4

Second-order elliptic boundary value problems in polygons

4.1 Foreword

The purpose of this chapter is to investigate the properties of the second derivatives of the solutions of boundary value problems for the Laplace operator in a plane domain with a polygonal boundary. Here, we just consider classical polygons, i.e. the union of a finite number of linear segments $\bar{\Gamma}_j$, $1 \leqslant j \leqslant N$ (it is convenient to assume that Γ_j is an open linear segment). We also fix a partition of $\{1, 2, \ldots, N\}$ into two subsets $\mathcal{N}$ and $\mathcal{D}$. The union of the Γ_j with $j \in \mathcal{D}$ is going to be the part of the boundary where we consider a Dirichlet boundary condition. We shall consider first-order boundary conditions (either Neumann or oblique) on the other sides. Accordingly, our main problem will be the following. Given $f \in L_p(\Omega)$, we look for $u \in W_p^2(\Omega)$, a solution of

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \gamma_j u = 0 & \text{on } \Gamma_j, \quad j \in \mathcal{D} \\ \gamma_j \dfrac{\partial u}{\partial \nu_j} + \beta_j \dfrac{\partial}{\partial \tau_j} \gamma_j u = 0 & \text{on } \Gamma_j, \quad j \in \mathcal{N} \end{cases} \tag{4,1,1}$$

where $\boldsymbol{\nu}_j$ denotes the unit normal on Γ_j, while $\boldsymbol{\tau}_j$ denotes the unit tangent vector on Γ_j (following the direct orientation; finally β_j, $j \in \mathcal{N}$ are given real numbers).

The first step in solving (4,1,1) is the proof of *a priori* bounds for solutions in $W_p^2(\Omega)$. Actually, we shall prove the existence of a constant C depending on Ω, p, $\mathcal{D}$ and β_j ($j \in \mathcal{N}$) such that

$$\|u\|_{2,p,\Omega} \leqslant C\{\|\Delta u\|_{0,p,\Omega} + \|u\|_{0,p,\Omega}\} \tag{4,1,2}$$

for all $u \in W_p^2(\Omega)$ fulfilling the boundary conditions in (4,1,1).

Curiously enough the inequality (4,1,2) always holds when $p = 2$, while it does not hold for some exceptional values of the numbers β_j ($j \in \mathcal{N}$) when $p \neq 2$ (a detailed investigation of some exceptional cases can be found in Fabes *et al.* (1977)). Actually our methods of proof when $p=2$ and when $p \neq 2$ are quite different. Our proof for $p=2$ starts from a particular case of identity (3,1,1,10) which we shall prove directly by performing integration by parts. On the other hand, when $p \neq 2$, we shall use a local method which, together with the same change of variables as in Kondratiev (1967), reduces our problem to a boundary value problem in an infinite plane strip. The main advantage is that such a strip has a smooth boundary. There, we essentially use the same techniques as in Subsection 2.3.2. That is, we write the solution as a double layer potential which is estimated by applying Mikhlin's multipliers theorem.

Inequalities like (4,1,2) when $p=2$ for the Dirichlet problem have been proven in Aronszajn (1951) and Hanna and Smith (1967). General boundary conditions are dealt with in Grisvard (1972).

The second step in solving (4,1,1) is the following. The *a priori* bound (4,1,2) implies that the Laplace operator Δ has a closed range in $L_p(\Omega)$ when we look at it, as an unbounded operator whose domain is the subspace of $W_p^2(\Omega)$ defined by the boundary conditions in (4,1,1). Therefore, the annihilator of the range is a space of functions in $L_q(\Omega)$ (where $p^{-1}+q^{-1}=1$) which are, in some weak sense, solutions of the homogeneous adjoint problem. Using separation of variables in polar coordinates, we shall be able to derive precise expansions of those functions near the corners. Then it will be easy to calculate the codimension of the range of the Laplace operator in $L_p(\Omega)$. This is carried out in Section 4.4.

Such results for $p \neq 2$ were first proven by Merigot (1972), who makes use of quite different methods. A comprehensive detailed study of problem (4,1,1) has been carried out independently by Lorenzi (1978) and Moussaoui (1977). Here is a simplified version of their work.

The reader interested only by the $p=2$ case may skip Section 4.2 and Subsection 4.3.2.

Here are some additional notation. We denote by ω_j the measure (we allow $\omega_j=\pi$ in order to consider also mixed problems along a flat boundary) of the angle at S_j and we set

$$\boldsymbol{\mu}_j=\begin{cases}\boldsymbol{\nu}_j+\beta_j\boldsymbol{\tau}_j, & j\in\mathcal{N}\\ \boldsymbol{\tau}_j & j\in\mathcal{D}\end{cases} \tag{4,1,3}$$

Accordingly we have $\gamma_j(\partial u/\partial \mu_j)=0$ for all j when u fulfils the boundary conditions in (4,1,1). Finally, we define Φ_j by

$$\begin{cases}\tan\Phi_j=\beta_j & j\in\mathcal{N}\\ \Phi_j=\pi/2 & j\in\mathcal{D},\end{cases} \tag{4,1,4}$$

thus Φ_j is the angle of the vectors $\boldsymbol{\nu}_j$ and $\boldsymbol{\mu}_j$.

Finally, let us mention that all the results in this section hold for domains with holes. Considering domains with holes (or domains which are not connected) just leads to more complicated notation.

4.2 *A priori* estimates for a problem in an infinite strip

This whole section is devoted to the proof of bounds for solutions of the following boundary value problem in the infinite strip $B=\mathbb{R}\times]0,h[$ $(h>0)$. We shall denote by x and y the coordinates in $\mathbb{R}^2$ and thus we have

$$B=\{(x,y)\mid x\in\mathbb{R}, 0<y<h\}.$$

We shall deal with a boundary value problem for the operator L defined by

$$Lu=D_x^2u+D_y^2u+aD_xu+bu,$$

where a and b are real numbers. The boundary conditions involve the operators

$$M_ju=\alpha_jD_yu+\beta_jD_xu+\lambda_ju, \qquad j=0,1.$$

where α_j, β_j and λ_j are real numbers $j=0$, 1. (Actually, we shall only need, in the forthcoming sections, these two special cases: either $\alpha_j=1$ or $\alpha_j=\beta_j=0$ and $\lambda_j=1$.) Precisely, we look at $u\in W_p^2(B)$, a solution of

$$\begin{cases} Lu=f & \text{in } B\\ \gamma_jM_ju=0 & \text{on } F_j, \quad j=0,1\end{cases} \tag{4,2,1}$$

where $F_0=\{(x,0)\mid x\in\mathbb{R}\}$, $F_1=\{(x,h)\mid x\in\mathbb{R}\}$ and γ_j denotes the trace operator on F_j, $j=0,1$.

We shall look for conditions on the coefficients a, b, c, α_j, β_j, λ_j, $j=0$, 1, ensuring the existence of a constant C such that

$$\|u\|_{2,p,B}\leq C\,\|f\|_{0,p,B}. \tag{4,2,2}$$

For that purpose we shall calculate explicitly a solution u of (4,2,1) by a Fourier transform in x. Then the explicit solution will be suitably estimated by using Mikhlin's theorem.

4.2.1 Explicit solution by Fourier transform and consequences

As in Subsection 2.3.2 we denote by $\hat{u}$ (respectively $\hat{f}$) the partial Fourier transform of u (respectively f) with respect to x, i.e.

$$\hat{u}(\xi,y)=\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{+\infty}e^{-ix\xi}u(x,y)\,dx,$$

for $\xi \in \mathbb{R}$ and $y \in]0, h[$. Actually, in order to deal only with Fourier transforms which are functions, we shall always assume that $u \in H^2(B)$. Eventually, we shall take advantage of density theorems for extending our results to the whole of $W_p^2(B)$.

After performing the Fourier transform, problem (4,2,1) becomes a two-point boundary value problem in the interval $]0, h[$ depending on the parameter ξ. Namely this problem is

$$\begin{cases} \hat{u}'' + (-\xi^2 + ia\xi + b)\hat{u} = \hat{f} & \text{in }]0, h[\\ \alpha_0 \hat{u}'(\xi, 0) + (i\beta_0\xi + \lambda_0)\hat{u}(\xi, 0) = 0 \\ \alpha_1 \hat{u}'(\xi, h) + (i\beta_1\xi + \lambda_1)\hat{u}(\xi, h) = 0 \end{cases} \tag{4,2,1,1}$$

for all $\xi \in \mathbb{R}$, where the superscript prime denotes differentiation with respect to y.

As is well known for two-point boundary value problems, problem (4,2,1,1) is well posed if and only if the only solution of the corresponding homogeneous boundary value problem is zero. This possibly leads to exceptional values of ξ. More precisely we have the following.

Theorem 4.2.1.1 *Assuming $b > 0$, $a \neq 0$, then the problem* (4,2,1,1) *is well posed unless*

$$\begin{aligned} \sin \rho h(\alpha_0\alpha_1\rho^2 - \beta_0\beta_1\xi^2 + \lambda_0\lambda_1 + i[\beta_0\lambda_1 + \beta_1\lambda_0]\xi) \\ = \rho \cos \rho h([\alpha_0\lambda_1 - \alpha_1\lambda_0] + i[\alpha_0\beta_1 - \alpha_1\beta_0]\xi), \end{aligned} \tag{4,2,1,2}$$

where $\rho = (b + ia\xi - \xi^2)^{1/2}$.†

The case when $b > 0$, $a \neq 0$ is the only one that we need in the sequel.

Proof A fundamental system of solutions for the differential equation in (4,2,1,1) is the couple of functions

$$v_1(y) = \sin \rho y, \qquad v_2(y) = \cos \rho y$$

where $\rho = (b + ia\xi - \xi^2)^{1/2}$, for $\xi \in \mathbb{R}$. Thus any solution of the homogeneous equation

$$v'' + (b + ia\xi - \xi^2)v = 0$$

has the form

$$v = \mu_1 v_1 + \mu_2 v_2.$$

The boundary conditions in problem (4,2,1,1) are fulfilled if and only if

$$\begin{cases} \alpha_0(\mu_1 v_1'(0) + \mu_2 v_2'(0)) + (i\beta_0\xi + \lambda_0)(\mu_1 v_1(0) + \mu_2 v_2(0)) = 0 \\ \alpha_1(\mu_1 v_1'(h) + \mu_2 v_2'(h)) + (i\beta_1\xi + \lambda_1)(\mu_1 v_1(h) + \mu_2 v_2(h)) = 0. \end{cases}$$

† We define the square root of a complex number by placing the cut on the negative real axis.

This is a linear system in (μ_1, μ_2) whose only solution is zero, provided its determinant d is different from zero. Actually we have

$$\begin{aligned} d &= \{\alpha_0 v_1'(0) + (\mathrm{i}\beta_0\xi + \lambda_0)v_1(0)\}\{\alpha_1 v_2'(h) + (\mathrm{i}\beta_1\xi + \lambda_1)v_2(h)\} \\ &\quad - \{\alpha_0 v_2'(0) + (\mathrm{i}\beta_0\xi + \lambda_0)v_2(0)\}\{\alpha_1 v_1'(h) + (\mathrm{i}\beta_1\xi + \lambda_1)v_1(h)\} \\ &= \alpha_0\rho\{-\alpha_1\rho \sin \rho h + (\mathrm{i}\beta_1\xi + \lambda_1)\cos \rho h\} \\ &\quad - (\mathrm{i}\beta_0\xi + \lambda_0)\{\alpha_1\rho \cos \rho h + (\mathrm{i}\beta_1\xi + \lambda_1)\sin \rho h\} \\ &= -\sin \rho h\{\alpha_0\alpha_1\rho^2 + (\mathrm{i}\beta_0\xi + \lambda_0)(\mathrm{i}\beta_1\xi + \lambda_1)\} \\ &\quad + \rho \cos \rho h\{\alpha_0(\mathrm{i}\beta_1\xi + \lambda_1) - \alpha_1(\mathrm{i}\beta_0\xi + \lambda_0)\}. \end{aligned}$$

The condition that $d = 0$ is exactly (4,2,1,2). It is the one that allows the homogeneous problem corresponding to problem (4,2,1,1) to admit non-zero solutions. ■

We observe that condition (4,2,1,2) involves only analytic functions of ξ. Thus, it is fulfilled for countably many exceptional values of ξ. In addition, due to the asymptotic behaviour, for large $|\xi|$, of both members of equation (4,2,1,2), there is only a finite number of exceptional values on the real axis. From now on we shall assume that problem (4,2,1,1) is well posed for all $\xi \in \mathbb{R}$. In other words, we assume that equation (4,2,1,2) has no real root.

Theorem 4.2.1.2 *Assuming $b > 0$, $a \neq 0$ and that ξ is not a root of* (4,2,1,2), *the solution of problem* (4,2,1,1) *is*

$$\hat{u}(\xi, y) = \int_0^h K(\xi, y, z)\hat{f}(\xi, z)\,\mathrm{d}z, \tag{4,2,1,3}$$

where, for $z \geq y$,

$$\begin{aligned} K(\xi, y, z) &= \frac{1}{\delta}\left[\alpha_0 \cos \rho y - \frac{\mathrm{i}\beta_0\xi + \lambda_0}{\rho} \sin \rho y\right] \\ &\quad \times \left[\alpha_1 \cos \rho(z-h) - \frac{\mathrm{i}\beta_1\xi + \lambda_1}{\rho} \sin \rho(z-h)\right] \end{aligned}$$

and, for $z \leq y$,

$$\begin{aligned} K(\xi, y, z) &= \frac{1}{\delta}\left[\alpha_1 \cos \rho(y-h) - \frac{\mathrm{i}\beta_1\xi + \lambda_1}{\rho} \sin \rho(y-h)\right] \\ &\quad \times \left[\alpha_0 \cos \rho z - \frac{\mathrm{i}\beta_0\xi + \lambda_0}{\rho} \sin \rho z\right] \end{aligned}$$

where $\rho = (b + \mathrm{i}a\xi - \xi^2)^{1/2}$ and

$$\begin{aligned} \delta &= [\alpha_1(\mathrm{i}\beta_0\xi + \lambda_0) - \alpha_0(\mathrm{i}\beta_1\xi + \lambda_1)]\cos \rho h \\ &\quad + [\alpha_0\alpha_1\rho^2 + (\mathrm{i}\beta_0\xi + \lambda_0)(\mathrm{i}\beta_1\xi + \lambda_1)]\frac{\sin \rho h}{\rho}. \end{aligned}$$

Although these formulas are consequences of general procedures for solving two-point boundary value problems, it is simpler to check them directly by verifying that (4,2,1,3) actually gives the solution of problem (4,2,1,1). This is straightforward.

We shall now use identity (4,2,1,3) to show the existence of a constant C_0 such that

$$\|u\|_{0,p,B} \leq C_0 \|Lu\|_{0,p,B} \tag{4,2,1,4}$$

for all $u \in W_p^2(B) \cap H^2(B)$ which fulfils the boundary conditions in (4,2,1), assuming the problem is well posed. We shall need the following lemma.

Lemma 4.2.1.3 *Let $\xi, y, z \mapsto K(\xi, y, z)$ be a smooth function such that*

$$\max_{y \in]0,h[} \int_0^h \max_{\xi \in \mathbb{R}} \{|K(\xi, y, z)| + |\xi|\, |D_\xi K(\xi, y, z)|\}\, dz < +\infty \tag{4,2,1,5}$$

and

$$\max_{z \in]0,h[} \int_0^h \max_{\xi \in \mathbb{R}} \{|K(\xi, y, z)| + |\xi|\, |D_\xi K(\xi, y, z)|\}\, dy < +\infty; \tag{4,2,1,6}$$

then the mapping $u \mapsto f$ defined by

$$\hat{u}(\xi, y) = \int_0^h K(\xi, y, z) \hat{f}(\xi, z)\, dz$$

is continuous in $L_p(B)$ for p such that $1 < p < \infty$.

Proof Let us denote by M the function

$$M(y, z) = \max_{\xi \in \mathbb{R}} \{|K(\xi, y, z)| + |\xi|\, |D_\xi K(\xi, y, z)|\}.$$

Applying Mikhlin's theorem (see Theorem 2.3.2.1) we know that there exists a constant C such that

$$\left(\int_{-\infty}^{+\infty} |u(x, y)|^p\, dx\right) \leq C \int_0^h M(y, z) \left(\int_{-\infty}^{+\infty} |f(x, z)|^p\, dx\right)^{1/p} dz.$$

We conclude by applying a classical result on integral operators (see for instance Widom (1965). ■

Now we have to check the conditions in Lemma 4.2.1.3 on the kernel defined in Theorem 4.2.1.2. It is easy to derive the following bounds for K when the problem is well posed (i.e. when (4,2,1,2) has no real root). Indeed there exists a constant L such that

$$|K(\xi, y, z)| \leq \frac{L}{|\rho|} \exp |\rho|\, (y + z - 2h)$$

$$|\xi|\, |D_\xi K(\xi, y, z)| \leq L \exp |\rho|\, (y + z - 2h),$$

while the asymptotic behaviour of ρ is like $\pm i\xi$, when $|\xi|\to\infty$. It follows that inequalities (4,2,1,5) and (4,2,1,6) hold and consequently we have proved inequality (4,2,1,4).

Summing up, we have proved the following result.

Theorem 4.2.1.4 *Let us assume that $b>0$, $a\neq 0$ and that equation* (4,2,1,2) *has no real root, then there exists a constant C_0 such that* (4,2,1,4) *holds for all $u\in W_p^2(B)\cap H^2(B)$ fulfilling the boundary conditions in* (4,2,1).

Actually, with a little more care, we should be able to bound the first derivatives of u in $L_p(B)$, by using the same Lemma 4.2.1.3. Unfortunately, this does not allow us to bound the second derivatives, which is our real goal.

Finally in most cases, a simple density argument allows us to extend the previous result to all $u\in W_p^2(B)$ fulfilling the boundary conditions.

Corollary 4.2.1.5 *Under the assumptions of Theorem* 4.2.1.4, *inequality* (4,2,1,4) *holds for all $u\in W_p^2(B)$ fulfilling the boundary conditions in* (4,2,1), *provided either $\alpha_j=1$ or $\alpha_j=\beta_j=0$, $\lambda_j=1$ for each $j=0,1$.*

This is deduced from Theorem 4.2.1.4 with the help of the following lemma.

Lemma 4.2.1.6 *Assuming $\alpha_j=1$ or $\alpha_j=\beta_j=0$, $\lambda_j=1$ for each $j=0,1$, the subspace of $W_p^2(B)\cap H^2(B)$ defined by the boundary conditions $\gamma_jM_ju=0$ on F_j is dense in the subspace of $W_p^2(B)$ defined by the same boundary conditions.*

Proof Due to the trace theorems in Section 1.5 we can view $W_p^2(B)$ as

$$\mathring{W}_p^2(B)\times\prod_{j=0}^{1}\{W_p^{2-1/p}(F_j)\times W_p^{1-1/p}(F_j)\}$$

and $H^2(B)$ as

$$\mathring{H}^2(B)\times\prod_{j=0}^{1}\{H^{3/2}(F_j)\times H^{1/2}(F_j)\}.$$

Since $\mathring{H}^2(B)\cap\mathring{W}_p^2(B)$ is dense in $\mathring{W}_p^2(B)$, it will be enough to prove that for each j the space $Z_2\cap Z_p$ is dense in Z_p, where Z_q is defined for all q as follows:

$$Z_q=\{(k,l)\mid k\in W_q^{2-1/q}(F_j),\ l\in W_q^{1-1/q}(F_j),\ \alpha_jl+\beta_jD_xk+\lambda_jk=0\}.$$

In the case when $\alpha_j=\beta_j=0$ and $\lambda_j=1$, the space Z_q reduces to $0\times W_q^{1-1/q}(F_j)$ and thus $Z_2\cap Z_p$ is obviously dense in Z_p.

In the case when $\alpha_j = 1$, the space Z_q is nothing but

$$\{(k, -\beta_j D_x k - \lambda_j k) \mid k \in W_q^{2-1/q}(F_j)\},$$

which is isomorphic to $W_q^{2-1/q}(F_j)$. Again $Z_2 \cap Z_p$ is obviously dense in Z_p. ■

4.2.2 L_p bounds for the second derivatives of the solution

In order to be able to bound the second derivatives of u, the solution of problem (4,2,1), we replace it by a slightly different problem:

$$\begin{cases} L_1 u = g & \text{in } B \\ \gamma_j M_j u = 0 & \text{on } F_j, \quad j = 0,1, \end{cases} \tag{4,2,2,1}$$

where

$$L_1 u = D_x^2 u + D_y^2 u - u = (\Delta - 1)u.$$

The reason for doing this is that $(1 - \Delta)$ has an elementary solution E with good properties in the Sobolev spaces. This was not true for L.

We shall prove in this subsection that there exists a constant C_1 such that

$$\|D^\alpha u\|_{0,p,B} \leq C_1 \|(1-\Delta)u\|_{0,p,B} \tag{4,2,2,2}$$

for all $|\alpha| = 2$ and $u \in W_p^2(B) \cap H^2(B)$ which fulfils the boundary conditions in (4,2,2,1). Then it will be very easy to deduce (4,2,2) from (4,2,1,4) combined with (4,2,2,2).

