Traces of Differential Forms on Lipschitz Domains, the Boundary De Rham Complex, and Hodge Decompositions

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ABSTRACT. We study the extent to which classical trace and extension theorems for scalar-valued functions can be extended to differential forms of higher-degree. For maximum applicability, this is done in the context of Lipschitz subdomains of Riemannian manifolds, and on the scale of Besov and Triebel-Lizorkin spaces. A key ingredient in this regard is the boundary De Rham complex, which we consider in the geometric and analytic setting above. Applications to Hodge-decompositions and interpolation with constraints are also presented.

1. INTRODUCTION

A basic prerequisite in the study of boundary value problems associated with a differential operator in a domain Ω is the availability of suitable trace and extension theorems. In the context when the smoothness of the functions in question is measured on the scales of Sobolev (potential) spaces, L^p_s, and Besov spaces, $B^{p,p}_s$, two fundamental results in this regard are as follows.

Theorem A. *If Ω is a bounded, Lipschitz domain in $\mathbb{R}^n$ and $1 < p < \infty$, $\alpha \in \mathbb{R}$, then the operator of restriction to Ω, mapping $L^p_\alpha(\mathbb{R}^n)$ onto $L^p_\alpha(\Omega)$, has a linear, continuous, right inverse. That is, there exists a linear, bounded operator*

$$\mathcal{E} : L^p_\alpha(\Omega) \longrightarrow L^p_\alpha(\mathbb{R}^n) \tag{1.1}$$

such that $(\mathcal{E}u)|_\Omega = u$ for each $u \in L^p_\alpha(\Omega)$.

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Theorem B. *Assume that Ω is a bounded, Lipschitz domain in $\mathbb{R}^n$ and that $1 < p < \infty$, $\alpha \in (1/p, 1 + 1/p)$. Then the restriction to the boundary operator, originally defined from $C^\infty(\bar{\Omega})$ to* $\mathrm{Lip}(\partial\Omega)$ *extends to a bounded mapping*

$$\mathrm{Tr} : L^p_\alpha(\Omega) \longrightarrow B^{p,p}_{\alpha-1/p}(\partial\Omega). \tag{1.2}$$

Furthermore, this operator is onto; indeed, it has a bounded, linear right inverse.

These results have a rich history and have received a great deal of attention in the literature. Here we would like to mention that some of the pioneering work has been done by E. Gagliardo [7], and A.P. Calderón [4], in the 50's and early 60's; see also the excellent monographs [34] by E.M. Stein, [21] by V.G. Maz'ya, and [14] by A. Jonsson and H. Wallin, and the informative survey by V. Burenkov in [3]. Other versions and extensions, as well as more references, can be found P. Jones [13], D. Jerison and C. Kenig [12], S. Mayboroda and M. Mitrea [20], V.G. Maz'ya, M. Mitrea and T. Shaposhnikova [22], and S. Rychkov [31].

In this paper we are interested in proving suitable versions of Theorems A–B in the case when differential forms are considered in place of scalar functions. Carrying out this program is justified given that a great many boundary value problems in mathematical physics (Maxwell equations, elasticity, hydrodynamics, etc.) involve working with vector fields or, more generally, differential forms; the reader is referred to the monographs [5], [6], [9], [17], [18], [36], and [39]. Another area of mathematics where such a theory plays a significant role is the study of the $\bar{\partial}$-Neumann problem in several complex variables and its real counterpart, the d-Neumann problem. Note that these problems, as well as many others, are naturally formulated in the framework of Riemannian manifolds. For maximum applicability, it is therefore important to consider, as we do in the present paper, differential forms of arbitrary degrees in a Lipschitz subdomain Ω of a given Riemannian manifold M.

One distinctive feature, naturally inherent to this setting, is that the definitions of Sobolev-like and Besov-like smoothness spaces must be adapted to the exterior derivative operator d on the manifold M, much as the standard, scalar Sobolev and Besov spaces in the Euclidean context are adapted to the gradient operator. In order to consider a concrete example, let TM stand for the tangent bundle of M (and set Λ^ℓ for its ℓ-th exterior power). Then since the scalar Sobolev (potential) spaces $L^p_{s+1}(\Omega)$ can be described as the collection of functions $u \in L^p_s(\Omega)$ with the property that $du \in L^p_s(\Omega) \otimes T^*M$, at the level of ℓ-forms it is natural to consider the space

(1.3) $D_\ell(d; L^p_s(\Omega)) :=$ ℓ-forms u with coefficients in $L^p_s(\Omega)$ for which
du has also coefficients in $L^p_s(\Omega)$.

Since the operator of restriction to Ω maps $D_\ell(d;L^p_s(M))$ into $D_\ell(d;L^p_s(\Omega))$, the issue arises whether this mapping is onto. In this regard, we shall prove the following.

Theorem C. *If Ω is a Lipschitz subdomain of M and $1 < p < \infty$, $-1 + 1/p < s < 1/p$, then there exists a linear, bounded operator*

$$\mathcal{E} : D_\ell(d;L^p_s(\Omega)) \longrightarrow D_\ell(d;L^p_s(M)) \tag{1.4}$$

such that $(\mathcal{E}u)|_\Omega = u$ for each $u \in D_\ell(d;L^p_s(\Omega))$.

The trace operator naturally associated with the space (1.3) is the assignment $u \mapsto \nu \wedge u$, where $\nu \in T^*M$ is the unit conormal to $\partial\Omega$ and the wedge stands for the exterior product of forms. It is then possible to check that, with δ denoting the formal adjoint of d, any element ξ in the range of this map satisfies

$$\begin{aligned} &\xi \in B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1}) \text{ and there exists } \eta \in B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+2}) \\ &\qquad \text{such that } \langle \xi, (\delta f)|_{\partial\Omega}\rangle = \langle \eta, f|_{\partial\Omega}\rangle, \quad \forall f \in C^\infty(M, \Lambda^{\ell+2}). \end{aligned} \tag{1.5}$$

Let $NB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^\ell)$ denote the space of all differential forms ξ as in (1.5). One of the main results proved in this paper states that this is the smallest space in which the trace map $D_\ell(d;L^p_s(\Omega)) \ni u \mapsto \nu \wedge u \in B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell+1})$ takes values. More precisely, we have the following result.

Theorem D. *Assume that Ω is a Lipschitz subdomain of the Riemannian manifold M and that $1 < p < \infty$ and $-1 + 1/p < s < 1/p$. Then*

$$\nu \wedge \cdot : D_\ell(d;L^p_s(\Omega)) \longrightarrow NB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell+1}) \tag{1.6}$$

is well-defined, linear, bounded and onto. Furthermore, it has a bounded, linear right inverse.

Theorem D is the key ingredient in the proof of Theorem C. Indeed, our strategy is to extend a given $u \in D_\ell(d;L^p_s(\Omega))$ to M by constructing a differential form in $M \setminus \bar{\Omega}$ whose trace (in the sense of Theorem D) coincides with that of u. More specifically, we take

$$\mathcal{E}(u) := \begin{cases} u \text{ in } \Omega, \\ \\ \mathrm{Ex}(\nu \wedge u) \text{ in } M \setminus \bar{\Omega}, \end{cases} \tag{1.7}$$

where $\mathrm{Ex} : NB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^\ell) \to D_\ell(d;L^p_s(M \setminus \bar{\Omega}))$ is a right-inverse for the trace operator $\nu \wedge \cdot : D_\ell(d;L^p_s(M \setminus \bar{\Omega})) \to NB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^\ell)$.

Going further, a result similar in spirit to Theorem D is valid for the interior product map

$$(1.8)\qquad \nu \vee \cdot : D_\ell(\delta; L^p_s(\Omega)) \longrightarrow TB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell-1})$$

(see the body of the paper for the relevant definitions). This and the observation that the restriction to the boundary of any differential form $u \in C^\infty(M, \Lambda^\ell)$ can be decomposed as $u|_{\partial\Omega} = \nu \vee (\nu \wedge u) + \nu \wedge (\nu \vee u)$ allow us to finally define a *global* trace map

$$\mathrm{tr} : D_\ell(d; L^p_s(\Omega)) \cap D_\ell(\delta; L^p_s(\Omega)) \to (TB^{p',p'}_{s-1/p'}(\partial\Omega, \Lambda^\ell) \oplus NB^{p',p'}_{s-1/p'}(\partial\Omega, \Lambda^\ell))^*$$

$$\langle \mathrm{tr}\, u,\, f \oplus g\rangle := \langle \nu \vee (\nu \wedge u),\, f\rangle + \langle \nu \wedge (\nu \vee u),\, g\rangle,$$

which we prove (cf. Theorem 5.6) to be well-defined, linear, bounded, and onto whenever $1 < p, p' < \infty$, $1/p + 1/p' = 1$, and $-1 + 1/p < s < 1/p$. Again, this trace operator has a linear, bounded, right inverse.

In the context of (1.5), define

$$B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell) \ni \xi \mapsto d_\partial \xi := \eta \in B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1})$$

and note that $d_\partial \circ d_\partial = 0$. This gives rise to a sequence of homomorphisms, referred to in this paper as the *boundary De Rham complex*,

$$\cdots \xrightarrow{d_\partial} NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell-1}) \xrightarrow{d_\partial} NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell) \xrightarrow{d_\partial} NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1}) \xrightarrow{d_\partial} \cdots,$$

in which the image of each arrow is contained in the kernel of the subsequent one. We study the cohomology groups associated with this complex and establish that for each $\ell = 1, 2, \ldots, n-1$,

$$(1.9)\qquad \frac{\{\xi \in NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1}) : d_\partial \xi = 0\}}{\{d_\partial \zeta : \zeta \in NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell)\}} \cong H^\ell_{\mathrm{sing}}(\partial\Omega; \mathbb{R}),$$

the ℓ-th singular homology group of $\partial\Omega$ over the field of real numbers. The proof of (1.9) relies on elements of sheaf theory, suitably adapted to the current setting. In this regard, the crux of the matter is proving that the aforementioned complex is locally exact, i.e.,

$$(1.10)\qquad \text{if } \xi \in NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1}) \text{ is such that } d_\partial \xi = 0 \text{ near } x_0 \in \partial\Omega$$
$$\Rightarrow\ \exists\, \zeta \in NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell) \text{ such that } d_\partial \zeta = \xi \text{ near } x_0.$$

It is precisely at this stage that the ontoness of the trace (1.6) in Theorem D is most useful, as it allows us to shift focus from differential forms defined on $\partial\Omega$ to differential forms defined in Ω. See the proof of Theorem 6.2 for details.

Another notable corollary of Theorem D is that the family of spaces $NB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^\ell)$, $1 < p < \infty$, $-1 + 1/p < s < 1/p$, is stable under complex interpolation. This observation then allows us to deduce, for the first time, Hodge decompositions for differential forms with coefficients in $L^p_s(\Omega)$, for an arbitrary Lipschitz domain $\Omega \subset M$, when $p \in (2-\varepsilon,\ 2+\varepsilon)$ and $s \in (-\varepsilon, \varepsilon)$, extending work in [24] where the case $s = 0$ has been considered.

The particular case of Theorem C has been proved in [1] when $M = \mathbb{R}^n$, $\ell = n - 1$, $p = 2$, and $s = 0$, by reducing matters to solving a suitable boundary value problem for the Laplacian. Another approach, which eliminates all the above restrictions on the indices involved with the exception of $s = 0$, has been developed in [24]. This requires (locally) flattening the boundary of Ω via bi-Lipschitz maps. We would now like to briefly comment on the nature of this approach and explain why its applicability is limited to the L^p scale alone. Specifically, for a differential form f of degree ℓ,

$$f = \sum_{|I|=\ell} f_I\, dx^I, \quad f_I : \mathcal{O} \to \mathbb{R}, \tag{1.11}$$

where $\mathcal{O}$ is an open subset of $\mathbb{R}^n$, and a Lipschitz map $\Phi = (\Phi_1, \dots, \Phi_n) : \mathcal{O}' \to \mathcal{O}$ where $\mathcal{O}'$ is another open subset of $\mathbb{R}^n$, the pull-back of f by Φ is given by

$$\begin{aligned} \Phi^* f &= \sum_{I=(i_1,\dots,i_\ell)} (f_I \circ \Phi)\, d\Phi_{i_1} \wedge d\Phi_{i_2} \wedge \cdots \wedge d\Phi_{i_\ell} \\ &= \sum_{I=(i_1,\dots,i_\ell)} \sum_{J=(j_1,\dots,j_\ell)} (f_I \circ \Phi) \frac{D(\Phi_{i_1}, \Phi_{i_2}, \dots, \Phi_{i_\ell})}{D(x_{j_1}, x_{j_2}, \dots, x_{j_\ell})} \\ &\qquad\qquad dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_\ell}. \end{aligned} \tag{1.12}$$

Since, in general, the determinants $D(\Phi_{i_1}, \Phi_{i_2}, \dots, \Phi_{i_\ell})/D(x_{j_1}, x_{j_2}, \dots, x_{j_\ell})$ are merely L^∞ functions, it is only the space of forms with L^p coefficients which is stable under pull-back via Lipschitz maps. This observation, which has been employed in, e.g., [11], [30], [38], [37], allows one to do analysis on a Lipschitz manifold Σ in the context of differential forms with L^p coefficients. When Σ is a Lipschitz submanifold of codimension one of a smooth manifold M, it is possible to take advantage of the smooth ambient structure of M in order to define classes of differential forms on Σ whose coefficients are smoother than just L^p. In particular, this is the case when $\Sigma = \partial\Omega$ where Ω is a Lipschitz subdomain of M. However, much as before, these classes are not invariant under Lipschitz pull-back.

