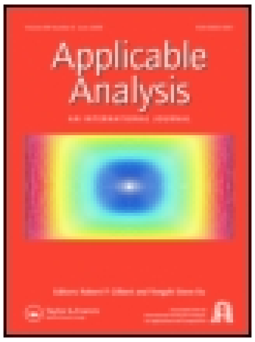

Applicable Analysis
An International Journal

ISSN: 0003-6811 (Print) 1563-504X (Online) Journal homepage: http://www.tandfonline.com/loi/gapa20

The divDiv-complex and applications to biharmonic equations

Dirk Pauly & Walter Zulehner

To cite this article: Dirk Pauly & Walter Zulehner (2018): The divDiv-complex and applications to biharmonic equations, Applicable Analysis, DOI: 10.1080/00036811.2018.1542685

To link to this article: https://doi.org/10.1080/00036811.2018.1542685

Published online: 30 Nov 2018.

APPLICABLE ANALYSIS
https://doi.org/10.1080/00036811.2018.1542685

The divDiv-complex and applications to biharmonic equations

Dirk Pauly[a] and Walter Zulehner[b]

[a]Fakultät für Mathematik, Universität Duisburg-Essen, Essen, Germany; [b]Institut für Numerische Mathematik, Johannes Kepler Universität Linz, Linz, Austria

ABSTRACT

It is shown that the first biharmonic boundary value problem on a topologically trivial domain in 3D is equivalent to three (consecutively to solve) second-order problems. This decomposition result is based on a Helmholtz-like decomposition of an involved non-standard Sobolev space of tensor fields and a proper characterization of the operator divDiv acting on this space. Similar results for biharmonic problems in 2D and their impact on the construction and analysis of finite element methods have been recently published in Krendl et al. [A decomposition result for biharmonic problems and the Hellan–Herrmann–Johnson method. Electron Trans Numer Anal. 2016;45:257–282]. The discussion of the kernel of divDiv leads to (de Rham-like) closed and exact Hilbert complexes, the divDiv-complex and its adjoint the Gradgrad-complex, involving spaces of trace-free and symmetric tensor fields. For these tensor fields, we show Helmholtz type decompositions and, most importantly, new compact embedding results. Almost all our results hold and are formulated for general bounded strong Lipschitz domains of arbitrary topology. There is no reasonable doubt that our results extend to strong Lipschitz domains in $\mathbb{R}^N$.

ARTICLE HISTORY
Received 19 June 2017
Accepted 26 October 2018

COMMUNICATED BY
J. Zou

KEYWORDS
Biharmonic equations; Helmholtz decomposition; Hilbert complexes

1991 MATHEMATICS SUBJECT CLASSIFICATIONS
35G15; 58A14

1. Introduction

In [1] it was shown that the fourth-order biharmonic boundary value problem

$$\Delta^2 u = f \quad \text{in } \Omega, \qquad u = \partial_n u = 0 \quad \text{on } \Gamma, \tag{1}$$

where Ω is a bounded and simply connected domain in $\mathbb{R}^2$ with a (strong) Lipschitz boundary[1] Γ, f is a given right-hand side, Δ and ∂_n denote the Laplace operator and the derivative in direction of the outward normal vector $\mathbf{n}$, respectively, can be decomposed into three second-order problems. The first problem is a Dirichlet–Poisson problem for an auxiliary scalar field p

$$\Delta p = f \quad \text{in } \Omega, \qquad p = 0 \quad \text{on } \Gamma,$$

the second problem is a linear elasticity Neumann problem for an auxiliary vector field $\mathbf{v}$

$$\operatorname{Div}(\operatorname{symGrad} \mathbf{v}) = -\operatorname{grad} p \quad \text{in } \Omega, \qquad (\operatorname{symGrad} \mathbf{v})\, \mathbf{n} = -p\, \mathbf{n} = 0 \quad \text{on } \Gamma,$$

and, finally, the third problem is again a Dirichlet–Poisson problem for the original scalar field u

$$\Delta u = 2p + \operatorname{div} \mathbf{v} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma.$$

CONTACT Dirk Pauly dirk.pauly@uni-due.de

Here the differential operators grad, Grad, div, and Div denote the gradient of a scalar field, the gradient of a vector field, the divergence of a vector field, and the divergence of a tensor field, respectively. The prefix sym is used for the symmetric part of a matrix.

This decomposition is of triangular structure, i.e. the first problem is a well-posed second-order problem in p, the second problem is a well-posed second-order problem in $\mathbf{v}$ for given p, and the third problem is a well-posed second-order problem in u for given p and $\mathbf{v}$. This allows to solve them consecutively analytically or numerically by means of techniques for second-order problems.

This is – in the first place – a new analytic result for fourth-order problems. But it also has interesting implications for discretization methods applied to (1). It allows to re-interpret known finite element methods as well as to construct new discretization methods for (1) by exploiting the decomposable structure of the problem. In particular, it was shown in [1] that the Hellan–Herrmann–Johnson mixed method (see [2–4]) for (1) allows a similar decomposition as the continuous problem, which leads to a new and simpler assembling procedure for the discretization matrix and to more efficient solution techniques for the discretized problem. Moreover, a novel conforming variant of the Hellan–Herrmann–Johnson mixed method was found based on the decomposition.

We will see that the situation in $\mathbb{R}^3$ is much more complicated. The main application of this paper is to derive a similar decomposition result for biharmonic problems (1) on bounded and topologically trivial three-dimensional domains $\Omega \subset \mathbb{R}^3$ with a (strong) Lipschitz boundary Γ. For this we proceed as in [1] and reformulate (1) using

$$\Delta^2 = \operatorname{divDiv} \operatorname{Gradgrad}$$

as a mixed problem by introducing the (negative) Hessian of the original scalar field u as an auxiliary tensor field

$$\mathbf{M} = -\operatorname{Gradgrad} u. \tag{2}$$

Then the biharmonic differential equation reads

$$-\operatorname{divDiv} \mathbf{M} = f \quad \text{in } \Omega. \tag{3}$$

For an appropriate non-standard Sobolev space for $\mathbf{M}$ it can be shown that the mixed problem in $\mathbf{M}$ and u is well-posed, see (31)–(32). Then the decomposition of the biharmonic problem follows from a regular decomposition of this non-standard Sobolev space, see Lemma 3.21. This part of the analysis carries over completely from the two-dimensional case to the three-dimensional case and is recalled in Section 4. To efficiently utilize this regular decomposition for the decomposition of the biharmonic problem an appropriate characterization of the kernel of the operator divDiv is required, which is well understood for the two-dimensional case, see, e.g. [1, 5, 6]. Its extension to the three-dimensional case is one of the central topics of this paper. We expect – as in the two-dimensional case – similar interesting implications for the study of appropriate discretization methods for fourth-order problems in the three-dimensional case.

Another application comes from the theory for general relativity and gravitational waves. There, the so-called linearized Einstein–Bianchi system reads as the Maxwell's equations

$$\partial_t \mathbf{E} + \operatorname{Curl} \mathbf{B} = \mathbf{F}, \quad \operatorname{Div} \mathbf{E} = \mathbf{f} \quad \text{in } \Omega,$$
$$\partial_t \mathbf{B} - \operatorname{Curl} \mathbf{E} = \mathbf{G}, \quad \operatorname{Div} \mathbf{B} = \mathbf{g} \quad \text{in } \Omega,$$

but with symmetric and deviatoric (trace-free) tensor fields $\mathbf{E}$ and $\mathbf{B}$, where Curl denotes the rotation of a tensor field, see [7] for more details, especially on the modeling.

The paper is organized as follows: in Section 1.1 we summarize some basic notations from linear algebra and introduce several differential operators, which enable us to present in Section 1.2 some of the main analytical results in a non-rigorous way and the application of these results to the three-dimensional biharmonic equation, i.e. to (1) for $\Omega \subset \mathbb{R}^3$. The mathematically rigorous part, where also all precise definitions can be found, begins with preliminaries in Section 2 and introduces our general functional analytical setting. Then we will discuss the relevant unbounded linear operators, show closed and exact Hilbert complex properties, and present a suitable representation of the kernel of divDiv for the three-dimensional case in Section 3.1 for topologically trivial domains. In Section 3.2 we extend our results to (strong) Lipschitz domains with arbitrary topology based on two new and crucial compact embeddings. In the final Section 4 we give a detailed study of the application of our results to the three-dimensional biharmonic equation from Section 1.2. The proofs of some useful identities are presented in an Appendix.

1.1. Notations

Throughout the paper lower-case standard letters are used for denoting scalars and scalar functions, lower-case boldface letters for vectors and vector fields, and upper-case boldface letters for matrices/tensors and tensor fields.

We will use the following standard notations from linear algebra. For vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{3\times 3}$ the expressions

$$\mathbf{a} \cdot \mathbf{b} \quad \text{and} \quad \mathbf{A} : \mathbf{B}$$

denote the Euclidean inner product of vectors and the Frobenius inner product of matrices, respectively. The exterior product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{a} \times \mathbf{b}$. The exterior product $\mathbf{a} \times \mathbf{B}$ of a vector $\mathbf{a} \in \mathbb{R}^3$ and a matrix $\mathbf{B} \in \mathbb{R}^{3\times 3}$ is defined as the matrix which is obtained by applying the exterior product row-wise. With the help of the Levi-Civita symbol ε_{ijk} the exterior products can be expressed in the following way:

$$\mathbf{a} \times \mathbf{b} = \left(\sum_{j,k=1}^{3} \varepsilon_{ijk}\, \mathbf{a}_j\, \mathbf{b}_k \right)_{i=1,2,3} \qquad \text{and} \qquad \mathbf{a} \times \mathbf{B} = \left(\sum_{k,\ell=1}^{3} \varepsilon_{jk\ell}\, \mathbf{a}_k\, \mathbf{B}_{i\ell} \right)_{i,j=1,2,3}.$$

For a vector $\mathbf{a} \in \mathbb{R}^3$ the matrix $\operatorname{spn} \mathbf{a} \in \mathbb{R}^{3\times 3}$ is defined by

$$\operatorname{spn} \mathbf{a} = \begin{bmatrix} 0 & -\mathbf{a}_3 & \mathbf{a}_2 \\ \mathbf{a}_3 & 0 & -\mathbf{a}_1 \\ -\mathbf{a}_2 & \mathbf{a}_1 & 0 \end{bmatrix}.$$

Note that spn is a bijective mapping from $\mathbb{R}^3$ to the set of skew-symmetric matrices in $\mathbb{R}^{3\times 3}$ with the inverse mapping spn^{-1}. Observe that $\mathbf{a} \times \mathbf{b} = (\operatorname{spn} \mathbf{a})\, \mathbf{b}$ and $\mathbf{a} \times \mathbf{B} = \mathbf{B} \operatorname{spn} \mathbf{a}^\top = -\mathbf{B} \operatorname{spn} \mathbf{a}$.

We use

$$\operatorname{sym} \mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^\top), \quad \operatorname{skw} \mathbf{A} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^\top), \quad \operatorname{tr} \mathbf{A} = \sum_{i=1}^{3} \mathbf{A}_{ii}, \quad \text{and} \quad \operatorname{dev} \mathbf{A} = \mathbf{A} - \frac{1}{3}(\operatorname{tr} \mathbf{A})\, \mathbf{I}$$

for denoting the symmetric part, the skew-symmetric part, the trace, and the deviatoric part of a matrix $\mathbf{A}$, respectively, where $\mathbf{I}$ is the identity matrix. Finally we introduce the sets

$$\mathbb{S} = \{\mathbf{A} \in \mathbb{R}^{3\times 3} : \mathbf{A}^\top = \mathbf{A}\} \quad \text{and} \quad \mathbb{T} = \{\mathbf{A} \in \mathbb{R}^{3\times 3} : \operatorname{tr} \mathbf{A} = 0\}$$

of symmetric matrices and deviatoric (trace-free) matrices in $\mathbb{R}^{3\times 3}$.

For the convenience of the reader we summarize the definitions for differential operators used in this paper: Let φ be a scalar field, $\boldsymbol{\phi}$ be a vector field, and $\boldsymbol{\Phi}$ be a tensor field. In strong form the gradient of φ and the gradient of $\boldsymbol{\phi}$ are given by

$$\operatorname{grad}\varphi = (\partial_i\varphi)_{i=1,2,3}, \quad \operatorname{Grad}\boldsymbol{\phi} = (\partial_j\boldsymbol{\phi}_i)_{i,j=1,2,3},$$

the divergence of $\boldsymbol{\phi}$ and divergence of $\boldsymbol{\Phi}$ are given by

$$\operatorname{div}\boldsymbol{\phi} = \sum_{i=1}^{3}\partial_i\boldsymbol{\phi}_i, \quad \operatorname{Div}\boldsymbol{\Phi} = \left(\sum_{j=1}^{3}\partial_j\boldsymbol{\Phi}_{ij}\right)_{i=1,2,3},$$

and the rotation of $\boldsymbol{\phi}$ and rotation of $\boldsymbol{\Phi}$ are given by

$$\operatorname{curl}\boldsymbol{\phi} = \left(\sum_{j,k=1}^{3}\varepsilon_{ijk}\partial_j\boldsymbol{\phi}_k\right)_{i=1,2,3}, \quad \operatorname{Curl}\boldsymbol{\Phi} = \left(\sum_{k,\ell=1}^{3}\varepsilon_{jk\ell}\partial_k\boldsymbol{\Phi}_{i\ell}\right)_{i,j=1,2,3}.$$

Note that the capitalized differential operators Grad, Div, and Curl result from the row-wise application of grad to a vector field, div and curl to a tensor field. We will also use the differential operators of the form of algebraic modifications of the respective differential operators, like symGrad and devGrad, which are given in strong form component-wise by

$$[\operatorname{symGrad}\boldsymbol{\phi}]_{i,j} = \tfrac{1}{2}(\partial_j\boldsymbol{\phi}_i + \partial_i\boldsymbol{\phi}_j), \quad [\operatorname{devGrad}\boldsymbol{\phi}]_{i,j} = \partial_j\boldsymbol{\phi}_i - \tfrac{1}{3}(\operatorname{div}\boldsymbol{\phi})\delta_{ij},$$

where δ_{ij} denotes the Kronecker symbol. Second-order operators used in this papers are

$$\operatorname{Gradgrad} u = (\partial_j\partial_i u)_{i,j=1,2,3}, \quad \operatorname{divDiv}\boldsymbol{\Phi} = \sum_{i,j=1}^{3}\partial_i\partial_j\boldsymbol{\Phi}_{ij}.$$

Additionally we need the differential operators symCurl and, on one occasion only, $\operatorname{CurlCurl}^\top$, given by

$$\operatorname{symCurl}\boldsymbol{\Phi} = \tfrac{1}{2}(\operatorname{Curl}\boldsymbol{\Phi} + (\operatorname{Curl}\boldsymbol{\Phi})^\top), \quad \operatorname{CurlCurl}^\top\boldsymbol{\Phi} = \operatorname{Curl}\left((\operatorname{Curl}\boldsymbol{\Phi})^\top\right),$$

for which we omit the rather lengthy expressions for the components, since they do not provide any additional insight for the results in this paper. Depending on the context, all these differential operators are understood in the distributional or in the weak sense.

Remark 1.1: For simplicity we will discuss here only scalar/vector/tensor fields with components in $\mathbb{R}$. The extension of the results to scalar/vector/tensor fields with components in $\mathbb{C}$ is straightforward.

1.2. Some main results

Let $\Omega \subset \mathbb{R}^3$ be a bounded and topologically trivial strong Lipschitz domain. Based on a decomposition result of the non-standard Hilbert space for the auxiliary variable $\mathbf{M}$ a decomposition of the three-dimensional biharmonic problem (1) into three (consecutively to solve) second-order problems will be rigorously derived in Section 4. Written in strong form, the three resulting second-order equations are a Dirichlet–Poisson problem for the auxiliary scalar function p

$$\Delta p = f \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \Gamma,$$

a second-order Neumann type Curl symCurl –Div-system for the auxiliary tensor field $\mathbf{E}$

$$\operatorname{tr}\mathbf{E} = 0, \quad \operatorname{Curl}\operatorname{symCurl}\mathbf{E} = \operatorname{spn}\operatorname{grad} p, \quad \operatorname{Div}\mathbf{E} = 0 \quad \text{in } \Omega,$$

$$\mathbf{n}\times\operatorname{symCurl}\mathbf{E}=p\operatorname{spn}\mathbf{n}=0,\quad \mathbf{E}\,\mathbf{n}=0\quad\text{on }\Gamma,$$

and, finally, a Dirichlet–Poisson problem for the original scalar function u

$$\Delta u=3p+\operatorname{tr}\operatorname{symCurl}\mathbf{E}=\operatorname{tr}(p\,\mathbf{I}+\operatorname{symCurl}\mathbf{E})\quad\text{in }\Omega,\qquad u=0\quad\text{on }\Gamma.$$

Our results rely on the study of the Hilbert complexes

$$\mathsf{H}^2(\Omega)\xrightarrow{\operatorname{Gradgrad}}\mathcal{H}_{\mathbb{S}}(\operatorname{Curl},\Omega)\xrightarrow{\operatorname{Curl}}\mathcal{H}_{\mathbb{T}}(\operatorname{Div},\Omega)\xrightarrow{\operatorname{Div}}\mathbf{L}^2(\Omega)$$

and

$$\mathsf{L}^2(\Omega)\xleftarrow{\operatorname{divDiv}}\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega)\xleftarrow{\operatorname{symCurl}}\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega)\xleftarrow{-\operatorname{devGrad}}\mathbf{H}^1(\Omega).$$

The involved Hilbert spaces are standard Lebesgue and Sobolev spaces $\mathsf{L}^2(\Omega)$, $\mathsf{H}^2(\Omega)$ of real-valued functions, $\mathbf{L}^2(\Omega)$, $\mathbf{H}^1(\Omega)$ of vector fields, Sobolev spaces $\mathcal{H}_{\mathbb{S}}(\operatorname{Curl},\Omega)$ and $\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega)$ of symmetric tensor fields $\boldsymbol{\Phi}$ with $\operatorname{Curl}\boldsymbol{\Phi}\in\mathcal{L}^2(\Omega)$ (the space of square integrable tensor fields) and $\operatorname{divDiv}\boldsymbol{\Phi}\in\mathsf{L}^2(\Omega)$, respectively, as well as Sobolev spaces $\mathcal{H}_{\mathbb{T}}(\operatorname{Div},\Omega)$ and $\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega)$ of deviatoric (trace-free) tensor fields $\boldsymbol{\Phi}$ with $\operatorname{Div}\boldsymbol{\Phi}\in\mathbf{L}^2(\Omega)$ and $\operatorname{symCurl}\boldsymbol{\Phi}\in\mathcal{L}^2(\Omega)$, respectively. We call these complexes the Gradgrad-complex and the divDiv-complex, respectively. Up to standard modifications concerning boundary conditions these complexes are dual or adjoint to each other. In this contribution we will study all important tools and properties for these complexes, such as Helmholtz type decompositions, potentials, regular decompositions, regular potentials, Poincaré type estimates, closed ranges, exactness, and, most importantly, the key property that certain canonical embeddings are compact.

In principle, such results are known in simpler situations, e.g. in electro-magnetic theory (Maxwell's equations), where one has to deal with the de Rham complex (grad-curl-div-complex)

$$\mathsf{H}^1(\Omega)\xrightarrow{\operatorname{grad}}\mathbf{H}(\operatorname{curl},\Omega)\xrightarrow{\operatorname{curl}}\mathbf{H}(\operatorname{div},\Omega)\xrightarrow{\operatorname{div}}\mathsf{L}^2(\Omega).$$

In linear elasticity the elasticity complex ($\operatorname{CurlCurl}^\top$-complex)

$$\mathbf{H}^1(\Omega)\xrightarrow{\operatorname{symGrad}}\mathcal{H}_{\mathbb{S}}(\operatorname{CurlCurl}^\top,\Omega)\xrightarrow{\operatorname{CurlCurl}^\top}\mathcal{H}_{\mathbb{S}}(\operatorname{Div},\Omega)\xrightarrow{\operatorname{Div}}\mathbf{L}^2(\Omega)$$

plays an important role. Note that the de Rham and the elasticity complex admit certain symmetries, which is not the case for the Gradgrad-complex and the divDiv-complex. Furthermore, note that the elasticity complex as well as the Gradgrad-complex and the divDiv-complex involve first-order and also second-order differential operators.

2. Preliminaries

We start by recalling some basic concepts and abstract results from functional analysis concerning Helmholtz decompositions, closed ranges, Friedrichs/Poincaré type estimates, and bounded or even compact inverse operators. Since we will need both the Banach space setting for bounded linear operators as well as the Hilbert space setting for (possibly unbounded) closed and densely defined linear operators, we will shortly recall these two variants.

2.1. Functional analysis toolbox

Let X and Y be real Banach spaces. With $BL(\mathsf{X},\mathsf{Y})$ we introduce the space of bounded linear operators mapping X to Y. The dual spaces of X and Y are denoted by $\mathsf{X}':=BL(\mathsf{X},\mathbb{R})$ and $\mathsf{Y}':=BL(\mathsf{Y},\mathbb{R})$. For a given $\mathrm{A}\in BL(\mathsf{X},\mathsf{Y})$ we write $\mathrm{A}'\in BL(\mathsf{Y}',\mathsf{X}')$ for its Banach space dual or adjoint operator defined by $\mathrm{A}'\, y'(x) := y'(\mathrm{A}\, x)$ for all $y' \in \mathsf{Y}'$ and all $x \in \mathsf{X}$. Norms and duality in X resp. X' are denoted by $|\cdot|_{\mathsf{X}}, |\cdot|_{\mathsf{X}'}$, and $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{X}'}$.

Suppose H_1 and H_2 are Hilbert spaces. For a (possibly unbounded) densely defined linear operator $\mathrm{A}: D(\mathrm{A}) \subset \mathsf{H}_1 \to \mathsf{H}_2$ we recall that its Hilbert space dual or adjoint $\mathrm{A}^*: D(\mathrm{A}^*) \subset \mathsf{H}_2 \to \mathsf{H}_1$ can be defined via its Banach space adjoint A' and the Riesz isomorphisms of H_1 and H_2 or directly as follows: $y \in D(\mathrm{A}^*)$ if and only if $y \in \mathsf{H}_2$ and

$$\exists f \in \mathsf{H}_1 \quad \forall\, x \in D(\mathrm{A}) \quad \langle \mathrm{A}\, x, y\rangle_{\mathsf{H}_2} = \langle x, f\rangle_{\mathsf{H}_1}.$$

In this case, we define $\mathrm{A}^*\, y := f$. We note that A^* has maximal domain of definition and that A^* is characterized by

$$\forall\, x \in D(\mathrm{A}) \quad \forall\, y \in D(\mathrm{A}^*) \quad \langle \mathrm{A}\, x, y\rangle_{\mathsf{H}_2} = \langle x, \mathrm{A}^* y\rangle_{\mathsf{H}_1}.$$

Here $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{H}}$ denotes the scalar product in a Hilbert space H and D is used for the domain of definition of a linear operator. Additionally, we introduce the notation N for the kernel or null space and R for the range of a linear operator.

Let $\mathrm{A}: D(\mathrm{A}) \subset \mathsf{H}_1 \to \mathsf{H}_2$ be a (possibly unbounded) closed and densely defined linear operator on two Hilbert spaces H_1 and H_2 with adjoint $\mathrm{A}^*: D(\mathrm{A}^*) \subset \mathsf{H}_2 \to \mathsf{H}_1$. Note $(\mathrm{A}^*)^* = \overline{\mathrm{A}} = \mathrm{A}$, i.e. $(\mathrm{A}, \mathrm{A}^*)$ is a dual pair. By the projection theorem the Helmholtz type decompositions

$$\mathsf{H}_1 = N(\mathrm{A}) \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}^*)}, \quad \mathsf{H}_2 = N(\mathrm{A}^*) \oplus_{\mathsf{H}_2} \overline{R(\mathrm{A})} \tag{4}$$

hold and we can define the reduced operators

$$\begin{aligned} \mathcal{A} &:= \mathrm{A}\,|_{\overline{R(\mathrm{A}^*)}} : D(\mathcal{A}) \subset \overline{R(\mathrm{A}^*)} \to \overline{R(\mathrm{A})}, \quad D(\mathcal{A}) := D(\mathrm{A}) \cap N(\mathrm{A})^{\perp_{\mathsf{H}_1}} = D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)}, \\ \mathcal{A}^* &:= \mathrm{A}^*|_{\overline{R(\mathrm{A})}} : D(\mathcal{A}^*) \subset \overline{R(\mathrm{A})} \to \overline{R(\mathrm{A}^*)}, \quad D(\mathcal{A}^*) := D(\mathrm{A}^*) \cap N(\mathrm{A}^*)^{\perp_{\mathsf{H}_2}} = D(\mathrm{A}^*) \cap \overline{R(\mathrm{A})}, \end{aligned}$$

which are also closed and densely defined linear operators. We note that $\mathcal{A}$ and $\mathcal{A}^*$ are indeed adjoint to each other, i.e. $(\mathcal{A}, \mathcal{A}^*)$ is a dual pair as well. Now the inverse operators

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A}), \quad (\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$$

exist and they are bijective, since $\mathcal{A}$ and $\mathcal{A}^*$ are injective by definition. Furthermore, by (4) we have the refined Helmholtz type decompositions

$$D(\mathrm{A}) = N(\mathrm{A}) \oplus_{\mathsf{H}_1} D(\mathcal{A}), \quad D(\mathrm{A}^*) = N(\mathrm{A}^*) \oplus_{\mathsf{H}_2} D(\mathcal{A}^*) \tag{5}$$

and thus we obtain for the ranges

$$R(\mathrm{A}) = R(\mathcal{A}), \quad R(\mathrm{A}^*) = R(\mathcal{A}^*). \tag{6}$$

By the closed range theorem and the closed graph theorem we get immediately the following.

Lemma 2.1: *The following assertions are equivalent:*

(i) $\exists c_{\mathrm{A}} \in (0, \infty) \quad \forall\, x \in D(\mathcal{A}) \quad |x|_{\mathsf{H}_1} \le c_{\mathrm{A}} |\mathrm{A}\, x|_{\mathsf{H}_2}$,
(i*) $\exists\, c_{\mathrm{A}^*} \in (0, \infty) \quad \forall\, y \in D(\mathcal{A}^*)\ |y|_{\mathsf{H}_2} \le c_{\mathrm{A}^*} |\mathrm{A}^* y|_{\mathsf{H}_1}$,
(ii) $R(\mathrm{A}) = R(\mathcal{A})$ *is closed in* H_2,
(ii*) $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ *is closed in* H_1,
(iii) $\mathcal{A}^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ *is continuous and bijective with norm bounded by* $(1 + c_{\mathrm{A}}^2)^{1/2}$,
(iii*) $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ *is continuous and bijective with norm bounded by* $(1 + c_{\mathrm{A}^*}^2)^{1/2}$.

In case that one of the assertions of Lemma 2.1 is true, e.g. $R(\mathrm{A})$ is closed, we have

$$\begin{aligned} \mathsf{H}_1 &= N(\mathrm{A}) \oplus_{\mathsf{H}_1} R(\mathrm{A}^*), & \mathsf{H}_2 &= N(\mathrm{A}^*) \oplus_{\mathsf{H}_2} R(\mathrm{A}),\\ D(\mathrm{A}) &= N(\mathrm{A}) \oplus_{\mathsf{H}_1} D(\mathcal{A}), & D(\mathrm{A}^*) &= N(\mathrm{A}^*) \oplus_{\mathsf{H}_2} D(\mathcal{A}^*),\\ D(\mathcal{A}) &= D(\mathrm{A}) \cap R(\mathrm{A}^*), & D(\mathcal{A}^*) &= D(\mathrm{A}^*) \cap R(\mathrm{A}). \end{aligned}$$

For the 'best' constants $c_{\mathrm{A}}, c_{\mathrm{A}^*}$ we have the following lemma.

Lemma 2.2: *The Rayleigh quotients*

$$\frac{1}{c_{\mathrm{A}}} := \inf_{0\neq x\in D(\mathcal{A})} \frac{|\mathrm{A}\,x|_{\mathsf{H}_2}}{|x|_{\mathsf{H}_1}} = \inf_{0\neq y\in D(\mathcal{A}^*)} \frac{|\mathrm{A}^*y|_{\mathsf{H}_1}}{|y|_{\mathsf{H}_2}} =: \frac{1}{c_{\mathrm{A}^*}}$$

coincide, i.e. $c_{\mathrm{A}} = c_{\mathrm{A}^*}$*, if either* c_{A} *or* c_{A^*} *exists in* $(0,\infty)$*. Otherwise they also coincide, i.e. it holds* $c_{\mathrm{A}} = c_{\mathrm{A}^*} = \infty$.

From now on and throughout this paper, we always pick the best possible constants in the various Friedrichs/Poincaré type estimates.

A standard indirect argument shows the following.

Lemma 2.3: *Let* $D(\mathcal{A}) = D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)} \hookrightarrow \mathsf{H}_1$ *be compact. Then the assertions of Lemma* 2.1 *hold. Moreover, the inverse operators*

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to R(\mathrm{A}^*), \quad (\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to R(\mathrm{A})$$

are compact with norms $\left|\mathcal{A}^{-1}\right|_{R(\mathrm{A}),R(\mathrm{A}^*)} = \left|(\mathcal{A}^*)^{-1}\right|_{R(\mathrm{A}^*),R(\mathrm{A})} = c_{\mathrm{A}}$.

Moreover, we have:

Lemma 2.4: $D(\mathcal{A}) \hookrightarrow \mathsf{H}_1$ *is compact, if and only if* $D(\mathcal{A}^*) \hookrightarrow \mathsf{H}_2$ *is compact.*

Now, let $\mathrm{A}_0 : D(\mathrm{A}_0) \subset \mathsf{H}_0 \to \mathsf{H}_1$ and $\mathrm{A}_1 : D(\mathrm{A}_1) \subset \mathsf{H}_1 \to \mathsf{H}_2$ be (possibly unbounded) closed and densely defined linear operators on three Hilbert spaces H_0, H_1 and H_2 with adjoints $\mathrm{A}_0^* : D(\mathrm{A}_0^*) \subset \mathsf{H}_1 \to \mathsf{H}_0$ and $\mathrm{A}_1^* : D(\mathrm{A}_1^*) \subset \mathsf{H}_2 \to \mathsf{H}_1$ as well as reduced operators $\mathcal{A}_0, \mathcal{A}_0^*$, and $\mathcal{A}_1, \mathcal{A}_1^*$. Furthermore, we assume the sequence or complex property of A_0 and A_1, that is, $\mathrm{A}_1\,\mathrm{A}_0 = 0$, i.e.

$$R(\mathrm{A}_0) \subset N(\mathrm{A}_1). \tag{7}$$

Then also $\mathrm{A}_0^*\,\mathrm{A}_1^* = 0$, i.e. $R(\mathrm{A}_1^*) \subset N(\mathrm{A}_0^*)$. The Helmholtz type decompositions of (4) for $\mathrm{A} = \mathrm{A}_1$ and $\mathrm{A} = \mathrm{A}_0$ read

$$\mathsf{H}_1 = N(\mathrm{A}_1) \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)}, \quad \mathsf{H}_1 = N(\mathrm{A}_0^*) \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_0)} \tag{8}$$

and by (7) we see

$$N(\mathrm{A}_0^*) = N_{0,1} \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)}, \quad N(\mathrm{A}_1) = N_{0,1} \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_0)}, \quad N_{0,1} := N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \tag{9}$$

yielding the refined Helmholtz type decomposition

$$\mathsf{H}_1 = \overline{R(\mathrm{A}_0)} \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)}, \quad R(\mathrm{A}_0) = R(\mathcal{A}_0), \quad R(\mathrm{A}_1^*) = R(\mathcal{A}_1^*). \tag{10}$$

The previous results of this section imply immediately the following.

Lemma 2.5: *Let* A_0, A_1 *be as introduced before with* $\mathrm{A}_1\,\mathrm{A}_0 = 0$, *i.e.* (7). *Moreover, let* $R(\mathrm{A}_0)$ *and* $R(\mathrm{A}_1)$ *be closed. Then, the assertions of Lemmas* 2.1 *and* 2.2 *hold for* A_0 *and* A_1. *Moreover, the refined Helmholtz type decompositions*

$$
\begin{aligned}
\mathsf{H}_1 &= R(\mathrm{A}_0) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} R(\mathrm{A}_1^*), &\quad N_{0,1} &= N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*),\\
N(\mathrm{A}_1) &= R(\mathrm{A}_0) \oplus_{\mathsf{H}_1} N_{0,1}, & N(\mathrm{A}_0^*) &= N_{0,1} \oplus_{\mathsf{H}_1} R(\mathrm{A}_1^*),\\
D(\mathrm{A}_1) &= R(\mathrm{A}_0) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} D(\mathcal{A}_1), & D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} R(\mathrm{A}_1^*),\\
D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus_{\mathsf{H}_1} N_{0,1} \oplus_{\mathsf{H}_1} D(\mathcal{A}_1)
\end{aligned}
$$

hold. Especially, $R(\mathrm{A}_0)$, $R(\mathrm{A}_0^*)$, $R(\mathrm{A}_1)$, *and* $R(\mathrm{A}_1^*)$ *are closed, the respective inverse operators, i.e.*

$$
\begin{aligned}
\mathcal{A}_0{}^{-1} &: R(\mathrm{A}_0) \to D(\mathcal{A}_0), &\quad \mathcal{A}_1{}^{-1} &: R(\mathrm{A}_1) \to D(\mathcal{A}_1),\\
(\mathcal{A}_0^*)^{-1} &: R(\mathrm{A}_0^*) \to D(\mathcal{A}_0^*), & (\mathcal{A}_1^*)^{-1} &: R(\mathrm{A}_1^*) \to D(\mathcal{A}_1^*),
\end{aligned}
$$

are continuous, and there exist positive constants c_{A_0}, c_{A_1}, *such that the Friedrichs/Poincaré type estimates*

$$
\begin{aligned}
&\forall\, x \in D(\mathcal{A}_0) \quad |x|_{\mathsf{H}_0} \le c_{\mathrm{A}_0} |\mathrm{A}_0\, x|_{\mathsf{H}_1}, &\quad &\forall\, y \in D(\mathcal{A}_1) \quad |y|_{\mathsf{H}_1} \le c_{\mathrm{A}_1} |\mathrm{A}_1\, y|_{\mathsf{H}_2},\\
&\forall\, y \in D(\mathcal{A}_0^*) \quad |y|_{\mathsf{H}_1} \le c_{\mathrm{A}_0} |\mathrm{A}_0^*\, y|_{\mathsf{H}_0}, & &\forall\, z \in D(\mathcal{A}_1^*) \quad |z|_{\mathsf{H}_2} \le c_{\mathrm{A}_1} |\mathrm{A}_1^*\, z|_{\mathsf{H}_1}
\end{aligned}
$$

hold.

Remark 2.6: Note that $R(\mathrm{A}_0)$ resp. $R(\mathrm{A}_1)$ is closed, if, e.g. $D(\mathcal{A}_0) \hookrightarrow \mathsf{H}_0$ resp. $D(\mathcal{A}_1) \hookrightarrow \mathsf{H}_1$ is compact. In this case, the respective inverse operators, i.e.

$$
\begin{aligned}
\mathcal{A}_0{}^{-1} &: R(\mathrm{A}_0) \to R(\mathrm{A}_0^*), &\quad \mathcal{A}_1{}^{-1} &: R(\mathrm{A}_1) \to R(\mathrm{A}_1^*),\\
(\mathcal{A}_0^*)^{-1} &: R(\mathrm{A}_0^*) \to R(\mathrm{A}_0), & (\mathcal{A}_1^*)^{-1} &: R(\mathrm{A}_1^*) \to R(\mathrm{A}_1),
\end{aligned}
$$

are compact.

Observe $D(\mathcal{A}_1) = D(\mathrm{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \subset D(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) \subset D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*)$. Utilizing the Helmholtz type decompositions of Lemma 2.5 we immediately have:

Lemma 2.7: *The embeddings* $D(\mathcal{A}_0) \hookrightarrow \mathsf{H}_0$, $D(\mathcal{A}_1) \hookrightarrow \mathsf{H}_1$, *and* $N_{0,1} \hookrightarrow \mathsf{H}_1$ *are compact, if and only if the embedding* $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1$ *is compact. In this case,* $N_{0,1}$ *has finite dimension.*

Remark 2.8: The assumptions in Lemma 2.5 on A_0 and A_1 are equivalent to the assumption that

$$
D(\mathrm{A}_0) \subset \mathsf{H}_0 \xrightarrow{\ \mathrm{A}_0\ } D(\mathrm{A}_1) \subset \mathsf{H}_1 \xrightarrow{\ \mathrm{A}_1\ } \mathsf{H}_2
$$

is a closed Hilbert complex, meaning that the ranges are closed. As a result of the previous lemmas, the adjoint complex

$$
\mathsf{H}_0 \xleftarrow{\ \mathrm{A}_0^*\ } D(\mathrm{A}_0^*) \subset \mathsf{H}_1 \xleftarrow{\ \mathrm{A}_1^*\ } D(\mathrm{A}_1^*) \subset \mathsf{H}_2
$$

is a closed Hilbert complex as well.

We can summarize.

Theorem 2.9: *Let* A_0, A_1 *be as introduced before, i.e. having the complex property* $\mathrm{A}_1\,\mathrm{A}_0 = 0$, *i.e.* $R(\mathrm{A}_0) \subset N(\mathrm{A}_1)$. *Moreover, let* $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1$ *be compact. Then the assertions of Lemma* 2.5

hold, $N_{0,1}$ is finite dimensional and the corresponding inverse operators are continuous resp. compact. Especially, all ranges are closed and the corresponding Friedrichs/Poincaré type estimates hold.

A special situation is the following.

Lemma 2.10: *Let $\mathrm{A}_0, \mathrm{A}_1$ be as introduced before with $R(\mathrm{A}_0) = N(\mathrm{A}_1)$ and $R(\mathrm{A}_1)$ closed in H_2. Then $R(\mathrm{A}_0^*)$ and $R(\mathrm{A}_1^*)$ are closed as well, and the simplified Helmholtz type decompositions*

$$\begin{aligned} \mathsf{H}_1 &= R(\mathrm{A}_0) \oplus_{\mathsf{H}_1} R(\mathrm{A}_1^*), \quad & N_{0,1} &= \{0\}, \\ N(\mathrm{A}_1) &= R(\mathrm{A}_0) = R(\mathcal{A}_0), & N(\mathrm{A}_0^*) &= R(\mathrm{A}_1^*) = R(\mathcal{A}_1^*), \\ D(\mathrm{A}_1) &= R(\mathrm{A}_0) \oplus_{\mathsf{H}_1} D(\mathcal{A}_1), & D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus_{\mathsf{H}_1} R(\mathrm{A}_1^*), \\ D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) &= D(\mathcal{A}_0^*) \oplus_{\mathsf{H}_1} D(\mathcal{A}_1) \end{aligned}$$

are valid. Moreover, the respective inverse operators are continuous and the corresponding Friedrichs/ Poincaré type estimates hold.

Remark 2.11: Note that $R(\mathrm{A}_1^*) = N(\mathrm{A}_0^*)$ and $R(\mathrm{A}_0^*)$ closed are equivalent assumptions for Lemma 2.10 to hold.

Lemma 2.12: *Let $\mathrm{A}_0, \mathrm{A}_1$ be as introduced before with the sequence property* (7)*, i.e. $R(\mathrm{A}_0) \subset N(\mathrm{A}_1)$. If the embedding $D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) \hookrightarrow \mathsf{H}_1$ is compact and $N_{0,1} = \{0\}$, then the assumptions of Lemma* 2.10 *are satisfied.*

Remark 2.13: The assumptions in Lemma 2.10 on A_0 and A_1 are equivalent to the assumption that

$$D(\mathrm{A}_0) \subset \mathsf{H}_0 \xrightarrow{\mathrm{A}_0} D(\mathrm{A}_1) \subset \mathsf{H}_1 \xrightarrow{\mathrm{A}_1} \mathsf{H}_2$$

is a closed and exact Hilbert complex. By Lemma 2.10 the adjoint complex

$$\mathsf{H}_0 \xleftarrow{\mathrm{A}_0^*} D(\mathrm{A}_0^*) \subset \mathsf{H}_1 \xleftarrow{\mathrm{A}_1^*} D(\mathrm{A}_1^*) \subset \mathsf{H}_2$$

is a closed and exact Hilbert complex as well.

