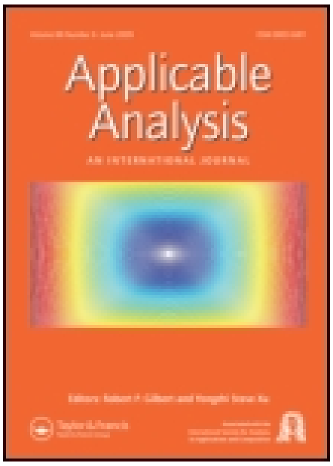

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Asymptotic behavior of solution to the Cahn-Hillard equation

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Asymptotic Behavior of Solution to the Cahn-Hillard Equation

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Abstract The asymptotic behavior of solution to the initial boundary value problem of nonlinear Cahn-Hilliard equation and the associated stationary problem have been extensively studied. In particular, it is proved that in the one space dimensional case the associated stationary problem has exactly 2N + 1 numbers of solutions and the solution of evolution equation converges to certain equilibrium solution as $t \to +\infty$.

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§1. INTRODUCTION

The initial boundary value problem of the nonlinear Cahn-Hilliard equation

$$\begin{cases} u_t + \gamma\Delta^2 u = \Delta\phi(u), \text{ in } \Omega \quad R^n, \ n \le 3 \\ \frac{\partial u}{\partial n}\Big|_\Gamma = \frac{\partial \Delta u}{\partial n}\Big|_\Gamma = 0 , \\ u\big|_{t=0} = u_0(x) \end{cases} \tag{1.1}$$

where

$$\phi(u) = \gamma_2 u^3 + \gamma_1 u^2 - u \tag{1.2}$$

and γ, γ_1, γ_2 are constants with $\gamma > 0$, Ω is a bounded domain in R^n, $n \le 3$ with smooth boundary Γ, arises in the study of phase

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separation in cooling binary solutions such as alloys, glasses and polymer mixtures, as well as in the study of mathematical biology, see Cahn & Hilliard [4], Novick-Cohen & Segel [15], Novick-Cohen [14], Cohen & Murray [7] and the references cited therein.

It has been proved in [9] that the sign of γ_2 is crucial for the global existence or blow up in a finite time of solution of (1.1). when $\gamma_2 > 0$, problem (1.1) admits a unique global solution $u \in H^{4,1}(Q_T) \cap C([0,T],H^2)$ for any $T > 0$ and any initial data

$$u_0 \in H_E^2 = \{u | u \in H^2, \frac{\partial u}{\partial n}|_\Gamma = 0\}.$$

On the other hand, if $\gamma_2 < 0$, then the solution of (1.1) must blow up in a finite time provided the initial data u_0 is large enough.

In this paper we are interested in the asymptotic behavior of solution as $t \to +\infty$ and the associated stationary problem in the case $\gamma_2 > 0$.

It is proved in section 2 that for any $u_0 \in H_E^4 = \{u | u \in H^4, \frac{\partial u}{\partial n}|_\Gamma = \frac{\partial \Delta u}{\partial n}|_\Gamma = 0\}$ the trajectory of solution of (1.1) is relatively compact in H^2 and as $t \to +\infty$, $u(x,t)$ converges to $\omega(u_0)$, where $\omega(u_0)$ is the ω-limit set of u_0. Furthermore, the ω-limit set $\omega(u_0)$ is a compact, connected subset of H^2 and is positive invariant under the nonlinear semigroup $T(t)u_0$ defined by solution $u(x,t)$. It is also proved that each element of $\omega(u_0)$ is an equilibrium solution.

In section 3, the associated stationary problem, which is different from the usual Dirichlet, Neumann boundary value problems, is extensively studied. It is proved that if γ or $|M|$, where

$$M = \frac{1}{|\Omega|} \int_\Omega u_0 dx, \tag{1.3}$$

is sufficiently large, then the stationary problem has only trivial solution $v \equiv M$. Therefore, the solution of (1.1) converges in $H^2(\Omega)$ to M as $t \to \infty$.

In particular, in one space dimensional case we prove in section 3 that the stationary problem has exactly 2N + 1 number of solutions which can be considered as an inversion of Chafee-Infante's results [6] for Dirichlet problem. As the result, solution $u(x,t)$ of (1.1) converges in H^2 to certain equilibrium solutions as $t \to +\infty$.