As already alluded to, the main results proved in this paper are particularly relevant in the context of boundary value problems involving vector fields in Lipschitz domains, such as those arising in electromagnetic scattering by rough obstacles. See, e.g., the monographs [5] by M. Cessenat and [6] R. Dautray and J.-L. Lions, as well as the articles [2] by A. Buffa, M. Costabel and C. Schwab, [29] by L. Paquet, and [35] by L. Tartar. Here we only want to remark that [35] contains a proof of the case Theorem D for $M = \mathbb{R}^3$, $\ell = 1$, $s = 0$, and $p = 2$, via an argument which requires flattening the boundary (cf. also Section 7 in [2] for a discussion), and that [29] proves Theorem D for subdomains with a C^∞ boundary of smooth manifolds, when $p = 2$, $s = 0$, via Fourier methods.

The organization of the remainder of the paper is as follows.

1. The geometrical setting
2. Sobolev and Besov spaces on Lipschitz domains
3. Differential forms with Sobolev-Besov coefficients
4. Traces of differential forms
5. The boundary De Rham complex
6. Extending differential forms from Ω to M
7. Hodge decompositions

2. THE GEOMETRICAL SETTING

Let M be a smooth, compact, oriented manifold of real dimension n, equipped with a smooth metric tensor, $\sum_{j,k} g_{jk}\, dx_j \otimes dx_k$. Denote by TM and T^*M the tangent and cotangent bundles to M, respectively. We shall frequently identify $T^*M \equiv \Lambda^1$ canonically, via the metric. Set Λ^ℓ for the ℓ-th exterior power of TM. Sections in this latter vector bundle are ℓ-differential forms. The Hermitian structure on TM extends naturally to $T^*M := \Lambda^1$ and, further, to Λ^ℓ. We denote by $\langle \cdot, \cdot \rangle$ the corresponding (pointwise) inner product. The volume form on M, dV, is the unique unitary, positively oriented differential form of maximal degree on M. In local coordinates, $dV := [\det(g_{jk})]^{1/2} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$.

Going further, we denote by $* : \Lambda^\ell \to \Lambda^{n-\ell}$ the Hodge star operator. The interior product between a 1-form ν and an ℓ-form u is then defined by

$$\nu \vee u := (-1)^{\ell(n+1)} * (\nu \wedge *u). \tag{2.1}$$

Let d stand for the (exterior) derivative operator and denote by δ its formal adjoint (with respect to the metric introduced above). For further reference, some basic properties of these objects are summarized below.

Proposition 2.1. *For arbitrary* 1*-form* ν, ℓ*-form* u, $(n-\ell)$*-form* v, *and* $(\ell+1)$*-form* w, *the following are true:*

(1) $\langle u, *v \rangle = (-1)^{\ell(n-\ell)} \langle *u, v \rangle$ *and* $\langle *u, *v \rangle = \langle u, v \rangle$. *Also,* $**u = (-1)^{\ell(n-\ell)} u$;

(2) $\langle \nu \wedge u, w \rangle = \langle u, \nu \vee w \rangle$;

(3) $*(\nu \wedge u) = (-1)^{\ell}\nu \vee (*u)$ *and* $*(\nu \vee u) = (-1)^{\ell+1}\nu \wedge (*u)$;
(4) $*\delta = (-1)^{\ell} d*$, $\delta * = (-1)^{\ell+1} * d$, *and* $\delta = (-1)^{n(\ell+1)+1} * d *$ *on* ℓ*-forms.*

Let Ω be a *Lipschitz* subdomain of M. That is, $\partial\Omega$ can be described in appropriate local coordinates by means of graphs of Lipschitz functions (cf., e.g., [28]). Then the unit conormal $\nu \in T^*M$ is defined a.e., with respect to the surface measure $d\sigma$, on $\partial\Omega$. For any two sufficiently well-behaved differential forms (of compatible degrees) u, w we then have the integration by parts formula

$$\begin{aligned}\int_\Omega \langle du, w\rangle \, dV &= \int_\Omega \langle u, \delta w\rangle \, dV + \int_{\partial\Omega} \langle \nu \wedge u, w\rangle \, d\sigma \\ &= \int_\Omega \langle u, \delta w\rangle \, dV + \int_{\partial\Omega} \langle u, \nu \vee w\rangle \, d\sigma.\end{aligned} \tag{2.2}$$

We conclude with a brief discussion of a number of notational conventions used throughout the paper. By $C^k(\Omega)$, $k \in \mathbb{N}_0 \cup \{\infty\}$, we shall denote the space of functions of class C^k in Ω, and by $C_c^\infty(\Omega)$ the subspace of $C^\infty(\Omega)$ consisting of compactly supported functions. When viewed as a topological space, the latter is equipped with the usual inductive limit topology and its dual, i.e., the space of distributions in Ω, is denoted by $D'(\Omega) := (C_c^\infty(\Omega))'$. We also denote by $\mathrm{Lip}(\partial\Omega)$ the class of real-valued Lipschitz functions defined on $\partial\Omega$ and set $C^k(\Omega, \Lambda^\ell) := C^k(\Omega) \otimes \Lambda^\ell$, $\mathrm{Lip}(\Omega, \Lambda^\ell) := \mathrm{Lip}(\Omega) \otimes \Lambda^\ell$, etc. Finally, we would like to alert the reader that, besides denoting the pointwise inner product of forms, $\langle \cdot, \cdot \rangle$ is also used as a duality bracket between a topological space and its dual (in each case, the spaces in question should be clear from the context).

3. Sobolev and Besov Spaces on Lipschitz Domains

The Sobolev (potential) scale in $\mathbb{R}^n$ can be defined as

$$L_s^p(\mathbb{R}^n) := (I - \Delta)^{-s/2} L^p(\mathbb{R}^n), \quad 1 < p < \infty, \ s \in \mathbb{R}. \tag{3.1}$$

The Besov spaces can then be introduced via real interpolation, i.e.,

$$B_s^{p,q}(\mathbb{R}^n) := (L_{s_0}^{p_0}(\mathbb{R}^n), L_{s_1}^{p_1}(\mathbb{R}^n))_{\theta,q} \tag{3.2}$$

if $1 < p, q < \infty$, $s \in \mathbb{R}$, and $1 < p_j < \infty$, $s_j \in \mathbb{R}$, $j = 0, 1$, $\theta \in (0,1)$ are such that $1/p = (1-\theta)/p_0 + \theta/p_1$, $s = (1-\theta)s_0 + \theta s_1$. Above, $(\cdot\,,\,\cdot)_{\theta,q}$ stands for the real interpolation bracket.

Next, the classes $L_s^p(M)$, $B_s^{p,q}(M)$, $1 < p, q < \infty$, $s \in \mathbb{R}$, are obtained by lifting the corresponding Euclidean scales to M via a C^∞ partition of unity and pull-back. Given an arbitrary open subset Ω of M, we denote by $f|_\Omega \in D'(\Omega)$ the restriction of a distribution f on M to Ω. For $1 < p, q \le \infty$ and $s \in \mathbb{R}$ we then set

$$L^p_s(\Omega) := \Big\{ f \in D'(\Omega) : \exists\, F \in L^p_s(M) \text{ such that } F|_\Omega = f \Big\}, \tag{3.3a}$$

$$\|f\|_{L^p_s(\Omega)} := \inf\Big\{ \|F\|_{L^p_s(M)} : F \in L^p_s(M),\ F|_\Omega = f \Big\}, \quad f \in L^p_s(\Omega), \tag{3.3b}$$

and

$$B^{p,q}_s(\Omega) := \Big\{ f \in D'(\Omega) : \exists\, F \in B^{p,q}_s(M) \text{ such that } F|_\Omega = f \Big\}, \tag{3.4a}$$

$$\|f\|_{B^{p,q}_s(\Omega)} := \inf\Big\{ \|F\|_{B^{p,q}_s(M)} : F \in B^{p,q}_s(M),\ F|_\Omega = f \Big\},\ f \in B^{p,q}_s(\Omega). \tag{3.4b}$$

For the remainder of this section we assume that Ω is a Lipschitz subdomain of M. In this case, according to [31], there exists a universal linear extension operator. More specifically, we have the following result.

Proposition 3.1. *If Ω is a Lipschitz subdomain of M, then there exists a linear operator $\mathcal{E}$ mapping $C^\infty_c(\Omega)$ into distributions on M, and such that for any $1 < p < \infty$ and $s \in \mathbb{R}$,*

$$\mathcal{E} : L^p_s(\Omega) \longrightarrow L^p_s(M) \tag{3.5}$$

boundedly, and

$$(\mathcal{E} f)|_\Omega = f, \quad \forall\, f \in L^p_s(\Omega). \tag{3.6}$$

Other properties of interest are summarized in the propositions below; cf., e.g., [40] for proofs.

Proposition 3.2. *Assume that $1 < p_j < \infty$, $s_j \in \mathbb{R}$, $j \in \{0,1\}$, $\theta \in (0,1)$ and that $1/p = (1-\theta)/p_0 + \theta/p_1$, $s = (1-\theta)s_0 + \theta s_1$. Then*

$$[L^{p_0}_{s_0}(\Omega), L^{p_1}_{s_1}(\Omega)]_\theta = L^p_s(\Omega), \tag{3.7}$$

where $[\cdot\,,\,\cdot]_\theta$ stands for the complex interpolation bracket.

Proposition 3.3. *If $p \in (1,\infty)$, $1/p + 1/p' = 1$, then*

$$(L^p_s(\Omega))^* = L^{p'}_{-s}(\Omega), \quad \forall\, s \in (-1+1/p,\ 1/p). \tag{3.8}$$

Furthermore, for each $s \in \mathbb{R}$ and $1 < p < \infty$ the space $L^p_s(\Omega)$ is reflexive.

Proposition 3.4. *If $1 < p < \infty$ and $-1 + 1/p < s < 1/p$, then*

$$C^\infty_c(\Omega) \hookrightarrow L^p_s(\Omega) \quad \text{densely}. \tag{3.9}$$

Also, for the same range of indices, the operator $C_c^\infty(\Omega) \ni u \mapsto \tilde{u} \in C^\infty(M)$, *where* $\tilde{u}$ *is the extension of* u *to* M *by zero outside* Ω, *extends to a linear, bounded, one-to-one operator*

$$\tilde{\cdot} : L_s^p(\Omega) \longrightarrow L_s^p(M). \tag{3.10}$$

In fact, (3.10) *has a left-inverse, given by the restriction operator*

$$\cdot\,|_\Omega : L_s^p(M) \longrightarrow L_s^p(\Omega). \tag{3.11}$$

The operators (3.10)–(3.11) *are further related by*

$$\langle \tilde{u}, w \rangle = \langle u, w\,|_\Omega \rangle, \quad \forall\, u \in L_s^p(\Omega),\ \forall\, w \in L_{-s}^{p'}(M). \tag{3.12}$$

Turning to spaces defined on Lipschitz boundaries, assume $1 < p, q < \infty$, $0 < s < 1$, and that Ω is the unbounded region in $\mathbb{R}^n$ lying above the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. We then define $B_s^{p,q}(\partial\Omega)$ as the space of locally integrable functions g for which the assignment $\mathbb{R}^{n-1} \ni x \mapsto g(x', \varphi(x'))$ belongs to $B_s^{p,q}(\mathbb{R}^{n-1})$. The above definition readily adapts to the case of a Lipschitz subdomain of the manifold M, via a standard partition of unity argument. The resulting space is reflexive. Having defined Besov spaces on $\partial\Omega$ with a positive, sub-unitary amount of smoothness, we then set

$$B_{-s}^{p,q}(\partial\Omega) := (B_s^{p',q'}(\partial\Omega))^*, \tag{3.13}$$
$$1 < p, q < \infty,\ 1/p + 1/p' = 1/q + 1/q' = 1,\ 0 < s < 1.$$

The next result is proved in [12], [14].