Parts of Lemma 2.10 hold also in the Banach space setting. As a direct consequence of the closed range theorem and the closed graph theorem the following abstract result holds.

Lemma 2.14: *Let $\mathsf{X}_0, \mathsf{X}_1, \mathsf{X}_2$ be Banach spaces and suppose $\mathrm{A}_0 \in BL(\mathsf{X}_0, \mathsf{X}_1)$, $\mathrm{A}_1 \in BL(\mathsf{X}_1, \mathsf{X}_2)$ with $R(\mathrm{A}_0) = N(\mathrm{A}_1)$ and that $R(\mathrm{A}_1)$ is closed in X_2. Then $R(\mathrm{A}_0')$ is closed in X_0' and $R(\mathrm{A}_1') = N(\mathrm{A}_0')$. Moreover, $(\mathrm{A}_1')^{-1} \in BL(R(\mathrm{A}_1'), R(\mathrm{A}_1)')$.*

Note that in the latter context we consider the operators

$$\mathrm{A}_1 : \mathsf{X}_1 \longrightarrow R(\mathrm{A}_1), \quad \mathrm{A}_1' : R(\mathrm{A}_1)' \longrightarrow R(\mathrm{A}_1') \quad (\mathrm{A}_1')^{-1} : R(\mathrm{A}_1') \longrightarrow R(\mathrm{A}_1)',$$

with $N(\mathrm{A}_1') = R(\mathrm{A}_1)^\circ = \{0\}$.

Remark 2.15: The conditions on A_0 and A_1 in Lemma 2.14 are identical to the assumption that

$$\mathsf{X}_0 \xrightarrow{\mathrm{A}_0} \mathsf{X}_1 \xrightarrow{\mathrm{A}_1} \mathsf{X}_2$$

is a closed and exact complex of Banach spaces. The consequences of Lemma 2.14 can be rephrased as follows. The adjoint complex of Banach spaces

$$\mathsf{X}_0' \xleftarrow{\mathrm{A}_0'} \mathsf{X}_1' \xleftarrow{\mathrm{A}_1'} \mathsf{X}_2'$$

is closed and exact as well.

Lemma 2.16: $(\mathrm{A}_1')^{-1} \in BL(R(\mathrm{A}_1'), R(\mathrm{A}_1)')$ *is equivalent to*

$$\exists\, c_{\mathrm{A}_1'} > 0 \quad \forall\, y' \in R(\mathrm{A}_1)' \quad |y'|_{R(\mathrm{A}_1)'} \le c_{\mathrm{A}_1'} |\mathrm{A}_1'\, y'|_{\mathsf{X}_1'}. \tag{11}$$

For the best constant $c_{\mathrm{A}_1'}$, (11) *is equivalent to the general inf–sup-condition*

$$0 < \frac{1}{c_{\mathrm{A}_1'}} = \inf_{0 \ne y' \in R(\mathrm{A}_1)'} \sup_{0 \ne x \in \mathsf{X}_1} \frac{\langle y', \mathrm{A}_1\, x\rangle_{R(\mathrm{A}_1)'}}{|y'|_{R(\mathrm{A}_1)'} |x|_{\mathsf{X}_1}}. \tag{12}$$

In the special case that $\mathsf{X}_2 = \mathsf{H}_2$ *is a Hilbert space the closed subspace* $R(\mathrm{A}_1)$ *is isometrically isomorphic to* $R(\mathrm{A}_1)'$ *and we obtain the following form of the inf–sup-condition:*

$$0 < \frac{1}{c_{\mathrm{A}_1'}} = \inf_{0 \ne y \in R(\mathrm{A}_1)} \sup_{0 \ne x \in \mathsf{X}_1} \frac{\langle y, \mathrm{A}_1\, x\rangle_{\mathsf{H}_2}}{|y|_{\mathsf{H}_2} |x|_{\mathsf{X}_1}}. \tag{13}$$

The results collected in this section are well known in functional analysis. We refer to [8] for a presentation of some results of this section from a numerical analysis perspective.

2.2. Sobolev spaces

Next we introduce our notations for several classes of Sobolev spaces of real-valued functions and vector fields on a bounded domain $\Omega \subset \mathbb{R}^3$. Let $m \in \mathbb{N}_0$. We denote by $\mathsf{L}^2(\Omega)$ and $\mathsf{H}^m(\Omega)$ the standard Lebesgue and Sobolev spaces of real-valued functions and write $\mathsf{H}^0(\Omega) = \mathsf{L}^2(\Omega)$. For the Lebesgue and Sobolev spaces of vector fields we use the corresponding notations in boldface letters $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^m(\Omega)$. For the rotation and divergence we define the Sobolev spaces

$$\mathbf{H}(\mathrm{curl}, \Omega) := \left\{\mathbf{v} \in \mathbf{L}^2(\Omega) : \mathrm{curl}\,\mathbf{v} \in \mathbf{L}^2(\Omega)\right\}, \quad \mathbf{H}(\mathrm{div}, \Omega) := \left\{\mathbf{v} \in \mathbf{L}^2(\Omega) : \mathrm{div}\,\mathbf{v} \in \mathsf{L}^2(\Omega)\right\}$$

with the respective graph norms, where curl and div have to be understood in the distributional or weak sense. We introduce spaces with homogeneous boundary conditions in the weak sense naturally by

$$\overset{\circ}{\mathsf{H}}{}^m(\Omega) := \overline{\overset{\circ}{\mathsf{C}}{}^\infty(\Omega)}^{\mathsf{H}^m(\Omega)}$$

and

$$\overset{\circ}{\mathbf{H}}{}^m(\Omega) := \overline{\overset{\circ}{\mathbf{C}}{}^\infty(\Omega)}^{\mathbf{H}^m(\Omega)}, \quad \overset{\circ}{\mathbf{H}}(\mathrm{curl}, \Omega) := \overline{\overset{\circ}{\mathbf{C}}{}^\infty(\Omega)}^{\mathbf{H}(\mathrm{curl},\Omega)}, \quad \overset{\circ}{\mathbf{H}}(\mathrm{div}, \Omega) := \overline{\overset{\circ}{\mathbf{C}}{}^\infty(\Omega)}^{\mathbf{H}(\mathrm{div},\Omega)},$$

i.e. as closures of test functions or test vector fields under the respective standard and graph norms, which generalizes homogeneous scalar and vectorial, tangential and normal boundary conditions, respectively. We also introduce the well-known dual spaces

$$\mathsf{H}^{-m}(\Omega) := \left(\overset{\circ}{\mathsf{H}}{}^m(\Omega)\right)'$$

with the standard dual or operator norm defined by

$$|u|_{\mathsf{H}^{-m}(\Omega)} := \sup_{0 \ne \varphi \in \overset{\circ}{\mathsf{H}}{}^m(\Omega)} \frac{\langle u, \varphi\rangle_{\mathsf{H}^{-m}(\Omega)}}{|\varphi|_{\overset{\circ}{\mathsf{H}}{}^m(\Omega)}} \quad \text{for } u \in \mathsf{H}^{-m}(\Omega),$$

where we recall the duality pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{H}^{-m}(\Omega)}$ in $(\mathsf{H}^{-m}(\Omega),\mathring{\mathsf{H}}^{m}(\Omega))$. Analogous notations for norms and duality products are used for the dual spaces

$$\mathbf{H}^{-m}(\Omega) := \left(\mathring{\mathbf{H}}^{m}(\Omega)\right)'.$$

Moreover, we define with respective graph norms

$$\mathbf{H}^{-m}(\mathrm{curl},\Omega) := \left\{\mathbf{v}\in\mathbf{H}^{-m}(\Omega) : \mathrm{curl}\,\mathbf{v}\in\mathbf{H}^{-m}(\Omega)\right\},$$
$$\mathbf{H}^{-m}(\mathrm{div},\Omega) := \left\{\mathbf{v}\in\mathbf{H}^{-m}(\Omega) : \mathrm{div}\,\mathbf{v}\in\mathsf{H}^{-m}(\Omega)\right\}.$$

A vanishing differential operator will be indicated by a zero after the operator in notations for spaces, e.g.

$$\mathbf{H}(\mathrm{curl}\,0,\Omega) = \{\mathbf{v}\in\mathbf{H}(\mathrm{curl},\Omega) : \mathrm{curl}\,\mathbf{v}=0\},\qquad \mathring{\mathbf{H}}(\mathrm{div}\,0,\Omega) = \left\{\mathbf{v}\in\mathring{\mathbf{H}}(\mathrm{div},\Omega) : \mathrm{div}\,\mathbf{v}=0\right\},$$
$$\mathbf{H}^{-m}(\mathrm{curl}\,0,\Omega) = \left\{\mathbf{v}\in\mathbf{H}^{-m}(\mathrm{curl},\Omega) : \mathrm{curl}\,\mathbf{v}=0\right\},$$
$$\mathbf{H}^{-1}(\mathrm{div}\,0,\Omega) = \left\{\mathbf{v}\in\mathbf{H}^{-1}(\mathrm{div},\Omega) : \mathrm{div}\,\mathbf{v}=0\right\}.$$

Let us also introduce

$$\mathsf{L}^2_0(\Omega) := \left\{u\in\mathsf{L}^2(\Omega) : u\perp_{\mathsf{L}^2(\Omega)}\mathbb{R}\right\} = \left\{u\in\mathsf{L}^2(\Omega) : \int_\Omega u = 0\right\},$$

where $\perp_{\mathsf{L}^2(\Omega)}$ denotes orthogonality in $\mathsf{L}^2(\Omega)$. Finally, the restrictions of the differential operators

$$\mathrm{grad},\quad \mathrm{curl},\quad \text{and}\quad \mathrm{div},$$

originally defined on $\mathsf{H}^1(\Omega)$, $\mathbf{H}(\mathrm{curl},\Omega)$, and $\mathbf{H}(\mathrm{div},\Omega)$, to the subspaces $\mathring{\mathsf{H}}^1(\Omega)$, $\mathring{\mathbf{H}}(\mathrm{curl},\Omega)$, $\mathring{\mathbf{H}}(\mathrm{div},\Omega)$ are denoted by

$$\mathring{\mathrm{grad}},\quad \mathring{\mathrm{curl}},\quad \text{and}\quad \mathring{\mathrm{div}}.$$

2.3. General assumptions

We will impose the following regularity and topology assumptions on our domain Ω.

Definition 2.17: Let Ω be an open subset of $\mathbb{R}^3$ with boundary $\Gamma := \partial\,\Omega$. We will call Ω

(i) strong Lipschitz, if Γ is locally a graph of a Lipschitz function $\psi : U\subset\mathbb{R}^2\to\mathbb{R}$,
(ii) topologically trivial, if Ω is simply connected with connected boundary Γ.

General Assumption 2.18:
From now on and throughout this paper it is assumed that $\Omega\subset\mathbb{R}^3$ is a bounded strong Lipschitz domain.

If the domain Ω has to be topologically trivial, we will always indicate this in the respective result. Note that several results will hold for arbitrary open subsets Ω of $\mathbb{R}^3$. All results are valid for bounded and topologically trivial strong Lipschitz domains $\Omega\subset\mathbb{R}^3$. Nevertheless, most of the results will remain true for bounded strong Lipschitz domains $\Omega\subset\mathbb{R}^3$.

2.4. Vector analysis

In this last part of the preliminary section, we summarize and prove several results related to scalar and vector potentials of various smoothness, corresponding Friedrichs/Poincaré type estimates, and related Helmholtz decompositions of $\mathsf{L}^2(\Omega)$ and other Hilbert and Sobolev spaces. This is a first application of the functional analysis toolbox Section 2.1 for the operators $\mathring{\operatorname{grad}}$, $\mathring{\operatorname{curl}}$, $\mathring{\operatorname{div}}$, and their adjoints $-\operatorname{div}$, curl, $-\operatorname{grad}$. Although these are well-known facts, we recall and collect them here, as we will use later similar techniques to obtain related results for the more complicated operators $\operatorname{Grad}\mathring{\operatorname{grad}}$, $\mathring{\operatorname{Curl}}_{\mathbb{S}}$, $\mathring{\operatorname{Div}}_{\mathbb{T}}$, and their adjoints $\operatorname{divDiv}_{\mathbb{S}}$, $\operatorname{symCurl}_{\mathbb{T}}$, $-\operatorname{devGrad}$, introduced in Section 3. Let

$$
\begin{aligned}
\mathrm{A}_0 &:= \mathring{\operatorname{grad}} : \mathring{\mathsf{H}}^1(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega),\\
\mathrm{A}_1 &:= \mathring{\operatorname{curl}} : \mathring{\mathbf{H}}(\operatorname{curl},\Omega) \subset \mathbf{L}^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega),\\
\mathrm{A}_2 &:= \mathring{\operatorname{div}} : \mathring{\mathbf{H}}(\operatorname{div},\Omega) \subset \mathbf{L}^2(\Omega) \longrightarrow \mathsf{L}^2(\Omega).
\end{aligned}
$$

Then A_0, A_1, and A_2 are unbounded, densely defined, and closed linear operators with adjoints

$$
\begin{aligned}
\mathrm{A}_0^* &= \mathring{\operatorname{grad}}^* = -\operatorname{div} : \mathbf{H}(\operatorname{div},\Omega) \subset \mathbf{L}^2(\Omega) \longrightarrow \mathsf{L}^2(\Omega),\\
\mathrm{A}_1^* &= \mathring{\operatorname{curl}}^* = \operatorname{curl} : \mathbf{H}(\operatorname{curl},\Omega) \subset \mathbf{L}^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega),\\
\mathrm{A}_2^* &= \mathring{\operatorname{div}}^* = -\operatorname{grad} : \mathsf{H}^1(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega)
\end{aligned}
$$

and the sequence or complex properties

$$
\begin{aligned}
R(\mathrm{A}_0) &= \mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega) \subset \mathring{\mathbf{H}}(\operatorname{curl}0,\Omega) = N(\mathrm{A}_1),\\
R(\mathrm{A}_1^*) &= \operatorname{curl}\mathbf{H}(\operatorname{curl},\Omega) \subset \mathbf{H}(\operatorname{div}0,\Omega) = N(\mathrm{A}_0^*),\\
R(\mathrm{A}_1) &= \mathring{\operatorname{curl}}\,\mathring{\mathbf{H}}(\operatorname{curl},\Omega) \subset \mathring{\mathbf{H}}(\operatorname{div}0,\Omega) = N(\mathrm{A}_2),\\
R(\mathrm{A}_2^*) &= \operatorname{grad}\mathsf{H}^1(\Omega) \subset \mathbf{H}(\operatorname{curl}0,\Omega) = N(\mathrm{A}_1^*)
\end{aligned}
$$

hold. Note $N(\mathrm{A}_0) = \{0\}$ and $N(\mathrm{A}_2^*) = \mathbb{R}$. Moreover, the embeddings

$$
\begin{aligned}
D(\mathrm{A}_1) \cap D(\mathrm{A}_0^*) &= \mathring{\mathbf{H}}(\operatorname{curl},\Omega) \cap \mathbf{H}(\operatorname{div},\Omega) \hookrightarrow \mathbf{L}^2(\Omega),\\
D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*) &= \mathring{\mathbf{H}}(\operatorname{div},\Omega) \cap \mathbf{H}(\operatorname{curl},\Omega) \hookrightarrow \mathbf{L}^2(\Omega)
\end{aligned}
$$

are compact. The latter compact embeddings are called Maxwell compactness properties or Weck's selection theorems. The first proof for strong Lipschitz domains (uniform cone like domains) avoiding smoothness of Γ was given by Weck in [9]. Generally, Weck's selection theorems hold, e.g. for weak Lipschitz domains, see [10], or even for more general domains with p-cusps or antennas, see [11, 12]. See also [13] for a different proof in the case of a strong Lipschitz domain. Weck's selection theorem for mixed boundary conditions has been proved in [14] for strong Lipschitz domains and recently in [15] for weak Lipschitz domains. Similar to Rellich's selection theorem, i.e. the compact embedding of $\mathring{\mathsf{H}}^1(\Omega)$ resp. $\mathsf{H}^1(\Omega)$ into $\mathsf{L}^2(\Omega)$, it is crucial that the domain Ω is bounded. Finally, the kernels

$$
\begin{aligned}
N(\mathrm{A}_1) \cap N(\mathrm{A}_0^*) &= \mathring{\mathbf{H}}(\operatorname{curl}0,\Omega) \cap \mathbf{H}(\operatorname{div}0,\Omega) =: \mathscr{H}_{\mathsf{D}},\\
N(\mathrm{A}_2) \cap N(\mathrm{A}_1^*) &= \mathring{\mathbf{H}}(\operatorname{div}0,\Omega) \cap \mathbf{H}(\operatorname{curl}0,\Omega) =: \mathscr{H}_{\mathsf{N}},
\end{aligned}
$$

are finite dimensional, as the unit balls are compact, i.e. the spaces of Dirichlet resp. Neumann fields are finite dimensional. More precisely, the dimension of the Dirichlet resp. Neumann fields depends on the topology or cohomology of Ω, i.e. equals the second resp. first Betti number, see, e.g. [16, 17]. Especially we have

$$\mathscr{H}_{\mathsf{D}} = \{0\}, \text{if } \Gamma \text{ is connected}, \quad \mathscr{H}_{\mathsf{N}} = \{0\}, \text{if } \Omega \text{ is simply connected}.$$

Remark 2.19: Our general assumption on Ω to be bounded and strong Lipschitz ensures that Weck's selection theorems (and thus also Rellich's) hold. The additional assumption that Ω is also topologically trivial excludes the existence of non-trivial Dirichlet or Neumann fields, as Ω is simply connected with a connected boundary Γ.

By the results of the functional analysis toolbox Section 2.1 we see that all ranges are closed with

$$R(\mathrm{A}_0) = R(\mathcal{A}_0), \quad R(\mathrm{A}_1) = R(\mathcal{A}_1), \quad R(\mathrm{A}_2) = R(\mathcal{A}_2),$$
$$R(\mathrm{A}_0^*) = R(\mathcal{A}_0^*), \quad R(\mathrm{A}_1^*) = R(\mathcal{A}_1^*), \quad R(\mathrm{A}_2^*) = R(\mathcal{A}_2^*),$$

i.e. the ranges

$$\begin{aligned}
&\mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega), \quad \operatorname{grad}\mathsf{H}^1(\Omega) = \operatorname{grad}\left(\mathsf{H}^1(\Omega)\cap\mathsf{L}_0^2(\Omega)\right),\\
&\mathring{\operatorname{curl}}\,\mathring{\mathbf{H}}(\operatorname{curl},\Omega) = \mathring{\operatorname{curl}}\left(\mathring{\mathbf{H}}(\operatorname{curl},\Omega)\cap\operatorname{curl}\mathbf{H}(\operatorname{curl},\Omega)\right),\\
&\operatorname{curl}\mathbf{H}(\operatorname{curl},\Omega) = \operatorname{curl}\left(\mathbf{H}(\operatorname{curl},\Omega)\cap\mathring{\operatorname{curl}}\,\mathring{\mathbf{H}}(\operatorname{curl},\Omega)\right),\\
&\mathring{\operatorname{div}}\,\mathring{\mathbf{H}}(\operatorname{div},\Omega) = \mathring{\operatorname{div}}\left(\mathring{\mathbf{H}}(\operatorname{div},\Omega)\cap\operatorname{grad}\mathsf{H}^1(\Omega)\right), \quad \operatorname{div}\mathbf{H}(\operatorname{div},\Omega) = \operatorname{div}\left(\mathbf{H}(\operatorname{div},\Omega)\cap\mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega)\right)
\end{aligned} \tag{14}$$

are closed, and the reduced operators are

$$\begin{aligned}
\mathcal{A}_0 &= \mathring{\operatorname{grad}} : \mathring{\mathsf{H}}^1(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega),\\
\mathcal{A}_1 &= \mathring{\operatorname{curl}} : \mathring{\mathbf{H}}(\operatorname{curl},\Omega)\cap\operatorname{curl}\mathbf{H}(\operatorname{curl},\Omega) \subset \operatorname{curl}\mathbf{H}(\operatorname{curl},\Omega) \longrightarrow \operatorname{curl}\mathring{\mathbf{H}}(\operatorname{curl},\Omega),\\
\mathcal{A}_2 &= \mathring{\operatorname{div}} : \mathring{\mathbf{H}}(\operatorname{div},\Omega)\cap\operatorname{grad}\mathsf{H}^1(\Omega) \subset \operatorname{grad}\mathsf{H}^1(\Omega) \longrightarrow \mathsf{L}_0^2(\Omega),\\
\mathcal{A}_0^* &= -\operatorname{div} : \mathbf{H}(\operatorname{div},\Omega)\cap\operatorname{grad}\mathring{\mathsf{H}}^1(\Omega) \subset \operatorname{grad}\mathring{\mathsf{H}}^1(\Omega) \longrightarrow \mathsf{L}^2(\Omega),\\
\mathcal{A}_1^* &= \operatorname{curl} : \mathbf{H}(\operatorname{curl},\Omega)\cap\mathring{\operatorname{curl}}\,\mathring{\mathbf{H}}(\operatorname{curl},\Omega) \subset \mathring{\operatorname{curl}}\,\mathring{\mathbf{H}}(\operatorname{curl},\Omega) \longrightarrow \operatorname{curl}\mathbf{H}(\operatorname{curl},\Omega),\\
\mathcal{A}_2^* &= -\operatorname{grad} : \mathsf{H}^1(\Omega)\cap\mathsf{L}_0^2(\Omega) \subset \mathsf{L}_0^2(\Omega) \longrightarrow \operatorname{grad}\mathsf{H}^1(\Omega).
\end{aligned}$$

Moreover, we have the following well-known Helmholtz decompositions of $\mathbf{L}^2$-vector fields into irrotational and solenoidal vector fields, corresponding Friedrichs/Poincaré type estimates and continuous or compact inverse operators.

Lemma 2.20: *The Helmholtz decompositions*

$$\mathsf{L}^2(\Omega) = \mathring{\operatorname{div}}\,\mathring{\mathbf{H}}(\operatorname{div},\Omega) \oplus_{\mathsf{L}^2(\Omega)} \mathbb{R}, \quad \mathring{\operatorname{div}}\,\mathring{\mathbf{H}}(\operatorname{div},\Omega) = \mathsf{L}_0^2(\Omega),$$
$$\mathsf{L}^2(\Omega) = \operatorname{div}\mathbf{H}(\operatorname{div},\Omega),$$

$$\begin{aligned}
\mathsf{L}^2(\Omega) &= \mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega) \oplus_{\mathsf{L}^2(\Omega)} \mathsf{H}(\operatorname{div} 0, \Omega)\\
&= \mathring{\mathsf{H}}(\operatorname{curl} 0, \Omega) \oplus_{\mathsf{L}^2(\Omega)} \operatorname{curl} \mathsf{H}(\operatorname{curl}, \Omega)\\
&= \mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega) \oplus_{\mathsf{L}^2(\Omega)} \mathscr{H}_{\mathsf{D}} \oplus_{\mathsf{L}^2(\Omega)} \operatorname{curl} \mathsf{H}(\operatorname{curl}, \Omega),\\
\mathsf{L}^2(\Omega) &= \operatorname{grad} \mathsf{H}^1(\Omega) \oplus_{\mathsf{L}^2(\Omega)} \mathring{\mathsf{H}}(\operatorname{div} 0, \Omega)\\
&= \mathsf{H}(\operatorname{curl} 0, \Omega) \oplus_{\mathsf{L}^2(\Omega)} \mathring{\operatorname{curl}}\,\mathring{\mathsf{H}}(\operatorname{curl}, \Omega)\\
&= \operatorname{grad} \mathsf{H}^1(\Omega) \oplus_{\mathsf{L}^2(\Omega)} \mathscr{H}_{\mathsf{N}} \oplus_{\mathsf{L}^2(\Omega)} \mathring{\operatorname{curl}}\,\mathring{\mathsf{H}}(\operatorname{curl}, \Omega)
\end{aligned}$$

hold. Moreover, (14) *is true for the respective ranges and the 'better' potentials in* (14) *are uniquely determined and depend continuously in the right-hand sides. If* Γ *is connected, it holds* $\mathscr{H}_{\mathsf{D}} = \{0\}$ *and, e.g.*

$$\mathsf{L}^2(\Omega) = \mathring{\mathsf{H}}(\operatorname{curl} 0, \Omega) \oplus \mathsf{H}(\operatorname{div} 0, \Omega),$$

$$\mathring{\mathsf{H}}(\operatorname{curl} 0, \Omega) = \mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega), \quad \mathsf{H}(\operatorname{div} 0, \Omega) = \operatorname{curl} \mathsf{H}(\operatorname{curl}, \Omega) = \operatorname{curl}\Big(\mathsf{H}(\operatorname{curl}, \Omega) \cap \mathring{\mathsf{H}}(\operatorname{div} 0, \Omega)\Big).$$

If Ω *is simply connected, it holds* $\mathscr{H}_{\mathsf{N}} = \{0\}$ *and, e.g.*

$$\mathsf{L}^2(\Omega) = \mathsf{H}(\operatorname{curl} 0, \Omega) \oplus \mathring{\mathsf{H}}(\operatorname{div} 0, \Omega),$$

$$\mathsf{H}(\operatorname{curl} 0, \Omega) = \operatorname{grad} \mathsf{H}^1(\Omega), \quad \mathring{\mathsf{H}}(\operatorname{div} 0, \Omega) = \mathring{\operatorname{curl}}\,\mathring{\mathsf{H}}(\operatorname{curl}, \Omega) = \mathring{\operatorname{curl}}\Big(\mathring{\mathsf{H}}(\operatorname{curl}, \Omega) \cap \mathsf{H}(\operatorname{div} 0, \Omega)\Big).$$

Lemma 2.21: *The following Friedrichs/Poincaré type estimates hold. There exist positive constants* c_{g}*,* c_{r}*,* c_{d}*, such that*

$$\begin{aligned}
&\forall\, u \in \mathring{\mathsf{H}}^1(\Omega) && |u|_{\mathsf{L}^2(\Omega)} \le c_{\mathsf{g}}\, |\operatorname{grad} u|_{\mathsf{L}^2(\Omega)},\\
&\forall\, \mathbf{v} \in \mathsf{H}(\operatorname{div}, \Omega) \cap \mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega) && |\mathbf{v}|_{\mathsf{L}^2(\Omega)} \le c_{\mathsf{g}}\, |\operatorname{div} \mathbf{v}|_{\mathsf{L}^2(\Omega)},\\
&\forall\, \mathbf{v} \in \mathring{\mathsf{H}}(\operatorname{curl}, \Omega) \cap \operatorname{curl} \mathsf{H}(\operatorname{curl}, \Omega) && |\mathbf{v}|_{\mathsf{L}^2(\Omega)} \le c_{\mathsf{r}}\, |\operatorname{curl} \mathbf{v}|_{\mathsf{L}^2(\Omega)},\\
&\forall\, \mathbf{v} \in \mathsf{H}(\operatorname{curl}, \Omega) \cap \mathring{\operatorname{curl}}\,\mathring{\mathsf{H}}(\operatorname{curl}, \Omega) && |\mathbf{v}|_{\mathsf{L}^2(\Omega)} \le c_{\mathsf{r}}\, |\operatorname{curl} \mathbf{v}|_{\mathsf{L}^2(\Omega)},\\
&\forall\, \mathbf{v} \in \mathring{\mathsf{H}}(\operatorname{div}, \Omega) \cap \operatorname{grad} \mathsf{H}^1(\Omega) && |\mathbf{v}|_{\mathsf{L}^2(\Omega)} \le c_{\mathsf{d}}\, |\operatorname{div} \mathbf{v}|_{\mathsf{L}^2(\Omega)},\\
&\forall\, u \in \mathsf{H}^1(\Omega) \cap \mathsf{L}^2_0(\Omega) && |u|_{\mathsf{L}^2(\Omega)} \le c_{\mathsf{d}}\, |\operatorname{grad} u|_{\mathsf{L}^2(\Omega)}.
\end{aligned}$$

Moreover, the reduced versions of the operators

$$\mathring{\operatorname{grad}}, \quad \mathring{\operatorname{curl}}, \quad \mathring{\operatorname{div}}, \quad \operatorname{grad}, \quad \operatorname{curl}, \quad \operatorname{div}$$

have continuous resp. compact inverse operators

$$\begin{aligned}
\mathring{\operatorname{grad}}^{-1} &: \mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega) \longrightarrow \mathring{\mathsf{H}}^1(\Omega), & \mathring{\operatorname{grad}}^{-1} &: \mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega) \longrightarrow \mathsf{L}^2(\Omega),\\
\operatorname{div}^{-1} &: \mathsf{L}^2(\Omega) \longrightarrow \mathsf{H}(\operatorname{div}, \Omega) \cap \mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega), & \operatorname{div}^{-1} &: \mathsf{L}^2(\Omega) \longrightarrow \mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega) \subset \mathsf{L}^2(\Omega),\\
\mathring{\operatorname{curl}}^{-1} &: \mathring{\operatorname{curl}}\,\mathring{\mathsf{H}}(\operatorname{curl}, \Omega) \longrightarrow \mathring{\mathsf{H}}(\operatorname{curl}, \Omega) \cap \operatorname{curl} \mathsf{H}(\operatorname{curl}, \Omega), & \mathring{\operatorname{curl}}^{-1} &: \mathring{\operatorname{curl}}\,\mathring{\mathsf{H}}(\operatorname{curl}, \Omega) \longrightarrow \operatorname{curl} \mathsf{H}(\operatorname{curl}, \Omega) \subset \mathsf{L}^2(\Omega),
\end{aligned}$$

$$\operatorname{curl}^{-1}:\operatorname{curl}\mathbf{H}(\operatorname{curl},\Omega)\longrightarrow\mathbf{H}(\operatorname{curl},\Omega)\cap\mathring{\operatorname{curl}}\,\mathring{\mathbf{H}}(\operatorname{curl},\Omega),\qquad \operatorname{curl}^{-1}:\operatorname{curl}\mathbf{H}(\operatorname{curl},\Omega)\longrightarrow\mathring{\operatorname{curl}}\,\mathring{\mathbf{H}}(\operatorname{curl},\Omega)\subset\mathbf{L}^2(\Omega),$$
$$\mathring{\operatorname{div}}^{-1}:\mathsf{L}^2_0(\Omega)\longrightarrow\mathring{\mathbf{H}}(\operatorname{div},\Omega)\cap\operatorname{grad}\mathsf{H}^1(\Omega),\qquad \mathring{\operatorname{div}}^{-1}:\mathsf{L}^2_0(\Omega)\longrightarrow\operatorname{grad}\mathsf{H}^1(\Omega)\subset\mathbf{L}^2(\Omega),$$
$$\operatorname{grad}^{-1}:\operatorname{grad}\mathsf{H}^1(\Omega)\longrightarrow\mathsf{H}^1(\Omega)\cap\mathsf{L}^2_0(\Omega),\qquad \operatorname{grad}^{-1}:\operatorname{grad}\mathsf{H}^1(\Omega)\longrightarrow\mathsf{L}^2_0(\Omega),$$

with norms $(1+c_{\mathsf{g}}^2)^{1/2}$, $(1+c_{\mathsf{r}}^2)^{1/2}$, $(1+c_{\mathsf{d}}^2)^{1/2}$ *resp.* c_{g}, c_{r}, c_{d}. *In other words, the operators*

$$\begin{aligned}\mathring{\operatorname{grad}}:\mathring{\mathsf{H}}^1(\Omega)&\longrightarrow\mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega), & \operatorname{div}:\mathbf{H}(\operatorname{div},\Omega)\cap\mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega)&\longrightarrow\mathsf{L}^2(\Omega),\\ u&\longmapsto\operatorname{grad}u & \mathbf{v}&\longmapsto\operatorname{div}\mathbf{v}\\ \mathring{\operatorname{curl}}:\mathring{\mathbf{H}}(\operatorname{curl},\Omega)\cap\operatorname{curl}\mathbf{H}(\operatorname{curl},\Omega)&\longrightarrow\mathring{\operatorname{curl}}\,\mathring{\mathbf{H}}(\operatorname{curl},\Omega), & \operatorname{curl}:\mathbf{H}(\operatorname{curl},\Omega)\cap\mathring{\operatorname{curl}}\,\mathring{\mathbf{H}}(\operatorname{curl},\Omega)&\longrightarrow\operatorname{curl}\mathbf{H}(\operatorname{curl},\Omega),\\ \mathbf{v}&\longmapsto\operatorname{curl}\mathbf{v} & \mathbf{v}&\longmapsto\operatorname{curl}\mathbf{v}\\ \mathring{\operatorname{div}}:\mathring{\mathbf{H}}(\operatorname{div},\Omega)\cap\operatorname{grad}\mathsf{H}^1(\Omega)&\longrightarrow\mathsf{L}^2_0(\Omega), & \operatorname{grad}:\mathsf{H}^1(\Omega)\cap\mathsf{L}^2_0(\Omega)&\longrightarrow\operatorname{grad}\mathsf{H}^1(\Omega),\\ \mathbf{v}&\longmapsto\operatorname{div}\mathbf{v} & u&\longmapsto\operatorname{grad}u\end{aligned}$$

are topological isomorphisms. If Ω *is topologically trivial, then*

$$\begin{aligned}\mathring{\operatorname{grad}}:\mathring{\mathsf{H}}^1(\Omega)&\longrightarrow\mathring{\mathbf{H}}(\operatorname{curl}0,\Omega), & \operatorname{div}:\mathbf{H}(\operatorname{div},\Omega)\cap\mathring{\mathbf{H}}(\operatorname{curl}0,\Omega)&\longrightarrow\mathsf{L}^2(\Omega),\\ u&\longmapsto\operatorname{grad}u & \mathbf{v}&\longmapsto\operatorname{div}\mathbf{v}\\ \mathring{\operatorname{curl}}:\mathring{\mathbf{H}}(\operatorname{curl},\Omega)\cap\mathbf{H}(\operatorname{div}0,\Omega)&\longrightarrow\mathring{\mathbf{H}}(\operatorname{div}0,\Omega), & \operatorname{curl}:\mathbf{H}(\operatorname{curl},\Omega)\cap\mathring{\mathbf{H}}(\operatorname{div}0,\Omega)&\longrightarrow\mathbf{H}(\operatorname{div}0,\Omega),\\ \mathbf{v}&\longmapsto\operatorname{curl}\mathbf{v} & \mathbf{v}&\longmapsto\operatorname{curl}\mathbf{v}\\ \mathring{\operatorname{div}}:\mathring{\mathbf{H}}(\operatorname{div},\Omega)\cap\mathbf{H}(\operatorname{curl}0,\Omega)&\longrightarrow\mathsf{L}^2_0(\Omega), & \operatorname{grad}:\mathsf{H}^1(\Omega)\cap\mathsf{L}^2_0(\Omega)&\longrightarrow\mathbf{H}(\operatorname{curl}0,\Omega),\\ \mathbf{v}&\longmapsto\operatorname{div}\mathbf{v} & u&\longmapsto\operatorname{grad}u\end{aligned}\tag{15}$$

are topological isomorphisms.

Remark 2.22: Recently it has been shown in [18–20] that for bounded and convex $\Omega\subset\mathbb{R}^3$ it holds

$$c_{\mathsf{r}}\le c_{\mathsf{d}}\le\frac{\operatorname{diam}\Omega}{\pi},$$

i.e. the Maxwell constant c_{r} can be estimated from above by the Poincaré constant c_{d}.

Remark 2.23: Some of the previous results can be formulated equivalently in terms of complexes: The sequence

$$\{0\}\xrightarrow{\;0\;}\mathring{\mathsf{H}}^1(\Omega)\xrightarrow{\;\mathring{\operatorname{grad}}\;}\mathring{\mathbf{H}}(\operatorname{curl},\Omega)\xrightarrow{\;\mathring{\operatorname{curl}}\;}\mathring{\mathbf{H}}(\operatorname{div},\Omega)\xrightarrow{\;\mathring{\operatorname{div}}\;}\mathsf{L}^2(\Omega)\xrightarrow{\;\pi_{\mathbb{R}}\;}\mathbb{R}$$

and thus also its dual or adjoint sequence

$$\{0\}\xleftarrow{\;0\;}\mathsf{L}^2(\Omega)\xleftarrow{\;-\operatorname{div}\;}\mathbf{H}(\operatorname{div},\Omega)\xleftarrow{\;\operatorname{curl}\;}\mathbf{H}(\operatorname{curl},\Omega)\xleftarrow{\;-\operatorname{grad}\;}\mathsf{H}^1(\Omega)\xleftarrow{\;\iota_{\mathbb{R}}\;}\mathbb{R}$$

are closed Hilbert complexes. Here $\pi_{\mathbb{R}}:\mathsf{L}^2(\Omega)\to\mathbb{R}$ denotes the orthogonal projector onto $\mathbb{R}$ with adjoint $\pi_{\mathbb{R}}^*=\iota_{\mathbb{R}}:\mathbb{R}\to\mathsf{L}^2(\Omega)$, the canonical embedding. If Ω is additionally topologically trivial, then the complexes are also exact. These complexes are widely known as de Rham complexes.

Let Ω be additionally topologically trivial. For irrotational vector fields in $\mathring{\mathbf{H}}^m(\Omega)$ resp. $\mathbf{H}^m(\Omega)$ we have smooth potentials, which follows immediately by $\mathring{\mathbf{H}}(\operatorname{curl}0,\Omega)=\mathring{\operatorname{grad}}\,\mathring{\mathsf{H}}^1(\Omega)$ resp. $\mathbf{H}(\operatorname{curl}0,\Omega)=\operatorname{grad}\mathsf{H}^1(\Omega)$ from the previous lemma.

Lemma 2.24: *Let Ω be additionally topologically trivial and $m \in \mathbb{N}_0$. Then*

$$\mathring{\mathbf{H}}^{m}(\Omega) \cap \mathring{\mathbf{H}}(\operatorname{curl} 0, \Omega) = \mathring{\operatorname{grad}}\, \mathring{\mathsf{H}}^{m+1}(\Omega), \quad \mathbf{H}^{m}(\Omega) \cap \mathbf{H}(\operatorname{curl} 0, \Omega) = \operatorname{grad} \mathsf{H}^{m+1}(\Omega)$$

hold with linear and continuous potential operators $\mathrm{P}_{\mathring{\operatorname{grad}}}$, $\mathrm{P}_{\operatorname{grad}}$.

So, for each $\mathbf{v} \in \mathring{\mathbf{H}}^{m}(\Omega) \cap \mathring{\mathbf{H}}(\operatorname{curl} 0, \Omega)$, we have $\mathbf{v} = \mathring{\operatorname{grad}}\, u$ for the potential $u = \mathrm{P}_{\mathring{\operatorname{grad}}} \mathbf{v} \in \mathring{\mathsf{H}}^{m+1}(\Omega)$ and, analogously, for each $\mathbf{v} \in \mathbf{H}^{m}(\Omega) \cap \mathbf{H}(\operatorname{curl}, \Omega)$, it holds $\mathbf{v} = \operatorname{grad} u$ for the potential $u = \mathrm{P}_{\operatorname{grad}} \mathbf{v} \in \mathsf{H}^{m+1}(\Omega)$. Note that the potential in $\mathsf{H}^{m+1}(\Omega)$ is uniquely determined only up to a constant.

For solenoidal vector fields in $\mathring{\mathbf{H}}^{m}(\Omega)$ resp. $\mathbf{H}^{m}(\Omega)$ we have smooth potentials, too.

Lemma 2.25: *Let Ω be additionally topologically trivial and $m \in \mathbb{N}_0$. Then*

$$\mathring{\mathbf{H}}^{m}(\Omega) \cap \mathring{\mathbf{H}}(\operatorname{div} 0, \Omega) = \mathring{\operatorname{curl}}\, \mathring{\mathbf{H}}^{m+1}(\Omega), \quad \mathbf{H}^{m}(\Omega) \cap \mathbf{H}(\operatorname{div} 0, \Omega) = \operatorname{curl} \mathbf{H}^{m+1}(\Omega)$$

hold with linear and continuous potential operators $\mathrm{P}_{\mathring{\operatorname{curl}}}$, $\mathrm{P}_{\operatorname{curl}}$.

For a proof see, e.g. [21, Corollary 4.7] or with slight modifications the generalized lifting lemma [22, Corollary 5.4] for the case $d = 3, k = m, l = 2$. Moreover, the potential in $\mathring{\mathsf{H}}^{m+1}(\Omega)$ resp. $\mathsf{H}^{m+1}(\Omega)$ is no longer uniquely determined.

For the divergence operator we have the following result.

