Some studies on mathematical models for general elastic multi-structures

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Abstract The aim of this paper is to study the static problem about a general elastic multi-structure composed of an arbitrary number of elastic bodies, plates and rods. The mathematical model is derived by the variational principle and the principle of virtual work in a vector way. The unique solvability of the resulting problem is proved by the Lax-Milgram lemma after the presentation of a generalized Korn's inequality on general elastic multi-structures. The equilibrium equations are obtained rigorously by only assuming some reasonable regularity of the solution. An important identity is also given which is essential in the finite element analysis for the problem.

Keywords: elastic multi-structures, mathematical models, unique solvability, generalized Korn's inequality, equilibrium equations.

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Elastic multi-structures usually consist of a number of elastic substructures with the same or different dimensions (three-dimensional bodies, plates, rods, etc.) coupled by some proper junctions, which are widely used in engineering applications. In the past few decades, much work has been done about elastic multi-structure problems. Feng and Shi[1,2] discussed it by the variational principle together with reasonable mechanical interpretation for interface conditions. Ciarlet and his colleagues[3] have derived proper junctions among substructures by the technique of asymptotic analysis, which is important in obtaining reasonable mathematical models of such structures. Kozlov et al.[4] established an asymptotic theory of boundary values problems in multi-structures with junctions between one-dimensional and three-dimensional substructures, as well as presented a spectral analysis of the related problems.

The numerical analysis has been studied by Bernadou et al.[5] using a conforming finite element, and by Wang[6−8] using some nonconforming finite elements and the TRUNC element. There appeared some efficient domain decomposition methods for solving linear systems arising from the previous finite element methods[9,10].

However, except the work in refs. [1, 2], all other results merely treated a simple elastic multi-structure composed of only two elastic members. This is far from actual applications.

In this paper, we intend to consider a general elastic multi-structure composed of an arbitrary number of elastic bodies, plates and rods. As in ref. [2] the mathematical model is established by the variational principle and the principle of virtual work, under the assumption that classical models are used for individual elastic members and there exist rigid connections among interfaces of substructures. However, the derivation here is carried out in a vector way, which makes the deduction more intuitive and easy. The unique solvability of the resulting problem is proved by the Lax-Milgram lemma after the presentation of a generalized Korn's inequality on general elastic multi-structures. The equilibrium equations are obtained rigorously by only assuming some reasonable regularity of the solution. It should be mentioned that in existing literatures[1−4], the equilibrium equations were obtained by integration by parts under the assumption that the solution is sufficiently smooth. So considering the regularity of the actual solution, this kind of equations is only formal and can hardly be used in further theoretical studies. Otherwise, all derivations are also formal and not rigorous[6−8]. By virtue of these equilibrium equations, we finally derive an important identity which is essential in our finite element analysis presented in the forthcoming paper.

1 Mathematical model

Let there be given N_3 body members $\Omega^3 := \{\alpha_1, \cdots, \alpha_{N_3}\}$, N_2 plate members $\Omega^2 := \{\beta_1, \cdots, \beta_{N_2}\}$, and N_1 rod members $\Omega^1 := \{\gamma_1, \cdots, \gamma_{N_1}\}$, which are rigidly connected to form an elastic multi-structure:

$$\Omega = \{\alpha_1, \cdots, \alpha_{N_3}; \beta_1, \cdots, \beta_{N_2}; \gamma_1, \cdots, \gamma_{N_1}\}.$$

For simplicity, we assume that

(1) each body member α is a bounded polyhedron and each plate member β is a bounded polygon;

(2) Ω is geometrically connected in the sense that for any two points in Ω, one can connect them by a continuous path consisting of a finite number of line segments each of which belongs to some elastic member in Ω;

(3) for any two adjacent elastic members $\mathcal{A}$ and $\mathcal{B}$, the dimension of the intersection $\bar{\mathcal{A}} \cap \bar{\mathcal{B}}$ can only differ from the dimensions of these two members by one dimension at most, for example, a body member can only have body or plate members as its adjacent elastic members;

(4) Ω is geometrically conforming in the sense that if $\mathcal{A}$ and $\mathcal{B}$ are two adjacent elastic members in Ω with the same dimension, then $\partial\mathcal{A} \cap \partial\mathcal{B}$ should be the common boundary of $\mathcal{A}$ and $\mathcal{B}$.

We mention that the first condition is given only for simplicity of presentation, and the second one is satisfied generally for practical problems. However, the remaining two conditions may not be satisfied for some elastic multi-structure. In this case, one can transform the original structure into a new one which satisfies such conditions by adding or changing some individual elastic members, we refer to ref. [2] for details along this line.

It is evident that the boundary surface of a body member need not be a plate member but may be a proper boundary area element. We denote all such ones by $\Gamma^2 := \{\beta_{N_2+1}, \cdots, \beta_{N_2'}\}$, and let $\Gamma^2 = \Gamma_1^2 \cup \Gamma_2^2$ with $\Gamma_1^2 := \{\beta_{N_2+1}, \cdots, \beta_{N_2+M_2}\}$ and $\Gamma_2^2 := \{\beta_{N_2+M_2+1}, \cdots, \beta_{N_2'}\}$. Here Γ_1^2 consists of all external proper boundary area elements while Γ_2^2 consists of all interfaces of bodies. Likewise, we denote all the proper boundary lines by $\Gamma^1 := \{\gamma_{N_1+1}, \cdots, \gamma_{N_1'}\}$, and let $\Gamma^1 = \Gamma_1^1 \cup \Gamma_2^1$ with $\Gamma_1^1 := \{\gamma_{N_1+1}, \cdots, \gamma_{N_1+M_1}\}$ and $\Gamma_2^1 := \{\gamma_{N_1+M_1+1}, \cdots, \gamma_{N_1'}\}$. Here Γ_1^1 consists of all external boundary lines while Γ_2^1 consists of all interfaces of plates. We denote all boundary points (a common point is counted only once) of the rod members by $\Gamma^0 := \{\delta_1, \cdots, \delta_{N_0}\}$, denote all corner points of proper boundaries of plate members by $\Gamma_3^0 := \{\delta_{N_0+1}, \cdots, \delta_{N_0'}\}$. Let $\Gamma^0 = \Gamma_1^0 \cup \Gamma_2^0$ with $\Gamma_1^0 := \{\delta_1, \cdots, \delta_{M_0}\}$ and $\Gamma_2^0 := \{\delta_{M_0+1}, \cdots, \delta_{N_0}\}$. Here Γ_1^0 consists of all external boundary points while Γ_2^0 consists of all common boundary points.

The elements of Ω^3, $\Omega^2 \cup \Gamma^2$, $\Omega^1 \cup \Gamma^1$ and $\Gamma^0 \cup \Gamma_3^0$ are called respectively the body, area, line and point elements. Geometrically all these elements are viewed as open sets.

For any two elements $\beta \in \Omega^2 \cup \Gamma^2$ and $\alpha \in \Omega^3$, $\alpha \in \partial^{-1}\beta$ means that β is a boundary element of α. The symbols $\gamma \in \partial^{-1}\delta$, $\alpha \in \partial^{-1}\gamma$ and $\beta \in \partial^{-1}\gamma$ are defined in the same manners.

We fix a right-handed orthogonal system (x_1, x_2, x_3) in the space, called the global coordinates. Its orthonormal basis vectors are $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$. For each element ω, we adopt the following convention to introduce a local right-handed coordinate system $(x_1^\omega, x_2^\omega, x_3^\omega)$, with $\{\vec{e}_1^\omega, \vec{e}_2^\omega, \vec{e}_3^\omega\}$ being the resulting orthonormal basis vectors. The local coordinates of a body member α or a point element δ are taken to be just the global coordinates. For an area element β, x_1^β and x_2^β are its longitudinal directions, and x_3^β the transverse direction. For a line element γ, x_1^γ is the longitudinal direction, x_2^γ and x_3^γ are the transverse directions, and the origin of the local coordinates is located at an endpoint of γ. Moreover, along the boundary $\partial\beta$ of an area element β we choose a unit tangent vector $\vec{t}^\beta$ such that $\{\vec{n}^\beta, \vec{t}^\beta, \vec{e}_3^\beta\}$ forms a right-handed coordinate system, where $\vec{n}^\beta$ denotes the unit outward normal to $\partial\beta$ in the longitudinal plane, and $\vec{e}_3^\beta$ the unit transverse vector of the area element.

For an area element $\beta \in \partial\alpha$, let $\vec{n}^\alpha$ be the unit outward normal to the boundary $\partial\alpha$ of the body element α. We then define

$$\varepsilon(\alpha,\beta) := \begin{cases} 0 & \text{if } \beta \notin \partial\alpha, \\ 1 & \text{if } \beta \in \partial\alpha,\ \vec{n}^\alpha \text{ and } \vec{e}_3^\beta \text{ have the same direction on } \beta, \\ -1 & \text{if } \beta \in \partial\alpha,\ \vec{n}^\alpha \text{ and } \vec{e}_3^\beta \text{ have the opposite direction on } \beta. \end{cases}$$

Similarly,

$$\varepsilon(\beta,\gamma) := \begin{cases} 0 & \text{if } \gamma \notin \partial\beta, \\ 1 & \text{if } \gamma \in \partial\beta,\ \vec{t}^\beta \cdot \vec{e}_1^\gamma > 0 \text{ on } \gamma, \\ -1 & \text{if } \gamma \in \partial\beta,\ \vec{t}^\beta \cdot \vec{e}_1^\gamma < 0 \text{ on } \gamma, \end{cases}$$

$$\varepsilon(\gamma,\delta) := \begin{cases} 0 & \text{if } \delta \notin \partial\gamma, \\ 1 & \text{if } \delta \in \partial\gamma, \vec{e}_1^\gamma \text{ is directed toward } \delta, \\ -1 & \text{if } \delta \in \partial\gamma, \vec{e}_1^\gamma \text{ is directed outward } \delta. \end{cases}$$

Now we are ready to propose a mathematical model describing the generalized displacement field of the equilibrium configuration of Ω,

$$\vec{u} := \{\{\vec{u}^\alpha\}_{\alpha\in\Omega^3},\ \{\vec{u}^\beta\}_{\beta\in\Omega^2},\ \{\vec{u}^\gamma\}_{\gamma\in\Omega^1},\ \{u_4^\gamma\}_{\gamma\in\Omega^1}\},$$

under the action of the applied generalized load field

$$\vec{f} := \{\{\vec{f}^\alpha\}_{\alpha\in\Omega^3},\ \{\vec{f}^\beta\}_{\beta\in\Omega^2},\ \{\vec{f}^\gamma\}_{\gamma\in\Omega^1},\ \{f_4^\gamma\}_{\gamma\in\Omega^1}\}.$$

Here

$$\vec{u}^\alpha := \sum_{i=1}^3 u_i^\alpha \vec{e}_i^\alpha, \quad \vec{u}^\beta := \sum_{i=1}^3 u_i^\beta \vec{e}_i^\beta, \quad \vec{u}^\gamma := \sum_{i=1}^3 u_i^\gamma \vec{e}_i^\gamma,$$

with u_i^ω denoting the displacement along the direction $\vec{e}_i^\omega$, $1 \leqslant i \leqslant 3$, $\omega = \alpha$, β, γ, and u_4^γ stands for the rotational angle along the axis $\vec{e}_1^\gamma$. Similarly,

$$\vec{f}^\alpha := \sum_{i=1}^3 f_i^\alpha \vec{e}_i^\alpha, \quad \vec{f}^\beta := \sum_{i=1}^3 f_i^\beta \vec{e}_i^\beta, \quad \vec{f}^\gamma := \sum_{i=1}^3 f_i^\gamma \vec{e}_i^\gamma,$$

with f_i^ω being the force load along $\vec{e}_i^\omega$, $1 \leqslant i \leqslant 3$, applied on the elastic member $\omega = \alpha$, β, γ, and f_4^γ is the moment load around $\vec{e}_1^\gamma$, applied on some rod member γ.

For brevity, we assume that $\gamma_{N_1+1} \in \partial\beta_1$ and there hold the clamped conditions:

$$\vec{u}^{\beta_1} = \vec{0},\ \partial_{\vec{n}^{\beta_1}} u_3^{\beta_1} = 0 \quad \text{on } \gamma_{N_1+1}. \tag{1.1}$$

We impose the force and moment free conditions on all kinds of proper boundaries of Ω except γ_{N_1+1}.

