Some Results in the Analysis of Thin Shell Structures

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Abstract

This paper represents a progress report of some of the research that we are conducting in the finite element analysis of thin shell structures. We consider our isoparametric displacement-rotation thin shell element and our discrete-Kirchhoff-theory (DKT) plate/shell element, which we are continuously refining for accurate and effective geometric and materially nonlinear analysis. In the paper we briefly discuss the locking phenomenon of the isoparametric element, the use of this element as a transition element between shell surfaces and in shell-solid transitions, and we give some results using the DKT element.

1. INTRODUCTION

Much research has been conducted during the last two decades in the development of thin shell finite elements. However, despite the large amount of research effort there do not exist as yet what may be considered to be cost-effective, reliable and general thin shell analysis capabilities. We believe that an effective thin shell element should satisfy the following criteria:

1) The element should yield accurate solutions when modeling any shell geometry and under all boundary and loading conditions. In particular, the element should not contain any spurious zero energy modes, so that reliable results can always be expected. The theory of the element formulation must be well-understood and should not contain any "numerical fudge factors." These considerations are most important and many elements that have been published do not satisfy this criterion. Such element developments can represent

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interesting research but should not be used in actual engineering analyses, because the generated analysis results cannot be interpreted with confidence.

2) We should be able to use the element in the modeling of general shell structures with beam stiffeners, cut-outs, intersections, and so on.

3) The element should be cost-effective in linear as well as in nonlinear, static and dynamic analysis. In nonlinear analysis, the element should be applicable to large displacement, large rotation, and materially nonlinear conditions.

Various approaches have been advocated for the development of thin shell elements [1-2]; however, based on the above objectives we have concentrated our research efforts onto two procedures:

a) The use of displacement/rotation isoparametric elements that can be employed with a variable number of nodes, but are usually used with 9 or 16 nodes.

b) The use of a simple triangular element which is obtained by superimposing the bending and membrane behaviors.

We have reported our formulations and first experiences with these elements in previous publications [3-5]. Our objective in this contribution is to present additional results that we have obtained recently in our continued research on the element formulations and implementations. The aim in our research is to improve and refine the performance of the elements to the maximum extent possible but always subject to the constraints summarized in 1) to 3) above.

2. SOME RECENT RESULTS WITH OUR ISOPARAMETRIC DISPLACEMENT-ROTATION ELEMENT

The use of an isoparametric displacement-rotation shell element was earlier proposed by Ahmad et al. [6] for linear analysis and Ramm [7] and Kråkeland [8] for nonlinear analysis. We have used the same concepts but refined the formulation of the element in some details to enhance its practical usage. The element formulation is very general because no specific shell theory is used; instead, only the two following basic assumptions are employed on the shell behavior:

(i) Particles originally on a straight line in the direction of the "normal vector" to the midsurface of the shell element remain on a straight line during the deformation of the shell.

(ii) The stress in the direction of the "normal vector" to the shell midsurface is zero.

The "normal vector" at any point of the midsurface of the element is used in the description of the shell geometry and displacements and is defined by the direction cosines of the "normal vectors" at the nodal points. These direction cosines are interpolated over the midsurface of the shell elements. As previously discussed [3], the element formulation reduces in the analysis of plates to the finite element discretization of the Mindlin plate theory. The formulation includes shear deformations which, however, are assumed to be constant over the plate thickness.

We have implemented the element with a variable number of nodes, but in practical analysis the element is usually most effective using the 9-node (parabolic) or 16-node (cubic) Lagrangian interpolations, as discussed further below. The element can also be employed as a transition element between solid and shell element idealizations or different shell surfaces. Figures 1 and 2 illustrate some of the features of the element.

The major advantage of the element is its general applicability, but it is also recognized that the use of the higher-order elements can be expensive. For this reason

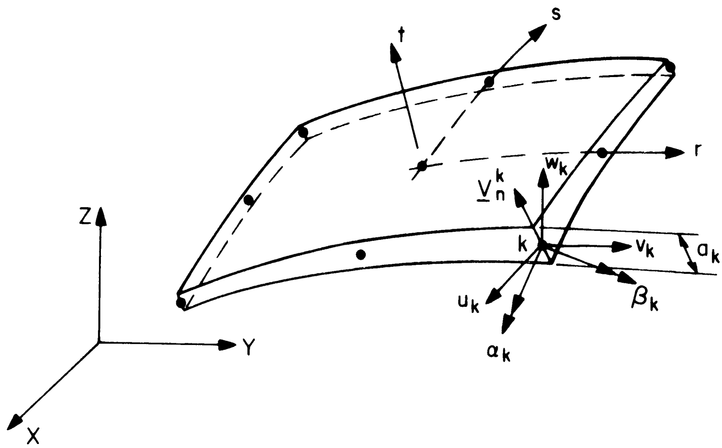


Figure 1 Nine-Node Isoparametric Shell Element

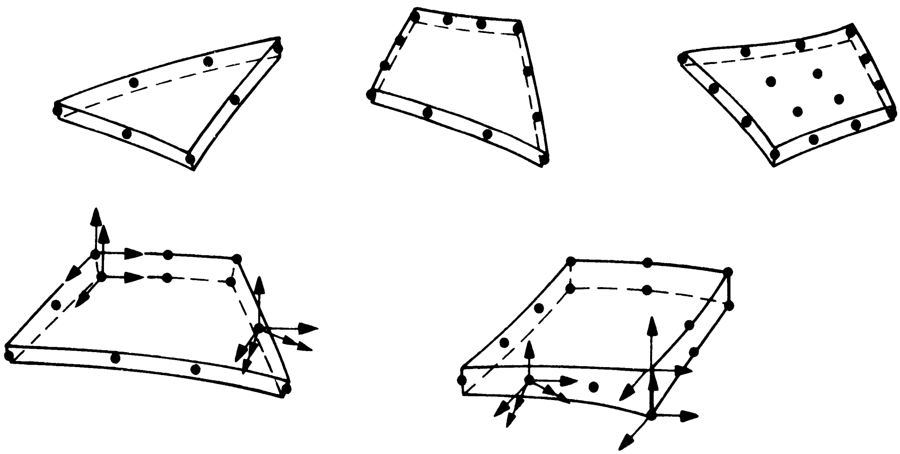


Figure 2 Shell Elements with Variable Number of Nodes and Transition Elements

much attention has been given to the use of the low-order elements. However, if the stiffness matrices of these elements are fully integrated, the elements display a "locking phenomenon" and therefore some reduced or selective integration techniques have been proposed [1,2,9,10]. These techniques, which can also be related to mixed formulations [11, 12, 5] , relieve the "locking behavior" of the elements but because of the various difficulties encountered have not resulted as yet into a shell element that satisfies our three criteria of an effective practical shell solution capability. Indeed, based on the experience available so far it appears that for the development of low-order shell elements the approach used in the discrete Kirchhoff theory is more effective [5].

In the following sections we consider two important ingredients of the isoparametric shell element: the convergence of the higher-order elements (when fully integrated) to a finite element discretization of the Kirchhoff plate theory and the use of the transition element.

Although the results of a number of investigations have been published in which the locking phenomenon in conjunction with reduced integration procedures was considered, there is still a need for further insight into the problem of whether a mesh will lock or not. The objective in our work was to address this question first for the simpler case when using fully integrated elements. We are currently extending the results obtained to elements that are not fully integrated.