Lemma 2.26: *Let $m \in \mathbb{N}_0$. Then*

$$\mathring{\mathsf{H}}^{m}(\Omega) \cap \mathsf{L}^2_0(\Omega) = \mathring{\operatorname{div}}\, \mathring{\mathbf{H}}^{m+1}(\Omega), \quad \mathsf{H}^{m}(\Omega) = \operatorname{div} \mathbf{H}^{m+1}(\Omega)$$

hold with linear and continuous potential operators $\mathrm{P}_{\mathring{\operatorname{div}}}$, $\mathrm{P}_{\operatorname{div}}$.

Again, the potential in $\mathring{\mathbf{H}}^{m+1}(\Omega)$ resp. $\mathbf{H}^{m+1}(\Omega)$ is no longer uniquely determined. Also Lemma 2.24 resp. Lemma 2.26 has been proved in [21, Corollary 4.7(b)] and in [22, Corollary 5.4] for the case $d = 3, k = m, l = 1$ resp. $d = 3, k = m, l = 3$.

Remark 2.27: Lemma 2.26, which shows a classical result on the solvability and on the properties of the solution operator of the divergence equation, is an important tool in fluid dynamics, i.e. in the theory of Stokes or Navier–Stokes equations. The potential operator is often called Bogovskii operator, see [23, 24] for the original works and also [[25, p.179, Theorem III.3.3], [26, Lemma 2.1.1]]. Moreover, there are also versions of Lemmas 2.24 and 2.25, if Ω is not topologically trivial, which we will not need in the paper at hand.

Remark 2.28: A closer inspection of Lemmas 2.24 and 2.25 and their proofs shows that these results extend to general topologies as well. More precisely we have:

(i) It holds

$$\mathring{\mathbf{H}}^{m}(\Omega) \cap \mathring{\operatorname{grad}}\, \mathring{\mathsf{H}}^{1}(\Omega) = \mathring{\mathbf{H}}^{m}(\Omega) \cap \mathring{\mathbf{H}}(\operatorname{curl} 0, \Omega) \cap \mathscr{H}_{\mathsf{D}}^{\perp} = \mathring{\operatorname{grad}}\, \mathring{\mathsf{H}}^{m+1}(\Omega),$$
$$\mathbf{H}^{m}(\Omega) \cap \operatorname{grad} \mathsf{H}^{1}(\Omega) = \mathbf{H}^{m}(\Omega) \cap \mathbf{H}(\operatorname{curl} 0, \Omega) \cap \mathscr{H}_{\mathsf{N}}^{\perp} = \operatorname{grad} \mathsf{H}^{m+1}(\Omega)$$

with linear and continuous potential operators $\mathrm{P}_{\mathring{\operatorname{grad}}}$, $\mathrm{P}_{\operatorname{grad}}$.

(ii) It holds

$$\mathring{\mathsf{H}}^{m}(\Omega)\cap\mathring{\operatorname{curl}}\,\mathring{\mathsf{H}}(\operatorname{curl},\Omega)=\mathring{\mathsf{H}}^{m}(\Omega)\cap\mathring{\mathsf{H}}(\operatorname{div}0,\Omega)\cap\mathscr{H}_{\mathsf{N}}^{\perp}=\mathring{\operatorname{curl}}\,\mathring{\mathsf{H}}^{m+1}(\Omega),$$
$$\mathsf{H}^{m}(\Omega)\cap\operatorname{curl}\mathsf{H}(\operatorname{curl},\Omega)=\mathsf{H}^{m}(\Omega)\cap\mathsf{H}(\operatorname{div}0,\Omega)\cap\mathscr{H}_{\mathsf{D}}^{\perp}=\operatorname{curl}\mathsf{H}^{m+1}(\Omega)$$

with linear and continuous potential operators $\mathrm{P}_{\mathring{\operatorname{curl}}}$, $\mathrm{P}_{\operatorname{curl}}$.

Using the latter three results and Lemma 2.14, irrotational and solenoidal vector fields in $\mathsf{H}^{-m}(\Omega)$ can be characterized.

Corollary 2.29: *Let Ω be additionally topologically trivial and $m\in\mathbb{N}$. Then*

$$\mathsf{H}^{-m}(\operatorname{curl}0,\Omega)=\operatorname{grad}\mathsf{H}^{-m+1}(\Omega)=\operatorname{grad}\left(\mathring{\mathsf{H}}^{m-1}(\Omega)\cap\mathsf{L}_0^2(\Omega)\right)'$$

is closed in $\mathsf{H}^{-m}(\Omega)$ with continuous inverse, i.e. $\operatorname{grad}^{-1}\in BL(\mathsf{H}^{-m}(\operatorname{curl}0,\Omega),(\mathring{\mathsf{H}}^{m-1}(\Omega)\cap\mathsf{L}_0^2(\Omega))')$*. Especially for $m=1$,*

$$\mathsf{H}^{-1}(\operatorname{curl}0,\Omega)=\operatorname{grad}\mathsf{L}^2(\Omega)=\operatorname{grad}\mathsf{L}_0^2(\Omega)$$

is closed in $\mathsf{H}^{-1}(\Omega)$ with continuous inverse $\operatorname{grad}^{-1}\in BL(\mathsf{H}^{-1}(\operatorname{curl}0,\Omega),\mathsf{L}_0^2(\Omega))$ *and uniquely determined potential in $\mathsf{L}_0^2(\Omega)$. Moreover,*

$$\exists\,c_{\mathrm{g},-1}>0\quad\forall\,u\in\mathsf{L}_0^2(\Omega)\quad|u|_{\mathsf{L}^2(\Omega)}\leq c_{\mathrm{g},-1}|\operatorname{grad}u|_{\mathsf{H}^{-1}(\Omega)}\leq\sqrt{3}\,c_{\mathrm{g},-1}|u|_{\mathsf{L}^2(\Omega)}$$

and the inf–sup-condition

$$0<\frac{1}{c_{\mathrm{g},-1}}=\inf_{0\neq u\in\mathsf{L}_0^2(\Omega)}\frac{|\operatorname{grad}u|_{\mathsf{H}^{-1}(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)}}=\inf_{0\neq u\in\mathsf{L}_0^2(\Omega)}\sup_{0\neq\mathbf{v}\in\mathring{\mathsf{H}}^{1}(\Omega)}\frac{\langle u,\operatorname{div}\mathbf{v}\rangle_{\mathsf{L}^2(\Omega)}}{|u|_{\mathsf{L}^2(\Omega)}|\operatorname{Grad}\mathbf{v}|_{\mathsf{L}^2(\Omega)}}$$

holds.

Proof: Let $\mathsf{X}_0:=\mathring{\mathsf{H}}^{m+1}(\Omega)$, $\mathsf{X}_1:=\mathring{\mathsf{H}}^{m}(\Omega)$, $\mathsf{X}_2:=\mathring{\mathsf{H}}^{m-1}(\Omega)$ and

$$\mathrm{A}_0:=\mathring{\operatorname{curl}}:\mathring{\mathsf{H}}^{m+1}(\Omega)\to\mathring{\mathsf{H}}^{m}(\Omega),\quad\mathrm{A}_1:=-\mathring{\operatorname{div}}:\mathring{\mathsf{H}}^{m}(\Omega)\to\mathring{\mathsf{H}}^{m-1}(\Omega).$$

These linear operators are bounded, $R(\mathrm{A}_0)=\mathring{\operatorname{curl}}\,\mathring{\mathsf{H}}^{m+1}(\Omega)=\mathring{\mathsf{H}}^{m}(\Omega)\cap\mathring{\mathsf{H}}(\operatorname{div}0,\Omega)=N(\mathrm{A}_1)$ by Lemma 2.25, and $R(\mathrm{A}_1)=\mathring{\operatorname{div}}\mathring{\mathsf{H}}^{m}(\Omega)=\mathring{\mathsf{H}}^{m-1}(\Omega)\cap\mathsf{L}_0^2(\Omega)$ by Lemma 2.26. Therefore, $R(\mathrm{A}_1)$ is closed. For the adjoint operators we get

$$\mathrm{A}_0'=\operatorname{curl}=\mathring{\operatorname{curl}}':\mathsf{H}^{-m}(\Omega)\to\mathsf{H}^{-m-1}(\Omega),\quad\mathrm{A}_1'=\operatorname{grad}=-\mathring{\operatorname{div}}':\mathsf{H}^{-m+1}(\Omega)\to\mathsf{H}^{-m}(\Omega)$$

and obtain from Lemma 2.14 that

$$\mathsf{H}^{-m}(\operatorname{curl}0,\Omega)=N(\mathrm{A}_0')=R(\mathrm{A}_1')=\operatorname{grad}\mathsf{H}^{-m+1}(\Omega)$$

is closed and

$$\operatorname{grad}^{-1}=(\mathrm{A}_1')^{-1}\in BL\left(R(\mathrm{A}_1'),R(\mathrm{A}_1)'\right)=BL\left(\mathsf{H}^{-m}(\operatorname{curl}0,\Omega),(\mathring{\mathsf{H}}^{m-1}(\Omega)\cap\mathsf{L}_0^2(\Omega))'\right),$$

which completes the proof for general m. If $m=1$, we get the assertions about the Friedrichs/Poincaré/Nečas inequality and inf–sup-condition by Lemma 2.16, i.e. (11) and (13). ∎

Corollary 2.30: *Let Ω be additionally topologically trivial and $m \in \mathbb{N}$. Then*

$$\mathsf{H}^{-m}(\operatorname{div} 0, \Omega) = \operatorname{curl} \mathsf{H}^{-m+1}(\Omega) = \operatorname{curl} \left(\mathring{\mathsf{H}}^{m-1}(\Omega) \cap \mathring{\mathsf{H}}(\operatorname{div} 0, \Omega) \right)'$$

is closed in $\mathsf{H}^{-m}(\Omega)$ with continuous inverse, i.e. $\operatorname{curl}^{-1} \in BL(\mathsf{H}^{-m}(\operatorname{div} 0, \Omega), (\mathring{\mathsf{H}}^{m-1}(\Omega) \cap \mathring{\mathsf{H}}(\operatorname{div} 0, \Omega))')$. *Especially for $m = 1$,*

$$\mathsf{H}^{-1}(\operatorname{div} 0, \Omega) = \operatorname{curl} \mathsf{L}^2(\Omega) = \operatorname{curl} \mathring{\mathsf{H}}(\operatorname{div} 0, \Omega)$$

is closed in $\mathsf{H}^{-1}(\Omega)$ with continuous inverse $\operatorname{curl}^{-1} \in BL(\mathsf{H}^{-1}(\operatorname{div} 0, \Omega), \mathring{\mathsf{H}}(\operatorname{div} 0, \Omega))$ *and uniquely determined potential in* $\mathring{\mathsf{H}}(\operatorname{div} 0, \Omega)$. *Moreover,*

$$\exists\, c_{\mathsf{r},-1} > 0 \quad \forall\, \mathbf{v} \in \mathring{\mathsf{H}}(\operatorname{div} 0, \Omega) \quad |\mathbf{v}|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{r},-1} |\operatorname{curl} \mathbf{v}|_{\mathsf{H}^{-1}(\Omega)} \leq \sqrt{2}\, c_{\mathsf{r},-1} |\mathbf{v}|_{\mathsf{L}^2(\Omega)}$$

and the inf–sup-condition

$$0 < \frac{1}{c_{\mathsf{r},-1}} = \inf_{0 \neq \mathbf{v} \in \mathring{\mathsf{H}}(\operatorname{div} 0, \Omega)} \frac{|\operatorname{curl} \mathbf{v}|_{\mathsf{H}^{-1}(\Omega)}}{|\mathbf{v}|_{\mathsf{L}^2(\Omega)}} = \inf_{0 \neq \mathbf{v} \in \mathring{\mathsf{H}}(\operatorname{div} 0, \Omega)} \sup_{0 \neq \mathbf{w} \in \mathring{\mathsf{H}}^1(\Omega)} \frac{\langle \mathbf{v}, \operatorname{curl} \mathbf{w} \rangle_{\mathsf{L}^2(\Omega)}}{|\mathbf{v}|_{\mathsf{L}^2(\Omega)} |\operatorname{Grad} \mathbf{w}|_{\mathcal{L}^2(\Omega)}}$$

holds.

Proof: Let $\mathsf{X}_0 := \mathring{\mathsf{H}}^{m+1}(\Omega)$, $\mathsf{X}_1 := \mathring{\mathsf{H}}^{m}(\Omega)$, $\mathsf{X}_2 := \mathring{\mathsf{H}}^{m-1}(\Omega)$ and

$$\mathrm{A}_0 := \mathring{\operatorname{grad}} : \mathring{\mathsf{H}}^{m+1}(\Omega) \to \mathring{\mathsf{H}}^{m}(\Omega), \quad \mathrm{A}_1 := \mathring{\operatorname{curl}} : \mathring{\mathsf{H}}^{m}(\Omega) \to \mathring{\mathsf{H}}^{m-1}(\Omega).$$

These linear operators are bounded, $R(\mathrm{A}_0) = \mathring{\operatorname{grad}}\, \mathring{\mathsf{H}}^{m+1}(\Omega) = \mathring{\mathsf{H}}^{m}(\Omega) \cap \mathring{\mathsf{H}}(\operatorname{curl} 0, \Omega) = N(\mathrm{A}_1)$ by Lemma 2.24, and $R(\mathrm{A}_1) = \operatorname{curl} \mathring{\mathsf{H}}^{m}(\Omega) = \mathring{\mathsf{H}}^{m-1}(\Omega) \cap \mathring{\mathsf{H}}(\operatorname{div} 0, \Omega)$ by Lemma 2.25. Therefore, $R(\mathrm{A}_1)$ is closed. For the adjoint operators we get

$$\mathrm{A}_0' = -\operatorname{div} = \mathring{\operatorname{grad}}' : \mathsf{H}^{-m}(\Omega) \to \mathsf{H}^{-m-1}(\Omega), \quad \mathrm{A}_1' = \operatorname{curl} = \mathring{\operatorname{curl}}' : \mathsf{H}^{-m+1}(\Omega) \to \mathsf{H}^{-m}(\Omega)$$

and obtain from Lemma 2.14 that

$$\mathsf{H}^{-m}(\operatorname{div} 0, \Omega) = N(\mathrm{A}_0') = R(\mathrm{A}_1') = \operatorname{curl} \mathsf{H}^{-m+1}(\Omega)$$

is closed and

$$\operatorname{curl}^{-1} = (\mathrm{A}_1')^{-1} \in BL\left(R(\mathrm{A}_1'), R(\mathrm{A}_1)'\right) = BL\left(\mathsf{H}^{-m}(\operatorname{div} 0, \Omega), (\mathring{\mathsf{H}}^{m-1}(\Omega) \cap \mathring{\mathsf{H}}(\operatorname{div} 0, \Omega))' \right),$$

which completes the proof for general m. If $m = 1$, we get the assertions about the Friedrichs/Poincaré/Nečas inequality and inf–sup-condition by Lemma 2.16, i.e. (11) and (13). ■

Let us present the corresponding result for the divergence as well.

Corollary 2.31: *Let Ω be additionally topologically trivial and $m \in \mathbb{N}$. Then*

$$\mathsf{H}^{-m}(\Omega) = \operatorname{div} \mathbf{H}^{-m+1}(\Omega) = \operatorname{div} \left(\overset{\circ}{\mathbf{H}}{}^{m-1}(\Omega) \cap \overset{\circ}{\mathbf{H}}(\operatorname{curl} 0, \Omega) \right)'$$

(is closed in $\mathsf{H}^{-m}(\Omega)$) with continuous inverse, i.e. $\operatorname{div}^{-1} \in BL(\mathsf{H}^{-m}(\Omega), (\overset{\circ}{\mathbf{H}}{}^{m-1}(\Omega) \cap \overset{\circ}{\mathbf{H}}(\operatorname{curl} 0, \Omega))')$. *Especially for $m = 1$,*

$$\mathsf{H}^{-1}(\Omega) = \operatorname{div} \mathsf{L}^2(\Omega) = \operatorname{div} \overset{\circ}{\mathbf{H}}(\operatorname{curl} 0, \Omega)$$

(is closed in $\mathsf{H}^{-1}(\Omega)$) with continuous inverse $\operatorname{div}^{-1} \in BL(\mathsf{H}^{-1}(\Omega), \overset{\circ}{\mathbf{H}}(\operatorname{curl} 0, \Omega))$ *and uniquely determined potential in $\overset{\circ}{\mathbf{H}}(\operatorname{curl} 0, \Omega)$. Moreover,*

$$\exists\, c_{\mathsf{d},-1} > 0 \quad \forall\, \mathbf{v} \in \overset{\circ}{\mathbf{H}}(\operatorname{curl} 0, \Omega) \quad |\mathbf{v}|_{\mathbf{L}^2(\Omega)} \le c_{\mathsf{d},-1} |\operatorname{div} \mathbf{v}|_{\mathsf{H}^{-1}(\Omega)} \le c_{\mathsf{d},-1} |\mathbf{v}|_{\mathbf{L}^2(\Omega)}$$

and the inf–sup-condition

$$0 < \frac{1}{c_{\mathsf{d},-1}} = \inf_{0 \ne \mathbf{v} \in \overset{\circ}{\mathbf{H}}(\operatorname{curl} 0, \Omega)} \frac{|\operatorname{div} \mathbf{v}|_{\mathsf{H}^{-1}(\Omega)}}{|\mathbf{v}|_{\mathbf{L}^2(\Omega)}} = \inf_{0 \ne \mathbf{v} \in \overset{\circ}{\mathbf{H}}(\operatorname{div} 0, \Omega)} \sup_{0 \ne u \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)} \frac{\langle \mathbf{v}, \operatorname{grad} u \rangle_{\mathbf{L}^2(\Omega)}}{|\mathbf{v}|_{\mathbf{L}^2(\Omega)} |\operatorname{grad} u|_{\mathbf{L}^2(\Omega)}}$$

holds.

Proof: Let $\mathsf{X}_1 := \overset{\circ}{\mathsf{H}}{}^m(\Omega)$, $\mathsf{X}_2 := \overset{\circ}{\mathbf{H}}{}^{m-1}(\Omega)$ and $\mathrm{A}_1 := -\overset{\circ}{\operatorname{grad}} : \overset{\circ}{\mathsf{H}}{}^m(\Omega) \to \overset{\circ}{\mathbf{H}}{}^{m-1}(\Omega)$. A_1 is linear and bounded with $R(\mathrm{A}_1) = \operatorname{grad} \overset{\circ}{\mathsf{H}}{}^m(\Omega) = \overset{\circ}{\mathbf{H}}{}^{m-1}(\Omega) \cap \overset{\circ}{\mathbf{H}}(\operatorname{curl} 0, \Omega)$ by Lemma 2.24. Therefore, $R(\mathrm{A}_1)$ is closed. The adjoint is $\mathrm{A}_1' = \operatorname{div} = -\overset{\circ}{\operatorname{grad}}{}' : \mathbf{H}^{-m+1}(\Omega) \to \mathsf{H}^{-m}(\Omega)$ with closed range $R(\mathrm{A}_1') = \operatorname{div} \mathbf{H}^{-m+1}(\Omega)$ by the closed range theorem. Moreover, $N(\mathrm{A}_1) = \{0\}$. Hence A_1' is surjective as A_1 is injective, i.e.

$$\mathsf{H}^{-m}(\Omega) = N(\mathrm{A}_1)^\circ = R(\mathrm{A}_1') = \operatorname{div} \mathbf{H}^{-m+1}(\Omega).$$

As A_1 is also surjective onto its range, $\mathrm{A}_1' = \operatorname{div} : \mathbf{H}^{-m+1}(\Omega) \to R(\mathrm{A}_1')$ is bijective. By the bounded inverse theorem we get

$$\operatorname{div}^{-1} = (\mathrm{A}_1')^{-1} \in BL\left(R(\mathrm{A}_1'), R(\mathrm{A}_1)'\right) = BL\left(\mathsf{H}^{-m}(\Omega), (\overset{\circ}{\mathbf{H}}{}^{m-1}(\Omega) \cap \overset{\circ}{\mathbf{H}}(\operatorname{curl} 0, \Omega))'\right),$$

which completes the proof for general m. If $m = 1$, we get the assertions about the Friedrichs/Poincaré/Nečas inequality and inf–sup-condition by Lemma 2.16, i.e. (11) and (13). ■

Remark 2.32: The results of the latter three lemmas and corollaries can be formulated equivalently in terms of complexes: Let Ω be additionally topologically trivial. Then the sequence

$$\overset{\circ}{\mathsf{H}}{}^{m+1}(\Omega) \xrightarrow{\overset{\circ}{\operatorname{grad}}} \overset{\circ}{\mathbf{H}}{}^{m}(\Omega) \xrightarrow{\overset{\circ}{\operatorname{curl}}} \overset{\circ}{\mathbf{H}}{}^{m-1}(\Omega) \xrightarrow{\overset{\circ}{\operatorname{div}}} \overset{\circ}{\mathsf{H}}{}^{m-2}(\Omega)$$

and thus also its dual or adjoint sequence

$$\mathsf{H}^{-m-1}(\Omega) \xleftarrow{-\operatorname{div}} \mathbf{H}^{-m}(\Omega) \xleftarrow{\operatorname{curl}} \mathbf{H}^{-m+1}(\Omega) \xleftarrow{-\operatorname{grad}} \mathsf{H}^{-m+2}(\Omega)$$

are closed and exact Banach complexes.

3. The Gradgrad- and divDiv-complexes

So far we have used notations of the form

$$X^{ord}(\Omega) \quad \text{and} \quad Y(L,\Omega) \tag{16}$$

for function spaces, where X is one of the letters C, L, H for spaces of real-valued functions, and one of the corresponding boldface letters $\mathbf{C}$, $\mathbf{L}$, $\mathbf{H}$ for spaces of vector fields, *ord* describes the order of differentiability, Y stands for the symbol $\mathbf{H}$ and L is one of the differential operators div and curl. The modifier $\circ$ on top of X and Y were used for denoting the corresponding spaces of functions with vanishing boundary trace leading to

$$\overset{\circ}{X}{}^{ord}(\Omega) \quad \text{and} \quad \overset{\circ}{Y}(L,\Omega).$$

Another modifier 0 after L in $Y(L,\Omega)$ was used to denote the subspace of functions from $Y(L,\Omega)$ in the kernel of L leading to

$$Y(L0,\Omega).$$

We extend now this notational system by enlarging the possible symbols for X by the calligraphic letters $\mathcal{C}$, $\mathcal{L}$, $\mathcal{H}$ for denoting the corresponding spaces of tensor fields and the possible symbols for Y by H and $\mathcal{H}$ for denoting the corresponding spaces of scalar and tensor fields, and enlarging the possible differential operators L by Gradgrad for scalar fields, Grad, symGrad, devGrad for vector fields, and Curl, symCurl, Div, and divDiv for tensor fields. In addition to the two modifiers $\circ$ and 0 described above, which also make sense for this extended notation system in an obvious way, we introduce two more modifiers $\mathbb{S}$ and $\mathbb{T}$ as subscripts of X and Y for denoting the corresponding spaces of symmetric tensor fields and tensor fields with vanishing matrix trace leading to

$$X^{ord}_{\mathbb{S}}(\Omega), \quad Y_{\mathbb{S}}(\Omega) \quad \text{and} \quad X^{ord}_{\mathbb{T}}(\Omega), \quad Y_{\mathbb{T}}(L,\Omega),$$

respectively. The meaning of the use of any combination of these modifiers $\circ$, $\mathbb{S}$, $\mathbb{T}$ for X and Y and 0 after L for denoting function spaces is meant in a cumulative sense. Finally, the symbol L modified by any combination of $\circ$, $\mathbb{S}$, $\mathbb{T}$ denotes the restriction of a differential operator L to the corresponding subspace of $Y(L,\Omega)$ described by the same combination of these modifiers. In other words, the restricted differential operator inherits the modifiers from the function space.

To make this notational system for function spaces and operators more transparent, we present some examples: the following spaces of tensor fields:

$$\overset{\circ}{\mathcal{C}}{}^{\infty}(\Omega),\ \mathcal{L}^2(\Omega),\ \mathcal{H}^m(\Omega),\ \overset{\circ}{\mathcal{H}}{}^m(\Omega),\ \mathcal{H}^{-m}(\Omega),\ \mathcal{H}(\mathrm{Curl},\Omega),\ \mathcal{H}(\mathrm{Curl}\,0,\Omega),\ \overset{\circ}{\mathcal{H}}(\mathrm{Curl},\Omega),\ \overset{\circ}{\mathcal{H}}(\mathrm{Div},\Omega)$$

are counterparts of the corresponding spaces of vector fields $\overset{\circ}{\mathbf{C}}{}^{\infty}(\Omega)$, $\mathbf{L}^2(\Omega)$, $\mathbf{H}^m(\Omega)$, $\overset{\circ}{\mathbf{H}}{}^m(\Omega)$, $\mathbf{H}^{-m}(\Omega)$, $\mathbf{H}(\mathrm{curl},\Omega)$, $\mathbf{H}(\mathrm{curl}\,0,\Omega)$, $\overset{\circ}{\mathbf{H}}(\mathrm{curl},\Omega)$, $\overset{\circ}{\mathbf{H}}(\mathrm{div},\Omega)$. Additionally, we will need spaces allowing for a deviatoric gradient, a symmetric rotation, and a double divergence, i.e.

$$\begin{aligned}
\mathbf{H}(\mathrm{devGrad},\Omega) &:= \left\{\mathbf{v} \in \mathbf{L}^2(\Omega) : \mathrm{devGrad}\,\mathbf{v} \in \mathcal{L}^2(\Omega)\right\}, & \mathbf{H}(\mathrm{devGrad}\,0,\Omega) &:= \left\{\mathbf{v} \in \mathbf{L}^2(\Omega) : \mathrm{devGrad}\,\mathbf{v} = 0\right\},\\
\mathcal{H}(\mathrm{symCurl},\Omega) &:= \left\{\mathbf{E} \in \mathcal{L}^2(\Omega) : \mathrm{symCurl}\,\mathbf{E} \in \mathcal{L}^2(\Omega)\right\}, & \mathcal{H}(\mathrm{symCurl}\,0,\Omega) &:= \left\{\mathbf{E} \in \mathcal{L}^2(\Omega) : \mathrm{symCurl}\,\mathbf{E} = 0\right\},\\
\mathcal{H}(\mathrm{divDiv},\Omega) &:= \left\{\mathbf{M} \in \mathcal{L}^2(\Omega) : \mathrm{divDiv}\,\mathbf{M} \in \mathsf{L}^2(\Omega)\right\}, & \mathcal{H}(\mathrm{divDiv}\,0,\Omega) &:= \left\{\mathbf{M} \in \mathcal{L}^2(\Omega) : \mathrm{divDiv}\,\mathbf{M} = 0\right\}.
\end{aligned}$$

We will use the following spaces of symmetric tensor fields:

$$\mathcal{L}^2_{\mathbb{S}}(\Omega) := \{\mathbf{M} \in \mathcal{L}^2(\Omega) \,:\, \mathbf{M}^\top = \mathbf{M}\}, \quad \mathcal{H}_{\mathbb{S}}(\mathrm{divDiv},\Omega) = \mathcal{H}(\mathrm{divDiv},\Omega) \cap \mathcal{L}^2_{\mathbb{S}}(\Omega)$$

and

$$\mathring{\mathcal{H}}^1_{\mathbb{S}}(\Omega) := \overline{\mathring{\mathcal{C}}^\infty(\Omega) \cap \mathcal{L}^2_{\mathbb{S}}(\Omega)}^{\mathcal{H}^1(\Omega)}, \quad \mathring{\mathcal{H}}_{\mathbb{S}}(\mathrm{Curl}, \Omega) := \overline{\mathring{\mathcal{C}}^\infty(\Omega) \cap \mathcal{L}^2_{\mathbb{S}}(\Omega)}^{\mathcal{H}(\mathrm{Curl},\Omega)},$$

as well as spaces of tensor fields with vanishing matrix trace

$$\mathcal{L}^2_{\mathbb{T}}(\Omega) := \{\mathbf{E} \in \mathcal{L}^2(\Omega) \,:\, \mathrm{tr}\,\mathbf{E} = 0\}, \quad \mathcal{H}^1_{\mathbb{T}}(\Omega) = \mathcal{H}^1(\Omega) \cap \mathcal{L}^2_{\mathbb{T}}(\Omega)$$

and

$$\mathring{\mathcal{H}}^1_{\mathbb{T}}(\Omega) := \overline{\mathring{\mathcal{C}}^\infty(\Omega) \cap \mathcal{L}^2_{\mathbb{T}}(\Omega)}^{\mathcal{H}^1(\Omega)}, \quad \mathring{\mathcal{H}}_{\mathbb{T}}(\mathrm{Div}, \Omega) := \overline{\mathring{\mathcal{C}}^\infty(\Omega) \cap \mathcal{L}^2_{\mathbb{T}}(\Omega)}^{\mathcal{H}(\mathrm{Div},\Omega)}.$$

Of particular interest are the differential operators

$$\mathrm{Grad}\mathring{\mathrm{g}}\mathrm{rad}, \quad \mathrm{Cu}\mathring{\mathrm{r}}\mathrm{l}_{\mathbb{S}}, \quad \mathrm{D}\mathring{\mathrm{i}}\mathrm{v}_{\mathbb{T}}, \quad \mathrm{divDiv}_{\mathbb{S}}, \quad \mathrm{symCurl}_{\mathbb{T}}$$

which are the restrictions of the differential operators Gradgrad, Curl, Div, divDiv, symCurl to the spaces $\mathring{\mathsf{H}}(\mathrm{Gradgrad}, \Omega)$, $\mathring{\mathcal{H}}_{\mathbb{S}}(\mathrm{Curl}, \Omega)$, $\mathring{\mathcal{H}}_{\mathbb{T}}(\mathrm{Div}, \Omega)$, $\mathcal{H}_{\mathbb{S}}(\mathrm{divDiv}, \Omega)$, $\mathcal{H}_{\mathbb{T}}(\mathrm{symCurl}, \Omega)$, respectively.

We note that

$$\mathring{\mathcal{H}}^1_{\mathbb{S}}(\Omega) = \mathrm{sym}\,\mathring{\mathcal{H}}^1(\Omega) = \mathring{\mathcal{H}}^1(\Omega) \cap \mathcal{L}^2_{\mathbb{S}}(\Omega), \quad \mathring{\mathcal{H}}^1_{\mathbb{T}}(\Omega) = \mathrm{dev}\,\mathring{\mathcal{H}}^1(\Omega) = \mathring{\mathcal{H}}^1(\Omega) \cap \mathcal{L}^2_{\mathbb{T}}(\Omega),$$

but generally only

$$\mathring{\mathcal{H}}_{\mathbb{S}}(\mathrm{Curl}, \Omega) \subset \mathring{\mathcal{H}}(\mathrm{Curl}, \Omega) \cap \mathcal{L}^2_{\mathbb{S}}(\Omega), \quad \mathring{\mathcal{H}}_{\mathbb{T}}(\mathrm{Div}, \Omega) \subset \mathring{\mathcal{H}}(\mathrm{Div}, \Omega) \cap \mathcal{L}^2_{\mathbb{T}}(\Omega), \quad \ldots .$$

Let us also mention that trivially

$$\mathrm{devGrad}\,\mathsf{H}(\mathrm{devGrad}, \Omega) \subset \mathcal{L}^2_{\mathbb{T}}(\Omega), \quad \mathrm{symCurl}\,\mathcal{H}(\mathrm{symCurl}, \Omega) \subset \mathcal{L}^2_{\mathbb{S}}(\Omega)$$

hold. This can be seen as follows. Pick $\mathbf{v} \in \mathsf{H}(\mathrm{devGrad}, \Omega)$ with $\mathbf{E} := \mathrm{devGrad}\,\mathbf{v}$ and $\mathbf{N} \in \mathcal{H}(\mathrm{symCurl}, \Omega)$ with $\mathbf{M} := \mathrm{symCurl}\,\mathbf{N}$. Then for all $\varphi \in \mathring{\mathsf{C}}^\infty(\Omega)$ and $\mathbf{\Phi} \in \mathring{\mathcal{C}}^\infty(\Omega)$

$$\langle \mathrm{tr}\,\mathbf{E}, \varphi \rangle_{\mathsf{L}^2(\Omega)} = \langle \mathbf{E}, \varphi\,\mathbf{I} \rangle_{\mathcal{L}^2(\Omega)} = -\langle \mathbf{v}, \mathrm{Div}\,\mathrm{dev}\,\varphi\,\mathbf{I} \rangle_{\mathcal{L}^2(\Omega)} = 0,$$

$$\langle \mathrm{skw}\,\mathbf{M}, \mathbf{\Phi} \rangle_{\mathcal{L}^2(\Omega)} = \langle \mathbf{M}, \mathrm{skw}\,\mathbf{\Phi} \rangle_{\mathcal{L}^2(\Omega)} = \langle \mathbf{N}, \mathrm{Curl}\,\mathrm{sym}\,\mathrm{skw}\,\mathbf{\Phi} \rangle_{\mathcal{L}^2(\Omega)} = 0.$$

Before we proceed we need a few technical lemmas.

Lemma 3.1: *For any distributional vector field* $\mathbf{v}$ *it holds for* $i, j, k = 1, \ldots, 3$

$$\partial_k(\mathrm{Grad}\,\mathbf{v})_{ij} = \begin{cases} \partial_k(\mathrm{devGrad}\,\mathbf{v})_{ij} & \text{if } i \neq j, \\ \partial_j(\mathrm{devGrad}\,\mathbf{v})_{ik} & \text{if } i \neq k, \\ \dfrac{3}{2}\,\partial_i(\mathrm{devGrad}\,\mathbf{v})_{ii} + \dfrac{1}{2}\displaystyle\sum_{l \neq i} \partial_l(\mathrm{devGrad}\,\mathbf{v})_{li} & \text{if } i = j = k. \end{cases}$$

Proof: Let $\boldsymbol{\phi} \in \mathring{\mathbf{C}}^\infty(\mathbb{R}^3)$ be a vector field. We want to express the second derivatives of $\boldsymbol{\phi}$ by the derivatives of the deviatoric part of the Jacobian, i.e. of $\mathrm{devGrad}\,\boldsymbol{\phi}$. Recall that we have $\mathrm{dev}\,\mathbf{E} = \mathbf{E} - \frac{1}{3}(\mathrm{tr}\,\mathbf{E})\,\mathbf{I}$ for a tensor $\mathbf{E}$. Hence $\mathrm{devGrad}\,\boldsymbol{\phi}$ coincides with $\mathrm{Grad}\,\boldsymbol{\phi}$ outside the diagonal entries, i.e. we observe $(\mathrm{Grad}\,\boldsymbol{\phi})_{ij} = (\mathrm{devGrad}\,\boldsymbol{\phi})_{ij}$ for $i \neq j$. Hence, looking at second derivatives, we see immediately

$$\partial_k\,\partial_j\,\boldsymbol{\phi}_i = \partial_k(\mathrm{Grad}\,\boldsymbol{\phi})_{ij} = \partial_k(\mathrm{devGrad}\,\boldsymbol{\phi})_{ij} \quad \text{for } i \neq j,$$

$$\partial_k\,\partial_j\,\boldsymbol{\phi}_i = \partial_j\,\partial_k\,\boldsymbol{\phi}_i = \partial_j(\operatorname{Grad}\boldsymbol{\phi})_{ik} = \partial_j(\operatorname{devGrad}\boldsymbol{\phi})_{ik} \quad \text{for } i \neq k.$$

Thus it remains to represent $\partial_i^2\,\boldsymbol{\phi}_i$ by the derivatives of $\operatorname{devGrad}\boldsymbol{\phi}$. By

$$\partial_i^2\,\boldsymbol{\phi}_i = \partial_i(\operatorname{Grad}\boldsymbol{\phi})_{ii} = \partial_i(\operatorname{devGrad}\boldsymbol{\phi})_{ii} + \tfrac{1}{3}\,\partial_i\operatorname{div}\boldsymbol{\phi}$$

we get

$$\frac{2}{3}\,\partial_i^2\,\boldsymbol{\phi}_i = \partial_i(\operatorname{devGrad}\boldsymbol{\phi})_{ii} + \frac{1}{3}\sum_{l\neq i}\partial_i\,\partial_l\,\boldsymbol{\phi}_l = \partial_i(\operatorname{devGrad}\boldsymbol{\phi})_{ii} + \frac{1}{3}\sum_{l\neq i}\partial_l(\operatorname{devGrad}\boldsymbol{\phi})_{li},$$

yielding the stated result for test vector fields. Testing extends the formulas to distributions, which finishes the proof. ∎

We note that the latter trick is similar to the well-known fact that second derivatives of a vector field can always be written as derivatives of the symmetric gradient of the vector field, leading by Nečas estimate to Korn's second and first inequalities. We will now do the same for the operator devGrad.

Lemma 3.2: *It holds:*

(i) *There exists $c > 0$, such that for all vector fields $\mathbf{v} \in \mathsf{H}^1(\Omega)$*

$$|\operatorname{Grad}\mathbf{v}|_{\mathcal{L}^2(\Omega)} \leq c\,\Big(|\mathbf{v}|_{\mathsf{L}^2(\Omega)} + |\operatorname{devGrad}\mathbf{v}|_{\mathcal{L}^2(\Omega)}\Big).$$

(ii) $\mathsf{H}(\operatorname{devGrad},\Omega) = \mathsf{H}^1(\Omega)$.

(iii) *For* $\operatorname{devGrad} : \mathsf{H}(\operatorname{devGrad},\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathcal{L}^2_{\mathbb{T}}(\Omega)$ *it holds*

$$D(\operatorname{devGrad}) = \mathsf{H}(\operatorname{devGrad},\Omega) = \mathsf{H}^1(\Omega),$$

and the kernel of devGrad *equals the space of (global) shape functions of the lowest order Raviart–Thomas elements, i.e.*

$$N(\operatorname{devGrad}) = \mathsf{H}(\operatorname{devGrad}0,\Omega) = \mathsf{RT}_0 := \{\mathbf{p} \,:\, \mathbf{p}(\mathbf{x}) = a\,\mathbf{x} + \mathbf{b},\ a \in \mathbb{R},\ \mathbf{b} \in \mathbb{R}^3\},$$

which dimension is $\dim\mathsf{RT}_0 = 4$.