Since Ω is rigidly connected, we have from ref. [2] that, for any area element $\beta \in \Gamma_2^2$ and any two body members $\alpha, \alpha' \in \partial^{-1}\beta$,

$$\vec{u}^\alpha = \vec{u}^{\alpha'} \ \text{ on } \beta; \tag{1.2}$$

for any line element $\gamma \in \Gamma_2^1$ and any two plate members $\beta, \beta' \in \partial^{-1}\gamma$,

$$\vec{u}^\beta = \vec{u}^{\beta'}, \quad \varepsilon(\beta,\gamma)\partial_{\vec{n}^\beta} u_3^\beta = \varepsilon(\beta',\gamma)\partial_{\vec{n}^{\beta'}} u_3^{\beta'} \quad \text{on } \gamma; \tag{1.3}$$

for any point element $\delta \in \Gamma_2^0$ and any two rod members $\gamma, \gamma' \in \partial^{-1}\delta$,

$$\sum_{i=1}^3 u_i^\gamma \vec{e}_i^\gamma = \sum_{i=1}^3 u_i^{\gamma'} \vec{e}_i^{\gamma'}, \quad \sum_{i=1}^3 u_{i+3}^\gamma \vec{e}_i^\gamma = \sum_{i=1}^3 u_{i+3}^{\gamma'} \vec{e}_i^{\gamma'} \quad \text{on } \delta, \tag{1.4}$$

where u_{i+3}^γ $(1 \leqslant i \leqslant 3)$ denotes the rotational angle along the axis $\vec{e}_i^\gamma$, with u_4^γ showed above, and $u_5^\gamma := -du_3^\gamma/dx_1^\gamma$, $u_6^\gamma := du_2^\gamma/dx_1^\gamma$. Moreover, for any plate member β and any body member $\alpha \in \partial^{-1}\beta$,

$$\vec{u}^\alpha = \vec{u}^\beta \quad \text{on } \beta; \tag{1.5}$$

for any rod member γ and any plate member $\beta \in \partial^{-1}\gamma$,

$$\vec{u}^\beta = \vec{u}^\gamma, \quad -\varepsilon(\beta,\gamma)\partial_{\vec{n}^\beta} u_3^\beta = u_4^\gamma \quad \text{on } \gamma. \tag{1.6}$$

We use classical models to describe the deformation of individual elastic members[2]. Then the admissible space for the generalized displacement field on Ω is chosen as

$$\vec{V} := \left\{ \vec{v} \in \prod_{\alpha\in\Omega^3} \vec{W}(\alpha) \times \prod_{\beta\in\Omega^2} \vec{W}(\beta) \times \prod_{\gamma\in\Omega^1} \vec{W}(\gamma) \times \prod_{\gamma\in\Omega^1} H^1(\gamma);\ \vec{v} \text{ satisfies (1.1)–(1.6)} \right\}, \tag{1.7}$$

where

$$\vec{v} = \{\{\vec{v}^\alpha\}_{\alpha\in\Omega^3}, \{\vec{v}^\beta\}_{\beta\in\Omega^2}, \{\vec{v}^\gamma\}_{\gamma\in\Omega^1}, \{v_4^\gamma\}_{\gamma\in\Omega^1}\}$$

and

$$\vec{W}(\alpha) := (H^1(\alpha))^3,\ \vec{W}(\beta) := (H_*^1(\beta))^2 \times H_*^2(\beta),\ \vec{W}(\gamma) := H^1(\gamma) \times (H^2(\gamma))^2,$$
$$H_*^1(\beta_1) := H_0^1(\beta_1; \gamma_{N_1+1}),\ H_*^2(\beta_1) := H_0^2(\beta_1; \gamma_{N_1+1}),$$
$$H_*^1(\beta) := H^1(\beta),\ H_*^2(\beta) := H^2(\beta) \text{ for each } \beta \in \Omega^2 \backslash \beta_1.$$

It is noted that we adopt the standard definitions for Sobolev spaces[11−13], and the equations used in (1.1)–(1.6) for the definition of (1.7) are understood in the trace sense of Sobolev spaces. Moreover, $\vec{v}^\beta \in (H_*^1(\beta))^2 \times H_*^2(\beta)$ indicates that $v_I^\beta \in H_*^1(\beta)$, $1 \leqslant I \leqslant 2$, and $v_3^\beta \in H_*^2(\beta)$. Similar conventions hold for other corresponding quantities.

Under the virtual generalized displacement field

$$\vec{v} = \{\{\vec{v}^\alpha\}_{\alpha\in\Omega^3}, \{\vec{v}^\beta\}_{\beta\in\Omega^2}, \{\vec{v}^\gamma\}_{\gamma\in\Omega^1}, \{v_4^\gamma\}_{\gamma\in\Omega^1}\},$$

the total potential energy of the elastic multi-structure Ω is[2]

$$J(\vec{v}) := \frac{1}{2} D(\vec{v}, \vec{v}) - F(\vec{v}),$$

where

$$F(\vec{v}) := \sum_{\alpha\in\Omega^3} F^\alpha(\vec{v}) + \sum_{\beta\in\Omega^2} F^\beta(\vec{v}) + \sum_{\gamma\in\Omega^1} F^\gamma(\vec{v}),$$

$$F^\alpha(\vec{v}) := \int_\alpha \vec{f}^\alpha \cdot \vec{v}^\alpha d\alpha, \quad F^\beta(\vec{v}) := \int_\beta \vec{f}^\beta \cdot \vec{v}^\beta d\beta,\ F^\gamma(\vec{v}) := \int_\gamma \vec{f}^\gamma \cdot \vec{v}^\gamma d\gamma + \int_\gamma f_4^\gamma v_4^\gamma d\gamma.$$

For $\vec{w} = \{\{\vec{w}^\alpha\}_{\alpha\in\Omega^3}, \{\vec{w}^\beta\}_{\beta\in\Omega^2}, \{\vec{w}^\gamma\}_{\gamma\in\Omega^1}, \{w_4^\gamma\}_{\gamma\in\Omega^1}\}$, we define

$$D(\vec{v},\vec{w}) := \sum_{\alpha\in\Omega^3} D^\alpha(\vec{v},\vec{w}) + \sum_{\beta\in\Omega^2} D^\beta(\vec{v},\vec{w}) + \sum_{\gamma\in\Omega^1} D^\gamma(\vec{v},\vec{w}),$$

where

$$D^\alpha(\vec{v},\vec{w}) := \int_\alpha \sigma_{ij}^\alpha(\vec{v})\varepsilon_{ij}^\alpha(\vec{w})d\alpha, \tag{1.8}$$

$$\begin{aligned} &\varepsilon_{ij}^\alpha(\vec{v}) := (\partial_i v_j^\alpha + \partial_j v_i^\alpha)/2, \quad \partial_i v_j^\alpha := v_{j,i}^\alpha = \partial v_j^\alpha/\partial x_i^\alpha, \\ &\sigma_{ij}^\alpha(\vec{v}) := \frac{E_\alpha}{1+\nu_\alpha}\varepsilon_{ij}^\alpha(\vec{v}) + \frac{E_\alpha\nu_\alpha}{(1+\nu_\alpha)(1-2\nu_\alpha)}(\varepsilon_{ll}^\alpha(\vec{v}))\delta_{ij}, \ 1\leqslant i,j\leqslant 3, \end{aligned} \tag{1.9}$$

$$D^\beta(\vec{v},\vec{w}) := \int_\beta \mathcal{Q}_{IJ}^\beta(\vec{v})\varepsilon_{IJ}^\beta(\vec{w})d\beta + \int_\beta \mathcal{M}_{IJ}^\beta(\vec{v})\mathcal{K}_{IJ}^\beta(\vec{w})d\beta, \tag{1.10}$$

$$\begin{aligned} &\varepsilon_{IJ}^\beta(\vec{v}) := (\partial_I v_J^\beta + \partial_J v_I^\beta)/2, \ \partial_I v_J^\beta := v_{J,I}^\beta = \frac{\partial v_J^\beta}{\partial x_I^\beta}, \\ &\mathcal{Q}_{IJ}^\beta(\vec{v}) := \frac{E_\beta h_\beta}{1-\nu_\beta^2}((1-\nu_\beta)\varepsilon_{IJ}^\beta(\vec{v}) + \nu_\beta(\varepsilon_{LL}^\beta(\vec{v}))\delta_{IJ}), \ 1\leqslant I,J\leqslant 2, \end{aligned} \tag{1.11}$$

$$\begin{aligned} &\mathcal{K}_{IJ}^\beta(\vec{v}) := -\partial_{IJ}v_3^\beta = -\frac{\partial^2 v_3^\beta}{\partial x_I^\beta \partial x_J^\beta}, \\ &\mathcal{M}_{IJ}^\beta(\vec{v}) := \frac{E_\beta h_\beta^3}{12(1-\nu_\beta^2)}((1-\nu_\beta)\mathcal{K}_{IJ}^\beta(\vec{v}) + \nu_\beta(\mathcal{K}_{LL}^\beta(\vec{v}))\delta_{IJ}), \end{aligned} \tag{1.12}$$

$$D^\gamma(\vec{v},\vec{w}) := \int_\gamma \mathcal{Q}_1^\gamma(\vec{v})\varepsilon_{11}^\gamma(\vec{w})d\gamma + \int_\gamma \mathcal{M}_i^\gamma(\vec{v})\mathcal{K}_i^\gamma(\vec{w})d\gamma, \tag{1.13}$$

$$\varepsilon_{11}^\gamma(\vec{v}) := dv_1^\gamma/dx_1^\gamma, \quad \mathcal{Q}_1^\gamma(\vec{v}) := E_\gamma A_\gamma \varepsilon_{11}^\gamma(\vec{v}), \tag{1.14}$$

$$\begin{aligned} &\mathcal{K}_2^\gamma(\vec{v}) := -d^2v_3^\gamma/(dx_1^\gamma)^2, \quad \mathcal{K}_3^\gamma := d^2v_2^\gamma/(dx_1^\gamma)^2, \\ &\mathcal{M}_2^\gamma(\vec{v}) := E_\gamma I_{22}^\gamma \mathcal{K}_2^\gamma(\vec{v}) + E_\gamma I_{23}^\gamma \mathcal{K}_3^\gamma(\vec{v}), \\ &\mathcal{M}_3^\gamma(\vec{v}) := E_\gamma I_{32}^\gamma \mathcal{K}_2^\gamma(\vec{v}) + E_\gamma I_{33}^\gamma \mathcal{K}_3^\gamma(\vec{v}), \\ &I_{23}^\gamma = I_{32}^\gamma, \end{aligned} \tag{1.15}$$

$$\mathcal{K}_1^\gamma(\vec{v}) := dv_4^\gamma/dx_1^\gamma, \quad \mathcal{M}_1^\gamma(\vec{v}) := \frac{E_\gamma}{2(1+\nu_\gamma)}J_\gamma \mathcal{K}_1^\gamma(\vec{v}). \tag{1.16}$$

Here $E_\omega > 0, \nu_\omega \in (0, \frac{1}{2})$ denote Young's modulus, Poisson's ratio of the elastic member $\omega = \alpha, \beta, \gamma$, respectively; h_β is the thickness of plate β; A_γ is the area of the cross section, I_{ij} the moment of inertia of the cross section, and J_γ the geometric torsional rigidity of the cross section; δ_{ij} and δ_{IJ} stand for the usual Kronecker delta. Unless stated to the contrary, Latin indices i, j, l take their values in the set $\{1, 2, 3\}$, while the capital Latin indices I, J, L (resp. K) take their values in the set $\{1, 2\}$ (resp. $\{2, 3\}$). For clarity, we also use the summation convention whereby summation is implied when a Latin index (or a capital Latin index) is repeated exactly two times.

According to the variational principle, the generalized displacement field of the equilibrium configuration $\vec{u}$ under the prescribed geometric constraint must minimize the total potential energy $J(\vec{u})$. Therefore, our mathematical model for the general elastic multi-structure Ω is to find $\vec{u} = \{\{\vec{u}^\alpha\}_{\alpha\in\Omega^3}, \{\vec{u}^\beta\}_{\beta\in\Omega^2}, \{\vec{u}^\gamma\}_{\gamma\in\Omega^1}, \{u_4^\gamma\}_{\gamma\in\Omega^1}\} \in \vec{V}$ such that

$$J(\vec{u}) = \min_{\vec{v}\in\vec{V}} J(\vec{v}), \tag{1.17}$$

or equivalently,

$$D(\vec{u},\vec{v}) = F(\vec{v}), \quad \forall \vec{v} \in \vec{V} \tag{1.18}$$

by the principle of virtual work.

2 Unique solvability of the problem

Throughout this paper, we use the usual notations for Sobolev norms and seminorms, and we also use "$\lesssim \cdots$" (resp. "$\gtrsim \cdots$") to denote "$\leqslant C\cdots$" (resp. "$\geqslant C\cdots$") with a generic positive constant C independent of corresponding parameters and functions under considerations.

Lemma 2.1. Let α be a body member in Ω and Γ_2 a subset of $\partial\alpha$ with $\mathrm{meas}(\Gamma_2) > 0$. Then for each $\vec{u}^\alpha \in \vec{W}(\alpha)$,

$$\sum_{i=1}^{3} \|u_i^\alpha\|_{1,\alpha}^2 \lesssim \sum_{i,j=1}^{3} \|\varepsilon_{ij}^\alpha(\vec{u})\|_{0,\alpha}^2 + \sum_{i=1}^{3} \int_{\Gamma_2} |u_i^\alpha|^2 d\Gamma_2. \tag{2.1}$$

This result can be obtained by the compactness argument[12] along with the well-known Korn's inequality in three dimensions[13]. The next result can be proved in the similar manners.

