^{|\gamma|}\,\mathrm{Tr}\,[D^\gamma F_\varepsilon] \to f_\gamma \quad \text{in } L^1_p(\partial\Omega) \quad \text{as } \varepsilon \to 0^+. \tag{7.109}$$

From (7.109) and (7.70) it follows that for each $\alpha \in \mathbb{N}^n_0$ with $|\alpha| = \ell + 1$,

$$f_\alpha := \nu^\alpha g_\ell + i^{\ell+1} \lim_{\varepsilon\to 0^+} \sum_{|\beta|=\ell+1} \frac{(\ell+1)!}{\beta!}\nu^\beta\Big(\nu^\beta\mathrm{Tr}[D^\alpha F_\varepsilon] - \nu^\alpha\mathrm{Tr}[D^\beta F_\varepsilon]\Big) \quad \text{in } L_p(\partial\Omega). \tag{7.110}$$

Next, fix an arbitrary $\alpha \in \mathbb{N}^n_0$ with $|\alpha| = \ell$, choose $j, k \in \{1,\dots,n\}$, and consider the identity (7.110) written twice, with $\alpha + e_k$ and $\alpha + e_j$, respectively, in place of α. If we multiply the first such identity by ν_j, the second one by ν_k and then subtract them from one another, we arrive at

$$\nu_j f_{\alpha+e_k} - \nu_k f_{\alpha+e_j} = i^\ell \lim_{\varepsilon\to 0^+} \sum_{|\beta|=\ell+1} \frac{(\ell+1)!}{\beta!}\, \nu^{2\beta}\, \frac{\partial}{\partial\tau_{jk}} \mathrm{Tr}\,[D^\alpha F_\varepsilon]. \tag{7.111}$$

By (7.109), the above limit is $i^{-\ell}\partial f_\alpha/\partial\tau_{jk}$ and this finishes the proof of the induction step. Thus (7.82) holds and, as a result, $\dot{f} := \{f_\alpha\}_{|\alpha|\leq m-1} \in \dot{B}_p^{m-1+s}(\partial\Omega)$, as desired. As for (7.107), if we set

$$\mathcal{U} := \mathcal{E}\dot{f} \in W_p^{m,a}(\Omega), \tag{7.112}$$

it follows from (7.25), (7.75) and (7.70) that

$$f_\alpha = \nu^\alpha g_k + \sum_{|\beta|=k} \frac{k!}{\beta!}\,\nu^\beta\,(\nu^\beta f_\alpha - \nu^\alpha f_\beta), \quad \forall\,\alpha\,:\,|\alpha| = k, \tag{7.113}$$

from which we deduce that $\nu^\alpha g_k = \nu^\alpha \sum_{|\beta|=k}\frac{k!}{\beta!}\nu^\beta f_\beta$ for each multi-index α of length k. Multiplying both sides of this equality by $\frac{k!}{\alpha!}\nu^\alpha$ and summing over all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$ finally yields (7.107).

Proceeding further, from (7.112), (7.107) and (1.5) we may conclude that $\{g_k\}_{0\leq k\leq m-1} = \mathrm{Tr}_{m-1}\,\mathcal{U}$, which proves that the family $\{g_k\}_{0\leq k\leq m-1}$ belongs to the image of the mapping (7.73). Conversely, if $\{g_k\}_{0\leq k\leq m-1} = \mathrm{Tr}_{m-1}\,\mathcal{U}$ for some function $\mathcal{U} \in W_p^{m,a}(\Omega)$, it follows from (7.70) and (7.75) that $f_\alpha = i^{|\alpha|}\mathrm{Tr}[D^\alpha\mathcal{U}]$ for $|\alpha| \leq m-1$. Consequently, $f_\alpha \in B_p^s(\partial\Omega)$ if $|\alpha| \leq m-1$ and $f_\alpha \in L_p^1(\partial\Omega)$ for $|\alpha| \leq m-2$, thanks to (7.69). This finishes the proof of the fact that (7.76) characterizes the image of the operator (7.73). That the space (7.77) is independent of the choice of polynomials $P_{\gamma jk}^{\alpha\beta}$ satisfying (7.70) is implicit in the above reasoning. Finally, the results in §7.1 imply that the operator (7.79) is bounded. Since as a byproduct of the above proof, the assignment