2.1 On the Use of the Higher-Order Elements

As shown earlier the low-order elements and the higher-order serendipity elements can greatly overestimate the stiffness of a thin plate when the element stiffness matrices are fully integrated [9, 13]. This phenomenon referred to as "element locking" can be explained by considering the variational indicator of the Mindlin plate theory,

$$\Pi = \frac{h^3}{2}\left[\underbrace{\int_A \underline{\kappa}^T \, \underline{C}_b \, \underline{\kappa} \, dA}_{\text{TERM 1}} + \alpha \underbrace{\int_A \underline{\gamma}^T \, \underline{C}_s \, \underline{\gamma} \, dA}_{\text{TERM 2}}\right] \qquad (1)$$

$$-(\text{potential of external loads})$$

where

$$\underline{\kappa} = \begin{bmatrix} \frac{\partial \beta_x}{\partial x} \\ -\frac{\partial \beta_y}{\partial y} \\ \frac{\partial \beta_x}{\partial y} - \frac{\partial \beta_y}{\partial x} \end{bmatrix} \; ; \quad \underline{\gamma} = \begin{bmatrix} \frac{\partial w}{\partial y} - \beta_y \\ \frac{\partial w}{\partial x} + \beta_x \end{bmatrix} \qquad (2)$$

$$\underline{C}_b = \frac{E}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \; ; \; \underline{C}_s = \frac{Ek}{2(1+\nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (3)$$

and

w = transverse displacement of plate

β_y, β_x = section rotations about x and y-axes, respectively

h = thickness of plate (assumed constant)

k = shear correction factor

E, ν = Young's modulus and Poisson ratio

$\alpha = \left(\frac{L}{h}\right)^2$; L = characteristic length

The expression labelled "TERM 1" corresponds to the bending strain energy and the expression labelled "TERM 2" corresponds to the shear strain energy. We notice that for thin plates the constant α is large and can be regarded as a penalty parameter, which enforces that the shear strains are small and converge to zero as $\frac{h}{L}$ decreases. Hence, a Mindlin plate theory analytical solution converges to the Kirchhoff plate theory solution when the thickness of the plate becomes very small.

Considering now a finite element discretization of the variational indicator, we can directly conclude that for the finite element scheme to be applicable to the analysis of thin plates, we must have that, with the interpolations used on w, β_x and β_y, the two expressions for the shear strains

$$\gamma_{zy} = \frac{\partial w}{\partial y} - \beta_y \tag{4}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \beta_x \tag{5}$$

admit very small values. Also, since we want to use full numerical integration--to avoid the problem of generating spurious zero energy modes in the element formulation--we must have that the shear strain expressions in Eqs. (4) and (5) can be very small <u>throughout</u> the element. Hence, our criterion on whether an element or a patch of elements will lock is simply that the finite element interpolations with the available degrees of freedom must enable the conditions

$$\gamma_{zy} = 0 \tag{6}$$

$$\gamma_{zx} = 0 \tag{7}$$

throughout the element or patch of elements. However, this means that a simple condition for a mesh of rectangular plate elements not to lock is:

$$2q - k > np \tag{8}$$

where

q = total number of nodal points,

k = number of constrained degrees-of-freedom, maximum corresponding to rotations about x and y-axes,

n = number of finite elements,

p = number of nodes (basis functions) per element.

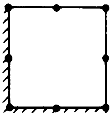
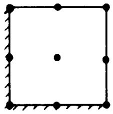
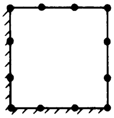
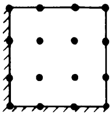
The relation in Eq. (8) is derived by considering Eqs. (6) and (7) when using the finite element displacement and rotation interpolations. The maximum number of individual polynomial terms that can occur in the shear expressions is equal to np. If the relation in Eq. (8) is satisfied, the coefficients of these terms can all be individually equal to zero, which represents the worst case. Hence, the relation in Eq. (8) is a sufficient but not necessary condition for the element mesh not to lock. Figures 3 and 4 illustrate the application of the condition in Eq. (8) for various analysis cases and give computed results using ADINA [14] (in each case one quarter of the plate was modeled).

The relation in Eq. (8) is a sufficient but not necessary condition because the np equations derived as described above may be linearly dependent. In such case the patch of elements may not lock although Eq. (8) is violated. Figure 5 summarizes some results obtained using the parabolic Lagrangian element, for which the condition in Eq. (8) predicted locking but the element meshes proved adequate in the finite element solutions using ADINA. Hence, in summary, the relation in Eq. (8) is a conservative rule for telling whether a mesh locks.

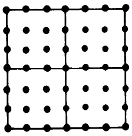
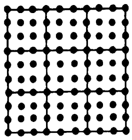
We should emphasize that the simple rule in Eq. (8) is strictly only applicable to rectangular (undistorted) plate elements. The development of a similar criterion for general plate and shell elements appears to be much more difficult, since the element distortions (measured on the natural element coordinates) enter into the calculation of the shear strains.

2.2 On the Use of the Transition Element

The transition element can be employed to model shell-solid transitions and the intersections of different shell

ELEMENT TYPE	NO OF D.O.F (2q-k)	NO OF POLYNOMIAL TERMS (np)	ADINA RESULTS
	5	8	LOCKS
	7	9	LOCKS
	9	12	LOCKS
	17	16	DOES NOT LOCK

(a) Analysis of a Clamped Square Plate Modeled with a Single Shell Element

ELEMENT MESH CONSIDERED	CASE	NO OF D.O.F (2q-k)	NO OF POLYNOMIAL TERMS (np)	ADINA RESULTS
	SIMPLY SUPPORTED	78	64	DOES NOT LOCK
	CLAMPED	71	64	DOES NOT LOCK
	SIMPLY SUPPORTED	171	144	DOES NOT LOCK
	CLAMPED	161	144	DOES NOT LOCK

(b) Analysis of a Square Plate Modeled with Cubic Elements

Figure 3 Application of the Locking Criterion

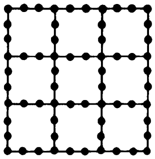
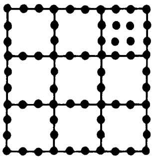
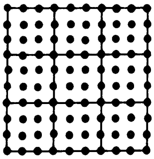
ELEMENT MESH CONSIDERED	NO OF D O F (2q-k)	NO OF POLYNOMIAL TERMS (n p)	ADINA RESULTS
	89	108	LOCKS
	97	112	LOCKS (12 - NODE ELEMENTS ARE LOCKING)
	161	144	DOES NOT LOCK

Figure 4 Application of the Locking Criterion to the Analysis of a Clamped Square Plate Modeled with Different Elements

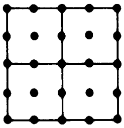
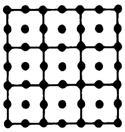
ELEMENT MESH CONSIDERED	CASE	NO OF D O F (2q-k)	NO OF POLYNOMIAL TERMS (n p)	ADINA RESULTS
	SIMPLY SUPPORTED	36	36	DOES NOT LOCK
	CLAMPED	31	36	DOES NOT LOCK
	SIMPLY SUPPORTED	78	81	DOES NOT LOCK
	CLAMPED	71	81	DOES NOT LOCK

Figure 5 Application of the Locking Criterion to the Analysis of a Square Plate Modeled with Parabolic Elements

surfaces[2]. The use of the element is relatively easy because no special constraint equations need be written at the transition regions and yet full compatibility between the elements is preserved. However, an important question must be whether the element predicts displacements and stresses accurately enough in the transition regions.