Lemma 2.2. Let β be a plate member in Ω and Γ_1 a subset of $\partial\beta$ with $\mathrm{meas}(\Gamma_1) > 0$. Then for each $\vec{u}^\beta \in \vec{W}(\beta)$,

$$\begin{aligned} \sum_{I=1}^{2} \|u_I^\beta\|_{1,\beta}^2 + \|u_3^\beta\|_{2,\beta}^2 \lesssim & \sum_{I,J=1}^{2} (\|\varepsilon_{IJ}^\beta(\vec{u})\|_{0,\beta}^2 + \|\mathcal{K}_{IJ}^\beta(\vec{u})\|_{0,\beta}^2) + \sum_{i=1}^{3} \int_{\Gamma_1} |u_i^\beta|^2 d\Gamma_1 \\ & + \int_{\Gamma_1} |\partial_{\vec{n}^\beta} u_3^\beta|^2 d\Gamma_1. \end{aligned} \tag{2.2}$$

We now equip the function space $\vec{V}$ with a norm given by

$$\begin{aligned} \|\vec{v}\|_{\vec{V}} := \Bigg\{ & \sum_{\alpha\in\Omega^3} \sum_{i=1}^{3} \|v_i^\alpha\|_{1,\alpha}^2 + \sum_{\beta\in\Omega^2} \left(\sum_{I=1}^{2} \|v_I^\beta\|_{1,\beta}^2 + \|v_3^\beta\|_{2,\beta}^2 \right) \\ & + \sum_{\gamma\in\Omega^1} \left(\|v_1^\gamma\|_{1,\gamma}^2 + \sum_{K=2}^{3} \|v_K^\gamma\|_{2,\gamma}^2 + \|v_4^\gamma\|_{1,\gamma}^2 \right) \Bigg\}^{1/2} \end{aligned} \tag{2.3}$$

for all $\vec{v} \in \vec{V}$.

Theorem 2.1. The function space $\vec{V}$ equipped with the norm (2.3) is a Hilbert space. Moreover, for each $\vec{v} \in \vec{V}$,

$$\begin{aligned}\|\vec{v}\|_{\vec{V}}^2 \lesssim \sum_{\alpha\in\Omega^3}\sum_{i,j=1}^{3}\|\varepsilon_{ij}^{\alpha}(\vec{v})\|_{0,\alpha}^2 + \sum_{\beta\in\Omega^2}\sum_{I,J=1}^{2}(\|\varepsilon_{IJ}^{\beta}(\vec{v})\|_{0,\beta}^2 + \|\mathcal{K}_{IJ}^{\beta}(\vec{v})\|_{0,\beta}^2) \\ + \sum_{\gamma\in\Omega^1}(\|\varepsilon_{11}^{\gamma}(\vec{v})\|_{0,\gamma}^2 + \sum_{i=1}^{3}\|\mathcal{K}_{i}^{\gamma}(\vec{v})\|_{0,\gamma}^2)\end{aligned} \tag{2.4}$$

and

$$\|\vec{v}\|_{\vec{V}}^2 \lesssim D(\vec{v},\vec{v}). \tag{2.5}$$

(2.4) can be viewed as a generalized Korn's inequality on general elastic multi-structures Ω.

Proof. At first, it is easy to check that $\vec{V}$ is a Hilbert space. We now verify (2.4) by contradiction. Assume that (2.4) is not valid. Then there exists a sequence $\{\vec{v}^k\}$ in $\vec{V}$ such that

$$\begin{aligned}\|\vec{v}^k\|_{\vec{V}}^2 = \sum_{\alpha\in\Omega^3}\sum_{i=1}^{3}\|v_i^{\alpha,k}\|_{1,\alpha}^2 + \sum_{\beta\in\Omega^2}\left(\sum_{I=1}^{2}\|v_I^{\beta,k}\|_{1,\beta}^2 + \|v_3^{\beta,k}\|_{2,\beta}^2\right) \\ + \sum_{\gamma\in\Omega^1}\left(\|v_1^{\gamma,k}\|_{1,\gamma}^2 + \sum_{K=2}^{3}\|v_K^{\gamma,k}\|_{2,\gamma}^2 + \|v_4^{\gamma,k}\|_{1,\gamma}^2\right) = 1\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\sum_{\alpha\in\Omega^3}\sum_{i,j=1}^{3}\|\varepsilon_{ij}^{\alpha}(\vec{v}^k)\|_{0,\alpha}^2 + \sum_{\beta\in\Omega^2}\sum_{I,J=1}^{2}(\|\varepsilon_{IJ}^{\beta}(\vec{v}^k)\|_{0,\beta}^2 + \|\mathcal{K}_{IJ}^{\beta}(\vec{v}^k)\|_{0,\beta}^2) \\ + \sum_{\gamma\in\Omega^1}\left(\|\varepsilon_{11}^{\gamma}(\vec{v}^k)\|_{0,\gamma}^2 + \sum_{i=1}^{3}\|\mathcal{K}_{i}^{\gamma}(\vec{v}^k)\|_{0,\gamma}^2\right) < \frac{1}{k}.\end{aligned} \tag{2.7}$$

By the Rellich-Kondrachov compact embedding theorem[11] and noting the assumption (2.6), we know that there exists a subsequence $\{\vec{v}^k\}$ (we use the same notation for simplicity) such that for each $\alpha \in \Omega^3$, $\beta \in \Omega^2$ and $\gamma \in \Omega^1$,

$$\begin{aligned}&v_i^{\alpha,k} \to v_i^{\alpha,*} \quad \text{in} \quad L^2(\alpha),\ v_I^{\beta,k} \to v_I^{\beta,*} \quad \text{in} \quad L^2(\beta),\ v_3^{\beta,k} \to v_3^{\beta,*} \quad \text{in} \quad H^1(\beta); \\ &v_1^{\gamma,k} \to v_1^{\gamma,*} \quad \text{in} \quad L^2(\gamma),\ v_K^{\gamma,k} \to v_K^{\gamma,*} \quad \text{in} \quad H^1(\gamma),\ v_4^{\gamma,k} \to v_4^{\gamma,*} \quad \text{in} \quad L^2(\gamma).\end{aligned}$$

Therefore, we have by Korn's inequality and (2.7) that

$$\vec{v}^k \to \vec{v}^* \quad \text{in} \quad \vec{V}.$$

Combining this with (2.6) and (2.7) leads to

$$\begin{aligned}\sum_{\alpha\in\Omega}\sum_{i=1}^{3}\|v_i^{\alpha,*}\|_{1,\alpha}^2 + \sum_{\beta\in\Omega^2}\left(\sum_{I=1}^{2}\|v_I^{\beta,*}\|_{1,\beta}^2 + \|v_3^{\beta,*}\|_{2,\beta}^2\right) \\ + \sum_{\gamma\in\Omega^1}\left(\|v_1^{\gamma,*}\|_{1,\gamma}^2 + \sum_{K=2}^{3}\|v_K^{\gamma,*}\|_{2,\gamma}^2 + \|v_4^{\gamma,*}\|_{1,\gamma}^2\right) = 1\end{aligned} \tag{2.8}$$

and

$$\sum_{\alpha\in\Omega^3}\sum_{i,j=1}^{3}\|\varepsilon_{ij}^{\alpha}(\vec{v}^*)\|_{0,\alpha}^2+\sum_{\beta\in\Omega^2}\sum_{I,J=1}^{2}(\|\varepsilon_{IJ}^{\beta}(\vec{v}^*)\|_{0,\beta}^2+\|\mathcal{K}_{IJ}^{\beta}(\vec{v}^*)\|_{0,\beta}^2)$$
$$+\sum_{\gamma\in\Omega^1}\left(\|\varepsilon_{11}^{\gamma}(\vec{v}^*)\|_{0,\gamma}^2+\sum_{i=1}^{3}\|\mathcal{K}_{i}^{\gamma}(\vec{v}^*)\|_{0,\gamma}^2\right)=0. \tag{2.9}$$

Equation (2.9) shows that for each $\alpha\in\Omega^3$, $\beta\in\Omega^2$ and $\gamma\in\Omega^1$,

$$\varepsilon_{ij}^{\alpha}(\vec{v}^*)=0,\ \varepsilon_{IJ}^{\beta}(\vec{v}^*)=0,\ \mathcal{K}_{IJ}^{\beta}(\vec{v}^*)=0,\ \varepsilon_{11}^{\gamma}(\vec{v}^*)=0,\ \mathcal{K}_{i}^{\gamma}(\vec{v}^*)=0. \tag{2.10}$$

On the other hand, we have by the second and third equations of (2.10) that $\vec{v}^{\beta,*}$ must take the form[2]

$$\begin{bmatrix} v_1^{\beta,*}\\ v_2^{\beta,*}\end{bmatrix}=\begin{bmatrix} a_1^{\beta}\\ a_2^{\beta}\end{bmatrix}+\begin{bmatrix} 0 & -b_3^{\beta}\\ b_3^{\beta} & 0\end{bmatrix}\begin{bmatrix} x_1^{\beta}\\ x_2^{\beta}\end{bmatrix}, \tag{2.11}$$

$$v_3^{\beta,*}=a_3^{\beta}-b_1^{\beta}x_1^{\beta}+b_2^{\beta}x_2^{\beta}, \tag{2.12}$$

where a_i^{β} and b_i^{β}, $1\leqslant i\leqslant 3$, are six scalar constants. Thanks to the boundary condition (1.1), it follows from the last two equations that

$$\vec{v}^{\beta_1,*}\equiv\vec{0}\ \text{ on }\ \beta_1. \tag{2.13}$$

Since Ω is geometrically connected, there must exist at least one elastic member adjacent to β_1. If there exists a body member $\alpha\in\Omega^3$ with $\alpha\in\partial^{-1}\beta_1$, then due to (1.5), (2.1), (2.10) and (2.13) we know

$$\vec{v}^{\alpha,*}\equiv\vec{0}\ \text{ in }\ \alpha. \tag{2.14}$$

If there exists a plate member $\beta'\in\Omega^2$ such that $\gamma'=\partial\beta'\cap\partial\beta_1\in\Gamma_2^1$, we have by (1.3), (2.2), (2.10) and (2.13) that

$$\vec{v}^{\beta',*}\equiv\vec{0}\ \text{ on }\ \beta'. \tag{2.15}$$

If there exists a rod member $\gamma\in\Omega^1$ with $\gamma\in\partial\beta_1$, we have by (1.6) and (2.13) that

$$\vec{v}^{\gamma,*}\equiv\vec{0}\text{ and }v_4^{\gamma,*}\equiv 0\text{ on }\ \gamma. \tag{2.16}$$

Hence, the generalized displacements are identically zero on all elastic members adjacent to β_1.

Let $\mathcal{U}$ be the set of all elastic members adjacent to β_1 and $\mathcal{U}'$ the set of all elastic members each of which has an adjacent member in $\mathcal{U}$. We consider the generalized displacement field on $\mathcal{U}'$ in the following steps. If $\beta\in\mathcal{U}$, using the same argument as the above, we easily know that the generalized displacements are identically zero on all members adjacent to β. If $\alpha\in\mathcal{U}$, it may only have body members α' or plate members β' as the adjacent members. For the first case, letting $\beta=\partial\alpha\cap\partial\alpha'\in\Gamma_2^2$, we have by (1.2), (2.1), (2.10) and (2.14) that

$$\vec{v}^{\alpha',*}\equiv\vec{0}\ \text{ on }\ \alpha'; \tag{2.17}$$

for the second case, letting $\beta''\in\Omega^2$ with $\beta''\in\partial\alpha$, we have by (1.5), (2.10) and (2.14) that

$$\vec{v}^{\beta'',*}\equiv\vec{0}\ \text{ on }\ \beta''. \tag{2.18}$$

It remains to consider a rod member $\gamma \in \mathcal{U}$ which may only have plate members or rod members as the adjacent members. For the first case, let $\beta''' \in \Omega^2$ with $\beta''' \in \partial^{-1}\gamma$. Employing the similar argument as the above, we easily have

$$\vec{v}^{\beta''',*} \equiv \vec{0} \quad \text{on} \quad \beta'''. \tag{2.19}$$

For the second case that γ'' is a line element with $\delta = \partial\gamma'' \cap \partial\gamma \in \Gamma_2^0$, it follows from the last two equations of (2.10) that $\vec{v}^{\gamma'',*}$ must take the form[2]

$$v_1^{\gamma'',*} = a_1^{\gamma''},\ v_4^{\gamma'',*} = b_1^{\gamma''},\ \begin{bmatrix} v_2^{\gamma'',*} \\ v_3^{\gamma'',*} \end{bmatrix} = \begin{bmatrix} a_2^{\gamma''} + b_3^{\gamma''} x_1^{\gamma''} \\ a_3^{\gamma''} - b_2^{\gamma''} x_1^{\gamma''} \end{bmatrix}, \tag{2.20}$$

where $a_i^{\gamma''}$ and $b_i^{\gamma''}, 1 \leqslant i \leqslant 3$, are six scalar constants. However, (1.4) and (2.16) imply

$$\sum_{i=1}^{3} v_i^{\gamma'',*} \vec{e}_i^{\gamma''} = \vec{0},\ \sum_{i=1}^{3} v_{i+3}^{\gamma'',*} \vec{e}_i^{\gamma'',*} = \vec{0} \quad \text{on} \quad \delta. \tag{2.21}$$

So it follows from (2.20)–(2.21) that

$$\vec{v}^{\gamma'',*} \equiv \vec{0} \text{ and } v_4^{\gamma'',*} = 0 \text{ on } \ \gamma''. \tag{2.22}$$

Therefore the generalized displacement field $\vec{v}^*$ is identically zero on $\mathcal{U}'$.