Now we proceed exactly as in Subsection 2.3.2. The elementary solution for $\Delta - 1$, namely

$$E = -F^{-1}[1 + |\xi|^2]^{-1}$$

is linear continuous from $L_p(\mathbb{R}^2)$ into $W_p^2(\mathbb{R}^2)$ according to Theorem 2.3.2.1. Then let us denote by v the function

$$v = u - E * \tilde{g}; \tag{4,2,2,3}$$

we obviously have

$$\begin{cases} (1-\Delta)v = 0 & \text{in } B \\ \gamma_j M_j v = h_j & \text{on } F_j, \quad j = 0, 1 \end{cases} \tag{4,2,2,4}$$

where

$$h_j = -\gamma_j M_j E * \tilde{g}, \qquad j = 0, 1. \tag{4,2,2,5}$$

Thus, denoting by d_j the order of $M_j = \alpha_j D_y + \beta_j D_x + \lambda_j$, we obviously have

$$h_j \in W_p^{2-d_j-1/p}(F_j) \cap H^{2-d_j-1/2}(F_j), \qquad j = 0, 1.$$

We shall calculate explicitly the partial Fourier transform in x of v. It is a solution of

$$\begin{cases} -\hat{v}''+(1+\xi^2)\hat{v}=0 \quad \text{in} \quad]0,h[\\ \alpha_0\hat{v}(\xi,0)+(i\beta_0\xi+\lambda_0)\hat{v}(\xi,0)=\hat{h}_0 \\ \alpha_1\hat{v}'(\xi,h)+(i\beta_1\xi+\lambda_1)\hat{v}(\xi,h)=\hat{h}_1 \end{cases}$$

for almost all $\xi\in\mathbb{R}$.

Consequently we have $\hat{v}(\xi,y)=\alpha(\xi)\cosh ry+\beta(\xi)\sinh ry$ where $r=\sqrt{(1+\xi^2)}$ and

$$\alpha(\xi)=\frac{1}{d}\{\alpha_0 r\hat{h}_1-[\alpha_1 r\cosh rh+(i\beta_1\xi+\lambda_1)\sinh rh]\hat{h}_0\}$$

$$\beta(\xi)=\frac{1}{d}\{-(i\beta_0\xi+\lambda_0)\hat{h}_1+[\alpha_1 r\sinh rh+(i\beta_1\xi+\lambda_1)\cosh rh]\hat{h}_0\}$$

$$\begin{aligned} d={}&r\cosh rh[\alpha_0(i\beta_1\xi+\lambda_1)-\alpha_1(i\beta_0\xi+\lambda_0)] \\ &+\sinh rh[r^2\alpha_0\alpha_1-(i\beta_0\xi+\lambda_0)(i\beta_1\xi+\lambda_1)]. \end{aligned} \tag{4,2,2,6}$$

Of course these identities make sense only if we assume that d does not vanish.

From these formulas, we shall deduce the traces

$$k_j=\gamma_j v \quad \text{on} \quad F_j, \qquad l_j=\gamma_j D_y v \quad \text{on} \quad F_j, \qquad j=0,1.$$

Actually we have

$$\begin{aligned} &\hat{k}_0(\xi)=\alpha(\xi), \qquad \hat{l}_0(\xi)=r(\xi)\beta(\xi) \\ &\hat{k}_1(\xi)=\alpha(\xi)\cosh rh+\beta(\xi)\sinh rh \\ &\hat{l}_1(\xi)=\alpha(\xi)r\sinh rh+\beta(\xi)\,r\cosh rh. \end{aligned}$$

Lemma 4.2.2.1 *Assume that d defined by* (4,2,2,6) *does not vanish for any $\xi\in\mathbb{R}$, then there exists a constant C such that*

$$\sum_{j=0}^{1}\{\|k_j\|_{2-1/p,p,F_j}+\|l_j\|_{1-1/p,p,F_j}\}\leq C\sum_{j=0}^{1}\|h_j\|_{2-d_j-1/p,p,F_j}.$$

Proof This is just a repeated application of Lemma 2.3.2.5. ■

Then we look at $\tilde{v}$, the continuation of v by zero outside B. It is a solution of

$$(1-\Delta)\tilde{v}=-k_0\otimes\delta_0'+k_1\otimes\delta_h'-l_0\otimes\delta_0+l_1\otimes\delta_h, \tag{4,2,2,7}$$

where δ_0 (respectively δ_h) is the Dirac measure at zero (respectively h).

Consequently we have

$$u = E*\{\tilde{g} - k_0\otimes\delta_0' + k_1\otimes\delta_h' - l_0\otimes\delta_0 + l_1\otimes\delta_h\}.$$

From this representation of u we shall deduce the following basic result.

Theorem 4.2.2.2 *Assume that d defined by* (4,2,2,6) *does not vanish for any real ξ, then there exists a constant C_1 such that*

$$\|u\|_{2,p,B} \leq C_1 \|g\|_{0,p,B} \tag{4,2,2,8}$$

for all $u \in W_p^2(B)\cap H^2(B)$ which are solutions of problem (4,2,2,1).

Proof This is mainly the same proof as for Theorem 2.3.2.7. Indeed we know that there exists a constant C such that

$$\|E*\tilde{g}\|_{2,p,\mathbb{R}^2} \leq C \|g\|_{0,p,\mathbb{R}^2}.$$

Then, let us consider one typical term $E*(k_0\otimes\delta_0')$. From (4,2,2,5) and Theorems 2.3.2.1 and 1.5.1.1, we know that there exists a constant C such that

$$\|h_j\|_{2-d_j-1/p,p,F_j} \leq C \|g\|_{0,p,B}.$$

Then combining this inequality with Lemma 4.2.2.1 we have (with possibly a larger constant)

$$\|k_0\|_{2-1/p,p,F_0} \leq C \|g\|_{0,p,B}.$$

Now Lemma 2.3.2.2 shows that

$$k_0\otimes\delta_0', \qquad D_x k_0\otimes\delta_0', \qquad D_x^2 k_0\otimes\delta_0' \in W_p^{-2}(\mathbb{R}^2)$$

and depend continuously on $k_0 \in W_p^{2-1/p}(F_0)$. Then setting

$$u_0 = E*(k_0\otimes\delta_0')$$

and applying Lemma 2.3.2.5, we see that

$$u_0,\ D_x u_0,\ D_x^2 u_0 \in L_p(\mathbb{R}^2)$$

and depend continuously on $k_0 \in W_p^{2-1/p}(F_0)$.

Then we write

$$D_y u_0 = D_y^2 E*(k_0\otimes\delta_0) = (1-D_x^2)E*(k_0\otimes\delta_0)$$

in the half plane $y>0$. Thus

$$D_y u_0 = E*(k_0\otimes\delta_0) - E*(D_x^2 k_0\otimes\delta_0)$$

and

$$D_x D_y u_0 = E*(D_x k_0\otimes\delta_0) - D_x E*(D_x^2 k_0\otimes\delta_0).$$

Applying again Lemmas 2.3.2.2 and 2.3.2.5 we show that

$$D_y u_0,\ D_x D_y u_0 \in L_p(B)$$

and depend continuously on $k_0 \in W_p^{2-1/p}(F_0)$.

Finally we write that

$$D_y^2 u_0 = u_0 - D_x^2 u_0 \qquad \text{in } B.$$

Consequently $D_y^2 u_0 \in L_p(B)$ and depend continuously on $k_0 \in W_p^{2-1/p}(F_0)$.

Summing up we have proved that

$$\|u_0\|_{2,p,B} \leq C\, \|g\|_{0,p,B}$$

for some constant C. The other terms in (4,2,2,7) are estimated by the same techniques. ■

Again using the density result of Lemma 4.2.1.6, we can extend inequality (4,2,2,8) to all $u \in W_p^2(B)$ in most cases.

Corollary 4.2.2.3 *Under the assumptions of Theorem* 4.2.2.2, *inequality* (4,2,2,8) *holds for all* $u \in W_p^2(B)$ *which are solutions of problem* (4,2,2,1) *provided either* $\alpha_j = 1$ *or* $\alpha_j = \beta_j = 0$ *and* $\lambda_j = 1$ *for each* $j = 0, 1$.

Theorem 4.2.2.4 *Assume that* $b > 0$, $a \neq 0$ *and that for each* $j = 0$ *or* 1 *we have either* $\alpha_j = 1$ *or* $\alpha_j = \beta_j = 0$ *and* $\lambda_j = 1$. *Assume in addition that equation* (4,2,1,2) *has no real root. Then there exists a constant C such that inequality* (4,2,2) *holds for* $u \in W_p^2(B)$ *the solution of problem* (4,2,1).

In the proof, we shall need this auxiliary lemma which we will prove later.

Lemma 4.2.2.5 *Assume that* $b > 0$, $a \neq 0$ *and that for each j we have either* $\alpha_j = 1$ *or* $\alpha_j = \beta_j = 0$. *Then it is possible to find* μ_j, $j = 0, 1$, *such that*

$$r \cosh rh[\alpha_0(i\beta_1\xi + \mu_1) - \alpha_1(i\beta_0\xi + \mu_0)] \\ + \sinh rh[r^2\alpha_0\alpha_1 - (i\beta_0\xi + \mu_0)(i\beta_1\xi + \mu_1)],$$

where $r = \sqrt{(1+\xi^2)}$ *does not vanish for* $\xi \in \mathbb{R}$.

Proof of Theorem 4.2.2.4 Due to the assumption that equation (4,2,1,2) has no real root, inequality (4,2,1,4) holds. Next we shall use inequality (4,2,2,8) with λ_j replaced by μ_j. From Corollary 4.2.2.3 and the trace theorems in Subsection 1.5, we deduce the existence of a constant C_1 such that

$$\|u\|_{2,p,B} \leq C_1\Big(\|-\Delta u + u\|_{0,p,B} + \sum_{j=0}^{1} \|\gamma_j(\alpha_j D_y u + \beta_j D_x u + \mu_j u)\|_{2-d_j-1/p,p,F_j}\Big)$$

for all $u \in W_p^2(B)$. In particular when u satisfies the boundary condition

$$\gamma_j(\alpha_j D_y u + \beta_j D_x u + \lambda_j u) = 0 \tag{4,2,2,9}$$

on F_j, we have

$$\|u\|_{2,p,B} \leq C_1\Big\{\|-\Delta u + u\|_{0,p,B} + \sum_{d_j=1} |\lambda_j - \mu_j|\, \|\gamma_j u\|_{1-1/p,p,F_j}\Big\}. \tag{4,2,2,10}$$

We observe that we have no boundary terms in the case when $d_j = 0$, since $\gamma_j u = 0$.

From (4,2,2,10) it follows that

$$\|u\|_{2,p,B} \leq C_1\Big\{\|Lu\|_{0,p,B} + \|aD_x u + (b+1)u\|_{0,p,B} + \sum_{d_j=1} |\lambda_j - \mu_j|\, \|\gamma_j u\|_{1-1/p,p,F_j}\Big\}.$$

In other words, we have

$$\|u\|_{2,p,B} \leq C_2\{\|Lu\|_{0,p,B} + \|u\|_{1,p,B}\}.$$

Next we take advantage of the inequality

$$\|u\|_{1,p,B} \leq \varepsilon\, \|u\|_{2,p,B} + \frac{K}{\varepsilon}\, \|u\|_{0,p,B}$$

for all $\varepsilon < 1$. If we choose ε to be $1/2C_2$, we obtain

$$\|u\|_{2,p,B} \leq 2C_2\Big\{\|Lu\|_{0,p,B} + \frac{K}{\varepsilon}\, \|u\|_{0,p,B}\Big\}.$$

Finally, using inequality (4,2,1,4) we conclude that there exists a constant C_3 such that

$$\|u\|_{2,p,B} \leq C_3\, \|Lu\|_{0,p,B}$$

for all $u \in W_p^2(B)$ which fulfils the boundary conditions (4,2,3,9). ■

Proof of Lemma 4.2.2.5 Let us consider first the case when $\alpha_j = \beta_j = 0$ for $j = 0, 1$. Then the requirement is that equation

$$\mu_0\mu_1 \sinh \sqrt{(1+\xi^2)}\, h = 0$$

should have no real root ξ. This is achieved provided $\mu_j \neq 0$, $j = 0, 1$.

Next we consider the mixed case when $\alpha_0 = \beta_0 = 0$ and $\alpha_1 = 1$. Then the requirement is that equation

$$\mu_0\sqrt{(1+\xi^2)} \cosh \sqrt{(1+\xi^2)}\, h + \mu_0(i\beta_1\xi + \mu_1) \sinh \sqrt{(1+\xi^2)}\, h = 0$$

should have no real root. The real part of the equation is

$$\mu_0\{\sqrt{(1+\xi^2)} \cosh \sqrt{(1+\xi^2)}\, h + \mu_1 \sinh \sqrt{(1+\xi^2)}\, h\} = 0.$$

The condition is achieved if we choose $\mu_0 \neq 0$ and $\mu_1 = 0$, for instance. Of course in the case $\alpha_1 = \beta_1 = 0$ and $\alpha_0 = 1$ we choose $\mu_0 = 0$ and $\mu_1 \neq 0$.

Finally, let us consider the case when $\alpha_0 = \alpha_1 = 1$. Then we require that

$$\sqrt{(1+\xi^2)} \cosh \sqrt{(1+\xi^2)}\, h[\mathrm{i}(\beta_1 - \beta_0)\xi + (\mu_1 - \mu_0)] \\ + \sinh \sqrt{(1+\xi^2)}\, h[(1+\xi^2) - (\mathrm{i}\beta_0\xi + \mu_0)(\mathrm{i}\beta_1\xi + \mu_1)] = 0$$

should have no real root. The real part of this equation is

$$(\mu_1 - \mu_0)\sqrt{(1+\xi^2)} \cosh \sqrt{(1+\xi^2)}\, h \\ + [1 + \xi^2\{1 + \beta_0\beta_1\} - \mu_0\mu_1] \sinh \sqrt{(1+\xi^2)}\, h = 0.$$

When $1 + \beta_0\beta_1 \geqslant 0$, we can choose $\mu_0 = \mu_1 = 0$; then the equation becomes

$$[1 + \xi^2\{1 + \beta_0\beta_1\}] \sinh \sqrt{(1+\xi^2)}\, h = 0,$$

which has no real root. On the other hand, when $1 + \beta_0\beta_1 < 0$, we can choose $\mu_0 = \mu_1 = 2$; then the equation becomes

$$[-3 + \xi^2\{1 + \beta_0\beta_1\}] \sinh \sqrt{(1+\xi^2)}\, h,$$

which has no real root either. ■

4.3 Bounds in a polygon

4.3.1 The L_2 case

We prove here inequality (4,1,2) when $p = 2$. We follow word by word the proof in Grisvard (1972). The principle is the same as in Theorem 3.1.1.2 plus an additional density result. Again here, it is technically more convenient to work with

$$v = D_x u, \qquad w = D_y u.$$

The boundary conditions for v and w are the following.

Lemma 4.3.1.1 *Let $u \in H^2(\Omega)$ fulfil the boundary conditions in* (4,1,1); *then for all j there exist two real numbers λ_j and μ_j such that*

$$\lambda_j \gamma_j v + \mu_j \gamma_j w = 0 \qquad on\ \Gamma_j \tag{4,3,1,1}$$

and $\lambda_j^2 + \mu_j^2 \neq 0$.

Proof The condition (4,3,1,1) means that ∇u is orthogonal to some nonzero vector whose components are λ_j and μ_j. Indeed, in the notation of Section 4.1, we assume that

$$\gamma_j\, \nabla u \cdot (\boldsymbol{\nu}_j + \beta_j \boldsymbol{\tau}_j) = 0$$

on Γ_j for $j \in \mathcal{N}$, and that

$$\gamma_j \nabla u \cdot \boldsymbol{\tau}_j = 0$$

on Γ_j for $j \in \mathcal{D}$. ■

From now on, we denote by $G^s(\Omega)$ the space

$$\{(v, w) \in H^s(\Omega) \times H^s(\Omega) \mid \gamma_j(\lambda_j v + \mu_j w) = 0 \quad \text{on} \quad \Gamma_j,\ 1 \leqslant j \leqslant N\}.$$

Lemma 4.3.1.2 *The identity*

$$\int_\Omega D_x v D_y w \,\mathrm{d}x\,\mathrm{d}y = \int_\Omega D_y v D_x w \,\mathrm{d}x\,\mathrm{d}y \tag{4,3,1,2}$$

holds for all $(v, w) \in G^2(\Omega)$.

Proof Integrating by parts twice, we obtain

$$\int_\Omega D_x v D_y w \,\mathrm{d}x\,\mathrm{d}y - \int_\Omega D_y v D_x w \,\mathrm{d}x\,\mathrm{d}y = \int_\Gamma \gamma v \,\mathrm{d}\gamma w,$$

owing to the Green formula.

Next we split the boundary integral into pieces. We have

$$\int_\Gamma \gamma v \,\mathrm{d}\gamma w = \sum_{j=1}^N \int_{\Gamma_j} \gamma_j v \,\mathrm{d}\gamma_j w.$$

We assume that the Γ_j have been numbered according to the positive orientation of the boundary. We denote by S_j the terminal point of Γ_j; thus S_{j-1} is the origin of Γ_j for $j > 1$, while S_N is the origin of Γ_1. (It is obviously convenient to set $S_N = S_0$, $\Gamma_{N+1} = \Gamma_1$.)

When $\mu_j \neq 0$, we write $\gamma_j w = -(\lambda_j/\mu_j)\gamma_j v$, and consequently

$$\int_{\Gamma_j} \gamma_j v \,\mathrm{d}\gamma_j w = -\frac{\lambda_j}{\mu_j} \int_{\Gamma_j} \gamma_j v \,\mathrm{d}\gamma_j v = -\frac{\lambda_j}{2\mu_j} \{(\gamma_j v)^2(S_j) - (\gamma_j v)^2(S_{j-1})\}.$$

This identity is meaningful since $v \in H^2(\Omega)$ and consequently $v \in C(\bar{\Omega})$ by the Sobolev imbedding theorem.

When $\mu_j = 0$ we just have $\gamma_j v = 0$ on Γ_j and accordingly

$$\int_{\Gamma_j} \gamma_j v \,\mathrm{d}\gamma_j w = 0.$$

Next we observe that $(\gamma_j v)(S_j) = (\gamma_{j+1} v)(S_j)$ due to Theorem 1.5.2.8. Thus, we have

$$2\int_\Gamma \gamma v \, d\gamma w = \sum_{\mu_j\mu_{j+1}\neq 0}\left(\frac{\lambda_{j+1}}{\mu_{j+1}}-\frac{\lambda_j}{\mu_j}\right)(\gamma_j v)^2(S_j)$$
$$+\sum_{\substack{\mu_j=0\\ \mu_{j+1}\neq 0}}\frac{\lambda_{j+1}}{\mu_{j+1}}(\gamma_j v)^2(S_j)-\sum_{\substack{\mu_j\neq 0\\ \mu_{j+1}=0}}\frac{\lambda_j}{\mu_j}(\gamma_j v)^2(S_j).$$

We shall now check that $(\gamma_j v)(S_j)=0$ for all j such that $\lambda_{j+1}\mu_j\neq\lambda_j\mu_{j+1}$ and consequently that

$$\int_\Gamma \gamma v \, d\gamma w = 0.$$

Indeed at S_j the boundary conditions corresponding to Γ_j and Γ_{j+1} hold together by continuity. Thus, we have

$$\begin{cases}\lambda_j(\gamma_j v)(S_j)+\mu_j(\gamma_j w)(S_j)=0\\ \lambda_{j+1}(\gamma_j v)(S_j)+\mu_{j+1}(\gamma_j w)(S_j)=0.\end{cases}$$

This implies that $\gamma_j v(S_j)=0$ when $\lambda_j\mu_{j+1}\neq\lambda_{j+1}\mu_j$. This is the claim.

Summing up, we have proved that

$$\int_\Omega D_x v D_y w \, dx \, dy = \int_\Omega D_y v D_x w \, dx \, dy$$

for all $(v, w)\in G^2(\Omega)$. ■

In order to extend the previous result to the whole of $G^1(\Omega)$, we need a density lemma.

Lemma 4.3.1.3 *$G^2(\Omega)$ is dense in $G^1(\Omega)$; consequently* (4,3,1,2) *holds for all $\{v, w\}$ in $G^1(\Omega)$.*

Proof The trace Theorem 1.5.1.3 allows us to consider $H^1(\Omega)$ as the direct sum of $\mathring{H}^1(\Omega)$ with the space $H^{1/2}(\Gamma)$. Thus any continuous linear form l on $G^1(\Omega)$ may be represented as

$$\langle l; \{v, w\}\rangle = \langle S, v-\rho\gamma v\rangle + \langle T, w-\rho\gamma w\rangle + \langle g, \gamma v\rangle + \langle h, \gamma w\rangle$$

where $S, T\in H^{-1}(\Omega)$ and $g, h\in H^{-1/2}(\Gamma)$ and where ρ is a right inverse for the trace operator γ.