Throughout this paper we denote

$$\overset{\circ}{H}{}^2_E = \left\{u \mid u \in H^2(\Omega),\ \frac{\partial u}{\partial n}\Big|_\Gamma = 0,\ \int_\Omega u dx = 0\right\}, \tag{1.4}$$

$$\overset{\circ}{H}{}^4_E = \left\{u \mid u \in H^4(\Omega),\ \frac{\partial u}{\partial n}\Big|_\Gamma = \frac{\partial \Delta u}{\partial n}\Big|_\Gamma = 0,\ \int_\Omega u dx = 0\right\}, \tag{1.5}$$

$$H^{4,1}(Q_T) = \left\{u \mid u \in L^2\big([0,T]; H^4(\Omega)\big),\ u_t \in L^2\big([0,T]; L^2(\Omega)\big)\right\}. \tag{1.6}$$

We also denote by $\| \ \|$ the $L^2(\Omega)$ norm and denote by C the universal constant.

§2. ω-LIMIT SET

By translation of unknown function

$$\tilde{u} = u - M \tag{2.1}$$

problem (1.1) is reduced to

$$\begin{cases} \tilde{u}_t + \gamma\Delta^2\tilde{u} = \Delta\tilde{\phi}(\tilde{u}), \\ \dfrac{\partial \tilde{u}}{\partial n}\Big|_\Gamma = \dfrac{\partial \Delta\tilde{u}}{\partial n}\Big|_\Gamma = 0, \\ \tilde{u}\big|_{t=0} = \tilde{u}_0(x) \overset{\Delta}{=} u_0(x) - M,\ \int_\Omega \tilde{u}_0 dx = 0 \end{cases} \tag{2.2}$$

where

$$\tilde{\phi}(\tilde{u}) = \gamma_2\tilde{u}^3 + \tilde{\gamma}_1\tilde{u}^2 + \tilde{\gamma}_0\tilde{u} \tag{2.3}$$

$$\tilde{\gamma}_1 = (3\gamma_2 M + \gamma_1),\ \tilde{\gamma}_0 = (3\gamma_2 M^2 + 2\gamma_1 M - 1) \tag{2.4}$$

It can be seen from (2.3), (2.4) that although $\tilde{\gamma}_0$ can be positive for some M,

$$\tilde{\gamma}_1^2 - 3\gamma_2\tilde{\gamma}_0 = \gamma_1^2 + 3\gamma_2 > 0 \tag{2.5}$$

shows there still exists an interval (a,b) such that $\tilde{\gamma}'(\tilde{u}) < 0$ for $\tilde{u} \in (a,b)$.

Instead of (1.1), we work on problem (2.2) and still denote $\tilde{u}$, $\tilde{\phi}$, $\tilde{u}_0$ by u, ϕ, u_0, etc. It has been proved in [9] that for any $u_0 \in \overset{\circ}{H}{}^2_E$ and any $T > 0$, problem (2.2) admits a unique global solution $u(x,t) \in H^{4,1}(Q_T) \cap C([0,T];\overset{\circ}{H}{}^2_E)$. Therefore, the nonlinear semigroup $T(t)u_0$ defined by solution u(x,t) is strongly continuous. Moreover, we have

THEOREM 2.1.

Suppose $u_0 \in \overset{\circ}{H}{}^4_E$, then the orbit $\bigcup_{t>0} T(t)u_0$ is relatively compact in $\overset{\circ}{H}{}^2_E$.

Proof. It suffices to prove that for all $t \geq 0$

$$\|u(t)\|_{H^3} \leq C \tag{2.6}$$

Step 1. H^1 norm estimate

Let

$$F(u) = \int_\Omega [\frac{\gamma}{2} |\nabla u|^2 + H(u)]dx \tag{2.7}$$

where

$$H(u) = \int_0^u \phi(s)ds \tag{2.8}$$

It has been proved in [9] that

$$\frac{dF(u)}{dt} + \|\nabla(\gamma\Delta u - \phi(u))\|^2 = 0 \tag{2.9}$$

Thus

$$F(u(t)) \leq F(u_0) \tag{2.10}$$

Since $\gamma_2 > 0$, (2.10) and the Young's inequality give

$$\|u(t)\|_{H^1} \leq C_1\,, \qquad \forall\, t \geq 0 \tag{2.11}$$

where C depends only on $u_0, \gamma, \gamma_2, \gamma_1, \gamma_0$ but is independent of t.

By the Sobolev inequality, we have

$$\|u(t)\|_{L^6} \leq C, \quad (n \leq 3) \tag{2.12}$$

Step 2. H^2 norm estimate.