We can carry out the procedure repeatedly and finally obtain

$$\vec{v}^* \equiv \vec{0} \quad \text{on} \quad \Omega,$$

which contradicts (2.8), and the estimate (2.4) then follows.

We next prove (2.5). We have from (1.9) that

$$\sigma_{ij}^{\alpha}(\vec{v})\varepsilon_{ij}^{\alpha}(\vec{v}) = \frac{E_\alpha}{1+\nu_\alpha} \sum_{i,j=1}^{3} \varepsilon_{ij}^{\alpha}(\vec{v})^2 + \frac{E_\alpha \nu_\alpha}{(1+\nu_\alpha)(1-2\nu_\alpha)} (\varepsilon_{ll}^{\alpha}(\vec{v}))^2 \gtrsim \sum_{i,j=1}^{3} (\varepsilon_{ij}^{\alpha}(\vec{v}))^2. \tag{2.23}$$

Similarly, it follows from (1.11) and (1.12) that

$$\mathcal{Q}_{IJ}^{\beta}(\vec{v})\varepsilon_{IJ}^{\beta}(\vec{v}) \gtrsim \sum_{I,J=1}^{2} (\varepsilon_{IJ}^{\beta}(\vec{v}))^2,\ \mathcal{M}_{IJ}^{\beta}(\vec{v})\mathcal{K}_{IJ}^{\beta}(\vec{v}) \gtrsim \sum_{I,J=1}^{2} (\mathcal{K}_{IJ}^{\beta}(\vec{v}))^2. \tag{2.24}$$

For a rod member $\gamma \in \Omega^1$, by (1.15) and noting that $I_{22}I_{33} > (I_{23})^2$, we see

$$\mathcal{M}_K^{\gamma}(\vec{v})\mathcal{K}_K^{\gamma}(\vec{v}) = E_\gamma (I_{22}^{\gamma} \mathcal{K}_2^{\gamma}(\vec{v})^2 + 2I_{23}^{\gamma}\mathcal{K}_2^{\gamma}(\vec{v})\mathcal{K}_3^{\gamma}(\vec{v}) + I_{33}^{\gamma}\mathcal{K}_3^{\gamma}(\vec{v})^2) \gtrsim \sum_{K=2}^{3} (\mathcal{K}_K^{\gamma}(\vec{v}))^2. \tag{2.25}$$

Therefore, in terms of (1.14), (1.16) and (2.23)–(2.25), and noting the definitions (1.8), (1.10) and (1.13), we get

$$\begin{aligned} D(\vec{v},\vec{v}) \gtrsim & \sum_{\alpha\in\Omega^3} \sum_{i,j=1}^{3} \|\varepsilon_{ij}^{\alpha}(\vec{v})\|_{0,\alpha}^2 + \sum_{\beta\in\Omega^2} \sum_{I,J=1}^{2} (\|\varepsilon_{IJ}^{\beta}(\vec{v})\|_{0,\beta}^2 + \|K_{IJ}^{\beta}(\vec{v})\|_{0,\beta}^2) \\ & + \sum_{\gamma\in\Omega^1} (\|\varepsilon_{11}^{\gamma}(\vec{v})\|_{0,\gamma}^2 + \sum_{i=1}^{3} \|\mathcal{K}_i^{\gamma}(\vec{v})\|_{0,\gamma}^2). \end{aligned}$$

This with (2.4) implies (2.5). □

After obtaining Theorem 2.1, we can prove the following result directly by the Lax-Milgram lemma.

Theorem 2.2. Let $\vec{f} := \{\{\vec{f}^\alpha\}_{\alpha\in\Omega^3}, \{\vec{f}^\beta\}_{\beta\in\Omega^2}, \{\vec{f}^\gamma\}_{\gamma\in\Omega^1}, \{f_4^\gamma\}_{\gamma\in\Omega^1}\}$ be the applied generalized load field on the elastic multi-structure Ω, with $\vec{f}^\alpha \in (L^2(\alpha))^3$, $\vec{f}^\beta \in (L^2(\beta))^3$, $\vec{f}^\gamma \in (L^2(\gamma))^3$ and $f_4^\gamma \in L^2(\gamma)$. Then there exists a unique function $\vec{u} \in \vec{V}$ satisfying (1.18) (equivalently, (1.17)).

From now on, we will always use $\vec{u} = \{\{\vec{u}^\alpha\}_{\alpha\in\Omega^3}, \{\vec{u}^\beta\}_{\beta\in\Omega^2}, \{\vec{u}^\gamma\}_{\gamma\in\Omega^1}, \{u_4^\gamma\}_{\gamma\in\Omega^1}\}$ to denote the solution of (1.18), and assume that $\vec{f}^\alpha \in (L^2(\alpha))^3$, $\vec{f}^\beta \in (L^2(\beta))^3$, $\vec{f}^\gamma \in (L^2(\gamma))^3$ and $f_4^\gamma \in L^2(\gamma)$ for all $\alpha \in \Omega^3$, $\beta \in \Omega^2$ and $\gamma \in \Omega^1$.

3 Equilibrium equations

In this section, we will derive by (1.18) all the equilibrium equations, except those ones corresponding to point elements, by only assuming that $\vec{u}^\alpha \in (H^2(\alpha))^3$, $\vec{u}^\beta \in (H^2(\beta))^2 \times H^3(\beta)$, $\vec{u}^\gamma \in H^2(\gamma) \times (H^3(\gamma))^2$ and $u_4^\gamma \in H^2(\gamma)$ for all $\alpha \in \Omega^3$, $\beta \in \Omega^2$ and $\gamma \in \Omega^1$ respectively. Such regularity assumptions are reasonable in the sense that they are valid for individual elastic members when the occupied regions are convex[14−16].

3.1 Equilibrium equations on body members and area elements

To begin with, we present several identities which can be proved by integration by parts directly. For $\vec{u}^\alpha \in (H^2(\alpha))^3$, $\vec{v}^\alpha \in (H^1(\alpha))^3$,

$$\int_\alpha \sigma_{ij}^\alpha(\vec{u})\varepsilon_{ij}^\alpha(\vec{v})d\alpha = -\int_\alpha \sigma_{ij,j}^\alpha(\vec{u})v_i^\alpha d\alpha + \sum_{\beta\in\partial\alpha}\int_\beta \sigma_{ij}^\alpha(\vec{u})n_j^\alpha v_i^\alpha d\beta. \tag{3.1}$$

For $\vec{u}^\beta \in (H^2(\beta))^2 \times H^3(\beta)$, $\vec{v}^\beta \in (H^1(\beta))^2 \times H^2(\beta)$,

$$\int_\beta \mathcal{Q}_{IJ}^\beta(\vec{u})\varepsilon_{IJ}^\beta(\vec{v})d\beta = -\int_\beta \mathcal{Q}_{IJ,J}^\beta(\vec{u})v_I^\beta d\beta + \sum_{\gamma\in\partial\beta}\int_\gamma \mathcal{Q}_{IJ}^\beta(\vec{u})n_J^\beta v_I^\beta d\gamma \tag{3.2}$$

and

$$\begin{aligned}\int_\beta \mathcal{M}_{IJ}^\beta(\vec{u})\mathcal{K}_{IJ}^\beta(\vec{v})d\beta = &\int_\beta \mathcal{M}_{IJ,J}^\beta(\vec{u})\partial_I v_3^\beta d\beta\\ &-\sum_{\gamma\in\partial\beta}\int_\gamma\{\mathcal{M}_{\vec{n}\vec{n}}^\beta(\vec{u})\partial_{\vec{n}^\beta}v_3^\beta + \mathcal{M}_{\vec{n}\vec{t}}^\beta(\vec{u})\partial_{\vec{t}^\beta}v_3^\beta\}d\gamma,\end{aligned} \tag{3.3}$$

where $\vec{n}^\beta := n_I^\beta\vec{e}_I^\beta$, $\vec{t}^\beta := t_I^\beta\vec{e}_I^\beta$ and $\mathcal{M}_{\vec{n}\vec{n}}^\beta := \mathcal{M}_{IJ}^\beta n_I^\beta n_J^\beta, \mathcal{M}_{\vec{n}\vec{t}}^\beta := \mathcal{M}_{IJ}^\beta n_I^\beta t_J^\beta$. For $\vec{u}^\gamma \in H^2(\gamma) \times (H^3(\gamma))^2$, $\vec{v}^\gamma \in H^1(\gamma) \times (H^2(\gamma))^2$,

$$\int_\gamma \mathcal{Q}_1^\gamma(\vec{u})\varepsilon_{11}^\gamma(\vec{v})d\gamma = -\int_\gamma \mathcal{Q}_{1,1}^\gamma(\vec{u})v_1^\gamma d\gamma + \sum_{\delta\in\partial\gamma}\varepsilon(\gamma,\delta)(\mathcal{Q}_1^\gamma(\vec{u})v_1^\gamma)(\delta) \tag{3.4}$$

and

$$\int_\gamma \mathcal{M}_K^\gamma(\vec{u})\mathcal{K}_K^\gamma(\vec{v})d\gamma = \int_\gamma \mathcal{Q}_K^\gamma(\vec{u})(v_K^\gamma)'d\gamma + \sum_{\delta\in\partial\gamma}\varepsilon(\gamma,\delta)(\mathcal{M}_K^\gamma(\vec{u})v_{K+3}^\gamma)(\delta), \tag{3.5}$$

where

$$v_5^\gamma := -\frac{dv_3^\gamma}{dx_1^\gamma}, \quad v_6^\gamma := \frac{dv_2^\gamma}{dx_1^\gamma}; \quad \mathcal{Q}_2^\gamma(\vec{u}) := -\frac{d\mathcal{M}_3^\gamma(\vec{u})}{dx_1^\gamma}, \quad \mathcal{Q}_3^\gamma(\vec{u}) := \frac{d\mathcal{M}_2^\gamma(\vec{u})}{dx_1^\gamma}.$$

For $u_4^\gamma \in H^2(\gamma)$ and $v_4^\gamma \in H^1(\gamma)$,

$$\int_\gamma \mathcal{M}_1^\gamma(\vec{u})\mathcal{K}_1^\gamma(\vec{v})d\gamma = -\int_\gamma \mathcal{M}_{1,1}^\gamma(\vec{u})v_4^\gamma d\gamma + \sum_{\delta\in\partial\gamma}\varepsilon(\gamma,\delta)(\mathcal{M}_1^\gamma(\vec{u})v_4^\gamma)(\delta). \quad (3.6)$$

Now we are ready to get equilibrium equations on body members and area elements. For each $\alpha \in \Omega^3$, we choose $\vec{v}^\alpha$ to be an arbitrary vector-valued function in $(C_0^\infty(\alpha))^3$ and $\vec{v}^{\alpha'} \equiv \vec{0}$ for all other $\alpha' \in \Omega^3$. For $\beta \in \Omega^2$, we choose $\vec{v}^\beta \equiv \vec{0}$ on β, and for $\gamma \in \Omega^1$, $\vec{v}^\gamma \equiv \vec{0}$ and $v_4^\gamma \equiv 0$ on γ. It is clear that such a $\vec{v}$ is in $\vec{V}$, and substituting it into (1.18) yields

$$\int_\alpha \sigma_{ij}^\alpha(\vec{u})\varepsilon_{ij}^\alpha(\vec{v})d\alpha = \int_\alpha f_i^\alpha v_i^\alpha d\alpha,$$

which with (3.1) implies

$$-\int_\alpha \sigma_{ij,j}^\alpha(\vec{u})v_i^\alpha d\alpha = \int_\alpha f_i^\alpha v_i^\alpha d\alpha.$$

Therefore,

$$-\sigma_{ij,j}^\alpha(\vec{u}) = f_i^\alpha \quad \text{in } L^2(\alpha). \quad (3.7)$$

For each $\beta \in \Omega^2$, we choose $\vec{v}^\beta$ to be an arbitrary vector-valued function in $(C_0^\infty(\beta))^3$. By the trace theorem for Sobolev spaces, we know that for each $\alpha \in \partial^{-1}\beta$, there exists a $\vec{v}^\alpha \in (H^1(\alpha))^3$ such that

$$v_i^\alpha \vec{e}_i^\alpha = v_i^\beta \vec{e}_i^\beta \quad \text{on } \beta, \quad \vec{v}^\alpha \equiv \vec{0} \quad \text{on any other } \beta' \in \partial\alpha. \quad (3.8)$$