Let us assume that l vanishes on $G^2(\Omega)$. Then, in particular, it vanishes on $\mathring{H}^2(\Omega)\times\mathring{H}^2(\Omega)$ and, therefore, we have

$$\langle S, v\rangle + \langle T, w\rangle = 0$$

for all $v, w\in\mathring{H}^2(\Omega)$. This implies that $S=T=0$. We have thus shown that any continuous linear form on $G^1(\Omega)$ which vanishes on $G^2(\Omega)$ may be represented as

$$\langle l; \{v, w\}\rangle = \langle g; \gamma v\rangle + \langle h; \gamma w\rangle,$$

where g and $h \in H^{-1/2}(\Gamma)$.

Now, in order to prove that $G^2(\Omega)$ is dense in $G^1(\Omega)$, we have to check that any such l is identically zero. In view of the above representation formula, it is therefore enough to prove that the space $Z^2(\Gamma)$ of the traces of the functions in $G^2(\Omega)$ is dense in the space $Z^1(\Gamma)$ of the traces of the functions in $G^1(\Omega)$.

A first step is to describe these spaces, taking advantage of Theorem 1.6.1.5. Let us begin with $Z^1(\Gamma)$. This is a subspace of $\{\prod_{j=1}^N H^{1/2}(\Gamma_j)\}^2$. An element belonging to $Z^1(\Gamma)$ will be denoted

$$\{g_j, h_j\}_{j=1}^N$$

where g_j stands for $\gamma_j v$ and h_j stands for $\gamma_j w$. According to this notation $Z^1(\Gamma)$ is the subspace defined by

$$\begin{cases} \lambda_j g_j + \mu_j h_j = 0 \quad \text{on} \quad \Gamma_j, & 1 \le j \le N \\ \displaystyle\int_0^{\delta_j} \frac{|h_{j+1}(x_j(\sigma)) - h_j(x_j(-\sigma))|^2}{\sigma}\, d\sigma < \infty, & 1 \le j \le N \\ \displaystyle\int_0^{\delta_j} \frac{|g_{j+1}(x_j(\sigma)) - g_j(x_j(-\sigma))|^2}{\sigma}\, d\sigma < \infty, & 1 \le j \le N. \end{cases} \tag{4,3,1,3}$$

Now we choose N pairs of real numbers (ξ_j, η_j) with $\xi_j^2 + \eta_j^2 \ne 0$, $1 \le j \le N$ and such that

(a) $\xi_j\mu_j - \eta_j\lambda_j \ne 0, \qquad 1 \le j \le N$

(b) $(\xi_j, \eta_j) = (\xi_{j+1}, \eta_{j+1}) \qquad$ for all j such that $\lambda_j\mu_{j+1} - \lambda_{j+1}\mu_j = 0$.

In other words, we require the vector $\mathbf{v}_j = (\xi_j; \eta_j)$ to be linearly independent of the vector $(\lambda_j; \mu_j)$ and in addition we require $\mathbf{v}_j$ to be equal to $\mathbf{v}_{j+1}$ whenever (λ_j, μ_j) and $(\lambda_{j+1}, \mu_{j+1})$ are linearly dependent. Such a choice of vectors $\mathbf{v}_j$ is obviously possible. Next, let us define

$$\varphi_j = \xi_j g_j + \eta_j h_j, \qquad 1 \le j \le N. \tag{4,3,1,4}$$

It is easy to check that $Z^1(\Gamma)$ is isomorphic to the subspace of those

$$\{\varphi_j\}_{j=1}^N \in \prod_{j=1}^N H^{1/2}(\Gamma_j)$$

such that

$$\begin{cases} \displaystyle\int_0^{\delta_j} |\varphi_j(x_j(-\sigma))|^2 \frac{d\sigma}{\sigma} < \infty \\ \displaystyle\int_0^{\delta_j} |\varphi_{j+1}(x_j(\sigma))|^2 \frac{d\sigma}{\sigma} < \infty \end{cases} \tag{4,3,1,5}$$

when $\lambda_j\mu_{j+1}-\lambda_{j+1}\mu_j \neq 0$ and such that

$$\int_0^{\delta_j} |\varphi_j(x_j(-\sigma)) - \varphi_{j+1}(x_j(\sigma))|^2 \frac{d\sigma}{\sigma} < \infty \tag{4,3,1,6}$$

when $\lambda_j\mu_{j+1}-\lambda_{j+1}\mu_j = 0$.

In the same way we describe the space $Z^2(\Gamma)$. It is the subspace of $\{\prod_{j=1}^N H^{3/2}(\Gamma_j)\}^2$ defined by

$$\begin{cases} \lambda_j g_j + \mu_j h_j = 0 \quad \text{on} \quad \Gamma_j, & 1 \leq j \leq N \\ h_{j+1}(S_j) = h_j(S_j), & 1 \leq j \leq N \\ g_{j+1}(S_j) = g_j(S_j), & 1 \leq j \leq N. \end{cases} \tag{4,3,1,7}$$

Then introducing again φ_j defined by (4,3,1,4), we check that $Z^2(\Gamma)$ is isomorphic to the subspace of those $\{\varphi_j\}_{j=1}^N \in \prod_{j=1}^N H^{3/2}(\Gamma_j)$ such that

$$\varphi_j(S_j) = \varphi_{j+1}(S_j) = 0 \tag{4,3,1,8}$$

when $\lambda_j\mu_{j+1}-\lambda_{j+1}\mu_j \neq 0$ and such that

$$\varphi_j(S_j) = \varphi_{j+1}(S_j) \tag{4,3,1,9}$$

when $\lambda_j\mu_{j+1}-\lambda_{j+1}\mu_j = 0$.

Finally let us consider $\{\varphi_j\}_{j=1}^N \in Z^1(\Gamma)$. Due to the density of $\mathscr{D}(\Gamma_j)$ in $\tilde{H}^{1/2}(\Gamma_j)$ and of $\mathscr{D}(\bar{\Gamma}_j)$ in $H^{1/2}(\Gamma_j)$ (see Subsection 1.4.2), we can approximate φ_j by $\varphi_{j,m} \in H^{3/2}(\Gamma_j)$, $m = 1, 2, \ldots$ such that, for each m, (4,3,1,8) and (4,3,1,9) hold with φ_j replaced by $\varphi_{j,m}$. This can be achieved in such a way that

$$\varphi_{j,m} \to \varphi_j$$

in the norm of $H^{1/2}(\Gamma_j)$ and, in addition, that

$$\int_0^{\delta_j} |(\varphi_j - \varphi_{j,m})(x_j(-\sigma))|^2 \frac{d\sigma}{\sigma} \to 0$$

$$\int_0^{\delta_j} |(\varphi_{j+1} - \varphi_{j+1,m})(x_j(\sigma))|^2 \frac{d\sigma}{\sigma} \to 0$$

when $\lambda_j\mu_{j+1}-\lambda_{j+1}\mu_j \neq 0$, while

$$\int_0^{\delta_j} |(\varphi_j - \varphi_{j,m})(x_j(-\sigma)) - (\varphi_{j+1} - \varphi_{j+1,m})(x_j(\sigma))|^2 \frac{d\sigma}{\sigma} \to 0$$

when $\lambda_j\mu_{j+1}-\lambda_{j+1}\mu_j = 0$. This completes the proof of Lemma 4.3.1.3. ■

We are now able to prove our main result.

Theorem 4.3.1.4 *Let Ω be a bounded open subset of $\mathbb{R}^2$ with a (strictly)* *polygonal boundary Γ. Then there exists a constant C such that*

$$\|u\|_{2,2,\Omega} \leq C\{\|f\|_{0,2,\Omega} + \|u\|_{0,2,\Omega}\} \tag{4,3,1,10}$$

for all $u \in H^2(\Omega)$ which are solutions of problem (4,1,1).

Proof This is proved via a very straightforward calculation. As in Section 3.1.2 we calculate the following integral

$$\int_\Omega |\Delta u|^2 \,dx\,dy = \int_\Omega |D_x^2 u + D_y^2 u|^2 \,dx\,dy$$
$$= \int_\Omega |D_x^2 u|^2 \,dx\,dy + \int_\Omega |D_y^2 u|^2 \,dx\,dy + 2\int_\Omega D_x^2 u D_y^2 u \,dx\,dy \tag{4,3,1,11}$$

Then applying Lemma 4.3.1.3 to $v = D_x u$, $w = D_y u$, we obtain

$$\int_\Omega \{|D_x^2 u|^2 + |D_y^2 u|^2 + 2\,|D_x D_y u|^2\} \,dx\,dy = \int_\Omega |\Delta u|^2 \,dx\,dy.$$

Consequently we have

$$\|u\|_{2,2,\Omega}^2 \leq \|f\|_{0,2,\Omega}^2 + \|u\|_{1,2,\Omega}^2$$

and the result follows by inequality (1,4,3,2). ■

Remark 4.3.1.5 Here we have a very precise control of the constant in inequality (4,3,1,10). Indeed it depends only on the best constant K in inequality (1,4,3,2); in this particular case this is

$$\|u\|_{1,2,\Omega} \leq \varepsilon\, \|u\|_{2,2,\Omega} + K\varepsilon^{-1}\, \|u\|_{0,2,\Omega}.$$

In most practical cases, given a plane domain Ω with a polygonal boundary, the constant K can be determined explicitly as a function of Ω.

Remark 4.3.1.6 Let us observe for further reference that we have proved identity (4,3,1,11) for all functions $u \in H^2(\Omega)$ which fulfil the boundary conditions in (4,1,1).

Remark 4.3.1.7 So far we have excluded the domains Ω with cuts. However, the inequality (4,3,1,10) remains valid if one allows Ω to have cuts. Indeed the only modification of the proof occurs in Lemma 4.3.1.3. An application of Theorem 1.7.3 must be substituted for the application of Theorem 1.6.1.5 at the appropriate corners.

4.3.2 The L_p case ($p \neq 2$)

We shall now derive inequality (4,1,2) for $p \neq 2$. The method that we shall use here is quite different from the method of Subsection 4.3.1. Curiously enough, the method used here does not work when $p = 2$. (See, however, Section 4 in Kondratiev (1967a) who deals with the case $p = 2$.) It relies essentially on the estimates proved in Section 4.2.

We shall need some new weighted spaces similar to those introduced by Kondratiev (1967a) in the case when $p = 2$. We shall denote by $\rho(x, y)$ the distance from the point (x, y) to the vertices (S_j, $1 \leq j \leq N$) of Ω.

Definition 4.3.2.1 *We denote by $P_p^m(\Omega)$ the space of all functions u defined in Ω such that*

$$\rho^{|\alpha|-m} D^\alpha u \in L_p(\Omega)$$

for all $|\alpha| \leq m$.

Obviously we can define a Banach norm on $P_p^m(\Omega)$ by setting

$$\|u\|_{P_p^m(\Omega)} = \sum_{|\alpha| \leq m} \|\rho^{|\alpha|-m} D^\alpha u\|_{0,p,\Omega}.$$

The inclusion of $P_p^m(\Omega)$ into $W_p^m(\Omega)$ is obvious. We shall actually need a converse statement. Fortunately the converse inclusion holds, up to the addition of a finite-diensional space, at least when $p \neq 2$.

Theorem 4.3.2.2 *Let $u \in W_p^m(\Omega)$ be such that*

$$D^\alpha u(S_j) = 0 \qquad \textit{for} \quad |\alpha| < m - \frac{2}{p} \dagger$$

$j = 1, 2, \ldots, N$ and $p \neq 2$; then $u \in P_p^m(\Omega)$.

Proof By induction on k, we shall prove that

$$\rho^{-k} D^\alpha u \in L_p(\Omega) \tag{4,3,2,1}$$

for $|\alpha| \leq m - k$. Thus we assume that inclusion (4,3,2,1) holds for a given k and we derive the same inclusion where k is replaced by $k+1$.

Let $|\alpha| \leq m - k - 1$ and set $v = D^\alpha u$. Thus we know that

$$\rho^{-k} v, \qquad \rho^{-k}\, \nabla v \in L_p(\Omega).$$

In addition we know that $v(S_j) = 0$ when $p > 2$ or when $k \geq 1$. We shall show that this implies that

$$\rho^{-k-1} v \in L_p(\Omega).$$

Let us first look at the case where $k = 0$ and $p < 2$. We observe that the condition that $\rho^{-k-1} v$ belongs to $L_p(\Omega)$ is relevant only near the vertices.

† Let l be the greatest integer $< m - 2/p$; it follows from Sobolev's imbedding theorem, that $u \in C^l(\bar{\Omega})$ and consequently the condition $D^\alpha u(S_j) = 0$ is meaningful for $|\alpha| < m - 2/p$.

This allows us to localize the problem. Let $\eta_j \in \mathscr{D}(\mathbb{R}^2)$ be such that $\eta_j \equiv 1$ near S_j and $\eta_j \equiv 0$ outside a small circle centred at S_j which contains no other vertex of Ω. Let us denote its radius by δ_j. Now we only need to show that

$$\rho^{-1}\eta_j v \in L_p(\Omega)$$

for all j. Using polar coordinates centred at S_j, we can write

$$(\eta_j v)(\rho e^{i\theta}) = \int_\rho^{\delta_j} \frac{\partial}{\partial\sigma}(\eta_j v)(\sigma e^{i\theta})\, d\sigma.$$

Equivalently, we have

$$\rho^{-1}(\eta_j v)(\rho e^{i\theta}) = \frac{1}{\rho}\int_\rho^{\infty} \frac{\partial}{\partial\sigma}(\eta_j v)(\sigma e^{i\theta})\, d\sigma.$$

From the assumptions on v it follows that $|(\partial/\partial\sigma)(\eta_j v)|^p$ is integrable with respect to the measure $\sigma\, d\sigma\, d\theta$. By Hardy's inequality (see Subsection 1.4.4) it follows that $|\rho^{-1}\eta_j v|^p$ is integrable with respect to the measure $\rho\, d\rho\, d\theta$. This is the claim.

Let us now consider the case where either $p > 2$ or $k \geq 1$. Then with the same notation we write

$$\rho^{-1}(\eta_j v)(\rho e^{i\theta}) = \frac{1}{\rho}\int_0^{\rho} \frac{\partial}{\partial\sigma}(\eta_j v)(\sigma e^{i\theta})\, d\sigma,$$

and again the inequality mentioned above shows that $|\rho^{-1}\eta_j v|^p$ is integrable with respect to the measure $\rho\, d\rho\, d\theta$.

The basic result in this subsection is the following

Theorem 4.3.2.3 *There exists a constant C such that*

$$\|u\|_{P_p^2(\Omega)} \leq C\{\|f\|_{0,p,\Omega} + \|u\|_{1,p,\Omega}\}$$

for all $u \in P_p^2(\Omega)$ which are solutions of problem (4,1,1), *provided*

$$\frac{1}{\pi}\left(\Phi_{j+1} - \Phi_j - \frac{2}{q}\,\omega_j\right)$$

is not an integer for any j, where $\Phi_j = \arctan \beta_j$ for $j \in \mathcal{N}$ and $\Phi_j = \pi/2$ for $j \in \mathscr{D}$.

Proof Again we can consider our problem locally. We fix a partition of unity $\{\eta\}_j$, $j = 0, \ldots, N$ on $\bar{\Omega}$ such that $\eta_j \in \mathscr{D}(\mathbb{R}^2)$ for each j and

(a) the support of η_0 does not contain any vertex of Ω,
(b) the support of η_j contains S_j and does not contain any other vertex; in addition the support of η_j does not intersect Γ_k for $k \neq j$ and $k \neq j+1$.

(c) $\partial\eta_j/\partial\nu_k + \beta_k(\partial\eta_j/\partial\tau_k) = 0$ on Γ_k for $k = j$ if $j \in \mathcal{N}$ and for $k = j+1$ if $j+1 \in \mathcal{N}$.

It follows that there exists K_j such that

$$\|\Delta(\eta_j u) - \eta_j f\|_{0,p,\Omega} \leq K_j \|u\|_{1,p,\Omega} \tag{4,3,2,2}$$

and

$$\begin{cases} \eta_j u = 0 & \text{on } \Gamma_k, \quad k \in \mathcal{D} \\ \dfrac{\partial}{\partial \nu_k}(\eta_j u) + \beta_k \dfrac{\partial}{\partial \tau_k}(\eta_j u) = 0 & \text{on } \Gamma_k, \quad k \in \mathcal{N} \end{cases}$$

for $j = 1, 2, \ldots, N$.

The results in Subsection 2.3.3 imply that

$$\|\eta_0 u\|_{2,p,\Omega} \leq C_0\{\|f\|_{0,p,\Omega} + \|u\|_{1,p,\Omega}\} \tag{4,3,2,3}$$

since the support of $\eta_0 u$ is at a strictly positive distance from the corners.

We are now left with estimating $\eta_j u$ for $j = 1, \ldots, N$. For that purpose we use local coordinates as follows.

We fix j once and for all and choose polar coordinates with origin at S_j, and such that $\theta = 0$ on Γ_{j+1}, while $\theta = \omega_j$ on Γ_j. We also denote by G the

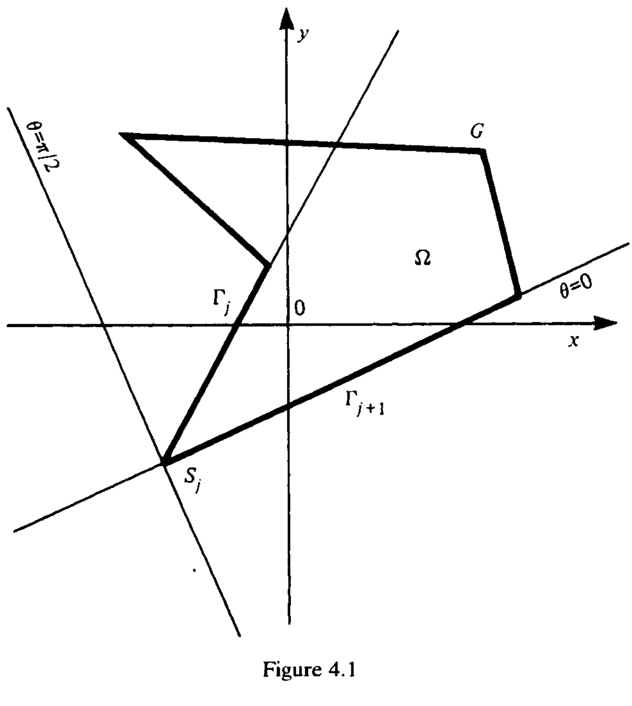

Figure 4.1

infinite sector defined by the half lines with origin at S_j and which contain Γ_j and Γ_{j+1} respectively.

With this notation the function $v = \widetilde{\eta_j u}$ is a solution of

$$\Delta v = g \qquad \text{in } G$$

where

$$\|g\|_{0,p,G} \leq K\{\|f\|_{0,p,\Omega} + \|u\|_{1,p,\Omega}\}$$

with the following boundary conditions. On the line $\theta = 0$ we have

$$v = 0$$

if $j+1 \in \mathscr{D}$ and we have

$$\frac{1}{r}\frac{\partial v}{\partial \theta} - \beta_{j+1}\frac{\partial v}{\partial r} = 0$$

if $j+1 \in \mathscr{N}$. In the same way, on $\theta = \omega_j$ we have

$$v = 0$$

if $j \in \mathscr{D}$ and we have

$$\frac{1}{r}\frac{\partial v}{\partial \theta} - \beta_j\frac{\partial v}{\partial r} = 0$$

if $j \in \mathscr{N}$.

Finally we set

$$w(t, \theta) = e^{-(2/q)t} v(e^{t+i\theta}), \tag{4,3,2,4}$$

where we make the following abuse of notation whose meaning is obvious: we denote by $v(e^{t+i\theta})$ the value of v at the point whose polar coordinates are e^t and θ. Then w is solution of a boundary value problem in the strip

$$B = \mathbb{R} \times]0, \omega_j[.$$

The equation is

$$D_t^2 w + D_\theta^2 w + \frac{4}{q} D_t w + \frac{4}{q^2} w = k \tag{4,3,2,5}$$

in B, where we have set

$$k(t; \theta) = e^{-(2/q)t}\{e^{2t} g(e^{t+i\theta})\}. \tag{4,3,2,6}$$

The boundary condition at $\theta = \omega_j$ is as follows:

$$w = 0 \tag{4,3,2,7a}$$

if $j \in \mathcal{D}$ and

$$D_\theta w - \beta_j D_t w - \frac{2}{q}\beta_j w = 0 \tag{4,3,2,7b}$$

if $j \in \mathcal{N}$. In the same way, the boundary condition at $\theta = 0$ is as follows

$$w = 0 \tag{4,3,2,8a}$$

if $j+1 \in \mathcal{D}$ and

$$D_\theta w - \beta_{j+1} D_t w - \frac{2}{q}\beta_{j+1} w = 0 \tag{4,3,2,8b}$$

if $j+1 \in \mathcal{N}$.