Figure 6 shows a simple folded roof structure that we analyzed using the transition element. Figure 7 lists the various finite element idealizations used. Since no analytical or accurate numerical solution of the stresses in the transition region could be located we used the idealization in Fig. 7(a) to obtain an accurate prediction of the stress components. The model in Fig. 7(b) corresponds to the usual procedure of modeling shell intersections, whereas the model in Fig. 7(c) may be attractive because of ease of program input of only one normal. The definition of the model in Fig. 7(d) is equally effective, but we need to question the accuracy of the results of the stress predictions.

Figures 8 and 9 give the stresses predicted in the analyses. It is interesting to note that the results with an average normal at the shell intersection are not far from the results using the usual modeling procedure for shell intersections (the model in Fig. 7(b)). The results obtained with the transition element are different near the transition region, but they show the trend of the 2-D fine mesh results. Also, they indicate that the transition element should be used to model only the actual transition region.

Figure 10 shows the model used in the next analysis of the folded roof, which was established based on the above considerations. Figures 11 and 12 give the results obtained in the analysis of this model. It is seen that the predicted stresses are significantly more accurate than those calculated with the usual shell idealization (in Fig. 7(b)). We should note that the number of degrees-of-freedom in the analysis was about $\frac{1}{5}$ th of those used in the two-dimensional model. Hence, although more detailed studies of the

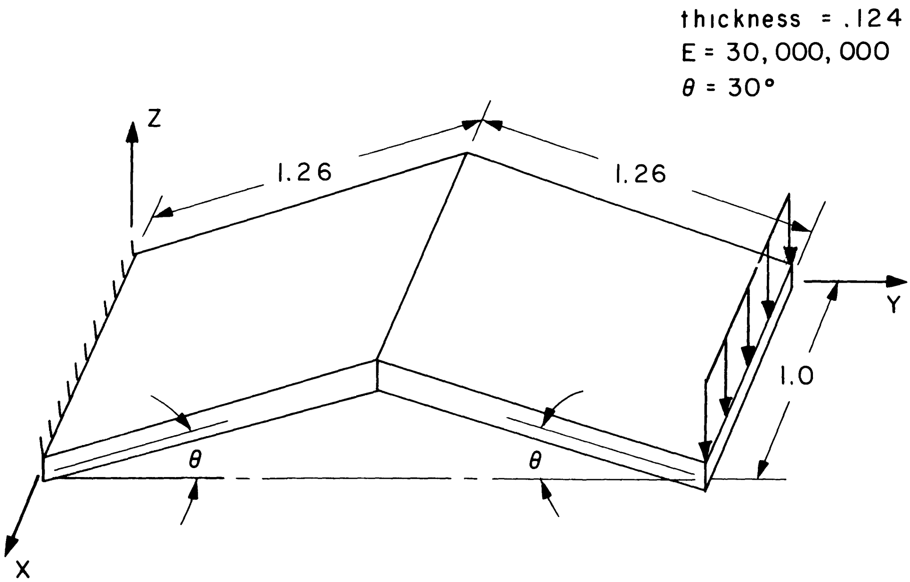
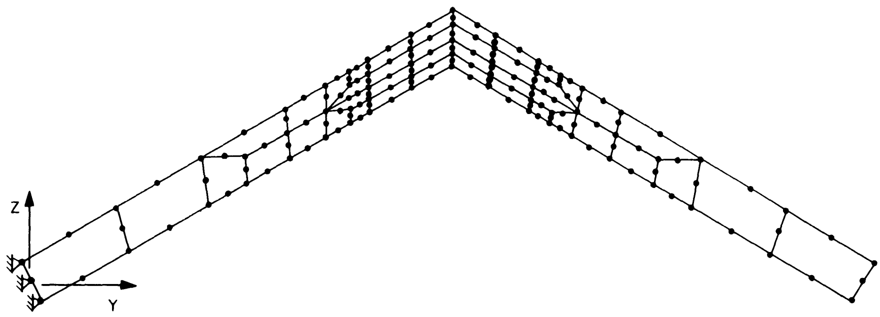
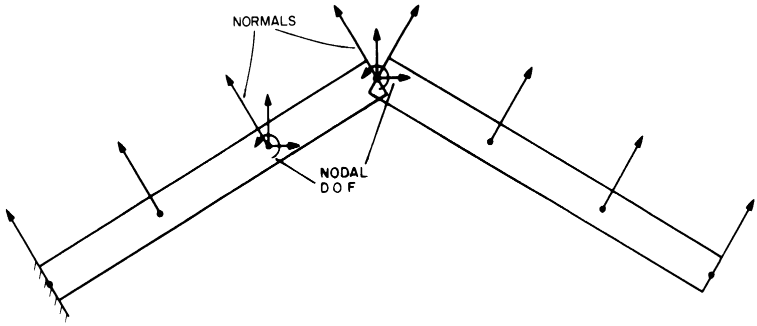


Figure 6 Folded Cantilever Plate Subjected to a Line Load at Its Tip

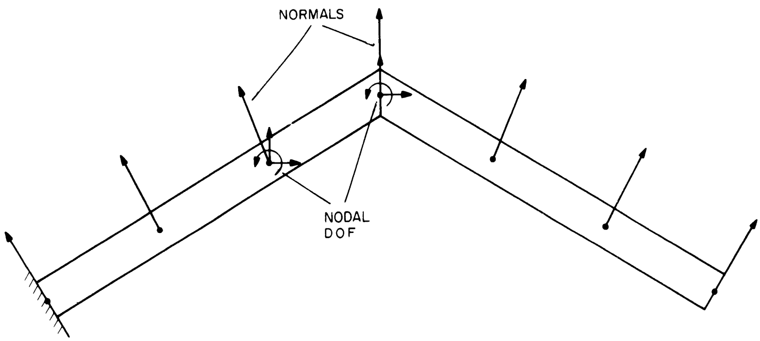


(a) Two-Dimensional Finite Element Mesh

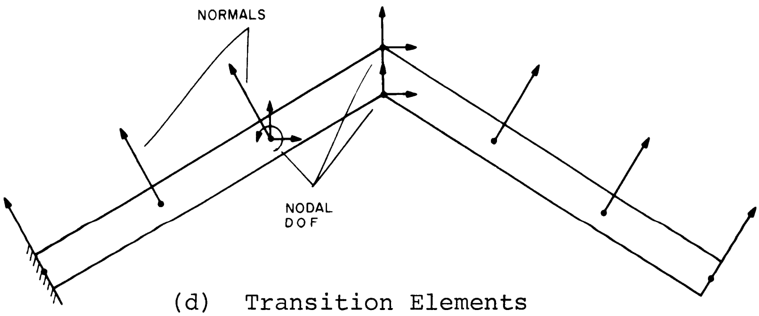
Figure 7 Models Used for the Analysis of the Folded Cantilever Plate



(b) Shell Elements Using Constraint Equations



(c) Shell Elements Using the Average Normal Technique



(d) Transition Elements

Figure 7 (Continued)