For $\alpha \in \Omega^3$ with $\varepsilon(\alpha,\beta) = 0$, we choose $\vec{v}^\alpha \equiv \vec{0}$ on α, and for all other $\beta' \in \Omega^2\backslash\beta$, $\vec{v}^{\beta'} \equiv \vec{0}$ on β'. For $\gamma \in \Omega^1$, we choose $\vec{v}^\gamma \equiv \vec{0}$ and $v_4^\gamma \equiv 0$ on γ. It is clear that such a $\vec{v} \in \vec{V}$, and substituting it into (1.18) yields

$$\sum_{\alpha\in\partial^{-1}\beta}\int_\alpha \sigma_{ij}^\alpha(\vec{u})\varepsilon_{ij}^\alpha(\vec{v})d\alpha + \int_\beta \mathcal{Q}_{IJ}^\beta(\vec{u})\varepsilon_{IJ}^\beta(\vec{v})d\beta + \int_\beta \mathcal{M}_{IJ}^\beta(\vec{u})\mathcal{K}_{IJ}^\beta(\vec{v})d\beta$$
$$= \sum_{\alpha\in\partial^{-1}\beta}\int_\alpha f_i^\alpha v_i^\alpha d\alpha + \int_\beta f_i^\beta v_i^\beta d\beta.$$

This with (3.1)–(3.3) and the construction of $\vec{v}$ implies

$$\sum_{\alpha\in\partial^{-1}\beta}\int_\beta \sigma_{ij}^\alpha(\vec{u})n_j^\alpha v_i^\alpha d\beta - \int_\beta \mathcal{Q}_{IJ,J}^\beta(\vec{u})v_I^\beta d\beta + \int_\beta \mathcal{M}_{IJ,J}^\beta(\vec{u})\partial_I v_3^\beta d\beta = \int_\beta f_i^\beta v_i^\beta d\beta. \quad (3.9)$$

On the other hand, for all $\alpha \in \partial^{-1}\beta$ we know from (3.8) that $v_i^\alpha \vec{e}_i^\alpha = v_i^\beta \vec{e}_i^\beta$ on β. Hence, (3.9) is equivalent to

$$\left\langle -\mathcal{Q}_{IJ,J}^\beta(\vec{u})\vec{e}_I^\beta - \mathcal{M}_{IJ,IJ}^\beta(\vec{u})\vec{e}_3^\beta + \sum_{\alpha\in\partial^{-1}\beta}\sigma_{ij}^\alpha(\vec{u})n_j^\alpha \vec{e}_i^\alpha - f_i^\beta \vec{e}_i^\beta, v_l^\beta \vec{e}_l^\beta \right\rangle_{H^{-1}(\beta)\times H_0^1(\beta)} = 0,$$

which leads to

$$-\mathcal{Q}_{IJ,J}^\beta(\vec{u})\vec{e}_I^\beta - \mathcal{M}_{IJ,IJ}^\beta(\vec{u})\vec{e}_3^\beta + \sum_{\alpha\in\partial^{-1}\beta}\sigma_{ij}^\alpha(\vec{u})n_j^\alpha \vec{e}_i^\alpha = f_i^\beta \vec{e}_i^\beta \quad \text{in } (L^2(\beta))^3, \quad (3.10)$$

due to the arbitrariness of $v_l^\beta \vec{e}_l^\beta$. Here and in what follows, to simplify the presentation, we write $\langle\cdot,\cdot\rangle_{H^{-1}(\beta)\times H_0^1(\beta)}$ for $\langle\cdot,\cdot\rangle_{(H^{-1}(\beta))^3\times(H_0^1(\beta))^3}$ which is frequently used for the couple of vector-valued functions.

Applying the similar argument as above, we know that for each $\beta \in \Gamma^2$, it holds

$$\sum_{\alpha\in\partial^{-1}\beta} \sigma_{ij}^{\alpha}(\vec{u})n_j^{\alpha}\vec{e}_i^{\alpha} = \vec{0} \quad \text{in } (H^{1/2}(\beta))^3. \tag{3.11}$$

3.2 Some auxiliary results

Lemma 3.1. Let $\beta \in \Omega^2$ be a plate member in Ω. Assume that $\vec{u}^{\beta} \in (H^2(\beta))^2 \times H^3(\beta)$, and for each $\alpha \in \partial^{-1}\beta$, $\vec{u}^{\alpha} \in (H^2(\alpha))^3$. Then for each $\vec{v}^{\beta} \in (H^1(\beta))^3$,

$$-\int_{\beta} \mathcal{Q}_{IJ,J}^{\beta}(\vec{u})v_I^{\beta}d\beta + \int_{\beta} \mathcal{M}_{IJ,J}^{\beta}(\vec{u})\partial_I v_3^{\beta}d\beta + \int_{\beta}\sum_{\alpha\in\partial^{-1}\beta} \sigma_{ij}^{\alpha}(\vec{u})n_j^{\alpha}\vec{e}_i^{\alpha}\cdot v_l^{\beta}\vec{e}_l^{\beta}d\beta$$
$$-\int_{\beta} f_i^{\beta}v_i^{\beta}d\beta = \langle \mathcal{M}_{IJ,J}^{\beta}(\vec{u})n_I^{\beta}, v_3^{\beta}\rangle_{H^{-1/2}(\partial\beta)\times H^{1/2}(\partial\beta)}. \tag{3.12}$$

Proof. Since $\vec{u}^{\beta} \in (H^2(\beta))^2 \times H^3(\beta)$ and $\vec{f}^{\beta} \in (L^2(\beta))^3$, $\{\mathcal{M}_{IJ,J}^{\beta}(\vec{u})\}_{I=1}^2 \in \mathrm{H}(\mathrm{div};\beta)$ in terms of (3.10). Therefore we know from ref. [17] that $\mathcal{M}_{IJ,J}^{\beta}(\vec{u})n_I^{\beta} \in H^{-1/2}(\partial\beta)$, and the following identity holds:

$$\int_{\beta} \mathcal{M}_{IJ,IJ}^{\beta}(\vec{u})v_3^{\beta}d\beta + \int_{\beta} \mathcal{M}_{IJ,J}^{\beta}(\vec{u})\partial_I v_3^{\beta}d\beta = \langle \mathcal{M}_{IJ,J}^{\beta}(\vec{u})n_I^{\beta}, v_3^{\beta}\rangle_{H^{-1/2}(\partial\beta)\times H^{1/2}(\partial\beta)}.$$

This with (3.10) leads to (3.12) after some simple computation. □

We next introduce a trace space $H(\partial\beta)$ for each $\beta \in \Omega^2$ as follows.

$$H(\partial\beta_1) := \{v \in H^1(\partial\beta_1);\ v = 0 \text{ on } \gamma_{N_1+1}\}, \tag{3.13}$$

and for each $\beta \in \Omega^2\backslash\beta_1$,

$$H(\partial\beta) := H^1(\partial\beta). \tag{3.14}$$

Lemma 3.2. Let β be a plate member in Ω^2. Then for any $u_3^{\beta} \in H^3(\beta)$ and $v_3^{\beta} \in H(\partial\beta)$,

$$\begin{aligned}&\langle \partial_{\vec{t}^{\beta}}\mathcal{M}_{\vec{n}\vec{t}}^{\beta}(\vec{u}), v_3^{\beta}\rangle_{H^{-1}(\partial\beta)\times H^1(\partial\beta)}\\ =&-\langle \mathcal{M}_{\vec{n}\vec{t}}^{\beta}(\vec{u}), \partial_{\vec{t}^{\beta}}v_3^{\beta}\rangle_{L^2(\partial\beta)\times L^2(\partial\beta)}\\ &+\sum_{k=1}^{N_{\beta}}\left\{\sum_{\gamma\in\partial^{-1}\delta_k^{\beta}} \varepsilon(\beta,\gamma)\varepsilon(\gamma,\delta_k^{\beta})\mathcal{M}_{\vec{n}\vec{t}}^{\beta}(\vec{u})|_{\gamma}(\delta_k^{\beta})\right\}v_3^{\beta}(\delta_k^{\beta}),\end{aligned} \tag{3.15}$$

where $\{\delta_k^{\beta}\}_{k=1}^{N_{\beta}}$ are all point elements on $\partial\beta$, and $\mathcal{M}_{\vec{n}\vec{t}}^{\beta}(\vec{u})|_{\gamma}$ denotes the restriction of $\mathcal{M}_{\vec{n}\vec{t}}^{\beta}(\vec{u})$ to line element $\gamma \in \partial\beta$.

It deserves to point out that the identity (3.15) makes sense due to the important fact[18] that $\mathcal{M}_{\vec{n}\vec{t}}^{\beta}(\vec{u}) \in H^q(\partial\beta)$ for all $q < \frac{1}{2}$, and it implies $\partial_{\vec{t}^{\beta}}\mathcal{M}_{\vec{n}\vec{t}}^{\beta}(\vec{u}) \in H^{q-1}(\partial\beta)$.

Proof of Lemma 3.2. We introduce a functional of $v_3^{\beta} \in H(\partial\beta)$ by

$$L_{\partial\beta}(v_3^{\beta}) := \langle \partial_{\vec{t}^{\beta}}\mathcal{M}_{\vec{n}\vec{t}}^{\beta}(\vec{u}), v_3^{\beta}\rangle_{H^{-1}(\partial\beta)\times H^1(\partial\beta)} + \langle \mathcal{M}_{\vec{n}\vec{t}}^{\beta}(\vec{u}), \partial_{\vec{t}^{\beta}}v_3^{\beta}\rangle_{L^2(\partial\beta)\times L^2(\partial\beta)}. \tag{3.16}$$

For each $\gamma \in \partial\beta$ and $v_3^\beta \in H_0^1(\gamma)$, it is easy to check

$$L_{\partial\beta}(v_3^\beta) = 0. \tag{3.17}$$

Now let $\phi_k^{\partial\beta}$ ($1 \leqslant k \leqslant N_\beta$) be a function in $H(\partial\beta)$ such that it is a linear function on each line element $\gamma \in \partial\beta$ satisfying $\phi_k^{\partial\beta}(\delta_s^\beta) = \delta_{ks}$, $1 \leqslant s \leqslant N_\beta$. Then for each $v \in H(\partial\beta)$, we have

$$v = \sum_{k=1}^{N_\beta} v(\delta_k^\beta)\phi_k^{\partial\beta} + \left(v - \sum_{k=1}^{N_\beta} v(\delta_k^\beta)\phi_k^{\partial\beta}\right) = \sum_{k=1}^{N_\beta} v(\delta_k^\beta)\phi_k^{\partial\beta} + \sum_{\gamma\in\partial\beta} v^{\partial\beta,\gamma}$$

with $v^{\partial\beta,\gamma} \in H_0^1(\gamma)$. Hence it follows from (3.17) that

$$L_{\partial\beta}(v_3^\beta) = L_{\partial\beta}\left(\sum_{k=1}^{N_\beta} v_3^\beta(\delta_k^\beta)\phi_k^{\partial\beta}\right) = \sum_{k=1}^{N_\beta} a_{\delta_k^\beta} v_3^\beta(\delta_k^\beta),$$

where $a_{\delta_k^\beta} := L_{\partial\beta}(\phi_k^{\partial\beta})$. Combining this with (3.16) implies

$$\langle \partial_{\vec{t}^\beta}\mathcal{M}_{\vec{n}\vec{t}}^\beta(\vec{u}), v_3^\beta\rangle_{H^{-1}(\partial\beta)\times H^1(\partial\beta)} + \langle \mathcal{M}_{\vec{n}\vec{t}}^\beta(\vec{u}), \partial_{\vec{t}^\beta} v_3^\beta\rangle_{L^2(\partial\beta)\times L^2(\partial\beta)} = \sum_{k=1}^{N_\beta} a_{\delta_k^\beta} v_3^\beta(\delta_k^\beta). \tag{3.18}$$

On the other hand, when u_3^β is sufficiently smooth, by a direct computation we find

$$a_{\delta_k^\beta} = \sum_{\gamma\in\partial^{-1}\delta_k^\beta} \varepsilon(\beta,\gamma)\varepsilon(\gamma,\delta_k^\beta)\mathcal{M}_{\vec{n}\vec{t}}^\beta(\vec{u})|_\gamma(\delta_k^\beta). \tag{3.19}$$

For $u_3^\beta \in H^3(\beta)$, (3.19) should be understood in the extension procedure as follows. By virtue of the density theory for Sobolev spaces[15], there exists a sequence $u_3^{\beta,s} \in \mathcal{D}(\overline{\beta})$ such that

$$\lim_{s\to\infty} \|u_3^{\beta,s} - u_3^\beta\|_{3,\beta} = 0. \tag{3.20}$$

To clarify the presentation, we temporarily write $L_{\partial\beta}(u_3^\beta,\cdot)$ for $L_{\partial\beta}(\cdot)$ to show its dependence on u_3^β. Using the trace theorem in a polygonal domain and Theorem 1.4.4.6 in ref. [15], we know that for each $v_3^\beta \in H(\partial\beta)$,

$$\begin{aligned} |L_{\partial\beta}(u_3^{\beta,s}, v_3^\beta) - L_{\partial\beta}(u_3^\beta, v_3^\beta)| &\leqslant \|\partial_{\vec{t}^\beta}\mathcal{M}_{\vec{n}\vec{t}}^\beta(\vec{u} - \vec{u}^s)\|_{-1,\partial\beta}\|v_3^\beta\|_{1,\partial\beta} \\ &\quad + \|\mathcal{M}_{\vec{n}\vec{t}}^\beta(\vec{u} - \vec{u}^s)\|_{0,\partial\beta}\|\partial_{\vec{t}^\beta} v_3^\beta\|_{0,\partial\beta} \\ &\lesssim \|u_3^{\beta,s} - u_3^\beta\|_{3,\beta}\|v_3^\beta\|_{1,\partial\beta}, \end{aligned}$$

hence we have by (3.20) that

$$\lim_{s\to\infty} L_{\partial\beta}(u_3^{\beta,s}, v_3^\beta) = L_{\partial\beta}(u_3^\beta, v_3^\beta). \tag{3.21}$$