The boundary value problem (4,3,2,5) (4,3,2,7) (4,3,2,8) is one of those that we have studied in Section 4.2. Applying Theorem 4.2.2.4, we know that inequality (4,2,2) holds provided the following equation has no real root. When j and $j+1 \in \mathcal{N}$ the equation is

$$\sin \rho\omega_j (1 + \beta_j\beta_{j+1}) = \cos \rho\omega_j(\beta_{j+1} - \beta_j), \tag{4,3,2,9a}$$

where $\rho = 2/q + i\xi$, $\xi \in \mathbb{R}$. When $j \in \mathcal{N}$ and $j+1 \in \mathcal{D}$ the equation is

$$\beta_j \sin \rho\omega_j = \cos \rho\omega_j. \tag{4,3,2,9b}$$

When $j \in \mathcal{D}$ and $j+1 \in \mathcal{N}$ the equation is

$$\beta_{j+1} \sin \rho\omega_j = -\cos \rho\omega_j. \tag{4,3,2,9c}$$

Finally, when j and $j+1 \in \mathcal{D}$, the equation is just

$$\sin \rho\omega_j = 0. \tag{4,3,2,9d}$$

Separating the real part and the imaginary part in equations (4,3,2,9) we obtain the following systems of equations:

$$\begin{cases} (1+\beta_j\beta_{j+1}) \sin \frac{2}{q}\omega_j \cosh \xi\omega_j = (\beta_{j+1} - \beta_j) \cos \frac{2}{q}\omega_j \cosh \xi\omega_j \\ (1+\beta_j\beta_{j+1}) \cos \frac{2}{q}\omega_j \sinh \xi\omega_j = (\beta_j - \beta_{j+1}) \sin \frac{2}{q}\omega_j \sinh \xi\omega_j \end{cases}$$

when j and $j+1$ belong to $\mathcal{N}$ and

$$\begin{cases} \beta_j \sin \frac{2}{q}\omega_j \cosh \xi\omega_j = \cos \frac{2}{q}\omega_j \cosh \xi\omega_j \\ \beta_j \cos \frac{2}{q}\omega_j \sinh \xi\omega_j = -\sin \frac{2}{q}\omega_j \sinh \xi\omega_j \end{cases}$$

when $j \in \mathcal{N}$ and $j+1 \in \mathcal{D}$ and

$$\begin{cases} \beta_{j+1} \sin \dfrac{2}{q}\, \omega_j \cosh \xi\omega_j = -\cos \dfrac{2}{q}\, \omega_j \cosh \xi\omega_j \\[2ex] \beta_{j+1} \cos \dfrac{2}{q}\, \omega_j \sinh \xi\omega_j = \sin \dfrac{2}{q}\, \omega_j \sinh \xi\omega_j \end{cases}$$

when $j \in \mathcal{D}$ and $j+1 \in \mathcal{N}$ and finally

$$\begin{cases} \sin \dfrac{2}{q}\, \omega_j \cosh \xi\omega_j = 0 \\[2ex] \cos \dfrac{2}{q}\, \omega_j \sinh \xi\omega_j = 0 \end{cases}$$

when j and $j+1$ belong to $\mathcal{D}$.

In each of the previous systems of equations, $\xi = 0$ is a root of the second equation, while the first equation can be divided by $\cosh \xi\omega_j$. It follows that equation (4,3,2,9) has no real root iff

$$(1+\beta_j\beta_{j+1}) \sin \frac{2}{q}\, \omega_j \neq (\beta_{j+1} - \beta_j) \cos \frac{2}{q}\, \omega_j$$

when j and $j+1$ belong to $\mathcal{N}$ and

$$\beta_j \sin \frac{2}{q}\, \omega_j \neq \cos \frac{2}{q}\, \omega_j$$

when $j \in \mathcal{N}$ and $j+1 \in \mathcal{D}$ and

$$\beta_{j+1} \sin \frac{2}{q}\, \omega_j \neq -\cos \frac{2}{q}\, \omega_j$$

when $j \in \mathcal{D}$ and $j+1 \in \mathcal{N}$ and finally

$$\sin \frac{2}{q}\, \omega_j \neq 0$$

when j and $j+1$ belong to $\mathcal{D}$. If we define Φ_j by the equation

$$\tan \Phi_j = \beta_j$$

when $j \in \mathcal{N}$ and set $\Phi_j = \pi/2$ when $j \in \mathcal{D}$, then all the previous conditions can be summarized as

$$\frac{\Phi_{j+1} - \Phi_j + k\pi}{\omega_j} \neq \frac{2}{q}. \tag{4,3,2,10}$$

for all $k \in \mathbb{Z}$ (i.e. k an integer).

When condition (4,3,2,10) is fulfilled then inequality (4,2,2) holds for our problem and this means the existence of a constant C_j such that

$$\|w\|_{2,p,B} \leq C_j \|k\|_{0,p,B}.$$

Finally, performing the inverse change of variables in (4,3,2,4) and (4,3,2,6), we see that there exists another constant C_j' such that

$$\|v\|_{P_p^2(G)} \leq C_j' \|g\|_{0,p,G}$$

and consequently

$$\|\eta_j u\|_{P^2(G)} \leq C_j' K\{\|f\|_{0,p,\Omega} + \|u\|_{1,p,\Omega}\}. \tag{4,3,2,11}$$

These last inequalities together with inequality (4,3,2,3) imply the claim in Theorem 4.3.2.3. ■

An easy consequence of Theorems 4.3.2.2 and 4.3.2.3 is the following theorem.

Theorem 4.3.2.4 *Assume that $p \neq 2$ and that*

$$\frac{1}{\pi}\left(\Phi_{j+1} - \Phi_j - \frac{2}{q}\,\omega_j\right)$$

is not an integer for any j where $\Phi_j = \arctan \beta_j$ for $j \in \mathcal{N}$ and $\Phi_j = \pi/2$ for $j \in \mathcal{D}$. Then there exists a constant C such that

$$\|u\|_{2,p,\Omega} \leq C\{\|\Delta u\|_{0,p,\Omega} + \|u\|_{1,p,\Omega}\} \tag{4,3,2,12}$$

for all $u \in W_p^2(\Omega)$ which are solutions of the problem (4,1,1).

Proof Let us denote by E the space of all $u \in W_p^2(\Omega)$ which fulfil the boundary conditions of problem (4,1,1). Let us also denote by F the subspace of E defined by the conditions

$$u(S_j) = 0, \qquad 1 \leq j \leq N$$

(This is not an extra condition when j or $j+1$ belongs to $\mathcal{D}$) and

$$\nabla u(S_j) = 0, \qquad 1 \leq j \leq N \quad \text{when } p > 2.$$

(This is not an extra condition when j and $j+1$ belong to $\mathcal{N}$ and $\boldsymbol{\nu}_j + \beta_j \boldsymbol{\tau}_j$ and $\boldsymbol{\nu}_{j+1} + \boldsymbol{\beta}_{j+1}\boldsymbol{\tau}_{j+1}$ are linearly independent. It is not an extra condition either when $j \in \mathcal{D}$, $j+1 \in \mathcal{N}$ and $\boldsymbol{\tau}_j$ and $\boldsymbol{\nu}_{j+1} + \boldsymbol{\beta}_{j+1}\boldsymbol{\tau}_{j+1}$ are independent (and, *mutatis mutandis*, when $j \in \mathcal{N}$ and $j+1 \in \mathcal{D}$).) The codimension of F in E is finite and due to Theorem 4.3.2.2, F is a subspace of $P_p^2(\Omega)$. Thus inequality (4,3,2,12) holds for all $u \in F$, by Theorem 4.3.2.3.

Now let us denote by Π any projection from E onto F. It is clear that

$$\Pi u = u - \sum_{j=1}^{N} u(S_j)\varphi_j$$

when $p<2$ and

$$\Pi u = u - \sum_{i=1}^{N} \{u(S_i)\varphi_i + D_x u(S_i)\psi_i + D_y u(S_i)\xi_i\}$$

when $p>2$ where φ_i, ψ_i, ξ_i belong to $W_p^2(\Omega)$. This representation shows that Π is also a linear continuous operator in the norm of $W_p^s(\Omega)$ provided

$$\frac{2}{p}<s\leqslant 2 \qquad \text{when} \quad p<2$$

and

$$1+\frac{2}{p}<s\leqslant 2 \qquad \text{when} \quad p>2. \tag{4,3,2,13}$$

We can now prove the desired inequality. Let $u\in E$, then we have

$$\begin{aligned}\|u\|_{2,p,\Omega} &\leqslant \|u-\Pi u\|_{2,p,\Omega} + \|\Pi u\|_{2,p,\Omega}\\ &\leqslant \|u-\Pi u\|_{2,p,\Omega} + C\{\|\Delta \Pi u\|_{0,p,\Omega} + \|\Pi u\|_{1,p,\Omega}\}\end{aligned}$$

since we can apply Theorem 4.3.2.3 to Πu. It follows that

$$\begin{aligned}\|u\|_{2,p,\Omega} &\leqslant \|u-\Pi u\|_{2,p,\Omega} + C\{\|\Delta u\|_{0,p,\Omega} + \|\Delta(u-\Pi u)\|_{0,p,\Omega}\\ &\qquad + \|u\|_{1,p,\Omega} + \|u-\Pi u\|_{1,p,\Omega}\}\\ &\leqslant C_1\|u-\Pi u\|_{2,p,\Omega} + C\{\|\Delta u\|_{0,p,\Omega} + \|u\|_{1,p,\Omega}\}.\end{aligned}$$

Next, we observe that on the finite dimensional space $(1-\Pi)E$, the norms of $W_p^2(\Omega)$ and of $W_p^s(\Omega)$ are equivalent for $s<2$. We choose s such that condition (4,3,2,13) is fulfilled so that Π is continuous in the $W_p^s(\Omega)$ norm. Therefore we have

$$\begin{aligned}\|u\|_{2,p,\Omega} &\leqslant C_2\|u-\Pi u\|_{s,p,\Omega} + C\{\|\Delta u\|_{0,p,\Omega} + \|u\|_{1,p,\Omega}\}\\ &\leqslant C_3\|u\|_{s,p,\Omega} + C\|\Delta u\|_{0,p,\Omega}.\end{aligned}$$

We conclude by taking advantage of inequality (1,4,3,2) which implies that

$$\|u\|_{s,p,\Omega} \leqslant \varepsilon\|u\|_{2,p,\Omega} + K\varepsilon^{(1-s)/(2-s)}\|u\|_{1,p,\Omega}$$

for every $\varepsilon\in]0,1[$. Choosing $C_3\varepsilon=\frac{1}{2}$, we conclude that

$$\|u\|_{2,p,\Omega} \leqslant 2C\|\Delta u\|_{0,p,\Omega} + C_3K\varepsilon^{(1-s)/(2-s)}\|u\|_{1,p,\Omega}.$$

This is inequality (4,3,2,12). ∎

Remark 4.3.2.5 The inequality (4,1,2) follows plainly from inequality

(4,3,2,12) by applying again inequality (1,4,3,2):

$$\|u\|_{1,p,\Omega} \leq \varepsilon \|u\|_{2,p,\Omega} + K\varepsilon^{-1} \|u\|_{0,p,\Omega}$$

for every $\varepsilon \in]0, 1[$.

Remark 4.3.2.6 Here we have very poor control of the constant in inequality (4,3,2,12) in particular because of the abstract functional analysis procedure that we used for dealing with the equivalences of norms on finite-dimensional spaces. In that respect we have much less information in the case $p \neq 2$ in comparison with the case $p = 2$ (see Remark 4.3.1.5).

Remark 4.3.2.7 The inequality (4,3,2,12) remains valid for domains with cuts (i.e. we allow $\omega_j = 2\pi$ for some j). Indeed the results in Section 4.2 have been derived without any limitation on the width h of the strip B. Thus the only modification lies in the proof of Theorem 4.3.2.4. There the imbedding (1,7,4) has to replace the usual Sobolev imbedding (1,4,4,6) in the definition of the space E.

4.4 The Fredholm alternative

In this section we shall derive the consequences of the inequality (4,1,2). An immediate consequence is that our problem has the semi-Fredholm property. Then in most cases we shall be able to prove the uniqueness of the solution by very straightforward arguments. Studying the range of the Laplace operator under the given boundary conditions will require much more work. A careful study of the orthogonal of the range will allow us to calculate exactly the index of our problem.

4.4.1 The semi-Fredholm properties

We first need a classical result of functional analysis.

Lemma 4.4.1.1 *Let E_1 and E_2 be two Banach spaces such that E_1 is compactly imbedded in E_2. Assume that A is a continuous linear operator from E_1 into E_2 and that there exists a constant C such that*

$$\|x\|_{E_1} \leq C\{\|Ax\|_{E_2} + \|x\|_{E_2}\} \qquad (4,4,1,1)$$

for all $x \in E_1$. Then A has a finite-dimensional kernel and a closed range.

In other words, A is a semi-Fredholm operator. We now apply this result to $A = \Delta$ considered as an operator from

$$E_1 = \{u \in W_p^2(\Omega);\ \gamma_j u = 0 \text{ on } \Gamma_j,\ j \in \mathcal{D} \text{ and}$$
$$\gamma_j(\partial u/\partial \nu_j) + \beta_j(\partial/\partial \tau_j)\gamma_j u = 0 \text{ on } \Gamma_j,\ j \in \mathcal{N}\}$$

into $E_2 = L_p(\Omega)$. Due to Theorem 1.4.3.2, E_1 is compactly imbedded in E_2 and inequality (4,1,2) is nothing but inequality (4,4,1,1). Thus Lemma 4.4.1.1 shows that the space of the solutions $u \in W_p^2(\Omega)$ of problem (4,1,1) for $f = 0$ is finite-dimensional. In addition, the subspace of all $f \in L_p(\Omega)$, for which problem (4,1,1) has a solution $u \in W_p^2(\Omega)$, is closed in $L_p(\Omega)$.

We shall now investigate the uniqueness of u. We shall state two kinds of results corresponding to two different methods of proof. We first look at problems for which uniqueness (possibly up to a constant) follows from the consideration of

$$\int_\Omega \Delta u\, u \,dx\,dy.$$

Theorem 4.4.1.2 *Assume that $\beta_j \leq \beta_{j-1}$ whenever $j-1$ and $j \in \mathcal{N}$. Then problem* (4,1,1) *has at most one solution $u \in W_p^2(\Omega)$ defined up to an additive constant. If in addition, $\mathcal{D}$ is non-empty, then problem* (4,1,1) *has at most one solution in $W_p^2(\Omega)$.*

In other words the kernel of Δ, considered as an operator from E_1 to E_2, is either one-dimensional (when $\mathcal{D} = \emptyset$) or zero (when $\mathcal{D} \neq \emptyset$).

Proof Let us assume that $u \in W_p^2(\Omega)$ is a solution of problem (3,1,1) with $f = 0$. We use the classical identity

$$-\int_\Omega \Delta u\, u\,dx\,dy = \int_\Omega |\nabla u|^2\,dx\,dy - \sum_{j=1}^{N} \int_{\Gamma_j} \gamma_j \frac{\partial u}{\partial \nu_j}\, \gamma_j u\,d\sigma. \tag{4,4,1,2}$$

Such an identity obviously holds for functions in $W_p^2(\Omega) \cap W_q^2(\Omega)$ with $1/p + 1/q = 1$ (see Lemma 1.5.3.3). We therefore consider a sequence u_m, $m = 1, 2, \ldots$ of functions belonging to $W_p^2(\Omega) \cap W_q^2(\Omega)$ and such that

$$u_m \to u$$

in $W_p^2(\Omega)$ when $m \to +\infty$. We have

$$-\int_\Omega \Delta u_m\, u_m\,dx\,dy = \int_\Omega |\nabla u_m|^2\,dx\,dy - \sum_{j=1}^{N} \int_{\Gamma_j} \gamma_j \frac{\partial u_m}{\partial \nu_j}\, \gamma_j u_m\,d\sigma.$$

for all m, and we can take the limit in m, since by Sobolev's imbedding theorem (see Subsection 1.4.4), we have

$$\begin{aligned}
u_m &\to u && \text{in } L_q(\Omega)\\
\nabla u_m &\to \nabla u && \text{in } L_2(\Omega)\\
\gamma_j \frac{\partial u_m}{\partial \nu_j} &\to \gamma_j \frac{\partial u}{\partial \nu_j} && \text{in } L_p(\Gamma_j)\\
\gamma_j u_m &\to \gamma_j u && \text{in } L_q(\Gamma_j).
\end{aligned}$$

This proves identity (4,4,1,2).

Then since $\Delta u = 0$ and u fulfils the boundary conditions, we have

$$\begin{aligned}
0 &= \int_\Omega |\nabla u|^2 \, dx \, dy + \sum_{j=1}^N \beta_j \int_{\Gamma_j} \frac{\partial}{\partial \tau_j} \gamma_j u \gamma_j u \, d\sigma\\
&= \int_\Omega |\nabla u|^2 \, dx \, dy + \sum_{j=1}^N \frac{(\beta_j - \beta_{j+1})}{2} u(S_j)^2.
\end{aligned}$$

This identity is meaningful since u is continuous up to the boundary of Ω. The assumption that $\beta_j \geqslant \beta_{j+1}$ for all j implies

$$\int_\Omega |\nabla u|^2 \, dx \, dy = 0$$

and therefore u is a constant (since Ω is connected). This constant is zero if and only if $\mathscr{D}$ is nonempty. ■

Next we consider problems for which uniqueness follows from the consideration of

$$\int_\Omega |\Delta u|^2 \, dx \, dy.$$

Theorem 4.4.1.3 *Assume that $p \geqslant 2$ and that at least two of the vectors $\boldsymbol{\mu}_j$ are linearly independent. Then problem* (4,1,1) *has at most one solution $u \in W_p^2(\Omega)$ defined up to the addition of a constant. If in addition, $\mathscr{D}$ is nonempty, then problem* (4,1,1) *has at most one solution in $W_p^2(\Omega)$.*

In other words, the kernel of Δ as an operator from E_1 to E_2 is either one-dimensional (when $\mathscr{D} = \varnothing$) or zero (when $\mathscr{D} \neq \varnothing$.)

Proof Let $u \in W_p^2(\Omega)$ be a solution of problem (3,1,1) with $f = 0$. Then since we assume that $p \geqslant 2$, we have

$$u \in H^2(\Omega).$$

It follows that

$$0=\int_{\Omega}|\Delta u|^2\,dx\,dy=\int_{\Omega}\{|D_x^2u|^2+|D_y^2u|^2+2\,|D_xD_yu|^2\}\,dx\,dy$$

by Remark 4.3.1.6. This shows that u is a polynomial of degree less than or equal to 1.

Let us assume from now on that $u=\xi x+\eta y+\alpha$. The boundary condition on Γ_j means that the vector whose components are ξ and η, is orthogonal to μ_j. Since two of these vectors are linearly independent by assumption, this implies that $u=\alpha$, a constant. The constant α is zero if and only if $\mathscr{D}$ is nonempty. ■

The above investigation of the kernel of Δ as an operator from E_1 to E_2 is conclusive in most of the practical cases. We turn now to studying the range of Δ. Taking advantage of the fact that it is closed, we shall instead investigate its annihilator which is a subspace of $L_q(\Omega)$ with $1/p+1/q=1$. Naturally this is, in some sense, the space of the solutions of a homogeneous adjoint problem. This will be stated in a precise way with the aid of Theorem 1.5.3.6.

From now on we shall denote by N_q the subspace of all functions $v\in L_q(\Omega)$ such that

$$\int_{\Omega} fv\,dx\,dy=0$$

for all $f\in L_p(\Omega)$ such that there exists $u\in W_p^2(\Omega)$ satisfying (4,1,1). This is the annihilator of the image of Δ. Obviously N_q is a space of harmonic functions. Indeed, for all $u\in\mathscr{D}(\Omega)$ we have

$$\int_{\Omega}\Delta u\,v\,dx\,dy=0,$$

and consequently $\Delta v=0$ in the sense of distributions. This implies that

$$v\in D(\Delta\,;L_q(\Omega)),$$

a space defined in Subsection 1.5.3. Therefore by Theorem 1.5.3.4 the traces of v and $\partial v/\partial\nu_j$ are well defined on each of the sides Γ_j, $1\leqslant j\leqslant N$. Precisely, we have

$$\gamma_j v\in W_q^{-1/q}(\Gamma_j),\qquad \gamma_j\frac{\partial v}{\partial\nu_j}\in W_q^{-1-1/q}(\Gamma_j)$$

when $p\neq 2$ and

$$\gamma_j v\in(\tilde{H}^{1/2}(\Gamma_j))^*,\qquad \gamma_j\frac{\partial v}{\partial\nu_j}\in(\tilde{H}^{3/2}(\Gamma_j))^*$$

when $p=2$. Accordingly the following statement is meaningful.