Proposition 3.5. *The restriction to the boundary,* $C^\infty(\bar{\Omega}) \ni u \mapsto u|_{\partial\Omega} \in \mathrm{Lip}(\partial\Omega)$, *extends to a bounded, linear trace operator*

$$\mathrm{Tr} : L_s^p(\Omega) \longrightarrow B_{s-1/p}^{p,p}(\partial\Omega), \tag{3.14}$$

whenever $1 < p < \infty$ *and* $1/p < s < 1 + 1/p$. *This operator is also onto and, in fact, has a bounded, linear, right-inverse*

$$\mathrm{Ext} : B_{s-1/p}^{p,p}(\partial\Omega) \longrightarrow L_s^p(\Omega). \tag{3.15}$$

Finally, for the same range of indices,

$$C_c^\infty(\Omega) \hookrightarrow \{u \in L_s^p(\Omega) : \mathrm{Tr}\, u = 0\} \quad \text{densely}. \tag{3.16}$$

4. DIFFERENTIAL FORMS WITH SOBOLEV-BESOV COEFFICIENTS

We shall work with certain nonstandard smoothness spaces which are naturally adapted to the type of differential operators we intend to study. Specifically, fix a Lipschitz domain $\Omega \subset M$ and, for $1 < p < \infty$, $s \in \mathbb{R}$, consider the spaces

$$D_\ell(d; L^p_s(\Omega)) := \{u \in L^p_s(\Omega, \Lambda^\ell) : du \in L^p_s(\Omega, \Lambda^{\ell+1})\}, \tag{4.1}$$

$$D_\ell(\delta; L^p_s(\Omega)) := \{u \in L^p_s(\Omega, \Lambda^\ell) : \delta u \in L^p_s(\Omega, \Lambda^{\ell-1})\}, \tag{4.2}$$

equipped with the natural graph norms. Throughout the paper, all derivatives are taken in the sense of distributions.

Let us now assume (as we shall do for the remainder of this section) that $\Omega \subseteq M$ is an arbitrary Lipschitz domain with outward unit conormal $\nu \in T^*M \equiv \Lambda^1$, and that $1 < p < \infty$, $1/p + 1/p' = 1$, and $-1 + 1/p < s < 1/p$. Also, let $\ell \in \{0, 1, \dots, n\}$. Inspired by (2.2), for each $u \in D_\ell(d; L^p_s(\Omega))$ we can define $\nu \wedge u$ as a functional on $\partial\Omega$ by setting

$$\langle \nu \wedge u, \psi \rangle := \langle du, \Psi \rangle - \langle u, \delta\Psi \rangle \tag{4.3}$$

whenever $\psi \in B^{p',p'}_{1-1/p'-s}(\partial\Omega, \Lambda^{\ell+1})$ is arbitrary and $\Psi \in L^{p'}_{1-s}(\Omega, \Lambda^{\ell+1})$ is so that $\operatorname{Tr}\Psi = \psi$.

Proposition 4.1. *The normal trace operator*

$$\nu \wedge \cdot : D_\ell(d; L^p_s(\Omega)) \longrightarrow B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1}) \tag{4.4}$$

introduced above is well-defined, linear and bounded.

Proof. Given $u \in D_\ell(d; L^p_s(\Omega))$, we first note that $\xi := \nu \wedge u$ is unambiguously defined as an element in $B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1})$.

To see this, assume that, for a given $\psi \in B^{p',p'}_{1-1/p'-s}(\partial\Omega, \Lambda^{\ell+1})$, two differential forms $\Psi_1, \Psi_2 \in L^{p'}_{1-s}(\Omega, \Lambda^{\ell+1})$ are selected such that $\operatorname{Tr}\Psi_1 = \operatorname{Tr}\Psi_2 = \psi$. We aim to show that $\langle du, \Psi_1 \rangle - \langle u, \delta\Psi_1 \rangle = \langle du, \Psi_2 \rangle - \langle u, \delta\Psi_2 \rangle$. With this in mind, observe that $\Psi := \Psi_1 - \Psi_2 \in L^{p'}_{1-s}(\Omega, \Lambda^{\ell+1})$ satisfies $\operatorname{Tr}\Psi = 0$ so that, by (3.16), there exists a sequence $\Psi_j \in C^\infty_c(\Omega)$ which converges to Ψ in $L^{p'}_{1-s}(\Omega, \Lambda^{\ell+1})$. This and the distributional definition of d then imply that $\langle du, \Psi \rangle - \langle u, \delta\Psi \rangle = \lim_j(\langle du, \Psi_j \rangle - \langle u, \delta\Psi_j \rangle) = 0$, from which the desired conclusion follows.

Next, we notice that the estimate

$$\|\xi\|_{B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell+1})} \leq C(\|u\|_{L^p_s(\Omega,\Lambda^\ell)} + \|du\|_{L^p_s(\Omega,\Lambda^{\ell+1})}) \tag{4.5}$$

follows from (3.13), (4.3), (3.8), plus the fact that, given $\psi \in B^{p',p'}_{1-1/p'-s}(\partial\Omega, \Lambda^{\ell+1})$, one is able to choose some $\Psi \in L^{p'}_{1-s}(\Omega, \Lambda^{\ell+1})$ such that $\operatorname{Tr}\Psi = \psi$ with the additional property that $\|\Psi\|_{L^{p'}_{1-s}(\Omega,\Lambda^{\ell+1})} \leq C\|\psi\|_{B^{p',p'}_{1-1/p'-s}(\partial\Omega,\Lambda^{\ell+1})}$, where $C = C(\Omega, p, s) > 0$. See Proposition 3.5 for the latter claim. $\square$

An immediate corollary of the proposition above and of Definition (4.3) is the following useful integration by parts formula.

Corollary 4.2. *For any $u \in D_\ell(d; L^p_s(\Omega))$ and $\Psi \in L^{p'}_{1-s}(\Omega, \Lambda^{\ell+1})$, there holds*

$$\langle du, \Psi\rangle = \langle u, \delta\Psi\rangle + \langle \nu \wedge u, \operatorname{Tr}\Psi\rangle. \tag{4.6}$$

Finally, a similar set of results are valid for *tangential* traces of differential forms. Below, we record the main result in this regard.

Proposition 4.3. *There exists a linear, bounded, tangential trace operator*

$$\nu \vee \cdot : D_\ell(\delta; L^p_s(\Omega)) \longrightarrow B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell-1}) \tag{4.7}$$

such that

$$\langle u, d\Phi\rangle = \langle \delta u, \Phi\rangle + \langle \nu \vee u, \operatorname{Tr}\Phi\rangle \tag{4.8}$$

for any $u \in D_\ell(\delta; L^p_s(\Omega))$ and any $\Phi \in L^{p'}_{1-s}(\Omega, \Lambda^{\ell-1})$.

5. Traces of Differential Forms

Assume that

$$1 < p < \infty, \quad -1 + 1/p < s < 1/p, \quad 1/p + 1/p' = 1, \tag{5.1}$$

and, for a fixed Lipschitz domain $\Omega \subset M$, consider the space

$$NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell) := \Big\{\xi \in B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell) : \exists\, \eta \in B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1}) \\ \text{such that } \langle \xi, \operatorname{Tr}(\delta f)\rangle = \langle \eta, \operatorname{Tr} f\rangle,\ \forall\, f \in D_{\ell+1}(\delta; L^{p'}_{1-s}(\Omega))\Big\} \tag{5.2}$$

equipped with the natural graph norm, i.e.,

$$\|\xi\|_{NB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^\ell)} := \|\xi\|_{B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^\ell)} + \|\eta\|_{B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell+1})}. \tag{5.3}$$

Our first result shows that there is no ambiguity in defining the norm (5.3) and that the space $NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell)$ depends exclusively on $\partial\Omega$ and not on Ω itself.

Proposition 5.1. *The definition above is meaningful and, in fact,*

$$NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell) = \Big\{\xi \in B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell) : \exists\, \eta \in B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1}) \\ \text{such that } \langle \xi, (\delta f)\,|_{\partial\Omega}\rangle = \langle \eta, f\,|_{\partial\Omega}\rangle,\ \forall\, f \in C^\infty(M, \Lambda^{\ell+1})\Big\}. \tag{5.4}$$

Proof. We note that, in the context of (5.2), the differential form ξ determines η uniquely. Indeed, if η_1, $\eta_2 \in B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1})$ are such that

$$\langle \eta_1, \mathrm{Tr}\, f\rangle = \langle \eta_2, \mathrm{Tr}\, f\rangle, \quad \forall\, f \in D_{\ell+1}(\delta; L^{p'}_{1-s}(\Omega)), \tag{5.5}$$

then $\eta := \eta_1 - \eta_2 \in B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1})$ satisfies

$$\langle \eta, f\,|_{\partial\Omega}\rangle = 0 \quad \forall\, f \in C^\infty(\bar{\Omega}, \Lambda^{\ell+1}). \tag{5.6}$$

Since $\mathrm{Tr} : C^\infty(\bar{\Omega}, \Lambda^{\ell+1}) \to B^{p',p'}_{-s+1-1/p'}(\partial\Omega, \Lambda^{\ell+1})$ has dense range, it follows that $\eta = 0$, i.e., $\eta_1 = \eta_2$, as claimed.

Second, the equality (5.4) is a consequence of a density result, proved in [27], to the effect that if $f \in D_{\ell+1}(\delta; L^{p'}_{1-s}(\Omega))$, then there exists a sequence $f_j \in C^\infty(\bar{\Omega}, \Lambda^{\ell+1})$, $j = 1, 2, \dots$, such that

$$\begin{aligned} f_j &\to f && \text{in } L^{p'}_{1-s}(\Omega, \Lambda^{\ell+1}), \\ \delta f_j &\to \delta f && \text{in } L^{p'}_{1-s}(\Omega, \Lambda^{\ell}), \end{aligned} \qquad \text{as } j \to \infty. \tag{5.7}$$

This conclusion of the proof of the proposition. $\square$

Remarks. (i) It follows from (5.2) that $NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell)$ can be identified with a closed subspace of $B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell) \oplus B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1})$ and, hence, this space is reflexive.

(ii) The membership of a differential form $\xi \in B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell)$ to $NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell)$ ensures that ξ *is normal to* $\partial\Omega$ *in a weak sense*, i.e., it satisfies

$$\langle \xi, \mathrm{Tr}(\delta f)\rangle = 0, \quad \forall\, f \in D_{\ell+1}(\delta; L^{p'}_{1-s}(\Omega)) \text{ with } \mathrm{Tr}\, f = 0. \tag{5.8}$$

In particular, whenever $\xi \in B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell)$ is such that (5.8) holds, the functional

$$C^\infty(M, \Lambda^{\ell+1})\,|_{\partial\Omega} \ni f\,|_{\partial\Omega} \mapsto \langle \xi, (\delta f)\,|_{\partial\Omega}\rangle \in \mathbb{R} \tag{5.9}$$

is well defined.

After this preamble, we are now ready to state and prove the first major result of this section.

Theorem 5.2. *The trace operator*

$$\nu \wedge \cdot : D_\ell(d; L^p_s(\Omega)) \longrightarrow NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1}) \tag{5.10}$$

is well defined, linear, bounded, and onto. In fact, there exists a bounded, linear operator

$$\mathrm{Ex} : NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1}) \longrightarrow D_\ell(d; L^p_s(\Omega)) \tag{5.11}$$

such that

$$\nu \wedge (\mathrm{Ex}\,\xi) = \xi, \quad \forall\, \xi \in NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1}). \tag{5.12}$$

Proof. We shall proceed in a series of steps, starting with Step I.

Step I: The operator (5.10) *is well defined, linear and bounded.*

If $u \in D_\ell(d; L^p_s(\Omega))$, then $du \in D_{\ell+1}(d; L^p_s(\Omega))$ and, hence, $\eta := -\nu \wedge (du) \in B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+2})$ and

$$\|\eta\|_{B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell+2})} \leq C\|du\|_{L^p_s(\Omega,\Lambda^{\ell+1})}, \tag{5.13}$$

by Proposition 4.1. Furthermore, if $u \in D_\ell(d; L^p_s(\Omega))$, then $\xi := \nu \wedge u \in B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1})$ by Proposition 4.1. In addition, for each $f \in D_{\ell+2}(\delta; L^{p'}_{1-s}(\Omega))$ we may write, based on repeated integrations by parts (cf. Corollary 4.2):

$$\begin{aligned} \langle \xi, \mathrm{Tr}(\delta f)\rangle &= \langle \nu \wedge u, \mathrm{Tr}(\delta f)\rangle = \langle du, \delta f\rangle \\ &= -\langle \nu \wedge (du), \mathrm{Tr}\, f\rangle = \langle \eta, \mathrm{Tr}\, f\rangle \end{aligned} \tag{5.14}$$

which shows that $\xi \in NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1})$, as desired.

Step II: Localization. Set

$$\Omega_+ := \Omega, \quad \Omega_- := M \setminus \bar{\Omega}. \tag{5.15}$$

Since $NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell)$ is a module over $C^\infty(M)$, there is no loss of generality in assuming that we seek to extend forms $\xi \in NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell)$ with the additional property that $\operatorname{supp}\xi \subset \mathcal{O}$ where $\mathcal{O}$ is an open coordinate chart on M such that, when viewed as a subset of the Euclidean space, $\mathcal{O} \cap \Omega_-$ becomes a bounded Lipschitz domain which is star-like with respect to a ball.