$$\dot{B}_p^{m-1+s}(\partial\Omega) \ni \dot{f} = \{f_\alpha\}_{|\alpha|\leq m-1} \mapsto \Big\{\sum_{|\alpha|=k}\frac{k!}{\alpha!}\,\nu^\alpha\,f_\alpha\Big\}_{0\leq k\leq m-1} \in \dot{W}_p^{m-1+s}(\partial\Omega) \tag{7.114}$$

is an isomorphism, we may take Ext in (7.80) to be the composition between the operator (7.24) and the inverse of the mapping (7.114). This finishes the proof of the theorem. □

A specific implementation of the algorithm (7.74)–(7.75) is discussed below.

Corollary 7.10. *Assume that Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and fix $1 < p < \infty$ and $-1/p < a < 1-1/p$, $s := 1-a-1/p \in (0,1)$, $m \in \mathbb{N}$. For a family $(g_0, g_1, \ldots, g_{m-1}) \in L_p(\partial\Omega)$ set $f_{(0,\ldots,0)} := g_0$ and, inductively, if $\{f_\gamma\}_{|\gamma|\leq\ell-1}$ have already been defined for some $\ell \in \{1,\ldots,m-1\}$, set*

$$f_\alpha := \nu^\alpha g_\ell + \frac{\alpha!}{\ell!}\sum_{\substack{\mu+\delta+e_j=\alpha\\ |\theta|=|\delta|}} \frac{|\delta|!}{\delta!}\,\frac{|\mu|!}{\mu!}\,\frac{|\theta|!}{\theta!}\,\nu^{\delta+\theta}\,(\nabla_{\tan} f_{\mu+\theta})_j, \quad \forall\,\alpha\in\mathbb{N}_0^n\,:\,|\alpha| = \ell, \tag{7.115}$$

where $(\cdot)_j$ *is the* j*-th component. Then* $\dot{g} = (g_0, g_1, \dots, g_{m-1})$ *belongs to* $\dot{W}_p^{m-1+s}(\partial\Omega)$ *if and only if* $\dot{f} := \{f_\alpha\}_{|\alpha|\le m-1}$ *belongs to* $\dot{B}_p^{m-1+s}(\partial\Omega)$*, in which case* (7.107) *also holds.*

Proof. For any two multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ of length ℓ written as $\alpha = e_{j_1} + \cdots + e_{j_\ell}$ and $\beta = e_{k_1} + \cdots + e_{k_\ell}$, a direct calculation yields

$$\nu^\beta D^\alpha - \nu^\alpha D^\beta = i^\ell \sum_{r=0}^{\ell-1} \nu_{k_1} \cdots \nu_{k_{\ell-r-1}} \nu_{j_{\ell-r+1}} \cdots \nu_{j_\ell} \frac{\partial}{\partial \tau_{k_{\ell-r} j_{\ell-r}}} \partial_{j_1} \cdots \partial_{j_{\ell-r-1}} \partial_{k_{\ell-r+1}} \cdots \partial_{j_\ell}. \tag{7.116}$$

In order to be able to re-write (7.116) in multi-index notation, it is convenient to symmetrize the right-hand side of this identity by adding up all its versions obtained by permuting the indices $j_1, \dots, j_\ell$ and $k_1, \dots, k_\ell$. In this fashion, we obtain

$$\nu^\beta D^\alpha - \nu^\alpha D^\beta = \frac{1}{i} \frac{\alpha!}{\ell!} \frac{\beta!}{\ell!} \sum_{r=0}^{\ell-1} \sum_{\substack{\mu+\delta+e_j=\alpha,\, |\delta|=r \\ \gamma+\theta+e_k=\beta,\, |\theta|=r}} \frac{(\ell-r-1)!}{\mu!} \frac{r!}{\delta!} \frac{(\ell-r-1)!}{\gamma!} \frac{r!}{\theta!} \nu^{\gamma+\delta} \frac{\partial}{\partial \tau_{kj}} D^{\mu+\theta}. \tag{7.117}$$