Multiplying equation (2.2) by $\Delta^2 u$, integrating with respect to x, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + \gamma\|\Delta^2 u\|^2 &= \int_\Omega \Delta^2 u \Delta\phi\, dx \\ &\leq \frac{\gamma}{2}\|\Delta^2 u\|^2 + \frac{1}{2\gamma}\|\Delta\phi\|^2 \end{aligned} \tag{2.13}$$

$$\begin{aligned} \|\Delta\phi\|^2 &\leq 2\Big(\int_\Omega |\phi'|^2 |\Delta u|^2 dx + \int_\Omega |\phi''|^2 |\nabla u|^4 dx\Big) \\ &\leq 2\Big[\Big(\int_\Omega |\phi'|^3 dx\Big)^{2/3}\Big(\int_\Omega |\Delta u|^6 dx\Big)^{1/3} \\ &\quad + \Big(\int_\Omega |\phi''|^6 dx\Big)^{1/3}\Big(\int_\Omega |\nabla u|^6 dx\Big)^{2/3}\Big] \\ &\leq C\Big[\Big(\int_\Omega |\Delta u|^6 dx\Big)^{1/3} + \Big(\int_\Omega |\nabla u|^6 dx\Big)^{2/3}\Big] \end{aligned} \tag{2.14}$$

By the Nirenberg inequality [1], we have

$$\|Du\|_{L^6} \leq C_1 \|D^4 u\|^{n/9} \|Du\|^{1-\frac{n}{9}} + C_2\|Du\|, \quad (n \leq 3) \tag{2.15}$$

$$\|D^2u\|_{L^6} \leq C_1\|D^4u\|^{\frac{3+n}{9}}\|Du\|^{1-\frac{3+n}{9}} + C_2\|Du\|, \quad (n \leq 3) \tag{2.16}$$

Thus, by (2.9) and Young's inequality, we obtain

$$\|\nabla u\|_{L^6}^4 \leq C_1'\|D^4u\|^{4/3} + C_2' \leq \varepsilon\,\|D^4u\|^2 + C_\varepsilon \tag{2.17}$$

$$\|\Delta u\|_{L^6}^2 \leq C_1'\|D^4u\|^{4/3} + C_2' \leq \varepsilon\|D^4u\|^2 + C_\varepsilon \tag{2.18}$$

Since for $t > 0$, $u \in \overset{\circ}{H}{}^4_E$, by the regularity theorem of elliptic operator, we have

$$\|D^4u(t)\|^2 \leq C\|\Delta^2u(t)\|^2 \tag{2.19}$$

where C depends only on Ω.

Taking ε small enough, we finally arrive at

$$\frac{d}{dt}\|\Delta u\|^2 + \frac{\gamma}{2}\|\Delta^2u\|^2 \leq C_2 \tag{2.20}$$

Thus, it follows from (2.19), (2.20) that

$$\frac{d}{dt}\|\Delta u\|^2 + \frac{\gamma}{2C}\|\Delta u\|^2 \leq C_2 \tag{2.21}$$

which gives

$$\begin{aligned} \|\Delta u\|^2 &\leq e^{-\frac{\gamma}{2C}t}\left(\|\Delta u_0\|^2 + C_2\int_0^t e^{\frac{\gamma}{2C}t}dt\right) \\ &= e^{-\frac{\gamma}{2C}t}\left(\|\Delta u_0\|^2 + \frac{2CC_2}{\gamma}(e^{\frac{\gamma}{2C}t} - 1)\right) \\ &\leq C_3 \end{aligned} \tag{2.22}$$

Again applying regularity theorem of elliptic operator for $u \in \overset{\circ}{H}{}^4_E$, from (2.22) we obtain

$$\|u(t)\|_{H^2} \leq C_4 \tag{2.23}$$

By Sobolev's imbedding theorem, we have

$$\|u(t)\|_{L^\infty} \le C_5 \tag{2.24}$$

Step 3. H^3 norm estimate.

Since $u_0 \in \overset{\circ}{H}{}^4_E$, by the bootstrap argument we have [13]

$$u \in L^2([0,T];H^6),\ u_t \in L^2([0,T];H^2) \tag{2.25}$$

Thus, we can act Δ on equation (2.2) and obtain

$$\frac{\partial \Delta u}{\partial t} + \gamma\Delta^3 u = \Delta^2\phi(u) \tag{2.26}$$

From the equation and boundary conditions of (2.2) it follows

$$\frac{\partial}{\partial n}\left(\gamma\Delta^2 u - \Delta\phi(u)\right)\Big|_\Gamma = 0 \tag{2.27}$$

Multiplying equation (2.26) by $-\Delta^2 u$ and integrating with respect to x, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\Delta u\|^2 + \gamma\int_\Omega |\nabla\Delta^2 u|^2 dx - \int_\Omega \nabla\Delta\phi(u)\cdot\nabla\Delta u dx = 0 \tag{2.28}$$