If $\Delta_\ell := -\delta d - d\delta$ is the Hodge Laplacian on ℓ-forms, then

$$\Delta_\ell - 1 : L^2_1(M, \Lambda^\ell) \longrightarrow L^2_{-1}(M, \Lambda^\ell) \tag{5.16}$$

has an inverse, $(\Delta_\ell - 1)^{-1}$, for each $\ell \in \{0, \ldots, n\}$ whose Schwartz kernel, $\Gamma_\ell(x, y)$, is a symmetric double form of bidegree (ℓ, ℓ). The commutation relations $d\Delta_\ell = \Delta_{\ell+1} d$ and $\delta\Delta_\ell = \Delta_{\ell-1}\delta$ translate into

$$\delta_x \Gamma_{\ell+1}(x, y) = d_y \Gamma_\ell(x, y), \quad d_x \Gamma_\ell(x, y) = \delta_y \Gamma_{\ell+1}(x, y). \tag{5.17}$$

Next, denote by $S_\ell^\pm$ the single layer potential operators associated with $\Omega_\pm$, i.e.,

$$S_\ell^\pm f(x) := \langle \Gamma_\ell(x,\cdot), f\rangle, \ \ f \in B_{s-1/p}^{p,p}(\partial\Omega, \Lambda^\ell), \quad x \in \Omega_\pm. \tag{5.18}$$

Note that $(\Delta_\ell - 1)S_\ell^\pm f = 0$ in $\Omega_\pm$. Mapping properties for these operators have been established in Theorem 7.1 of [26], where it has been proved that

$$S_\ell^\pm : B_{s-1/p}^{p,p}(\partial\Omega, \Lambda^\ell) \to L_{s+1}^p(\Omega_\pm, \Lambda^\ell), \qquad 1 < p < \infty, \ -1 + 1/p < s < 1/p, \tag{5.19}$$

boundedly. Let us also set

$$S_\ell := \mathrm{Tr} \circ S_\ell^+ = \mathrm{Tr} \circ S_\ell^- : B_{s-1/p}^{p,p}(\partial\Omega, \Lambda^\ell) \to B_{s+1-1/p}^{p,p}(\partial\Omega, \Lambda^\ell), \tag{5.20}$$

where Tr is the trace on $\partial\Omega$. In particular,

$$S_\ell f := S_\ell^\pm f \text{ in } \Omega_\pm \ \Rightarrow \ S_\ell : B_{s-1/p}^{p,p}(\partial\Omega, \Lambda^\ell) \to L_{s+1}^p(M, \Lambda^\ell), \tag{5.21}$$

boundedly, whenever $1 < p < \infty$, $-1 + 1/p < s < 1/p$. Finally, define the Newtonian (volume) potential

$$\Pi_\ell u(x) := \langle \Gamma_\ell(x,\cdot), u\rangle, \quad x \in M, \tag{5.22}$$

which is a classical pseudo-differential operator of order -2. Then

$$\Pi_\ell : L_s^p(M, \Lambda^\ell) \longrightarrow L_{s+2}^p(M, \Lambda^\ell), \quad 1 < p < \infty, \ s \in \mathbb{R}, \tag{5.23}$$

$$(\Delta_\ell - 1)\Pi_\ell = I, \quad \text{the identity operator.} \tag{5.24}$$

Step III. Fix an arbitrary $\xi \in NB_{s-1/p}^{p,p}(\partial\Omega, \Lambda^{\ell+1})$ and let $\eta \in B_{s-1/p}^{p,p}(\partial\Omega, \Lambda^{\ell+2})$ be the differential form associated with ξ as in (5.4). Finally, define

$$u^\pm := \delta S_{\ell+1}^\pm \xi \quad \text{in } \Omega_\pm. \tag{5.25}$$

Then $u^\pm \in D_\ell(d; L_s^p(\Omega_\pm))$ and

$$du^\pm = -\delta S_{\ell+2}^\pm(\eta) - S_{\ell+1}^\pm \xi, \quad \text{in } \Omega_\pm. \tag{5.26}$$

Indeed, we have

$$du^+ = d\,\delta S_{\ell+1}^+ \xi = -\delta\, dS_{\ell+1}^+ \xi - S_{\ell+1}^+ \xi, \tag{5.27}$$

and for each fixed $x \in \Omega_+$ we may write

$$(5.28) \qquad (dS^+_{\ell+1}\xi)(x) = \langle d_x\Gamma_{\ell+1}(x,\cdot),\xi\rangle = \langle \delta_y\Gamma_{\ell+2}(x,y),\xi(y)\rangle$$
$$= \langle \mathrm{Tr}\cdot(\delta\cdot\Gamma_{\ell+2}(x,\cdot)),\xi\rangle = \langle \Gamma_{\ell+2}(x,\cdot),\eta\rangle = S^+_{\ell+1}\eta,$$

since $\Gamma_{\ell+2}(x,\cdot) \in C^\infty(\bar{\Omega}_-,\Lambda^{\ell+2})$. The case when the superscript $+$ is replaced by $-$ is analogous.

Step IV. With ξ, η, u^+ and u^- as in Step III,

$$(5.29) \qquad \nu\wedge u^+ - \nu\wedge u^- = \xi.$$

To justify (5.29), for an arbitrary $\Phi \in C^\infty(M,\Lambda^{\ell+1})$ we write (using the fact that the outward unit normal for Ω_- is $-\nu$):

$$(5.30) \qquad \begin{aligned} &\langle \nu\wedge u^+ - \nu\wedge u^-,\mathrm{Tr}\,\Phi\rangle \\ &\quad = \langle du^+,\Phi\rangle - \langle u^+,\delta\Phi\rangle + \langle du^-,\Phi\rangle - \langle u^-,\delta\Phi\rangle \\ &\quad =: I + II + III + IV. \end{aligned}$$

Next, (5.26) and (4.7) yield

$$(5.31) \qquad \begin{aligned} I &= -\langle \delta S^+_{\ell+2}\eta,\Phi|_{\Omega_+}\rangle - \langle S^+_{\ell+1}\xi,\Phi|_{\Omega_+}\rangle \\ &= -\int_{\Omega_+}\langle S^+_{\ell+2}\eta,d\Phi\rangle\,\mathrm{d}V - \int_{\Omega_+}\langle S^+_{\ell+1}\xi,\Phi\rangle\,\mathrm{d}V + \int_{\partial\Omega}\langle \nu\vee S_{\ell+2}\eta,\Phi\rangle\,\mathrm{d}\sigma, \end{aligned}$$

$$(5.32) \qquad \begin{aligned} III &= -\langle \delta S^-_{\ell+2}\eta,\Phi|_{\Omega_-}\rangle - \langle S^-_{\ell+1}\xi,\Phi|_{\Omega_-}\rangle \\ &= -\int_{\Omega_-}\langle S^-_{\ell+2}\eta,d\Phi\rangle\,\mathrm{d}V - \int_{\Omega_-}\langle S^-_{\ell+1}\xi,\Phi\rangle\,\mathrm{d}V - \int_{\partial\Omega}\langle \nu\vee S_{\ell+2}\eta,\Phi\rangle\,\mathrm{d}\sigma. \end{aligned}$$

Consequently, since the boundary integrals in (5.32) and (5.31) have opposite signs, we may use Fubini's Theorem, (5.17) and the definition of η in order to write

$$(5.33) \qquad \begin{aligned} I + III &= -\int_M\langle S_{\ell+2}\eta,d\Phi\rangle\,\mathrm{d}V - \int_M\langle S_{\ell+1}\xi,\Phi\rangle\,\mathrm{d}V \\ &= -\langle \eta,\mathrm{Tr}\,\Pi_{\ell+2}(d\Phi)\rangle - \langle \xi,\mathrm{Tr}\,\Pi_{\ell+1}\Phi\rangle \\ &= -\langle \xi,\mathrm{Tr}(\delta\Pi_{\ell+2}(d\Phi))\rangle - \langle \xi,\mathrm{Tr}\,\Pi_{\ell+1}\Phi\rangle \\ &= -\langle \xi,\mathrm{Tr}(\delta d\Pi_{\ell+1}\Phi + \Pi_{\ell+1}\Phi)\rangle. \end{aligned}$$

Going further,

$$II = -\langle \delta S^{+}_{\ell+1}\xi, \delta\Phi|_{\Omega_+}\rangle = -\int_{\Omega_+}\langle S^{+}_{\ell+1}\xi, d\,\delta\Phi\rangle\,\mathrm{d}V + \int_{\partial\Omega}\langle \nu\vee S_{\ell+1}\xi, \delta\Phi\rangle\,\mathrm{d}\sigma, \tag{5.34}$$

$$IV = -\langle \delta S^{-}_{\ell+1}\xi, \delta\Phi|_{\Omega_-}\rangle = -\int_{\Omega_-}\langle S^{-}_{\ell+1}\xi, d\,\delta\Phi\rangle\,\mathrm{d}V - \int_{\partial\Omega}\langle \nu\vee S_{\ell+1}\xi, \delta\Phi\rangle\,\mathrm{d}\sigma, \tag{5.35}$$

so that

$$II + IV = -\int_M \langle S_{\ell+1}\xi, d\,\delta\Phi\rangle\,\mathrm{d}V = -\langle \xi, \mathrm{Tr}(d\,\delta\Pi_{\ell+1}\Phi)\rangle. \tag{5.36}$$

Since

$$d\,\delta\Pi_\ell + \delta\, d\Pi_\ell + \Pi_\ell = -I \quad \text{on } M, \tag{5.37}$$

it follows from (5.30)–(5.37) that

$$\langle \nu\wedge u^+ - \nu\wedge u^-,\ \mathrm{Tr}\,\Phi\rangle = \langle \xi, \mathrm{Tr}\,\Phi\rangle. \tag{5.38}$$

Finally, we note that (5.29) follows from this and from the fact that the trace operator

$$\mathrm{Tr} : C^\infty(M, \Lambda^{\ell+1}) \to B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1})$$

has a dense range. This completes the proof of Step IV.

Step V. There exists $w \in L^p_{s+1}(M, \Lambda^\ell)$ with $dw = du^-$ in $\mathcal{O}\cap\Omega_-$.

Under the current assumptions on $\mathcal{O}\cap\Omega_-$, it has been proved in Section 4 of [27] that there exist $\theta \in C^\infty_c(\Omega)$ and a family of linear operators

$$K_\ell : (C^\infty_c(\mathcal{O}\cap\Omega_-, \Lambda^\ell))' \longrightarrow (C^\infty_c(\mathcal{O}\cap\Omega_-, \Lambda^{\ell-1}))', \quad 1 \le \ell \le n, \tag{5.39}$$

such that

$$\forall\, w \in (C^\infty_c(\mathcal{O}\cap\Omega_-, \Lambda^\ell))' \Rightarrow w = \begin{cases} K_1(dw) + \langle w, \theta\rangle & \text{if } \ell = 0, \\ d(K_\ell w) + K_{\ell+1}(dw) & \text{if } 1 \le \ell \le n-1, \\ d(K_n w) & \text{if } \ell = n. \end{cases} \tag{5.40}$$

and for which

$$K_\ell : L^p_s(\mathcal{O}\cap\Omega_-,\Lambda^\ell) \longrightarrow L^p_{s+1}(\mathcal{O}\cap\Omega_-,\Lambda^{\ell-1}) \tag{5.41}$$

in a bounded fashion. Thus, we may define the differential form w as the extension of $K_\ell(du^-) \in L^p_{s+1}(\mathcal{O}\cap\Omega_-,\Lambda^{\ell-1})$ to an element in $L^p_{s+1}(M,\Lambda^{\ell-1})$.

Step VI. There exists $\omega \in L^p_{s+1}(M,\Lambda^{\ell-1})$ such that $d\omega = u^- - w$ in $\mathcal{O}\cap\Omega_-$.

This time, take ω to be the extension of $K_\ell(u^- - w) \in L^p_{s+1}(\mathcal{O}\cap\Omega_-,\Lambda^{\ell-1})$ to an element in $L^p_{s+1}(M,\Lambda^{\ell-1})$.

Step VII. Retaining the notation used in the previous steps, we have

$$v := w + d\omega \in D_\ell(d; L^p_s(M)). \tag{5.42}$$

Moreover, if $\psi \in C^\infty_c(\mathcal{O})$ is such that $\psi \equiv 1$ on $\operatorname{supp}\xi$, then $u := \psi(u^+ - v)|_\Omega$ satisfies $u \in D_\ell(d; L^p_s(\Omega))$ and $\nu \wedge u = \xi$.

To see this, we compute

$$\nu \wedge (\psi v\,|_{\Omega_+}) = \nu \wedge (\psi v\,|_{\Omega_-}) = \psi\nu \wedge (w + d\omega)\,|_{\Omega_-} = \psi\nu \wedge u^-. \tag{5.43}$$

Thus,

$$\nu \wedge u = \psi\nu \wedge u^+ - \psi\nu \wedge (v\,|_{\Omega_+}) = \psi(\nu \wedge u^+ - \nu \wedge u^-) = \psi\xi = \xi, \tag{5.44}$$

by Step IV.