This is a particular version of (7.70), where the intervening polynomials are identified explicitly. If we now implement the algorithm (7.74)–(7.75), for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = \ell$ we arrive at

$$f_\alpha = \nu^\alpha g_\ell + \frac{\alpha!}{\ell!} \sum_{|\beta|=\ell} \sum_{r=0}^{\ell-1} \sum_{\substack{\mu+\delta+e_j=\alpha,\, |\delta|=r \\ \gamma+\theta+e_k=\beta,\, |\theta|=r}} \frac{(\ell-r-1)!}{\mu!} \frac{r!}{\delta!} \frac{(\ell-r-1)!}{\gamma!} \frac{r!}{\theta!} \nu^{\gamma+\delta} \frac{\partial f_{\mu+\theta}}{\partial \tau_{kj}}. \tag{7.118}$$

Next, we replace β by $\gamma + \theta + e_k$, eliminating the sum over β, and make use of the identities

$$\sum_{|\gamma|=\ell-r-1} \frac{(\ell-r-1)!}{\gamma!} \nu^{2\gamma} = 1, \qquad \sum_{k=1}^{n} \nu_k \frac{\partial f}{\partial \tau_{kj}} = (\nabla_{\tan} f)_j, \tag{7.119}$$

in order to transform (7.118) into

$$f_\alpha = \nu^\alpha g_\ell + \frac{\alpha!}{\ell!} \sum_{r=0}^{\ell-1} \sum_{\substack{\mu+\delta+e_j=\alpha \\ |\theta|=|\delta|=r}} \frac{(\ell-r-1)!}{\mu!} \frac{r!}{\delta!} \frac{r!}{\theta!} \nu^{\delta+\theta} (\nabla_{\tan} f_{\mu+\theta})_j, \tag{7.120}$$

which is equivalent to (7.115). □

The space (7.77) takes a particularly simple form when $m = 2$. Indeed, as a direct consequence of (7.115) in which we take $\ell = 1$ we have:

Corollary 7.11. *For each Lipschitz domain $\Omega \subset \mathbb{R}^n$ and each $1 < p < \infty$, $s \in (0,1)$,*

$$\dot{W}_p^{1+s}(\partial\Omega) = \{(g_0, g_1) \in L_p^1(\partial\Omega) \oplus L_p(\partial\Omega) : \nu g_1 + \nabla_{\tan} g_0 \in B_p^s(\partial\Omega)\}. \tag{7.121}$$

This has been conjectured to hold (when $s = 1 - 1/p$) by A. Buffa and G. Geymonat on p. 703 of [8].

Finally, we comment on how (7.77) relates to more classical spaces of higher order traces when $\Omega \subset \mathbb{R}^n$ has a smoother boundary than mere Lipschitz. Specifically, fix $m \in \mathbb{N}$, $m \geq 2$, $p \in (1,\infty)$, $s \in (0,1)$ and assume that $\partial\Omega$ is locally given by graphs of Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ with the additional property that $\nabla\varphi$ belongs to $MB_p^{m-2+s}(\mathbb{R}^{n-1})$, the space of (pointwise) multipliers for the Besov space $B_p^{m-2+s}(\mathbb{R}^{n-1})$ (cf. [39], [41]). Then, for each non-integer $1 < \mu \leq m-1+s$, one can coherently define the space $B_p^\mu(\partial\Omega)$ by starting from $B_p^\mu(\mathbb{R}^{n-1})$ and then transporting it to $\partial\Omega$ via a smooth partition of unity argument and by locally flattening the boundary. In fact, we arrive at the same space by taking the image of the trace operator on $\partial\Omega$, acting from $B_p^{\mu+1/p}(\mathbb{R}^n)$.

Proposition 7.12. *Assume that $p \in (1,\infty)$, $s \in (0,1)$ and that $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain whose boundary is locally described by means of graphs of real-valued functions in $\mathbb{R}^{n-1}$ whose gradients belong to $MB_p^{m-2+s}(\mathbb{R}^{n-1})$. Then*

$$\dot{W}_p^{m-1+s}(\partial\Omega) = \prod_{k=0}^{m-1} B_p^{m-1-k+s}(\partial\Omega). \tag{7.122}$$

In particular, this is the case if $\partial\Omega \in C^{m-1,\theta}$ for some $\theta > s$.