Owing to (2.23), (2.24) and

$$\nabla\Delta\phi(u) = \nabla\left(\phi'(u)\Delta u + \phi''(u)|\nabla u|^2\right) \tag{2.29}$$

we obtain

$$\left|\int_\Omega \nabla\Delta\phi(u)\cdot\nabla\Delta u dx\right| \le C\left(\|\nabla\Delta u\|^2 + \|\nabla u\Delta u\|^2 + \|\nabla u\|^6_{L^6}\right) \le C\left(\|\nabla\Delta u\|^2 + \|\nabla u\Delta u\|^2 + C'\right) \tag{2.30}$$

$$\|\nabla\Delta u\|^2 = -\int_\Omega \Delta u\Delta^2 u dx \le \varepsilon\|\Delta^2 u\|^2 + C_\varepsilon \tag{2.31}$$

$$\|\nabla u\Delta u\|^2 \le |\nabla u|^2_{L^\infty}\|\Delta u\|^2 \le C|\nabla u|^2_{L^\infty} \tag{2.32}$$

By Nirenberg's inequality, we have

$$\begin{aligned}|Du|^2_{L^\infty} &\le C\left(\|D^4u\|^{n/3}\|Du\|^{2-\frac{n}{3}} + \|Du\|^2\right) \\ &\le \varepsilon\ \|D^4u\|^2 + C_\varepsilon \le C\varepsilon\ \|\Delta^2 u\|^2 + C_\varepsilon, \quad (n \le 3)\end{aligned} \tag{2.33}$$

Thus

$$\frac{d}{dt}\|\nabla\Delta u\|^2 + 2\gamma\ \|\nabla\Delta^2 u\|^2 \le C\varepsilon\ \|\Delta^2 u\|^2 + C' \tag{2.34}$$

Adding (2.20), (2.34) together and taking ε small enough gives

$$\frac{d}{dt}\left(\|\Delta u\|^2 + \|\nabla\Delta u\|^2\right) + \frac{\gamma}{4}\|\Delta^2 u\|^2 \le C \tag{2.35}$$

which also implies

$$\frac{d}{dt}\left(\|\Delta u\|^2 + \|\nabla\Delta u\|^2\right) + \frac{C''\gamma}{4}\left(\|\Delta u\|^2 + \|\nabla\Delta u\|^2\right) \le C \tag{2.36}$$

Thus, solving the above differential inequality in the same way as (2.21) leads to

$$\|\nabla\Delta u\|^2 \le C_6 \tag{2.37}$$

Since for $u \in \overset{\circ}{H}{}^4_E$, $\|\nabla\Delta u\|$ is equivalent to H^3 norm, (2.37) gives the desired estimate. Thus the proof is completed. ∎

The ω-limit set of u_0 is defined as follows.

$$\omega(u_0) = \{v(x) \mid \exists\, t_n,\ \text{s.t.}\ \ u(x,t_n) \xrightarrow{H^2} v(x)\} \tag{2.38}$$

We now have

THEOREM 2.2.

For any $u_0 \in \overset{\circ}{H}{}^4_E$, the ω-limit set of u_0 is a compact connected subset in $\overset{\circ}{H}{}^2_E$. Furthermore,

(i) $\omega(u_0)$ is positive invariant under $T(t)$ defined by solution $u(x,t)$, i.e., $T(t)\omega(u_0) \subset \omega(u_0)$ for any $t \geq 0$.

(ii) $F(u)$ defined by (2.7) is constant on $\omega(u_0)$.

Proof. By Theorem 2.1 and (2.4), this is a consequence of the known results of [8] (Proposition 2.1, 2.2). ∎

It can be seen from (2.2) that the stationary problem associated with (2.2) is the following:

$$(P)\quad \begin{aligned} &\gamma\Delta v = \phi(v) - \sigma, \\ &\frac{\partial v}{\partial n}\Big|_{\Gamma} = 0, \\ &\int_{\Omega} v dx = 0 \end{aligned} \tag{2.39}$$

where σ is a constant to be determined along with function $v(x)$.

Owing to (2.9), as a direct consequence of Theorem 2.2, we immediately have

THEOREM 2.3.

Each element $v \in \omega(u_0)$ is an equilibrium solution of (2.20), i.e. a solution of problem (P). ∎

§3. STATIONARY PROBLEM

The stationary problem (P) associated with (2.2) is quite different from the usual Dirichlet, Neumann, etc. boundary value problems. In the one-dimensional case, i.e., $\Omega = (-L,L)$, it has been extensively investigated by Carr, Gurtin and Slemrod in [5]. They especially focussed their attention on the problem of phase transition as $\gamma \to 0$. But in this paper we want to consider this problem from different aspects related to the asymptotic behavior of the solution of (2.2). More precisely, we are interested in, for fixed γ, the existence and number of solutions of problem (P).

It is easy to see that $v \equiv 0$ ($\sigma = 0$) is a (trivial) solution of problem (P). It is interesting to know whether there exist nontrivial solutions.