This concludes the proof of the claim in Step VII and finishes the proof of the theorem. ∎

Under the assumptions (5.1), we shall also consider the space

$$\begin{aligned} &TB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^\ell) \\ &\quad := \Big\{\xi \in B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^\ell) : \exists\, \eta \in B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell+1}) \text{ such that} \\ &\qquad\qquad \langle \xi, \operatorname{Tr}(df)\rangle = \langle \eta, \operatorname{Tr} f\rangle,\ \forall\, f \in D_{\ell-1}(d; L^{p'}_{1-s}(\Omega))\Big\}. \end{aligned} \tag{5.45}$$

once again, equipped with the natural graph norm.

As with the space $NB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^\ell)$, this definition in unambiguous and the following alternative description holds:

$$\begin{aligned} &TB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^\ell) \\ &\quad := \Big\{\xi \in B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^\ell) : \exists\, \eta \in B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell+1}) \\ &\qquad\qquad \text{such that } \langle \xi, \operatorname{Tr}(df)\rangle = \langle \eta, \operatorname{Tr} f\rangle,\ \forall\, f \in C^\infty(M,\Lambda^{\ell-1})\Big\}. \end{aligned} \tag{5.46}$$

Moreover, the mapping induced by the Hodge-star isomorphism

$$(5.47) \qquad * : NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell) \longrightarrow TB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{n-\ell})$$

is an isomorphism.

Corollary 5.3. *Let* $1 < p < \infty$ *and* $-1 + 1/p < s < 1/p$. *Then the operator*

$$(5.48) \qquad \nu \vee \cdot : D_\ell(\delta; L^p_s(\Omega)) \longrightarrow TB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell-1})$$

is well-defined, linear, bounded, and has a linear, bounded right inverse. In particular, it is onto.

Proof. This follows from Theorem 5.2, definitions, and Hodge duality. □

The spaces (5.2), (5.45) are further related as described in the proposition below.

Proposition 5.4. *The mapping*

$$(5.49) \qquad \nu \wedge \cdot : TB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell) \longrightarrow (NB^{p',p'}_{s-1/p'}(\partial\Omega, \Lambda^{\ell+1}))^*$$

defined by

$$(5.50) \qquad \langle \nu \wedge f, g \rangle := \langle u, dw \rangle - \langle \delta u, w \rangle$$

for $u \in D_{\ell+1}(\delta; L^p_s(\Omega))$, *with* $f = \nu \vee u$ *and* $w \in D_\ell(d; L^{p'}_{-s}(\Omega))$ *with* $g = \nu \wedge w$ *is well-defined, in fact an isomorphism, for each* $\ell \in \{0, 1, \dots, n\}$.

Furthermore, the adjoint of (5.49)–(5.50) *is the operator*

$$(5.51) \qquad \nu \vee \cdot : NB^{p',p'}_{s-1/p'}(\partial\Omega, \Lambda^{\ell+1}) \longrightarrow (TB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell))^*$$

defined as

$$(5.52) \qquad \langle \nu \vee f, g \rangle := \langle du, w \rangle - \langle u, \delta w \rangle$$

for $u \in D_\ell(d; L^{p'}_{-s}(\Omega))$, *with* $f = \nu \wedge u$ *and* $w \in D_{\ell+1}(\delta; L^p_s(\Omega))$ *with* $g = \nu \vee w$.

Proof. Proving that the map (5.49)–(5.50) is well-defined comes down to checking the following claim: if $u \in D_{\ell+1}(\delta; L^p_s(\Omega))$ and $w \in D_\ell(d; L^{p'}_{-s}(\Omega))$ are such that either $\nu \vee u = 0$ or $\nu \wedge w = 0$, then $\langle u, dw \rangle - \langle \delta u, w \rangle = 0$. In turn, this is an easy consequence of a density result, proved in [27], according to which if $w \in D_\ell(d; L^p_s(\Omega))$ has $\nu \wedge w = 0$, then there exists a sequence $w_j \in C^\infty_c(\Omega, \Lambda^\ell)$, $j = 1, 2, \dots$, such that

$$(5.53) \quad w_j \to w \text{ in } L^p_s(\Omega, \Lambda^\ell) \quad \text{and} \quad dw_j \to dw \text{ in } L^p_s(\Omega, \Lambda^{\ell+1}) \quad \text{as } j \to \infty.$$

Having established the well-definiteness of the map (5.49)–(5.50), we next remark that this map is bounded, thanks to (3.8), as well as linear and one-to-one. There remains to show that it is also onto. To this end, pick an arbitrary $\theta \in (NB^{p',p'}_{s-1/p'}(\partial\Omega, \Lambda^{\ell+1}))^*$ and consider $\hat{\theta} : D_\ell(d; L^{p'}_{-s}(\Omega)) \to \mathbb{R}$ defined by $\hat{\theta}(u) := \theta(\nu \wedge u)$. Regarding $D_\ell(d; L^{p'}_{-s}(\Omega))$ as a (closed) subspace of $L^{p'}_{-s}(\Omega, \Lambda^\ell) \oplus L^{p'}_{-s}(\Omega, \Lambda^{\ell+1})$ via the identification $u \mapsto (u, du)$, the Hahn-Banach theorem in concert with (3.8) allow us to conclude that there exist $v_1 \in L^p_s(\Omega, \Lambda^{\ell+1})$ and $v_2 \in L^p_s(\Omega, \Lambda^\ell)$ such that

$$\hat{\theta}(u) = \langle v_1, du \rangle - \langle v_2, u \rangle, \quad \forall\, u \in D_\ell(d; L^{p'}_{-s}(\Omega)). \tag{5.54}$$

Note that choosing $u \in C^\infty_c(\Omega, \Lambda^\ell)$ yields $\delta v_1 = v_2$. In particular, $v_1 \in D_\ell(\delta; L^p_s(\Omega))$. Utilizing this identity back in (5.54) gives that $\theta(\nu \wedge u) = \langle \nu \wedge (\nu \vee v_1), \nu \wedge u \rangle$, for each $u \in D_\ell(d; L^{p'}_{-s}(\Omega))$ which, in turn, entails $\nu \wedge (\nu \vee v_1) = \theta$. Hence, the map (5.49) is onto and this finishes the proof of the claims made in the first part of the proposition.

Finally, the claim that (5.51) is the adjoint of the operator (5.49) follows by comparing (5.52) with (5.50). ◻

The analysis above allows us to deduce a rather general integration by parts formula for differential forms on Lipschitz domains.

Corollary 5.5. *Under the assumptions* (5.1), *the following identities hold:*

$$\langle du, w \rangle = \langle u, \delta w \rangle + \langle \nu \wedge u, \nu \wedge (\nu \vee w) \rangle, \tag{5.55}$$

$$\langle du, w \rangle = \langle u, \delta w \rangle + \langle \nu \vee (\nu \wedge u), \nu \vee w \rangle, \tag{5.56}$$

for any $u \in D_\ell(d; L^p_s(\Omega))$ *and* $w \in D_{\ell+1}(\delta; L^p_s(\Omega))$.

The corollary above is an immediate consequence of Proposition 5.4. Here we only want to point out that, in the context of (5.55), $\nu \vee w \in TB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell)$ by (5.48) so that, further, $\nu \wedge (\nu \vee w) \in (NB^{p',p'}_{s-1/p'}(\partial\Omega, \Lambda^{\ell+1}))^*$ by Proposition 5.4. In particular, since $\nu \wedge u$ belongs to $NB^{p',p'}_{s-1/p'}(\partial\Omega, \Lambda^{\ell+1})$, the last pairing in the right-hand side of (5.55) is well defined. Of course, the same considerations apply to (5.56).

Finally, we are now in a position to discuss *global* traces of differential forms in the space $D_\ell(d; L^p_s(\Omega)) \cap D_\ell(\delta; L^p_s(\Omega))$.

Theorem 5.6. *The assignments*

$$D_\ell(d; L^p_s(\Omega)) \ni u \mapsto u_{\tan} := \nu \vee (\nu \wedge u) \in (TB^{p',p'}_{s-1/p'}(\partial\Omega, \Lambda^\ell))^*, \tag{5.57}$$

$$D_\ell(\delta; L^p_s(\Omega)) \ni u \mapsto u_{\text{nor}} := \nu \wedge (\nu \vee u) \in (NB^{p',p'}_{s-1/p'}(\partial\Omega, \Lambda^\ell))^*, \tag{5.58}$$

are linear, bounded and onto. In fact, each has a bounded, linear, right inverse. Furthermore, they are compatible with the mappings

$$\begin{aligned}(5.59)\qquad \mathrm{Tr}: L^p_{s+1}(\Omega,\Lambda^\ell) &\to B^{p,p}_{s+1-1/p}(\partial\Omega)\\ &= (B^{p',p'}_{-s-1/p'}(\partial\Omega))^* \hookrightarrow (TB^{p',p'}_{-s-1/p'}(\partial\Omega,\Lambda^\ell))^*\end{aligned}$$

$$\begin{aligned}(5.60)\qquad \mathrm{Tr}: L^p_{s+1}(\Omega,\Lambda^\ell) &\to B^{p,p}_{s+1-1/p}(\partial\Omega)\\ &= (B^{p',p'}_{-s-1/p'}(\partial\Omega))^* \hookrightarrow (NB^{p',p'}_{-s-1/p'}(\partial\Omega,\Lambda^\ell))^*\end{aligned}$$

in the sense that $L^p_{s+1}(\Omega,\Lambda^\ell) \hookrightarrow D_\ell(d;L^p_s(\Omega)) \cap D_\ell(\delta;L^p_s(\Omega))$ *and the actions of* (5.57), (5.58) *agree with those of* (5.59) *and* (5.60), *respectively.*

Finally, the global trace map

$$\begin{aligned}(5.61)\qquad \mathrm{tr}: D_\ell(d;L^p_s(\Omega)) \cap D_\ell(\delta;L^p_s(\Omega))&\\ \longrightarrow (TB^{p',p'}_{s-1/p'}(\partial\Omega,\Lambda^\ell) &\oplus NB^{p',p'}_{s-1/p'}(\partial\Omega,\Lambda^\ell))^*\end{aligned}$$

$$\begin{aligned}(5.62)\qquad \langle \mathrm{tr}\,u, f\oplus g\rangle &:= \langle u_{\mathrm{tan}}, f\rangle + \langle u_{\mathrm{nor}}, g\rangle,\\ &\forall\, u \in D_\ell(d;L^p_s(\Omega)) \cap D_\ell(\delta;L^p_s(\Omega)),\\ &\forall\, f \in TB^{p',p'}_{s-1/p'}(\partial\Omega,\Lambda^\ell),\ \forall\, g \in NB^{p',p'}_{s-1/p'}(\partial\Omega,\Lambda^\ell)\end{aligned}$$

is linear, bounded, and has a linear, bounded right inverse. In particular, it is onto.

Furthermore, the action of the map (5.61)–(5.62) *is compatible with that of*

$$\begin{aligned}(5.63)\qquad \mathrm{Tr}: L^p_{s+1}(\Omega,\Lambda^\ell) &\to B^{p,p}_{s+1-1/p}(\partial\Omega,\Lambda^\ell)\\ &\overset{\iota}{\hookrightarrow} (TB^{p',p'}_{s-1/p'}(\partial\Omega,\Lambda^\ell) \oplus NB^{p',p'}_{s-1/p'}(\partial\Omega,\Lambda^\ell))^*\end{aligned}$$

where the inclusion acts above according to $\langle \iota(\xi), f\oplus g\rangle := \langle \xi, f\rangle + \langle \xi, g\rangle$, *in the sense that* $L^p_{s+1}(\Omega,\Lambda^\ell) \hookrightarrow D_\ell(d;L^p_s(\Omega)) \cap D_\ell(\delta;L^p_s(\Omega))$ *and the action of* (5.63) *agrees with that of* (5.61)–(5.62).

Hence, from this point of view, the global trace map for differential forms introduced in (5.61)–(5.62) *can be regarded as an extension of the ordinary componentwise trace map from* (3.14).

Proof. The claim that the mappings (5.57), (5.58) are well-defined, linear, bounded and that each has a linear, bounded, right inverse, follows from Proposition 5.4 and Theorem 5.2.