Proof. In one direction, (7.69) and lifting theorems imply that if $\{f_\alpha\}_{|\alpha|\leq m-1} \in \dot{B}_p^{m-1+s}(\partial\Omega)$ then $f_\alpha \in B_p^{m-1-|\alpha|+s}(\partial\Omega)$ for each α with $|\alpha| \leq m-1$. Hence, $g_k := \sum_{|\alpha|=k} \frac{k!}{\alpha!} \nu^\alpha f_\alpha \in B_p^{m-1-k+s}(\partial\Omega)$ for each $k \in \{0,\dots,m-1\}$, so the left-to-right inclusion in (7.122) follows from the fact that (7.114) is an isomorphism.

As for the opposite implication, given $\{g_k\}_{0\leq k\leq m-1} \in \bigoplus_{k=0}^{m-1} B_p^{m-1-k+s}(\partial\Omega)$, define $\{f_\alpha\}_{|\alpha|\leq m-1}$ as in (7.74)–(7.75). Granted the current assumptions on $\partial\Omega$, an argument based on induction and the fact that $\nu \in MB_p^{m-2+s}(\partial\Omega)$ shows that $f_\alpha \in B_p^{m-1-|\alpha|+s}(\partial\Omega)$ for each $|\alpha| \leq m-1$. In particular, (7.76) holds, which proves that $\{g_k\}_{0\leq k\leq m-1} \in \dot{W}_p^{m-1+s}(\partial\Omega)$. This shows that the right-to-left inclusion in (7.122) also holds, thus finishing the proof of the proposition. $\square$

8 Proof of the main result

8.1 The inhomogeneous Dirichlet problem. Theorem 1.1 is a particular case of Theorem 8.1 below, concerning the solvability of the inhomogeneous Dirichlet problem

$$\begin{cases} \mathcal{A}(X, D_X)\mathcal{U} = \mathcal{F} & \text{in } \Omega, \\ \frac{\partial^k \mathcal{U}}{\partial \nu^k} = g_k \quad \text{on } \partial\Omega, \quad 0 \le k \le m-1, \end{cases} \tag{8.1}$$

in the space $W_p^{m,a}(\Omega)$. Note that for any operator $\mathcal{A}$ as in §6.1 we have

$$\mathcal{A}(X, D_X) : W_p^{m,a}(\Omega) \longrightarrow V_p^{-m,a}(\Omega) \tag{8.2}$$

boundedly. Thus, granted the membership of $\mathcal{U}$ solving (8.1) to $W_p^{m,a}(\Omega)$, it follows from Theorem 7.8 that necessarily $\mathcal{F} \in V_p^{-m,a}(\Omega)$ and $g := \{g_k\}_{0\le k\le m-1} \in \dot{W}_p^{m-1+s}(\partial\Omega)$. Moreover,

$$\|g\|_{\dot{W}_p^{m-1+s}(\partial\Omega)} + \|\mathcal{F}\|_{V_p^{-m,a}(\Omega)} \le C\|\mathcal{U}\|_{W_p^{m,a}(\Omega)}. \tag{8.3}$$

The converse direction provides the object of the theorem below.

Theorem 8.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that the operator $\mathcal{A}$ is as in §6.1. Then there exists $c > 0$ such that if* (1.12) *is satisfied, then the Dirichlet problem* (8.1) *has a unique solution $\mathcal{U} \in W_p^{m,a}(\Omega)$ for any given $\mathcal{F} \in V_p^{-m,a}(\Omega)$ and $g := \{g_k\}_{0\le k\le m-1} \in \dot{W}_p^{m-1+s}(\partial\Omega)$. Furthermore, there exists $C = C(\partial\Omega, \mathcal{A}, p, s) > 0$ such that*

$$\|\mathcal{U}\|_{W_p^{m,a}(\Omega)} \le C\Big(\|g\|_{\dot{W}_p^{m-1+s}(\partial\Omega)} + \|\mathcal{F}\|_{V_p^{-m,a}(\Omega)}\Big). \tag{8.4}$$