(iv) *There exists $c > 0$, such that for all vector fields $\mathbf{v} \in \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$*

$$|\mathbf{v}|_{\mathsf{H}^1(\Omega)} \leq c\,|\operatorname{devGrad}\mathbf{v}|_{\mathcal{L}^2(\Omega)}.$$

Proof: Let $\mathbf{v} \in \mathsf{H}^1(\Omega)$. By the latter lemma and Nečas estimate, i.e.

$$\exists\, c > 0 \quad \forall\, u \in \mathsf{L}^2(\Omega) \quad c\,|u|_{\mathsf{L}^2(\Omega)} \leq |\operatorname{grad} u|_{\mathsf{H}^{-1}(\Omega)} + |u|_{\mathsf{H}^{-1}(\Omega)} \leq (\sqrt{3}+1)|u|_{\mathsf{L}^2(\Omega)},$$

we get

$$\begin{aligned}
|\operatorname{Grad}\mathbf{v}|_{\mathcal{L}^2(\Omega)} &\leq c\left(\sum_{k=1}^{3}|\partial_k\operatorname{Grad}\mathbf{v}|_{\mathcal{H}^{-1}(\Omega)} + |\operatorname{Grad}\mathbf{v}|_{\mathcal{H}^{-1}(\Omega)}\right)\\
&\leq c\left(\sum_{k=1}^{3}|\partial_k\operatorname{devGrad}\mathbf{v}|_{\mathcal{H}^{-1}(\Omega)} + |\operatorname{Grad}\mathbf{v}|_{\mathcal{H}^{-1}(\Omega)}\right)
\end{aligned}$$

$$\leq c\left(|\operatorname{devGrad}\mathbf{v}|_{\mathcal{L}^2(\Omega)} + |\mathbf{v}|_{\mathsf{L}^2(\Omega)}\right),$$

which shows (i). As Ω has the segment property and by standard mollification we obtain that restrictions of $\overset{\circ}{\mathsf{C}}^{\infty}(\mathbb{R}^3)$-vector fields are dense in $\mathsf{H}(\operatorname{devGrad},\Omega)$. Especially $\mathsf{H}^1(\Omega)$ is dense in $\mathsf{H}(\operatorname{devGrad},\Omega)$. Let $\mathbf{v}\in\mathsf{H}(\operatorname{devGrad},\Omega)$ and $(\mathbf{v}_n)\subset\mathsf{H}^1(\Omega)$ with $\mathbf{v}_n\to\mathbf{v}$ in $\mathsf{H}(\operatorname{devGrad},\Omega)$. By (i) $(\mathbf{v}_n)$ is a Cauchy sequence in $\mathsf{H}^1(\Omega)$ converging to $\mathbf{v}$ in $\mathsf{H}^1(\Omega)$, which proves $\mathbf{v}\in\mathsf{H}^1(\Omega)$ and hence (ii). For $\mathbf{p}\in\mathsf{RT}_0$ it holds $\operatorname{devGrad}\mathbf{p}=a\operatorname{dev}\mathbf{I}=0$. Let $\operatorname{devGrad}\mathbf{v}=0$ for some vector field $\mathbf{v}\in\mathsf{H}(\operatorname{devGrad},\Omega)=\mathsf{H}^1(\Omega)$. By Lemma 3.1 we get $\partial_k\operatorname{Grad}\mathbf{v}=0$ for all $k=1,\ldots,3$, and therefore $\mathbf{v}(\mathbf{x})=\mathbf{A}\,\mathbf{x}+\mathbf{b}$ for some matrix $\mathbf{A}\in\mathbb{R}^{3\times3}$ and vector $\mathbf{b}\in\mathbb{R}^3$. Then $0=\operatorname{devGrad}\mathbf{v}=\operatorname{dev}\mathbf{A}$, if and only if $\mathbf{A}=\frac{1}{3}(\operatorname{tr}\mathbf{A})\,\mathbf{I}$, which shows (iii). If (iv) was wrong, there exists a sequence $(\mathbf{v}_n)\subset\mathsf{H}^1(\Omega)\cap\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$ with $|\mathbf{v}_n|_{\mathsf{H}^1(\Omega)}=1$ and $\operatorname{devGrad}\mathbf{v}_n\to0$. As $(\mathbf{v}_n)$ is bounded in $\mathsf{H}^1(\Omega)$, by Rellich's selection theorem there exists a subsequence, again denoted by $(\mathbf{v}_n)$, and some $\mathbf{v}\in\mathsf{L}^2(\Omega)$ with $\mathbf{v}_n\to\mathbf{v}$ in $\mathsf{L}^2(\Omega)$. By (i), $(\mathbf{v}_n)$ is a Cauchy sequence in $\mathsf{H}^1(\Omega)$. Hence $\mathbf{v}_n\to\mathbf{v}$ in $\mathsf{H}^1(\Omega)$ and $\mathbf{v}\in\mathsf{H}^1(\Omega)\cap\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$. As $0\leftarrow\operatorname{devGrad}\mathbf{v}_n\to\operatorname{devGrad}\mathbf{v}$, we have by (iii) $\mathbf{v}\in\mathsf{RT}_0\cap\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}=\{0\}$, a contradiction to $1=|\mathbf{v}_n|_{\mathsf{H}^1(\Omega)}\to0$. The proof is complete. ■

We recall the following well-known result for the spaces

$$\mathsf{H}(\operatorname{Gradgrad},\Omega):=\{u\in\mathsf{L}^2(\Omega)\;:\;\operatorname{Gradgrad}u\in\mathcal{L}^2(\Omega)\},\quad \overset{\circ}{\mathsf{H}}(\operatorname{Gradgrad},\Omega):=\overline{\overset{\circ}{\mathsf{C}}^{\infty}(\Omega)}^{\mathsf{H}(\operatorname{Gradgrad},\Omega)}.$$

Lemma 3.3: *It holds* $\overset{\circ}{\mathsf{H}}(\operatorname{Gradgrad},\Omega)=\overset{\circ}{\mathsf{H}}{}^2(\Omega)$ *and* $\overset{\circ}{\mathsf{H}}(\operatorname{Gradgrad}0,\Omega)=\{0\}$, *and there exists $c>0$ such that for all* $u\in\overset{\circ}{\mathsf{H}}{}^2(\Omega)$

$$|u|_{\mathsf{H}^2(\Omega)}\leq c\,|\operatorname{Gradgrad}u|_{\mathcal{L}^2(\Omega)}=c\,|\Delta u|_{\mathsf{L}^2(\Omega)},\quad c\leq\sqrt{1+c_{\mathsf{g}}^2(1+c_{\mathsf{g}}^2)}\leq1+c_{\mathsf{g}}^2.$$

By straightforward calculations and standard arguments for distributions, see the Appendix, we get the following.

Lemma 3.4: *It holds:*

(i) $\operatorname{skw}\operatorname{Gradgrad}\mathsf{H}^2(\Omega)=0$, *i.e. Hessians are symmetric.*
(ii) $\operatorname{tr}\operatorname{Curl}\mathcal{H}_{\mathbb{S}}(\operatorname{Curl},\Omega)=0$, *i.e. rotations of symmetric tensors are trace free.*

These formulas extend to distributions as well.

With Lemmas 3.3 and 3.4 let us now consider the linear operators

$$\mathrm{A}_0:=\overset{\circ}{\operatorname{Gradgrad}}:\overset{\circ}{\mathsf{H}}(\operatorname{Gradgrad},\Omega)=\overset{\circ}{\mathsf{H}}{}^2(\Omega)\subset\mathsf{L}^2(\Omega)\longrightarrow\mathcal{L}^2_{\mathbb{S}}(\Omega),\quad u\mapsto\operatorname{Gradgrad}u,\tag{17}$$

$$\mathrm{A}_1:=\overset{\circ}{\operatorname{Curl}}_{\mathbb{S}}:\overset{\circ}{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega)\subset\mathcal{L}^2_{\mathbb{S}}(\Omega)\longrightarrow\mathcal{L}^2_{\mathbb{T}}(\Omega),\quad \mathbf{M}\mapsto\operatorname{Curl}\mathbf{M},\tag{18}$$

$$\mathrm{A}_2:=\overset{\circ}{\operatorname{Div}}_{\mathbb{T}}:\overset{\circ}{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega)\subset\mathcal{L}^2_{\mathbb{T}}(\Omega)\longrightarrow\mathsf{L}^2(\Omega),\quad \mathbf{E}\mapsto\operatorname{Div}\mathbf{E}.\tag{19}$$

These are well and densely defined and closed. Closedness is clear. For densely definedness we look, e.g. at $\overset{\circ}{\operatorname{Curl}}_{\mathbb{S}}$. For $\mathbf{M}\in\mathcal{L}^2_{\mathbb{S}}(\Omega)$ pick $(\mathbf{\Phi}_n)\subset\overset{\circ}{\mathcal{C}}^{\infty}(\Omega)$ with $\mathbf{\Phi}_n\to\mathbf{M}$ in $\mathcal{L}^2(\Omega)$. Then

$$|\mathbf{M}-\operatorname{sym}\mathbf{\Phi}_n|^2_{\mathcal{L}^2(\Omega)}+|\operatorname{skw}\mathbf{\Phi}_n|^2_{\mathcal{L}^2(\Omega)}=|\mathbf{M}-\mathbf{\Phi}_n|^2_{\mathcal{L}^2(\Omega)}\to0,$$

showing $(\operatorname{sym}\boldsymbol{\Phi}_n)\subset\mathring{\mathcal{C}}^\infty(\Omega)\cap\mathcal{L}^2_{\mathbb{S}}(\Omega)\subset\mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega)$ and $\operatorname{sym}\boldsymbol{\Phi}_n\to\mathbf{M}$ in $\mathcal{L}^2_{\mathbb{S}}(\Omega)$. By Lemma 3.3 the kernels are

$$N(\mathring{\operatorname{Gradgrad}})=\mathring{\mathsf{H}}(\operatorname{Gradgrad}0,\Omega)=\{0\},\quad N(\mathring{\operatorname{Curl}}_{\mathbb{S}})=\mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl}0,\Omega),$$
$$N(\mathring{\operatorname{Div}}_{\mathbb{T}})=\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}0,\Omega).$$

Lemma 3.5: *The adjoints of* (17), (18), (19) *are*

$$\mathrm{A}_0^*=(\mathring{\operatorname{Gradgrad}})^*=\operatorname{divDiv}_{\mathbb{S}}:\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega)\subset\mathcal{L}^2_{\mathbb{S}}(\Omega)\longrightarrow\mathsf{L}^2(\Omega),\quad\mathbf{M}\mapsto\operatorname{divDiv}\mathbf{M},$$
$$\mathrm{A}_1^*=(\mathring{\operatorname{Curl}}_{\mathbb{S}})^*=\operatorname{symCurl}_{\mathbb{T}}:\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega)\subset\mathcal{L}^2_{\mathbb{T}}(\Omega)\longrightarrow\mathcal{L}^2_{\mathbb{S}}(\Omega),\quad\mathbf{E}\mapsto\operatorname{symCurl}\mathbf{E},$$
$$\mathrm{A}_2^*=(\mathring{\operatorname{Div}}_{\mathbb{T}})^*=-\operatorname{devGrad}:\mathbf{H}(\operatorname{devGrad},\Omega)=\mathbf{H}^1(\Omega)\subset\mathbf{L}^2(\Omega)\longrightarrow\mathcal{L}^2_{\mathbb{T}}(\Omega),\ \mathbf{v}\mapsto-\operatorname{devGrad}\mathbf{v}$$

with kernels

$$N(\operatorname{divDiv}_{\mathbb{S}})=\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}0,\Omega),\quad N(\operatorname{symCurl}_{\mathbb{T}})=\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}0,\Omega),\quad N(\operatorname{devGrad})=\mathsf{RT}_0.$$

Proof: We have $\mathbf{M}\in D((\mathring{\operatorname{Gradgrad}})^*)\subset\mathcal{L}^2_{\mathbb{S}}(\Omega)$ and $(\mathring{\operatorname{Gradgrad}})^*\mathbf{M}=u\in\mathsf{L}^2(\Omega)$, if and only if $\mathbf{M}\in\mathcal{L}^2_{\mathbb{S}}(\Omega)$ and there exists $u\in\mathsf{L}^2(\Omega)$, such that

$$\begin{aligned}&\forall\,\varphi\in D(\mathring{\operatorname{Gradgrad}})=\mathring{\mathsf{H}}^2(\Omega)\langle\operatorname{Gradgrad}\varphi,\mathbf{M}\rangle_{\mathcal{L}^2_{\mathbb{S}}(\Omega)}=\langle\varphi,u\rangle_{\mathsf{L}^2(\Omega)}\\ \Leftrightarrow&\forall\,\varphi\in\mathring{\mathsf{C}}^\infty(\Omega)\langle\operatorname{Gradgrad}\varphi,\mathbf{M}\rangle_{\mathcal{L}^2(\Omega)}=\langle\varphi,u\rangle_{\mathsf{L}^2(\Omega)},\end{aligned}$$

if and only if $\mathbf{M}\in\mathcal{H}(\operatorname{divDiv},\Omega)\cap\mathcal{L}^2_{\mathbb{S}}(\Omega)=\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega)$ and $\operatorname{divDiv}\mathbf{M}=u$. Moreover, we observe that $\mathbf{E}\in D((\mathring{\operatorname{Curl}}_{\mathbb{S}})^*)\subset\mathcal{L}^2_{\mathbb{T}}(\Omega)$ and $(\mathring{\operatorname{Curl}}_{\mathbb{S}})^*\mathbf{E}=\mathbf{M}\in\mathcal{L}^2_{\mathbb{S}}(\Omega)$, if and only if $\mathbf{E}\in\mathcal{L}^2_{\mathbb{T}}(\Omega)$ and there exists $\mathbf{M}\in\mathcal{L}^2_{\mathbb{S}}(\Omega)$, such that (note $\operatorname{sym}^2=\operatorname{sym}$)

$$\begin{aligned}&\forall\,\boldsymbol{\Phi}\in D(\mathring{\operatorname{Curl}}_{\mathbb{S}})=\mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega)&&\langle\operatorname{Curl}\boldsymbol{\Phi},\mathbf{E}\rangle_{\mathcal{L}^2_{\mathbb{T}}(\Omega)}=\langle\boldsymbol{\Phi},\mathbf{M}\rangle_{\mathcal{L}^2_{\mathbb{S}}(\Omega)}\\ \Leftrightarrow\quad&\forall\,\boldsymbol{\Phi}\in\mathring{\mathcal{C}}^\infty(\Omega)\cap\mathcal{L}^2_{\mathbb{S}}(\Omega)&&\langle\operatorname{Curl}\operatorname{sym}\boldsymbol{\Phi},\mathbf{E}\rangle_{\mathcal{L}^2(\Omega)}=\langle\operatorname{sym}\boldsymbol{\Phi},\mathbf{M}\rangle_{\mathcal{L}^2(\Omega)}\\ \Leftrightarrow\quad&\forall\,\boldsymbol{\Phi}\in\mathring{\mathcal{C}}^\infty(\Omega)&&\langle\operatorname{Curl}\operatorname{sym}\boldsymbol{\Phi},\mathbf{E}\rangle_{\mathcal{L}^2(\Omega)}=\langle\operatorname{sym}\boldsymbol{\Phi},\mathbf{M}\rangle_{\mathcal{L}^2(\Omega)}\\ \Leftrightarrow\quad&\forall\,\boldsymbol{\Phi}\in\mathring{\mathcal{C}}^\infty(\Omega)&&\langle\operatorname{Curl}\operatorname{sym}\boldsymbol{\Phi},\mathbf{E}\rangle_{\mathcal{L}^2(\Omega)}=\langle\boldsymbol{\Phi},\mathbf{M}\rangle_{\mathcal{L}^2(\Omega)},\end{aligned}$$

if and only if $\mathbf{E}\in\mathcal{H}(\operatorname{symCurl},\Omega)\cap\mathcal{L}^2_{\mathbb{T}}(\Omega)=\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega)$ and $\operatorname{symCurl}\mathbf{E}=\mathbf{M}$. Similarly, we see that $\mathbf{v}\in D((\mathring{\operatorname{Div}}_{\mathbb{T}})^*)\subset\mathbf{L}^2(\Omega)$ and $(\mathring{\operatorname{Div}}_{\mathbb{T}})^*\mathbf{v}=\mathbf{E}\in\mathcal{L}^2_{\mathbb{T}}(\Omega)$, if and only if $\mathbf{v}\in\mathbf{L}^2(\Omega)$ and there exists $\mathbf{E}\in\mathcal{L}^2_{\mathbb{T}}(\Omega)$, such that (note $\operatorname{dev}^2=\operatorname{dev}$)

$$\begin{aligned}&\forall\,\boldsymbol{\Phi}\in D(\mathring{\operatorname{Div}}_{\mathbb{S}})=\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega)&&\langle\operatorname{Div}\boldsymbol{\Phi},\mathbf{v}\rangle_{\mathbf{L}^2(\Omega)}=\langle\boldsymbol{\Phi},\mathbf{E}\rangle_{\mathcal{L}^2_{\mathbb{T}}(\Omega)}\\ \Leftrightarrow\quad&\forall\,\boldsymbol{\Phi}\in\mathring{\mathcal{C}}^\infty(\Omega)\cap\mathcal{L}^2_{\mathbb{T}}(\Omega)&&\langle\operatorname{Div}\operatorname{dev}\boldsymbol{\Phi},\mathbf{v}\rangle_{\mathbf{L}^2(\Omega)}=\langle\operatorname{dev}\boldsymbol{\Phi},\mathbf{E}\rangle_{\mathcal{L}^2(\Omega)}\\ \Leftrightarrow\quad&\forall\,\boldsymbol{\Phi}\in\mathring{\mathcal{C}}^\infty(\Omega)&&\langle\operatorname{Div}\operatorname{dev}\boldsymbol{\Phi},\mathbf{v}\rangle_{\mathbf{L}^2(\Omega)}=\langle\operatorname{dev}\boldsymbol{\Phi},\mathbf{E}\rangle_{\mathcal{L}^2(\Omega)}\\ \Leftrightarrow\quad&\forall\,\boldsymbol{\Phi}\in\mathring{\mathcal{C}}^\infty(\Omega)&&\langle\operatorname{Div}\operatorname{dev}\boldsymbol{\Phi},\mathbf{v}\rangle_{\mathbf{L}^2(\Omega)}=\langle\boldsymbol{\Phi},\mathbf{E}\rangle_{\mathcal{L}^2(\Omega)},\end{aligned}$$

if and only if $\mathbf{v}\in\mathbf{H}(\operatorname{devGrad},\Omega)=\mathbf{H}^1(\Omega)$ and $-\operatorname{devGrad}\mathbf{v}=\mathbf{E}$ using Lemma 3.2. Lemma 3.2 also shows $N(\operatorname{devGrad})=\mathbf{H}(\operatorname{devGrad}0,\Omega)=\mathsf{RT}_0$, completing the proof. ∎

Remark 3.6: Note that, e.g. the second-order operator $\mathring{\operatorname{Gradgrad}}$ is 'one' operator and not a composition of the two first order operators $\mathring{\operatorname{Grad}}$ and $\mathring{\operatorname{grad}}$. Similarly the operator $\operatorname{divDiv}_{\mathbb{S}}$, $\operatorname{symCurl}_{\mathbb{T}}$, resp. devGrad has to be understood as 'one' operator.

We observe the following complex properties for A_0, A_1, A_2, and A_0^*, A_1^*, A_2^*.

Lemma 3.7: *It holds*

$$\mathring{\operatorname{Curl}}_{\mathbb{S}}\mathring{\operatorname{Gradgrad}} = 0, \quad \mathring{\operatorname{Div}}_{\mathbb{T}}\mathring{\operatorname{Curl}}_{\mathbb{S}} = 0, \quad \operatorname{divDiv}_{\mathbb{S}}\operatorname{symCurl}_{\mathbb{T}} = 0, \quad \operatorname{symCurl}_{\mathbb{T}} \operatorname{devGrad} = 0,$$

i.e.

$$R(\mathring{\operatorname{Gradgrad}}) \subset N(\mathring{\operatorname{Curl}}_{\mathbb{S}}), \quad R(\operatorname{symCurl}_{\mathbb{T}}) \subset N(\operatorname{divDiv}_{\mathbb{S}}),$$
$$R(\mathring{\operatorname{Curl}}_{\mathbb{S}}) \subset N(\mathring{\operatorname{Div}}_{\mathbb{T}}), \quad R(\operatorname{devGrad}) \subset N(\operatorname{symCurl}_{\mathbb{T}}).$$

Proof: For $\mathbf{E} = \operatorname{Curl}\mathbf{M} \in R(\mathring{\operatorname{Curl}}_{\mathbb{S}})$ with $\mathbf{M} \in D(\mathring{\operatorname{Curl}}_{\mathbb{S}})$ there exists a sequence $(\mathbf{M}_n) \subset \mathring{\mathcal{C}}^{\infty}(\Omega) \cap \boldsymbol{\mathcal{L}}^2_{\mathbb{S}}(\Omega)$ such that $\mathbf{M}_n \to \mathbf{M}$ in the graph norm of $D(\mathring{\operatorname{Curl}}_{\mathbb{S}})$. As

$$\operatorname{Curl}\left(\mathring{\mathcal{C}}^{\infty}(\Omega) \cap \boldsymbol{\mathcal{L}}^2_{\mathbb{S}}(\Omega)\right) \subset \mathring{\mathcal{C}}^{\infty}(\Omega) \cap \boldsymbol{\mathcal{L}}^2_{\mathbb{T}}(\Omega) \cap \boldsymbol{\mathcal{H}}(\operatorname{Div} 0, \Omega) \subset N(\mathring{\operatorname{Div}}_{\mathbb{T}})$$

we have $\mathbf{E} \in N(\mathring{\operatorname{Div}}_{\mathbb{T}})$ since $\mathbf{E} \leftarrow \operatorname{Curl}\mathbf{M}_n \in N(\mathring{\operatorname{Div}}_{\mathbb{T}})$. Hence $R(\mathring{\operatorname{Curl}}_{\mathbb{S}}) \subset N(\mathring{\operatorname{Div}}_{\mathbb{T}})$, i.e. $\mathring{\operatorname{Div}}_{\mathbb{T}}\mathring{\operatorname{Curl}}_{\mathbb{S}} = 0$ and for the adjoints we have $\operatorname{symCurl}_{\mathbb{T}} \operatorname{devGrad} = 0$. Analogously, we see the other two inclusions. ∎

Remark 3.8: The latter considerations show that the sequence

$$\{0\} \xrightarrow{\;0\;} \mathring{\mathsf{H}}^2(\Omega) \xrightarrow{\mathring{\operatorname{Gradgrad}}} \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \xrightarrow{\mathring{\operatorname{Curl}}_{\mathbb{S}}} \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div}, \Omega) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}} \mathsf{L}^2(\Omega) \xrightarrow{\pi_{\mathsf{RT}_0}} \mathsf{RT}_0$$

and thus also its dual or adjoint sequence

$$\{0\} \xleftarrow{\;0\;} \mathsf{L}^2(\Omega) \xleftarrow{\operatorname{divDiv}_{\mathbb{S}}} \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) \xleftarrow{\operatorname{symCurl}_{\mathbb{T}}} \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \xleftarrow{-\operatorname{devGrad}} \mathsf{H}^1(\Omega) \xleftarrow{\iota_{\mathsf{RT}_0}} \mathsf{RT}_0$$

are Hilbert complexes. Here $\pi_{\mathsf{RT}_0} : \mathsf{L}^2(\Omega) \to \mathsf{RT}_0$ denotes the orthogonal projector onto RT_0 with adjoint $\pi^*_{\mathsf{RT}_0} = \iota_{\mathsf{RT}_0} : \mathsf{RT}_0 \to \mathsf{L}^2(\Omega)$, the canonical embedding. The first complex might be called $\mathring{\operatorname{Gradgrad}}$-complex and the second one divDiv-complex.

3.1. Topologically trivial domains

We start with a useful lemma, which will be shown in the Appendix, collecting a few differential identities, which will be utilized in the proof of the subsequent main theorem.

Lemma 3.9: *Let u, $\mathbf{v}$, and $\mathbf{E}$ be distributional scalar, vector, and tensor fields. Then*

(i) $2 \operatorname{skw} \operatorname{Grad} \mathbf{v} = \operatorname{spn} \operatorname{curl} \mathbf{v}$,
(ii) $\operatorname{Curl} \operatorname{spn} \mathbf{v} = (\operatorname{div} \mathbf{v})\, \mathbf{I} - (\operatorname{Grad} \mathbf{v})^{\top}$ *and, as a consequence,* $\operatorname{tr} \operatorname{Curl} \operatorname{spn} \mathbf{v} = 2 \operatorname{div} \mathbf{v}$,
(iii) $\operatorname{Div}(u\,\mathbf{I}) = \operatorname{grad} u$ *and* $\operatorname{Curl}(u\,\mathbf{I}) = -\operatorname{spn} \operatorname{grad} u$,
(iv) $2 \operatorname{grad} \operatorname{div} \mathbf{v} = 3 \operatorname{Div}(\operatorname{dev} (\operatorname{Grad} \mathbf{v})^{\top})$,
(v) $\operatorname{skw} \operatorname{Curl} \mathbf{E} = \operatorname{spn} \mathbf{w}$ *and* $\operatorname{Div}(\operatorname{symCurl} \mathbf{E}) = \operatorname{curl} \mathbf{w}$ *with* $2\mathbf{w} = \operatorname{Div} \mathbf{E}^{\top} - \operatorname{grad}(\operatorname{tr} \mathbf{E})$,

(vi) $\operatorname{Div}(\operatorname{spn}\mathbf{v}) = -\operatorname{curl}\mathbf{v}$.

Observe that we already know that $N(\mathring{\operatorname{Gradgrad}}) = \{0\}$ and $N(\operatorname{devGrad}) = \mathsf{RT}_0$. If the topology of the underlying domain is trivial, we will now characterize the remaining kernels and the ranges of the linear operators $\mathring{\operatorname{Gradgrad}}$, $\mathring{\operatorname{Curl}}_{\mathbb{S}}$, $\mathring{\operatorname{Div}}_{\mathbb{T}}$, and devGrad, $\operatorname{symCurl}_{\mathbb{T}}$, $\operatorname{divDiv}_{\mathbb{S}}$.

Theorem 3.10: *Let Ω be additionally topologically trivial. Then*

$$\begin{aligned}
&\text{(i)} && \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl}0,\Omega) = N(\mathring{\operatorname{Curl}}_{\mathbb{S}}) = R(\mathring{\operatorname{Gradgrad}}) = \operatorname{Gradgrad}\mathring{\mathsf{H}}^{2}(\Omega),\\
&\text{(ii)} && \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}0,\Omega) = N(\mathring{\operatorname{Div}}_{\mathbb{T}}) = R(\mathring{\operatorname{Curl}}_{\mathbb{S}}) = \operatorname{Curl}\mathring{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega),\\
&\text{(iii)} && \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} = N(\pi_{\mathsf{RT}_0}) = R(\mathring{\operatorname{Div}}_{\mathbb{T}}) = \operatorname{Div}\mathring{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega),\\
&\text{(iv)} && \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}0,\Omega) = N(\operatorname{symCurl}_{\mathbb{T}}) = R(\operatorname{devGrad}) = \operatorname{devGrad}\mathsf{H}^{1}(\Omega),\\
&\text{(v)} && \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}0,\Omega) = N(\operatorname{divDiv}_{\mathbb{S}}) = R(\operatorname{symCurl}_{\mathbb{T}}) = \operatorname{symCurl}\mathcal{H}^{1}_{\mathbb{T}}(\Omega),\\
&\text{(vi)} && \mathsf{L}^2(\Omega) = N(0) = R(\operatorname{divDiv}_{\mathbb{S}}) = \operatorname{divDiv}\mathcal{H}^{2}_{\mathbb{S}}(\Omega).
\end{aligned}$$

Especially, all latter ranges are closed and admit regular H^1-potentials. The corresponding linear and continuous (regular) potential operators are given by

$$\begin{aligned}
\mathrm{P}_{\mathring{\operatorname{Gradgrad}}} &= \mathrm{P}_{\mathring{\operatorname{grad}}}\mathrm{P}_{\mathring{\operatorname{Grad}}} : \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl}0,\Omega) \longrightarrow \mathring{\mathsf{H}}^{2}(\Omega),\\
\mathrm{P}_{\mathring{\operatorname{Curl}}_{\mathbb{S}}} &= \operatorname{sym}\Big(1 - 2\operatorname{Grad}\mathrm{P}_{\mathring{\operatorname{curl}}}\operatorname{spn}^{-1}\operatorname{skw}\Big)\mathrm{P}_{\mathring{\operatorname{Curl}}} : \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}0,\Omega) \longrightarrow \mathring{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega),\\
\mathrm{P}_{\mathring{\operatorname{Div}}_{\mathbb{T}}} &= \operatorname{dev}\Big(1 + \tfrac{1}{2}\operatorname{Grad}^{\top}\mathrm{P}_{\mathring{\operatorname{div}}}\operatorname{tr}\Big)\mathrm{P}_{\mathring{\operatorname{Div}}} : \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} \longrightarrow \mathring{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega),\\
\mathrm{P}_{\operatorname{devGrad}} &= \operatorname{Grad}^{-1}\Big(1 + \tfrac{1}{2}(\operatorname{grad}^{-1}\operatorname{Div}(\,\cdot\,)^{\top})\,\mathbf{I}\Big) : \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}0,\Omega) \longrightarrow \mathsf{H}^{1}(\Omega),\\
\mathrm{P}_{\operatorname{symCurl}_{\mathbb{T}}} &= \operatorname{dev}\mathrm{P}_{\operatorname{Curl}}\big(1 + \operatorname{spn}\operatorname{curl}^{-1}\operatorname{Div}\big) : \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}0,\Omega) \longrightarrow \mathcal{H}^{1}_{\mathbb{T}}(\Omega),\\
\mathrm{P}_{\operatorname{divDiv}_{\mathbb{S}}} &= \operatorname{sym}\mathrm{P}_{\operatorname{Div}}\mathrm{P}_{\operatorname{div}} : \mathsf{L}^{2}(\Omega) \longrightarrow \mathcal{H}^{2}_{\mathbb{S}}(\Omega).
\end{aligned}$$

Remark 3.11: It holds

$$\mathcal{H}^{1}_{\mathbb{S}}(\Omega) = \operatorname{sym}\mathcal{H}^{1}(\Omega),\quad \mathcal{H}^{1}_{\mathbb{T}}(\Omega) = \operatorname{dev}\mathcal{H}^{1}(\Omega),\quad \mathring{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega) = \operatorname{sym}\mathring{\mathcal{H}}^{1}(\Omega),\quad \mathring{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega) = \operatorname{dev}\mathring{\mathcal{H}}^{1}(\Omega)$$

as, e.g. $\operatorname{dev}\mathcal{H}^1(\Omega) \subset \mathcal{H}^1_{\mathbb{T}}(\Omega) = \operatorname{dev}\mathcal{H}^1_{\mathbb{T}}(\Omega) \subset \operatorname{dev}\mathcal{H}^1(\Omega)$. The same holds for the corresponding spaces of skew-symmetric tensor fields as well. Moreover:

(i) Theorem 3.10 holds also for the other set of canonical boundary conditions, which follows directly from the proof.
(ii) A closer inspection shows that for (iii) and (vi), i.e. $\mathrm{P}_{\mathring{\operatorname{Div}}_{\mathbb{T}}}$ and $\mathrm{P}_{\operatorname{divDiv}_{\mathbb{S}}}$, only the potential operators corresponding to the divergence, i.e. $\mathrm{P}_{\mathring{\operatorname{div}}}$, $\mathrm{P}_{\mathring{\operatorname{Div}}}$, $\mathrm{P}_{\operatorname{Div}}$, $\mathrm{P}_{\operatorname{div}}$, are involved. As Lemma 2.26 does not need any topological assumptions, (iii) and (vi), together with the representations of the potential operators, hold for general topologies as well.

Proof of Theorem* 3.10*: Note that by Lemmas 3.2 (iii), 3.3, and 3.7 all inclusions of the type $R(\ldots) \subset N(\ldots)$ easily follow. Therefore it suffices to show that $N(\ldots)$ is included in the corresponding space appearing at the end of each line in (i)–(vi), which itself is obviously included in $R(\ldots)$. Throughout the proof, we will frequently use the formulas of Lemma 3.9.

ad (i): Let $\mathbf{M} \in \overset{\circ}{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) = N(\overset{\circ}{\operatorname{Curl}}_{\mathbb{S}})$. Applying Lemma 2.24 for $m=0$ row-wise, there is a vector field $\mathbf{v} := \mathrm{P}_{\overset{\circ}{\operatorname{Grad}}} \mathbf{M} \in \overset{\circ}{\mathbf{H}}{}^{1}(\Omega)$ with $\mathbf{M} = \operatorname{Grad} \mathbf{v}$. Since $\operatorname{skw} \mathbf{M} = 0$ and $2 \operatorname{skw} \operatorname{Grad} \mathbf{v} = \operatorname{spn} \operatorname{curl} \mathbf{v}$, it follows that $\operatorname{curl} \mathbf{v} = 0$. By Lemma 2.24 for $m=1$ there is a function $u := \mathrm{P}_{\overset{\circ}{\operatorname{grad}}} \mathbf{v} \in \overset{\circ}{\mathsf{H}}{}^{2}(\Omega)$ with $\mathbf{v} = \operatorname{grad} u$. Hence $\mathbf{M} = \operatorname{Grad} \mathbf{v} = \operatorname{Gradgrad} u \in \operatorname{Grad} \operatorname{grad} \overset{\circ}{\mathsf{H}}{}^{2}(\Omega)$. So $\overset{\circ}{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) \subset \operatorname{Gradgrad} \overset{\circ}{\mathsf{H}}{}^{2}(\Omega)$, which completes the proof of (i). Note that

$$\mathrm{P}_{\overset{\circ}{\operatorname{Gradgrad}}} \mathbf{M} := u = \mathrm{P}_{\overset{\circ}{\operatorname{grad}}} \mathrm{P}_{\overset{\circ}{\operatorname{Grad}}} \mathbf{M} \in \overset{\circ}{\mathsf{H}}{}^{2}(\Omega),$$

from which it directly follows that $\mathrm{P}_{\overset{\circ}{\operatorname{Gradgrad}}}$ is linear and bounded.

ad (ii): Let $\mathbf{E} \in \overset{\circ}{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega) = N(\overset{\circ}{\operatorname{Div}}_{\mathbb{T}})$. Then there is a tensor field $\mathbf{N} := \mathrm{P}_{\overset{\circ}{\operatorname{Curl}}} \mathbf{E} \in \overset{\circ}{\mathcal{H}}{}^{1}(\Omega)$ with $\mathbf{E} = \operatorname{Curl} \mathbf{N}$, see Lemma 2.25 for $m=0$ applied row-wise. Since $\operatorname{tr} \mathbf{E} = 0$ and $\operatorname{tr} \operatorname{Curl} \operatorname{sym} \mathbf{N} = 0$, it follows that $\operatorname{tr} \operatorname{Curl} \operatorname{skw} \mathbf{N} = 0$. Now let $\mathbf{v} := \operatorname{spn}^{-1} \operatorname{skw} \mathbf{N} \in \overset{\circ}{\mathbf{H}}{}^{1}(\Omega)$, i.e. $\operatorname{skw} \mathbf{N} = \operatorname{spn} \mathbf{v}$. Since $\operatorname{tr} \operatorname{Curl} \operatorname{spn} \mathbf{v} = 2 \operatorname{div} \mathbf{v}$, it follows that $\operatorname{div} \mathbf{v} = 0$. Therefore, there is a vector field $\mathbf{w} := \mathrm{P}_{\overset{\circ}{\operatorname{curl}}} \mathbf{v} \in \overset{\circ}{\mathbf{H}}{}^{2}(\Omega)$ such that $\mathbf{v} = \operatorname{curl} \mathbf{w}$, see Lemma 2.25 for $m=1$. So we have

$$\operatorname{Curl} \operatorname{skw} \mathbf{N} = \operatorname{Curl} \operatorname{spn} \operatorname{curl} \mathbf{w} = 2 \operatorname{Curl} \operatorname{skw} \operatorname{Grad} \mathbf{w} = -2 \operatorname{Curl} \operatorname{sym} \operatorname{Grad} \mathbf{w}.$$

Hence

$$\mathbf{E} = \operatorname{Curl} \mathbf{N} = \operatorname{Curl} \operatorname{sym} \mathbf{N} + \operatorname{Curl} \operatorname{skw} \mathbf{N} = \operatorname{Curl} \mathbf{M}, \quad \mathbf{M} := \operatorname{sym} \mathbf{N} - 2 \operatorname{sym} \operatorname{Grad} \mathbf{w} \in \overset{\circ}{\mathcal{H}}{}^{1}_{\mathbb{S}}(\Omega).$$

So $\overset{\circ}{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega) \subset \operatorname{Curl} \overset{\circ}{\mathcal{H}}{}^{1}_{\mathbb{S}}(\Omega)$, which completes the proof of (ii). Note that

$$\begin{aligned}
\mathrm{P}_{\overset{\circ}{\operatorname{Curl}}_{\mathbb{S}}} \mathbf{E} := \mathbf{M} &= \operatorname{sym} \mathrm{P}_{\overset{\circ}{\operatorname{Curl}}} \mathbf{E} - 2 \operatorname{sym} \operatorname{Grad} \left(\mathrm{P}_{\overset{\circ}{\operatorname{curl}}} \operatorname{spn}^{-1} \operatorname{skw} \mathrm{P}_{\overset{\circ}{\operatorname{Curl}}} \mathbf{E} \right) \\
&= \operatorname{sym} \left(1 - 2 \operatorname{Grad} \mathrm{P}_{\overset{\circ}{\operatorname{curl}}} \operatorname{spn}^{-1} \operatorname{skw} \right) \mathrm{P}_{\overset{\circ}{\operatorname{Curl}}} \mathbf{E} \in \overset{\circ}{\mathcal{H}}{}^{1}_{\mathbb{S}}(\Omega),
\end{aligned}$$

from which it directly follows that $\mathrm{P}_{\overset{\circ}{\operatorname{Curl}}_{\mathbb{S}}}$ is linear and bounded.

ad (iii): Let $\mathbf{v} \in \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} = N(\pi_{\mathsf{RT}_0})$. As $\mathbf{v} \in (\mathbb{R}^3)^{\perp_{\mathsf{L}^2(\Omega)}}$, there is a tensor field $\mathbf{F} = \mathrm{P}_{\overset{\circ}{\operatorname{Div}}} \mathbf{v} \in \overset{\circ}{\mathcal{H}}{}^{1}(\Omega)$ with $\mathbf{v} = \operatorname{Div} \mathbf{F}$, see Lemma 2.26 for $m=0$ applied row-wise. We have $\operatorname{Div} \mathbf{F} \in \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$ as well as $\operatorname{Div} \operatorname{dev} \mathbf{F} \in \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$. Hence $\operatorname{grad}(\operatorname{tr} \mathbf{F}) = \operatorname{Div}((\operatorname{tr} \mathbf{F})\, \mathbf{I}) \in \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$, which implies $\operatorname{tr} \mathbf{F} \in \overset{\circ}{\mathsf{H}}{}^{1}(\Omega) \cap \mathsf{L}^2_0(\Omega)$. Therefore, there is a vector field $\mathbf{w} := \mathrm{P}_{\overset{\circ}{\operatorname{div}}} \operatorname{tr} \mathbf{F} \in \overset{\circ}{\mathbf{H}}{}^{2}(\Omega)$ with $\operatorname{tr} \mathbf{F} = \operatorname{div} \mathbf{w}$, see Lemma 2.26 for $m=1$. Thus

$$\operatorname{Div}((\operatorname{tr} \mathbf{F})\, \mathbf{I}) = \operatorname{grad} \operatorname{div} \mathbf{w} = \tfrac{3}{2} \operatorname{Div} \left(\operatorname{dev} (\operatorname{Grad} \mathbf{w})^{\top} \right).$$

Hence

$$\mathbf{v} = \operatorname{Div} \mathbf{F} = \operatorname{Div} \operatorname{dev} \mathbf{F} + \tfrac{1}{3} \operatorname{Div}((\operatorname{tr} \mathbf{F}) \mathbf{I}) = \operatorname{Div} \mathbf{E}, \quad \mathbf{E} := \operatorname{dev} \left(\mathbf{F} + \tfrac{1}{2} (\operatorname{Grad} \mathbf{w})^{\top} \right) \in \overset{\circ}{\mathcal{H}}{}^{1}_{\mathbb{T}}(\Omega).$$

So $\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} \subset \operatorname{Div} \overset{\circ}{\mathcal{H}}{}^{1}_{\mathbb{T}}(\Omega)$, which completes the proof of (iii). Note that

$$\begin{aligned}
\mathrm{P}_{\overset{\circ}{\operatorname{Div}}_{\mathbb{T}}} \mathbf{v} := \mathbf{E} &= \operatorname{dev} \left(\mathrm{P}_{\overset{\circ}{\operatorname{Div}}} \mathbf{v} + \tfrac{1}{2} (\operatorname{Grad} \mathrm{P}_{\overset{\circ}{\operatorname{div}}} \operatorname{tr} \mathrm{P}_{\overset{\circ}{\operatorname{Div}}} \mathbf{v})^{\top} \right) \\
&= \operatorname{dev} \left(1 + \tfrac{1}{2} \operatorname{Grad}^{\top} \mathrm{P}_{\overset{\circ}{\operatorname{div}}} \operatorname{tr} \right) \mathrm{P}_{\overset{\circ}{\operatorname{Div}}} \mathbf{v} \in \overset{\circ}{\mathcal{H}}{}^{1}_{\mathbb{T}}(\Omega),
\end{aligned}$$

from which it directly follows that $\mathrm{P}_{\overset{\circ}{\operatorname{Div}}_{\mathbb{T}}}$ is linear and bounded.

ad (iv): Let $\mathbf{E} \in \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl} 0, \Omega) = N(\operatorname{symCurl}_{\mathbb{T}})$. Then (trivially) $\operatorname{Div} \operatorname{symCurl} \mathbf{E} = 0$ and it follows

$$\operatorname{curl} \mathbf{w} = 0 \quad \text{with} \quad \mathbf{w} := \tfrac{1}{2}\Big(\operatorname{Div} \mathbf{E}^{\top} - \operatorname{grad}(\operatorname{tr} \mathbf{E})\Big) = \tfrac{1}{2} \operatorname{Div} \mathbf{E}^{\top}$$

and

$$\operatorname{skw} \operatorname{Curl} \mathbf{E} = \operatorname{spn} \mathbf{w}. \tag{20}$$