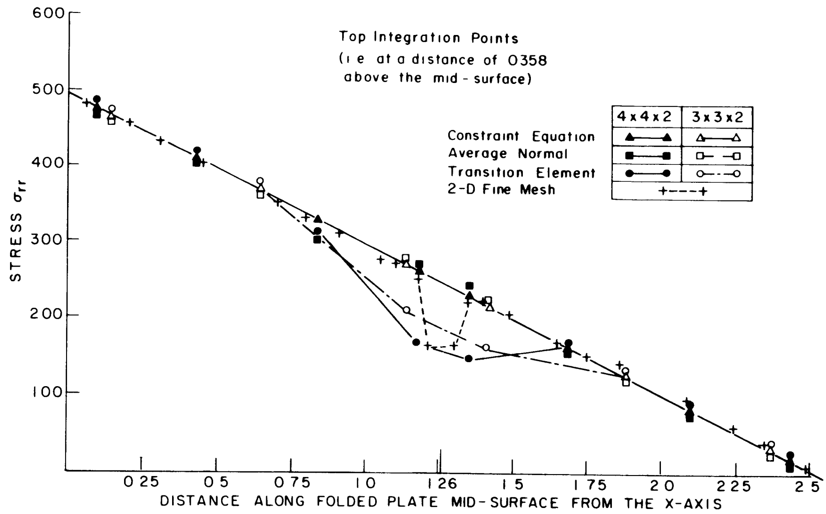


Figure 8 Predicted Stresses with Folded Plate Models of Fig. 7 (Gauss Integration Used)

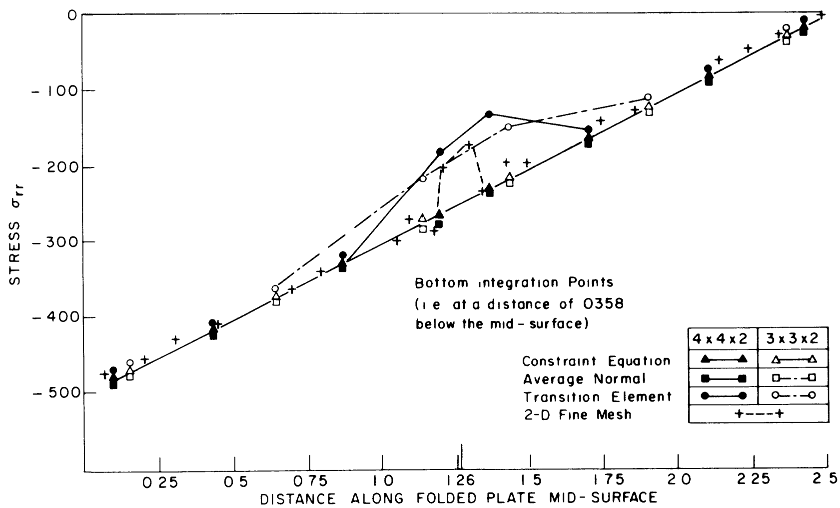


Figure 9 Predicted Stresses with Folded Plate Models of Fig. 7 (Gauss Integration Used)

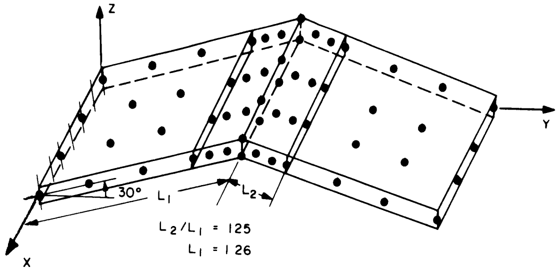


Figure 10 A Refined Model of the Folded Plate with Transition and Shell Elements

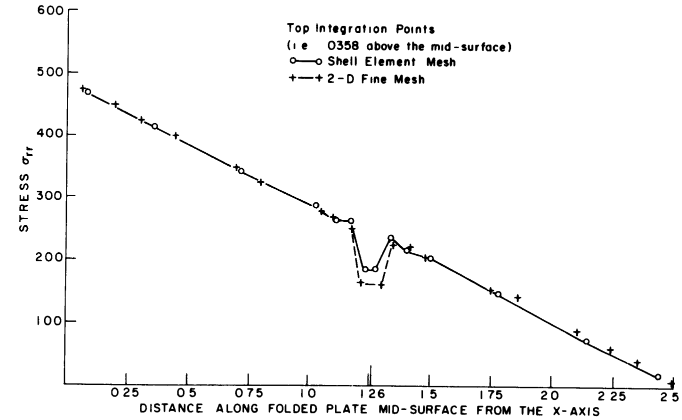


Figure 11 Predicted Stresses with Folded Plate Models of Figs. 7(a) and 10

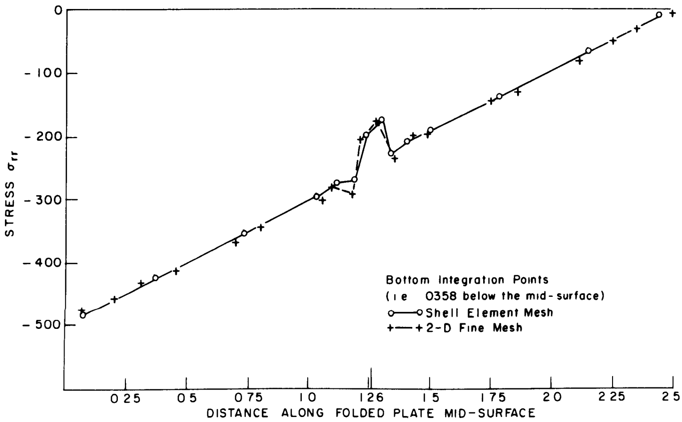


Figure 12 Predicted Stresses with Folded Plate Models of Figs. 7(a) and 10

transition element are still necessary, this example analysis shows the potential in the practical use of the element.

3. SOME RECENT RESULTS WITH OUR DISCRETE-KIRCHHOFF-THEORY TRIANGULAR ELEMENT

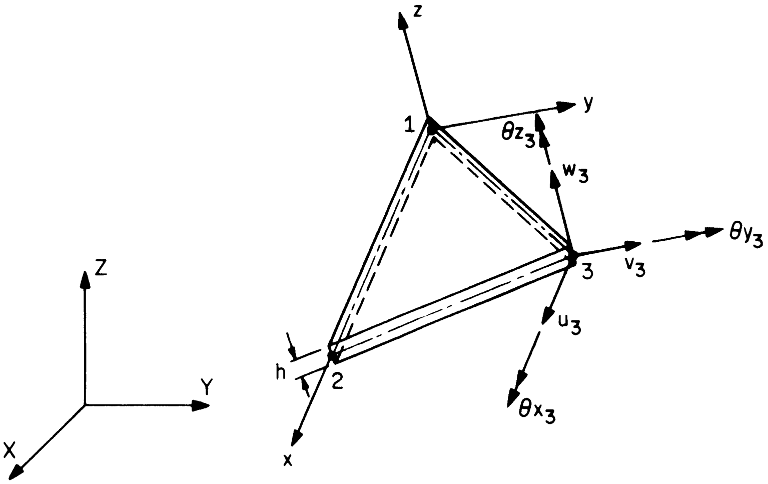
The triangular element developed has three corner nodes and six degrees-of-freedom per node as shown in Fig. 13. The element is flat and therefore a shell surface is modeled with the element as an assemblage of flat facets in the way some of the earliest shell analyses were carried out [1,2,15]. The primary objective in the development of the element was to have a very simple and cost-effective element of good accuracy as an alternative to the higher-order isoparametric elements. The three-node DKT shell element stiffness matrix is constructed as follows [16]:

1) The bending behavior is described by a discrete Kirchhoff formulation. This description has been found to yield the most effective three-node plate bending element of a large number of elements considered in a survey study [4, 5]. The element has high accuracy, has no spurious zero energy modes, is computationally very inexpensive and the programming is very simple (less than 100 Fortran statements). The element is very likely the most effective simple plate bending element currently available.