Proof. We seek a solution for (8.1) in the form $\mathcal{U} = \mathrm{Ext}(g) + \mathcal{W}$, where Ext denotes the extension operator from Theorem 7.8 and $\mathcal{W} \in V_p^{m,a}(\Omega)$. Note that, by Theorem 7.8, this membership automatically entails $\partial^k\mathcal{W}/\partial\nu^k = 0$ on $\partial\Omega$ for $k = 0, 1, \dots, m-1$, so it suffices to take

$$\mathcal{W} := \mathcal{A}(X, D_X)^{-1}\Big(\mathcal{F} - \mathcal{A}(X, D_X)\mathrm{Ext}\,(g)\Big) \in V_p^{m,a}(\Omega) \tag{8.5}$$

which, by (8.2) and Theorem 6.3, is meaningful. As for uniqueness, let $\mathcal{U} \in W_p^{m,a}(\Omega)$ solve (8.1) with $\mathcal{F} = 0$ and $g_k = 0$, $0 \le k \le m-1$. Then the function $\mathcal{U}$ belongs to $V_p^{m,a}(\Omega)$, thanks to Theorem 7.8, and is a null-solution of $\mathcal{A}(X, D_X)$. In turn, Theorem 6.3 gives that $\mathcal{U} = 0$, as desired. Finally, (8.4) is a consequence of the results in §7. $\square$

We conclude this subsection with a remark pertaining to the presence of lower order terms. More specifically, granted Theorem 8.1, a standard perturbation argument (cf., e.g., [27]) proves the following. Assume that

$$\mathcal{A}(X,D_X)\mathcal{U} := \sum_{0\leq|\alpha|,|\beta|\leq m} D^\alpha(\mathcal{A}_{\alpha\beta}(X)\, D^\beta\mathcal{U}), \quad X\in\Omega, \tag{8.6}$$

where the top part of $\mathcal{A}(X,D_X)$ satisfies the hypotheses made in Theorem 1.1 and the lower order terms are bounded. Then, assuming that either (8.9) or (1.12) holds, the Dirichlet problem (8.1) is Fredholm with index zero, in the sense that the operator

$$W_p^{m,a}(\Omega)\ni\mathcal{U}\mapsto\Big(\mathcal{A}(X,D_X)\mathcal{U},\{\partial^k\mathcal{U}/\partial\nu^k\}_{0\leq k\leq m-1}\Big)\in V_p^{-m,a}(\Omega)\oplus\dot W_p^{m-1+s}(\partial\Omega) \tag{8.7}$$

is so. Furthermore, the estimate

$$\|\mathcal{U}\|_{W_p^{m,a}(\Omega)}\leq C\Big(\|\mathcal{F}\|_{V_p^{-m,a}(\Omega)}+\|g\|_{\dot W_p^{m-1+s}(\partial\Omega)}+\|\mathcal{U}\|_{L_p(\Omega)}\Big) \tag{8.8}$$

holds for any solution $\mathcal{U}\in W_p^{m,a}(\Omega)$ of (8.1).

8.2 Further comments and the sharpness of Theorem 8.1. A byproduct of our proof of Theorem 8.1 is the following. Assume that the domain $\Omega\subset\mathbb{R}^n$ and the operator $\mathcal{A}$ are as in the statement of Theorem 8.1. Then there exists $\varepsilon>0$, depending only on the L_∞-norm of the coefficients and the ellipticity constant of $\mathcal{A}$, with the property that the Dirichlet problem (8.1) with data from $\dot W_p^{m-1+s}(\partial\Omega)$ has a unique solution in $W_p^{m,a}(\Omega)$ granted that