Let

$$H_E^1 = \{v(x) | v \in H^1(\Omega), \int_\Omega v dx = 0\} \tag{3.1}$$

equipped with the usual H^1 norm. Then it is a Hilbert space.

LEMMA 3.1.

Problem (P) is equivalent to the problem of finding the critical points of functional $F(v)$ defined by (2.7) over H_E^1.

Proof. If (v,σ) is a solution of problem (P), then for any $w \in H_E^1$, by multiplying (2.39) by w and integrating with respect to x, we obtain

$$\int_\Omega (\gamma\nabla v\nabla w + \phi(v)w)dx = 0, \quad \forall\, w \in H_E^1 \tag{3.2}$$

This means v is a critical point of F over H_E^1.

On the other hand, if v is a critical point, i.e., (3.2) is satisfied, then for any $w \in H^1$ which implies $w - \frac{1}{|\Omega|}\int_\Omega w dx \in H_E^1$, we have

$$\int_\Omega \left[\gamma\nabla v\nabla w + (\phi(v) - \frac{1}{|\Omega|}\int_\Omega \phi dx)w\right]dx = 0, \quad \forall\, w \in H^1 \tag{3.2'}$$

i.e., v is a weak solution of equation (2.39) with Neumann boundary condition and $\sigma = \frac{1}{|\Omega|}\int_\Omega \phi dx$.

By the usual bootstrap argument we can conclude that v is a classical solution of problem (P). ∎

LEMMA 3.2.

If γ is large enough (lower bound will be given in the proof), then problem (P) has no nontrivial solutions.

Proof. Since any critical point $v \in H_E^1$ of $F(v)$ must satisfy (3.2), by taking $w = v$, we obtain

$$\int_\Omega (\gamma|\nabla v|^2 + \phi(v)v)dx = 0 \tag{3.3}$$

By Poincaré's inequality we have

$$\|v\|^2 \leq C_1\|\nabla v\|^2, \qquad \forall\, v \in H_E^1 \tag{3.4}$$

where C_1 is a constant depending only on Ω.

From

$$\left|\int_\Omega \gamma_1 v^3 dx\right| \leq \gamma_2 \int_\Omega v^4 dx + \frac{\gamma_1^2}{4\gamma_2}\int_\Omega v^2 dx \tag{3.5}$$

and (3.3) it follows that

$$\begin{aligned} 0 &= \int_\Omega (\gamma|\nabla v|^2 + \gamma_2 v^4 + \gamma_1 v^3 + \gamma_0 v^2)dx \\ &\geq \left(\frac{\gamma}{C_1} - \frac{\gamma_1^2}{4\gamma_2} - |\gamma_0|\right)\|v\|^2 \end{aligned} \tag{3.6}$$

If $\gamma > C_1\left(\frac{\gamma_1^2}{4\gamma_2} + |\gamma_0|\right)$, then (3.6) gives

$$v \equiv 0 \tag{3.7}$$

This completes the proof. ∎

THEOREM 3.1.

If $\gamma > C_1\left(\frac{\gamma_1^2}{4\gamma_2} + |\gamma_0|\right)$, then for any $u_0 \in \overset{\circ}{H}{}_E^4$, the solution $u(x,t)$ of (2.2) converges in H^2 to zero as $t \to +\infty$.

Proof. This is a corollary of Theorems 2.2., 2.3 and Lemma 3.2. ∎

On the other hand, for problem (1.1) if $|M| = \left|\frac{1}{|\Omega|}\int_\Omega u_0 dx\right|$ is large enough, no matter how small γ is, we still have

THEOREM 3.2.

If $|M|$ is large enough, then the solution $u(x,t)$ of (1.1) converges in H^2 to M as $t \to +\infty$.

Proof. First, we claim that when $|M|$ is large enough, for $\tilde{\phi}(\tilde{u})$ defined by (2.3) we have

$$\tilde{\phi}(\tilde{u})\tilde{u} = (\gamma_2\tilde{u}^2 + \tilde{\gamma}_1\tilde{u} + \tilde{\gamma}_0)\tilde{u}^2 \geq 0 \tag{3.8}$$

Indeed, this follows from (2.4) and

$$\tilde{\gamma}_1^2 - 4\gamma_2\tilde{\gamma}_0 = -3\gamma_2^2M^2 - 2\gamma_1\gamma_2M + \gamma_1^2 + 4\gamma_2 < 0 \tag{3.9}$$

as $|M|$ large enough.

Thus, (3.3), (3.8) imply that zero (M) is the only solution of problem (P) (corresponding stationary problem of (1.1)). We complete the proof by applying Theorems 2.2, 2.3. ∎

In general, problem (P) may have nontrivial solutions.

THEOREM 3.3.

If $\gamma_0 < 0$ and γ is small enough, then problem (P) admits at least one nontrivial solution.