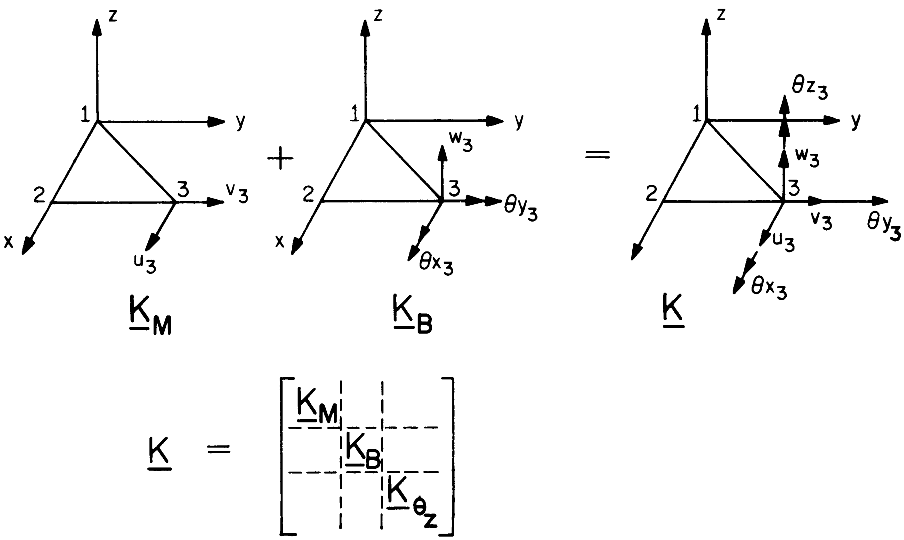
2) The membrane behavior is currently described by the simple constant strain triangle. However, it would be desirable to increase the order of accuracy of the membrane description.

3) The above stiffnesses are combined in a local coordinate system and the normal rotational stiffness is arbitrarily set to $(\frac{1}{10000})$ x (smallest bending stiffness) in order to obtain 6 stiffness degrees-of-freedom per node.

The element can be used very effectively in geometric nonlinear analysis because it remains flat during the response history. Also, for elastic-plastic analysis it is effective



(a) Configuration and Degrees-of-Freedom



(b) Element Stiffness Matrix Components

Figure 13 The DKT Three-Node Shell Element to use in the material description stress resultants, which, as in elastic analysis, circumvents the numerical integration through the element thickness.

To indicate the effectiveness of the element we are presenting in the following the results obtained in the analyses of three problems.

3.1 Analysis of a Square Plate

A square plate of side lengths 2a with clamped and simply-supported edges was analyzed [5]. Because of the symmetry conditions only one quarter of the plate was considered. Figure 14 gives the finite element meshes used in the analyses for $N = 2, 4$ and 8. Figure 15 summarizes the results obtained for the centre deflection of the plate. We may note that, considering the computational effort in the solution, these results should be compared with 2x2, 4x4 and 8x8 mesh idealizations using four-node square elements or 1x1, 2x2 and 4x4 meshes with nine-node square elements. Such evaluation and a comparison with other 3-node elements show that the 3-node DKT element is indeed very effective.

3.2 Analysis of a Pinched Cylindrical Shell

The structure analyzed and a typical finite element idealization used are shown in Fig. 16. In the figure the 10x10 mesh used is shown, but the analysis was also carried out with 4x4, 6x6, 8x8 and 16x16 mesh topologies. Figures 17 to 19 give calculated displacement and stress resultant distributions along the lines DC, BC and AD of the shell, respectively. It is seen that the finite element predictions converge rapidly to the analytical solution as a reasonable number of shell elements is employed in the structural idealization. Table 1 summarizes the solution times used in the analysis of the shell structure.

3.3 Large Displacement Analysis of a Simply - Supported Plate

The plate was subjected to a uniform pressure loading q. Figure 20 shows the finite element idealization used for the plate and