However,

$$\begin{aligned} L_{\partial\beta}(u_3^{\beta,s}, v_3^\beta) &= \sum_{k=1}^{N_\beta}\left\{\sum_{\gamma\in\partial^{-1}\delta_k^\beta} \varepsilon(\beta,\gamma)\varepsilon(\gamma,\delta_k^\beta)\mathcal{M}_{\vec{n}\vec{t}}^\beta(\vec{u}^s)|_\gamma(\delta_k^\beta)\right\} v_3^\beta(\delta_k^\beta), \\ L_{\partial\beta}(u_3^\beta, v_3^\beta) &= \sum_{k=1}^{N_\beta} a_{\delta_k^\beta} v_3^\beta(\delta_k^\beta). \end{aligned} \tag{3.22}$$

By the arbitrariness of v_3^β, it follows from (3.21)-(3.22) that

$$\lim_{s\to\infty}\sum_{\gamma\in\partial^{-1}\delta_k^\beta}\varepsilon(\beta,\gamma)\varepsilon(\gamma,\delta_k^\beta)\mathcal{M}^\beta_{\vec{n}\vec{t}}(\vec{u}^s)|_\gamma(\delta_k^\beta)=a_{\delta_k^\beta}. \tag{3.23}$$

In this sense, we simply write $\sum_{\gamma\in\partial^{-1}\delta_k^\beta}\varepsilon(\beta,\gamma)\varepsilon(\gamma,\delta_k^\beta)\mathcal{M}^\beta_{\vec{n}\vec{t}}(\vec{u})|_\gamma(\delta_k^\beta)$ for

$$\lim_{s\to\infty}\sum_{\gamma\in\partial^{-1}\delta_k^\beta}\varepsilon(\beta,\gamma)\varepsilon(\gamma,\delta_k^\beta)\mathcal{M}^\beta_{\vec{n}\vec{t}}(\vec{u}^s)|_\gamma(\delta_k^\beta).$$

Combining (3.18) and (3.23) immediately gives the identity (3.15) for $u_3^\beta\in H^3(\beta)$. □

In order to obtain the equilibrium equations on line elements, we also require a simple result about the extension for smooth functions.

Lemma 3.3. Let γ be an edge of a bounded polygon β. Then for each function $v\in C_0^\infty(\gamma)$, there exist three functions $w_i\in C_0^\infty(\beta\cup\gamma)$, $1\leqslant i\leqslant 3$, such that on γ,

$$w_1=v;\ w_2=v \text{ and } \partial_{\vec{n}}w_2=0;\ w_3=0 \text{ and } \partial_{\vec{n}}w_3=v,$$

where $\vec{n}$ denotes the unit outward normal to $\partial\beta$.

Proof. We only give the construction of w_2 in details, and the other two can be obtained in the similar manners. Without loss of generality, we assume that the edge γ lies in the x_1-axis with $\gamma=(0,1)\times\{0\}$. Let $v=v(x_1)\in C_0^\infty(0,1)$ with the support in the interval $[a,b]$ with $a>0$ and $b<1$. It is clear that there exists $r>0$ such that $R=[a,b]\times[0,r]$ (or $R=[a,b]\times[-r,0]$) belongs to $\beta\cup\gamma$. It suffices to consider the first case as follows. By the usual mollifying technique in the density theory for Sobolev spaces[11], there exists a function $\psi\in C_0^\infty(\beta\cup\gamma)$ such that $\psi\equiv 1$ on R. Let $\varphi(x_2)$ be the polynomial of order 3 satisfying the conditions: $\varphi(r)=\varphi'(0)=\varphi'(r)=0$ and $\varphi(0)=1$. Then, by a direct computation we know that $w_2=\varphi(x_2)\psi(x_1,x_2)v(x_1)$ is the desired function. □

3.3 Equilibrium equations on line elements

For each $\gamma\in\Omega^1$, we choose an arbitrary vector-valued function $\vec{v}^\gamma\in(C_0^\infty(\gamma))^3$ and let $v_4^\gamma\equiv 0$ on γ. We have from Lemma 3.3 that for each $\beta\in\partial^{-1}\gamma$, there exists a function $\vec{v}^\beta\in(C_0^\infty(\beta\cup\gamma))^3$ such that

$$v_i^\beta\vec{e}_i^{\,\beta}=v_i^\gamma\vec{e}_i^{\,\gamma}\quad\text{on }\gamma,\ \vec{v}^\beta\equiv\vec{0}\quad\text{on }\gamma'\in\partial\beta\backslash\gamma\text{ and }\partial_{\vec{n}^\beta}v_3^\beta=0\quad\text{on }\partial\beta. \tag{3.24}$$

For the other area elements $\beta\in\Omega^2\cup\Gamma^2$, we choose $\vec{v}^\beta$ to be zero functions on β. For such a vector-valued function $\{\vec{v}^\beta\}_{\beta\in\Omega^2\cup\Gamma^2}$, it is easy to show its restriction to the boundary $\partial\alpha$ of each $\alpha\in\Omega^3$ is in $(H^1(\partial\alpha))^3$. Hence, we have by the inverse trace theorem for Sobolev spaces[11,15,16] that there exists a function $\vec{v}^\alpha\in(H^1(\alpha))^3$ with the trace $\{\vec{v}^\beta\}_{\beta\in\Omega^2\cup\Gamma^2}$ on $\partial\alpha$. That means,

$$v_i^\alpha\vec{e}_i^{\,\alpha}=v_i^\beta\vec{e}_i^{\,\beta}\quad\text{on }\beta\in\partial^{-1}\gamma,\ \vec{v}^\alpha\equiv\vec{0}\quad\text{on any other }\beta\in\partial\alpha. \tag{3.25}$$

For each $\alpha'\in\Omega^3$ with $\varepsilon(\alpha',\beta)=0$ for all $\beta\in\partial^{-1}\gamma$, we can directly choose

$$\vec{v}^{\alpha'}\equiv\vec{0}\quad\text{on }\alpha', \tag{3.26}$$

since the function $\{\vec{v}^\beta\}_{\beta\in\Omega^2\cup\Gamma^2}$ is identically zero on $\partial\alpha$ in this case. For all other $\gamma'\in\Omega^1\backslash\gamma$, we choose

$$\vec{v}^\gamma\equiv\vec{0}\text{ and }v_4^\gamma\equiv 0\quad\text{on }\gamma'. \tag{3.27}$$

It is clear that such a $\vec{v}\in\vec{V}$, and substituting it into (1.18) gives

$$\sum_{\alpha\in\partial^{-1}\gamma}\int_\alpha\sigma_{ij}^\alpha(\vec{u})\varepsilon_{ij}^\alpha(\vec{v})d\alpha+\sum_{\beta\in\partial^{-1}\gamma}\bigg\{\int_\beta\mathcal{Q}_{IJ}^\beta(\vec{u})\varepsilon_{IJ}^\beta(\vec{v})d\beta+$$
$$\int_\beta\mathcal{M}_{IJ}^\beta(\vec{u})\mathcal{K}_{IJ}^\beta(\vec{v})d\beta\bigg\}+\int_\gamma\mathcal{Q}_1^\gamma(\vec{u})\varepsilon_{11}^\gamma(\vec{v})d\gamma+\int_\gamma\mathcal{M}_K^\gamma(\vec{u})\mathcal{K}_K^\gamma(\vec{v})d\gamma$$
$$=\sum_{\alpha\in\partial^{-1}\gamma}\int_\alpha f_i^\alpha v_i^\alpha d\alpha+\sum_{\beta\in\partial^{-1}\gamma}\int_\beta f_i^\beta v_i^\beta d\gamma+\int_\gamma f_i^\gamma v_i^\gamma d\gamma,$$

which together with (3.1)–(3.5), (3.7), (3.11)–(3.15) and (3.24)–(3.27) implies

$$\sum_{\beta\in\partial^{-1}\gamma}\bigg\{\int_\gamma\mathcal{Q}_{IJ}^\beta(\vec{u})n_J^\beta v_I^\beta d\gamma+\langle\partial_{\vec{t}^\beta}\mathcal{M}_{\vec{n}\vec{t}}^\beta(\vec{u})+\mathcal{M}_{IJ,J}^\beta(\vec{u})n_I^\beta,v_3^\beta\rangle_{H^{-1}(\gamma)\times H_0^1(\gamma)}\bigg\}$$
$$-\int_\gamma\mathcal{Q}_{1,1}^\gamma(\vec{u})v_1^\gamma d\gamma+\int_\gamma\mathcal{Q}_K^\gamma(\vec{u})(v_K^\gamma)'d\gamma=\int_\gamma f_i^\gamma v_i^\gamma d\gamma,$$

i.e.

$$\sum_{\beta\in\partial^{-1}\gamma}\langle\mathcal{Q}_{IJ}^\beta(\vec{u})n_J^\beta\vec{e}_I^\beta+(\partial_{\vec{t}^\beta}\mathcal{M}_{\vec{n}\vec{t}}^\beta(\vec{u})+\mathcal{M}_{IJ,J}^\beta(\vec{u})n_I^\beta)\vec{e}_3^\beta,v_i^\beta\vec{e}_i^\beta\rangle_{H^{-1}(\gamma)\times H_0^1(\gamma)}$$
$$-\int_\gamma\mathcal{Q}_{1,1}^\gamma(\vec{u})v_1^\gamma d\gamma+\int_\gamma\mathcal{Q}_K^\gamma(\vec{u})(v_K^\gamma)'d\gamma=\int_\gamma f_i^\gamma v_i^\gamma d\gamma. \tag{3.28}$$

On the other hand, for each $\beta\in\partial^{-1}\gamma$, we know from (3.24) that $v_i^\beta\vec{e}_i^\beta=v_i^\gamma\vec{e}_i^\gamma$ on γ. Hence, (3.28) can be rewritten as

$$\langle-\mathcal{Q}_{i,1}^\gamma(\vec{u})\vec{e}_i^\gamma+\sum_{\beta\in\partial^{-1}\gamma}\{\mathcal{Q}_{IJ}^\beta(\vec{u})n_J^\beta\vec{e}_I^\beta+(\partial_{\vec{t}^\beta}\mathcal{M}_{\vec{n}\vec{t}}^\beta(\vec{u})$$
$$+\mathcal{M}_{IJ,J}^\beta(\vec{u})n_I^\beta)\vec{e}_3^\beta\},v_l^\gamma\vec{e}_l^\gamma\rangle_{H^{-1}(\gamma)\times H_0^1(\gamma)}=\langle f_i^\gamma\vec{e}_i^\gamma,v_l^\gamma\vec{e}_l^\gamma\rangle_{H^{-1}(\gamma)\times H_0^1(\gamma)},$$

which leads to

$$-\mathcal{Q}_{i,1}^\gamma(\vec{u})\vec{e}_i^\gamma+\sum_{\beta\in\partial^{-1}\gamma}\{\mathcal{Q}_{IJ}^\beta(\vec{u})n_J^\beta\vec{e}_I^\beta+(\partial_{\vec{t}^\beta}\mathcal{M}_{\vec{n}\vec{t}}^\beta(\vec{u})+\mathcal{M}_{IJ,J}^\beta(\vec{u})n_I^\beta)\vec{e}_3^\beta\}=f_i^\gamma\vec{e}_i^\gamma \tag{3.29}$$

in $(H^{-1}(\gamma))^3$, due to the arbitrariness of $v_l^\gamma\vec{e}_l^\gamma$.

Similarly for each $\gamma\in\Gamma^1\backslash\gamma_{N_1+1}$, it holds

$$\sum_{\beta\in\partial^{-1}\gamma}\{\mathcal{Q}_{IJ}^\beta(\vec{u})n_J^\beta\vec{e}_I^\beta+(\partial_{\vec{t}^\beta}\mathcal{M}_{\vec{n}\vec{t}}^\beta(\vec{u})+\mathcal{M}_{IJ,J}^\beta(\vec{u})n_I^\beta)\vec{e}_3^\beta\}=\vec{0}\quad\text{in }(H^{-1}(\gamma))^3. \tag{3.30}$$

Using Lemma 3.3 and an argument similar to the one for obtaining (3.28)–(3.29), we also have

$$-\mathcal{M}_{1,1}^\gamma(\vec{u})+\sum_{\beta\in\partial^{-1}\gamma}\varepsilon(\beta,\gamma)\mathcal{M}_{\vec{n}\vec{n}}^\beta(\vec{u})=f_4^\gamma\quad\text{in }L^2(\gamma) \tag{3.31}$$

for all $\gamma\in\Omega^1$ and

$$\sum_{\beta\in\partial^{-1}\gamma}\varepsilon(\beta,\gamma)\mathcal{M}_{\vec{n}\vec{n}}^\beta(\vec{u})=0\quad\text{in }H^{1/2}(\gamma) \tag{3.32}$$

for all $\gamma\in\Gamma^1\backslash\gamma_{N_1+1}$.