Proof. Since $\gamma_2 > 0$, $F(u)$ is bounded from below.

Let u_n be a minimizing sequence:

$$F(u_n) \to m = \inf_{v \in H_E^1} F(v) \tag{3.10}$$

Thus u_n is bounded in H_E^1. Therefore, there exists a subsequence, still denoting by u_n, such that

$$\begin{aligned} &u_n \to u, \text{ weakly in } H^1_E \\ &u_n \to u, \text{ strongly in } L^p(\Omega),\ 1 < p < 6,\ \ n \le 3 \end{aligned} \tag{3.11}$$

which gives

$$\int_\Omega |\nabla u|^2 dx \le \underline{\lim}_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx \tag{3.12}$$

$$\int_\Omega H(u_n)dx \to \int_\Omega H(u)dx \tag{3.13}$$

Thus it follows from (3.10), (3.12), (3.13) that u is a minimum element of F, and, by Lemma 3.1, is also a solution of problem (P).

It remains to prove $m < 0$.

Let $\lambda_1 > 0$ be the first nonzero eigenvalue of the following eigenproblem

$$\begin{cases} -\Delta v = \lambda v \\ \left.\frac{\partial v}{\partial n}\right|_\Gamma = 0 \end{cases} \tag{3.14}$$

and ϕ_1 be the corresponding eigenfunction such that $\|\phi_1\| = 1$.

Then

$$F(\varepsilon\phi_1) = \frac{1}{2}(\lambda_1\gamma + \gamma_0)\varepsilon^2 + \frac{\gamma_2}{4}\int_\Omega \phi_1^4 dx\cdot\varepsilon^4 + \frac{\gamma_1}{3}\int_\Omega \phi_1^3 dx\cdot\varepsilon^3 \tag{3.15}$$

shows if $\gamma < \frac{|\gamma_0|}{\lambda_1}$, $\gamma_0 < 0$, and ε small enough, then $F(\varepsilon\phi_1) < 0$. This completes the proof. ∎

If $H(u)$ is an even function, i.e., $\gamma_1 = 0$, then by the critical point theory [2], [3], one may conclude that problem (P) admits at least $2N + 1$ solutions as γ small enough and $\gamma_0 < 0$. But it is not enough for our purpose. We are interested in at most how many solutions there are in order to exclude the possibility that $\omega(u_0)$ consists of infinitely many points.

In the one space dimensional case, we are going to give a satisfactory answer to the above problem.

We first notice that by translation and scaling of the unknown function and independent variable x, problem (P) can be deduced to

$$\begin{cases} \gamma \dfrac{d^2u}{dx^2} = u^3 - u - \sigma, \\ u_x\Big|_{x=-1,1} = 0, \\ \dfrac{1}{2}\int_{-1}^{1} u dx = M \end{cases} \tag{3.16}$$

The nonlinear term $u^3 - u$ is of Chafee-Infante type.

Secondly, we would like to point out that since the eigenfunctions of linearized problem of (3.16) may be two-dimensional, the theory recently developed in [10], which is applicable to the usual Dirichlet, Neumann boundary value problems, is not applicable to (3.16).

Novick-Cohen and Segel [15] found that the general solution of (9.16) can be expressed as a Jacobi elliptic function. Carr, Gurtin and Slemrod studied problem (3.16) especially from the aspect of phase transition. But, it is not clear from [5], [15] that for fixed small γ, how many solutions of (3.16) there exist.

In what follows we will use the same notations as in [5].

Let

$$\begin{cases} \gamma = 2\varepsilon^2, \ W(u) = \frac{1}{4}u^4 - \frac{1}{2}u^2, \ r = M \\ x = \varepsilon t, \ z(t) = u(\varepsilon t), \\ \Phi_\sigma(z) = W(z) - \sigma z = \frac{1}{4}z^4 - \frac{1}{2}z^2 - \sigma z \\ f_\Delta(z) = \Phi_\sigma(z) - b, \ \Delta = (\sigma, b) \end{cases} \tag{3.17}$$

The constants α_0, β_0, σ_0 defined by the Maxwell condition (see [5], (1.6)) are −1, 1, 0.

It has been proved in [5]:

LEMMA 3.3.

If $|r| = |M| \geq 1$, then problem (3.16) admits only trivial solution $u \equiv r = M$.

Actually, Lemma 3.3 is an inversion of Theorem 3.2.

We now have

THEOREM 3.4.

For fixed r, $-1 < r < 1$, and fixed $\varepsilon > 0$, problem (3.16) has only finite number of solutions. In particular, if $r = 0$ and $\exists$ integer N_0 such that $N_0 \leq \frac{\sqrt{2}}{\pi\varepsilon} < N_0 + 1$ $(N_0 \leq \frac{2}{\pi\sqrt{\gamma}} < N_0 + 1)$, then problem (3.16) has exactly $2N_0 + 1$ solutions.