So $\mathbf{w} \in \mathsf{H}^{-1}(\operatorname{curl} 0, \Omega)$. Therefore, there is a unique scalar field $u := \operatorname{grad}^{-1} \mathbf{w} \in \mathsf{L}^2_0(\Omega)$, such that

$$\mathbf{w} = \operatorname{grad} u,$$

see Corollary 2.29 for $m = 1$. As $\operatorname{Curl}(u \mathbf{I}) = -\operatorname{spn} \operatorname{grad} u$ implies $\operatorname{symCurl}(u \mathbf{I}) = 0$, we see

$$\mathbf{F} := \mathbf{E} + u \mathbf{I} \in \boldsymbol{\mathcal{H}}(\operatorname{symCurl} 0, \Omega).$$

Moreover, by (20)

$$\operatorname{skw} \operatorname{Curl} \mathbf{F} = \operatorname{skw} \operatorname{Curl} \mathbf{E} + \operatorname{skw} \operatorname{Curl}(u \mathbf{I}) = \operatorname{spn} \mathbf{w} - \operatorname{spn} \operatorname{grad} u = 0.$$

Hence $\mathbf{F} \in \boldsymbol{\mathcal{H}}(\operatorname{Curl} 0, \Omega)$. Therefore, there is a unique vector field $\mathbf{v} := \operatorname{Grad}^{-1} \mathbf{F} \in \mathsf{H}^1(\Omega) \cap \mathsf{L}^2_0(\Omega)$, such that $\mathbf{F} = \operatorname{Grad} \mathbf{v}$, see Lemma 2.24 for $m = 0$. So we have

$$\mathbf{E} = \operatorname{Grad} \mathbf{v} - u \mathbf{I}.$$

From the additional condition $\operatorname{tr} \mathbf{E} = 0$ it follows that $3u = \operatorname{tr} \operatorname{Grad} \mathbf{v} = \operatorname{div} \mathbf{v}$ leading to

$$\mathbf{E} = \operatorname{devGrad} \mathbf{v}, \quad \mathbf{v} \in \mathsf{H}^1(\Omega).$$

So $\boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl} 0, \Omega) \subset \operatorname{devGrad} \mathsf{H}^1(\Omega)$, which completes the proof of (iv). Note that

$$\begin{aligned} \mathrm{P}_{\operatorname{devGrad}} \mathbf{E} := \mathbf{v} &= \operatorname{Grad}^{-1}\Big(\mathbf{E} + \tfrac{1}{2}(\operatorname{grad}^{-1} \operatorname{Div} \mathbf{E}^{\top})\, \mathbf{I}\Big) \\ &= \operatorname{Grad}^{-1}\Big(1 + \tfrac{1}{2}(\operatorname{grad}^{-1} \operatorname{Div}(\,\cdot\,)^{\top})\, \mathbf{I}\Big) \mathbf{E} \in \mathsf{H}^1(\Omega), \end{aligned}$$

from which it directly follows that $\mathrm{P}_{\operatorname{devGrad}}$ is linear and bounded.

ad (v): Let $\mathbf{M} \in \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega) = N(\operatorname{divDiv}_{\mathbb{S}})$. So $\operatorname{Div} \mathbf{M} \in \mathsf{H}^{-1}(\operatorname{div} 0, \Omega)$ and there is a unique vector field $\mathbf{v} := \operatorname{curl}^{-1} \operatorname{Div} \mathbf{M} \in \mathring{\mathsf{H}}(\operatorname{div} 0, \Omega)$, such that

$$\operatorname{Div} \mathbf{M} = \operatorname{curl} \mathbf{v} = -\operatorname{Div}(\operatorname{spn} \mathbf{v}),$$

see Corollary 2.30 for $m = 1$. Hence $\operatorname{Div}(\mathbf{M} + \operatorname{spn} \mathbf{v}) = 0$, i.e. $\mathbf{M} + \operatorname{spn} \mathbf{v} \in \boldsymbol{\mathcal{H}}(\operatorname{Div} 0, \Omega)$, and by Lemma 2.25 there is a tensor field $\mathbf{F} := \mathrm{P}_{\operatorname{Curl}}(\mathbf{M} + \operatorname{spn} \mathbf{v}) \in \boldsymbol{\mathcal{H}}^1(\Omega)$, such that

$$\mathbf{M} + \operatorname{spn} \mathbf{v} = \operatorname{Curl} \mathbf{F}.$$

Observe that $\mathbf{M}$ is symmetric and $\operatorname{spn} \mathbf{v}$ is skew-symmetric. Thus

$$\mathbf{M} = \operatorname{symCurl} \mathbf{F} \quad \text{and} \quad \operatorname{spn} \mathbf{v} = \operatorname{skw} \operatorname{Curl} \mathbf{F}, \quad \mathbf{F} \in \boldsymbol{\mathcal{H}}^1(\Omega),$$

and hence

$$\mathbf{M} = \operatorname{symCurl} \mathbf{F} = \operatorname{symCurl} \mathbf{E} \quad \text{with} \quad \mathbf{E} := \operatorname{dev} \mathbf{F} \in \boldsymbol{\mathcal{H}}^1_{\mathbb{T}}(\Omega),$$

as $\operatorname{dev} \mathbf{F} = \mathbf{F} - \frac{1}{3}(\operatorname{tr} \mathbf{F})\, \mathbf{I}$ and $\operatorname{symCurl}((\operatorname{tr} \mathbf{F})\, \mathbf{I}) = 0$. So $\boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega) \subset \operatorname{symCurl} \boldsymbol{\mathcal{H}}^1_{\mathbb{T}}(\Omega)$, which completes the proof of (v). Note that

$$\mathrm{P}_{\operatorname{symCurl}_{\mathbb{T}}} \mathbf{M} := \mathbf{E} = \operatorname{dev} \mathrm{P}_{\operatorname{Curl}}\big(\mathbf{M} + \operatorname{spn} \operatorname{curl}^{-1} \operatorname{Div} \mathbf{M}\big)$$

$$= \operatorname{dev} \mathrm{P}_{\mathrm{Curl}} \left(1 + \operatorname{spn} \operatorname{curl}^{-1} \operatorname{Div}\right) \mathbf{M} \in \mathcal{H}^1_{\mathbb{T}}(\Omega),$$

from which it directly follows that $\mathrm{P}_{\mathrm{symCurl}_{\mathbb{T}}}$ is linear and bounded.

ad (vi): Let $u \in \mathsf{L}^2(\Omega) = N(0)$. Then there is a vector field $\mathbf{v} = \mathrm{P}_{\mathrm{div}}\, u \in \mathsf{H}^1(\Omega)$ with $u = \operatorname{div} \mathbf{v}$, see Lemma 2.26 for $m = 0$, and a tensor field $\mathbf{N} = \mathrm{P}_{\mathrm{Div}}\, \mathbf{v} \in \mathcal{H}^2(\Omega)$ such that $\mathbf{v} = \operatorname{Div} \mathbf{N}$, see Lemma 2.26 for $m = 1$ applied row-wise. Since $\operatorname{divDiv} \operatorname{skw} \mathbf{N} = 0$, it follows that

$$u = \operatorname{divDiv} \mathbf{N} = \operatorname{div} \operatorname{Div} \mathbf{M} \quad \text{with} \quad \mathbf{M} =: \operatorname{sym} \mathbf{N} \in \mathcal{H}^2_{\mathbb{S}}(\Omega).$$

So $\mathsf{L}^2(\Omega) \subset \operatorname{divDiv} \mathcal{H}^2_{\mathbb{S}}(\Omega)$, which completes the proof of (vi). Note that

$$\mathrm{P}_{\mathrm{divDiv}_{\mathbb{S}}}\, u := \mathbf{M} = \operatorname{sym} \mathrm{P}_{\mathrm{Div}}\, \mathrm{P}_{\mathrm{div}}\, u \in \mathcal{H}^2_{\mathbb{S}}(\Omega),$$

from which it directly follows that $\mathrm{P}_{\mathrm{divDiv}_{\mathbb{S}}}$ is linear and bounded. ■

Provided that the domain Ω has trivial topology, Theorem 3.10 implies that the densely defined, closed and unbounded linear operators $\mathring{\operatorname{Gradgrad}}$, $\mathring{\operatorname{Curl}}_{\mathbb{S}}$, $\mathring{\operatorname{Div}}_{\mathbb{T}}$, and their adjoints $\operatorname{divDiv}_{\mathbb{S}}$, $\operatorname{symCurl}_{\mathbb{T}}$, $\operatorname{devGrad}$ have closed ranges and that all relevant cohomology groups are trivial, as

$$\begin{aligned}
N(\mathring{\operatorname{Gradgrad}}) \cap N(0) &= \{0\} \cap \mathsf{L}^2(\Omega) = \{0\},\\
N(\mathring{\operatorname{Curl}}_{\mathbb{S}}) \cap N(\operatorname{divDiv}_{\mathbb{S}}) &= \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) \cap \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega) = \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) \cap \operatorname{symCurl} \mathcal{H}^1_{\mathbb{T}}(\Omega)\\
&= N(\mathring{\operatorname{Curl}}_{\mathbb{S}}) \cap R(\operatorname{symCurl}_{\mathbb{T}}) = \{0\},\\
N(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap N(\operatorname{symCurl}_{\mathbb{T}}) &= \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega) \cap \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl} 0, \Omega) = \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega) \cap \operatorname{devGrad} \mathsf{H}^1(\Omega)\\
&= N(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap R(\operatorname{devGrad}) = \{0\},\\
N(\pi_{\mathsf{RT}_0}) \cap N(\operatorname{devGrad}) &= \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} \cap \mathsf{RT}_0 = \{0\}.
\end{aligned}$$

In this case, the reduced operators are

$$\begin{aligned}
\mathcal{A}_0 = \mathring{\operatorname{Gradgrad}} : \mathring{\mathsf{H}}^2(\Omega) \subset \mathsf{L}^2(\Omega) &\longrightarrow \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega),\\
\mathcal{A}_1 = \mathring{\operatorname{Curl}}_{\mathbb{S}} : \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \cap \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega) \subset \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega) &\longrightarrow \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega),\\
\mathcal{A}_2 = \mathring{\operatorname{Div}}_{\mathbb{T}} : \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}, \Omega) \cap \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl} 0, \Omega) \subset \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl} 0, \Omega) &\longrightarrow \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}},\\
\mathcal{A}_0^* = \operatorname{divDiv}_{\mathbb{S}} : \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) \cap \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) \subset \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) &\longrightarrow \mathsf{L}^2(\Omega),\\
\mathcal{A}_1^* = \operatorname{symCurl}_{\mathbb{T}} : \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \cap \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega) \subset \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega) &\longrightarrow \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega),\\
\mathcal{A}_2^* = -\operatorname{devGrad} : \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} \subset \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} &\longrightarrow \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl} 0, \Omega)
\end{aligned}$$

as

$$R(\operatorname{divDiv}_{\mathbb{S}}) = \mathsf{L}^2(\Omega), \qquad R(\mathring{\operatorname{Div}}_{\mathbb{T}}) = \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}.$$

The functional analysis toolbox Section 2.1, e.g. Lemma 2.10, immediately lead to the following implications about Helmholtz type decompositions, Friedrichs/Poincaré type estimates and continuous inverse operators.

Theorem 3.12: *Let Ω be additionally topologically trivial. Then all occurring ranges are closed and all related cohomology groups are trivial. Moreover, the Helmholtz type decompositions*

$$\boldsymbol{\mathcal{L}}^2_{\mathbb{S}}(\Omega) = \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) \oplus_{\boldsymbol{\mathcal{L}}^2_{\mathbb{S}}(\Omega)} \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega), \quad \boldsymbol{\mathcal{L}}^2_{\mathbb{T}}(\Omega) = \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega) \oplus_{\boldsymbol{\mathcal{L}}^2_{\mathbb{T}}(\Omega)} \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl} 0, \Omega)$$

hold. The kernels can be represented by the following closed ranges:

$$\begin{aligned}
\mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) &= \operatorname{Gradgrad} \mathring{\mathsf{H}}^2(\Omega), \\
\operatorname{symCurl} \boldsymbol{\mathcal{H}}^1_{\mathbb{T}}(\Omega) = \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega) &= \operatorname{symCurl} \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) = \operatorname{symCurl} \left(\boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \cap \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega) \right), \\
\operatorname{Curl} \mathring{\boldsymbol{\mathcal{H}}}^1_{\mathbb{S}}(\Omega) = \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega) &= \operatorname{Curl} \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) = \operatorname{Curl} \left(\mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \cap \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega) \right), \\
\boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl} 0, \Omega) &= \operatorname{devGrad} \mathbf{H}^1(\Omega) = \operatorname{devGrad} \left(\mathbf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} \right),
\end{aligned}$$

and it holds

$$\begin{aligned}
\operatorname{divDiv} \boldsymbol{\mathcal{H}}^2_{\mathbb{S}}(\Omega) = \mathsf{L}^2(\Omega) &= \operatorname{divDiv} \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) = \operatorname{divDiv} \left(\boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) \cap \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) \right), \\
\operatorname{Div} \mathring{\boldsymbol{\mathcal{H}}}^1_{\mathbb{T}}(\Omega) = \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} = N(\pi_{\mathsf{RT}_0}) &= \operatorname{Div} \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div}, \Omega) = \operatorname{Div} \left(\mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div}, \Omega) \cap \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl} 0, \Omega) \right).
\end{aligned}$$

All potentials depend continuously on the data. The potentials on the very right-hand sides are uniquely determined. There exist positive constants c_{Gg}, c_{D}, c_{R} such that the Friedrichs/Poincaré type estimates

$$\begin{aligned}
&\forall\, u \in \mathring{\mathsf{H}}^2(\Omega) && |u|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{Gg}} \, |\operatorname{Gradgrad} u|_{\boldsymbol{\mathcal{L}}^2(\Omega)}, \\
&\forall\, \mathbf{M} \in \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) \cap \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) && |\mathbf{M}|_{\boldsymbol{\mathcal{L}}^2(\Omega)} \leq c_{\mathsf{Gg}} \, |\operatorname{divDiv} \mathbf{M}|_{\mathsf{L}^2(\Omega)}, \\
&\forall\, \mathbf{E} \in \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div}, \Omega) \cap \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl} 0, \Omega) && |\mathbf{E}|_{\boldsymbol{\mathcal{L}}^2(\Omega)} \leq c_{\mathsf{D}} \, |\operatorname{Div} \mathbf{E}|_{\mathbf{L}^2(\Omega)}, \\
&\forall\, \mathbf{v} \in \mathbf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} && |\mathbf{v}|_{\mathbf{L}^2(\Omega)} \leq c_{\mathsf{D}} \, |\operatorname{devGrad} \mathbf{v}|_{\boldsymbol{\mathcal{L}}^2(\Omega)}, \\
&\forall\, \mathbf{M} \in \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \cap \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega) && |\mathbf{M}|_{\boldsymbol{\mathcal{L}}^2(\Omega)} \leq c_{\mathsf{R}} \, |\operatorname{Curl} \mathbf{M}|_{\boldsymbol{\mathcal{L}}^2(\Omega)}, \\
&\forall\, \mathbf{E} \in \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \cap \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega) && |\mathbf{E}|_{\boldsymbol{\mathcal{L}}^2(\Omega)} \leq c_{\mathsf{R}} \, |\operatorname{symCurl} \mathbf{E}|_{\boldsymbol{\mathcal{L}}^2(\Omega)}
\end{aligned}$$

hold. Moreover, the reduced versions of the operators

$$\operatorname{Gra\mathring{d}grad}, \quad \operatorname{divDiv}_{\mathbb{S}}, \quad \mathring{\operatorname{Div}}_{\mathbb{T}}, \quad \operatorname{devGrad}, \quad \mathring{\operatorname{Curl}}_{\mathbb{S}}, \quad \operatorname{symCurl}_{\mathbb{T}}$$

have continuous inverse operators

$$\begin{aligned}
(\operatorname{Gra\mathring{d}grad})^{-1} : \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) &\longrightarrow \mathring{\mathsf{H}}^2(\Omega), \\
(\operatorname{divDiv}_{\mathbb{S}})^{-1} : \mathsf{L}^2(\Omega) &\longrightarrow \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) \cap \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega), \\
(\mathring{\operatorname{Div}}_{\mathbb{T}})^{-1} : \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} &\longrightarrow \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div}, \Omega) \cap \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl} 0, \Omega), \\
(\operatorname{devGrad})^{-1} : \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl} 0, \Omega) &\longrightarrow \mathbf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}, \\
(\mathring{\operatorname{Curl}}_{\mathbb{S}})^{-1} : \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega) &\longrightarrow \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \cap \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega),
\end{aligned}$$

$$(\operatorname{symCurl}_{\mathbb{T}})^{-1} : \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega) \longrightarrow \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \cap \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega)$$

with norms $(1 + c_{\mathsf{Gg}}^2)^{1/2}$, $(1 + c_{\mathsf{D}}^2)^{1/2}$, *resp.* $(1 + c_{\mathsf{R}}^2)^{1/2}$.

Remark 3.13: Let Ω be additionally topologically trivial. The Friedrichs/Poincaré type estimate for $\operatorname{Curl} \mathbf{M}$ in the latter theorem can be slightly sharpened. Utilizing Lemma 3.4 we observe $\operatorname{tr} \operatorname{Curl} \mathbf{M} = 0$ and thus $\operatorname{dev} \operatorname{Curl} \mathbf{M} = \operatorname{Curl} \mathbf{M}$ for $\mathbf{M} \in \mathcal{H}_{\mathbb{S}}(\operatorname{Curl}, \Omega)$. Hence

$$\forall\, \mathbf{M} \in \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \cap \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega) \quad |\mathbf{M}|_{\mathcal{L}^2(\Omega)} \leq c_{\mathsf{R}}\, |\operatorname{dev} \operatorname{Curl} \mathbf{M}|_{\mathcal{L}^2(\Omega)}.$$

Similarly and trivially we see

$$\forall\, u \in \mathring{\mathsf{H}}^2(\Omega) \quad |u|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{Gg}}\, |\operatorname{sym} \operatorname{Gradgrad} u|_{\mathcal{L}^2(\Omega)}.$$

Recalling Remark 3.8 we have the following result.

Remark 3.14: Let Ω be additionally topologically trivial. Theorems 3.10 and 3.12 easily lead to the following result in terms of complexes: The sequence

$$\{0\} \xrightarrow{\;0\;} \mathring{\mathsf{H}}^2(\Omega) \xrightarrow{\mathring{\operatorname{Gradgrad}}} \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \xrightarrow{\mathring{\operatorname{Curl}}_{\mathbb{S}}} \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}, \Omega) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}} \mathbf{L}^2(\Omega) \xrightarrow{\pi_{\mathsf{RT}_0}} \mathsf{RT}_0$$

and thus also its dual or adjoint sequence

$$\{0\} \xleftarrow{\;0\;} \mathsf{L}^2(\Omega) \xleftarrow{\operatorname{divDiv}_{\mathbb{S}}} \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) \xleftarrow{\operatorname{symCurl}_{\mathbb{T}}} \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \xleftarrow{-\operatorname{devGrad}} \mathbf{H}^1(\Omega) \xleftarrow{\iota_{\mathsf{RT}_0}} \mathsf{RT}_0$$

are closed and exact Hilbert complexes.

Remark 3.15: The part

$$\{0\} \xrightarrow{\;0\;} \mathring{\mathsf{H}}^2(\Omega) \xrightarrow{\mathring{\operatorname{Gradgrad}}} \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \xrightarrow{\mathring{\operatorname{Curl}}_{\mathbb{S}}} \mathcal{L}^2(\Omega)$$

of the Hilbert complex from above and the related adjoint complex

$$\{0\} \xleftarrow{\;0\;} \mathsf{L}^2(\Omega) \xleftarrow{\operatorname{divDiv}_{\mathbb{S}}} \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) \xleftarrow{\operatorname{symCurl}_{\mathbb{T}}} \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega)$$

have been discussed in [7] for problems in general relativity.

Remark 3.16: In 2D and under similar assumptions we obtain by completely analogous but much simpler arguments that the Hilbert complexes

$$\{0\} \xrightarrow{\;0\;} \mathring{\mathsf{H}}^2(\Omega) \xrightarrow{\mathring{\operatorname{Gradgrad}}} \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \xrightarrow{\mathring{\operatorname{Curl}}_{\mathbb{S}}} \mathbf{L}^2(\Omega) \xrightarrow{\pi_{\mathsf{RT}_0}} \mathsf{RT}_0\text{'}$$

$$\{0\} \xleftarrow{\;0\;} \mathsf{L}^2(\Omega) \xleftarrow{\operatorname{divDiv}_{\mathbb{S}}} \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) \xleftarrow{\operatorname{symCurl}} \mathbf{H}^1(\Omega) \xleftarrow{\iota_{\mathsf{RT}_0}} \mathsf{RT}_0$$

are dual to each other, closed and exact. Contrary to the 3D case, the operator $\mathring{\operatorname{Curl}}_{\mathbb{S}}$ maps a tensor field to a vector field and the operator $\operatorname{symCurl} \cong \operatorname{sym} \operatorname{Grad}$ is applied row-wise to a vector field and maps this vector field to a tensor field. The associated Helmholtz decomposition is

$$\mathcal{L}^2_{\mathbb{S}}(\Omega) = \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) \oplus_{\mathcal{L}^2_{\mathbb{S}}(\Omega)} \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega)$$

with

$$\mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) = \operatorname{Gradgrad} \mathring{\mathsf{H}}^2(\Omega), \quad \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega) = \operatorname{symCurl} \mathbf{H}^1(\Omega).$$

Theorem 3.10 leads to the following so-called regular decompositions.

Theorem 3.17: *Let Ω be additionally topologically trivial. Then the regular decompositions*

$$\begin{aligned}
\mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega) &= \mathring{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega) + \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl}0,\Omega), & \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl}0,\Omega) &= \operatorname{Gradgrad}\mathring{\mathsf{H}}^{2}(\Omega),\\
\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega) &= \mathring{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega) + \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}0,\Omega), & \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}0,\Omega) &= \operatorname{Curl}\mathring{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega),\\
\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega) &= \mathcal{H}^{1}_{\mathbb{T}}(\Omega) + \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}0,\Omega), & \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}0,\Omega) &= \operatorname{devGrad}\mathbf{H}^{1}(\Omega),\\
\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega) &= \mathcal{H}^{2}_{\mathbb{S}}(\Omega) + \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}0,\Omega), & \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}0,\Omega) &= \operatorname{symCurl}\mathcal{H}^{1}_{\mathbb{T}}(\Omega)
\end{aligned}$$

hold with linear and continuous (regular) decomposition resp. potential operators

$$\begin{aligned}
&\mathrm{P}_{\mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega),\mathring{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega)} : \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega) \longrightarrow \mathring{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega), && \mathrm{P}_{\mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega),\mathring{\mathsf{H}}^{2}(\Omega)} : \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega) \longrightarrow \mathring{\mathsf{H}}^{2}(\Omega),\\
&\mathrm{P}_{\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega),\mathring{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega)} : \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega) \longrightarrow \mathring{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega), && \mathrm{P}_{\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega),\mathring{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega)} : \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega) \longrightarrow \mathring{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega),\\
&\mathrm{P}_{\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega),\mathcal{H}^{1}_{\mathbb{T}}(\Omega)} : \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega) \longrightarrow \mathcal{H}^{1}_{\mathbb{T}}(\Omega), && \mathrm{P}_{\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega),\mathbf{H}^{1}(\Omega)} : \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega) \longrightarrow \mathbf{H}^{1}(\Omega),\\
&\mathrm{P}_{\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega),\mathcal{H}^{2}_{\mathbb{S}}(\Omega)} : \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega) \longrightarrow \mathcal{H}^{2}_{\mathbb{S}}(\Omega), && \mathrm{P}_{\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega),\mathcal{H}^{1}_{\mathbb{T}}(\Omega)} : \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega) \longrightarrow \mathcal{H}^{1}_{\mathbb{T}}(\Omega).
\end{aligned}$$

Proof: Let, e.g. $\mathbf{E} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega)$. Then

$$\operatorname{symCurl}\mathbf{E} \in \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}0,\Omega) = \operatorname{symCurl}\mathcal{H}^{1}_{\mathbb{T}}(\Omega)$$

with linear and continuous potential operator $\mathrm{P}_{\operatorname{symCurl}_{\mathbb{T}}} : \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}0,\Omega) \longrightarrow \mathcal{H}^{1}_{\mathbb{T}}(\Omega)$ by Theorem 3.10. Thus, there is $\tilde{\mathbf{E}} := \mathrm{P}_{\operatorname{symCurl}_{\mathbb{T}}}\operatorname{symCurl}\mathbf{E} \in \mathcal{H}^{1}_{\mathbb{T}}(\Omega)$ depending linearly and continuously on $\mathbf{E}$ with $\operatorname{symCurl}\tilde{\mathbf{E}} = \operatorname{symCurl}\mathbf{E}$. Hence,

$$\mathbf{E} - \tilde{\mathbf{E}} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}0,\Omega) = \operatorname{devGrad}\mathbf{H}^{1}(\Omega)$$

with linear and continuous potential operator $\mathrm{P}_{\operatorname{devGrad}} : \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}0,\Omega) \longrightarrow \mathbf{H}^{1}(\Omega)$ by Theorem 3.10. Hence, there exists $\mathbf{v} := \mathrm{P}_{\operatorname{devGrad}}(\mathbf{E} - \tilde{\mathbf{E}}) \in \mathbf{H}^{1}(\Omega)$ with $\operatorname{devGrad}\mathbf{v} = \mathbf{E} - \tilde{\mathbf{E}}$ and $\mathbf{v}$ depends linearly and continuously on $\mathbf{E}$. The other assertions are proved analogously. ■

Looking at the latter proof we see that the regular potential operators are given by

$$\begin{aligned}
\mathrm{P}_{\mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega),\mathring{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega)} &= \mathrm{P}_{\mathring{\operatorname{Curl}}_{\mathbb{S}}}\operatorname{Curl} : \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega) \longrightarrow \mathring{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega),\\
\mathrm{P}_{\mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega),\mathring{\mathsf{H}}^{2}(\Omega)} &= \mathrm{P}_{\mathring{\operatorname{Gradgrad}}}(1 - \mathrm{P}_{\mathring{\operatorname{Curl}}_{\mathbb{S}}}\operatorname{Curl}) : \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega) \longrightarrow \mathring{\mathsf{H}}^{2}(\Omega),\\
\mathrm{P}_{\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega),\mathring{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega)} &= \mathrm{P}_{\mathring{\operatorname{Div}}_{\mathbb{T}}}\operatorname{Div} : \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega) \longrightarrow \mathring{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega),\\
\mathrm{P}_{\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega),\mathring{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega)} &= \mathrm{P}_{\mathring{\operatorname{Curl}}_{\mathbb{S}}}(1 - \mathrm{P}_{\mathring{\operatorname{Div}}_{\mathbb{T}}}\operatorname{Div}) : \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega) \longrightarrow \mathring{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega), \qquad (21)\\
\mathrm{P}_{\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega),\mathcal{H}^{1}_{\mathbb{T}}(\Omega)} &= \mathrm{P}_{\operatorname{symCurl}_{\mathbb{T}}}\operatorname{symCurl} : \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega) \longrightarrow \mathcal{H}^{1}_{\mathbb{T}}(\Omega),\\
\mathrm{P}_{\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega),\mathbf{H}^{1}(\Omega)} &= \mathrm{P}_{\operatorname{devGrad}}(1 - \mathrm{P}_{\operatorname{symCurl}_{\mathbb{T}}}\operatorname{symCurl}) : \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega) \longrightarrow \mathbf{H}^{1}(\Omega),\\
\mathrm{P}_{\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega),\mathcal{H}^{2}_{\mathbb{S}}(\Omega)} &= \mathrm{P}_{\operatorname{divDiv}_{\mathbb{S}}}\operatorname{divDiv} : \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega) \longrightarrow \mathcal{H}^{2}_{\mathbb{S}}(\Omega),\\
\mathrm{P}_{\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega),\mathcal{H}^{1}_{\mathbb{T}}(\Omega)} &= \mathrm{P}_{\operatorname{symCurl}_{\mathbb{T}}}(1 - \mathrm{P}_{\operatorname{divDiv}_{\mathbb{S}}}\operatorname{divDiv}) : \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega) \longrightarrow \mathcal{H}^{1}_{\mathbb{T}}(\Omega).
\end{aligned}$$

Hence the regular decompositions of Theorem 3.17 can be slightly refined to even direct regular decompositions.

Corollary 3.18: *Let Ω be additionally topologically trivial. Then the direct regular decompositions*

$$\overset{\circ}{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega)=\mathrm{P}_{\overset{\circ}{\operatorname{Curl}}_{\mathbb{S}}}\overset{\circ}{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}0,\Omega)\dotplus\overset{\circ}{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl}0,\Omega),\quad \mathrm{P}_{\overset{\circ}{\operatorname{Curl}}_{\mathbb{S}}}\overset{\circ}{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}0,\Omega)\subset\overset{\circ}{\mathcal{H}}{}^{1}_{\mathbb{S}}(\Omega),$$

$$\overset{\circ}{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega)=\mathrm{P}_{\overset{\circ}{\operatorname{Div}}_{\mathbb{T}}}\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\dotplus\overset{\circ}{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}0,\Omega),\quad \mathrm{P}_{\overset{\circ}{\operatorname{Div}}_{\mathbb{T}}}\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\subset\overset{\circ}{\mathcal{H}}{}^{1}_{\mathbb{T}}(\Omega),$$

$$\begin{aligned}\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega)&=\mathrm{P}_{\operatorname{symCurl}_{\mathbb{T}}}\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}0,\Omega)\dotplus\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}0,\Omega),\\ &\qquad\mathrm{P}_{\operatorname{symCurl}_{\mathbb{T}}}\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}0,\Omega)\subset\mathcal{H}^1_{\mathbb{T}}(\Omega),\\ \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega)&=\mathrm{P}_{\operatorname{divDiv}_{\mathbb{S}}}\mathsf{L}^2(\Omega)\dotplus\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}0,\Omega),\quad \mathrm{P}_{\operatorname{divDiv}_{\mathbb{S}}}\mathsf{L}^2(\Omega)\subset\mathcal{H}^2_{\mathbb{S}}(\Omega)\end{aligned}$$

hold. More precisely

$$\begin{aligned}\overset{\circ}{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega)&=\mathrm{P}_{\overset{\circ}{\operatorname{Curl}}_{\mathbb{S}}}\overset{\circ}{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}0,\Omega)\dotplus\operatorname{Gradgrad}\mathrm{P}_{\overset{\circ}{\operatorname{Gradgrad}}}\overset{\circ}{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl}0,\Omega),\\ \overset{\circ}{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega)&=\mathrm{P}_{\overset{\circ}{\operatorname{Div}}_{\mathbb{T}}}\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\dotplus\operatorname{Curl}\mathrm{P}_{\overset{\circ}{\operatorname{Curl}}_{\mathbb{S}}}\overset{\circ}{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}0,\Omega),\\ \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega)&=\mathrm{P}_{\operatorname{symCurl}_{\mathbb{T}}}\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}0,\Omega)\dotplus\operatorname{devGrad}\mathrm{P}_{\operatorname{devGrad}}\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}0,\Omega),\\ \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega)&=\mathrm{P}_{\operatorname{divDiv}_{\mathbb{S}}}\mathsf{L}^2(\Omega)\dotplus\operatorname{symCurl}\mathrm{P}_{\operatorname{symCurl}_{\mathbb{T}}}\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}0,\Omega)\end{aligned}$$

with

$$\begin{aligned}&\mathrm{P}_{\overset{\circ}{\operatorname{Gradgrad}}}\overset{\circ}{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl}0,\Omega)\subset\overset{\circ}{\mathsf{H}}{}^{2}(\Omega),\quad \mathrm{P}_{\operatorname{devGrad}}\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}0,\Omega)\subset\mathbf{H}^1(\Omega),\\ &\mathrm{P}_{\overset{\circ}{\operatorname{Curl}}_{\mathbb{S}}}\overset{\circ}{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}0,\Omega)\subset\overset{\circ}{\mathcal{H}}{}^{1}_{\mathbb{S}}(\Omega),\quad \mathrm{P}_{\operatorname{symCurl}_{\mathbb{T}}}\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}0,\Omega)\subset\mathcal{H}^1_{\mathbb{T}}(\Omega).\end{aligned}$$

Here, $\dotplus$ denotes the direct sum.

Proof: For $\mathbf{M}\in\overset{\circ}{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl}0,\Omega)\cap\mathrm{P}_{\overset{\circ}{\operatorname{Curl}}_{\mathbb{S}}}\overset{\circ}{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}0,\Omega)$ we have $\mathbf{M}=\mathrm{P}_{\overset{\circ}{\operatorname{Curl}}_{\mathbb{S}}}\mathbf{E}$ with some $\mathbf{E}\in\overset{\circ}{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}0,\Omega)$. Thus $0=\operatorname{Curl}\mathbf{M}=\mathbf{E}$ showing $\mathbf{M}=0$ and hence the directness of the first regular decomposition. The directness of the others follows similarly. ∎

3.2. General bounded strong Lipschitz domains

In this section, we consider bounded strong Lipschitz domains Ω of general topology and we will extend the results of the previous section as follows. The $\overset{\circ}{\operatorname{Gradgrad}}$- and the divDiv-complexes remain closed and all associated cohomology groups are finite-dimensional. Moreover, the respective inverse operators are continuous and even compact, and corresponding Friedrichs/Poincaré type estimates hold. We will show this by verifying the compactness properties of Lemma 2.7 for the various linear operators of the complexes. Then Lemma 2.5, Remark 2.6, and Theorem 2.9 immediately lead to the desired results. Using Rellich's selection theorem, we have the following compact embeddings:

$$\begin{aligned}D(\overset{\circ}{\operatorname{Gradgrad}})\cap D(0)=\overset{\circ}{\mathsf{H}}{}^{2}(\Omega)&\overset{\mathsf{cpt}}{\hookrightarrow}\mathsf{L}^2(\Omega),\\ D(\pi_{\mathsf{RT}_0})\cap D(\operatorname{devGrad})=\mathbf{H}^1(\Omega)&\overset{\mathsf{cpt}}{\hookrightarrow}\mathsf{L}^2(\Omega).\end{aligned}$$

The two missing compactness results that would immediately lead to the desired results are

$$D(\overset{\circ}{\operatorname{Curl}}_{\mathbb{S}})\cap D(\operatorname{divDiv}_{\mathbb{S}})=\overset{\circ}{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega)\cap\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega)\overset{\mathsf{cpt}}{\hookrightarrow}\mathcal{L}^2_{\mathbb{S}}(\Omega),\tag{22}$$

$$D(\mathring{\operatorname{Div}}_{\mathbb{T}}) \cap D(\operatorname{symCurl}_{\mathbb{T}}) = \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div}, \Omega) \cap \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \overset{\mathsf{cpt}}{\hookrightarrow} \boldsymbol{\mathcal{L}}^2_{\mathbb{T}}(\Omega). \tag{23}$$

The main aim of this section is to show the compactness of the two crucial embeddings (22) and (23). As a first step we consider a trivial topology.