3.4 Equilibrium equations on point elements

It follows from (3.1)–(3.3), (3.7), (3.11)–(3.12) and (3.32) that for each $\vec{v}\in\vec{V}$, we have

$$\begin{aligned}&\sum_{\beta\in\Omega^2}\Big\{\langle\mathcal{M}^\beta_{IJ,J}(\vec{u})n^\beta_I,v^\beta_3\rangle_{H^{-1/2}(\partial\beta)\times H^{1/2}(\partial\beta)}+\sum_{\gamma\in\partial\beta}\int_\gamma(\mathcal{Q}^\beta_{IJ}(\vec{u})n^\beta_Jv^\beta_I\\&-\mathcal{M}^\beta_{\vec{n}\vec{t}}(\vec{u})\partial_{\vec{t}^\beta}v^\beta_3)d\gamma\Big\}+\sum_{\gamma\in\Omega^1}\sum_{\beta\in\partial^{-1}\gamma}\int_\gamma\varepsilon(\beta,\gamma)\mathcal{M}^\beta_{\vec{n}\vec{n}}(\vec{u})v^\gamma_4d\gamma\\&+\sum_{\gamma\in\Omega^1}\Big\{\int_\gamma\mathcal{Q}^\gamma_1(\vec{u})\varepsilon^\gamma_{11}(\vec{v})d\gamma+\int_\gamma\mathcal{M}^\gamma_i(\vec{u})\mathcal{K}^\gamma_i(\vec{v})d\gamma\Big\}\\=&\sum_{\gamma\in\Omega^1}\Big\{\int_\gamma f^\gamma_iv^\gamma_id\gamma+\int_\gamma f^\gamma_4v^\gamma_4d\gamma\Big\}.\end{aligned}\tag{3.33}$$

Now we assume that $\vec{u}^\alpha\in(H^2(\alpha))^3$, $\vec{u}^\beta\in(H^2(\beta))^2\times H^4(\beta)$, $\vec{u}^\gamma\in H^2(\gamma)\times(H^4(\gamma))^2$, and $u^\gamma_4\in H^2(\gamma)$ for all $\alpha\in\Omega^3$, $\beta\in\Omega^2$ and $\gamma\in\Omega^1$ respectively. In this case, it is easy to check that

$$\mathcal{Q}^\beta_{IJ}(\vec{u})n^\beta_J\vec{e}^\beta_I+(\partial_{\vec{t}^\beta}\mathcal{M}^\beta_{\vec{n}\vec{t}}(\vec{u})+\mathcal{M}^\beta_{IJ,J}(\vec{u})n^\beta_I)\vec{e}^\beta_3\in(L^2(\partial\beta))^3,\ -\mathcal{Q}^\gamma_{i,1}(\vec{u})\vec{e}^\gamma_i\in(L^2(\gamma))^3$$

and

$$\int_\gamma\mathcal{Q}^\gamma_K(\vec{u})(v^\gamma_K)'d\gamma=-\int_\gamma\mathcal{Q}^\gamma_{K,1}(\vec{u})v^\gamma_Kd\gamma+\sum_{\delta\in\partial\gamma}\varepsilon(\gamma,\delta)(\mathcal{Q}^\gamma_K(\vec{u})v^\gamma_K)(\delta).\tag{3.34}$$

Hence we have by (3.29)–(3.30) that for each $\gamma\in\Omega^1$,

$$-\mathcal{Q}^\gamma_{i,1}(\vec{u})\vec{e}^\gamma_i+\sum_{\beta\in\partial^{-1}\gamma}\{\mathcal{Q}^\beta_{IJ}(\vec{u})n^\beta_J\vec{e}^\beta_I+(\partial_{\vec{t}^\beta}\mathcal{M}^\beta_{\vec{n}\vec{t}}(\vec{u})+\mathcal{M}^\beta_{IJ,J}(\vec{u})n^\beta_I)\vec{e}^\beta_3\}=f^\gamma_i\vec{e}^\gamma_i\tag{3.35}$$

in $(L^2(\gamma))^3$, and for each $\gamma\in\Gamma^1\backslash\gamma_{N_1+1}$,

$$\sum_{\beta\in\partial^{-1}\gamma}\{\mathcal{Q}^\beta_{IJ}(\vec{u})n^\beta_J\vec{e}^\beta_I+(\partial_{\vec{t}^\beta}\mathcal{M}^\beta_{\vec{n}\vec{t}}(\vec{u})+\mathcal{M}^\beta_{IJ,J}(\vec{u})n^\beta_I)\vec{e}^\beta_3\}=\vec{0}\quad\text{in }(L^2(\gamma))^3.\tag{3.36}$$

Therefore, by (3.4)–(3.6), (3.15), (3.31) and (3.33)–(3.36) we find

$$\begin{aligned}&\sum_{\delta\in\Gamma^0}\sum_{\gamma\in\partial^{-1}\delta}\{\varepsilon(\gamma,\delta)\mathcal{Q}^\gamma_i(\vec{u})(\delta)\vec{e}^\gamma_i\cdot(v^\gamma_l\vec{e}^\gamma_l)(\delta)+\varepsilon(\gamma,\delta)\mathcal{M}^\gamma_i(\vec{u})(\delta)\vec{e}^\gamma_i\cdot(v^\gamma_{l+3}\vec{e}^\gamma_l)(\delta)\}\\&\quad-\sum_{\delta\in\Gamma^0}\sum_{\gamma\in\partial^{-1}\delta}\sum_{\beta\in\partial^{-1}\gamma}\varepsilon(\beta,\gamma)\varepsilon(\gamma,\delta)\mathcal{M}^\beta_{\vec{n}\vec{t}}(\vec{u})|_\gamma(\delta)\vec{e}^\beta_3\cdot(v^\beta_l\vec{e}^\beta_l)(\delta)\\&\quad-\sum_{\delta\in\Gamma^0_3\backslash\gamma_{N_1+1}}\sum_{\gamma\in\partial^{-1}\delta}\sum_{\beta\in\partial^{-1}\gamma}\varepsilon(\beta,\gamma)\varepsilon(\gamma,\delta)\mathcal{M}^\beta_{\vec{n}\vec{t}}(\vec{u})|_\gamma(\delta)\vec{e}^\beta_3\cdot(v^\beta_l\vec{e}^\beta_l)(\delta)=0.\end{aligned}\tag{3.37}$$

On the other hand, we know from (1.7) that for any $\gamma\in\Omega^1$, if $\beta\in\partial^{-1}\gamma$ and $\delta\in\partial\gamma$, then

$$v^\beta_i\vec{e}^\beta_i=v^\gamma_i\vec{e}^\gamma_i\quad\text{on }\delta.$$

So (3.37) can be rewritten as

$$\begin{aligned}
&\sum_{\delta\in\Gamma^0}\sum_{\gamma\in\partial^{-1}\delta}\Big\{\varepsilon(\gamma,\delta)\mathcal{Q}_i^\gamma(\vec{u})(\delta)\vec{e}_i^{\,\gamma}\\
&-\sum_{\beta\in\partial^{-1}\gamma}\varepsilon(\beta,\gamma)\varepsilon(\gamma,\delta)\mathcal{M}^\beta_{\vec{n}\vec{t}}(\vec{u})|_\gamma(\delta)\vec{e}_3^{\,\beta})\Big\}\cdot(v_l^\gamma\vec{e}_l^{\,\gamma})(\delta)\\
&+\sum_{\delta\in\Gamma^0}\sum_{\gamma\in\partial^{-1}\delta}\varepsilon(\gamma,\delta)\mathcal{M}_i^\gamma(\vec{u})(\delta)\vec{e}_i^{\,\gamma}\cdot(v_{l+3}^\gamma\vec{e}_l^{\,\gamma})(\delta)\\
&-\sum_{\delta\in\Gamma_3^0\backslash\gamma_{N_1+1}}\sum_{\gamma\in\partial^{-1}\delta}\sum_{\beta\in\partial^{-1}\gamma}\varepsilon(\beta,\gamma)\varepsilon(\gamma,\delta)\mathcal{M}^\beta_{\vec{n}\vec{t}}(\vec{u})|_\gamma(\delta)\vec{e}_3^{\,\beta}\cdot(v_l^\beta\vec{e}_l^{\,\beta})(\delta)=0,\quad\forall\vec{v}\in\vec{V},
\end{aligned}$$

which gives

$$\sum_{\gamma\in\partial^{-1}\delta}\varepsilon(\gamma,\delta)\mathcal{Q}_i^\gamma(\vec{u})(\delta)\vec{e}_i^{\,\gamma}-\sum_{\gamma\in\partial^{-1}\delta}\sum_{\beta\in\partial^{-1}\gamma}\varepsilon(\beta,\gamma)\varepsilon(\gamma,\delta)\mathcal{M}^\beta_{\vec{n}\vec{t}}(\vec{u})|_\gamma(\delta)\vec{e}_3^{\,\beta}=\vec{0} \tag{3.38}$$

and

$$\sum_{\gamma\in\partial^{-1}\delta}\varepsilon(\gamma,\delta)\mathcal{M}_i^\gamma(\vec{u})(\delta)\vec{e}_i^{\,\gamma}=\vec{0}, \tag{3.39}$$

for all point elements $\delta\in\Gamma^0$ and

$$\sum_{\gamma\in\partial^{-1}\delta}\sum_{\beta\in\partial^{-1}\gamma}\varepsilon(\beta,\gamma)\varepsilon(\gamma,\delta)\mathcal{M}^\beta_{\vec{n}\vec{t}}(\vec{u})|_\gamma(\delta)\vec{e}_3^{\,\beta}=\vec{0}, \tag{3.40}$$

for all point elements $\delta\in\Gamma_3^0\backslash\partial\gamma_{N_1+1}$.

Now we obtain all equilibrium equations from (1.18) described in the following result.

Theorem 3.1. Assume that $\vec{u}^\alpha\in(H^2(\alpha))^3$, $\vec{u}^\beta\in(H^2(\beta))^2\times H^3(\beta)$, $\vec{u}^\gamma\in H^2(\gamma)\times(H^3(\gamma))^2$ and $u_4^\gamma\in H^2(\gamma)$ for all $\alpha\in\Omega^3$, $\beta\in\Omega^2$ and $\gamma\in\Omega^1$ respectively. Then the solution to problem (1.18) satisfies equilibrium equations (3.7), (3.10)–(3.11) and (3.29)–(3.32). Furthermore, if $u_3^\beta\in H^4(\beta)$ and $u_K^\gamma\in H^4(\gamma)$, we also have the equilibrium equations (3.38)–(3.40).

It deserves to point out that the above equilibrium equations are presented in a vector form, not in componentwise form by Feng and Shi[2], which makes the theoretical derivation more intuitive and easy.

4 An important identity

We next derive an important identity for the solution $\vec{u}$ to (1.18) which is essential in our finite element error analysis presented in a forthcoming paper. In this case, we only assume that $\vec{u}^\alpha\in(H^2(\alpha))^3$, $\vec{u}^\beta\in(H^2(\beta))^2\times H^3(\beta)$, $\vec{u}^\gamma\in H^2(\gamma)\times(H^3(\gamma))^2$, and $u_4^\gamma\in H^2(\gamma)$ for all $\alpha\in\Omega^3$, $\beta\in\Omega^2$ and $\gamma\in\Omega^1$

respectively. It follows from (3.5), (3.6) and (3.31) that for each $\vec{v} \in \vec{V}$,

$$\begin{aligned}
&\sum_{\gamma\in\Omega^1}\sum_{\beta\in\partial^{-1}\gamma}\int_\gamma \varepsilon(\beta,\gamma)\mathcal{M}^{\beta}_{\vec{n}\vec{n}}(\vec{u})v_4^\gamma d\gamma + \sum_{\gamma\in\Omega^1}\int_\gamma \mathcal{M}_i^\gamma(\vec{u})\mathcal{K}_i^\gamma(\vec{v})d\gamma \\
&\quad - \sum_{\gamma\in\Omega^1}\int_\gamma \{f_4^\gamma v_4^\gamma + \mathcal{Q}_K^\gamma(\vec{u})(v_K^\gamma)'\}d\gamma \\
&= \sum_{\gamma\in\Omega^1}\int_\gamma \left\{\sum_{\beta\in\partial^{-1}\gamma}\varepsilon(\beta,\gamma)\mathcal{M}^\beta_{\vec{n}\vec{n}}(\vec{u}) - \mathcal{M}^\gamma_{1,1}(\vec{u}) - f_4^\gamma\right\}v_4^\gamma d\gamma \\
&\quad + \sum_{\gamma\in\Omega^1}\sum_{\delta\in\partial\gamma}\varepsilon(\gamma,\delta)\mathcal{M}_i^\gamma(\vec{u})(\delta)v^\gamma_{i+3}(\delta) \\
&= \sum_{\delta\in\Gamma^0}\left\{\sum_{\gamma\in\partial^{-1}\delta}\varepsilon(\gamma,\delta)\mathcal{M}_i^\gamma(\vec{u})(\delta)\vec{e}_i\right\}\cdot(v^\gamma_{l+3}(\delta)\vec{e}_l).
\end{aligned}$$