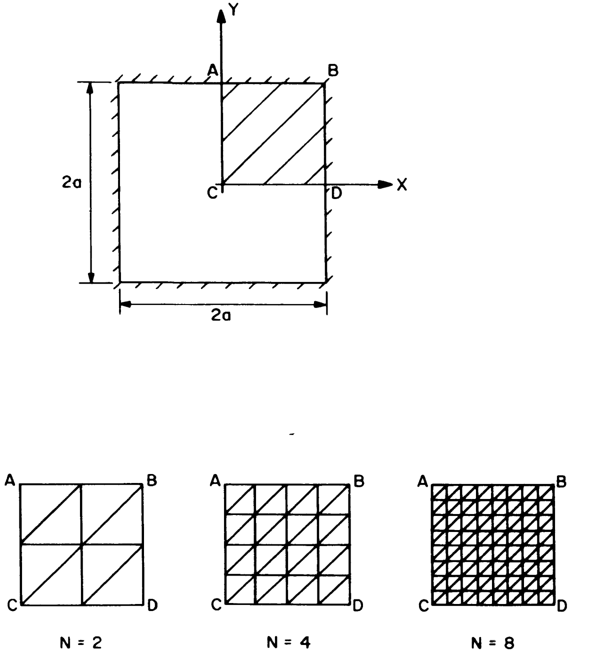


Figure 14 Element Meshes for the Analysis of a Square Plate Using DKT Elements

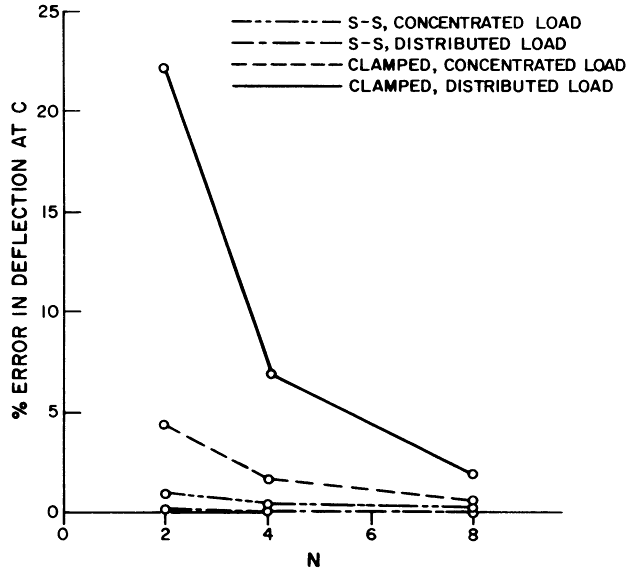
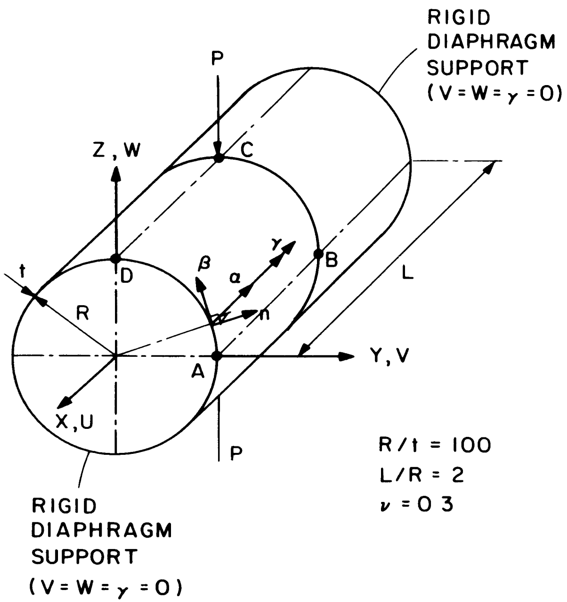
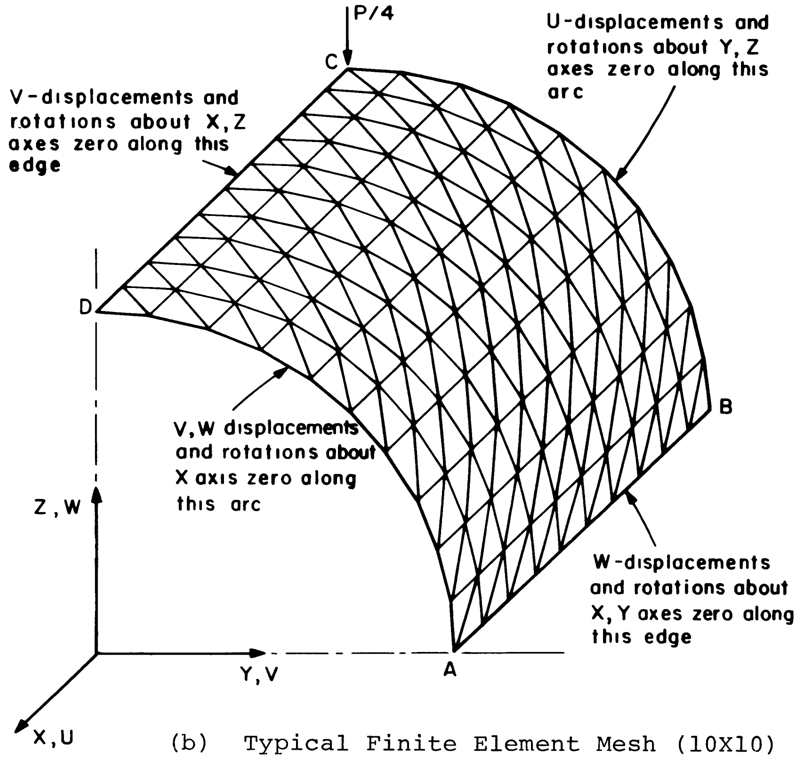


Figure 15 Percentage Error in the Predicted Centre Deflection of the Square Plate Using Element Meshes of Fig. 14

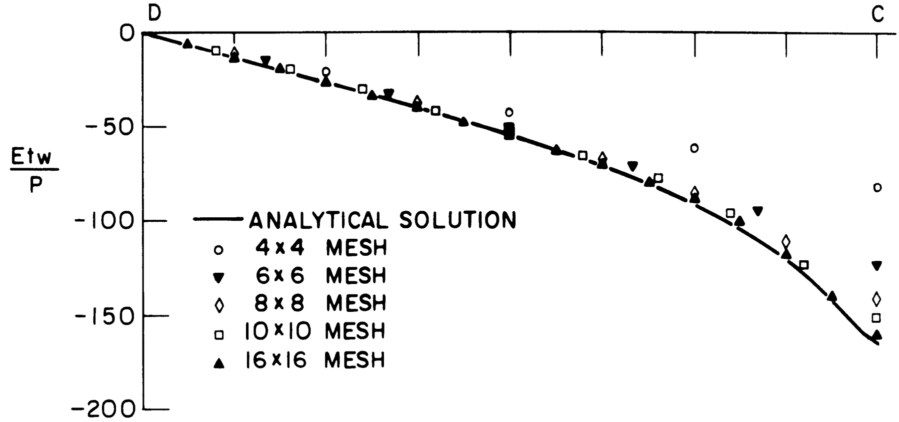


(a) Structure Considered



(b) Typical Finite Element Mesh (10X10) Used in the Analysis

Figure 16 Analysis of a Pinched Cylindrical Shell Structure Using DKT Elements



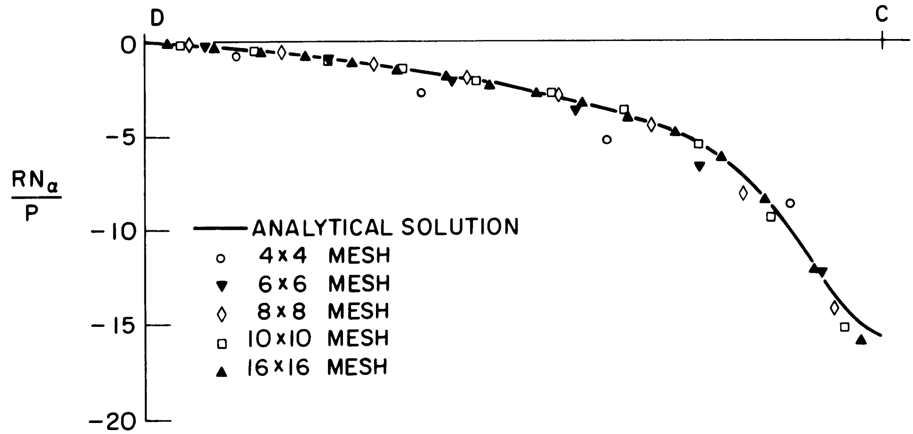
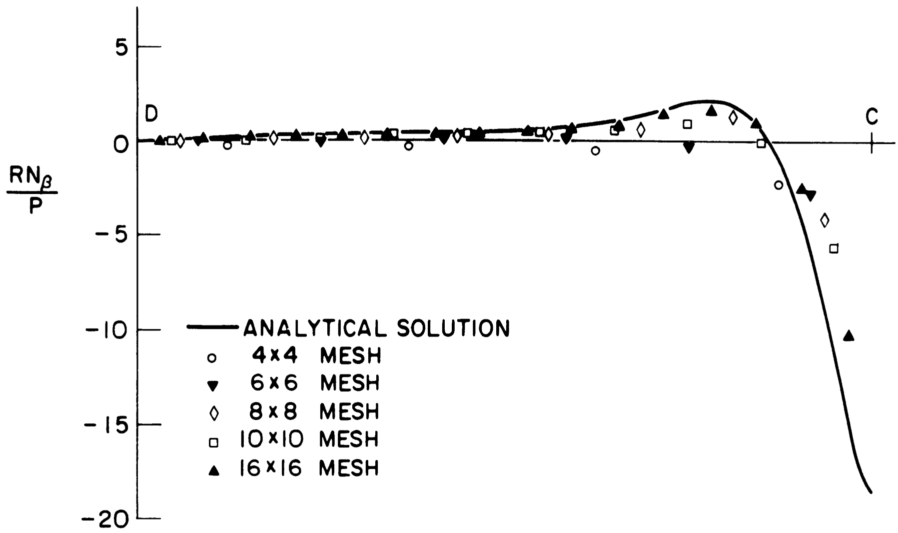