Lemma 3.19: *Let Ω be additionally topologically trivial. Then the embeddings* (22), (23) *are compact.*

Proof: Let $(\mathbf{M}_n)$ be a bounded sequence in $\mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \cap \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv}, \Omega)$. By Theorems 3.12 and 3.10 we have

$$\begin{aligned}\mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \cap \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) &= \Big(\mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) \cap \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv}, \Omega)\Big) \oplus_{\boldsymbol{\mathcal{L}}^2_{\mathbb{S}}(\Omega)} \\ &\qquad \Big(\mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \cap \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega)\Big),\\ \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) &= \operatorname{Gradgrad} \mathring{\mathsf{H}}^2(\Omega),\\ \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega) &= \operatorname{symCurl} \boldsymbol{\mathcal{H}}^1_{\mathbb{T}}(\Omega)\end{aligned}$$

with linear and continuous potential operators. Therefore, we can decompose

$$\begin{aligned}\mathbf{M}_n = \mathbf{M}_{n,\mathsf{r}} + \mathbf{M}_{n,\mathsf{d}} \in \Big(\mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) \cap \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv}, \Omega)\Big) \oplus_{\boldsymbol{\mathcal{L}}^2_{\mathbb{S}}(\Omega)}\\ \Big(\mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \cap \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega)\Big)\end{aligned}$$

with $\mathbf{M}_{n,\mathsf{r}} \in \operatorname{Gradgrad} \mathring{\mathsf{H}}^2(\Omega) \cap \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv}, \Omega)$, $\operatorname{Curl} \mathbf{M}_{n,\mathsf{d}} = \operatorname{Curl} \mathbf{M}_n$, and $\mathbf{M}_{n,\mathsf{r}} = \operatorname{Gradgrad} u_n$, $u_n \in \mathring{\mathsf{H}}^2(\Omega)$, as well as $\mathbf{M}_{n,\mathsf{d}} \in \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \cap \operatorname{symCurl} \boldsymbol{\mathcal{H}}^1_{\mathbb{T}}(\Omega)$, $\operatorname{divDiv} \mathbf{M}_{n,\mathsf{r}} = \operatorname{divDiv} \mathbf{M}_n$, and $\mathbf{M}_{n,\mathsf{d}} = \operatorname{symCurl} \mathbf{E}_n$, $\mathbf{E}_n \in \boldsymbol{\mathcal{H}}^1_{\mathbb{T}}(\Omega)$, and both u_n and $\mathbf{E}_n$ depend continuously on $\mathbf{M}_n$, i.e.

$$|u_n|_{\mathsf{H}^2(\Omega)} \le c\,|\mathbf{M}_{n,\mathsf{r}}|_{\boldsymbol{\mathcal{L}}^2(\Omega)} \le c\,|\mathbf{M}_n|_{\boldsymbol{\mathcal{L}}^2(\Omega)}, \qquad |\mathbf{E}_n|_{\boldsymbol{\mathcal{H}}^1(\Omega)} \le c\,|\mathbf{M}_{n,\mathsf{d}}|_{\boldsymbol{\mathcal{L}}^2(\Omega)} \le c\,|\mathbf{M}_n|_{\boldsymbol{\mathcal{L}}^2(\Omega)}.$$

By Rellich's selection theorem, there exist subsequences, again denoted by (u_n) and $(\mathbf{E}_n)$, such that (u_n) converges in $\mathsf{H}^1(\Omega)$ and $(\mathbf{E}_n)$ converges in $\boldsymbol{\mathcal{L}}^2(\Omega)$. Thus with $\mathbf{M}_{n,m} := \mathbf{M}_n - \mathbf{M}_m$, and similarly for $\mathbf{M}_{n,m,\mathsf{r}}$, $\mathbf{M}_{n,m,\mathsf{d}}$, $u_{n,m}$, $\mathbf{E}_{n,m}$, we see

$$\begin{aligned}|\mathbf{M}_{n,m,\mathsf{r}}|^2_{\boldsymbol{\mathcal{L}}^2(\Omega)} &= \langle \mathbf{M}_{n,m,\mathsf{r}}, \operatorname{Gradgrad} u_{n,m}\rangle_{\boldsymbol{\mathcal{L}}^2(\Omega)} = \langle \operatorname{divDiv} \mathbf{M}_{n,m,\mathsf{r}}, u_{n,m}\rangle_{\mathsf{L}^2(\Omega)}\\ &= \langle \operatorname{divDiv} \mathbf{M}_{n,m}, u_{n,m}\rangle_{\mathsf{L}^2(\Omega)} \le c\,|u_{n,m}|_{\mathsf{L}^2(\Omega)},\\ |\mathbf{M}_{n,m,\mathsf{d}}|^2_{\boldsymbol{\mathcal{L}}^2(\Omega)} &= \langle \mathbf{M}_{n,m,\mathsf{d}}, \operatorname{symCurl} \mathbf{E}_{n,m}\rangle_{\boldsymbol{\mathcal{L}}^2(\Omega)} = \langle \operatorname{Curl} \mathbf{M}_{n,m,\mathsf{d}}, \mathbf{E}_{n,m}\rangle_{\boldsymbol{\mathcal{L}}^2(\Omega)}\\ &= \langle \operatorname{Curl} \mathbf{M}_{n,m}, \mathbf{E}_{n,m}\rangle_{\boldsymbol{\mathcal{L}}^2(\Omega)} \le c\,|\mathbf{E}_{n,m}|_{\boldsymbol{\mathcal{L}}^2(\Omega)}.\end{aligned}$$

Hence, $(\mathbf{M}_n)$ is a Cauchy sequence in $\boldsymbol{\mathcal{L}}^2_{\mathbb{S}}(\Omega)$. So

$$\mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \cap \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) \hookrightarrow \boldsymbol{\mathcal{L}}^2_{\mathbb{S}}(\Omega)$$

is compact. To show the second compact embedding, let $(\mathbf{E}_n) \subset \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \cap \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div}, \Omega)$ be a bounded sequence. By Theorems 3.12 and 3.10 we have

$$\boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \cap \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div}, \Omega) = \Big(\boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl} 0, \Omega) \cap \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div}, \Omega)\Big) \oplus_{\boldsymbol{\mathcal{L}}^2_{\mathbb{T}}(\Omega)}$$

$$\left(\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega)\cap\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}0,\Omega)\right),$$

$$\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}0,\Omega)=\operatorname{devGrad}\mathsf{H}^{1}(\Omega),$$

$$\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}0,\Omega)=\operatorname{Curl}\mathring{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega)$$

with linear and continuous potential operators. Therefore, we can decompose

$$\mathbf{E}_n=\mathbf{E}_{n,\mathsf{r}}+\mathbf{E}_{n,\mathsf{d}}\in\left(\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}0,\Omega)\cap\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega)\right)\oplus_{\mathcal{L}^2_{\mathbb{T}}(\Omega)}$$
$$\left(\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega)\cap\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}0,\Omega)\right)$$

with $\mathbf{E}_{n,\mathsf{r}}\in\operatorname{devGrad}\mathsf{H}^1(\Omega)\cap\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega)$, $\operatorname{symCurl}\mathbf{E}_{n,\mathsf{d}}=\operatorname{symCurl}\mathbf{E}_n$, $\mathbf{E}_{n,\mathsf{r}}=\operatorname{devGrad}\mathbf{v}_n$, $\mathbf{v}_n\in\mathsf{H}^1(\Omega)$, as well as $\mathbf{E}_{n,\mathsf{d}}\in\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega)\cap\operatorname{Curl}\mathring{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega)$, $\operatorname{Div}\mathbf{E}_{n,\mathsf{r}}=\operatorname{Div}\mathbf{E}_n$, and $\mathbf{E}_{n,\mathsf{d}}=\operatorname{Curl}\mathbf{M}_n$, $\mathbf{M}_n\in\mathring{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega)$, and both $\mathbf{v}_n$ and $\mathbf{M}_n$ depend continuously on $\mathbf{E}_n$, i.e.

$$|\mathbf{v}_n|_{\mathsf{H}^1(\Omega)}\leq c\,|\mathbf{E}_{n,\mathsf{r}}|_{\mathcal{L}^2(\Omega)}\leq c\,|\mathbf{E}_n|_{\mathcal{L}^2(\Omega)},\quad |\mathbf{M}_n|_{\mathcal{H}^1(\Omega)}\leq c\,|\mathbf{E}_{n,\mathsf{d}}|_{\mathcal{L}^2(\Omega)}\leq c\,|\mathbf{E}_n|_{\mathcal{L}^2(\Omega)}.$$

By Rellich's selection theorem, there exist subsequences, again denoted by $(\mathbf{v}_n)$ and $(\mathbf{M}_n)$, such that $(\mathbf{v}_n)$ converges in $\mathsf{L}^2(\Omega)$ and $(\mathbf{M}_n)$ converges in $\mathcal{L}^2(\Omega)$. Thus with $\mathbf{E}_{n,m}:=\mathbf{E}_n-\mathbf{E}_m$, and similarly for $\mathbf{E}_{n,m,\mathsf{r}}$, $\mathbf{E}_{n,m,\mathsf{d}}$, $\mathbf{v}_{n,m}$, $\mathbf{M}_{n,m}$, we see

$$\begin{aligned}|\mathbf{E}_{n,m,\mathsf{r}}|^2_{\mathcal{L}^2(\Omega)}&=\langle\mathbf{E}_{n,m,\mathsf{r}},\operatorname{devGrad}\mathbf{v}_{n,m}\rangle_{\mathcal{L}^2(\Omega)}=-\langle\operatorname{Div}\mathbf{E}_{n,m,\mathsf{r}},\mathbf{v}_{n,m}\rangle_{\mathsf{L}^2(\Omega)}\\&=-\langle\operatorname{Div}\mathbf{E}_{n,m},\mathbf{v}_{n,m}\rangle_{\mathsf{L}^2(\Omega)}\leq c\,|\mathbf{v}_{n,m}|_{\mathsf{L}^2(\Omega)},\\|\mathbf{E}_{n,m,\mathsf{d}}|^2_{\mathcal{L}^2(\Omega)}&=\langle\mathbf{E}_{n,m,\mathsf{d}},\operatorname{Curl}\mathbf{M}_{n,m}\rangle_{\mathcal{L}^2(\Omega)}=\langle\operatorname{symCurl}\mathbf{E}_{n,m,\mathsf{d}},\mathbf{M}_{n,m}\rangle_{\mathcal{L}^2(\Omega)}\\&=\langle\operatorname{symCurl}\mathbf{E}_{n,m},\mathbf{M}_{n,m}\rangle_{\mathcal{L}^2(\Omega)}\leq c\,|\mathbf{M}_{n,m}|_{\mathcal{L}^2(\Omega)}.\end{aligned}$$

Note that here the symmetry of $\mathbf{M}_{n,m}$ is crucial. Finally, $(\mathbf{E}_n)$ is a Cauchy sequence in $\mathcal{L}^2_{\mathbb{T}}(\Omega)$. So

$$\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega)\cap\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega)\hookrightarrow\mathcal{L}^2_{\mathbb{T}}(\Omega)$$

is compact. ■

For general topologies, we will use a partition of unity argument. The next lemma, which we will prove in the Appendix, provides the necessary tools for this.

Lemma 3.20: *Let* $\varphi\in\mathring{\mathsf{C}}^{\infty}(\mathbb{R}^3)$.

(i) *If* $\mathbf{M}\in\mathring{\mathcal{H}}(\operatorname{Curl},\Omega)$ *resp.* $\mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega)$ *resp.* $\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Curl},\Omega)$, *then* $\varphi\mathbf{M}\in\mathring{\mathcal{H}}(\operatorname{Curl},\Omega)$ *resp.* $\mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega)$ *resp.* $\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Curl},\Omega)$ *and*

$$\operatorname{Curl}(\varphi\mathbf{M})=\varphi\operatorname{Curl}\mathbf{M}+\operatorname{grad}\varphi\times\mathbf{M}.\tag{24}$$

(ii) *If* $\mathbf{M}\in\mathcal{H}(\operatorname{Curl},\Omega)$ *resp.* $\mathcal{H}_{\mathbb{S}}(\operatorname{Curl},\Omega)$ *resp.* $\mathcal{H}_{\mathbb{T}}(\operatorname{Curl},\Omega)$, *then* $\varphi\mathbf{M}\in\mathcal{H}(\operatorname{Curl},\Omega)$ *resp.* $\mathcal{H}_{\mathbb{S}}(\operatorname{Curl},\Omega)$ *resp.* $\mathcal{H}_{\mathbb{T}}(\operatorname{Curl},\Omega)$ *and* (24) *holds.*

(iii) *If* $\mathbf{E}\in\mathring{\mathcal{H}}(\operatorname{Div},\Omega)$ *resp.* $\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega)$ *resp.* $\mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Div},\Omega)$, *then* $\varphi\mathbf{E}\in\mathring{\mathcal{H}}(\operatorname{Div},\Omega)$ *resp.* $\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega)$ *resp.* $\mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Div},\Omega)$ *and*

$$\operatorname{Div}(\varphi\mathbf{E})=\varphi\operatorname{Div}\mathbf{E}+\operatorname{grad}\varphi\cdot\mathbf{E}.\tag{25}$$

(iv) *If* $\mathbf{E} \in \mathcal{H}(\operatorname{Div}, \Omega)$ *resp.* $\mathcal{H}_{\mathbb{T}}(\operatorname{Div}, \Omega)$ *resp.* $\mathcal{H}_{\mathbb{S}}(\operatorname{Div}, \Omega)$, *then* $\varphi\mathbf{E} \in \mathcal{H}(\operatorname{Div}, \Omega)$ *resp.* $\mathcal{H}_{\mathbb{T}}(\operatorname{Div}, \Omega)$ *resp.* $\mathcal{H}_{\mathbb{S}}(\operatorname{Div}, \Omega)$ *and* (25) *holds.*

(v) *If* $\mathbf{E} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega)$, *then* $\varphi\mathbf{E} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega)$ *and*

$$\operatorname{symCurl}(\varphi\mathbf{E}) = \varphi \operatorname{symCurl}\mathbf{E} + \operatorname{sym}(\operatorname{grad}\varphi \times \mathbf{E}).$$

(vi) *If* $\mathbf{M} \in \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}, \Omega)$, *then* $\varphi\mathbf{M} \in \mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv}, \Omega)$ *and*

$$\operatorname{divDiv}(\varphi\mathbf{M}) = \varphi \operatorname{divDiv}\mathbf{M} + 2\operatorname{grad}\varphi \cdot \operatorname{Div}\mathbf{M} + \operatorname{tr}(\mathbf{M}\operatorname{Gradgrad}\varphi).$$

By mollifying these formulas extend to $\varphi \in \mathring{\mathsf{C}}^{0,1}(\mathbb{R}^3)$ *resp.* $\varphi \in \mathring{\mathsf{C}}^{1,1}(\mathbb{R}^3)$.

Here $\operatorname{grad}\varphi\,\times$ resp. $\operatorname{grad}\varphi\,\cdot$ is applied row-wise to a tensor $\mathbf{M}$ and we see $\operatorname{grad}\varphi \cdot \mathbf{M} = \mathbf{M}\operatorname{grad}\varphi$ as well as $\operatorname{grad}\varphi \times \mathbf{M} = -\mathbf{M}\operatorname{spn}(\operatorname{grad}\varphi)$. Moreover, we introduce the new space

$$\mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) := \{\mathbf{M} \in \mathcal{L}^2_{\mathbb{S}}(\Omega) \ : \ \operatorname{divDiv}\mathbf{M} \in \mathsf{H}^{-1}(\Omega)\}.$$

Another auxiliary result required for the compactness proof is presented in the next lemma.

Lemma 3.21: *The regular* (*type*) *decomposition*

$$\mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) = \mathring{\mathsf{H}}^{1}(\Omega) \cdot \mathbf{I} \dotplus \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega)$$

holds. More precisely, for $\mathbf{M} \in \mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv}, \Omega)$ *there are unique* $u \in \mathring{\mathsf{H}}^{1}(\Omega)$ *and* $\mathbf{M}_0 \in \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega)$ *such that* $\mathbf{M} = u\,\mathbf{I} + \mathbf{M}_0$. *The scalar function* $u \in \mathring{\mathsf{H}}^{1}(\Omega)$ *is given as the unique solution of the Dirichlet–Poisson problem*

$$\langle \operatorname{grad} u, \operatorname{grad}\varphi\rangle_{\mathsf{L}^2(\Omega)} = -\langle \operatorname{divDiv}\mathbf{M}, \varphi\rangle_{\mathsf{H}^{-1}(\Omega)} \quad \text{for all } \varphi \in \mathring{\mathsf{H}}^{1}(\Omega),$$

and the decomposition is continuous, more precisely there exists $c > 0$, *such that*

$$|u|_{\mathsf{H}^1(\Omega)} \leq c\,|\operatorname{divDiv}\mathbf{M}|_{\mathsf{H}^{-1}(\Omega)}, \quad |\mathbf{M} - u\,\mathbf{I}|_{\mathsf{L}^2(\Omega)} \leq c\,|\mathbf{M}|_{\mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv},\Omega)}.$$

Proof: The unique solution $u \in \mathring{\mathsf{H}}^{1}(\Omega)$ satisfies

$$\mathsf{H}^{-1}(\Omega) \ni \operatorname{divDiv} u\,\mathbf{I} = \operatorname{div}\operatorname{grad} u = \operatorname{divDiv}\mathbf{M},$$

i.e. $\mathbf{M}_0 := \mathbf{M} - u\,\mathbf{I} \in \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega)$, which shows the decomposition. Moreover,

$$|u|_{\mathsf{H}^1(\Omega)} \leq (1 + c_{\mathsf{g}}^2)^{1/2}\,|\operatorname{divDiv}\mathbf{M}|_{\mathsf{H}^{-1}(\Omega)}$$

shows that u depends continuously on $\mathbf{M}$ and hence also $\mathbf{M}_0$ since

$$|\mathbf{M}_0|_{\mathsf{L}^2(\Omega)} \leq |\mathbf{M}|_{\mathcal{L}^2(\Omega)} + |u|_{\mathsf{L}^2(\Omega)} \leq (2 + c_{\mathsf{g}}^2)^{1/2}\,|\mathbf{M}|_{\mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv},\Omega)}.$$

Let $u\,\mathbf{I} \in \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega)$ with $u \in \mathring{\mathsf{H}}^{1}(\Omega)$. Then $0 = \operatorname{divDiv} u\,\mathbf{I} = \operatorname{div}\operatorname{grad} u = \Delta u$, yielding $u = 0$. Hence, the decomposition is direct, completing the proof. ∎

Lemma 3.22: *The embeddings* (22) *and* (23) *are compact, i.e.*

$$\mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \cap \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) \overset{\mathsf{cpt}}{\hookrightarrow} \mathcal{L}^2_{\mathbb{S}}(\Omega), \quad \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \cap \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}, \Omega) \overset{\mathsf{cpt}}{\hookrightarrow} \mathcal{L}^2_{\mathbb{T}}(\Omega).$$

Proof: Let (U_i) be an open covering of $\overline{\Omega}$, such that $\Omega_i := \Omega \cap U_i$ is topologically trivial for all i. As $\overline{\Omega}$ is compact, there is a finite subcovering denoted by $(U_i)_{i=1,\dots,I}$ with $I \in \mathbb{N}$. Let (φ_i) with $\varphi_i \in \overset{\circ}{\mathsf{C}}{}^{\infty}(U_i)$ be a partition of unity subordinate to (U_i). Suppose $(\mathbf{E}_n) \subset \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\text{symCurl}, \Omega) \cap \overset{\circ}{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\text{Div}, \Omega)$ is a bounded sequence. Then $\mathbf{E}_n = \sum_{i=1}^{I} \varphi_i \mathbf{E}_n$ and $(\varphi_i \mathbf{E}_n) \subset \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\text{symCurl}, \Omega_i) \cap \overset{\circ}{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\text{Div}, \Omega_i)$ is a bounded sequence for all i by Lemma 3.20. As Ω_i is topologically trivial, there exists a subsequence, again denoted by $(\varphi_i \mathbf{E}_n)$, which is a Cauchy sequence in $\boldsymbol{\mathcal{L}}^2(\Omega_i)$ by Lemma 3.19. Picking successively subsequences yields that $(\varphi_i \mathbf{E}_n)$ is a Cauchy sequence in $\boldsymbol{\mathcal{L}}^2(\Omega_j)$ for all j. Hence $(\mathbf{E}_n)$ is a Cauchy sequence in $\boldsymbol{\mathcal{L}}^2(\Omega)$. So the second embedding of the lemma is compact. Let $(\mathbf{M}_n) \subset \overset{\circ}{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\text{Curl}, \Omega) \cap \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\text{divDiv}, \Omega)$ be a bounded sequence. Then $\mathbf{M}_n = \sum_{i=1}^{I} \varphi_i \mathbf{M}_n$ and $(\varphi_i \mathbf{M}_n) \subset \overset{\circ}{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\text{Curl}, \Omega_i) \cap \boldsymbol{\mathcal{H}}^{0,-1}_{\mathbb{S}}(\text{divDiv}, \Omega_i)$ is a bounded sequence for all i by Lemma 3.20 as $|\text{Div}\, \mathbf{M}_n|_{\boldsymbol{\mathcal{H}}^{-1}(\Omega)} \leq |\mathbf{M}_n|_{\boldsymbol{\mathcal{L}}^2(\Omega)}$. Using Lemma 3.21 we decompose

$$\varphi_i \mathbf{M}_n = u_{i,n}\, \mathbf{I} + \mathbf{M}_{0,i,n} \in \overset{\circ}{\mathsf{H}}{}^1(\Omega_i) \cdot \mathbf{I} \dotplus \Big(\overset{\circ}{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\text{Curl}, \Omega_i) \cap \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\text{divDiv}\, 0, \Omega_i) \Big).$$

Moreover, $(u_{i,n})$ is bounded in $\overset{\circ}{\mathsf{H}}{}^1(\Omega_i)$ and $(\mathbf{M}_{0,i,n})$ is bounded in $\overset{\circ}{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\text{Curl}, \Omega_i) \cap \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\text{divDiv}\, 0, \Omega_i)$. By Rellich's selection theorem and Lemma 3.19 as well as picking successively subsequences we get that $(\varphi_i \mathbf{M}_n)$ is a Cauchy sequence in $\boldsymbol{\mathcal{L}}^2(\Omega_j)$ for all j. Hence $(\mathbf{M}_n)$ is a Cauchy sequence in $\boldsymbol{\mathcal{L}}^2(\Omega)$, showing that the first embedding of the lemma is also compact and finishing the proof. ■

Utilizing the crucial compact embeddings of Lemma 3.22, we can apply the functional analysis toolbox Section 2.1 to the (linear, densely defined, and closed 'complex') operators A_0, A_1, A_2, A_0^*, A_1^*, A_2^*. In this general case the reduced operators are

$$\begin{aligned}
\mathcal{A}_0 &= \overset{\circ}{\text{Gradgrad}} : \overset{\circ}{\mathsf{H}}{}^2(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \overline{\text{Gradgrad}\, \overset{\circ}{\mathsf{H}}{}^2(\Omega)},\\
\mathcal{A}_1 &= \overset{\circ}{\text{Curl}}_{\mathbb{S}} : \overset{\circ}{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\text{Curl}, \Omega) \cap \overline{\text{symCurl}\, \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\text{symCurl}, \Omega)} \subset \overline{\text{symCurl}\, \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\text{symCurl}, \Omega)} \longrightarrow \overline{\text{Curl}\, \overset{\circ}{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\text{Curl}, \Omega)},\\
\mathcal{A}_2 &= \overset{\circ}{\text{Div}}_{\mathbb{T}} : \overset{\circ}{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\text{Div}, \Omega) \cap \overline{\text{devGrad}\, \mathbf{H}^1(\Omega)} \subset \overline{\text{devGrad}\, \mathbf{H}^1(\Omega)} \longrightarrow \mathsf{RT}_0^{\perp_{\mathbf{L}^2(\Omega)}},\\
\mathcal{A}_0^* &= \text{divDiv}_{\mathbb{S}} : \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\text{divDiv}, \Omega) \cap \overline{\text{Gradgrad}\, \overset{\circ}{\mathsf{H}}{}^2(\Omega)} \subset \overline{\text{Gradgrad}\, \overset{\circ}{\mathsf{H}}{}^2(\Omega)} \longrightarrow \mathsf{L}^2(\Omega),\\
\mathcal{A}_1^* &= \text{symCurl}_{\mathbb{T}} : \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\text{symCurl}, \Omega) \cap \overline{\text{Curl}\, \overset{\circ}{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\text{Curl}, \Omega)} \subset \overline{\text{Curl}\, \overset{\circ}{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\text{Curl}, \Omega)} \longrightarrow \overline{\text{symCurl}\, \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\text{symCurl}, \Omega)},\\
\mathcal{A}_2^* &= -\,\text{devGrad} : \mathbf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathbf{L}^2(\Omega)}} \subset \mathsf{RT}_0^{\perp_{\mathbf{L}^2(\Omega)}} \longrightarrow \overline{\text{devGrad}\, \mathbf{H}^1(\Omega)}
\end{aligned}$$

as

$$\begin{aligned}
\overline{\text{divDiv}\, \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\text{divDiv}, \Omega)} &= \overline{R(\text{divDiv}_{\mathbb{S}})} = N(\overset{\circ}{\text{Gradgrad}})^{\perp_{\mathsf{L}^2(\Omega)}} = \{0\}^{\perp_{\mathsf{L}^2(\Omega)}} = \mathsf{L}^2(\Omega),\\
\overline{\text{Div}\, \overset{\circ}{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\text{Div}, \Omega)} &= \overline{R(\overset{\circ}{\text{Div}}_{\mathbb{T}})} = N(\text{devGrad})^{\perp_{\mathbf{L}^2(\Omega)}} = \mathsf{RT}_0^{\perp_{\mathbf{L}^2(\Omega)}}.
\end{aligned}$$

Note that by the compact embeddings of Lemma 3.22 all ranges are actually closed and we can skip the closure bars. We obtain the following theorem.

Theorem 3.23: *It holds:*

(i) *The ranges*

$$\begin{aligned}
R(\overset{\circ}{\text{Gradgrad}}) &= \text{Gradgrad}\, \overset{\circ}{\mathsf{H}}{}^2(\Omega),\\
\mathsf{L}^2(\Omega) = R(\text{divDiv}_{\mathbb{S}}) &= \text{divDiv}\, \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\text{divDiv}, \Omega) = \text{divDiv}\Big(\boldsymbol{\mathcal{H}}_{\mathbb{S}}(\text{divDiv}, \Omega) \cap \text{Gradgrad}\, \overset{\circ}{\mathsf{H}}{}^2(\Omega) \Big),\\
R(\overset{\circ}{\text{Curl}}_{\mathbb{S}}) &= \text{Curl}\, \overset{\circ}{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\text{Curl}, \Omega) = \text{Curl}\Big(\overset{\circ}{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\text{Curl}, \Omega) \cap \text{symCurl}\, \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\text{symCurl}, \Omega) \Big),
\end{aligned}$$

$$R(\operatorname{symCurl}_{\mathbb{T}}) = \operatorname{symCurl} \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) = \operatorname{symCurl} \left(\boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \cap \operatorname{Curl} \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \right),$$

$$\mathsf{RT}_0^{\perp_{\mathbf{L}^2(\Omega)}} = R(\mathring{\operatorname{Div}}_{\mathbb{T}}) = \operatorname{Div} \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div}, \Omega) = \operatorname{Div} \left(\mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div}, \Omega) \cap \operatorname{devGrad} \mathbf{H}^1(\Omega) \right),$$

$$R(\operatorname{devGrad}) = \operatorname{devGrad} \mathbf{H}^1(\Omega) = \operatorname{devGrad} \left(\mathbf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathbf{L}^2(\Omega)}} \right)$$

are closed. The more regular potentials on the right-hand sides are uniquely determined and depend linearly and continuously on the data, see (v).

(ii) *The cohomology groups*

$$\mathscr{H}_{\mathsf{D},\mathbb{S}}(\Omega) := \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) \cap \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega), \quad \mathscr{H}_{\mathsf{N},\mathbb{T}}(\Omega) := \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega) \cap \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl} 0, \Omega)$$

are finite dimensional and may be called symmetric Dirichlet resp. deviatoric Neumann tensor fields.

(iii) *The Hilbert complexes from Remark* 3.8, *i.e.*

$$\{0\} \xrightarrow{0} \mathring{\mathsf{H}}^2(\Omega) \xrightarrow{\mathring{\operatorname{Gradgrad}}} \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \xrightarrow{\mathring{\operatorname{Curl}}_{\mathbb{S}}} \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div}, \Omega) \xrightarrow{\mathring{\operatorname{Div}}_{\mathbb{T}}} \mathbf{L}^2(\Omega) \xrightarrow{\pi_{\mathsf{RT}_0}} \mathsf{RT}_0$$

and its adjoint

$$\{0\} \xleftarrow{0} \mathsf{L}^2(\Omega) \xleftarrow{\operatorname{divDiv}_{\mathbb{S}}} \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) \xleftarrow{\operatorname{symCurl}_{\mathbb{T}}} \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \xleftarrow{-\operatorname{devGrad}} \mathbf{H}^1(\Omega) \xleftarrow{\iota_{\mathsf{RT}_0}} \mathsf{RT}_0,$$

are closed. They are also exact, if and only if $\mathscr{H}_{\mathsf{D},\mathbb{S}}(\Omega) = \{0\}$, $\mathscr{H}_{\mathsf{N},\mathbb{T}}(\Omega) = \{0\}$. *The latter holds, if* Ω *is topologically trivial.*

(iv) *The Helmholtz type decompositions*

$$\begin{aligned}
\boldsymbol{\mathcal{L}}^2_{\mathbb{S}}(\Omega) &= \operatorname{Gradgrad} \mathring{\mathsf{H}}^2(\Omega) \oplus_{\boldsymbol{\mathcal{L}}^2_{\mathbb{S}}(\Omega)} \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega)\\
&= \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) \oplus_{\boldsymbol{\mathcal{L}}^2_{\mathbb{S}}(\Omega)} \operatorname{symCurl} \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl}, \Omega)\\
&= \operatorname{Gradgrad} \mathring{\mathsf{H}}^2(\Omega) \oplus_{\boldsymbol{\mathcal{L}}^2_{\mathbb{S}}(\Omega)} \mathscr{H}_{\mathsf{D},\mathbb{S}}(\Omega) \oplus_{\boldsymbol{\mathcal{L}}^2_{\mathbb{S}}(\Omega)} \operatorname{symCurl} \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl}, \Omega),\\
\boldsymbol{\mathcal{L}}^2_{\mathbb{T}}(\Omega) &= \operatorname{Curl} \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \oplus_{\boldsymbol{\mathcal{L}}^2_{\mathbb{T}}(\Omega)} \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl} 0, \Omega)\\
&= \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega) \oplus_{\boldsymbol{\mathcal{L}}^2_{\mathbb{T}}(\Omega)} \operatorname{devGrad} \mathbf{H}^1(\Omega)\\
&= \operatorname{Curl} \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \oplus_{\boldsymbol{\mathcal{L}}^2_{\mathbb{T}}(\Omega)} \mathscr{H}_{\mathsf{N},\mathbb{T}}(\Omega) \oplus_{\boldsymbol{\mathcal{L}}^2_{\mathbb{T}}(\Omega)} \operatorname{devGrad} \mathbf{H}^1(\Omega)
\end{aligned}$$

are valid.

(v) *There exist positive constants* c_{Gg}, c_{D}, c_{R}, *such that the Friedrichs/Poincaré type estimates*

$$\begin{aligned}
&\forall\, u \in \mathring{\mathsf{H}}^2(\Omega) && |u|_{\mathsf{L}^2(\Omega)} \leq c_{\mathsf{Gg}}\, |\operatorname{Gradgrad} u|_{\boldsymbol{\mathcal{L}}^2(\Omega)},\\
&\forall\, \mathbf{M} \in \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) \cap \operatorname{Gradgrad} \mathring{\mathsf{H}}^2(\Omega) && |\mathbf{M}|_{\boldsymbol{\mathcal{L}}^2(\Omega)} \leq c_{\mathsf{Gg}}\, |\operatorname{divDiv} \mathbf{M}|_{\mathsf{L}^2(\Omega)},\\
&\forall\, \mathbf{E} \in \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div}, \Omega) \cap \operatorname{devGrad} \mathbf{H}^1(\Omega) && |\mathbf{E}|_{\boldsymbol{\mathcal{L}}^2(\Omega)} \leq c_{\mathsf{D}}\, |\operatorname{Div} \mathbf{E}|_{\mathbf{L}^2(\Omega)},\\
&\forall\, \mathbf{v} \in \mathbf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathbf{L}^2(\Omega)}} && |\mathbf{v}|_{\mathbf{L}^2(\Omega)} \leq c_{\mathsf{D}}\, |\operatorname{devGrad} \mathbf{v}|_{\boldsymbol{\mathcal{L}}^2(\Omega)},
\end{aligned}$$

$$\forall\, \mathbf{M} \in \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\mathrm{Curl}, \Omega) \cap \mathrm{symCurl}\, \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\mathrm{symCurl}, \Omega) \qquad |\mathbf{M}|_{\boldsymbol{\mathcal{L}}^2(\Omega)} \le c_{\mathsf{R}}\, |\mathrm{Curl}\, \mathbf{M}|_{\boldsymbol{\mathcal{L}}^2(\Omega)},$$
$$\forall\, \mathbf{E} \in \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\mathrm{symCurl}, \Omega) \cap \mathrm{Curl}\, \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\mathrm{Curl}, \Omega) \qquad |\mathbf{E}|_{\boldsymbol{\mathcal{L}}^2(\Omega)} \le c_{\mathsf{R}}\, |\mathrm{symCurl}\, \mathbf{E}|_{\boldsymbol{\mathcal{L}}^2(\Omega)}$$

hold[2].

(vi) *The inverse operators*

$$
\begin{aligned}
(\mathring{\mathrm{Gradgrad}})^{-1} &: \mathrm{Gradgrad}\, \mathring{\mathsf{H}}^2(\Omega) \longrightarrow \mathring{\mathsf{H}}^2(\Omega),\\
(\mathrm{divDiv}_{\mathbb{S}})^{-1} &: \mathsf{L}^2(\Omega) \longrightarrow \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\mathrm{divDiv}, \Omega) \cap \mathrm{Gradgrad}\, \mathring{\mathsf{H}}^2(\Omega),\\
(\mathring{\mathrm{Div}}_{\mathbb{T}})^{-1} &: \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} \longrightarrow \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\mathrm{Div}, \Omega) \cap \mathrm{devGrad}\, \mathbf{H}^1(\Omega),\\
(\mathrm{devGrad})^{-1} &: \mathrm{devGrad}\, \mathbf{H}^1(\Omega) \longrightarrow \mathbf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathbf{L}^2(\Omega)}},\\
(\mathring{\mathrm{Curl}}_{\mathbb{S}})^{-1} &: \mathrm{Curl}\, \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\mathrm{Curl}, \Omega) \longrightarrow \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\mathrm{Curl}, \Omega) \cap \mathrm{symCurl}\, \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\mathrm{symCurl}, \Omega),\\
(\mathrm{symCurl}_{\mathbb{T}})^{-1} &: \mathrm{symCurl}\, \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\mathrm{symCurl}, \Omega) \longrightarrow \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\mathrm{symCurl}, \Omega) \cap \mathrm{Curl}\, \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\mathrm{Curl}, \Omega)
\end{aligned}
$$

are continuous with norms $(1 + c_{\mathsf{Gg}}^2)^{1/2}$ *resp.* $(1 + c_{\mathsf{D}}^2)^{1/2}$, *resp.* $(1 + c_{\mathsf{R}}^2)^{1/2}$, *and their modifications*

$$
\begin{aligned}
(\mathring{\mathrm{Gradgrad}})^{-1} &: \mathrm{Gradgrad}\, \mathring{\mathsf{H}}^2(\Omega) \longrightarrow \mathring{\mathsf{H}}^1(\Omega) \subset \mathsf{L}^2(\Omega),\\
(\mathrm{divDiv}_{\mathbb{S}})^{-1} &: \mathsf{L}^2(\Omega) \longrightarrow \mathrm{Gradgrad}\, \mathring{\mathsf{H}}^2(\Omega) \subset \boldsymbol{\mathcal{L}}^2_{\mathbb{S}}(\Omega),\\
(\mathring{\mathrm{Div}}_{\mathbb{T}})^{-1} &: \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} \longrightarrow \mathrm{devGrad}\, \mathbf{H}^1(\Omega) \subset \boldsymbol{\mathcal{L}}^2_{\mathbb{T}}(\Omega),\\
(\mathrm{devGrad})^{-1} &: \mathrm{devGrad}\, \mathbf{H}^1(\Omega) \longrightarrow \mathsf{RT}_0^{\perp_{\mathbf{L}^2(\Omega)}} \subset \mathbf{L}^2(\Omega),\\
(\mathring{\mathrm{Curl}}_{\mathbb{S}})^{-1} &: \mathrm{Curl}\, \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\mathrm{Curl}, \Omega) \longrightarrow \mathrm{symCurl}\, \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\mathrm{symCurl}, \Omega) \subset \boldsymbol{\mathcal{L}}^2_{\mathbb{S}}(\Omega),\\
(\mathrm{symCurl}_{\mathbb{T}})^{-1} &: \mathrm{symCurl}\, \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\mathrm{symCurl}, \Omega) \longrightarrow \mathrm{Curl}\, \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\mathrm{Curl}, \Omega) \subset \boldsymbol{\mathcal{L}}^2_{\mathbb{T}}(\Omega)
\end{aligned}
$$

are compact with norms c_{Gg}, c_{D}, *resp.* c_{R}.

We note

$$
\begin{aligned}
\mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\mathrm{Curl}\, 0, \Omega) &= \mathrm{Gradgrad}\, \mathring{\mathsf{H}}^2(\Omega) \oplus_{\boldsymbol{\mathcal{L}}^2_{\mathbb{S}}(\Omega)} \mathscr{H}_{\mathsf{D},\mathbb{S}}(\Omega),\\
\boldsymbol{\mathcal{H}}_{\mathbb{S}}(\mathrm{divDiv}\, 0, \Omega) &= \mathrm{symCurl}\, \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\mathrm{symCurl}, \Omega) \oplus_{\boldsymbol{\mathcal{L}}^2_{\mathbb{S}}(\Omega)} \mathscr{H}_{\mathsf{D},\mathbb{S}}(\Omega),\\
\mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\mathrm{Div}\, 0, \Omega) &= \mathrm{Curl}\, \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\mathrm{Curl}, \Omega) \oplus_{\boldsymbol{\mathcal{L}}^2_{\mathbb{T}}(\Omega)} \mathscr{H}_{\mathsf{N},\mathbb{T}}(\Omega),\\
\boldsymbol{\mathcal{H}}_{\mathbb{T}}(\mathrm{symCurl}\, 0, \Omega) &= \mathrm{devGrad}\, \mathbf{H}^1(\Omega) \oplus_{\boldsymbol{\mathcal{L}}^2_{\mathbb{T}}(\Omega)} \mathscr{H}_{\mathsf{N},\mathbb{T}}(\Omega).
\end{aligned}
\tag{26}
$$

Finally, even parts of Theorems 3.10, 3.17, and Corollary 3.18, extend to the general case, i.e. we have regular potentials and regular decompositions for bounded strong Lipschitz domains as well.

Theorem 3.24: *The regular decompositions*

(i) $\mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\mathrm{Curl}, \Omega) = \mathring{\boldsymbol{\mathcal{H}}}^1_{\mathbb{S}}(\Omega) + \mathrm{Gradgrad}\, \mathring{\mathsf{H}}^2(\Omega)$,

(ii) $\mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\mathrm{Div}, \Omega) = \mathring{\boldsymbol{\mathcal{H}}}^1_{\mathbb{T}}(\Omega) + \mathrm{Curl}\, \mathring{\boldsymbol{\mathcal{H}}}^1_{\mathbb{S}}(\Omega)$,

(iii) $\boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl},\Omega) = \boldsymbol{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega) + \operatorname{devGrad}\mathsf{\mathbf{H}}^{1}(\Omega)$,
(iv) $\boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv},\Omega) = \boldsymbol{\mathcal{H}}^{2}_{\mathbb{S}}(\Omega) + \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv}0,\Omega)$

hold with linear and continuous (regular) decomposition resp. potential operators

$$\begin{aligned}
\tilde{\mathrm{P}}_{\mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl},\Omega),\mathring{\boldsymbol{\mathcal{H}}}^{1}_{\mathbb{S}}(\Omega)} &: \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl},\Omega) \longrightarrow \mathring{\boldsymbol{\mathcal{H}}}^{1}_{\mathbb{S}}(\Omega),\\
\tilde{\mathrm{P}}_{\mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl},\Omega),\mathring{\mathsf{H}}^{2}(\Omega)} &: \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl},\Omega) \longrightarrow \mathring{\mathsf{H}}^{2}(\Omega),\\
\tilde{\mathrm{P}}_{\mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div},\Omega),\mathring{\boldsymbol{\mathcal{H}}}^{1}_{\mathbb{T}}(\Omega)} &: \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div},\Omega) \longrightarrow \mathring{\boldsymbol{\mathcal{H}}}^{1}_{\mathbb{T}}(\Omega),\\
\tilde{\mathrm{P}}_{\mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div},\Omega),\mathring{\boldsymbol{\mathcal{H}}}^{1}_{\mathbb{S}}(\Omega)} &: \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{T}}(\operatorname{Div},\Omega) \longrightarrow \mathring{\boldsymbol{\mathcal{H}}}^{1}_{\mathbb{S}}(\Omega),\\
\tilde{\mathrm{P}}_{\boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl},\Omega),\boldsymbol{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega)} &: \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl},\Omega) \longrightarrow \boldsymbol{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega),\\
\tilde{\mathrm{P}}_{\boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl},\Omega),\mathsf{\mathbf{H}}^{1}(\Omega)} &: \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl},\Omega) \longrightarrow \mathsf{\mathbf{H}}^{1}(\Omega),\\
\tilde{\mathrm{P}}_{\boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv},\Omega),\boldsymbol{\mathcal{H}}^{2}_{\mathbb{S}}(\Omega)} &: \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv},\Omega) \longrightarrow \boldsymbol{\mathcal{H}}^{2}_{\mathbb{S}}(\Omega),\\
\tilde{\mathrm{P}}_{\boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv},\Omega),\boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv}0,\Omega)} &: \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv},\Omega) \longrightarrow \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\operatorname{divDiv}0,\Omega).
\end{aligned}$$

Proof: As in the proof of Lemma 3.22, let (U_i) be an open covering of $\overline{\Omega}$, such that $\Omega_i := \Omega \cap U_i$ is topologically trivial for all i. As $\overline{\Omega}$ is compact, there is a finite subcovering denoted by $(U_i)_{i=1,\dots,I}$ with $I \in \mathbb{N}$. Let (φ_i) with $\varphi_i \in \mathring{\mathsf{C}}^{\infty}(U_i)$ be a partition of unity subordinate to (U_i) and let additionally $\phi_i \in \mathring{\mathsf{C}}^{\infty}(U_i)$ with $\phi_i|_{\operatorname{supp}\varphi_i} = 1$. To prove (i), suppose $\mathbf{M} \in \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl},\Omega)$. By Lemma 3.20 and Theorem 3.17 we have

$$\varphi_i\mathbf{M} \in \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl},\Omega_i) = \mathring{\boldsymbol{\mathcal{H}}}^{1}_{\mathbb{S}}(\Omega_i) + \mathring{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\operatorname{Curl}0,\Omega_i) = \mathring{\boldsymbol{\mathcal{H}}}^{1}_{\mathbb{S}}(\Omega_i) + \operatorname{Gradgrad}\mathring{\mathsf{H}}^{2}(\Omega_i).$$

Hence, $\varphi_i\mathbf{M} = \mathbf{M}_i + \operatorname{Gradgrad} u_i$ with $\mathbf{M}_i \in \mathring{\boldsymbol{\mathcal{H}}}^{1}_{\mathbb{S}}(\Omega_i)$ and $u_i \in \mathring{\mathsf{H}}^{2}(\Omega_i)$. Let $\hat{\mathbf{M}}_i$ and $\hat{u}_i$ denote the extensions by zero of $\mathbf{M}_i$ and u_i. Then $\hat{\mathbf{M}}_i \in \mathring{\boldsymbol{\mathcal{H}}}^{1}_{\mathbb{S}}(\Omega)$ and $\hat{u}_i \in \mathring{\mathsf{H}}^{2}(\Omega)$. Thus

$$\mathbf{M} = \sum_i \varphi_i\mathbf{M} = \sum_i \hat{\mathbf{M}}_i + \operatorname{Gradgrad}\sum_i \hat{u}_i \in \mathring{\boldsymbol{\mathcal{H}}}^{1}_{\mathbb{S}}(\Omega) + \operatorname{Gradgrad}\mathring{\mathsf{H}}^{2}(\Omega),$$

and all applied operations are continuous. Similarly we prove (ii). To show (iii), let $\mathbf{E} \in \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl},\Omega)$. By Lemma 3.20 and Theorem 3.17 we have

$$\begin{aligned}
\varphi_i\mathbf{E} \in \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl},\Omega_i) = \boldsymbol{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega_i) + \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\operatorname{symCurl}0,\Omega_i) &= \boldsymbol{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega_i)\\
&\quad + \operatorname{devGrad}\mathsf{\mathbf{H}}^{1}(\Omega_i).
\end{aligned}$$

Hence, $\varphi_i\mathbf{E} = \mathbf{E}_i + \operatorname{devGrad}\mathbf{v}_i$ with $\mathbf{E}_i \in \boldsymbol{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega_i)$ and $\mathbf{v}_i \in \mathsf{\mathbf{H}}^{1}(\Omega_i)$. In Ω_i we observe

$$\begin{aligned}
\varphi_i\mathbf{E} = \phi_i\varphi_i\mathbf{E} &= \phi_i\mathbf{E}_i + \phi_i\operatorname{devGrad}\mathbf{v}_i\\
&= \phi_i\mathbf{E}_i - \operatorname{dev}(\mathbf{v}_i \cdot \operatorname{grad}^{\top}\phi_i) + \operatorname{devGrad}(\phi_i\mathbf{v}_i) \in \boldsymbol{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega_i) + \operatorname{devGrad}\mathsf{\mathbf{H}}^{1}(\Omega_i).
\end{aligned}$$

Let $\hat{\mathbf{E}}_i$ and $\hat{\mathbf{v}}_i$ denote the extensions by zero of $\phi_i\mathbf{E}_i - \operatorname{dev}(\mathbf{v}_i \cdot \operatorname{grad}^\top \phi_i)$ and $\phi_i\mathbf{v}_i$. Then $\hat{\mathbf{E}}_i \in \mathcal{H}^1_{\mathbb{T}}(\Omega)$ and $\hat{\mathbf{v}}_i \in \mathsf{H}^1(\Omega)$. Thus

$$\mathbf{E} = \sum_i \varphi_i\mathbf{E} = \sum_i \hat{\mathbf{E}}_i + \operatorname{devGrad}\sum_i \hat{\mathbf{v}}_i \in \mathcal{H}^1_{\mathbb{T}}(\Omega) + \operatorname{devGrad}\mathsf{H}^1(\Omega),$$

and all applied operations are continuous. To show (iv), let $\mathbf{M} \in \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega)$. Then $\operatorname{divDiv}\mathbf{M} \in \mathsf{L}^2(\Omega)$ and by Theorem 3.10 and Remark 3.11 (ii) there is some $\tilde{\mathbf{M}} \in \mathcal{H}^2_{\mathbb{S}}(\Omega)$, together with a linear and continuous potential operator, with $\operatorname{divDiv}\tilde{\mathbf{M}} = \operatorname{divDiv}\mathbf{M}$. Therefore, we have $\mathbf{M} - \tilde{\mathbf{M}} \in \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}0,\Omega)$, completing the proof. ■

Applying $\mathring{\operatorname{Curl}}_{\mathbb{S}}$, $\mathring{\operatorname{Div}}_{\mathbb{T}}$, and $\operatorname{symCurl}_{\mathbb{T}}$, $\operatorname{divDiv}_{\mathbb{S}}$ to the regular decompositions in Theorem 3.24 we get the following regular potentials.