Proof. We only give the proof for the case $r = 0$. The proof in the general case can be done in the same way.

As was shown in [5], the proof is reduced to finding admissible pair $\Delta = (\sigma, b)$ such that

$$I_0(\Delta) \triangleq \int_{z_1}^{z_2} f_\Delta^{-1/2}(z)dz = 2N^{-1}\varepsilon^{-1} \tag{3.18}$$

$$I_1(\Delta) \triangleq \int_{z_1}^{z_2} zf_\Delta^{-1/2}(z)dz = 2N^{-1}r\varepsilon^{-1} \tag{3.19}$$

where $z_1(\Delta)$, $z_2(\Delta)$ are roots of $f_\Delta(z) = 0$ and N is an integer (N^{-1} was misprinted to be N in [5]). Once we found the admissible pair satisfying (3.18), (3.19), we would have a pair of solutions $u(x)$ and $u(-x)$ of problem (3.16). For such solutions, u_x vanishes N times in $-1 \leq x < 1$, i.e., u has N transitions.

The admissible domain $\sum$ of $\Delta = (\sigma,b)$ is bounded by

$$\begin{cases} \partial_1\sum = \{(\sigma,b): b = \Phi_\sigma(\alpha_\sigma),\ 0 \le \sigma < \bar{\sigma} = \frac{2}{3\sqrt{3}}\} \\ \partial_2\sum = \{(\sigma,b): b = \Phi_\sigma(\beta_\sigma),\ -\frac{2}{3\sqrt{3}} = \underline{\sigma} < \sigma \le 0\} \\ \partial_3\sum = \{(\sigma,b): b = \Phi_\sigma(\xi_\sigma),\ -\frac{2}{3\sqrt{3}} = \underline{\sigma} \le \sigma \le \bar{\sigma} = \frac{2}{3\sqrt{3}}\} \end{cases} \tag{3.20}$$

and is shown in Figure 1.

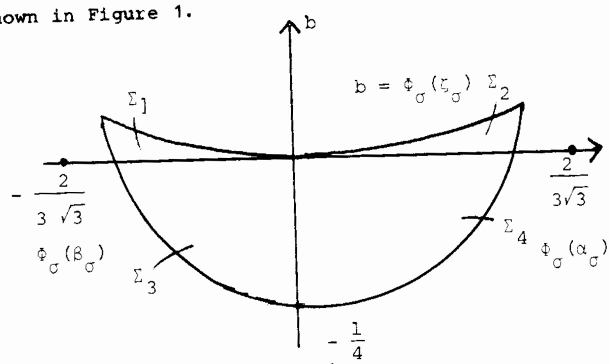

Fig. 1

where α_σ, ζ_σ, β_σ are roots of

$$z^3 - z - \sigma = 0 \tag{3.21}$$

It can be seen from (3.18), (3.19) that we only need to prove:

(i) $I_1(\Delta) = 0$ for $(0,b) \in \sum$
$I_1(\Delta) \ne 0$, for $(\sigma,b) \in \sum$, $\sigma \ne 0$

(ii) For $(0,b) \in \sum$, $I_0(\Delta)$ is a strictly monotone decreasing function of b.

(iii) Moreover,

$$\lim_{b\to -1/4} I_0(\Delta) = +\infty, \tag{3.22}$$

for $(0,b) \in \sum$

$$\lim_{b\to 0} I_0(\Delta) = \sqrt{2}\pi, \tag{3.23}$$

<u>Proof of (i).</u>

Since z_1, z_2 are roots of $f_\Delta(z)$, for $(0,b) \in \sum$, $I_1(\Delta) = 0$ follows from that

$$z_1(\Delta) = -z_2(\Delta), \quad \text{for } (0,b) \in \sum \tag{3.24}$$

and $f_\Delta(z)$ is an even function of z.

By Propositions 2.2 and 4.1 of [5], we have

$$\Phi_\sigma'(z_2) = z_2^3 - z_2 - \sigma < 0 \tag{3.25}$$

$$\Phi_\sigma'(z_1) = z_1^3 - z_1 - \sigma > 0 \tag{3.26}$$

Differentiating

$$f_\Delta(z_i) = 0 \qquad (i = 1,2) \tag{3.27}$$

with respect to σ, b and using (3.25), (3.26), we obtain for $\Delta \in \sum$

$$\frac{\partial z_2}{\partial b} = \frac{1}{z_2^3 - z_2 - \sigma} < 0, \quad \frac{\partial z_1}{\partial b} = \frac{1}{z_1^3 - z_1 - \sigma} > 0, \tag{3.28}$$

$$\frac{\partial z_i}{\partial \sigma} = \frac{z_i}{z_i^3 - z_i - \sigma}\ . \tag{3.29}$$