Next, let $u \in L^p_{s+1}(\Omega,\Lambda^\ell) \hookrightarrow D_\ell(d;L^p_s(\Omega))$ and consider the action of the functional

$$\nu \vee (\nu\wedge u) \in (TB^{p',p'}_{s-1/p'}(\partial\Omega,\Lambda^\ell))^*$$

on an arbitrary $f \in TB^{p',p'}_{s-1/p'}(\partial\Omega,\Lambda^\ell)$, i.e.,

$$(5.64)\qquad \langle \nu\vee(\nu\wedge u), f\rangle = \langle du, w\rangle - \langle u, \delta w\rangle$$

if $f = \nu \vee w$ for some $w \in D_\ell(\delta; L^{p'}_{1-s}(\Omega))$. Recalling the integration by parts formula (4.8), the expression in the right-hand side of (5.64) can be further expressed as

$$(5.65) \qquad \langle du, w\rangle - \langle u, \delta w\rangle = \langle \mathrm{Tr}\, u, \nu \vee w\rangle = \langle \mathrm{Tr}\, u, f\rangle.$$

Thus, all in all, $u_{\tan} = \mathrm{Tr}\, u$ as functionals on $TB^{p',p'}_{s-1/p'}(\partial\Omega, \Lambda^\ell)$ ($\hookrightarrow B^{p',p'}_{s-1/p'}(\partial\Omega, \Lambda^\ell)$), which proves that the mappings (5.57), (5.59) are compatible. The fact that (5.58) and (5.60) are also compatible is proved in a similar fashion.

Finally, the claims made about the map (5.61)–(5.62) follow from what we have proved so far in a straightforward manner. This finishes the proof of Theorem 5.6. □

6. The Boundary De Rham Complex

We continue to assume that $\Omega \subset M$ is an arbitrary Lipschitz domain. Here we take a closer look at the mapping $NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell) \ni \xi \mapsto \eta \in B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1})$ where ξ, η are as in the description of (5.2).

Proposition 6.1. *If* (5.1) *holds, then the operator*

$$(6.1) \qquad d_\partial : NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell) \longrightarrow NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1})$$

given by

$$(6.2) \qquad d_\partial \xi := \eta,$$

whenever ξ and η are as in (5.2), *is well defined, linear and bounded. Furthermore,*

$$(6.3) \qquad d_\partial \circ d_\partial = 0,$$

and

$$(6.4) \qquad d_\partial(\nu \wedge u) = -\nu \wedge du, \quad \forall\, u \in D_\ell(d; L^p_s(\Omega)).$$

Proof. If ξ, η are as in the right-hand side of (5.4), it follows that

$$(6.5) \qquad \langle \eta, \mathrm{Tr}(\delta f)\rangle = \langle \xi, \mathrm{Tr}(\delta^2 f)\rangle = 0, \quad \forall\, f \in C^\infty(M, \Lambda^{\ell+2}),$$

which proves that $\eta \in NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell+1})$. In particular, the operator (6.1)–(6.2) is well defined. It is also implicit in the above argument that $d_\partial \eta = 0$ from which the identity (6.3) follows. Finally, (6.4) is implicit in Step I of the proof of Theorem 5.2. □

The identity (6.3) suggests the consideration of the sequence of group homomorphisms

$$
\begin{aligned}
(6.6)\qquad \cdots \xrightarrow{d_\partial} NB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell-1}) \xrightarrow{d_\partial} NB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell}) \\
\xrightarrow{d_\partial} NB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell+1}) \xrightarrow{d_\partial} \cdots
\end{aligned}
$$

in which the image of each arrow is contained in the kernel of the next. Our goal is to study the cohomology of the boundary De Rham complex (6.6). To begin with, it is clear from definitions that the kernel of the operator (6.1) is given by

$$
(6.7)\qquad N_0B^{p,p}_s(\partial\Omega,\Lambda^{\ell}) := \{\xi \in NB^{p,p}_s(\partial\Omega,\Lambda^{\ell}) : d_\partial\xi = 0\}
$$

and that the image of d_∂ acting on $NB^{p,p}_s(\partial\Omega,\Lambda^{\ell})$ is a subspace of $N_0B^{p,p}_s(\partial\Omega,\Lambda^{\ell+1})$. The quotient group

$$
\frac{N_0B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell+1})}{d_\partial[NB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell})]}
$$

is studied in the theorem below.

To state it, we first recall that, for any reasonable topological space X, one associates $H^{\ell}_{\mathrm{sing}}(X;\mathbb{R})$, the classical ℓ-th singular homology group of X over the reals; see, e.g., [19]. Generally speaking, we set $b_\ell(X) := \dim H^{\ell}_{\mathrm{sing}}(X;\mathbb{R})$ and refer to it as the ℓ-th Betti number of X.

Theorem 6.2. *The operator*

$$
(6.8)\qquad d_\partial : NB^{p,p}_s(\partial\Omega,\Lambda^{\ell}) \longrightarrow N_0B^{p,p}_s(\partial\Omega,\Lambda^{\ell+1})
$$

has closed range and its cokernel is isomorphic to $H^{\ell}_{\mathrm{sing}}(\partial\Omega;\mathbb{R})$, the ℓ-th singular homology group of $\partial\Omega$ over the reals, i.e.,

$$
(6.9)\qquad \frac{N_0B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell+1})}{d_\partial[NB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell})]} \cong H^{\ell}_{\mathrm{sing}}(\partial\Omega;\mathbb{R}), \quad \ell = 1,2,\dots,n-1.
$$

In particular,

$$
(6.10)\qquad d_\partial : \frac{NB^{p,p}_s(\partial\Omega,\Lambda^{\ell})}{N_0B^{p,p}_s(\partial\Omega,\Lambda^{\ell})} \longrightarrow N_0B^{p,p}_s(\partial\Omega,\Lambda^{\ell+1})
$$

is a Fredholm operator with index $b_\ell(\partial\Omega) := \dim H^{\ell}_{\mathrm{sing}}(\partial\Omega;\mathbb{R})$, the ℓ-th Betti number of $\partial\Omega$.

Proof. We shall make use of a deep theorem of De Rham which we present below in an abstract form, well suited for our purposes.

De Rahm's Theorem. *Let X be a Hausdorff, para-compact topological space, and let $\mathcal{L}^0, \mathcal{L}^1, \dots$ be fine sheaves over X and, for $\ell = 0, 1, \dots,$ let $\vartheta_\ell : \mathcal{L}^\ell \to \mathcal{L}^{\ell+1}$ be sheaf homomorphisms such that the following is an exact complex:*

$$(6.11) \qquad 0 \longrightarrow \mathrm{LCF}_X \overset{\iota}{\hookrightarrow} \mathcal{L}^0 \xrightarrow{\vartheta_0} \mathcal{L}^1 \xrightarrow{\vartheta_1} \mathcal{L}^2 \xrightarrow{\vartheta_2} \cdots$$

(*hereafter, ι denotes inclusion*). *Then*

$$(6.12) \qquad H^\ell_{\rm sing}(X;\mathbb{R}) \cong \frac{\mathrm{Ker}(\vartheta_\ell : \mathcal{L}^\ell(X) \longrightarrow \mathcal{L}^{\ell+1}(X))}{\mathrm{Im}(\vartheta_{\ell-1} : \mathcal{L}^{\ell-1}(X) \longrightarrow \mathcal{L}^\ell(X))}, \quad \ell = 1, 2, \dots$$

See [41, Theorem 5.25], p. 185 for a proof; cf. also [10].

Next, we shall describe a setting in which this powerful machinery applies. To set the stage, we first need to define a local version of the space (5.2) (cf. also (5.4)) as well as of the operator d_∂. More specifically, for U an arbitrary open subset of $\partial\Omega$, we define $NB^{p,p}_{s-1/p}(U, \Lambda^\ell; \mathrm{loc})$ as the space consisting of functionals $f \in (\mathrm{Lip}(\partial\Omega, \Lambda^\ell))'$ enjoying the following properties.

(i) for each $\varphi \in \mathrm{Lip}(\partial\Omega)$ with $\mathrm{supp}\,\varphi \subset U$, it follows that $\varphi f \in B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell)$;

(ii) for each $x \in U$, there exist W, open neighborhood of x in M with $W \cap \partial\Omega \subset U$ and $u \in D_{\ell-1}(d; L^p_s(\Omega))$ such that $f|_{W\cap\partial\Omega} = (\nu \wedge u)|_{W\cap\partial\Omega}$.

Also, introduce

$$(6.13) \qquad d_\partial : NB^{p,p}_{s-1/p}(U, \Lambda^\ell; \mathrm{loc}) \longrightarrow NB^{p,p}_{s-1/p}(U, \Lambda^{\ell+1}; \mathrm{loc})$$

by setting $d_\partial f := -\nu \wedge du$ near x, if f is locally given by $\nu \wedge u$ near x. Clearly, each $NB^{p,p}_{s-1/p}(U, \Lambda^\ell; \mathrm{loc})$ is an additive Abelian group and also a module over the algebra $\mathrm{Lip}(\partial\Omega)$. It follows that the family $N^\ell B^{p,p}_{s-1/p} := \{NB^{p,p}_{s-1/p}(U, \Lambda^\ell; \mathrm{loc})\}_U$ indexed by open subsets in $\partial\Omega$, is a fine sheaf on the topological space $\partial\Omega$.

Going further, we observe that the operator d_∂ induces a natural sequence of sheaf morphisms

$$(6.14) \quad 0 \longrightarrow \mathrm{LCF} \xrightarrow{\iota} N^1 B^{p,p}_{s-1/p} \xrightarrow{d_\partial} N^2 B^{p,p}_{s-1/p} \xrightarrow{d_\partial} N^3 B^{p,p}_{s-1/p} \xrightarrow{d_\partial} \cdots$$

where LCF stands for the sheaf of germs of locally constant functions on $\partial\Omega$ and the embedding works according to $f \mapsto f\nu$. Since $d_\partial \circ d_\partial = 0$, the above is a complex. In fact, so we claim, (6.14) provides a fine resolution of the sheaf LCF. The essential ingredient in the proof of this claim is the *acyclicity* of the complex (6.14). Granted this, the so-called abstract De Rham theorem applies to our context and gives (6.9). With (6.9) in hand, all the claims made in the statement of the theorem follow easily.

Next, we aim to prove the acyclicity of the sheaf (6.14). It is not hard to see that this is equivalent to proving that, for each $1 \le \ell \le n$, the following claim is true:

$$\text{(6.15)} \quad \forall\, x_0 \in \partial\Omega \text{ and } \forall\, u \in D_\ell(d; L^p_s(\Omega)) \text{ with } \nu \wedge du = 0 \text{ near } x_0,$$
$$\exists\, v \in D_{\ell-1}(d; L^p_s(\Omega)) \text{ such that } \nu \wedge u = \nu \wedge dv \text{ near } x_0.$$

Since the case $\ell = n$ is trivial, below we shall focus on the proof of this claim when $1 \le \ell \le n-1$. For starters, fix $u \in D_\ell(d; L^p_s(\Omega))$ such that $\nu \wedge du = 0$ near x_0 and let $\mathcal{O} \subset M$ be an open neighborhood of x_0 with the following properties:

(i) $\mathcal{O}$ is a Lipschitz domain contained in a coordinate patch and such that, when viewed in local, Euclidean coordinates, $\mathcal{O}$ is star-like with respect to some ball $B \subset \mathcal{O} \setminus \bar{\Omega}$;
(ii) $\mathcal{O} \cap \Omega$ is a Lipschitz domain for which $b_\ell(\mathcal{O} \cap \Omega) = 0$, $\ell = 1, 2, \dots, n$;
(iii) $\nu \wedge du = 0$ on $\mathcal{O} \cap \partial\Omega$.