$$|2^{-1}-p^{-1}|<\varepsilon \quad\text{and}\quad |a|<\varepsilon. \tag{8.9}$$

To justify this claim, we rely on Theorem 7.8 and, using the same strategy as before, reduce matters to proving that the operator (6.28) is an isomorphism. When $a=0$ and $p=2$, our assumptions on $\mathcal{A}(X,D_X)$ and the classical Lax–Milgram lemma ensure that this is indeed the case. Then the stability theory from [32], [48] allows us to perturb this result, i.e., conclude that (6.28) is an isomorphism whenever (8.9) holds, as soon as we show that the scale $V_p^{m,a}(\Omega)$ is stable under complex interpolation. That is, if $1<p_i<\infty$, $-1/p_i<a_i<1-1/p_i$, $i\in\{0,1\}$, $\theta\in(0,1)$, $1/p=(1-\theta)/p_0+\theta/p_1$ and $a=(1-\theta)a_0+\theta a_1$, then

$$[V_{p_0}^{m,a_0}(\Omega),V_{p_1}^{m,a_1}(\Omega)]_\theta=V_p^{m,a}(\Omega), \tag{8.10}$$

where $[\cdot,\cdot]_\theta$ denotes the usual complex interpolation bracket. In the proof of (8.10) we may assume that Ω is a special Lipschitz domain and, further, that $\Omega=\mathbb{R}^n_+$, by making the change of variables described in §5.2–§5.3. In this latter setting, it will be useful to note that

$$[L_{p_0}(\mathbb{R}^n_+,\, x_n^{a_0p_0}\,dx), L_{p_1}(\mathbb{R}^n_+,\, x_n^{a_1p_1}\,dx)]_\theta = L_p(\mathbb{R}^n_+,\, x_n^{ap}\,dx), \tag{8.11}$$

granted that the indices involved are as before, which follows from well-known interpolation results for Lebesgue spaces with change of measure (cf. Theorem 5.5.3 on p. 120 in [6]). Then (8.10) follows easily from (8.11), the fact that for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = m$ the operator D^α maps the scale $V_p^{m,a}(\mathbb{R}^n_+)$ boundedly into the scale $L_p(\mathbb{R}^n_+,\, x_n^{ap}\,dx)$, and (4.23)–(4.25). This finishes the proof of the claim made at the beginning of this subsection.

In turn, the aforementioned result can be viewed as an extension of a well-known theorem of N. Meyers, who has treated the case $m = 1$, $l = 1$ in [42]. The example given in §5 of [42] shows that the membership of p to a small neighborhood of 2 is a necessary condition, even when $\partial\Omega$ is smooth, if the coefficients $A_{\alpha\beta}$ are merely bounded. For higher order operators we make use of an example originally due to V. G. Maz'ya [36] (cf. also the contemporary article by E. De Giorgi [16]). Specifically, when $m \in \mathbb{N}$ is even, consider the divergence-form equation

$$\Delta^{\frac{1}{2}m-1}\mathcal{L}_4\,\Delta^{\frac{1}{2}m-1}\mathcal{U} = 0 \quad \text{in } \Omega := \{X \in \mathbb{R}^n : |X| < 1\}, \tag{8.12}$$

where $\mathcal{L}_4$ is the fourth order operator

$$\begin{aligned}\mathcal{L}_4(X, D_X)\mathcal{U} :=& a\,\Delta^2\mathcal{U} + b\sum_{i,j=1}^n \Delta\Big(\frac{X_iX_j}{|X|^2}\partial_i\partial_j\,\mathcal{U}\Big) + b\sum_{i,j=1}^n \partial_i\partial_j\Big(\frac{X_iX_j}{|X|^2}\,\Delta\,\mathcal{U}\Big)\\ &+ c\sum_{i,j,k,l=1}^n \partial_k\partial_l\Big(\frac{X_iX_jX_kX_l}{|X|^4}\partial_i\partial_j\,\mathcal{U}\Big).\end{aligned} \tag{8.13}$$