Lemma 4.4.1.4 *Let $v \in N_q$; then v is solution of the following boundary value problem*

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \gamma_j v = 0 & \text{on } \Gamma_j,\ j \in \mathcal{D}, \\ \gamma_j \dfrac{\partial v}{\partial \nu_j} - \beta_j \dfrac{\partial}{\partial \tau_j} \gamma_j v = 0 & \text{on } \Gamma_j,\ j \in \mathcal{N}. \end{cases} \tag{4,4,1,3}$$

For convenience, in what follows, we shall denote by M_q the space of the solutions of problem (4,4,1,3) which belong to $L_q(\Omega)$.

Proof Let us first look at the case when $p \neq 2$. Given $\varphi_j \in \mathring{W}_p^{2-1/p}(\Gamma_j)$, $j \in \mathcal{N}$, and $\psi_j \in \mathring{W}_p^{1-1/p}(\Gamma_j)$, $j \in \mathcal{D}$, there exists $u \in W_p^2(\Omega)$ such that

$$\begin{cases} \gamma_j u = \varphi_j,\ \gamma_j \dfrac{\partial u}{\partial \nu_j} = -\beta_j \dfrac{\partial \varphi_j}{\partial \tau_j}, & j \in \mathcal{N} \\ \gamma_j u = 0,\ \gamma_j \dfrac{\partial u}{\partial \nu_j} = \psi_j, & j \in \mathcal{D}. \end{cases}$$

This is a direct application of Theorem 1.5.2.8. Indeed all the conditions (a) in this theorem are obviously fulfilled since both sides of the desired identities vanish (see Corollary 1.5.1.6).

We observe that $u(S_j) = 0$ for all j and in addition that

$$\nabla u(S_j) = 0 \qquad \text{for all } j$$

when $p > 2$. Consequently, we can apply Theorem 1.5.3.6 (the Green formula) to this function u and $v \in N_q$. We obtain

$$\sum_{j=1}^{N} \left\langle \gamma_j \frac{\partial u}{\partial \nu_j}; \gamma_j v \right\rangle - \left\langle \gamma_j u; \gamma_j \frac{\partial v}{\partial \nu_j} \right\rangle = 0,$$

i.e.

$$\sum_{j \in \mathcal{D}} \langle \psi_j; \gamma_j v \rangle - \sum_{j \in \mathcal{N}} \left\{ \left\langle -\beta_j \frac{\partial \varphi_j}{\partial \tau_j}; \gamma_j v \right\rangle - \left\langle \varphi_j, \gamma_j \frac{\partial v}{\partial \nu_j} \right\rangle \right\} = 0.$$

If we let φ_j vary in $\mathring{W}_p^{2-1/p}(\Gamma_j)$ and ψ_j vary in $\mathring{W}_p^{1-1/p}(\Gamma_j)$, this identity shows that

$$\gamma_j v = 0, \qquad j \in \mathcal{D},$$

$$\gamma_j \frac{\partial v}{\partial \nu_j} - \beta_j \frac{\partial}{\partial \tau_j} \gamma_j v = 0, \qquad j \in \mathcal{N}.$$

We have thus checked that v is solution of problem (4,4,1,3) when $p \neq 2$.

To conclude we must look at the case where $p = 2$. We just observe that $N_2 \subseteq N_q$ for $q<2$ and consequently $v \in N_2$ is also a solution of problem (4,4,1,3). ■

In the next subsection we shall show that M_q, the space of the solutions of problem (4,4,1,3), is a finite-dimensional subspace of $L_q(\Omega)$. Furthermore we shall be able to calculate its dimension in most cases. This will show that Δ is a Fredholm operator from E_1 to E_2. Actually, calculating its index will require some additional work since in many cases N_q happens to be a strict subspace of the space of the solutions of (4,4,1,3) in $L_q(\Omega)$. Indeed, let

$$\varphi_j \in \mathcal{D}(\bar{\Omega}), \qquad \boldsymbol{\psi}_j \in \mathcal{D}(\bar{\Omega})^2$$

be such that:

(a) The supports of φ_j and $\boldsymbol{\psi}_j$ do not meet $\bar{\Gamma}_l$ for $l \neq j$ and $j+1$ (in particular they do not contain S_l for $l \neq j$).

(b) $\varphi_j(S_j) = 1$, $D_x\boldsymbol{\psi}_j(S_j) = (1, 0)$, $D_y\psi_j(S_j) = (0, 1)$, $\nabla\varphi_j(S_j) = 0$, $\boldsymbol{\psi}_j(S_j) = 0$.

With this notation we can state the following lemma:

Lemma 4.4.1.5 *Let $v \in N_q$; then for all $u \in W_p^2(\Omega)$, fulfilling the boundary conditions in* (4,1,1) *and all j, we have*

$$\begin{aligned}\int_\Omega &\{u(S_j)\,\Delta\varphi_j + \nabla u(S_j)\cdot\Delta\boldsymbol{\psi}_j\}v \,dx\,dy\\ &= \sum_{k\in\mathcal{N}\cap\{j;j+1\}} \left\langle \left[\frac{\partial}{\partial\nu_k} + \beta_k\frac{\partial}{\partial\tau_k}\right]\{u(S_j)\varphi_j + \nabla u(S_j)\cdot\boldsymbol{\psi}_j\};\ \gamma_k v\right\rangle\\ &\quad - \sum_{k\in\mathcal{D}\cap\{j;j+1\}} \left\langle \{u(S_j)\varphi_j + \nabla u(S_j)\cdot\boldsymbol{\psi}_j\};\ \gamma_k\frac{\partial v}{\partial\nu_k}\right\rangle\end{aligned} \tag{4,4,1,4}$$

when $p>2$, and

$$\int_\Omega u(S_j)\Delta\varphi_j v\,dx\,dy = \sum_{k\in\mathcal{N}\cap\{j;j+1\}} \left\langle\left[\frac{\partial}{\partial\nu_k} + \beta_k\frac{\partial}{\partial\nu_k}\right]u(S_j)\varphi_j;\ \gamma_k v\right\rangle \tag{4,4,1,5}$$

when $p<2$.

Proof This is again an application of Theorem 1.5.3.6. Let us look at the case when $p>2$ first. Let $u \in W_p^2(\Omega)$ and set

$$w = u - \sum_{j=1}^{N} u(S_j)\varphi_j - \sum_{j=1}^{N} \nabla u(S_j)\cdot\boldsymbol{\psi}_j.$$

Then obviously $w \in W_p^2(\Omega)$ and

$$w(S_j) = 0, \qquad \nabla w(S_j) = 0. \tag{4,4,1,6}$$

for all j. Since $v \in N_q$ is also in $D(\Delta; L_q(\Omega))$, we can apply identity (1,5,3,6) to w and v.

Thus we have

$$\int_\Omega \Delta w\, v\, dx\, dy = \sum_{k=1}^N \left\{\left\langle \gamma_k \frac{\partial w}{\partial \nu_k}; \gamma_k v\right\rangle - \left\langle \gamma_k w; \gamma_k \frac{\partial v}{\partial \nu_k}\right\rangle\right\}$$

since $\Delta v = 0$. We then observe that (4,4,1,6) implies that $\gamma_k w \in \mathring{W}_p^{2-1/p}(\Gamma_k)$. On the other hand, we have proved in Lemma 4.4.1.4 that

$$\gamma_k \frac{\partial v}{\partial \nu_k} = \beta_k \frac{\partial}{\partial \tau_k} \gamma_k v$$

when $k \in \mathcal{N}$. It follows that

$$\left\langle \gamma_k w; \gamma_k \frac{\partial v}{\partial \nu_k}\right\rangle = \beta_k \left\langle \gamma_k w; \frac{\partial}{\partial \tau_k} \gamma_k v\right\rangle = -\beta_k \left\langle \frac{\partial}{\partial \tau_k} \gamma_k w; \gamma_k v\right\rangle.$$

Thus we have

$$\int_\Omega \Delta w\, v\, dx\, dy = \sum_{k \in \mathcal{N}} \left\langle \gamma_k \frac{\partial w}{\partial \nu_k} + \beta_k \frac{\partial}{\partial \tau_k} \gamma_k w; \gamma_k v\right\rangle - \sum_{k \in \mathcal{D}} \left\langle \gamma_k w; \gamma_k \frac{\partial v}{\partial \nu_k}\right\rangle.$$

Since

$$\int_\Omega \Delta u\, v\, dx\, dy = 0,$$

$$\gamma_k \frac{\partial u}{\partial \nu_k} + \beta_k \frac{\partial}{\partial \tau_k} \gamma_k u = 0 \qquad \text{on } \Gamma_k \quad \text{for} \quad k \in \mathcal{N}$$

and

$$\gamma_k u = 0 \qquad \text{on } \Gamma_k \quad \text{for} \quad k \in \mathcal{D},$$

it follows that

$$\begin{aligned}
&\int_\Omega \Delta \sum_{j=1}^N \{u(S_j)\varphi_j + \nabla u(S_j) \cdot \boldsymbol{\psi}_j\} v\, dx\, dy \\
&\quad = \sum_{k \in \mathcal{N}} \sum_{j=1}^N \left\langle \left[\frac{\partial}{\partial \nu_k} + \beta_k \frac{\partial}{\partial \tau_k}\right] \{u(S_j)\varphi_j + \nabla u(S_j) \cdot \boldsymbol{\psi}_j\}; \gamma_k v\right\rangle \\
&\qquad - \sum_{k \in \mathcal{D}} \sum_{j=1}^N \left\langle \{u(S_j)\varphi_j + \nabla u(S_j) \cdot \boldsymbol{\psi}_j\}; \gamma_k \frac{\partial v}{\partial \nu_k}\right\rangle.
\end{aligned}$$

Now if we let u vary, the values of $\{u(S_j), \nabla u(S_j)\}$ for different j are independent. Thus we have

$$\int_\Omega \Delta\{u(S_j)\varphi_j + \nabla u(S_j)\boldsymbol{\psi}_j\} v \, dx \, dy$$

$$= \sum_{k \in \mathcal{N}} \left\langle \left[\frac{\partial}{\partial \nu_k} + \beta_k \frac{\partial}{\partial \tau_k}\right]\{u(S_j)\varphi_j + \nabla u(S_j) \cdot \boldsymbol{\psi}_j\}; \gamma_k v \right\rangle$$

$$- \sum_{k \in \mathcal{D}} \left\langle \{u(S_j)\varphi_j + \nabla u(S_j) \cdot \boldsymbol{\psi}_j\}; \gamma_k \frac{\partial v}{\partial \nu_k} \right\rangle.$$

Due to the assumptions on the supports of φ_j and $\boldsymbol{\psi}_j$ the sum in k has to be extended to $k = j$ and $k = j + 1$ only. This proves (4,4,1,4).

In the case when $p < 2$, we make the same calculations defining w as

$$w = u - \sum_{j=1}^{N} u(S_j)\varphi_j.$$

The last sum

$$\sum_{k \in \mathcal{D} \cap \{j, j+1\}} u(S_j) \left\langle \varphi_j; \gamma_k \frac{\partial v}{\partial \nu_k} \right\rangle$$

is always zero since when $\mathcal{D} \cap \{j, j+1\} \neq \varnothing$ we must have $u(S_j) = 0$. ■

Remark 4.4.1.6 The meaning of these two lemmas is the following: In addition to being a solution of problem (4,4,1,3), every function $v \in N_q$ must fulfil a finite number of linear conditions defined by (4,4,1,4) or (4,4,1,5) (observe that it is not clear whether or not these conditions are independent). By the way, this shows that the adjoint problem to (4,1,1) is not exactly the adjoint boundary value problem as is always the case when the boundary of Ω is smooth.

Remark 4.4.1.7 The conditions on v expressed by (4,4,1,4) and (4,4,1,5) can be simplified in most cases. Let us first look at the case when $p < 2$; we have two possible cases:

(a) If j or $j + 1$ belongs to $\mathcal{D}$, we always have $u(S_j) = 0$ and (4,4,1,5) is not an additional condition on v.

(b) If j and $j + 1$ belong to $\mathcal{N}$, then $u(S_j)$ is any real number and consequently condition (4,4,1,5) is nothing but

$$\int_\Omega \Delta\varphi_j v \, dx \, dy = \sum_{k=j}^{j+1} \left\langle \left[\frac{\partial}{\partial \nu_k} + \beta_k \frac{\partial}{\partial \tau_k}\right]\varphi_j; \gamma_k v \right\rangle. \qquad (4,4,1,7)$$

Then when $p > 2$, there are many more cases.

(a) If j and $j + 1$ belong to $\mathcal{D}$ and the angle ω_j is not flat (the case of a flat angle with Dirichlet boundary conditions on both sides is irrelevant since flat angles are considered only when dealing with mixed boundary conditions), then we always have $u(S_j)$ and $\nabla u(S_j)=0$. Therefore, (4,4,1,4) is not an additional condition on v.

(b) If $j\in\mathcal{D}$ and $j+1\in\mathcal{N}$ and if we assume that $\boldsymbol{\tau}_j$ and $\boldsymbol{\mu}_{j+1}$ are linearly independent, we also have $u(S_j)=0$ and $\nabla u(S_j)=0$ and thus no additional condition on v.

(c) The same holds when $j\in\mathcal{N}$, $j+1\in\mathcal{D}$ and if $\boldsymbol{\tau}_{j+1}$ and $\boldsymbol{\mu}_j$ are linearly independent.

(d) If $j\in\mathcal{D}$ and $j+1\in\mathcal{N}$ and if $\boldsymbol{\tau}_j$ and $\boldsymbol{\mu}_{j+1}$ are parallel, we have only

$$u(S_j)=0 \quad\text{and}\quad \nabla u(S_j)\cdot\boldsymbol{\tau}_j=0.$$

Thus condition (4,4,1,4) is equivalent to

$$\int_\Omega \Delta\boldsymbol{\psi}_j\cdot\boldsymbol{\nu}_j\, v\, dx\, dy = \left\langle\left[\frac{\partial}{\partial\nu_{j+1}}+\beta_{j+1}\frac{\partial}{\partial\tau_{j+1}}\right]\boldsymbol{\psi}_j\cdot\boldsymbol{\nu}_j;\ \gamma_{j+1}v\right\rangle-\left\langle\boldsymbol{\psi}_j\cdot\boldsymbol{\nu}_j;\ \gamma_j\frac{\partial v}{\partial\nu_j}\right\rangle. \tag{4,4,1,8}$$

(e) A similar result holds *mutatis mutandis* when $j\in\mathcal{N}$ and $j+1\in\mathcal{D}$.

(f) When j and $j+1$ belong to $\mathcal{N}$ and $\boldsymbol{\mu}_j$ and $\boldsymbol{\mu}_{j+1}$ are linearly independent, then we have $\nabla u(S_j)=0$ and condition (4,4,1,4) reduces to (4,4,1,7) again.

(g) Finally, when j and $j+1$ belong to $\mathcal{N}$ and $\boldsymbol{\mu}_j$ is parallel to $\boldsymbol{\mu}_{j+1}$, we have only

$$\nabla u(S_j)\cdot\boldsymbol{\mu}_j=0$$

while $u(S_j)$ and $\nabla u(S_j)\cdot\boldsymbol{\tau}_j$ are any real numbers. Therefore, condition (4,4,1,4) is equivalent to condition (4,4,1,7) and the following condition:

$$\int_\Omega \Delta\boldsymbol{\psi}_j\cdot\boldsymbol{\tau}_j\, v\, dx\, dy=\sum_{k=j}^{j+1}\left\langle\left[\frac{\partial}{\partial\nu_k}+\beta_k\frac{\partial}{\partial\tau_k}\right]\boldsymbol{\psi}_j\cdot\boldsymbol{\tau}_j;\ \gamma_k v\right\rangle. \tag{4,4,1,9}$$

Summing up, we have proved the following theorem.

Theorem 4.4.1.8 *Let $p\neq 2$; then N_q is the space of all solutions $v\in L_q(\Omega)$ of problem* (4,4,1,3) *which in addition fulfil the following conditions*:

(a) (4,4,1,7) *for all j such that both j and $j+1$ belong to $\mathcal{N}$*;

and, when $p>2$:

(b) (4,4,1,8) *for all j such that $j\in\mathcal{D}$ and $j+1\in\mathcal{N}$ or such that $j\in\mathcal{N}$ and $j+1\in\mathcal{D}$ and $\boldsymbol{\mu}_j$ is parallel to $\boldsymbol{\mu}_{j+1}$*;

(c) (4,4,1,9) *for all j such that both j and $j+1$ belong to $\mathcal{N}$ and $\boldsymbol{\mu}_j$ is parallel to $\boldsymbol{\mu}_{j+1}$.*

Unfortunately we are unable to prove such a precise result when $p = 2$. The reason is that, for $u \in H^2(\Omega)$, it is not possible to apply the Green formula of Theorem 1.5.3.6 to the function

$$w = u - \sum_{j=1}^{N} u(S_j)\varphi_j.$$

Indeed, in general we have $\gamma_j w \in H^{3/2}(\Gamma_j)$, and in addition

$$\gamma_j w(S_{j-1}) = \gamma_j w(S_j) = 0,$$

but this is not enough to conclude that $\gamma_j w \in \tilde{H}^{3/2}(\Gamma_j)$ (see Subsection 1.5.1).

4.4.2 The adjoint problem

In this subsection we shall show that the dimension of N_q is finite in most cases. This will be achieved by studying thoroughly the behaviour of the solutions of problem (4,4,1,3) which belong to $L_q(\Omega)$. Sometimes, this will also allow us to calculate exactly the dimension of N_q.

Lemma 4.4.2.1 *Let $v \in M_q$, then $v \in C^\infty(\bar{\Omega} \backslash V)$, where V is any neighbourhood of the vertices of Ω.*

Proof Actually v is a harmonic function in Ω and it is well known that it is smooth inside Ω. We must prove the smoothness of v near any of the Γ_j. For that purpose we fix j and perform a change of coordinate axes such that the segment Γ_j is on the axis $\{x_2 = 0\}$ and such that Ω is above Γ_j. Then we introduce a cut-off function $\varphi \in \mathcal{D}(\bar{\Omega})$, whose support does not intersect any of the sides $\bar{\Gamma}_k$ with $k \neq j$ (consequently it does not contain any of the corners) and such that φ does not depend on x_2 for small values of x_2. We shall now investigate the smoothness of φv.

The function $w = \widetilde{\varphi v}$ belongs to $L_q(\mathbb{R}^2_+)$ where $\mathbb{R}^2_+ = \{x_2 > 0\}$. In addition, w is solution of

$$\begin{cases} -\Delta w + w = f & \text{in } \mathbb{R}^2_+ \\ \gamma \dfrac{\partial w}{\partial x_2} + \beta_j \gamma \dfrac{\partial w}{\partial x_1} = g & \text{on } \{x_2 = 0\} \quad \text{if } j \in \mathcal{N}, \\ \gamma w = 0 & \text{on } \{x_2 = 0\} \quad \text{if } j \in \mathcal{D}, \end{cases}$$

where

$$f = \{\varphi v - 2\nabla\varphi \cdot \nabla v - (\Delta\varphi)v\}$$

and

$$g = \left\{\left(\frac{\partial \varphi}{\partial x_2} + \beta_j \frac{\partial \varphi}{\partial x_1}\right)\gamma v\right\} \qquad \text{if } j \in \mathcal{N}.$$

At first sight we have $f \in W_q^{-1}(\mathbb{R}_+^2)$ and $g \in W_q^{-1/q}(\mathbb{R})$. However, f is actually a little better than this. Indeed, f is smooth for $x_2 > 0$, while for small values of x_2, we have

$$f = \left\{\varphi v - 2\frac{\partial \varphi}{\partial x_1}\frac{\partial v}{\partial x_1} - (\Delta\varphi)v\right\}^{\sim},$$

due to the fact that φ does not depend on x_2. It follows that

$$f \in L_q(\mathbb{R}_+; W_q^{-1}(\mathbb{R}))$$

if we agree to view f as a vector-valued function of x_2. This will allow us to show that $w \in W_q^1(\mathbb{R}_+^2)$ as a first step.

We replace w by Rw, where R is the inverse operator of $(1 - D_1^2)^{1/2}$; i.e.

$$Rw = F_1^{-1}(1 + \xi_1^2)^{-1/2} F_1 w,$$

where F_1 denotes the Fourier transform in x_1. It follows from Lemma 2.3.2.5 that $Rw \in L_q(\mathbb{R}_+^2)$ and that

$$\begin{cases} -\Delta Rw + Rw = Rf & \text{in } \mathbb{R}_+^2 \\ \gamma\left\{\dfrac{\partial Rw}{\partial x_2} + \beta_j \dfrac{\partial Rw}{\partial x_1}\right\} = Rg & \text{on } \{x_2 = 0\} \quad \text{if } j \in \mathcal{N} \\ \gamma Rw = 0 & \text{on } \{x_2 = 0\} \quad \text{if } j \in \mathcal{D}, \end{cases}$$

where $Rf \in L_q(\mathbb{R}_+^2)$ and $Rg \in W^{1-1/q}(\mathbb{R})$. We conclude by applying Proposition 2.5.2.4 when $j \in \mathcal{N}$ and Corollary 2.5.2.2 when $j \in \mathcal{D}$, replacing Ω by any domain Ω_1 with a smooth boundary containing the support of φ and such that $\Gamma_j \subset \partial\Omega_1$. It follows that

$$Rw\,|_{\Omega_1} \in W_q^2(\Omega_1)$$

and consequently we have $Rw \in W_q^2(\mathbb{R}_+^2)$ and $w \in W_q^1(\mathbb{R}_+^2)$. If we vary φ and j, we finally show that

$$v \in W_q^1(\Omega \backslash V),$$

where V is any neighbourhood of the vertices of Ω.