In this context, it has been shown in Section 4 of [27], that there exists a family of linear operators

$$\text{(6.16)} \qquad K_\ell : (C^\infty_c(\mathcal{O}, \Lambda^\ell))' \longrightarrow (C^\infty_c(\mathcal{O}, \Lambda^{\ell-1}))', \quad 1 \le \ell \le n,$$

such that (for some $\theta \in C^\infty_c(\mathcal{O} \setminus \bar{\Omega})$),

$$\text{(6.17)} \qquad \text{the homotopy formula (5.40) holds } \forall\, w \in (C^\infty_c(\mathcal{O}, \Lambda^\ell))',$$
$$\text{(6.18)} \qquad K_\ell : L^p_s(\mathcal{O}, \Lambda^\ell) \longrightarrow L^p_{s+1}(\mathcal{O}, \Lambda^{\ell-1}) \quad \text{boundedly},$$
$$\text{(6.19)} \qquad \operatorname{supp}(K_\ell w) \subset \mathcal{O} \cap \bar{\Omega} \quad \text{whenever } \operatorname{supp} w \subset \mathcal{O} \cap \bar{\Omega}.$$

From (iii) above, it follows that $\widetilde{du|_{\mathcal{O}\cap\Omega}} \in L^p_s(\mathcal{O}, \Lambda^{\ell+1})$ and $d(\widetilde{du|_{\mathcal{O}\cap\Omega}}) = 0$ in $\mathcal{O}$, where 'tilde' denotes the extension by zero of forms in $\mathcal{O} \cap \Omega$ to $\mathcal{O}$. Thus,

$$\text{(6.20)} \qquad \begin{aligned} &w := K_{\ell+1}(\widetilde{du|_{\mathcal{O}\cap\Omega}}) \in L^p_{s+1}(\mathcal{O}, \Lambda^\ell), \\ &dw = \widetilde{du|_{\mathcal{O}\cap\Omega}} \text{ in } \mathcal{O}, \quad \text{and } \operatorname{supp} w \subset \mathcal{O} \cap \bar{\Omega}. \end{aligned}$$

Since the trace of w on $\mathcal{O} \cap \partial\Omega$ vanishes, the above properties imply

$$\text{(6.21)} \qquad d(u - w) = 0 \text{ in } \mathcal{O} \cap \Omega \quad \text{and} \quad \nu \wedge w = 0 \text{ on } \mathcal{O} \cap \partial\Omega.$$

Next, bring in the isomorphism

$$\text{(6.22)} \qquad \frac{\{\omega \in D_\ell(d; L^p_s(U)) : d\omega = 0\}}{\{d\eta : \eta \in D_{\ell-1}(d; L^p_s(U))\}} \simeq H^\ell_{\text{sing}}(U; \mathbb{R}), \quad 1 \le \ell \le n,$$

proved in [27] for any Lipschitz subdomain U of M. Thanks to property (ii), it follows from (6.22) with $U := \mathcal{O} \cap \Omega$ that there exists $\eta \in D_{\ell-1}(d; L^p_s(\mathcal{O} \cap \Omega))$ such that $u - w = d\eta$ in $\mathcal{O} \cap \Omega$. In particular, on $\mathcal{O} \cap \partial\Omega$, we have $\nu \wedge d\eta = \nu \wedge (u - w) = \nu \wedge u$. Hence, if $\varphi \in C^\infty(M)$ is a scalar function such that $\operatorname{supp} \varphi \subset \mathcal{O}$ and $\varphi \equiv 1$ near x_0, then $v := \varphi\eta$ (viewed as a form in Ω) does the job advertised in (6.15). This finishes the proof of the theorem. □

We continue to assume that $1 < p < \infty$ and $-1 + 1/p < s < 1/p$. Upon recalling the space (5.45), we define the operator

$$\delta_\partial : TB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell) \longrightarrow TB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^{\ell-1}) \tag{6.23}$$

by setting

$$\delta_\partial \xi := \eta, \tag{6.24}$$

whenever ξ and η are as in (5.45). Much as before, this is well defined, linear and bounded. Let us also note here that, based on (6.4) and Hodge duality,

$$\delta_\partial(\nu \vee u) = -\nu \vee \delta u, \quad \forall\, u \in D_\ell(\delta; L^p_s(\Omega)). \tag{6.25}$$

The operators d_∂, δ_∂, are further related via the following duality result.

Proposition 6.3. *For each $f \in TB^{p',p'}_{s-1/p'}(\partial\Omega, \Lambda^\ell)$ and $g \in NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell)$, there holds*

$$\langle \delta_\partial f, \nu \vee g \rangle = -\langle \nu \wedge f, d_\partial g \rangle. \tag{6.26}$$

Proof.
By Theorem 5.2, we can choose $u \in D_{\ell+1}(\delta; L^{p'}_{-s}(\Omega))$, $w \in D_{\ell-1}(d; L^p_s(\Omega))$ such that $f = \nu \vee u$, $g = \nu \wedge w$. Also, from (6.25), (6.4) we have $\delta_\partial f = -\nu \vee \delta u$ and $d_\partial g = -\nu \wedge dw$. Thus, Corollary 5.5 gives

$$\langle \delta_\partial f, \nu \vee g \rangle = -\langle \nu \vee \delta u, \nu \vee (\nu \wedge w) \rangle = -\langle \delta u, dw \rangle. \tag{6.27}$$

In a similar fashion one can prove that $\langle \nu \wedge f, d_\partial g \rangle = \langle \delta u, dw \rangle$, so that (6.26) follows from this and (6.27). □

Define

$$T_0B^{p,p}_s(\partial\Omega, \Lambda^\ell) := \{\xi \in TB^{p,p}_s(\partial\Omega, \Lambda^\ell) : \delta_\partial \xi = 0\}. \tag{6.28}$$

Corollary 6.4. *The operator*

$$\delta_\partial : TB^{p,p}_s(\partial\Omega, \Lambda^\ell) \longrightarrow T_0B^{p,p}_s(\partial\Omega, \Lambda^{\ell-1}) \tag{6.29}$$

has closed range and its cokernel is isomorphic to $H^{n-\ell}_{\text{sing}}(\partial\Omega;\mathbb{R})$, *i.e.,*

$$\frac{T_0 B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell-1})}{\delta_\partial[TB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell})]} \cong H^{n-\ell}_{\text{sing}}(\partial\Omega;\mathbb{R}), \quad \ell = 1,2,\dots,n. \tag{6.30}$$

In particular,

$$\delta_\partial : \frac{TB^{p,p}_{s}(\partial\Omega,\Lambda^{\ell})}{T_0 B^{p,p}_{s}(\partial\Omega,\Lambda^{\ell})} \longrightarrow T_0 B^{p,p}_{s}(\partial\Omega,\Lambda^{\ell-1}) \tag{6.31}$$

is a Fredholm operator with index $b_{n-\ell}(\partial\Omega)$.

Proof. This is a simple consequence of Theorem 6.2 and Hodge duality. □

Corollary 6.5. *If* $b_{n-\ell}(\Omega) = 0$, *then the operator*

$$\nu \wedge \cdot : \{w \in L^p_s(\Omega,\Lambda^\ell) : dw = 0\} \longrightarrow N_0 B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell+1}) \tag{6.32}$$

is onto. Likewise, if $b_\ell(\Omega) = 0$, *then the operator*

$$\nu \vee \cdot : \{w \in L^p_s(\Omega,\Lambda^\ell) : \delta w = 0\} \longrightarrow T_0 B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell-1}) \tag{6.33}$$

is onto.

Proof. If $\xi \in N_0 B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell+1})$ is a given, arbitrary form, then Theorem 5.2 ensures that there exists $u \in D_\ell(d;L^p_s(\Omega))$ such that $\xi = \nu\wedge u$ and for which $\nu \wedge du = 0$. In order to continue, we now recall a general formula proved in [27], to the effect that

$$\dim\left[\frac{\{\eta \in D_\ell(d;L^p_s(\Omega)) : d\eta = 0 \text{ and } \nu\wedge\eta = 0\}}{\{d\omega : \omega \in D_{\ell-1}(d;L^p_s(\Omega)) \text{ and } \nu\wedge\omega = 0\}}\right] = b_{n-\ell}(\Omega), \tag{6.34}$$

for each $\ell = 0,\dots,n-1$. In the present context, this guarantees that there exists some $\omega \in D_{\ell-1}(d;L^p_s(\Omega))$ satisfying $\nu\wedge\omega = 0$ and such that $du = d\omega$. Then $\xi = \nu\wedge(u-\omega)$ and $u-\omega \in L^p_s(\Omega,\Lambda^\ell)$ has $d(u-\omega) = 0$. Hence, the map (6.32) is onto.

Finally, the claim about (6.33) is a consequence of this and Hodge duality. □

Corollary 6.6. *The space* $NB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^\ell)$ *is the completion of the space of forms* $\nu\wedge C^\infty(M,\Lambda^\ell)|_{\partial\Omega}$ *in the norm*

$$f \mapsto \|f\|_{B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^\ell)} + \|d_\partial f\|_{B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell+1})}.$$

In particular, the space $NB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^\ell)$ *is reflexive.*

Finally, similar results hold for the space $TB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^\ell)$.

Proof. It has been proved in [27] that $C^\infty(M,\Lambda^\ell)$ is a dense subspace of $D_\ell(d;L^p_s(\Omega))$. Granted this, the claim in the first part of the statement of the corollary is a direct consequence of Theorem 5.2. The last part of the corollary then follows from this and Hodge duality. $\square$

7. Extending Differential Forms from Ω to M

Throughout this section we shall assume that $1 < p < \infty$ and $-1 + 1/p < s < 1/p$. Given a Lipschitz domain $\Omega \subset M$, we wish to study the restriction operator

$$D_\ell(d;L^p_s(M)) \ni u \mapsto u|_\Omega \in D_\ell(d;L^p_s(\Omega)). \tag{7.1}$$

In the above context, this is clearly well defined, linear and bounded, and the issue arises whether this is onto as well. In this regard, we have the following.

Theorem 7.1. *There exists a bounded, linear operator*

$$E : D_\ell(d;L^p_s(\Omega)) \longrightarrow D_\ell(d;L^p_s(M)) \tag{7.2}$$

such that $(Eu)|_\Omega = u$ *for each* $u \in D_\ell(d;L^p_s(\Omega))$. *In particular, the operator* (7.1) *is onto.*

Proof. Set $\Omega_+ := \Omega$, $\Omega_- := M \setminus \bar{\Omega}$, and let

$$\mathrm{Ex}^+ : NB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell+1}) \longrightarrow D_\ell(d;L^p_s(\Omega_+)), \tag{7.3}$$

$$\mathrm{Ex}^- : NB^{p,p}_{s-1/p}(\partial\Omega,\Lambda^{\ell+1}) \longrightarrow D_\ell(d;L^p_s(\Omega_-)), \tag{7.4}$$

be the extension operators constructed in Theorem 5.2, corresponding to $\Omega_\pm$. Then, given an arbitrary differential form $u^+ \in D_\ell(d;L^p_s(\Omega))$, set

$$u := \widetilde{u^+} + \widetilde{u^-}, \tag{7.5}$$

where $u^- := \mathrm{Ex}^-(\nu \wedge u^+)$ and tilde denotes the extension by zero outside Ω. It follows that

$$u \in L^p_s(M,\Lambda^\ell), \quad u|_\Omega = u^+. \tag{7.6}$$

The proof is therefore completed as soon as we prove that $du \in L^p_s(M,\Lambda^{\ell+1})$. To this end, we shall actually show that, in fact,

$$du = \widetilde{du^+} + \widetilde{du^-} \in L^p_s(M,\Lambda^{\ell+1}). \tag{7.7}$$

In order to justify this, pick an arbitrary $\Phi \in C^\infty(M, \Lambda^\ell)$ and write

$$
\begin{aligned}
(7.8)\quad \langle du, \Phi\rangle &= \langle u, \delta\Phi\rangle = \langle \widetilde{u^+} + \widetilde{u^-}, \delta\Phi\rangle \\
&= \langle \widetilde{u^+}, \delta\Phi\rangle + \langle \widetilde{u^-}, \delta\Phi\rangle = \langle u^+, (\delta\Phi)|_{\Omega_+}\rangle + \langle u^-, (\delta\Phi)|_{\Omega_-}\rangle \\
&= \langle u^+, \delta(\Phi|_{\Omega_+})\rangle + \langle u^-, \delta(\Phi|_{\Omega_-})\rangle \\
&= \langle du^+, \Phi|_{\Omega_+}\rangle + \langle \nu\wedge u^+, \operatorname{Tr}\Phi\rangle + \langle du^-, \Phi|_{\Omega_-}\rangle - \langle \nu\wedge u^-, \operatorname{Tr}\Phi\rangle \\
&= \langle du^+, \Phi|_{\Omega_+}\rangle + \langle du^-, \Phi|_{\Omega_-}\rangle \\
&= \langle \widetilde{du^+}, \Phi\rangle + \langle \widetilde{du^-}, \Phi\rangle = \langle \widetilde{du^+} + \widetilde{du^-}, \Phi\rangle,
\end{aligned}
$$

where we have used the fact that, by design, $\nu \wedge u^+ = \nu \wedge u^-$. This justifies (7.7) and finishes the proof of the theorem. □

A similar result holds for spaces defined in connection with the operator δ. More specifically, we have the following result.

Corollary 7.2. *The restriction operator*

$$(7.9)\qquad D_\ell(\delta; L^p_s(M)) \ni u \mapsto u\big|_\Omega \in D_\ell(\delta; L^p_s(\Omega))$$

is well defined, linear, bounded and onto. In fact, there exists a bounded, linear operator

$$(7.10)\qquad E' : D_\ell(\delta; L^p_s(\Omega)) \longrightarrow D_\ell(\delta; L^p_s(M))$$

such that $(E'u)|_\Omega = u$ *for each* $u \in D_\ell(\delta; L^p_s(\Omega))$.

Proof. This is an immediate consequence of Theorem 7.1 and the properties of the Hodge star-isomorphism; cf. Proposition 2.1. □

Our last result in this section is a version of Theorem 7.1 for closed forms.

Theorem 7.3. *Given a Lipschitz domain* $\Omega \subset M$ *and a differential form* $u \in D_\ell(d; L^p_s(\Omega))$ *with* $du = 0$ *in* Ω, *there exist an open neighborhood* $\mathcal{O}$ *of* $\bar\Omega$ *and some* $v \in D_\ell(d; L^p_s(\mathcal{O}))$ *such that* $dv = 0$ *in* $\mathcal{O}$ *and* $v|_\Omega = u$.