For $(\sigma,0) \in \sum$, we have

$$\begin{cases} z_1 < z_2 = 0\ , & \sigma > 0 \\ 0 = z_1 < z_2\ , & \text{as } \sigma < 0 \end{cases} \tag{3.30}$$

$$\begin{cases} 0 > z_2 > z_1\ , \text{ for } \Delta \in \sum_2 = \{(\sigma,b) \in \sum,\ \sigma > 0,\ b > 0\} \\ 0 < z_1 < z_2\ , \text{ for } \Delta \in \sum_1 = \{(\sigma,b) \in \sum,\ \sigma < 0,\ b > 0\} \end{cases} \tag{3.31}$$

and

$$\frac{\partial z_i}{\partial \sigma} < 0 \ , \ \text{for } \Delta \in \textstyle\sum, \ b < 0 \tag{3.32}$$

It follows from (3.31) that

$$I_1(\Delta) < 0 \ , \quad \text{for } \Delta \in \textstyle\sum_2 \text{ or } (\sigma,0) \in \sum, \ \sigma > 0 \tag{3.33}$$

$$I_1(\Delta) > 0 \ , \quad \text{for } \Delta \in \textstyle\sum_1 \text{ or } (\sigma,0) \in \sum, \ \sigma < 0 \tag{3.34}$$

When $\Delta \quad \sum$, $b < 0$, simple computation shows

$$\begin{aligned} I_1 &= \int_{z_1}^{z_2} \frac{z}{\sqrt{\frac{1}{4}z^4 - \frac{1}{2}z^2 - \sigma z - b}}\, dz \\ &= z_2 \int_0^1 \frac{x}{\sqrt{(1-x)\left[(1+x)\left(\frac{1}{2} - \frac{z_2^2}{4}(1+x^2) + \frac{\sigma}{z_2}\right)\right]}}\, dx \\ &+ z_1 \int_0^1 \frac{x}{\sqrt{(1-x)\left[(1+x)\left(\frac{1}{2} - \frac{z_1^2}{4}(1+x^2) + \frac{\sigma}{z_1}\right)\right]}}\, dx \end{aligned} \tag{3.35}$$

Since for $(0,b) \in \sum$, $b < 0$, by (3.24)

$$z_1 + z_2 = 0 \tag{3.36}$$

we have from (3.32) that when $\Delta \in \sum_4$,

$$z_2 < |z_1| \tag{3.37}$$

Thus, (3.35) shows

$$I_1 < 0, \text{ for } \Delta \in \textstyle\sum_4 \tag{3.38}$$

Similarly, we have

$$I_1 > 0, \text{ for } \Delta \in \Sigma_3 \tag{3.39}$$

Thus, the proof of (i) is completed.

Proof of (ii), (iii).

For $(0,b) \in \Sigma$,

$$\begin{aligned} I_0 &= \int_{z_1}^{z_2} \frac{dz}{\sqrt{\frac{1}{4}z^4 - \frac{1}{2}z^2 - b}} = 2\int_0^{z_2} \frac{dz}{\sqrt{\frac{1}{4}z^4 - \frac{1}{2}z^2 - b}} \\ &= 2\int_0^1 \frac{dx}{\sqrt{(1 - x)[\frac{1}{2} - \frac{z_2^2}{4}(1 + x^2)]}} \end{aligned} \tag{3.40}$$

(3.28) shows z_2 is strictly monotone decreasing in b. Therefore, I_0 is a strictly monotone decreasing function of b. Simple computation shows

$$\begin{aligned} \lim_{b\to 0} I_0 &= \lim_{z_2\to 0} 2\int_0^1 \frac{dx}{\sqrt{(1 - x^2)[\frac{1}{2} - \frac{z_2^2}{4}(1 + x^2)]}} \\ &= 2\sqrt{2} \arcsin 1 = \sqrt{2}\pi \end{aligned} \tag{3.41}$$

$$\lim_{b\to -\frac{1}{4}} I_0 = \lim_{z_2\to 1} 2\int_0^1 \frac{dx}{\sqrt{(1 - x^2)[\frac{1}{2} - \frac{z_2^2}{4}(1 + x^2)]}} = +\infty \tag{3.42}$$

Thus, the proof is completed. ∎

As a corollary of Theorems 2.2, 2.3, 3.4 and Lemma 3.4 we immediately have

THEOREM 3.5.

In the one-dimensional case, for any $u_0 \in \overset{\circ}{H}{}^4_E$, the solution $u(x,t)$

of (1.1) converges in H^2 to certain equilibrium solutions of (1.1).

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