Now we retrace all the previous steps of the proof. Since we know that v belongs to $W_q^1(\Omega \backslash V)$, we also know that $f \in L_q(\mathbb{R}_+^2)$ and $g \in W_q^{1-1/q}(\mathbb{R}_+^2)$. Thus, applying Corollary 2.5.2.2 and Proposition 2.5.2.4 to w in this case (instead of Rw) shows that

$$w\,|_{\Omega_1} \in W_q^2(\Omega_1),$$

and consequently

$$v \in W_q^2(\Omega \setminus V),$$

where V is any neighbourhood of the corners of Ω.

Finally, repeated application of Theorem 2.5.1.1 with Ω replaced by Ω_1 as above, shows that

$$v \in W_q^{k+2}(\Omega \setminus V)$$

for every positive integer k. The Sobolev imbedding theorem (Subsection 1.4.4) implies that

$$v \in C^\infty(\bar{\Omega} \setminus V).$$

The proof of Lemma 4.4.2.1 is complete. ■

Now we shall study the behaviour of $v \in M_q$ near the corners. For simplicity we begin with those corners S_j which correspond to self-adjoint conditions. In other words, we assume that $\beta_j = 0$ if $j \in \mathcal{N}$ and that $\beta_{j+1} = 0$ if $j+1 \in \mathcal{N}$. For technical purposes, we shall need the eigenfunctions of the operator

$$\varphi \mapsto -\varphi''$$

under various boundary conditions in the interval $]0, \omega_j[$.

More precisely, let us define the unbounded operator Λ_j, in $\mathcal{H}_j = L_2(]0, \omega_j[)$ as follows:

$$\Lambda_j \varphi = -\varphi'',$$

where $\varphi \in D(\Lambda_j)$, the domain of Λ_j, given by

$$D(\Lambda_j) = \begin{cases} \{\varphi \in H^2(]0, \omega_j[) \mid \varphi(0) = \varphi(\omega_j) = 0\} & \text{if } j \text{ and } j+1 \in \mathcal{D} \\ \{\varphi \in H^2(]0, \omega_j[) \mid \varphi'(0) = \varphi'(\omega_j) = 0\} & \text{if } j \text{ and } j+1 \in \mathcal{N} \\ \{\varphi \in H^2(]0, \omega_j[) \mid \varphi(0) = \varphi'(\omega_j) = 0\} & \text{if } j \in \mathcal{N} \text{ and } j+1 \in \mathcal{D} \\ \{\varphi \in H^2(]0, \omega_j[) \mid \varphi'(0) = \varphi(\omega_j) = 0\} & \text{if } j \in \mathcal{D} \text{ and } j+1 \in \mathcal{N}. \end{cases}$$

This is a nonnegative self-adjoint operator with a discrete spectrum. We shall denote by $\varphi_{j,m}$, $m = 1, 2, \ldots$ the normalized eigenfunctions and by $\lambda_{j,m}^2$, $m = 1, 2, \ldots$ the corresponding eigenvalues in increasing order of magnitude. We thus have

$$-\varphi_{j,m}'' = \lambda_{j,m}^2 \varphi_{j,m}$$

where $\varphi_{j,m} \in D(\Lambda_j)$ for every m.

Of course, we have

$$\varphi_{j,m}(\theta)=\sqrt{\left(\frac{2}{\omega_j}\right)}\sin\frac{m\pi\theta}{\omega_j},\qquad \lambda_{j,m}=\frac{m\pi}{\omega_j}\quad\text{when}\quad j\text{ and }j+1\in\mathcal{D}$$

$$\varphi_{j,m}(\theta)=\begin{cases}1/\sqrt{\omega_j}\\ \sqrt{\left(\frac{2}{\omega_j}\right)}\cos\frac{(m-1)\pi\theta}{\omega_j},\end{cases}\qquad \lambda_{j,m}=\begin{cases}0 & m=1\\ \frac{(m-1)\pi}{\omega_j} & m\geq 2\end{cases}$$

when j and $j+1\in\mathcal{N}$

$$\varphi_{j,m}(\theta)=\sqrt{\left(\frac{2}{\omega_j}\right)}\sin\frac{(m-\frac{1}{2})\pi\theta}{\omega_j},\qquad \lambda_{j,m}=\frac{(m-\frac{1}{2})\pi}{\omega_j}$$

when $j\in\mathcal{N}$ and $j+1\in\mathcal{D}$

$$\varphi_{j,m}(\theta)=\sqrt{\left(\frac{2}{\omega_j}\right)}\sin\frac{(m-\frac{1}{2})\pi(\omega_j-\theta)}{\omega_j},\quad \lambda_{j,m}=\frac{(m-\frac{1}{2})\pi}{\omega_j}$$

when $j\in\mathcal{D}$ and $j+1\in\mathcal{N}$.

Using the polar coordinates with origin at S_j (introduced in Subsection 4.3.2), any $v\in M_q$ is a solution of

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}=0,\qquad 0<\theta<\omega_j,\quad 0<r<\rho,\tag{4,4,2,1}$$

where $\rho>0$ is small enough (chosen such that the disc whose centre is S_j and radius is ρ does not cut any side of Ω except Γ_j and Γ_{j+1}. We set $D_\rho=\Omega\cap\{0<r_j<\rho\}$). In addition it fulfils the following boundary conditions: at $\theta=0$

$$v=0\quad\text{if}\quad j+1\in\mathcal{D}\qquad\text{and}\qquad\frac{\partial v}{\partial\theta}=0\quad\text{if } j+1\in\mathcal{N},$$

and at $\theta=\omega_j$

$$v=0\quad\text{if } j\in\mathcal{D}\qquad\text{and}\qquad\frac{\partial v}{\partial\theta}=0\quad\text{if } j\in\mathcal{N}.$$

Since v is smooth for $r>0$ by Lemma 4.4.2.1, we have

$$v(re^{i\theta})\in H^2(]0,\omega_j[).$$

It follows that

$$v(re^{i\theta})\in D(\Lambda_j)$$

for each $r\in]0,\rho[$.

Consequently we have

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{1}{r^2}\Lambda_j v=0, \qquad 0<r<\rho. \tag{4,4,2,2}$$

This implies that v can be expanded in series of the eigenfunctions of Λ_j, in a very special fashion, which we describe now.

Proposition 4.4.2.2 *Let $v \in C^\infty(]0,\rho]; D(\Lambda_j))$ be a solution of Equation (4,4,2,2) and assume that $v \in L_q(D\rho)$. Then*

$$v(re^{i\theta})=v_1(r)\varphi_{j,1}(\theta)+\sum_{m\geq 2}\alpha_m r^{\lambda_{j,m}}\varphi_{j,m}(\theta)+\sum_{0<\lambda_{j,m}<2/q}\beta_m r^{-\lambda_{j,m}}\varphi_{j,m}(\theta),$$

where

$$v_1(r)=\alpha_1 r^{\lambda_{j,1}} \qquad \textit{if } \lambda_{j,1}>0$$

and

$$v_1(r)=\alpha_1+\beta_1\log r \qquad \textit{if } \lambda_{j,1}=0,$$

and where α_m and β_m are real numbers such that

$$|\alpha_m|\leq Lm^{1/q}\rho^{-\lambda_{j,m}}, \tag{4,4,2,3}$$

where L is a constant which depends only on v.

Proof Since the sequence $\varphi_{j,m}$, $m=1,2,\ldots$ is a basis of $\mathcal{H}_j$, we have

$$v(re^{i\theta})=\sum_{m\geq 1} v_m(r)\varphi_{j,m}(\theta), \tag{4,4,2,4}$$

where

$$v_m(r)=\int_0^{\omega_j} v(re^{i\theta})\varphi_{j,m}(\theta)\,d\theta. \tag{4,4,2,5}$$

However, since v is differentiable in r with values in $D(\Lambda_j)$, the differential equation (4,4,2,2) implies that

$$v_m''(r)+\frac{1}{r}v_m'(r)-\frac{\lambda_{j,m}^2}{r^2}v_m(r)=0, \qquad 0<r<\rho.$$

Accordingly, we have

$$v_m(r)=\alpha_m r^{\lambda_{j,m}}+\beta_m r^{-\lambda_{j,m}}$$

when $\lambda_{j,m}>0$, and

$$v_m(r)=\alpha_m+\beta_m\log r$$

when $\lambda_{j,m}=0$.

On the other hand, since v belongs to $L_q(D_\rho)$, it follows from identity (4,4,2,5) that

$$|v_m(r)|^q \leqslant \sqrt{2}^q \omega_j^{q/p-q/2} \int_0^{\omega_j} |v(re^{i\theta})|^q \, d\theta$$

and consequently

$$\int_0^\rho |v_m(r)|^q r \, dr \leqslant \sqrt{2}^q \omega_j^{q/p-q/2} \|v\|_{0,q,D_\rho}^q. \tag{4,4,2,6}$$

This implies that $\beta_m = 0$ when $\lambda_{j,m} \geqslant 2/q$, and in addition that

$$|\alpha_m|^q \int_0^\rho r^{\lambda_{j,m}q+1} \, dr = |\alpha_m|^q \frac{\rho^{\lambda_{j,m}q+2}}{\lambda_{j,m}q+2} \leqslant \sqrt{2}^q \omega_j^{q/p-q/2} \|v\|_{0,q,D_\rho}^q$$

for $\lambda_{j,m} \geqslant 2/q$. This completes the proof of Proposition 4.4.2.2.

We shall now show that $v \in M_q$ has an expansion near each corner (which looks very much like the expansion in Proposition 4.4.2.2) in the general case where β_j, β_{j+1} are possibly nonzero. We shall use here the eigenfunctions and eigenvalues of a different operator. Let us denote by Λ_j the unbounded operator in $\mathcal{H}_j = L_2(]0, \omega_j[) \times L_2(]0, \omega_j[)$ defined by

$$\Lambda_j\{v_1, v_2\} = \{v_2', -v_1'\}$$

where $\{v_1, v_2\} \in D(\Lambda_j)$ and

$$D(\Lambda_j) = \left\{ \{v_1, v_2\} \in H^1(]0, \omega_j[) \times H^1(]0, \omega_j[) \right. \\ \left. \times \begin{vmatrix} \cos \Phi_{j+1} v_2(0) - \sin \Phi_{j+1} v_1(0) = 0 \\ \cos \Phi_j v_2(\omega_j) - \sin \Phi_j v_1(\omega_j) = 0 \end{vmatrix} \right\}$$

It is obvious that Λ_j is a self-adjoint operator and has a discrete spectrum. The expansions in terms of eigenfunctions are as follows:

Lemma 4.4.2.3 *Every $\{v_1, v_2\} \in \mathcal{H}_j$ has an expansion of the following form*

$$v_1 = \sum_{-\infty}^{+\infty} \frac{1}{\sqrt{\omega_j}} \alpha_m \cos(\lambda_{j,m}\theta + \Phi_{j+1})$$

$$v_2 = \sum_{-\infty}^{+\infty} \frac{1}{\sqrt{\omega_j}} \beta_m \sin(\lambda_{j,m}\theta + \Phi_{j+1})$$

where

$$\lambda_{j,m} = \frac{\Phi_j - \Phi_{j+1} + m\pi}{\omega_j}$$

and

$$\alpha_m = \frac{1}{\sqrt{\omega_j}} \int_0^{\omega_j} [v_1(\theta)\cos(\lambda_{j,m}\theta + \Phi_{j+1}) + v_2(\theta)\sin(\lambda_{j,m}\theta + \Phi_{j+1})]\, d\theta$$

provided $(\Phi_j - \Phi_{j+1})/\pi$ *is not an integer.*

When $(\Phi_j - \Phi_{j+1})/\pi$ *is an integer* l, *the expansion is*

$$v_1 = \frac{\alpha_{-l}}{\sqrt{[\omega_j(1+\tan^2\Phi_j)]}} + \sum_{m\neq -l} \frac{1}{\sqrt{\omega_j}}\alpha_m \cos(\lambda_{j,m}\theta + \Phi_{j+1})$$

$$v_2 = \frac{\alpha_{-l}\tan\Phi_j}{\sqrt{[\omega_j(1+\tan^2\Phi_j)]}} + \sum_{m\neq -l} \frac{1}{\sqrt{\omega_j}}\alpha_m \sin(\lambda_{j,m}\theta + \Phi_{j+1})$$

where

$$\alpha_{-l} = \frac{1}{\sqrt{[\omega_j(1+\tan^2\Phi_j)]}} \int_0^{\omega_j} [v_1(\theta) + \tan\Phi_j v_2(\theta)]\, d\theta$$

and α_m, $m \neq -l$ *is as before.*

Proof It is easy to check that the eigenvalues of Λ_j are the numbers $\lambda_{j,m}$, $-\infty < m < +\infty$, m integer and that the corresponding eigenvectors are

$$\varphi_{j,m}(\theta) = \frac{1}{\sqrt{\omega_j}}\{\cos(\lambda_{j,m}\theta + \Phi_{j+1}); \sin(\lambda_{j,m}\theta + \Phi_{j+1})\}$$

for $\lambda_{j,m} \neq 0$ and

$$\varphi_{j,m}(\theta) = \frac{1}{\sqrt{[\omega_j(1+\tan^2\Phi_j)]}}\{1; \tan\Phi_j\}$$

for $\lambda_{j,m} = 0$. ∎

Using again the polar coordinates introduced in Subsection 4.3.2, we see that each $v \in M_q$ has the following features. First, by Lemma 4.4.2.1, v is a differentiable function of r with values in $H^2(]0, \omega_j[)$ for $r \in]0, \rho[$, where $\rho > 0$ is suitably small. Then, we have again

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0, \qquad 0 < \theta < \omega_j,\ 0 < r < \rho, \tag{4,4,2,7}$$

where v fulfils the boundary conditions

$$\begin{cases} \cos\Phi_{j+1}\dfrac{\partial v}{\partial \theta} + \sin\Phi_{j+1} r\dfrac{\partial v}{\partial r} = 0 & \text{(and } v = 0 \text{ if } j+1 \in \mathcal{D}\text{)},\\ & 0 < r < \rho,\ \theta = 0\\ \cos\Phi_j\dfrac{\partial v}{\partial \theta} + \sin\Phi_j r\dfrac{\partial v}{\partial r} = 0 & \text{(and } v = 0 \text{ if } j \in \mathcal{D}\text{)},\\ & 0 < r < \rho,\ \theta = \omega_j. \end{cases} \tag{4,4,2,8}$$

Let us set

$$w_1 = r\frac{\partial v}{\partial r}, \qquad w_2 = -\frac{\partial v}{\partial \theta},$$

then, obviously, $\mathbf{w} = \{w_1, w_2\}$ is a differentiable function of r with values in $D(\Lambda_j)$ for $0 < r \leqslant \rho$ and

$$r\frac{\partial \mathbf{w}}{\partial r} = \Lambda_j \mathbf{w}. \tag{4,4,2,9}$$

This implies the following:

Theorem 4.4.2.4 *Let* $v \in C^\infty(]0, \rho]; H^2(]0, \omega_j[))$ *be a solution of equation* (4,4,2,7) *fulfilling the boundary conditions* (4,4,2,8). *Assume, in addition, that* $v \in L_q(D_\rho)$. *Then*

$$v(re^{i\theta}) = \sum_{\lambda_{j,m} > -2/q} \frac{c_m}{\sqrt{\omega_j}} \frac{r^{\lambda_{j,m}}}{\lambda_{j,m}} \cos(\lambda_{j,m}\theta + \Phi_{j+1}) + \kappa$$

where c_m *and* κ *are real numbers such that*

$$|c_m| \leqslant L\rho^{-\lambda_{j,m}} \tag{4,4,2,10}$$

for some constant L *depending only on* v, *provided* $(\Phi_j - \Phi_{j+1})/\pi$ *is not an integer.*

When $(\Phi_j - \Phi_{j+1})/\pi$ *is an integer* l, *the expansion of* v *is*

$$v(re^{i\theta}) = c_{-l}\frac{\log r - \theta \tan \Phi_j}{\sqrt{[\omega_j(1 + \tan^2 \Phi_j)]}} + \sum_{\substack{\lambda_{j,m} > 2/q \\ m \neq -l}} \frac{c_m}{\sqrt{\omega_j}} \frac{r^{\lambda_{j,m}}}{\lambda_{j,m}} \cos(\lambda_{j,m}\theta + \Phi_{j+1}) + \kappa$$

with the same growth condition on the sequence c_m.

Proof The beginning of the proof is similar to that of Proposition 4.4.2.2. Indeed, the sequence $\varphi_{j,m}$ is a basis of $\mathcal{H}_j$, and thus we have

$$\mathbf{w}(re^{i\theta}) = \sum_{-\infty}^{+\infty} w_m(r)\varphi_{j,m}(\theta) \tag{4,4,2,11}$$

where

$$w_m(r) = \int_0^{\omega_j} \mathbf{w}(re^{i\theta}) \cdot \varphi_{j,m}(\theta)\, d\theta. \tag{4,4,2,12}$$

In other words, we have

$$r\frac{\partial v}{\partial r}(re^{i\theta}) = \sum_{-\infty}^{+\infty} w_m(r)\frac{1}{\sqrt{\omega_j}} \cos(\lambda_{j,m}\theta + \Phi_{j+1})$$

$$\frac{\partial v}{\partial \theta}(re^{i\theta}) = -\sum_{-\infty}^{+\infty} w_m(r)\frac{1}{\sqrt{\omega_j}} \sin(\lambda_{j,m}\theta + \Phi_{j+1}),$$

where

$$w_m(r) = \frac{1}{\sqrt{\omega_j}} \int_0^{\omega_j} \left\{ r \frac{\partial v}{\partial r}(re^{i\theta}) \cos(\lambda_{j,m}\theta + \Phi_{j+1}) - \frac{\partial v}{\partial \theta}(re^{i\theta}) \sin(\lambda_{j,m}\theta + \Phi_{j+1}) \right\} d\theta$$

with the obvious necessary modification when $(\Phi_j - \Phi_{j+1})/\pi$ happens to be an integer. Then the Equation (4,4,2,9) implies that

$$rw'_m(r) = \lambda_{j,m} w_m(r),$$

and, accordingly

$$w_m(r) = c_m r^{\lambda_{j,m}}.$$

Thus it follows that

$$c_m = r^{-\lambda_{j,m}} \frac{1}{\sqrt{\omega_j}} \int_0^{\omega_j} \left\{ r \frac{\partial v}{\partial r}(re^{i\theta}) \cos(\lambda_{j,m}\theta + \Phi_{j+1}) - \frac{\partial v}{\partial \theta}(re^{i\theta}) \sin(\lambda_{j,m}\theta + \Phi_{j+1}) \right\} d\theta$$

for every $r \in]0, \rho]$ and therefore there exists a constant $L(r)$ such that

$$|c_m| \leq L(r) r^{-\lambda_{j,m}}$$

for every $r \in]0, \rho]$. This implies the uniform convergence of the following series:

$$r \frac{\partial v}{\partial r}(re^{i\theta}) = \frac{1}{\sqrt{\omega_j}} \sum_{-\infty}^{+\infty} c_m r^{\lambda_{j,m}} \cos(\lambda_{j,m}\theta + \Phi_{j+1})$$

$$\frac{\partial v}{\partial \theta}(re^{i\theta}) = \frac{-1}{\sqrt{\omega_j}} \sum_{-\infty}^{+\infty} c_m r^{\lambda_{j,m}} \sin(\lambda_{j,m}\theta + \Phi_{j+1})$$

in the rectangles $Q_\varepsilon = \{(r, \theta);\ \varepsilon \leq r \leq \rho - \varepsilon,\ 0 \leq \theta \leq \omega_j\}$ for $\varepsilon > 0$.

Integrating, we obtain

$$v(re^{i\theta}) = \frac{1}{\sqrt{\omega_j}} \sum_{-\infty}^{+\infty} c_m \frac{r^{\lambda_{j,m}}}{\lambda_{j,m}} \cos(\lambda_{j,m}\theta + \Phi_{j+1}) + \kappa.$$

This expansion is valid in $\bigcup_{\varepsilon>0} Q_\varepsilon$, i.e. for $r \in]0, \rho]$ and $\theta \in [0, \omega_j]$ (here, for simplicity, we have assumed that none of the eigenvalues $\lambda_{j,m}$ vanish; the modifications for covering the general case are obvious). The condition that v belongs to $L_q(D_\rho)$ imply that $c_m = 0$ when $\lambda_{j,m} \leq -2/q$. ■

Remark 4.4.2.5 The results in Theorem 4.4.2.4 clearly imply those of Proposition 4.4.2.2.