Figure 17 Predicted Displacement and Stress Distributions along DC of Shell in Fig. 16



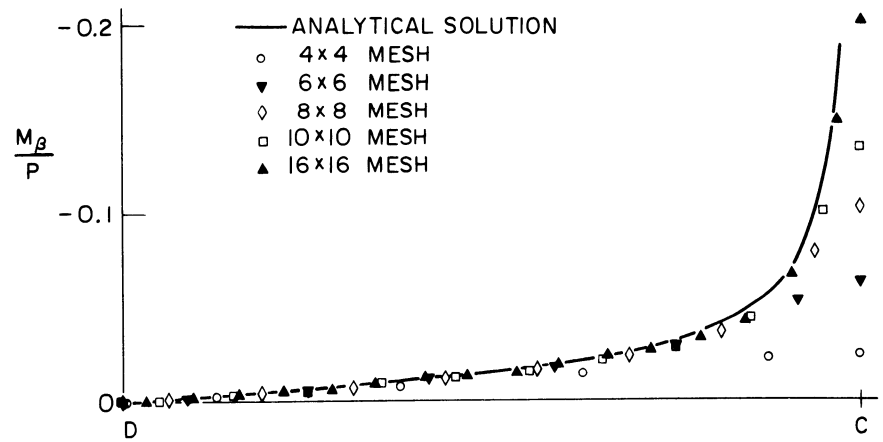
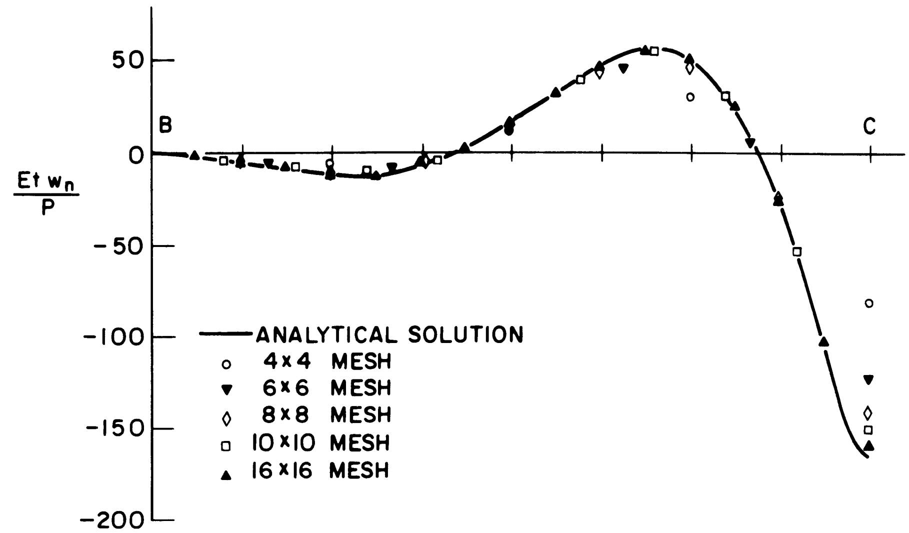


Figure 17 (Continued)



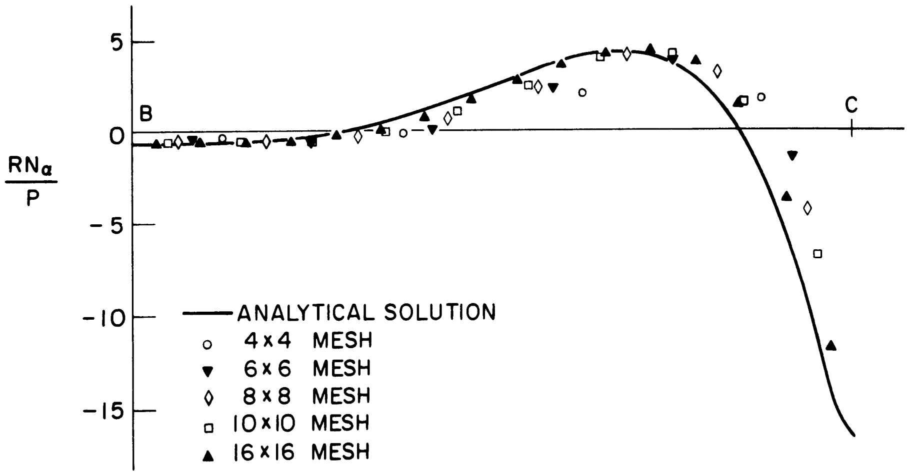
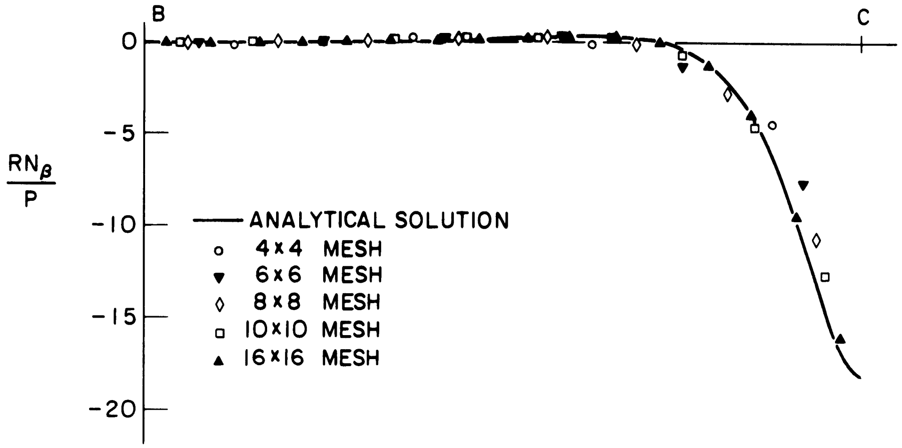


Figure 18 Predicted Displacement and Stress Distributions along BC of Shell in Fig. 16