Theorem 3.25: *It holds*

(i) $R(\mathring{\operatorname{Curl}}_{\mathbb{S}}) = \operatorname{Curl}\mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega) = \operatorname{Curl}\mathring{\mathcal{H}}^1_{\mathbb{S}}(\Omega)$,
(ii) $\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} = R(\mathring{\operatorname{Div}}_{\mathbb{T}}) = \operatorname{Div}\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega) = \operatorname{Div}\mathring{\mathcal{H}}^1_{\mathbb{T}}(\Omega)$,
(iii) $R(\operatorname{symCurl}_{\mathbb{T}}) = \operatorname{symCurl}\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega) = \operatorname{symCurl}\mathcal{H}^1_{\mathbb{T}}(\Omega)$,
(iv) $\mathsf{L}^2(\Omega) = R(\operatorname{divDiv}_{\mathbb{S}}) = \operatorname{divDiv}\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega) = \operatorname{divDiv}\mathcal{H}^2_{\mathbb{S}}(\Omega)$

with corresponding linear and continuous (*regular*) *potential operators* (*on the right-hand sides*).

Using Theorem 3.23, canonical linear and continuous regular potential operators in the latter theorem are given by

$$\begin{aligned}
\tilde{\mathrm{P}}_{\mathring{\operatorname{Curl}}_{\mathbb{S}}} &:= \tilde{\mathrm{P}}_{\mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega),\mathring{\mathcal{H}}^1_{\mathbb{S}}(\Omega)}(\mathring{\operatorname{Curl}}_{\mathbb{S}})^{-1} : \operatorname{Curl}\mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega) \longrightarrow \mathring{\mathcal{H}}^1_{\mathbb{S}}(\Omega),\\
\tilde{\mathrm{P}}_{\mathring{\operatorname{Div}}_{\mathbb{T}}} &:= \tilde{\mathrm{P}}_{\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega),\mathring{\mathcal{H}}^1_{\mathbb{T}}(\Omega)}(\mathring{\operatorname{Div}}_{\mathbb{T}})^{-1} : \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} \longrightarrow \mathring{\mathcal{H}}^1_{\mathbb{T}}(\Omega),\\
\tilde{\mathrm{P}}_{\operatorname{symCurl}_{\mathbb{T}}} &:= \tilde{\mathrm{P}}_{\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega),\mathcal{H}^1_{\mathbb{T}}(\Omega)}(\operatorname{symCurl}_{\mathbb{T}})^{-1} : \operatorname{symCurl}\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega) \longrightarrow \mathcal{H}^1_{\mathbb{T}}(\Omega),\\
\tilde{\mathrm{P}}_{\operatorname{divDiv}_{\mathbb{S}}} &:= \tilde{\mathrm{P}}_{\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega),\mathcal{H}^2_{\mathbb{S}}(\Omega)}(\operatorname{divDiv}_{\mathbb{S}})^{-1} : \mathsf{L}^2(\Omega) \longrightarrow \mathcal{H}^2_{\mathbb{S}}(\Omega).
\end{aligned} \tag{27}$$

We get the following direct regular decompositions.

Corollary 3.26: *The direct regular decompositions*

$$\begin{aligned}
\mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega) &= \tilde{\mathrm{P}}_{\mathring{\operatorname{Curl}}_{\mathbb{S}}}\operatorname{Curl}\mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl},\Omega) \dotplus \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl}0,\Omega),\\
\mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div},\Omega) &= \tilde{\mathrm{P}}_{\mathring{\operatorname{Div}}_{\mathbb{T}}}\mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} \dotplus \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div}0,\Omega),\\
\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega) &= \tilde{\mathrm{P}}_{\operatorname{symCurl}_{\mathbb{T}}}\operatorname{symCurl}\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega) \dotplus \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}0,\Omega),\\
\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega) &= \tilde{\mathrm{P}}_{\operatorname{divDiv}_{\mathbb{S}}}\mathsf{L}^2(\Omega) \dotplus \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}0,\Omega)
\end{aligned}$$

hold. Moreover,

$$\tilde{\mathrm{P}}_{\overset{\circ}{\mathrm{Curl}}_{\mathbb{S}}} \operatorname{Curl} \overset{\circ}{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\mathrm{Curl},\Omega) \subset \overset{\circ}{\boldsymbol{\mathcal{H}}}{}^{1}_{\mathbb{S}}(\Omega), \quad \tilde{\mathrm{P}}_{\mathrm{symCurl}_{\mathbb{T}}} \operatorname{symCurl} \boldsymbol{\mathcal{H}}_{\mathbb{T}}(\mathrm{symCurl},\Omega) \subset \boldsymbol{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega),$$

$$\tilde{\mathrm{P}}_{\overset{\circ}{\mathrm{Div}}_{\mathbb{T}}} \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} \subset \overset{\circ}{\boldsymbol{\mathcal{H}}}{}^{1}_{\mathbb{T}}(\Omega), \quad \tilde{\mathrm{P}}_{\mathrm{divDiv}_{\mathbb{S}}} \mathsf{L}^2(\Omega) \subset \boldsymbol{\mathcal{H}}^{2}_{\mathbb{S}}(\Omega).$$

Note that the second summands on the right-hand sides may be further decomposed by (26), Theorem 3.25, and (27).

Proof: For $\mathbf{M} \in \overset{\circ}{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\mathrm{Curl}\,0,\Omega) \cap \tilde{\mathrm{P}}_{\overset{\circ}{\mathrm{Curl}}_{\mathbb{S}}} \operatorname{Curl} \overset{\circ}{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\mathrm{Curl},\Omega)$ we have $\mathbf{M} = \tilde{\mathrm{P}}_{\overset{\circ}{\mathrm{Curl}}_{\mathbb{S}}} \mathbf{N}$ with some $\mathbf{N} \in \operatorname{Curl} \overset{\circ}{\boldsymbol{\mathcal{H}}}_{\mathbb{S}}(\mathrm{Curl},\Omega)$. Thus $0 = \operatorname{Curl} \mathbf{M} = \mathbf{N}$ showing $\mathbf{M} = 0$ and hence the directness of the first regular decomposition. The other assertions follow similarly. ■

Remark 3.27: While the results about the regular potentials in Theorem 3.25 hold in full generality for all operators, one may wonder that the regular decompositions from Theorem 3.24 hold in full generality only for (i)–(iii), but not for (iv), i.e. we just have in (iv)

$$\boldsymbol{\mathcal{H}}_{\mathbb{S}}(\mathrm{divDiv},\Omega) = \boldsymbol{\mathcal{H}}^{2}_{\mathbb{S}}(\Omega) + \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\mathrm{divDiv}\,0,\Omega) \supset \boldsymbol{\mathcal{H}}^{2}_{\mathbb{S}}(\Omega) + \operatorname{symCurl} \boldsymbol{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega).$$

The reason for the failure of the partition of unity argument from the proof of Theorem 3.24 is the following: Let $\mathbf{M} \in \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\mathrm{divDiv},\Omega)$. By Lemma 3.20 (vi) we just get $\varphi_i\mathbf{M} \in \boldsymbol{\mathcal{H}}^{0,-1}_{\mathbb{S}}(\mathrm{divDiv},\Omega_i)$, see also the proof of Lemma 3.22. Using Lemma 3.21 and Theorem 3.17 we can decompose

$$\varphi_i\mathbf{M} = u_i\,\mathbf{I} + \operatorname{symCurl} \mathbf{E}_i \in \overset{\circ}{\mathsf{H}}{}^{1}(\Omega_i)\cdot\mathbf{I} \dotplus \operatorname{symCurl} \boldsymbol{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega_i)$$

as $\boldsymbol{\mathcal{H}}_{\mathbb{S}}(\mathrm{divDiv}\,0,\Omega_i) = \operatorname{symCurl} \boldsymbol{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega_i)$. In Ω_i we observe

$$\begin{aligned}\varphi_i\mathbf{M} &= \phi_i\varphi_i\mathbf{M} = \phi_i u_i\,\mathbf{I} + \phi_i \operatorname{symCurl} \mathbf{E}_i \\ &= \phi_i u_i\,\mathbf{I} - \operatorname{sym}(\operatorname{grad}\phi_i \times \mathbf{E}_i) + \operatorname{symCurl}(\phi_i\mathbf{E}_i) \in \boldsymbol{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega_i) + \operatorname{symCurl} \boldsymbol{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega_i).\end{aligned}$$

Let $\hat{\mathbf{M}}_i$ and $\hat{\mathbf{E}}_i$ denote the extensions by zero of $\phi_i u_i\,\mathbf{I} - \operatorname{sym}(\operatorname{grad}\phi_i \times \mathbf{E}_i)$ and $\phi_i\mathbf{E}_i$. Then $\hat{\mathbf{M}}_i \in \boldsymbol{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega)$ and $\hat{\mathbf{E}}_i \in \boldsymbol{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega)$ and thus

$$\mathbf{M} = \sum_i \varphi_i\mathbf{M} = \sum_i \hat{\mathbf{M}}_i + \operatorname{symCurl} \sum_i \hat{\mathbf{E}}_i \in \boldsymbol{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega) + \operatorname{symCurl} \boldsymbol{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega),$$

and all applied operations are continuous. Therefore, we obtain

$$\begin{aligned}\boldsymbol{\mathcal{H}}^{2}_{\mathbb{S}}(\Omega) + \operatorname{symCurl} \boldsymbol{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega) \subset \boldsymbol{\mathcal{H}}^{2}_{\mathbb{S}}(\Omega) + \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\mathrm{divDiv}\,0,\Omega) = \boldsymbol{\mathcal{H}}_{\mathbb{S}}(\mathrm{divDiv},\Omega) \subset \boldsymbol{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega) \\ + \operatorname{symCurl} \boldsymbol{\mathcal{H}}^{1}_{\mathbb{T}}(\Omega).\end{aligned}$$

So we have lost one Sobolev order in the summand $\boldsymbol{\mathcal{H}}^{1}_{\mathbb{S}}(\Omega)$.

4. Application to biharmonic problems

By $\Delta^2 = \operatorname{divDiv}\operatorname{Gradgrad}$, a standard (primal) variational formulation of (1) in $\mathbb{R}^3$ reads as follows: For given $f \in \mathsf{H}^{-2}(\Omega)$, find $u \in \mathring{\mathsf{H}}^2(\Omega)$ such that

$$\langle \operatorname{Gradgrad} u, \operatorname{Gradgrad} \phi\rangle_{\mathcal{L}^2(\Omega)} = \langle f, \phi\rangle_{\mathsf{H}^{-2}(\Omega)} \quad \text{for all } \phi \in \mathring{\mathsf{H}}^2(\Omega). \tag{28}$$

Existence, uniqueness, and continuous dependence on f of a solution to (28) is guaranteed by the theorem of Lax–Milgram, see, e.g. [27, 28] or Lemma 3.3. Note that then

$$\mathbf{M} := \operatorname{Gradgrad} u \in \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) \ominus_{\mathcal{L}^2_{\mathbb{S}}(\Omega)} \mathscr{H}_{\mathsf{D},\mathbb{S}}(\Omega) \subset \mathcal{L}^2_{\mathbb{S}}(\Omega)$$

with $\operatorname{divDiv}\mathbf{M} = f \in \mathsf{H}^{-2}(\Omega)$. In other words the operator

$$\operatorname{divDiv} : \mathcal{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{H}^{-2}(\Omega) \tag{29}$$

is surjective and

$$\operatorname{divDiv} : \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl} 0, \Omega) \ominus_{\mathcal{L}^2_{\mathbb{S}}(\Omega)} \mathscr{H}_{\mathsf{D},\mathbb{S}}(\Omega) \to \mathsf{H}^{-2}(\Omega) \tag{30}$$

is bijective and even a topological isomorphism by the bounded inverse theorem. For our decomposition result we need the following variant of the Hilbert complex from Theorem 3.23:

$$\mathsf{RT}_0 \xrightarrow{\iota_{\mathsf{RT}_0}} \mathbf{H}^1(\Omega) \xrightarrow{-\operatorname{devGrad}} \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \xrightarrow{\operatorname{symCurl}_{\mathbb{T}}} \mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) \xrightarrow{\operatorname{divDiv}_{\mathbb{S}}} \mathsf{H}^{-1}(\Omega) \xrightarrow{0} \{0\},$$

where we recall $\mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv}, \Omega)$ from Lemma 3.21. This is obviously also a closed Hilbert complex as $\operatorname{divDiv} : \mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) \to \mathsf{H}^{-1}(\Omega)$ is surjective as well by (29). Observe that

$$\mathcal{H}^1_{\mathbb{S}}(\Omega) \subset \mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) \subset \mathcal{L}^2_{\mathbb{S}}(\Omega).$$

For right-hand sides $f \in \mathsf{H}^{-1}(\Omega)$ we consider the following mixed variational problem for u and the Hessian $\mathbf{M}$ of u: Find $\mathbf{M} \in \mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv}, \Omega)$ and $u \in \mathring{\mathsf{H}}^1(\Omega)$ such that

$$\langle \mathbf{M}, \boldsymbol{\Psi}\rangle_{\mathcal{L}^2(\Omega)} + \langle u, \operatorname{divDiv}\boldsymbol{\Psi}\rangle_{\mathsf{H}^{-1}(\Omega)} = 0 \quad \text{for all } \boldsymbol{\Psi} \in \mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv}, \Omega), \tag{31}$$

$$\langle \operatorname{divDiv}\mathbf{M}, \psi\rangle_{\mathsf{H}^{-1}(\Omega)} = -\langle f, \psi\rangle_{\mathsf{H}^{-1}(\Omega)} \quad \text{for all } \psi \in \mathring{\mathsf{H}}^1(\Omega). \tag{32}$$

The first row and the second row of this mixed problem are variational formulations of (2) and (3), respectively. We recall the following two results related to these mixed problems from [1].

Theorem 4.1: *Let $f \in \mathsf{H}^{-1}(\Omega)$. Then:*

(i) *Problem* (31)–(32) *is a well-posed saddle point problem.*

(ii) *The variational problems* (28) *and* (31)–(32) *are equivalent, i.e. if $u \in \mathring{\mathsf{H}}^2(\Omega)$ solves* (28), *then* $\mathbf{M} = -\operatorname{Gradgrad} u$ *lies in* $\mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv}, \Omega)$ *and* $(\mathbf{M}, u)$ *solves* (31)–(32). *And, vice versa, if* $(\mathbf{M}, u) \in \mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) \times \mathring{\mathsf{H}}^1(\Omega)$ *solves* (31)–(32), *then $u \in \mathring{\mathsf{H}}^2(\Omega)$ with* $\operatorname{Gradgrad} u = -\mathbf{M}$ *and u solves* (28).

Although only two-dimensional biharmonic problems were considered in [1], the proof of the latter theorem is completely identical for the three-dimensional case. The same holds for Lemma 3.21.

Proof: To show (i), we first note that $(\mathbf{\Phi}, \mathbf{\Psi}) \mapsto \langle \mathbf{\Phi}, \mathbf{\Psi} \rangle_{\mathcal{L}^2(\Omega)}$ is coercive over the kernel of (32), i.e. for $\mathbf{\Phi} \in \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega)$ we have $\langle \mathbf{\Phi}, \mathbf{\Phi} \rangle_{\mathcal{L}^2(\Omega)} = |\mathbf{\Phi}|^2_{\mathcal{L}^2(\Omega)} = |\mathbf{\Phi}|^2_{\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv},\Omega)} = |\mathbf{\Phi}|^2_{\mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv},\Omega)}$. Moreover, the inf–sup-condition holds, as

$$
\begin{aligned}
&\inf_{0\neq\varphi\in\overset{\circ}{\mathsf{H}}{}^1(\Omega)} \sup_{0\neq\mathbf{\Phi}\in\mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv},\Omega)} \frac{\langle \varphi, \operatorname{divDiv} \mathbf{\Phi} \rangle_{\mathsf{H}^{-1}(\Omega)}}{|\operatorname{grad}\varphi|_{\mathsf{L}^2(\Omega)} |\mathbf{\Phi}|_{\mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv},\Omega)}} \\
&\geq \inf_{0\neq\varphi\in\overset{\circ}{\mathsf{H}}{}^1(\Omega)} \frac{-\langle \varphi, \operatorname{divDiv}(\varphi\,\mathbf{I}) \rangle_{\mathsf{H}^{-1}(\Omega)}}{|\operatorname{grad}\varphi|_{\mathsf{L}^2(\Omega)} |\varphi\,\mathbf{I}|_{\mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv},\Omega)}} = \inf_{0\neq\varphi\in\overset{\circ}{\mathsf{H}}{}^1(\Omega)} \frac{|\operatorname{grad}\varphi|_{\mathsf{L}^2(\Omega)}}{\Big(|\varphi\,\mathbf{I}|^2_{\mathcal{L}^2(\Omega)} + |\operatorname{divDiv}(\varphi\,\mathbf{I})|^2_{\mathsf{H}^{-1}(\Omega)}\Big)^{1/2}} \\
&= \inf_{0\neq\varphi\in\overset{\circ}{\mathsf{H}}{}^1(\Omega)} \frac{|\operatorname{grad}\varphi|_{\mathsf{L}^2(\Omega)}}{\Big(3|\varphi|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{grad}\varphi|^2_{\mathsf{L}^2(\Omega)}\Big)^{1/2}} \geq (3\,c_{\mathsf{g}}^2 + 1)^{-1/2}
\end{aligned}
$$

by choosing $\mathbf{\Phi} := -\varphi\,\mathbf{I} \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)\cdot\mathbf{I} \subset \mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv},\Omega)$ and observing

$$
\begin{aligned}
-\langle \varphi, \operatorname{divDiv}(\varphi\,\mathbf{I}) \rangle_{\mathsf{H}^{-1}(\Omega)} &= -\langle \varphi, \operatorname{div}\operatorname{grad}\varphi \rangle_{\mathsf{H}^{-1}(\Omega)} = |\operatorname{grad}\varphi|^2_{\mathsf{L}^2(\Omega)}, \\
|\operatorname{divDiv}(\varphi\,\mathbf{I})|_{\mathsf{H}^{-1}(\Omega)} &= \sup_{0\neq\phi\in\overset{\circ}{\mathsf{H}}{}^1(\Omega)} \frac{\langle \phi, \operatorname{div}\operatorname{grad}\varphi \rangle_{\mathsf{H}^{-1}(\Omega)}}{|\operatorname{grad}\phi|_{\mathsf{L}^2(\Omega)}} \\
&= \sup_{0\neq\phi\in\overset{\circ}{\mathsf{H}}{}^1(\Omega)} \frac{\langle \operatorname{grad}\phi, \operatorname{grad}\varphi \rangle_{\mathsf{L}^2(\Omega)}}{|\operatorname{grad}\phi|_{\mathsf{L}^2(\Omega)}} = |\operatorname{grad}\varphi|_{\mathsf{L}^2(\Omega)}.
\end{aligned}
$$

Note that both the primal problem (28) and the mixed problem (31)–(32) are well-posed. So, it suffices to show the first part of (ii) only. The reverse direction follows then automatically. Let $u \in \overset{\circ}{\mathsf{H}}{}^2(\Omega)$ solve (28). Then $\mathbf{M} := -\operatorname{Gradgrad} u \in \mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv},\Omega)$ with $\operatorname{divDiv}\mathbf{M} = -f$ in $\mathsf{H}^{-2}(\Omega)$ and hence in $\mathsf{H}^{-1}(\Omega)$. Thus (32) holds. Moreover, for $\mathbf{\Psi} \in \mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv},\Omega)$ we see

$$
\langle \mathbf{M}, \mathbf{\Psi} \rangle_{\mathcal{L}^2(\Omega)} = -\langle \operatorname{Gradgrad} u, \mathbf{\Psi} \rangle_{\mathcal{L}^2(\Omega)} = -\langle u, \operatorname{divDiv}\mathbf{\Psi} \rangle_{\mathsf{H}^{-2}(\Omega)} = -\langle u, \operatorname{divDiv}\mathbf{\Psi} \rangle_{\mathsf{H}^{-1}(\Omega)}
$$

and hence (31) is true. Therefore, $(\mathbf{M}, u)$ solves (31)–(32). ∎

Remark 4.2: For convenience of the reader, we give additionally a proof of the other direction as well: if $(\mathbf{M}, u)$ in $\mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv},\Omega) \times \overset{\circ}{\mathsf{H}}{}^1(\Omega)$ solves (31)–(32), then $\operatorname{divDiv}\mathbf{M} = -f$ in $\mathsf{H}^{-1}(\Omega)$ and (31) holds. Especially, (31) holds for $\mathbf{\Psi} \in \mathcal{H}^2_{\mathbb{S}}(\Omega) \subset \mathcal{H}^1_{\mathbb{S}}(\Omega) \subset \mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv},\Omega)$, i.e.

$$
-\langle \mathbf{M}, \mathbf{\Psi} \rangle_{\mathcal{L}^2(\Omega)} = \langle u, \operatorname{divDiv}\mathbf{\Psi} \rangle_{\mathsf{H}^{-1}(\Omega)} = \langle u, \operatorname{divDiv}\mathbf{\Psi} \rangle_{\mathsf{L}^2(\Omega)}. \tag{33}
$$

But then (33) holds for all $\mathbf{\Psi} \in \mathcal{H}^2(\Omega)$ as $\operatorname{sym}\mathbf{\Psi} \in \mathcal{H}^2_{\mathbb{S}}(\Omega)$ and

$$
-\langle \mathbf{M}, \mathbf{\Psi} \rangle_{\mathcal{L}^2(\Omega)} = -\langle \mathbf{M}, \operatorname{sym}\mathbf{\Psi} \rangle_{\mathcal{L}^2(\Omega)} = \langle u, \operatorname{divDiv}\operatorname{sym}\mathbf{\Psi} \rangle_{\mathsf{L}^2(\Omega)} = \langle u, \operatorname{divDiv}\mathbf{\Psi} \rangle_{\mathsf{L}^2(\Omega)}, \tag{34}
$$

since $\operatorname{divDiv}\operatorname{skw}\mathbf{\Psi} = 0$ by

$$
\langle \operatorname{divDiv}\operatorname{skw}\mathbf{\Psi}, \phi \rangle_{\mathsf{L}^2(\Omega)} = \langle \operatorname{skw}\mathbf{\Psi}, \operatorname{Gradgrad}\phi \rangle_{\mathcal{L}^2(\Omega)} = 0
$$

for all $\phi \in \overset{\circ}{\mathsf{C}}^{\infty}(\Omega)$. (34) yields that $u \in \overset{\circ}{\mathsf{H}}^{2}(\Omega)$ with $\operatorname{Gradgrad} u = -\mathbf{M}$. Finally, for all $\phi \in \overset{\circ}{\mathsf{H}}^{2}(\Omega)$

$$\langle \operatorname{Gradgrad} u, \operatorname{Gradgrad} \phi \rangle_{\mathcal{L}^2(\Omega)} = -\langle \mathbf{M}, \operatorname{Gradgrad} \phi \rangle_{\mathcal{L}^2(\Omega)} = -\langle \operatorname{divDiv} \mathbf{M}, \phi \rangle_{\mathsf{H}^{-2}(\Omega)} = \langle f, \phi \rangle_{\mathsf{H}^{-2}(\Omega)},$$

showing that $u \in \overset{\circ}{\mathsf{H}}^{2}(\Omega)$ solves (28).

We note that the decomposition of $\mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv}, \Omega)$ in Lemma 3.21 is different to the Helmholtz type decomposition of the larger space $\mathcal{L}^2_{\mathbb{S}}(\Omega)$ in Theorems 3.12 and 3.23 and does not involve the Hessian of scalar functions in $\overset{\circ}{\mathsf{H}}^{2}(\Omega)$. Using the decomposition of $\mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv}, \Omega)$ in Lemma 3.21, we have the following decomposition result for the biharmonic problem. Let $(\mathbf{M}, u) \in \mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv}, \Omega) \times \overset{\circ}{\mathsf{H}}^{1}(\Omega)$ be the unique solution of (31)–(32). Using Lemma 3.21 we have the following direct decompositions for $\mathbf{M}, \boldsymbol{\Psi} \in \mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv}, \Omega)$

$$\mathbf{M} = p\,\mathbf{I} + \mathbf{M}_0, \quad \boldsymbol{\Psi} = \varphi\,\mathbf{I} + \boldsymbol{\Psi}_0, \quad p, \varphi \in \overset{\circ}{\mathsf{H}}^{1}(\Omega), \quad \mathbf{M}_0, \boldsymbol{\Psi}_0 \in \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega).$$

This allows to rewrite (31)–(32) equivalently in terms of $(p, \mathbf{M}_0, u)$ and for all $(\varphi, \boldsymbol{\Psi}_0, \psi)$, i.e.

$$\begin{aligned} \langle p\,\mathbf{I}, \varphi\,\mathbf{I} \rangle_{\mathcal{L}^2(\Omega)} + \langle \mathbf{M}_0, \boldsymbol{\Psi}_0 \rangle_{\mathcal{L}^2(\Omega)} + \langle p\,\mathbf{I}, \boldsymbol{\Psi}_0 \rangle_{\mathcal{L}^2(\Omega)} + \langle \mathbf{M}_0, \varphi\,\mathbf{I} \rangle_{\mathcal{L}^2(\Omega)} + \langle u, \operatorname{divDiv}(\varphi\,\mathbf{I}) \rangle_{\mathsf{H}^{-1}(\Omega)} &= 0, \\ \langle \operatorname{divDiv}(p\,\mathbf{I}), \psi \rangle_{\mathsf{H}^{-1}(\Omega)} &= -\langle f, \psi \rangle_{\mathsf{H}^{-1}(\Omega)} \end{aligned}$$

or equivalently

$$\begin{aligned} \langle \operatorname{grad} u, \operatorname{grad} \varphi \rangle_{\mathsf{L}^2(\Omega)} + 3\langle p, \varphi \rangle_{\mathsf{L}^2(\Omega)} + \langle \mathbf{M}_0, \boldsymbol{\Psi}_0 \rangle_{\mathcal{L}^2(\Omega)} + \langle p, \operatorname{tr} \boldsymbol{\Psi}_0 \rangle_{\mathsf{L}^2(\Omega)} + \langle \operatorname{tr} \mathbf{M}_0, \varphi \rangle_{\mathsf{L}^2(\Omega)} &= 0, \\ \langle \operatorname{grad} p, \operatorname{grad} \psi \rangle_{\mathsf{L}^2(\Omega)} &= -\langle f, \psi \rangle_{\mathsf{H}^{-1}(\Omega)}, \end{aligned}$$

which leads to the equivalent system

$$\begin{aligned} \langle \operatorname{grad} u, \operatorname{grad} \varphi \rangle_{\mathsf{L}^2(\Omega)} + 3\langle p, \varphi \rangle_{\mathsf{L}^2(\Omega)} + \langle \operatorname{tr} \mathbf{M}_0, \varphi \rangle_{\mathsf{L}^2(\Omega)} &= 0, \\ \langle \mathbf{M}_0, \boldsymbol{\Psi}_0 \rangle_{\mathcal{L}^2(\Omega)} + \langle p, \operatorname{tr} \boldsymbol{\Psi}_0 \rangle_{\mathsf{L}^2(\Omega)} &= 0, \\ \langle \operatorname{grad} p, \operatorname{grad} \psi \rangle_{\mathsf{L}^2(\Omega)} &= -\langle f, \psi \rangle_{\mathsf{H}^{-1}(\Omega)}. \end{aligned}$$

Theorem 4.3: *The variational problem* (31)–(32) *is equivalent to the following well-posed and uniquely solvable variational problem. For* $f \in \mathsf{H}^{-1}(\Omega)$ *find* $p \in \overset{\circ}{\mathsf{H}}^{1}(\Omega)$, $\mathbf{M}_0 \in \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega)$, *and* $u \in \overset{\circ}{\mathsf{H}}^{1}(\Omega)$ *such that*

$$\langle \operatorname{grad} u, \operatorname{grad} \varphi \rangle_{\mathsf{L}^2(\Omega)} + \langle \operatorname{tr} \mathbf{M}_0, \varphi \rangle_{\mathsf{L}^2(\Omega)} + 3\langle p, \varphi \rangle_{\mathsf{L}^2(\Omega)} = 0, \tag{35}$$

$$\langle \mathbf{M}_0, \boldsymbol{\Psi}_0 \rangle_{\mathcal{L}^2(\Omega)} + \langle p, \operatorname{tr} \boldsymbol{\Psi}_0 \rangle_{\mathsf{L}^2(\Omega)} = 0, \tag{36}$$

$$\langle \operatorname{grad} p, \operatorname{grad} \psi \rangle_{\mathsf{L}^2(\Omega)} = -\langle f, \psi \rangle_{\mathsf{H}^{-1}(\Omega)} \tag{37}$$

for all $\psi \in \overset{\circ}{\mathsf{H}}^{1}(\Omega)$, $\boldsymbol{\Psi}_0 \in \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega)$, *and* $\varphi \in \overset{\circ}{\mathsf{H}}^{1}(\Omega)$. *Moreover, the unique solution* $(\mathbf{M}, u)$ *of* (31)–(32) *is given by* $\mathbf{M} := p\,\mathbf{I} + \mathbf{M}_0$ *and* u *for the unique solution* $(p, \mathbf{M}_0, u)$ *of* (35)–(37).

If Ω is additionally topologically trivial, then by Theorem 3.12 or Theorem 3.23

$$\mathcal{H}_{\mathbb{S}}(\operatorname{divDiv} 0, \Omega) = \operatorname{symCurl} \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) = \operatorname{symCurl} \Big(\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \cap \overset{\circ}{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega) \Big)$$

and we obtain the following result.

Theorem 4.4: *Let Ω be additionally topologically trivial. The variational problem* (31)–(32) *is equivalent to the following well-posed and uniquely solvable variational problem. For $f \in \mathsf{H}^{-1}(\Omega)$ find $p \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)$, $\mathbf{E} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \cap \overset{\circ}{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega)$, and $u \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)$ such that*

$$\langle \operatorname{grad} u, \operatorname{grad} \varphi \rangle_{\mathbf{L}^2(\Omega)} + \langle \operatorname{tr} \operatorname{symCurl} \mathbf{E}, \varphi \rangle_{\mathsf{L}^2(\Omega)} + 3 \langle p, \varphi \rangle_{\mathsf{L}^2(\Omega)} = 0, \tag{38}$$

$$\langle \operatorname{symCurl} \mathbf{E}, \operatorname{symCurl} \mathbf{\Phi} \rangle_{\mathcal{L}^2(\Omega)} + \langle p, \operatorname{tr} \operatorname{symCurl} \mathbf{\Phi} \rangle_{\mathsf{L}^2(\Omega)} = 0, \tag{39}$$

$$\langle \operatorname{grad} p, \operatorname{grad} \psi \rangle_{\mathbf{L}^2(\Omega)} = -\langle f, \psi \rangle_{\mathsf{H}^{-1}(\Omega)} \tag{40}$$

for all $\psi \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)$, $\mathbf{\Phi} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \cap \overset{\circ}{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega)$, and $\varphi \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)$. Moreover, the unique solution $(\mathbf{M}, u)$ *of* (31)–(32) *is given by* $\mathbf{M} := p\,\mathbf{I} + \operatorname{symCurl} \mathbf{E}$ *and u for the unique solution* $(p, \mathbf{E}, u)$ *of* (38)–(40).

Note that, e.g. $\langle \operatorname{tr} \operatorname{symCurl} \mathbf{E}, \varphi \rangle_{\mathsf{L}^2(\Omega)} = \langle \operatorname{symCurl} \mathbf{E}, \varphi\,\mathbf{I} \rangle_{\mathcal{L}^2(\Omega)}$ and $3 \langle p, \varphi \rangle_{\mathsf{L}^2(\Omega)} = \langle p\,\mathbf{I}, \varphi\,\mathbf{I} \rangle_{\mathcal{L}^2(\Omega)}$.

Proof: (31)–(32) is equivalent to (35)–(37) and hence also to (38)–(40), if the latter system is well-posed. By Theorem 3.12 or Theorem 3.23 the bilinear form $\langle \operatorname{symCurl} \,\cdot\,, \operatorname{symCurl} \,\cdot\, \rangle_{\mathcal{L}^2(\Omega)}$ is coercive over $\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \cap \overset{\circ}{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega)$, which shows the consecutive unique solvability of (38)–(40). ■

The three problems in the previous theorem are weak formulations of the following three second-order problems in strong form. A Dirichlet–Poisson problem for the auxiliary scalar function p

$$\Delta p = f \quad \text{in } \Omega, \qquad p = 0 \quad \text{on } \Gamma,$$

a second-order Neumann type Curl symCurl –Div-system for the auxiliary tensor field $\mathbf{E}$

$$\begin{aligned} \operatorname{tr} \mathbf{E} = 0, \quad \operatorname{Curl} \operatorname{symCurl} \mathbf{E} = -\operatorname{Curl}(p\,\mathbf{I}) = \operatorname{spn} \operatorname{grad} p, \quad \operatorname{Div} \mathbf{E} = 0 \quad \text{in } \Omega, \\ \mathbf{n} \times \operatorname{symCurl} \mathbf{E} = -\mathbf{n} \times p\,\mathbf{I} = p \operatorname{spn} \mathbf{n} = 0, \quad \mathbf{E}\,\mathbf{n} = 0 \quad \text{on } \Gamma, \end{aligned}$$

and, finally, a Dirichlet–Poisson problem for the original scalar function u

$$\Delta u = 3p + \operatorname{tr}\operatorname{symCurl}\mathbf{E} = \operatorname{tr}(p\,\mathbf{I} + \operatorname{symCurl}\mathbf{E}) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma.$$

In other words, the system (38)–(40) has triangular structure

$$\begin{bmatrix} 3 & \operatorname{tr}\operatorname{symCurl}_{\mathbb{T}} & -\mathring{\Delta} \\ \mathring{\operatorname{Curl}}_{\mathbb{S}}(\,\cdot\,\mathbf{I}) & \mathring{\operatorname{Curl}}_{\mathbb{S}}\operatorname{symCurl}_{\mathbb{T}} & 0 \\ -\mathring{\Delta} & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ \mathbf{E} \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -f \end{bmatrix}$$

with $(\operatorname{tr}\operatorname{symCurl}_{\mathbb{T}})^* = \mathring{\operatorname{Curl}}_{\mathbb{S}}(\,\cdot\,\mathbf{I})$ and $\mathring{\Delta} = \operatorname{div}\mathring{\operatorname{grad}}$. Indeed, $\mathbf{E} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega) \cap \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0,\Omega)$ with

$$\langle \operatorname{symCurl}\mathbf{E}, \operatorname{symCurl}\boldsymbol{\Phi}\rangle_{\mathcal{L}^2(\Omega)} + \langle p, \operatorname{tr}\operatorname{symCurl}\boldsymbol{\Phi}\rangle_{\mathsf{L}^2(\Omega)} = 0$$

for all $\boldsymbol{\Phi} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega) \cap \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0,\Omega)$ is equivalent to $\mathbf{E} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega) \cap \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0,\Omega)$ and

$$\langle \operatorname{symCurl}\mathbf{E} + p\,\mathbf{I}, \operatorname{symCurl}\boldsymbol{\Phi}\rangle_{\mathcal{L}^2(\Omega)} = 0 \tag{41}$$

for all $\boldsymbol{\Phi} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega)$ as by Theorem 3.12

$$\operatorname{symCurl}\left(\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega) \cap \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0,\Omega)\right) = \operatorname{symCurl}\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega). \tag{42}$$

Now (41) shows that

$$\operatorname{symCurl}\mathbf{E} + p\,\mathbf{I} \in R(\operatorname{symCurl}_{\mathbb{T}})^{\perp_{\mathcal{L}^2(\Omega)}} = N(\operatorname{symCurl}_{\mathbb{T}}^*) = N(\mathring{\operatorname{Curl}}_{\mathbb{S}}) = \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl} 0,\Omega),$$

especially $\operatorname{Curl}(\operatorname{symCurl}\mathbf{E} + p\,\mathbf{I}) = 0$ in Ω and $\mathbf{n} \times (\operatorname{symCurl}\mathbf{E} + p\,\mathbf{I}) = 0$ on Γ.