4.4.3 The Fredholm alternative for variational problems

In this subsection, we restrict our purpose to those problems (4,1,1) which are variational. That is why we assume that

$$\beta_j = 0 \text{ unless both } j-1 \text{ and } j+1 \text{ belong to } \mathscr{D}. \tag{4,4,3,1}$$

Indeed we have the following statement

Lemma 4.4.3.1 *Assume that* (4,4,3,1) *holds. Then for every given* $f \in L_p(\Omega)$, *problem* (4,1,1) *has a unique solution* $u \in H^1(\Omega)$ *when* $\mathscr{D}$ *is nonempty. On the other hand, when* $\mathscr{D}$ *is empty, for every given* $f \in L_p(\Omega)$ *such that*

$$\int_\Omega f \,\mathrm{d}x\,\mathrm{d}y = 0,$$

the problem (4,1,1) *has a solution* $u \in H^1(\Omega)$, *which is unique up to an additive constant.*

Note that due to (4,4,3,1) this is a pure Neumann problem when $\mathscr{D} = \emptyset$.

Proof As usual, we define a variational solution of problem (4,1,1) as being any function

$$u \in V = \{u \in H^1(\Omega) \mid \gamma_j u = 0, \qquad \forall j \in \mathscr{D}\}$$

such that

$$a(u; v) = -\int_\Omega fv \,\mathrm{d}x\,\mathrm{d}y \tag{4,4,3,2}$$

for every $v \in V$, where

$$a(u; v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}y + \sum_{j \in \mathcal{N}} \beta_j \left\langle \frac{\partial}{\partial \tau_j} \gamma_j u; \gamma_j v \right\rangle. \tag{4,4,3,3}$$

We observe that the bilinear form a is continuous on $V \times V$ because the only boundary terms that actually occur (with $\beta_j \neq 0$) are such that

$$\gamma_j u \qquad \text{and} \qquad \gamma_j v \in \tilde{H}^{1/2}(\Gamma_j)$$

due to (4,4,3,1) and Theorem 1.5.2.3. On the other hand, by Remark 1.4.4.7 we know that $\partial/\partial\tau_j$ maps $H^{1/2}(\Gamma_j)$ into the dual of $\tilde{H}^{1/2}(\Gamma_j)$. Consequently all brackets in the right-hand side of (4,4,3,3) are continuous on $V \times V$.

Finally we observe that the form a is coercive (see Lemma 2.2.1.1) because we have

$$a(u; u) = \int_\Omega |\nabla u|^2 \, dx \, dy. \tag{4,4,3,4}$$

Indeed, for every $\varphi \in \mathscr{D}(\Gamma_j)$ we have obviously

$$\left\langle \frac{\partial \varphi}{\partial \tau_j}; \varphi \right\rangle = \int_{\Gamma_j} \frac{\partial \varphi}{\partial \tau_j} \varphi \, d\sigma = 0.$$

Then, since $\mathscr{D}(\Gamma_j)$ is dense in $\tilde{H}^{1/2}(\Gamma_j)$, we have also

$$\left\langle \frac{\partial \varphi}{\partial \tau_j}; \varphi \right\rangle = 0$$

for every $\varphi \in \tilde{H}^{1/2}(\Gamma_j)$.

From identity (4,4,3,4), the coerciveness of a follows with the aid of Poincaré's inequality when $\mathscr{D}$ is not empty. When $\mathscr{D}$ is empty, we have only shown that the form

$$\dot{u}, \dot{v} \mapsto a(u; v)$$

is coercive on $V = H^1(\Omega)/C$, where C denotes the subspace of the constant functions.

The existence and uniqueness of a solution $u \in V$ to problem (4,4,3,2) follows now by Lemma 2.2.1.1 when $\mathscr{D}$ is not empty. In the case when $\mathscr{D}$ is empty, we have existence and uniqueness in $H^1(\Omega)/C$, provided

$$\dot{v} \mapsto \int_\Omega fv \, dx \, dy$$

is a continuous linear form on $H^1(\Omega)/C$. This means that we have existence in $H^1(\Omega)$ up to an additive constant, provided

$$\int_\Omega f \, dx \, dy = 0.$$

We conclude by showing that our variational solution is actually a solution of problem (4,1,1). Indeed, restricting identity (4,4,3,2) to $v \in \mathscr{D}(\Omega)$ shows that $\Delta u = f$ in Ω in the sense of distributions. Accordingly, u belongs to $E(-\Delta; L_p(\Omega))$ (see Subsection 1.5.3) and $\gamma_j u$ and $\gamma_j \, \partial u/\partial \nu_j$ are well defined on each Γ_j by Theorem 1.5.3.10. Then the Green formula of Theorem 1.5.3.11 shows that

$$\gamma_j \frac{\partial u}{\partial \nu_j} + \beta_j \frac{\partial}{\partial \tau_j} \gamma_j u = 0$$

on Γ_j for every $j \in \mathcal{N}$. ■

We shall now try to calculate the dimension of M_q. The first technical step is the following. Here, again, η_j is any cut-off function which is 1 in a neighborhood of S_j, whose support does not intersect $\bar{\Gamma}_k$ for $k \neq j$ and $j+1$ and such that

$$\frac{\partial \eta_j}{\partial \nu_l} - \beta_l \frac{\partial \eta_j}{\partial \tau_l} = 0$$

on Γ_l when $l \in \mathcal{N} \cap \{j; j+1\}$.

Lemma 4.4.3.2 *For each j and each $\lambda_{j,m} \in]-2/q, 0]$ there exists $\sigma_{j,m} \in M_q$ such that*

$$\sigma_{j,m} - \eta_j u_{j,m} \in H^1(\Omega),$$

where

$$u_{j,m}(r_j e^{i\theta_j}) = \begin{cases} \dfrac{r_j^{\lambda_{j,m}}}{(\sqrt{\omega_j})\lambda_{j,m}} \cos(\lambda_{j,m}\theta_j + \Phi_{j+1}) & \text{if } \lambda_{j,m} < 0 \\ \dfrac{\log r_j - \theta_j \tan \Phi_j}{\sqrt{[\omega_j(1+\tan^2 \Phi_j)]}} & \begin{array}{l}\text{if } \lambda_{j,m} = 0 \\ \text{and } j \text{ and } j+1 \text{ are} \\ \text{not both in } \mathcal{D}.\end{array} \end{cases}$$

Here, again, r_j, θ_j denote the polar coordinates with origin at S_j.

Proof It is obvious that

$$\Delta \eta_j u_{j,m} = f_{j,m} \in C^\infty(\bar{\Omega})$$

and that

$$\begin{cases} \gamma_l \dfrac{\partial \eta_j u_{j,m}}{\partial \nu_l} - \beta_l \dfrac{\partial}{\partial \tau_l} \gamma_l \eta_j u_{j,m} = 0 & \text{on } \Gamma_l, \quad l \in \mathcal{N} \\ \gamma_l \eta_j u_{j,m} = 0 & \text{on } \Gamma_l, \quad l \in \mathcal{D}. \end{cases}$$

In addition $\int_\Omega f_{j,m}\, dx\, dy = 0$ when $\mathcal{D}$ is empty.

We can therefore apply Lemma 4.4.3.1 to prove the existence of $v_{j,m} \in H^1(\Omega)$, a solution of

$$\begin{cases} \Delta v_{j,m} = f_{j,m} & \text{in } \Omega \\ \gamma_l \dfrac{\partial v_{j,m}}{\partial \nu_l} - \beta_l \dfrac{\partial}{\partial \tau_l} \gamma_l v_{j,m} = 0 & \text{on } \Gamma_l, \quad l \in \mathcal{N} \\ \gamma_l v_{j,m} = 0 & \text{on } \Gamma_l, \quad l \in \mathcal{D}. \end{cases}$$

The conclusion of this lemma follows by setting

$$\sigma_{j,m} = \eta_j u_{j,m} - v_{j,m}. \quad \blacksquare$$

We are now able to state the key result of this subsection.

Theorem 4.4.3.3 *Under the assumption* (4,4,3,1) *and when*

$$(\Phi_j - \Phi_{j+1} + 2\omega_j/q)/\pi$$

is not an integer for any j, the dimension of the space of all solutions in $L_q(\Omega)$ *of problem* (4,4,1,3) *is*

$$\mu(\Omega) = \sum_{j,j+1\in\mathcal{D}} \operatorname{card}\left\{m \in \mathbb{Z} \;\middle|\; -\frac{2\omega_j}{q} < m\pi < 0\right\}$$

$$+ \sum_{j \text{ or } j+1\in\mathcal{N}} \operatorname{card}\left\{m \in \mathbb{Z} \;\middle|\; -\frac{2\omega_j}{q} < \Phi_j - \Phi_{j+1} + m\pi \leq 0\right\}$$

when $\mathcal{D}$ *is not empty and* $\mu(\Omega)+1$ *when* $\mathcal{D}$ *is empty.*

Proof Let $v \in L_q(\Omega)$ be solution of problem (4,4,1,3) and consider any fixed corner S_j. We apply Theorem 4.4.2.4 and Lemma 4.4.3.2 in the related disc D of radius ρ. It turns out that

$$v - \sum_{0\geq\lambda_{j,m}>-2/q} c_{j,m}\sigma_{j,m} - \sum_{\lambda_{j,m}>0} \frac{c_{j,m}}{\sqrt{\omega_j}} \frac{r_j^{\lambda_{j,m}}}{\lambda_{j,m}} \cos(\lambda_{j,m}\theta_j + \Phi_{j+1}) \in H^1(D) \tag{4,4,3,5}$$

with $c_{j,m} = 0$ in the particular case when $\lambda_{j,m} = 0$ and j and $j+1$ belong to $\mathcal{D}$.

We shall now show that the series in (4,4,3,5) belong to $H^1(D_1)$ for every disc of radius $\rho_1 < \rho$. Indeed, let us denote this series by w. We have

$$\frac{\partial w}{\partial r_j} = \sum_{\lambda_{j,m}>0} \frac{c_{j,m}}{\sqrt{\omega_j}} r_j^{\lambda_{j,m}-1} \cos(\lambda_{j,m}\theta_j + \Phi_{j+1})$$

$$\frac{1}{r_j}\frac{\partial v}{\partial \theta_j} = -\sum_{\lambda_{j,m}>0} \frac{c_{j,m}}{\sqrt{\omega_j}} r_j^{\lambda_{j,m}-1} \sin(\lambda_{j,m}\theta_j + \Phi_{j+1})$$

and consequently

$$|\nabla w| \leq \sum_{\lambda_{j,m}>0} |c_{j,m}|\, r_j^{\lambda_{j,m}-1} \frac{2}{\sqrt{\omega_j}}.$$

Then due to inequality (4,4,2,10), ∇w is bounded in D_1; indeed we have

$$|\nabla w(r_j e^{i\theta_j})| \leq \sum_{\lambda_{j,m}>0} \frac{2}{\sqrt{\omega_j}} L m^{1/q} \frac{\rho_1^{\lambda_{j,m}-1}}{\rho^{\lambda_{j,m}}}$$

and this last series is convergent since $\rho_1 > \rho$.

In other words, we have

$$v - \sum_{0\geq\lambda_{j,m}>-2/q} c_{j,m}\sigma_{j,m} \in H^1(D_1).$$

Such a smoothness result holds near each of the corners S_j. Then, with the help of Lemma 4.4.2.1, we conclude that

$$v - \sum_{\substack{j=1,2\cdots N\\ 0\geq\lambda_{j,m}>-2/q}} c_{j,m}\sigma_{j,m} \in H^1(\Omega) \tag{4,4,3,6}$$

where $c_{j,m}=0$ for $\lambda_{j,m}=0$, when both j and $j+1$ belong to $\mathscr{D}$.

To end the proof, let us denote by φ the function in (4,4,3,6). It is a solution of problem (4,4,1,3) and in addition it belongs to $H^1(\Omega)$. Thus Lemma 4.4.3.1 shows that $\varphi=0$, unless $\mathscr{D}$ is empty, where φ is a constant K. In other words, we have

$$v = \sum_{\substack{j=1,\ldots,N\\ 0\geq\lambda_{j,m}>-2/q}} c_{j,m}\sigma_{j,m} + K$$

where $K=0$ unless $\mathscr{D}$ is empty. The statement of Theorem 4.4.3.3 is an easy consequence. ■

Then, with the help of Theorem 4.4.1.6, we can derive a bound for the actual dimension of N_q.

Corollary 4.4.3.4 *Assume that* (4,4,3,1) *holds and that*

$$(\Phi_j-\Phi_{j+1}+2\omega_j/q)/\pi$$

is not an integer for any j. Then when $p<2$, the dimension of N_q is less than or equal to

$$\nu(\Omega)=\sum_{j=1}^{N} \operatorname{card}\left\{m\in\mathbb{Z}\;\middle|\; -\frac{2\omega_j}{q}<\Phi_j-\Phi_{j+1}+m\pi<0\right\}$$

if $\mathscr{D}$ is not empty and $\nu(\Omega)+1$ if $\mathscr{D}$ is empty. When $p>2$ the dimension of N_q is less than or equal to

$$\nu(\Omega)-\operatorname{card}\{j \mid \boldsymbol{\mu}_j \text{ is parallel to } \boldsymbol{\mu}_{j+1}\}$$

if $\mathscr{D}$ is not empty and $\nu(\Omega)+1$ again if $\mathscr{D}$ is empty.

Observe that when $\mathscr{D}$ is empty, we are just dealing with a pure Neumann problem, owing to (4,4,3,1).

Proof So far, we have shown that the $\sigma_{j,m}$, $1\leq j\leq N$, $-2/q<\lambda_{j,m}<0$ (if j and $j+1\in\mathscr{D}$), $-2/q<\lambda_{j,m}\leq 0$ (if j or $j+1\in\mathscr{N}$) are a basis of M_q (possibly up to the constant function).

We shall first show that any $\sigma_{j,m}$ corresponding to $\lambda_{j,m}=0$ does not belong to N_q. Due to assumption (4,4,3,1) $\lambda_{j,m}$ can vanish iff j and $j+1\in\mathscr{N}$ (and consequently $\Phi_j=\Phi_{j+1}=0$). The corresponding $\sigma_{j,m}$ is eliminated by condition (4,4,1,7) (see Theorem 4.4.1.8). Indeed we have (in the polar coordinates related to S_j)

$$\sigma_{j,m} = \eta \log r_j/\sqrt{(\omega_j)} + v,$$

where $v \in H^1(\Omega)$, $\eta(0)=1$, η depends only on r_j, $\eta(r_j)=1$ for $r_j \leqslant \rho_i$, $\eta(r_j)=0$ for $r_j \geqslant \rho_e$ where $0<\rho_i<\rho_e$ are chosen in such a way that the support of η does not meet Γ_k for $k \neq j$ and $j+1$. Then, in condition (4,4,1,7), we can choose $\varphi_j = \eta$. Accordingly, this condition reduces to

$$\int_\Omega \Delta\eta\sigma_{j,m}\, dx\, dy = 0.$$

Actually we have

$$\int_\Omega \Delta\eta v\, dx\, dy = \int_\Omega \eta\Delta v\, dx\, dy$$

since both η and v belong to $H^1(\Omega)$, η has a small support around S_j and both η and v fulfil a Neumann boundary condition on Γ_j and Γ_{j+1}.

On the other hand, we have

$$\int_\Omega \Delta\eta(\eta \log r_j)\, dx\, dy = \int_{\Omega'} \Delta\eta(\eta \log r_j)\, dx\, dy$$

where $\Omega' = \Omega \backslash \{r_j \leqslant \rho_i\}$, since $\Delta\eta$ vanishes in $\Omega\backslash\Omega'$. We can apply again Green's formula since both η and $\eta \log r_j$ are smooth in Ω'. We thus get

$$\int_\Omega \Delta\eta(\eta \log r_j)\, dx\, dy$$
$$= \int_{\Omega'} \eta\Delta(\eta \log r_j)\, dx\, dy + \int_\gamma \left\{\frac{\partial\eta}{\partial\nu}\eta \log r_j - \eta\frac{\partial}{\partial\nu}(\eta \log r_j)\right\} d\sigma$$

where $\gamma = \partial\Omega'\backslash\partial\Omega$. It follows that

$$\int_\Omega \Delta\eta(\eta \log r_j)\, dx\, dy/\sqrt{\omega_j}$$
$$= \int_\Omega \eta(\Delta\sigma_{j,m} - \Delta v)\, dx\, dy - \int_0^{\omega_j} \frac{1}{\rho_i}\rho_i\, d\theta/\sqrt{\omega_j} = -\int_\Omega \eta\Delta v\, dx\, dy - \sqrt{\omega_j}$$

since $\sigma_{j,m}$ is harmonic.

Finally, we have

$$\int_\Omega \Delta\eta\sigma_{j,m}\, dx\, dy = -\sqrt{\omega_j}$$

and this contradicts the condition (4,4,1,7). Accordingly, $\sigma_{j,m}$ does not belong to N_q. Consequently, the dimension of N_q is less than or equal to $\nu(\Omega)$ when $\mathscr{D}$ is not empty and to $\nu(\Omega)+1$ when $\mathscr{D}$ is empty.

To complete the proof of Corollary 4.4.3.4 we observe that any $\sigma_{j,m}$ corresponding to $\lambda_{j,m}=-1$ is eliminated from N_q by condition (4,4,1,8) or (4,4,1,9) in Theorem 4.4.1.6 when p is greater than 2. The calculations are very similar to the previous one, so that we do not need to repeat it. The condition $\lambda_{j,m}=-1$ for one integer m is fulfilled iff $\boldsymbol{\mu}_j$ is parallel to $\boldsymbol{\mu}_{j+1}$. ■

This result, together with Lemma 4.4.3.1, allows us to calculate the index of Δ as an operator from E_1 to E_2 (these spaces have been defined in Subsection 4.4.1). We shall also be able to conclude when $p=2$, due to the inclusion $N_2 \subseteq N_q$ which holds for $q<2$.

For that purpose, let us again use the polar coordinates with origin at S_j and let us consider the functions

$$S_{j,m}(r_j e^{i\theta_j}) = \frac{r_j^{-\lambda_{j,m}}}{(\sqrt{\omega_j})\lambda_{j,m}} \cos(\lambda_{j,m}\theta_j + \Phi_{j+1})\eta_j(r_j e^{i\theta_j}) \tag{4,4,3,7}$$

with $\lambda_{j,m}<0$, not an integer. Here are some properties of these functions.

Lemma 4.4.3.5 $S_{j,m} \in H^1(\Omega)\backslash W_p^2(\Omega)$ *for*

$$-\frac{2}{q} \leqslant \lambda_{j,m} < 0, \qquad 1 \leqslant j \leqslant N, \quad \lambda_{j,m} \neq -1$$

and in addition

$$\begin{cases} \Delta S_{j,m} \in C^\infty(\bar{\Omega}) \\ \gamma_l \dfrac{\partial S_{j,m}}{\partial \nu_l} + \beta_l \dfrac{\partial}{\partial \tau_l} \gamma_l S_{j,m} = 0 & \text{on } \Gamma_l \quad \text{if } l \in \mathscr{N} \\ \gamma_l S_{j,m} = 0 & \text{on } \Gamma_l \quad \text{if } l \in \mathscr{D}. \end{cases}$$

This is obvious. The following statement deserves a proof.

Lemma 4.4.3.6 *Assuming that* (4,4,3,1) *holds,* $\Delta S_{j,m}$ *is not orthogonal to* N_q *for*

$$-\frac{2}{q} \leqslant \lambda_{j,m} < 0, \qquad 1 \leqslant j \leqslant N, \quad \lambda_{j,m} \neq -1.$$

Proof This can be proved by contradiction. Thus, if we assume that $S_{j,m}$ is orthogonal to N_q, then there exists $w_{j,m} \in W_p^2(\Omega)$ fulfilling the boundary conditions in (4,1,1) such that

$$\Delta w_{j,m} = \Delta S_{j,m}.$$

Therefore $w_{j,m} - S_{j,m}$ is a solution of the homogeneous problem and belongs to $H^1(\Omega)$. By the uniqueness result of Lemma 4.4.3.1, this implies that $S_{j,m}$ belongs to $W_p^2(\Omega)$. This contradicts Lemma 4.4.3.5. ■

We are now able to conclude.