Obviously, the coefficients of $\mathcal{L}_4(X, D_X)$ are bounded, and if the parameters $a, b, c \in \mathbb{R}$, $a > 0$, are chosen such that $b^2 < ac$, then $\mathcal{L}_4$ along with $\Delta^{\frac{1}{2}m-1}\mathcal{L}_4\,\Delta^{\frac{1}{2}m-1}$ are strongly elliptic. Now, if W_p^s denotes the usual L_p-based Sobolev space of order s, it has been observed in [36] that the function $\mathcal{U}(X) := |X|^{\theta+m-2} \in W_2^m(\Omega)$ has $\mathrm{Tr}\,\mathcal{U} \in C^\infty(\partial\Omega)$ and is a weak solution of (8.12) for the choice

$$\theta := 2 - \frac{n}{2} + \sqrt{\frac{n^2}{4} - \frac{(n-1)(bn+c)}{a+2b+c}}. \tag{8.14}$$

Thus, if $a := (n-2)^2 + \varepsilon$, $b := n(n-2)$, $c := n^2$, $\varepsilon > 0$, the strong ellipticity condition is satisfied and $\theta = \theta(\varepsilon)$ becomes $2 - n/2 + n\,\varepsilon^{1/2}/2\sqrt{4(n-1)^2 + \varepsilon}$. However, $\mathcal{U} \in W_p^m(\Omega)$ if and only if $p < n/(2 - \theta(\varepsilon))$, and the bound $n/(2 - \theta(\varepsilon))$ approaches 2 when $\varepsilon \to 0^+$. An analogous example can be produced when $m > 1$

is odd, starting with a sixth order operator $\mathcal{L}_6(X, D_X)$ from [36]. In the above context, given that $W^1_n(\Omega) \hookrightarrow \mathrm{VMO}(\Omega)$, it is significant to point out that both for the example in [42], when $n = 2$, and for (8.12) when $n \geq 3$, the coefficients have their gradients in weak-L_n yet they fail to belong to $W^1_n(\Omega)$.

Of course, condition (1.3) ensures that the left-hand side of (1.12) is always finite, but it is its actual size which determines whether for a given pair of indices s, p, the problem (1.2), (1.6), (1.9) is well-posed. Note that the maximum value that the right-hand side of (1.12) takes for $0 < s < 1$ and $1 < p < \infty$ occurs precisely when $p = 2$ and $a := 1 - s - 1/p = 0$. As (1.12) shows, the set of pairs $(s, 1/p) \in (0,1) \times (0,1)$ for which (8.1) is well-posed in the context of Theorem 8.1 exhausts the entire square $(0,1) \times (0,1)$ as the distance from ν and the $A_{\alpha\beta}$'s to VMO tends to zero (while the Lipschitz constant of Ω and the ellipticity constant of $\mathcal{L}$ stay bounded). That the geometry of the Lipschitz domain Ω intervenes in this process through a condition such as (1.12) confirms a conjecture made by P. Auscher and M. Qafsaoui in [5].

While the main aim of the present work is the consideration of higher order operators with coefficients in L_∞, Theorem 8.1 (and, with it, Theorem 1.1) is new even in the case when $m = 1$ and $A_{\alpha\beta} \in \mathbb{C}^{l \times l}$ (i.e., for second order, constant coefficient systems). It provides a complete answer to the issue of well-posedness of the problem (8.1) in the sense that the small mean oscillation condition, depending on p and s, is in the nature of best possible if one insists on allowing arbitrary indices p and s. This can be seen by considering the following Dirichlet problem for the Laplacian in a domain $\Omega \subset \mathbb{R}^n$:

(8.15)
$$\Delta\mathcal{U} = 0 \,\text{in}\, \Omega, \quad \mathrm{Tr}\,\mathcal{U} = g \in B^s_p(\partial\Omega), \quad |\mathcal{U}| + |\nabla\mathcal{U}| \in L_p(\Omega,\, \rho(X)^{p(1-s)-1}\, dX).$$

It has long been known that, already in the case when $\partial\Omega$ exhibits one cone-like singularity, the well-posedness of (8.15) prevents $(s, 1/p)$ from being an arbitrary point in $(0,1) \times (0,1)$. At a more sophisticated level, the work of D. Jerison and C. Kenig in the 1990's shows that (8.15) is well-posed in an arbitrary, given Lipschitz domain Ω if and only if the point $(s, 1/p)$ belongs to a certain open subregion of $(0,1) \times (0,1)$, determined exclusively by the geometry of the domain Ω (cf. [29]).

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