Finally, we want to get rid of the complicated space $\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega) \cap \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0,\Omega)$ in the variational formulation in Theorem 4.4. For a given $p \in \mathring{\mathsf{H}}^1(\Omega)$ the part (39) of (38)–(40), i.e. find a tensor field $\mathbf{E} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega) \cap \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0,\Omega)$ such that

$$\langle \operatorname{symCurl}\mathbf{E}, \operatorname{symCurl}\boldsymbol{\Phi}\rangle_{\mathcal{L}^2(\Omega)} + \langle p, \operatorname{tr}\operatorname{symCurl}\boldsymbol{\Phi}\rangle_{\mathsf{L}^2(\Omega)} = 0 \tag{43}$$

for all $\boldsymbol{\Phi} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega) \cap \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0,\Omega)$, has also a saddle point structure. By Theorem 3.12 we have (42) as well as

$$\begin{aligned} \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0,\Omega) &= N(\mathring{\operatorname{Div}}_{\mathbb{T}}) = R(\mathring{\operatorname{Div}}_{\mathbb{T}}^*)^{\perp_{\mathcal{L}^2_{\mathbb{T}}(\Omega)}} = R(\operatorname{devGrad})^{\perp_{\mathcal{L}^2_{\mathbb{T}}(\Omega)}} \\ &= \left(\operatorname{devGrad}\left(\mathbf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\right)\right)^{\perp_{\mathcal{L}^2_{\mathbb{T}}(\Omega)}}. \end{aligned}$$

Hence (43) is equivalent to find $\mathbf{E} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl},\Omega)$ such that

$$\langle \operatorname{symCurl}\mathbf{E}, \operatorname{symCurl}\boldsymbol{\Phi}\rangle_{\mathcal{L}^2(\Omega)} + \langle p, \operatorname{tr}\operatorname{symCurl}\boldsymbol{\Phi}\rangle_{\mathsf{L}^2(\Omega)} = 0, \tag{44}$$

$$\langle \mathbf{E}, \operatorname{devGrad}\boldsymbol{\theta}\rangle_{\mathcal{L}^2(\Omega)} = 0 \tag{45}$$

for all $\boldsymbol{\Phi} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega)$ and $\boldsymbol{\theta} \in \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$. Observe that

$$(\mathbf{E}, \mathbf{v}) := (\mathbf{E}, 0) \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \times \left(\mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\right)$$

solves the modified variational system

$$\langle \operatorname{symCurl} \mathbf{E}, \operatorname{symCurl} \boldsymbol{\Phi} \rangle_{\mathcal{L}^2(\Omega)} + \langle \boldsymbol{\Phi}, \operatorname{devGrad} \mathbf{v} \rangle_{\mathcal{L}^2(\Omega)} = -\langle p, \operatorname{tr} \operatorname{symCurl} \boldsymbol{\Phi} \rangle_{\mathsf{L}^2(\Omega)}, \tag{46}$$

$$\langle \mathbf{E}, \operatorname{devGrad} \boldsymbol{\theta} \rangle_{\mathcal{L}^2(\Omega)} = 0 \tag{47}$$

for all $\boldsymbol{\Phi} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega)$ and $\boldsymbol{\theta} \in \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$. On the other hand, any solution

$$(\mathbf{E}, \mathbf{v}) \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \times \left(\mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}\right)$$

of (46)–(47) satisfies $\mathbf{v} = 0$, as (46) tested with

$$\boldsymbol{\Phi} := \operatorname{devGrad} \mathbf{v} \in \operatorname{devGrad} \mathsf{H}^1(\Omega) = \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl} 0, \Omega)$$

shows $\operatorname{devGrad} \mathbf{v} = 0$ and thus $\mathbf{v} \in \mathsf{RT}_0$ by Lemma 3.2, yielding $\mathbf{v} = 0$. Note that (46)–(47) has the saddle point structure

$$\begin{bmatrix} \mathring{\operatorname{Curl}}_{\mathbb{S}} \operatorname{symCurl}_{\mathbb{T}} & \operatorname{devGrad} \\ -\mathring{\operatorname{Div}}_{\mathbb{T}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} -\mathring{\operatorname{Curl}}_{\mathbb{S}}(v\,\mathbf{I}) \\ 0 \end{bmatrix}, \qquad (\operatorname{devGrad})^* = -\mathring{\operatorname{Div}}_{\mathbb{T}}.$$

We obtain the following final result.

Theorem 4.5: *Let Ω be additionally topologically trivial. The variational problem* (38)–(40) *is equivalent to the following well-posed and uniquely solvable variational system. For $f \in \mathsf{H}^{-1}(\Omega)$ find $p \in \mathring{\mathsf{H}}^1(\Omega)$, $\mathbf{E} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega)$, $\mathbf{v} \in \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$, and $u \in \mathring{\mathsf{H}}^1(\Omega)$ such that*

$$\langle \operatorname{grad} u, \operatorname{grad} \varphi \rangle_{\mathsf{L}^2(\Omega)} + \langle \operatorname{tr} \operatorname{symCurl} \mathbf{E}, \varphi \rangle_{\mathsf{L}^2(\Omega)} + 3\langle p, \varphi \rangle_{\mathsf{L}^2(\Omega)} = 0, \tag{48}$$

$$\langle \operatorname{symCurl} \mathbf{E}, \operatorname{symCurl} \boldsymbol{\Phi} \rangle_{\mathcal{L}^2(\Omega)} + \langle \boldsymbol{\Phi}, \operatorname{devGrad} \mathbf{v} \rangle_{\mathcal{L}^2(\Omega)} + \langle p, \operatorname{tr} \operatorname{symCurl} \boldsymbol{\Phi} \rangle_{\mathsf{L}^2(\Omega)} = 0, \tag{49}$$

$$\langle \mathbf{E}, \operatorname{devGrad} \boldsymbol{\theta} \rangle_{\mathcal{L}^2(\Omega)} = 0, \tag{50}$$

$$\langle \operatorname{grad} p, \operatorname{grad} \psi \rangle_{\mathsf{L}^2(\Omega)} = -\langle f, \psi \rangle_{\mathsf{H}^{-1}(\Omega)} \tag{51}$$

for all $\psi \in \mathring{\mathsf{H}}^1(\Omega)$, $\boldsymbol{\Phi} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega)$, $\boldsymbol{\theta} \in \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$, and $\varphi \in \mathring{\mathsf{H}}^1(\Omega)$. Moreover, the unique solution $(p, \mathbf{E}, \mathbf{v}, u)$ of (48)–(51) *satisfies $\mathbf{v} = 0$ and $(p, \mathbf{E}, u)$ is the unique solution of* (38)–(40).

Note that the system (48)–(51) has the block triangular saddle point structure

$$\begin{bmatrix} 3 & \operatorname{tr} \operatorname{symCurl}_{\mathbb{T}} & 0 & -\mathring{\Delta} \\ \mathring{\operatorname{Curl}}_{\mathbb{S}}(\,\cdot\,\mathbf{I}) & \mathring{\operatorname{Curl}}_{\mathbb{S}} \operatorname{symCurl}_{\mathbb{T}} & \operatorname{devGrad} & 0 \\ 0 & -\mathring{\operatorname{Div}}_{\mathbb{T}} & 0 & 0 \\ -\mathring{\Delta} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ \mathbf{E} \\ \mathbf{v} \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -f \end{bmatrix} \tag{52}$$

with $(\operatorname{tr} \operatorname{symCurl}_{\mathbb{T}})^* = \mathring{\operatorname{Curl}}_{\mathbb{S}}(\,\cdot\,\mathbf{I})$ and $(\operatorname{devGrad})^* = -\mathring{\operatorname{Div}}_{\mathbb{T}}$.

Proof: We only have to show well-posedness of the partial system (49)–(50). First note that by Theorem 3.12 the bilinear form $\langle \operatorname{symCurl} \cdot, \operatorname{symCurl} \cdot \rangle_{\mathcal{L}^2(\Omega)}$ is coercive over $\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \cap \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega)$, which equals the kernel of (50). Indeed it follows from (50) that

$$\mathbf{E} \in \left(\operatorname{devGrad} \left(\mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}} \right) \right)^{\perp_{\mathcal{L}^2_{\mathbb{T}}(\Omega)}} = \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega).$$

Moreover, the inf–sup-condition is satisfied as by picking for fixed $0 \neq \boldsymbol{\theta} \in \mathsf{H}^1(\Omega) \cap \mathsf{RT}_0^{\perp_{\mathsf{L}^2(\Omega)}}$ the tensor $\boldsymbol{\Phi} := \operatorname{devGrad} \boldsymbol{\theta} \in \operatorname{devGrad} \mathsf{H}^1(\Omega) = \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl} 0, \Omega)$ we have

$$\inf_{\substack{0 \neq \boldsymbol{\theta} \in \mathsf{H}^1(\Omega), \\ \boldsymbol{\theta} \perp_{\mathsf{L}^2(\Omega)} \mathsf{RT}_0}} \sup_{\boldsymbol{\Phi} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega)} \frac{\langle \boldsymbol{\Phi}, \operatorname{devGrad} \boldsymbol{\theta} \rangle_{\mathcal{L}^2(\Omega)}}{|\boldsymbol{\Phi}|_{\mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega)} |\boldsymbol{\theta}|_{\mathsf{H}^1(\Omega)}} \geq \inf_{\substack{0 \neq \boldsymbol{\theta} \in \mathsf{H}^1(\Omega), \\ \boldsymbol{\theta} \perp_{\mathsf{L}^2(\Omega)} \mathsf{RT}_0}} \frac{|\operatorname{devGrad} \boldsymbol{\theta}|_{\mathcal{L}^2(\Omega)}}{|\boldsymbol{\theta}|_{\mathsf{H}^1(\Omega)}} \geq \frac{1}{c}$$

by Lemma 3.2 (iv). ■

Remark 4.6: The corresponding result for the two-dimensional case is completely analogous with the exception that the tensor potential $\mathbf{E} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega) \cap \mathring{\mathcal{H}}_{\mathbb{T}}(\operatorname{Div} 0, \Omega)$ is to be replaced by a much simpler vector potential $\mathbf{w} \in \mathsf{H}^1(\Omega)$. Furthermore, observe that

$$\langle \operatorname{symCurl} \mathbf{w}, \operatorname{symCurl} \boldsymbol{\theta} \rangle_{\mathcal{L}^2(\Omega)} = \langle \operatorname{symGrad}^{\perp} \mathbf{w}, \operatorname{symGrad}^{\perp} \boldsymbol{\theta} \rangle_{\mathcal{L}^2(\Omega)}$$

holds for vector fields $\mathbf{w}, \boldsymbol{\theta} \in \mathsf{H}^1(\Omega)$. Here the superscript $\perp$ denotes the rotation of a vector field by 90°. Note that the complicated second-order Neumann type Curl symCurl –Div-system for the auxiliary tensor field $\mathbf{E}$ is replaced in 2D by a much simpler Neumann linear elasticity problem, where the standard Sobolev space $\mathsf{H}^1(\Omega)$ resp. $\mathsf{H}^1(\Omega) \cap \mathsf{RM}^{\perp_{\mathsf{L}^2(\Omega)}}$ can be used. Here RM denotes the space of rigid motions. This yields the decomposition result in [1] for the two-dimensional case, which was shortly mentioned in the introduction.

Notes

1. Γ is locally a graph of a Lipschitz function.
2. Note $\operatorname{Curl} \mathbf{M} = \operatorname{dev} \operatorname{Curl} \mathbf{M}$ for $\mathbf{M} \in \mathcal{H}_{\mathbb{S}}(\operatorname{Curl}, \Omega)$ and thus for all $\mathbf{M} \in \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl}, \Omega) \cap \operatorname{symCurl} \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega)$

$$|\mathbf{M}|_{\mathcal{L}^2(\Omega)} \leq c_{\mathsf{R}} \, |\operatorname{Curl} \mathbf{M}|_{\mathcal{L}^2(\Omega)} = c_{\mathsf{R}} \, |\operatorname{dev} \operatorname{Curl} \mathbf{M}|_{\mathcal{L}^2(\Omega)}.$$

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The research of the second author was supported by the Austrian Science Fund (FWF): project S11702-N23.

References

[1] Krendl W, Rafetseder K, Zulehner W. A decomposition result for biharmonic problems and the Hellan–Herrmann–Johnson method. Electron Trans Numer Anal. 2016;45:257–282.
[2] Hellan K. Analysis of elastic plates in flexure by a simplified finite element method. Acta Polytechnica Scandinavia, CI 46; 1967.
[3] Herrmann L. Finite element bending analysis for plates. J Eng Mech Div ASCE EM5. 1967;93:49–83.
[4] Johnson C. On the convergence of a mixed finite-element method for plate bending problems. Numer Math. 1973;21:43–62.

[5] Beirão da Veiga L, Niiranen J, Stenberg R. A posteriori error estimates for the Morley plate bending element. Numer Math. 2007;106(2):165–179.
[6] Huang J, Huang X, Xu Y. Convergence of an adaptive mixed finite element method for Kirchhoff plate bending problems. SIAM J Numer Anal. 2011;49(2):574–607.
[7] Quenneville-Bĺair V. A new approach to finite element simulation of general relativity [PhD thesis]. New York (NY): APAM, Columbia University; Minneapolis (MN): University of Minnesota; 2015.
[8] Arnold DN, Falk RS, Winther R. Finite element exterior calculus, homological techniques, and applications. Acta Numer. 2006;15:1–155.
[9] Weck N. Maxwell's boundary value problems on Riemannian manifolds with nonsmooth boundaries. J Math Anal Appl. 1974;46:410–437.
[10] Picard R. An elementary proof for a compact imbedding result in generalized electromagnetic theory. Math Z. 1984;187:151–164.
[11] Picard R, Weck N, Witsch K-J. Time-harmonic Maxwell equations in the exterior of perfectly conducting, irregular obstacles. Analysis (Munich). 2001;21:231–263.
[12] Witsch K-J. A remark on a compactness result in electromagnetic theory. Math Methods Appl Sci. 1993;16:123–129.
[13] Weber C. A local compactness theorem for Maxwell's equations. Math Methods Appl Sci. 1980;2:12–25.
[14] Jochmann F. A compactness result for vector fields with divergence and curl in $L^q(\Omega)$ involving mixed boundary conditions. Appl Anal. 1997;66:189–203.
[15] Bauer S, Pauly D, Schomburg M. The Maxwell compactness property in bounded weak Lipschitz domains with mixed boundary conditions. SIAM J Math Anal. 2016;48(4):2912–2943.
[16] Picard R. Zur Theorie der harmonischen Differentialformen. Manuscripta Math. 1979;27:31–45.
[17] Picard R. On the boundary value problems of electro- and magnetostatics. Proc R Soc Edinburgh Sect A. 1982;92:165–174.
[18] Pauly D. On constants in Maxwell inequalities for bounded and convex domains. Zapiski POMI. 2014;435:46–54 & J. Math. Sci. (N.Y.), 210(6):787–792, 2015.
[19] Pauly D. On Maxwell's and Poincaré's constants. Discrete Contin Dyn Syst Ser S. 2015;8(3):607–618.
[20] Pauly D. On the Maxwell constants in 3D. Math Methods Appl Sci. 2017;40(2):435–447.
[21] Costabel M, McIntosh A. On Bogovskii and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains. Math Z. 2010;265(2):297–320.
[22] Hiptmair R, Li J, Zou J. Universal extension for Sobolev spaces of differential forms and applications. J Funct Anal. 2012;263(2):364–382.
[23] Bogovskii ME. Solution of the first boundary value problem for an equation of continuity of an incompressible medium. Dokl Akad Nauk SSSR. 1979;248(5):1037–1040. Russian.
[24] Bogovskii ME. Solutions of some problems of vector analysis, associated with the operators div and grad. Theory of cubature formulas and the application of functional analysis to problems of mathematical physics. Akad Nauk SSSR Sibirsk Otdel Inst Mat Novosibirsk. 1980;1:5–40.Russian.
[25] Galdi GP. An introduction to the mathematical theory of the Navier–Stokes equations. Steady-state problems. 2nd ed. New York (NY): Springer; 2011. (Springer Monographs in Mathematics).
[26] Sohr H. The Navier–Stokes equations. Basel: Birkhäuser; 2001.
[27] Lions J-L, Magenes E. Non-homogeneous boundary value problems and applications. Vol. I. New York, Heidelberg: Springer-Verlag; 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.
[28] Nečas J. Direct methods in the theory of elliptic equations. Heidelberg: Springer; 2012. (Springer Monographs in Mathematics). Translated from the 1967 French original by Gerard Tronel and Alois Kufner.

Appendix. Proofs of some useful identities

Note that for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and $\mathbf{A} \in \mathbb{R}^{3\times 3}$

$$\operatorname{spn}\mathbf{a} : \operatorname{spn}\mathbf{b} = 2\,\mathbf{a}\cdot\mathbf{b}, \qquad \operatorname{skw}\mathbf{A} = \tfrac{1}{2}\operatorname{spn}\begin{bmatrix}\mathbf{A}_{32}-\mathbf{A}_{23}\\ \mathbf{A}_{13}-\mathbf{A}_{31}\\ \mathbf{A}_{21}-\mathbf{A}_{12}\end{bmatrix} \tag{A1}$$

hold and hence for skew-symmetric $\mathbf{A}$

$$\operatorname{spn}\mathbf{a} : A = \operatorname{spn}\mathbf{a} : \operatorname{spn}\operatorname{spn}^{-1}\mathbf{A} = 2\,\mathbf{a}\cdot\operatorname{spn}^{-1}\mathbf{A}, \tag{A2}$$

i.e. $\operatorname{spn}^* = 2\operatorname{spn}^{-1}$. Moreover, we have for two matrices $\mathbf{A}, \mathbf{B}$

$$\mathbf{A}^\top : \mathbf{B} = \operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA}) = \mathbf{B}^\top : \mathbf{A} = \mathbf{A} : \mathbf{B}^\top.$$

The assertions of Lemma 3.4 and 3.9 are contained in the assertions of the following lemma.

Lemma A.1: *For smooth functions, vector fields and tensor fields we have*

(i) $\operatorname{skw}\operatorname{Gradgrad} u = 0$,
(ii) $\operatorname{divDiv}\mathbf{M} = 0$, *if* $\mathbf{M}$ *is skew-symmetric,*
(iii) $\operatorname{Curl}(u\,\mathbf{I}) = -\operatorname{spn}\operatorname{grad} u$,
(iv) $\operatorname{tr}\operatorname{Curl}\mathbf{M} = 2\operatorname{div}(\operatorname{spn}^{-1}\operatorname{skw}\mathbf{M})$, *especially* $\operatorname{tr}\operatorname{Curl}\mathbf{M} = 0$, *if* $\mathbf{M}$ *is symmetric,*
(v) $\operatorname{Div}(u\,\mathbf{I}) = \operatorname{grad} u$,
(vi) $\operatorname{tr}\operatorname{Grad}\mathbf{v} = \operatorname{div}\mathbf{v}$,
(vii) $\operatorname{Div}(\operatorname{spn}\mathbf{v}) = -\operatorname{curl}\mathbf{v}$, *especially* $\operatorname{Div}(\operatorname{skw}\mathbf{M}) = -\operatorname{curl}\mathbf{v}$ *for* $\mathbf{v} = \operatorname{spn}^{-1}\operatorname{skw}\mathbf{M}$,
(viii) $\operatorname{Curl}(\operatorname{spn}\mathbf{v}) = (\operatorname{div}\mathbf{v})\,\mathbf{I} - (\operatorname{Grad}\mathbf{v})^\top$, *especially* $\operatorname{Curl}\operatorname{skw}\mathbf{M} = (\operatorname{div}\mathbf{v})\,\mathbf{I} - (\operatorname{Grad}\mathbf{v})^\top$ *for* $\mathbf{v} = \operatorname{spn}^{-1}\operatorname{skw}\mathbf{M}$,
(ix) $\operatorname{skw}\operatorname{Grad}\mathbf{v} = \frac{1}{2}\operatorname{spn}\operatorname{curl}\mathbf{v}$ *and* $\operatorname{Curl}(\operatorname{sym}\operatorname{Grad}\mathbf{v}) = -\operatorname{Curl}(\operatorname{skw}\operatorname{Grad}\mathbf{v}) = -\frac{1}{2}\operatorname{Curl}(\operatorname{spn}\operatorname{curl}\mathbf{v})$,
(x) $\operatorname{skw}\operatorname{Curl}\mathbf{M} = \operatorname{spn}\mathbf{v}$ *and* $\operatorname{Div}(\operatorname{symCurl}\mathbf{M}) = -\operatorname{Div}(\operatorname{skw}\operatorname{Curl}\mathbf{M}) = \operatorname{curl}\mathbf{v}$ *with* $\mathbf{v} = \frac{1}{2}(\operatorname{Div}\mathbf{M}^\top - \operatorname{grad}(\operatorname{tr}\mathbf{M}))$, *especially* $\operatorname{Div}(\operatorname{symCurl}\mathbf{M}) = -\operatorname{Div}(\operatorname{skw}\operatorname{Curl}\mathbf{M}) = \frac{1}{2}\operatorname{curl}\operatorname{Div}\mathbf{M}^\top$, *if* $\operatorname{tr}\mathbf{M} = 0$,
(xi) $\operatorname{grad}\operatorname{div}\mathbf{v} = \frac{3}{2}\operatorname{Div}\operatorname{dev}(\operatorname{Grad}\mathbf{v})^\top$.

These formulas hold for distributions as well.

Proof: (i)–(ix) and the first identity in (x) follow by elementary calculations. For the second identity in (x) observe that $0 = \operatorname{Div}\operatorname{Curl}\mathbf{M} = \operatorname{Div}(\operatorname{symCurl}\mathbf{M}) + \operatorname{Div}(\operatorname{skw}\operatorname{Curl}\mathbf{M})$ for $\mathbf{M} \in \mathring{\mathcal{C}}^\infty(\mathbb{R}^3)$ and hence, using the first identity in (x) and (vii), we obtain

$$\operatorname{Div}(\operatorname{symCurl}\mathbf{M}) = -\operatorname{Div}(\operatorname{skw}\operatorname{Curl}\mathbf{M}) = -\operatorname{Div}(\operatorname{spn}\mathbf{v}) = \operatorname{curl}\mathbf{v}.$$

To see (xi) we compute

$$\begin{aligned}0 = \operatorname{Div}\operatorname{Curl}\operatorname{spn}\mathbf{v} &= \operatorname{Div}((\operatorname{div}\mathbf{v})\,\mathbf{I}) - \operatorname{Div}(\operatorname{Grad}\mathbf{v})^\top \\ &= \operatorname{Div}((\operatorname{div}\mathbf{v})\,\mathbf{I}) - \operatorname{Div}\operatorname{dev}(\operatorname{Grad}\mathbf{v})^\top - \tfrac{1}{3}\operatorname{Div}\Big((\operatorname{tr}(\operatorname{Grad}\mathbf{v})^\top)\,\mathbf{I}\Big) \\ &= \tfrac{2}{3}\operatorname{Div}((\operatorname{div}\mathbf{v})\,\mathbf{I}) - \operatorname{Div}\operatorname{dev}(\operatorname{Grad}\mathbf{v})^\top = \tfrac{2}{3}\operatorname{grad}\operatorname{div}\mathbf{v} - \operatorname{Div}\operatorname{dev}(\operatorname{Grad}\mathbf{v})^\top.\end{aligned}$$

Therefore, the stated formulas hold in the smooth case. By density these formulas extend to u, $\mathbf{v}$, and $\mathbf{M}$ in respective Sobolev spaces. Let us give proofs for distributions as well. For this, let $m \in \mathbb{N}_0$ and $u \in \mathsf{H}^{-m}(\Omega)$, $\mathbf{v} \in \mathbf{H}^{-m}(\Omega)$, $\mathbf{M} \in \boldsymbol{\mathcal{H}}^{-m}(\Omega)$ and $\varphi \in \mathring{\mathsf{C}}^\infty(\Omega)$, $\boldsymbol{\theta} \in \mathring{\mathbf{C}}^\infty(\Omega)$, and $\boldsymbol{\Phi} \in \mathring{\mathcal{C}}^\infty(\Omega)$. By

$$\langle u, \partial_i\,\partial_j\,\varphi\rangle_{\mathsf{H}^{-m}(\Omega)} = \langle u, \partial_j\,\partial_i\,\varphi\rangle_{\mathsf{H}^{-m}(\Omega)}, \quad \text{or(with(ii))} \quad \langle u, \operatorname{divDiv}\operatorname{skw}\boldsymbol{\Phi}\rangle_{\mathsf{H}^{-m}(\Omega)} = 0,$$

we see that $\operatorname{Gradgrad} u \in \mathsf{H}^{-m-2}(\Omega)$ is symmetric and hence (i). Note that we observe formally $(\operatorname{skw}\operatorname{Gradgrad})^* = \operatorname{divDiv}\operatorname{skw}$. If $\mathbf{M}$ is skew-symmetric we have $\langle\mathbf{M}, \operatorname{Gradgrad}\varphi\rangle_{\boldsymbol{\mathcal{H}}^{-m}(\Omega)} = 0$, i.e. (ii). We compute with (iv)

$$\begin{aligned}\langle u\,\mathbf{I}, \operatorname{Curl}\boldsymbol{\Phi}\rangle_{\boldsymbol{\mathcal{H}}^{-m}(\Omega)} &= \langle u, \operatorname{tr}(\operatorname{Curl}\boldsymbol{\Phi})\rangle_{\mathsf{H}^{-m}(\Omega)} = 2\langle u, \operatorname{div}(\operatorname{spn}^{-1}\operatorname{skw}\boldsymbol{\Phi})\rangle_{\mathsf{H}^{-m}(\Omega)} \\ &= -\langle\operatorname{spn}\operatorname{grad} u, \operatorname{skw}\boldsymbol{\Phi}\rangle_{\boldsymbol{\mathcal{H}}^{-m-1}(\Omega)} = -\langle\operatorname{spn}\operatorname{grad} u, \boldsymbol{\Phi}\rangle_{\boldsymbol{\mathcal{H}}^{-m-1}(\Omega)},\end{aligned}$$

showing (iii). Formally, $(\operatorname{tr}\operatorname{Curl})^* = \operatorname{Curl}(\,\cdot\,\mathbf{I})$. Hence by (iii)

$$\begin{aligned}\langle\mathbf{M}, \operatorname{Curl}(\varphi\,\mathbf{I})\rangle_{\boldsymbol{\mathcal{H}}^{-m}(\Omega)} &= -\langle\mathbf{M}, \operatorname{spn}\operatorname{grad}\varphi\rangle_{\boldsymbol{\mathcal{H}}^{-m}(\Omega)} = -\langle\operatorname{skw}\mathbf{M}, \operatorname{spn}\operatorname{grad}\varphi\rangle_{\boldsymbol{\mathcal{H}}^{-m}(\Omega)} \\ &= -2\langle\operatorname{spn}^{-1}\operatorname{skw}\mathbf{M}, \operatorname{grad}\varphi\rangle_{\mathbf{H}^{-m}(\Omega)} = 2\langle\operatorname{div}\operatorname{spn}^{-1}\operatorname{skw}\mathbf{M}, \varphi\rangle_{\mathsf{H}^{-m-1}(\Omega)},\end{aligned}$$

yielding (iv). (v) follows by

$$-\langle u\,\mathbf{I}, \operatorname{Grad}\boldsymbol{\theta}\rangle_{\boldsymbol{\mathcal{H}}^{-m}(\Omega)} = -\langle u, \operatorname{tr}(\operatorname{Grad}\boldsymbol{\theta})\rangle_{\mathsf{H}^{-m}(\Omega)} = -\langle u, \operatorname{div}\boldsymbol{\theta}\rangle_{\mathsf{H}^{-m}(\Omega)}.$$

Formally, $(\operatorname{tr}\operatorname{Grad})^* = -\operatorname{Div}(\,\cdot\,\mathbf{I})$. Thus by (v)

$$-\langle\mathbf{v}, \operatorname{Div}(\varphi\,\mathbf{I})\rangle_{\mathbf{H}^{-m}(\Omega)} = -\langle\mathbf{v}, \operatorname{grad}\varphi\rangle_{\mathbf{H}^{-m}(\Omega)} = \langle\operatorname{div}\mathbf{v}, \varphi\rangle_{\mathsf{H}^{-m-1}(\Omega)},$$

yielding (vi). We have the formal adjoint $(\operatorname{Div}\operatorname{spn})^* = (\operatorname{Div}\operatorname{skw}\operatorname{spn})^* = -2\operatorname{spn}^{-1}\operatorname{skw}\operatorname{Grad}$, and by the formula $2\operatorname{skw}\operatorname{Grad}\boldsymbol{\theta} = \operatorname{spn}\operatorname{curl}\boldsymbol{\theta}$ from (ix), we obtain (vii), i.e.

$$-2\langle\mathbf{v}, \operatorname{spn}^{-1}\operatorname{skw}\operatorname{Grad}\boldsymbol{\theta}\rangle_{\mathbf{H}^{-m}(\Omega)} = -\langle\mathbf{v}, \operatorname{curl}\boldsymbol{\theta}\rangle_{\mathbf{H}^{-m}(\Omega)}.$$

Using the formal adjoint $(\operatorname{Curl}\operatorname{spn})^* = 2\operatorname{spn}^{-1}\operatorname{skw}\operatorname{Curl}$ we calculate with (x)

$$2\langle\mathbf{v}, \operatorname{spn}^{-1}\operatorname{skw}\operatorname{Curl}\boldsymbol{\Phi}\rangle_{\mathbf{H}^{-m}(\Omega)} = \langle\mathbf{v}, \operatorname{Div}\boldsymbol{\Phi}^\top - \operatorname{grad}(\operatorname{tr}\boldsymbol{\Phi})\rangle_{\mathbf{H}^{-m}(\Omega)}$$

$$= -\langle \operatorname{Grad} \mathbf{v}, \mathbf{\Phi}^\top \rangle_{\mathcal{H}^{-m-1}(\Omega)} + \langle \operatorname{div} \mathbf{v}, \operatorname{tr} \mathbf{\Phi} \rangle_{\mathsf{H}^{-m-1}(\Omega)},$$

i.e. (viii) holds. Formally $(\operatorname{skw}\operatorname{Grad})^* = -\operatorname{Div}\operatorname{skw}$. Using (vii) we see

$$-\langle \mathbf{v}, \operatorname{Div}\operatorname{skw}\mathbf{\Phi}\rangle_{\mathsf{H}^{-m}(\Omega)} = \langle \mathbf{v}, \operatorname{curl}\operatorname{spn}^{-1}\operatorname{skw}\mathbf{\Phi}\rangle_{\mathsf{H}^{-m}(\Omega)} = \tfrac{1}{2}\langle \operatorname{spn}\operatorname{curl}\mathbf{v}, \operatorname{skw}\mathbf{\Phi}\rangle_{\mathcal{H}^{-m-1}(\Omega)},$$

which proves (ix). We compute by (viii)

$$\begin{aligned}
\langle \mathbf{M}, \operatorname{Curl}\operatorname{skw}\mathbf{\Phi}\rangle_{\mathcal{H}^{-m}(\Omega)} &= \langle \operatorname{tr}\mathbf{M}, \operatorname{div}(\operatorname{spn}^{-1}\operatorname{skw}\mathbf{\Phi})\rangle_{\mathsf{H}^{-m}(\Omega)} - \langle \mathbf{M}^\top, \operatorname{Grad}(\operatorname{spn}^{-1}\operatorname{skw}\mathbf{\Phi})\rangle_{\mathcal{H}^{-m}(\Omega)}\\
&= -\langle \operatorname{grad}(\operatorname{tr}\mathbf{M}), \operatorname{spn}^{-1}\operatorname{skw}\mathbf{\Phi}\rangle_{\mathsf{H}^{-m-1}(\Omega)} + \langle \operatorname{Div}\mathbf{M}^\top, \operatorname{spn}^{-1}\operatorname{skw}\mathbf{\Phi}\rangle_{\mathsf{H}^{-m-1}(\Omega)}\\
&= -\tfrac{1}{2}\langle \operatorname{spn}(\operatorname{grad}\operatorname{tr}\mathbf{M}), \operatorname{skw}\mathbf{\Phi}\rangle_{\mathcal{H}^{-m-1}(\Omega)} + \tfrac{1}{2}\langle \operatorname{spn}\operatorname{Div}\mathbf{M}^\top, \operatorname{skw}\mathbf{\Phi}\rangle_{\mathcal{H}^{-m-1}(\Omega)},
\end{aligned}$$

showing the first formula in (x) and the second one follows by $\operatorname{Div}\operatorname{Curl} = 0$ and (vii). To prove (xi) we observe

$$\langle \mathbf{v}, \operatorname{Div}(\operatorname{devGrad}\boldsymbol{\theta})^\top\rangle_{\mathsf{H}^{-m}(\Omega)} = \langle \mathbf{v}, \operatorname{Div}\operatorname{dev}(\operatorname{Grad}\boldsymbol{\theta})^\top\rangle_{\mathsf{H}^{-m}(\Omega)} = \tfrac{2}{3}\langle \mathbf{v}, \operatorname{grad}\operatorname{div}\boldsymbol{\theta}\rangle_{\mathsf{H}^{-m}(\Omega)},$$

completing the proof. ∎

Proof of Lemma **3.20:** For $\mathbf{M} \in \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl}, \Omega)$ there exists a sequence $(\mathbf{\Phi}_n) \subset \mathring{\mathcal{C}}^\infty(\Omega) \cap \mathcal{L}^2_{\mathbb{S}}(\Omega)$ with $\mathbf{\Phi}_n \to \mathbf{M}$ in $\mathcal{H}(\operatorname{Curl}, \Omega)$. But then $(\varphi\mathbf{\Phi}_n) \subset \mathring{\mathcal{C}}^\infty(\Omega) \cap \mathcal{L}^2_{\mathbb{S}}(\Omega)$ with $\varphi\mathbf{\Phi}_n \to \varphi\mathbf{M}$ in $\mathcal{H}(\operatorname{Curl}, \Omega)$, proving $\varphi\mathbf{M} \in \mathring{\mathcal{H}}_{\mathbb{S}}(\operatorname{Curl}, \Omega)$, as we have $\operatorname{Curl}(\varphi\mathbf{\Phi}_n) = \varphi\operatorname{Curl}\mathbf{\Phi}_n + \operatorname{grad}\varphi \times \mathbf{\Phi}_n$. This formula also shows for $\mathbf{\Psi} \in \mathring{\mathcal{C}}^\infty(\Omega)$ (note that $\varphi\mathbf{\Psi} \in \mathring{\mathcal{C}}^\infty(\Omega)$)

$$\begin{aligned}
\langle \varphi\mathbf{M}, \operatorname{Curl}\mathbf{\Psi}\rangle_{\mathcal{L}^2(\Omega)} &= \langle \mathbf{M}, \varphi\operatorname{Curl}\mathbf{\Psi}\rangle_{\mathcal{L}^2(\Omega)} = \langle \mathbf{M}, \operatorname{Curl}(\varphi\mathbf{\Psi})\rangle_{\mathcal{L}^2(\Omega)} - \langle \mathbf{M}, \operatorname{grad}\varphi \times \mathbf{\Psi}\rangle_{\mathcal{L}^2(\Omega)}\\
&= \langle \operatorname{Curl}\mathbf{M}, \varphi\mathbf{\Psi}\rangle_{\mathcal{L}^2(\Omega)} + \langle \operatorname{grad}\varphi \times \mathbf{M}, \mathbf{\Psi}\rangle_{\mathcal{L}^2(\Omega)},
\end{aligned} \tag{A3}$$

and thus $\operatorname{Curl}(\varphi\mathbf{M}) = \varphi\operatorname{Curl}\mathbf{M} + \operatorname{grad}\varphi \times \mathbf{M}$. Analogously we prove the other cases of (i). Similarly we show (iii) using the formula $\operatorname{Div}(\varphi\mathbf{\Phi}_n) = \varphi\operatorname{Div}\mathbf{\Phi}_n + \operatorname{grad}\varphi \cdot \mathbf{\Phi}_n$. To show (ii), let $\mathbf{M} \in \mathcal{H}_{\mathbb{S}}(\operatorname{Curl}, \Omega)$. Then $\varphi\mathbf{M} \in \mathcal{L}^2_{\mathbb{S}}(\Omega)$ and (A.3) shows $\varphi\mathbf{M} \in \mathcal{H}_{\mathbb{S}}(\operatorname{Curl}, \Omega)$ with the desired formula. Analogously the other cases of (ii) follow. Similarly we prove (iv). Let $\mathbf{E} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega)$ and $\mathbf{\Phi} \in \mathring{\mathcal{C}}^\infty(\Omega)$. Then $\varphi\mathbf{E} \in \mathcal{L}^2_{\mathbb{T}}(\Omega)$ and with $\varphi\,\mathbf{\Phi} \in \mathring{\mathcal{C}}^\infty(\Omega)$ we get

$$\begin{aligned}
\langle \varphi\mathbf{E}, \operatorname{Curl}\operatorname{sym}\mathbf{\Phi}\rangle_{\mathcal{L}^2(\Omega)} &= \langle \mathbf{E}, \varphi\operatorname{Curl}\operatorname{sym}\mathbf{\Phi}\rangle_{\mathcal{L}^2(\Omega)} = \langle \mathbf{E}, \operatorname{Curl}\operatorname{sym}(\varphi\,\mathbf{\Phi})\rangle_{\mathcal{L}^2(\Omega)} - \langle \mathbf{E}, \operatorname{grad}\varphi \times \operatorname{sym}\mathbf{\Phi}\rangle_{\mathcal{L}^2(\Omega)}\\
&= \langle \operatorname{symCurl}\mathbf{E}, \varphi\mathbf{\Phi}\rangle_{\mathcal{L}^2(\Omega)} + \langle \operatorname{grad}\varphi \times \mathbf{E}, \operatorname{sym}\mathbf{\Phi}\rangle_{\mathcal{L}^2(\Omega)},
\end{aligned}$$

which shows $\varphi\mathbf{E} \in \mathcal{H}_{\mathbb{T}}(\operatorname{symCurl}, \Omega)$ and $\operatorname{symCurl}(\varphi\mathbf{E}) = \varphi\operatorname{symCurl}\mathbf{E} + \operatorname{sym}(\operatorname{grad}\varphi \times \mathbf{E})$ and hence (v). To prove (vi), let $\mathbf{M} \in \mathcal{H}_{\mathbb{S}}(\operatorname{divDiv}, \Omega)$ and $\phi \in \mathring{\mathsf{C}}^\infty(\Omega)$. Then $\varphi\mathbf{M} \in \mathcal{L}^2_{\mathbb{S}}(\Omega)$ and we compute by

$$\begin{aligned}
\operatorname{Gradgrad}(\varphi\,\phi) &= \varphi\operatorname{Gradgrad}\phi + \phi\operatorname{Gradgrad}\varphi + 2\operatorname{sym}\Big((\operatorname{grad}\varphi)(\operatorname{grad}\phi)^\top\Big),\\
(\operatorname{grad}\varphi)(\operatorname{grad}\phi)^\top &= \operatorname{Grad}(\phi\operatorname{grad}\varphi) - \phi\operatorname{Gradgrad}\varphi
\end{aligned}$$

the identity

$$\operatorname{Gradgrad}(\varphi\,\phi) = \varphi\operatorname{Gradgrad}\phi - \phi\operatorname{Gradgrad}\varphi + 2\operatorname{sym}\operatorname{Grad}(\phi\operatorname{grad}\varphi).$$

Finally with $\varphi\phi \in \mathring{\mathsf{C}}^\infty(\Omega)$ we get

$$\begin{aligned}
&\langle \varphi\mathbf{M}, \operatorname{Gradgrad}\phi\rangle_{\mathcal{L}^2(\Omega)} = \langle \mathbf{M}, \varphi\operatorname{Gradgrad}\phi\rangle_{\mathcal{L}^2(\Omega)}\\
&\quad= \big\langle \mathbf{M}, \operatorname{Gradgrad}(\varphi\,\phi)\big\rangle_{\mathcal{L}^2(\Omega)} + \langle \mathbf{M}, \phi\operatorname{Gradgrad}\varphi\rangle_{\mathcal{L}^2(\Omega)} - 2\big\langle \mathbf{M}, \operatorname{sym}\operatorname{Grad}(\phi\operatorname{grad}\varphi)\big\rangle_{\mathcal{L}^2(\Omega)}\\
&\quad= \langle \operatorname{divDiv}\mathbf{M}, \varphi\,\phi\rangle_{\mathsf{L}^2(\Omega)} + \langle \mathbf{M} : \operatorname{Gradgrad}\varphi, \phi\rangle_{\mathcal{L}^2(\Omega)} - 2\big\langle \mathbf{M}, \operatorname{Grad}(\phi\operatorname{grad}\varphi)\big\rangle_{\mathcal{L}^2(\Omega)}\\
&\quad= \langle \varphi\operatorname{divDiv}\mathbf{M}, \phi\rangle_{\mathsf{L}^2(\Omega)} + \big\langle \operatorname{tr}(\mathbf{M}\operatorname{Gradgrad}\varphi), \phi\big\rangle_{\mathsf{L}^2(\Omega)} + 2\underbrace{\langle \operatorname{Div}\mathbf{M}, \phi\operatorname{grad}\varphi\rangle_{\mathsf{H}^{-1}(\Omega)}}_{=\langle \operatorname{Div}\mathbf{M}\cdot\operatorname{grad}\varphi, \phi\rangle_{\mathsf{H}^{-1}(\Omega)}},
\end{aligned}$$

which shows (vi), i.e. $\varphi\mathbf{M} \in \mathcal{H}^{0,-1}_{\mathbb{S}}(\operatorname{divDiv}, \Omega)$ and

$$\operatorname{divDiv}(\varphi\mathbf{M}) = \varphi\operatorname{divDiv}\mathbf{M} + 2\operatorname{grad}\varphi \cdot \operatorname{Div}\mathbf{M} + \operatorname{tr}(\mathbf{M}\operatorname{Gradgrad}\varphi) \in \mathsf{H}^{-1}(\Omega).$$

The proof is finished. ∎