This with (3.33) implies that for each $\vec{v} \in \vec{V}$,

$$\begin{aligned}
&\sum_{\beta\in\Omega^2}\Bigg\{\langle\mathcal{M}^\beta_{IJ,J}(\vec{u})n_I^\beta, v_3^{\partial\beta}\rangle_{H^{-1/2}(\partial\beta)\times H^{1/2}(\partial\beta)} \\
&\quad + \sum_{\gamma\in\partial\beta}\int_\gamma(\mathcal{Q}^\beta_{IJ}(\vec{u})n_J^\beta v_I^{\partial\beta} - \mathcal{M}^\beta_{\vec{n}\vec{t}}(\vec{u})\partial_{\vec{t}^\beta}v_3^{\partial\beta})d\gamma\Bigg\} \\
&\quad + \sum_{\gamma\in\Omega^1}\int_\gamma \mathcal{Q}_i^\gamma(\vec{u})(v_i^\gamma)'d\gamma + \sum_{\delta\in\Gamma^0}\left\{\sum_{\gamma\in\partial^{-1}\delta}\varepsilon(\gamma,\delta)\mathcal{M}_i^\gamma(\vec{u})(\delta)\vec{e}_i\right\}\cdot(v^\gamma_{l+3}(\delta)\vec{e}_l) \\
&= \sum_{\gamma\in\Omega^1}\int_\gamma f_i^\gamma v_i^\gamma d\gamma. \qquad (4.1)
\end{aligned}$$

Let $\Omega^1 = \Omega^1_1 \cup \Omega^1_2$, where Ω^1_2 is the set of all rod members which have at least one plate as the adjacent members, while Ω^1_1 is the set of the remaining rod members. We introduce two auxiliary spaces as follows:

$$\begin{aligned}
\vec{H}(\Omega^1) := \{\vec{v}^{\Omega^1} = \{\vec{v}^\gamma\}_{\gamma\in\Omega^1} \in \prod_{\gamma\in\Omega^1}((H^1(\gamma))^3;\\
v_i^\gamma\vec{e}_i^\gamma = v_i^{\gamma'}\vec{e}_i^{\gamma'} \text{ on } \delta,\ \forall\delta\in\Gamma_2^0,\ \gamma,\gamma'\in\partial^{-1}\delta\}
\end{aligned}$$

and

$$\begin{aligned}
\vec{H}(\partial\Omega^2) := \{\vec{v}^{\partial\Omega^2} = \{\vec{v}^{\partial\beta}\}_{\beta\in\Omega^2} \in \prod_{\beta\in\Omega^2}(H(\partial\beta))^3;\\
v_i^{\partial\beta}\vec{e}_i^\beta = v_i^{\partial\beta'}\vec{e}_i^{\beta'} \text{ on } \gamma,\ \forall\gamma\in\Gamma_2^1,\ \beta,\beta'\in\partial^{-1}\gamma\}.
\end{aligned}$$

Here $H(\partial\beta)$ is defined by (3.13) and (3.14). Then we define a trace space on $\partial\Omega^2 \cup \Omega^1$ by

$$\begin{aligned}
\vec{H}(\partial\Omega^2\cup\Omega^1) := \{\vec{v} = (\vec{v}^{\partial\Omega^2},\ \vec{v}^{\Omega^1}) \in \vec{H}(\partial\Omega^2)\times\vec{H}(\Omega^1);\\
\vec{v}^{\partial\beta} = \vec{v}^\gamma \text{ on } \gamma,\ \forall\gamma\in\Omega_2^1,\ \beta\in\partial\gamma\}. \qquad (4.2)
\end{aligned}$$

We next introduce some auxiliary spaces for later uses.

$$\vec{H}_*(\Omega^1) := \{\vec{v}^{\Omega^1} = \{\vec{v}^\gamma\}_{\gamma\in\Omega^1} \in \prod_{\gamma\in\Omega^1} ((H^1_*(\gamma))^3;$$
$$v_i^\gamma \vec{e}_i^\gamma = v_i^{\gamma'} \vec{e}_i^{\gamma'} \text{ on } \delta,\ \forall\delta\in\Gamma_2^0,\ \gamma,\gamma'\in\partial^{-1}\delta\},$$

and

$$\vec{H}_*(\partial\Omega^2) := \{\vec{v}^{\partial\Omega^2} = \{\vec{v}^{\partial\beta}\}_{\beta\in\Omega^2} \in \prod_{\beta\in\Omega^2} (H_*(\partial\beta))^3;$$
$$v_i^{\partial\beta} \vec{e}_i^\beta = v_i^{\partial\beta'} \vec{e}_i^{\beta'} \text{ on } \gamma,\ \forall\gamma\in\Gamma_2^1,\ \beta,\beta'\in\partial^{-1}\gamma\},$$
$$\vec{H}_*(\partial\Omega^2\cup\Omega^1) := \{\vec{v} = (\vec{v}^{\partial\Omega^2},\ \vec{v}^{\Omega^1}) \in \vec{H}_*(\partial\Omega^2)\times\vec{H}_*(\Omega^1);$$
$$\vec{v}^{\partial\beta} = \vec{v}^\gamma \text{ on } \gamma,\ \forall\gamma\in\Omega_2^1,\ \beta\in\partial\gamma\}.$$

Here $H^1_*(\gamma)$ consists of all functions in $C^\infty(\bar\gamma)$ whose first-order derivatives are identically zero at two endpoints, while $H_*(\partial\beta)$ consists of all functions in $H(\partial\beta)$ which, when restricted on each line element $\gamma\in\partial\beta$, are the functions in $H^1_*(\gamma)$.

Lemma 4.1. $\vec{H}_*(\partial\Omega^2\cup\Omega^1)$ is dense in $\vec{H}(\partial\Omega^2\cup\Omega^1)$ in the norm

$$\|(\vec{v}^{\partial\Omega^2},\ \vec{v}^{\Omega^1})\|_{\vec{H}(\partial\Omega^2\cup\Omega^1)} := \left\{\sum_{\beta\in\Omega^2}\|\vec{v}^{\partial\beta}\|^2_{(H^1(\partial\beta))^3} + \sum_{\gamma\in\Omega^1}\|\vec{v}^\gamma\|^2_{(H^1(\gamma))^3}\right\}^{1/2}.$$

Proof. It suffices to verify that $H^1_*(\gamma)$ is dense in $H^1(\gamma)$ in the norm $\|\cdot\|_{1,\gamma}$. Or equivalently,

$$Z = \{v\in C^\infty[a_1,a_2];\ v'(a_1) = v'(a_2) = 0\}$$

is dense in $H^1(a_1,a_2)$ in the norm $\|\cdot\|_{H^1(a_1,a_2)}$. Here, we write (a_1,a_2) for γ for simplicity. The result is true intuitively, but the proof is not so trivial. Let $\bar{Z}$ be the completion of Z in the norm $\|\cdot\|_{H^1(a_1,a_2)}$. It is easy to check that $H^1_0(a_1,a_2)\subset\bar{Z}\subset H^1(a_1,a_2)$. If $\bar{Z}\neq H^1(a_1,a_2)$, then there exists a non-zero functional $f\in(H^1(a_1,a_2))'$ such that $f(v)$ is identically zero for each $v\in\bar{Z}$. On the other hand, let ϕ_I be a linear polynomial on $[a_1,a_2]$ such that $\phi_I(a_J)=\delta_{IJ}$, $1\leqslant I,\ J\leqslant 2$. Then, for each $v\in H^1(a_1,a_2)$ we can write

$$v = \bar{v} + \sum_{I=1}^2 v(a_I)\phi_I,$$

where $\bar{v} = v - \sum_{I=1}^2 v(a_I)\phi_I$. In other words, $H^1(a_1,a_2)$ is the direct sum of $H^1_0(a_1,a_2)$ and its trace space. Hence, there exist a functional $f_0\in H^{-1}(a_1,a_2)$, and two constants b_1, b_2 such that[15]

$$f(v) = f_0(\bar{v}) + b_1 v(a_1) + b_2 v(a_2), \tag{4.3}$$

for each $v\in H^1(a_1,a_2)$. However, it is clear from the above deduction that f_0 is a zero functional, and since we can find a function $v_I\in Z$ such that $v_I(a_J)=\delta_{IJ}$, it follows from (4.3) that $b_I=0$. This shows that f is also a zero functional, leading to a contradiction. The proof is completed. □

Theorem 4.1. Assume that $\vec{u}^\alpha\in(H^2(\alpha))^3$, $\vec{u}^\beta\in(H^2(\beta))^2\times H^3(\beta)$, $\vec{u}^\gamma\in H^2(\gamma)\times(H^3(\gamma))^2$ and $u_4^\gamma\in H^2(\gamma)$ for all $\alpha\in\Omega^3$, $\beta\in\Omega^2$, and $\gamma\in\Omega^1$

respectively. Then for each $(\vec{v}^{\partial\Omega^2}, \vec{v}^{\Omega^1}) \in \vec{H}(\partial\Omega^2 \cup \Omega^1)$, the identity

$$\begin{aligned}&\sum_{\beta\in\Omega^2}\Big\{\langle \mathcal{M}^\beta_{IJ,J}(\vec{u})n^\beta_I, v^{\partial\beta}_3\rangle_{H^{-1/2}(\partial\beta)\times H^{1/2}(\partial\beta)}\\&+\sum_{\gamma\in\partial\beta}\int_\gamma(\mathcal{Q}^\beta_{IJ}(\vec{u})n^\beta_J v^{\partial\beta}_I - \mathcal{M}^\beta_{\vec{n}\vec{t}}(\vec{u})\partial_{\vec{t}^\beta}v^{\partial\beta}_3)d\gamma\Big\} + \sum_{\gamma\in\Omega^1}\int_\gamma \mathcal{Q}^\gamma_i(\vec{u})(v^\gamma_i)'d\gamma \qquad (4.4)\\&=\sum_{\gamma\in\Omega^1}\int_\gamma f^\gamma_i v^\gamma_i d\gamma\end{aligned}$$

holds true.

Proof. We first show that the identity (4.4) holds for each $(\vec{v}^{\partial\Omega^2}, \vec{v}^{\Omega^1}) \in \vec{H}_*(\partial\Omega^2\cup\Omega^1)$ with $\vec{v}^{\partial\Omega^2} = \{\vec{v}^{\partial\beta}\}_{\beta\in\Omega^2}$ and $\vec{v}^{\Omega^1} = \{\vec{v}^\gamma\}_{\gamma\in\Omega^1}$. In fact, by the inverse trace theorem for Sobolev spaces in polygonal domains[16], for each $\beta \in \Omega^2$ we can find a vector-valued function $\vec{v}^\beta = v^\beta_i\vec{e}^\beta_i \in (H^1_*(\beta))^2 \times H^2_*(\beta)$ such that $\vec{v}^\beta = \vec{v}^{\partial\beta}$ and $\partial_{\vec{n}^\beta}v^\beta_3 = 0$ on $\partial\beta$. For each $\alpha \in \Omega^3$, observing the definition of $\vec{H}_*(\partial\Omega^2)$, we can get a function $\{\vec{v}^\beta\}_{\beta\in\partial\alpha}$ in $(H^1(\partial\alpha))^3$ such that it equals $\vec{v}^\beta$ on each $\beta \in \Omega^2$. After that, by the inverse trace theorem again we can find some $\vec{v}^\alpha \in (H^1(\alpha))^3$ with the trace $\{\vec{v}^\beta\}_{\beta\in\partial\alpha}$. For each $\gamma \in \Omega^1$, we simply take $v^\gamma_4 \equiv 0$ on γ. We then obtain a function $\vec{v} = \{\{\vec{v}^\alpha\}_{\alpha\in\Omega^3}, \{\vec{v}^\beta\}_{\beta\in\Omega^2}, \{\vec{v}^\gamma\}_{\gamma\in\Omega^1}, \{v^\gamma_4\}_{\gamma\in\Omega^1}\}$ defined on Ω. Moreover, it is easy to check by the definition of $\vec{H}_*(\partial\Omega^2 \cup \Omega^1)$ that such a function $\vec{v}$ satisfies the conditions (1.1)–(1.6), or equivalently $\vec{v} \in \vec{V}$. Therefore, $\vec{v}$ must satisfy the identity (4.1). On the other hand, the third term on the left-hand side of (4.1) vanishes according to the construction of $\vec{v}$. So $\vec{v}$ satisfies the identity (4.4) which implies our conclusion. Furthermore, by Lemma 4.1 and the usual density argument we know that the identity (4.4) also holds for each $(\vec{v}^{\partial\Omega^2}, \vec{v}^{\Omega^1}) \in \vec{H}(\partial\Omega^2 \cup \Omega^1)$. □

Remark 4.1. It follows from (4.1) and (4.4) that the equilibrium equation (3.39) also holds when $\vec{u}^\alpha \in (H^2(\alpha))^3$, $\vec{u}^\beta \in (H^2(\beta))^2 \times H^3(\beta)$, $\vec{u}^\gamma \in H^2(\gamma) \times (H^3(\gamma))^2$, and $u^\gamma_4 \in H^2(\gamma)$ for all $\alpha \in \Omega^3$, $\beta \in \Omega^2$ and $\gamma \in \Omega^1$ respectively.

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