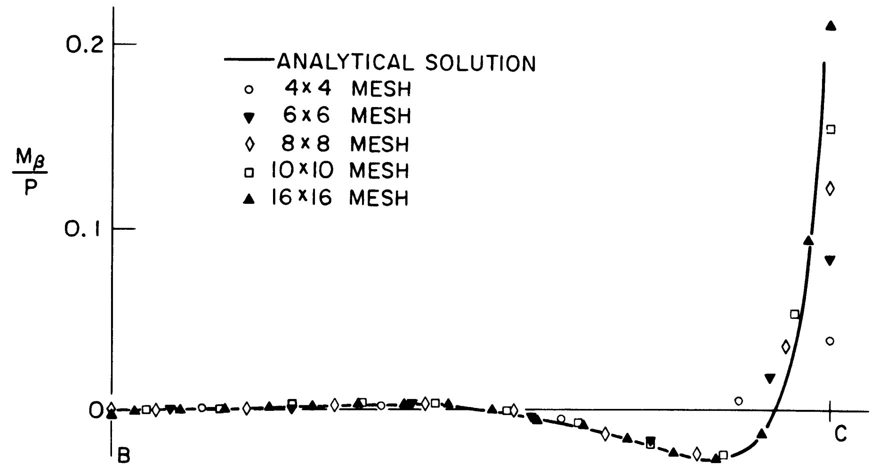
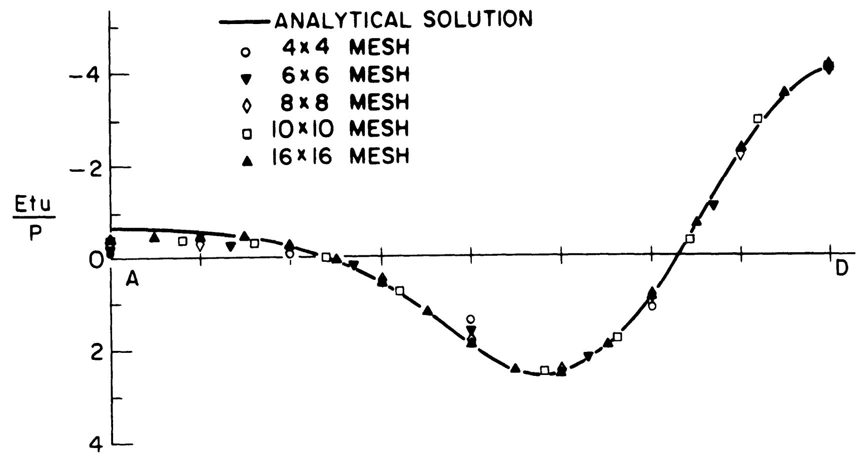


Figure 18 (Continued)



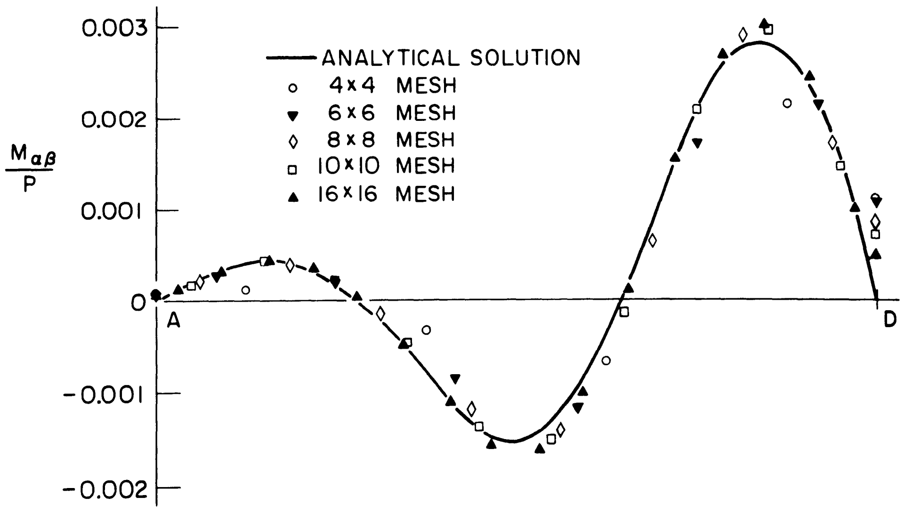


Figure 19 Predicted Displacement and Stress Distributions along AD of Shell in Fig. 16

the response predicted. In a first analysis, the edges were constrained not to move in the plane of the plate, and in a second analysis, the u and v edge displacements were left free. The predicted responses in these analyses are compared in Fig. 20 with other solutions reported earlier.

4. CONCLUDING REMARKS

In this paper we have reported some recent results of our research in finite element thin shell analysis capabilities. Our present analysis capabilities can be employed in linear analysis, in large displacement, large rotation and materially nonlinear analysis, and for static and dynamic solutions. A few demonstrative analysis results have been presented in the paper, and the reader may refer to ref. [3-5, 14, 16] for additional analysis results.

We are conducting our research to obtain increasingly more cost-effective and general solution procedures but we noted that primary emphasis is directed towards the reliability of the analysis methods. This is so, because we are convinced that an ultimate extensive usage of nonlinear finite element analysis capabilities will depend primarily on the reliability of the solution procedures available and much research work should be concentrated in this area.

Acknowledgment

We are grateful to the ADINA users group for supporting financially our research and development efforts, and we thank C. Keilers and U. Tsach for having carried out some of the analyses with ADINA.

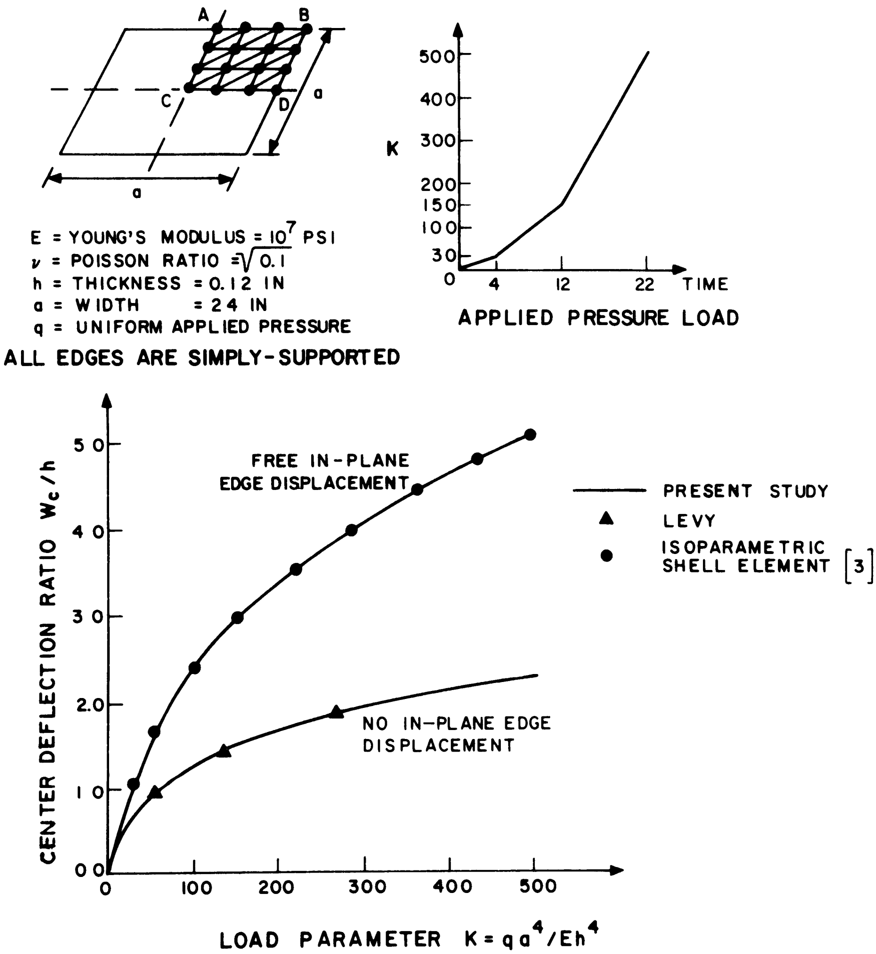


Figure 20 Large Displacement Analysis of a Simply-Supported Square Plate Subjected to Pressure Loading Using DKT Elements

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Table 1 Total Solution Times in Analysis of Pinched Cylindrical Shell (on a CDC Cyber 175)

Grid	Solution time (sec.)
4x4	1.00
6x6	1.86
8x8	3.19
10x10	5.73
16x16	21.94