Proof. Fix a form u as in the statement of the theorem and note that the manifold M can be altered away from $\bar\Omega$ as to ensure the existence of a Lipschitz domain $\mathcal{O}$ for which

$$(7.11)\qquad \bar\Omega \subset \mathcal{O}, \quad b_{n-\ell-1}(\mathcal{O} \setminus \bar\Omega) = 0.$$

Thanks to Theorem 7.1, there exists $w \in D_\ell(d; L^p_s(M))$ such that $w|_\Omega = u$. Multiplying w by a function $\psi \in C^\infty_c(\mathcal{O})$ such that $\psi \equiv 1$ near $\bar\Omega$, there is no loss of generality in assuming that $w \equiv 0$ near $\partial\mathcal{O}$. In particular, $\nu \wedge d(w|_{\mathcal{O}}) = 0$ on $\partial\mathcal{O}$. Since we also have

$$[\nu \wedge (dw|_{\mathcal{O}\setminus\bar{\Omega}})]|_{\partial\Omega} = \nu \wedge (dw|_{\Omega}) = \nu \wedge du = 0, \tag{7.12}$$

it ultimately follows that $\nu \wedge d(w|_{\mathcal{O}\setminus\bar{\Omega}}) = 0$.

Granted the current assumptions on $\mathcal{O}$, formula (6.34) written with $\mathcal{O}\setminus\bar{\Omega}$ in place of Ω guarantees the existence of a differential form $\omega \in D_\ell(d; L^p_s(\mathcal{O}\setminus\bar{\Omega}))$ such that

$$d\omega = dw \text{ in } \mathcal{O}\setminus\bar{\Omega} \quad \text{and} \quad \nu \wedge \omega = 0 \text{ on } \partial(\mathcal{O}\setminus\bar{\Omega}). \tag{7.13}$$

Thus, by the nature of our construction,

$$[\nu \wedge (w - \omega)]|_{\partial\Omega} = (\nu \wedge w)|_{\partial\Omega} = \nu \wedge (w|_{\Omega}) = \nu \wedge u. \tag{7.14}$$

Consequently, if $\tilde{u}$ is the extension of u by zero to $\mathcal{O}$ and $(\widetilde{w - \omega})|_{\mathcal{O}\setminus\bar{\Omega}}$ is the extension of $(w - \omega)|_{\mathcal{O}\setminus\bar{\Omega}}$ by zero to $\mathcal{O}$, then

$$v := \tilde{u} + (\widetilde{w - \omega})|_{\mathcal{O}\setminus\bar{\Omega}} \in L^p_s(\mathcal{O}, \Lambda^\ell) \tag{7.15}$$

satisfies $v|_{\Omega} = u$. Furthermore, thanks to (7.14), much as for the justification of (7.7), one can show that

$$dv := \widetilde{du} + (\widetilde{dw - d\omega})|_{\mathcal{O}\setminus\bar{\Omega}} = 0 \quad \text{in } \mathcal{O}. \tag{7.16}$$

Thus, v does the job advertised in the statement of the theorem. $\square$

Of course, there is a natural Hodge dual version of Theorem 7.3; we omit the details. In closing, we only wish to point out that this result extends the work of T. Kato, M. Mitrea, G. Ponce and M. Taylor in [16] where the authors have dealt with the case $M = \mathbb{R}^n$, $\ell = 1$, $s = 0$, via a conceptually different proof which requires the domain Ω to be suitably smooth if $s \neq 0$.

8. Hodge Decompositions

We first discuss a key preliminary result, which can be viewed as the analogue at the level of differential forms of the well-known fact that, for an arbitrary Lipschitz domain Ω, the scalar Besov scale $B^{p,p}_s(\partial\Omega)$ is stable under complex interpolation.

Theorem 8.1. *If* $1 < p < \infty$, $-1 + 1/p < s < 1/p$, $1 < p_j < \infty$, $-1 + 1/p_j < s_j < 1/p_j$, $j = 0, 1$, *and* $\theta \in (0, 1)$ *is such that* $1/p = (1-\theta)/p_0 + \theta/p_1$, $s = (1-\theta)s_0 + \theta s_1$, *then*

$$[NB^{p_0,p_0}_{s_0-1/p_0}(\partial\Omega, \Lambda^\ell), NB^{p_1,p_1}_{s_1-1/p_1}(\partial\Omega, \Lambda^\ell)]_\theta = NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell), \tag{8.1}$$

$$[TB^{p_0,p_0}_{s_0-1/p_0}(\partial\Omega, \Lambda^\ell), TB^{p_1,p_1}_{s_1-1/p_1}(\partial\Omega, \Lambda^\ell)]_\theta = TB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell), \tag{8.2}$$

for each $0 \le \ell \le n$.

Proof. By Hodge duality, it suffices to establish (8.1) only. In turn, the proof of this identity is divided into three steps.

Step I. Let X_j, Y_j, Z_j, $i = 0, 1$, be Banach spaces such that $X_0 \cap X_1$ is dense in both X_0 and X_1, and similarly for Z_0, Z_1. Suppose that $Y_j \hookrightarrow Z_j$, $j = 0$, 1 and there exists a linear operator D such that $D : X_j \to Z_j$ boundedly for $i = 0$, 1. Define the spaces

$$X_j(D) := \{u \in X_j : Du \in Y_j\}, \quad j = 0, 1, \tag{8.3}$$

equipped with the graph norm, i.e., $\|u\|_{X_j(D)} := \|u\|_{X_j} + \|Du\|_{Y_j}$, $i = 0, 1$. Finally, suppose that there exist continuous linear mappings $K : Z_j \to X_j$ and $R : Z_j \to Y_j$ with the property $D \circ K = I + R$ on the spaces Z_j for $j = 0, 1$. Then

$$[X_0(D), X_1(D)]_\theta = \{u \in [X_0, X_1]_\theta : Du \in [Y_0, Y_1]_\theta\}, \quad \theta \in (0, 1). \tag{8.4}$$

This is due to J.-L. Lions and E. Magenes [18, Theorem 14.3 on page 97]; cf. also [12].

Step II. If p, p_0, p_1, s, s_0, s_1, $\theta \in (0, 1)$ are as in the statement of the theorem, then

$$[D_\ell(d; L^{p_0}_{s_0}(\Omega)), D_\ell(d; L^{p_1}_{s_1}(\Omega))]_\theta = D_\ell(d; L^p_s(\Omega)), \tag{8.5}$$

for each $0 \le \ell \le n$.

The problem localizes, so there is no loss of generality in assuming that Ω is contained in a small coordinate patch such that, when viewed in local, Euclidean coordinates, Ω is star-like with respect to some ball. In this context, we shall implement the abstract interpolation result from Step I twice. First, the goal is to show that

$$\begin{aligned}[]
[\{u \in L^{p_0}_{s_0}(\Omega, \Lambda^\ell) : du = 0\}, \{u \in L^{p_1}_{s_1}(\Omega, \Lambda^\ell) : du = 0\}]_\theta \\
= \{u \in L^p_s(\Omega, \Lambda^\ell) : du = 0\}, \quad \forall\, \ell \in \{0, 1, \dots, n\},
\end{aligned} \tag{8.6}$$

so we find it convenient to take

$$\begin{aligned}
&X_j := L^{p_j}_{s_j}(\Omega, \Lambda^\ell), \quad Y_j := 0, && j = 0, 1, \\
&Z_j := \{u \in L^{p_j}_{s_j - 1}(\Omega, \Lambda^{\ell+1}) : du = 0\}, && j = 0, 1, \\
&D := d, \quad R := 0,
\end{aligned}$$

and $K := K_\ell$, the operator constructed in Section 4 of [27] where it has been shown that, if $1 \le \ell \le n - 1$,

$$K_\ell : L^p_{s-1}(\Omega, \Lambda^\ell) \longrightarrow L^p_s(\Omega, \Lambda^{\ell-1}) \quad \text{boundedly,} \tag{8.7}$$

and

$$\text{(8.8)} \quad u = dK_\ell u, \quad \forall\, u \in L^p_{s-1}(\Omega, \Lambda^\ell) \text{ with } du = 0 \text{ in } \Omega,$$
$$\text{whenever } 1 < p < \infty \text{ and } s < 1 + 1/p.$$

Since $X_j(D) = \{u \in L^{p_j}_{s_j}(\Omega, \Lambda^\ell) : du = 0\}$, $j = 0, 1$, the abstract result in Step I applies and yields (8.6), thanks to Proposition 3.2.

Our second implementation of the abstract interpolation result from Step I is when

$$\begin{aligned}
&X_j := L^{p_j}_{s_j}(\Omega, \Lambda^\ell), && j = 0, 1,\\
&Y_j := \{u \in L^{p_j}_{s_j}(\Omega, \Lambda^{\ell+1}) : du = 0\}, && j = 0, 1,\\
&Z_j := \{u \in L^{p_j}_{s_j-1}(\Omega, \Lambda^{\ell+1}) : du = 0\}, && j = 0, 1,\\
&D := d,\ R := 0, \text{ and } K := K_\ell, && \text{as in (8.7)–(8.8).}
\end{aligned}$$

Thanks to (8.6), this time $X_j(D) = D_\ell(d; L^{p_j}_{s_j}(\Omega))$, $j = 0, 1$, and an application of Step I yields (8.5).

Step III. Proof of (8.1). This identity now becomes an immediate consequence of Step II, Theorem 5.2 and properties of retractions. ☐

The case $s_0 = s_1 = s = 0$ of Theorem 8.1 has been first established in [24] via an approach which requires flattening the boundary. As explained in the introduction, this method is confined to the L^p scale, i.e., it does not allow the consideration of forms with coefficients in L^p_s when the smoothness index satisfies $s \neq 0$.

Nonetheless, once Theorem 8.1 has been established, the rest of the approach in [24], originally developed for forms with coefficients in L^p, can be adapted to the case when the coefficients are from L^p_s, at least if p is near 2 and s is sufficiently close to zero. Briefly, the genesis of the range $2 - \varepsilon < p < 2 + \varepsilon$, $-\varepsilon < s < \varepsilon$ is as follows. The crux of the approach developed in [24] is deriving Hodge decompositions as corollary of the solvability of certain Poisson-type problems for the Hodge Laplacian $\Delta = -d\delta - \delta d$ in Ω. In turn, these boundary value problems are reduced to the invertibility of a certain layer potential integral operator on the scale $NB^{p,p}_{s-1/p}(\partial\Omega, \Lambda^\ell)$. The case $s = 0$, $p = 2$ is special, as it naturally lends itself to Hilbert space methods. Once the invertibility has been established in this particular situation, perturbation methods yield a similar result for $|p - 2| < \varepsilon$, $|s| < \varepsilon$. Here, (8.1) is of paramount importance; see [15], [33].

In particular, all the main results from [24] have an analogue in this more general setting. For example, we have the following Hodge decomposition result for differential forms with coefficients in L^p_s:

Theorem 8.2. *Let Ω be an arbitrary Lipschitz subdomain of M. Then there exists some $\varepsilon = \varepsilon(\Omega) > 0$ with the following significance. For any $\ell \in \{0, 1, \dots, n\}$, the space*

$$\mathcal{H}_\wedge(\Omega, \Lambda^\ell) := \left\{u \in L^p_s(\Omega, \Lambda^\ell) : du = 0,\ \delta u = 0,\ \nu \wedge u = 0\right\} \tag{8.9}$$

is independent of $p \in (2 - \varepsilon, 2 + \varepsilon)$ and $s \in (-\varepsilon, \varepsilon)$. Furthermore, for such p and s, the dimension of (8.9) is $b_{n-\ell}(\Omega)$ and, if

$$\begin{aligned} \mathcal{H}_\vee(\Omega, \Lambda^\ell) &:= *\mathcal{H}_\wedge(\Omega, \Lambda^{n-\ell}) \\ &= \left\{u \in L^p_s(\Omega, \Lambda^\ell) : du = 0,\ \delta u = 0,\ \nu \vee u = 0\right\}, \end{aligned} \tag{8.10}$$

then

$$\begin{aligned} L^p_s(\Omega, \Lambda^\ell) &= \left\{du : u \in D_{\ell-1}(d; L^p_s(\Omega)),\ \nu \wedge u = 0\right\} \\ &\quad \oplus \left\{\delta w : w \in D_{\ell+1}(\delta; L^p_s(\Omega))\right\} \oplus \mathcal{H}_\wedge(\Omega, \Lambda^\ell), \end{aligned} \tag{8.11}$$

$$\begin{aligned} &= \left\{\delta u : u \in D_{\ell+1}(\delta; L^p_s(\Omega)),\ \nu \vee u = 0\right\} \\ &\quad \oplus \left\{dw : w \in D_{\ell-1}(d; L^p_s(\Omega))\right\} \oplus \mathcal{H}_\vee(\Omega, \Lambda^\ell), \end{aligned} \tag{8.12}$$

where the direct sums are topological.

We leave to the interested reader the formulation of other related results from [24] in the more general setting considered in this paper. Here we only want to point out that, when $n = 3$, the *sharp* range of indices p, s for which (8.11)–(8.12) hold has been identified in [25]. In the context of (8.11)–(8.12), what the optimal range is for $n \geq 4$ remains an open problem at the moment.

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