Theorem 4.4.3.7 *We assume that* (4,4,3,1) *holds and that*

$$(\Phi_j - \Phi_{j+1} + 2\omega_j/q)/\pi$$

is not an integer for any j, that in addition $\boldsymbol{\mu}_j$ is never parallel to $\boldsymbol{\mu}_{j+1}$, when $p=2$. Then for each $f \in L_p(\Omega)$, there exist unique real numbers $C_{j,m}$ and a unique u such that

$$u - \sum_{\substack{1 \leq j \leq N \\ -2/q < \lambda_{j,m} < 0 \\ \lambda_{j,m} \neq -1}} C_{j,m} S_{j,m} \in W_p^2(\Omega) \tag{4,4,3,8}$$

and u is solution of problem (4,1,1) *when $\mathscr{D}$ is not empty. Otherwise, when $\mathscr{D}$ is empty u is unique up to an additive constant and exists iff*

$$\int_\Omega f \, dx \, dy = 0.$$

Proof The functions $\Delta S_{j,m}$ corresponding to

$$-\frac{2}{q} < \lambda_{j,m} < 0, \qquad \lambda_{j,m} \neq -1, \quad j = 1, 2, \ldots, N$$

are in $L_p(\Omega)$ and are clearly linearly independent. Since they are not orthogonal to N_q, they do not belong to the image of E_1 through Δ. Moreover, their number is exactly the upper bound for the dimension of N_q (possibly minus one when $\mathscr{D}$ is empty) that we found in Corollary 4.4.3.4. Consequently, $L_p(\Omega)$ is the span of the image of E_1 through Δ and of these functions $\Delta S_{j,m}$. The claim follows by Lemma 4.4.3.1. ■

One could ask why there is a gap in the index of the problem corresponding to the eigenvalue $\lambda_{j,m} = -1$. Actually, there is no longer any gap when we consider nonhomogeneous boundary conditions:

Corollary 4.4.3.8 *Under the assumptions of Theorem* 4.4.3.7, *let $f \in L_p(\Omega)$ and $g_j \in W_p^{2-1/p}(\Gamma_j)$, $j \in \mathscr{D}$, $g_j \in W_p^{1-1/p}(\Gamma_j)$, $j \in \mathscr{N}$ be given such that*

$$g_j(S_j) = \frac{\partial g_{j+1}}{\partial \mu_j}(S_j) \text{ if } j \in \mathscr{N} \text{ and } j+1 \in \mathscr{D} \text{ or } g_{j+1}(S_j) = \frac{\partial g_j}{\partial \mu_{j+1}}(S_j)$$
$$\text{if } j \in \mathscr{D} \text{ and } j+1 \in \mathscr{N}, \tag{4,4,3,9}$$

whenever μ_j *is parallel to* μ_{j+1}, *and* $p>2$. *Then assuming that* $\mathscr{D}$ *is not empty, there exist unique real numbers* $C_{j,m}$ *and a unique* u *such that* (4,4,3,8) *holds and* u *is solution of*

$$\begin{cases} \Delta u = f, & (\Omega) \\ \gamma_j u = g_j, & j \in \mathscr{D} \\ \gamma_j \dfrac{\partial u}{\partial \nu_j} + \beta_j \dfrac{\partial}{\partial \tau_j} \gamma_j u = g_j, & j \in \mathscr{N}. \end{cases}$$

When $\mathscr{D}$ *is empty the condition* (4,4,3,9) *is void and* u *is unique up to the addition of a constant and exists iff*

$$\int_\Omega f \,\mathrm{d}x\,\mathrm{d}y - \sum_{j=1}^{N} \int_{\Gamma_j} g_j \,\mathrm{d}\sigma = 0.$$

This result follows from Theorem 4.4.3.7 and the trace theorems in Subsection 1.5.2. We observe that the number of extra conditions that we have added on the data in (4,4,3,9) is exactly

$$\sum_{j=1}^{N} \operatorname{card}\left\{\lambda_{j,m} \,\middle|\, -\frac{2}{q} < \lambda_{j,m} = -1\right\}.$$

4.4.4 The Fredholm alternative for nonvariational problems

Here, we try as far as possible, to deal with problem (4,1,1) in most cases. The existence and uniqueness result of Lemma 4.4.3.1 has been a basic tool in the study that we carried out in Subsection 4.4.3. Unfortunately, if we drop the assumption (4,4,3,1), it may happen that problem (4,1,1) could not be solved uniquely in $H^1(\Omega)$. This will make our analysis much more complicated.

On the one hand, we still have an existence result in $H^1(\Omega)$, which is an application of a lemma in Lions (1956). We recall this result with a slightly different proof.

Lemma 4.4.4.1 *Let* W *and* V *be a pair of Hilbert spaces with a continuous injection of* W *in* V *and let* a *be a continuous bilinear form on* $V \times W$. *Assume that there exists a constant* $\alpha > 0$ *such that*

$$a(v, v) \geq \alpha \|v\|_V^2 \tag{4,4,4,1}$$

for all $v \in W$. *Then for every continuous linear form* l *on* V, *there exists* $u \in V$, *possibly non-unique, such that*

$$a(u; v) = l(v) \tag{4,4,4,2}$$

for every $v \in W$.

This lemma is somewhat similar to Lemma 2.2.1.1 and is actually a consequence of it.

Proof For $\varepsilon > 0$, we introduce the form

$$a_\varepsilon(u, v) = a(u; v) + \varepsilon(u; v)_W, \qquad u, v \in W.$$

This is a continuous bilinear form on $W \times W$, which, in addition, is coercive (with coerciveness constant $\geqslant \varepsilon$). Consequently, by Lemma 2.2.1.1, there exists a unique $u_\varepsilon \in W$ such that

$$a_\varepsilon(u_\varepsilon, v) = l(v) \tag{4,4,4,3}$$

for every $v \in W$.

Using the coerciveness assumption on a and setting $v = u_\varepsilon$ in identity (4,4,4,3), we find bounds for u_ε:

$$\alpha\, \|u_\varepsilon\|_V^2 + \varepsilon\, \|u_\varepsilon\|_W^2 = l(u_\varepsilon) \leqslant \|l\|_{V^*}\, \|u_\varepsilon\|_V.$$

Consequently, we have

$$\begin{cases} \|u_\varepsilon\|_V \leqslant \alpha^{-1}\, \|l\|_{V^*}, \\ \|\sqrt{\varepsilon}\, u_\varepsilon\|_W \leqslant \alpha^{-1/2}\, \|l\|_{V^*}. \end{cases}$$

Due to the famous property of bounded sequences in Hilbert spaces, we can find a sequence ε_j, $j = 1, 2, \ldots$ converging to zero, together with $u \in V$ (clearly we cannot expect u to be unique in general) and $w \in W$ such that

$$\begin{cases} u_{\varepsilon_j} \rightharpoonup u \text{ weakly in } V \\ (\sqrt{\varepsilon_j})u_{\varepsilon_j} \rightharpoonup w \text{ weakly in } W. \end{cases}$$

Going back to identity (4,4,4,2), we have

$$a(u_{\varepsilon_j}, v) + \sqrt{\varepsilon_j}\, (\sqrt{\varepsilon_j}\, u_{\varepsilon_j}; v)_W = l(v)$$

for every $v \in W$. Taking the limit in j proves identity (4,4,4,2). ■

Lemma 4.4.4.1 will be applied as follows. Again, as in Subsection 4.4.3, we set

$$V = \{u \in H^1(\Omega) \mid \gamma_j u = 0, \qquad \forall j \in \mathscr{D}\}.$$

Then we set

$$W = \{u \in H^2(\Omega) \mid \gamma_j u = 0, \quad \forall j \in \mathscr{D} \quad \text{and} \quad u(S_j) = 0, \quad \forall j\};$$

this is a Hilbert space for the norm of $H^2(\Omega)$. Finally the form a is defined by

$$a(u; v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}y + \tfrac{1}{2} \sum_{j \in \mathscr{N}} \beta_j \left\{ \left\langle \frac{\partial}{\partial \tau_j} \gamma_j u; \gamma_j v \right\rangle - \left\langle \gamma_j u; \frac{\partial}{\partial \tau_j} \gamma_j v \right\rangle \right\}. \tag{4,4,4,4}$$

It is easy to check that a is well defined† and continuous on $V\times W$, since for $u\in V$ and $v\in W$, we have

$$\gamma_j u \in H^{1/2}(\Gamma_j), \qquad \frac{\partial}{\partial \tau_j}\gamma_j u \in \tilde{H}^{1/2}(\Gamma_j)^*$$

and

$$\gamma_j v \in H^{3/2}(\Gamma_j)\cap \mathring{H}^1(\Gamma_j)\subset \tilde{H}^{1/2}(\Gamma_j), \qquad \frac{\partial}{\partial \tau_j}\gamma_j v \in H^{1/2}(\Gamma_j).$$

The coerciveness of a in the sense of (4,4,4,1) follows obviously from Poincaré's inequality when $\mathcal{D}$ is not empty. When $\mathcal{D}$ is empty, we must replace everywhere V by V/C, where C denotes the space of constant functions in Ω.

Consequently, given $f\in L_p(\Omega)$, there exists at least one $u\in H^1(\Omega)$ such that

$$a(u;v) = -\int_\Omega fv\,dx\,dy \tag{4,4,4,5}$$

for every $v\in W$ (provided $\int_\Omega f\,dx\,dy=0$ when $\mathcal{D}$ is empty).

We must now make it clear in what sense such a u is the solution of problem (4,1,1). Obviously, we show that

$$\Delta u = f \qquad \text{in } \Omega,$$

by writing (4,4,4,5) with $v\in\mathcal{D}(\Omega)$. Therefore, u belongs to the space $E(\Delta; L_p(\Omega))$ defined in Subsection 1.5.3. Consequently, $\gamma_j\,\partial u/\partial\nu_j$ is well defined as an element of $\tilde{H}^{1/2}(\Gamma_j)^*$. Then, applying the Green formula (1,5,3,9), we deduce from (4,4,4,4) and (4,4,4,5) that

$$\sum_{j\in\mathcal{N}}\frac{\tan\Phi_j}{2}\left\{\left\langle\frac{\partial}{\partial\tau_j}\gamma_j u;\gamma_j v\right\rangle-\left\langle\gamma_j u;\frac{\partial}{\partial\tau_j}\gamma_j v\right\rangle\right\}=-\sum_{j\in\mathcal{N}}\left\langle\gamma_j\frac{\partial u}{\partial\nu_j};\gamma_j v\right\rangle$$

for every $v\in W$ (which is a subspace of the space of possible test-functions in Theorem 1.5.3.11). In other words, this identity holds for every

$$\gamma_j v\in H^{3/2}(\Gamma_j)\cap\mathring{H}^1(\Gamma_j), \qquad j\in\mathcal{N}.$$

† Observe that under assumption (4,4,3,1) the forms defined by (4,4,3,3) and (4,4,4,4) coincide, since

$$\left\langle\frac{\partial}{\partial\tau_j}\varphi;\psi\right\rangle=\left\langle\varphi;\frac{\partial\psi}{\partial\tau_j}\right\rangle$$

for every φ and $\psi\in\tilde{H}^{1/2}(\Gamma_j)$.

This is enough to prove that

$$\gamma_j \frac{\partial u}{\partial \nu_j} + \tan \Phi_j \frac{\partial}{\partial \tau_j} \gamma_j u = 0, \qquad j \in \mathcal{N}.$$

Summing up, we have proved the following statement.

Lemma 4.4.4.2 *Assume that $\mathcal{D}$ is not empty, then for every given $f \in L_p(\Omega)$, problem* (4,1,1) *has a (possibly nonunique) solution $u \in H^1(\Omega)$. When $\mathcal{D}$ is empty, the same result holds provided*

$$\int_\Omega f \,dx\,dy = 0.$$

Our main trouble now is that we have no uniqueness result in $H^1(\Omega)$. However, we have results in some particular cases, if we assume in addition that u is slightly more regular, namely $u \in W_p^1(\Omega)$ with $p > 2$.

In the first particular case, we assume that $\mathcal{D}$ is empty and that

$$\beta_2 \geqslant \cdots \geqslant \beta_j \geqslant \beta_{j+1} \geqslant \cdots \geqslant \beta_1, \qquad 2 \leqslant j \leqslant N \tag{4,4,4,6}$$

Lemma 4.4.4.3 *Let $u \in W_p^1(\Omega)$ with $p > 2$ be the solution of problem* (4,4,1) *with $f = 0$. Assume that* (4,4,4,6) *holds, then u is a constant.*

This will be proved as usual, by calculating the integral of Δu against u on Ω. Unfortunately, this cannot be done directly and we must approximate u by a sequence of smoother functions. This is the purpose of the following auxiliary lemma, whose proof is similar to that of Lemma 1.5.3.9 and Theorem 1.5.3.10.

Lemma 4.4.4.4 *$\mathcal{D}(\bar{\Omega})$ is dense in the space*

$$F(\Delta; L_p(\Omega)) = \{u \in W_p^1(\Omega); \Delta u \in L_p(\Omega)\}$$

equipped with the norm

$$u \mapsto \|u\|_{1,p,\Omega} + \|\Delta u\|_{0,p,\Omega}.$$

In addition $u \mapsto \gamma_j\, \partial u/\partial \nu_j$ has a continuous extension as an operator from $F(\Delta; L_p(\Omega))$ into $W_p^{-1/p}(\Gamma_j)$.

Proof of Lemma 4.4.4.3 We let u_m, $m = 1, 2, \ldots$ be a sequence of functions in $W_p^2(\Omega)$ such that

$$\begin{cases} u_m \to u & \text{in } W_p^1(\Omega) \\ \Delta u_m \to \Delta u & \text{in } L_p(\Omega) \end{cases}$$

where $m \to \infty$.

For $u_m \in W_p^2(\Omega)$ the usual Green formula holds. Thus we have

$$\begin{aligned}
&-\int_\Omega \Delta u_m u_m \,dx\,dy \\
&= \int_\Omega |\nabla u_m|^2 \,dx\,dy - \sum_{j=1}^N \int_{\Gamma_j} \gamma_j \frac{\partial u_m}{\partial \nu_j} \gamma_j u_m \,d\sigma \\
&= \int_\Omega |\nabla u_m|^2 \,dx\,dy - \sum_{j=1}^N \int_{\Gamma_j} \left(\gamma_j \frac{\partial u_m}{\partial \nu_j} + \beta_j \frac{\partial}{\partial \tau_j} \gamma_j u_m\right) \gamma_j u_m \,d\sigma \\
&\quad + \sum_{j=1}^N \frac{\beta_j}{2} \{u_m^2(S_j) - u_m^2(S_{j-1})\} \\
&= \int_\Omega |\nabla u_m|^2 \,dx\,dy - \sum_{j=1}^N \int_{\Gamma_j} \left(\gamma_j \frac{\partial u_m}{\partial \nu_j} + \beta_j \frac{\partial}{\partial \tau_j} \gamma_j u_m\right) \gamma_j u_m \,d\sigma \\
&\quad + \sum_{j=1}^N \frac{\beta_j - \beta_{j+1}}{2} u_m^2(S_j).
\end{aligned}$$

We can take the limit in m of this identity, due to the fact that p is strictly larger than 2. Thus we get

$$\begin{aligned}
-\int_\Omega \Delta u\, u \,dx\,dy &= \int_\Omega |\nabla u|^2 \,dx\,dy \\
&\quad - \sum_{j=1}^N \left\langle \gamma_j \frac{\partial u}{\partial \nu_j} + \beta_j \frac{\partial}{\partial \tau_j} \gamma_j u ; \gamma_j u \right\rangle + \sum_{j=1}^N \frac{\beta_j - \beta_{j+1}}{2} u^2(S_j).
\end{aligned} \tag{4,4,4,7}$$

We observe that the bracket on Γ_j is meaningful since

$$\gamma_j u \in W^{1-1/p}(\Gamma_j) \subset W_q^{1/p}(\Gamma_j) = \mathring{W}_q^{1/p}(\Gamma_j)$$

and

$$\gamma_j \frac{\partial u}{\partial \nu_j} + \beta_j \frac{\partial}{\partial \tau_j} \gamma_j u \in W_p^{-1/p}(\Gamma_j)$$

for $j = 1, \ldots, N$.

Actually the same identity holds with u replaced by $u - u(S_1)$. Thus we get

$$\begin{aligned}
&-\int_\Omega \Delta u (u - u(S_1)) \,dx\,dy \\
&= \int_\Omega |\nabla u|^2 \,dx\,dy - \sum_{j=1}^N \left\langle \gamma_j \frac{\partial u}{\partial \nu_j} + \beta_j \frac{\partial}{\partial \tau_j} \gamma_j u ; \gamma_j u - u(S_1) \right\rangle \\
&\quad + \sum_{j=2}^N \frac{\beta_j - \beta_{j+1}}{2} \{u(S_j) - u(S_1)\}^2.
\end{aligned}$$

Since u is harmonic and fulfils the boundary conditions in (4,4,1), we finally conclude that

$$0=\int_{\Omega} |\nabla u|^2 \, dx \, dy+\sum_{j=2}^{N} \frac{\beta_j-\beta_{j+1}}{2}\{u(S_j)-u(S_1)\}^2.$$

Since by assumption (4,4,4,6) we have

$$\beta_j-\beta_{j+1}\geq 0$$

for $j=2,\ldots,N$, it follows that u is a constant function. ∎

Another useful particular case is this: We assume that $\mathcal{D}=\{3,\ldots,N\}$ and that

$$\beta_1 \geq \beta_2. \tag{4,4,4,8}$$

Again we have a uniqueness result for solutions in $W_p^1(\Omega)$ with $p>2$.

Lemma 4.4.4.5 *Let $u \in W_p^1(\Omega)$, with $p>2$, be the solution of problem* (4,4,1) *with $f=0$. Assume that* (4,4,4,8) *holds, then u is zero.*

Proof Again identity (4,4,4,7) holds for u. Thus we have

$$0=\int_{\Omega} |\nabla u|^2 \, dx \, dy+\frac{\beta_1-\beta_2}{2}\, u(S_1)^2$$

and consequently u is zero. ∎

We shall now study the space M_q. First we observe that the analogue of Lemma 4.4.3.2 holds in the most general case.

Lemma 4.4.4.6 *For each j and each $\lambda_{j,m} \in]-2/q, 0]$ there exists $\sigma_{j,m} \in M_q$ such that*

$$w_{j,m}=\sigma_{j,m}-\eta_{j,m}u_{j,m} \in H^1(\Omega)$$

if $\lambda_{j,m}<0$ or if $\lambda_{j,m}=0$ and j and $j+1$ do not both belong to $\mathcal{D}$.

Proof This is quite similar to the proof of Lemma 4.4.3.2, since there we only used the existence result in Lemma 4.4.3.1. The corresponding existence result is now provided by Lemma 4.4.4.2. ∎

However, we can improve this result due to the fact that 0 is not a limit point of the set $\{\lambda_{j,m} \mid m \in \mathbb{Z}\}$.

Lemma 4.4.4.7 *There exists $p>2$ such that*

$$w_{j,m} \in W_p^1(\Omega)$$

for each $\lambda_{j,m} \in]-2/q, 0]$, provided $(\Phi_k - \Phi_{k+1} + 2\omega_k/q)/\pi$ is not an integer for any k.

Proof The function $w_{j,m}$ is one solution of the homogeneous problem near each corner S_k. In addition it belongs to $L_q(\Omega)$ and it is smooth in $\bar{\Omega}\backslash\{S_1, \ldots, S_N\}$ by Lemma 4.4.2.1. Thus it follows from Theorem 4.4.2.4 that

$$w_{j,m}(r_k e^{i\theta_k}) = \sum_{\lambda_{k,l} > -2/q} \frac{c_{k,l}}{\sqrt{\omega_k}} \frac{r_k^{\lambda_{k,l}}}{\lambda_{k,l}} \cos(\lambda_{k,l}\theta_k + \Phi_{k+1}) + \mathcal{K}$$

for r_k small enough, provided $(\Phi_k - \Phi_{k+1} + 2\omega_k/q)/\pi$ is not an integer.

Consequently, by Lemma 4.4.4.6, we have

$$\sum_{\lambda_{k,l} > -2/q} \frac{c_{k,l}}{\sqrt{\omega_k}} \frac{r_k^{\lambda_{k,l}}}{\lambda_{k,l}} \cos(\lambda_{k,l}\theta_k + \Phi_{k+1}) \in H^1(D) \tag{4,4,4,9}$$

where D is $\Omega \cap \{r_k < \rho\}$ for ρ small enough.

Now inequality (4,4,2,10) implies that

$$|c_{k,l}|\, l^{-1/q} \rho^{\lambda_{k,l}}$$

is bounded as $l \to +\infty$. It follows that

$$\sum_{\lambda_{k,l} > 0} \frac{c_{k,l}}{\sqrt{\omega_k}} \frac{r_k^{\lambda_{k,l}}}{\lambda_{k,l}} \cos(\lambda_{k,l}\theta_k + \Phi_{k+1}) \in W_p^1(D_1) \tag{4,4,4,10}$$

for each $D_1 = \Omega \cap \{r_k < \rho_1\}$, where $\rho_1 < \rho$. This implies, by difference between (4,4,4,9) and (4,4,4,10), that

$$\sum_{-2/q < \lambda_{k,l} \leq 0} \frac{c_{k,l}}{\sqrt{\omega_k}} \frac{r_k^{\lambda_{k,l}}}{\lambda_{k,l}} \cos(\lambda_{k,l}\theta_k + \Phi_{k+1}) \in H^1(D_1).$$

Consequently we have

$$c_{k,l} = 0 \qquad \text{for } \lambda_{k,l} \leq 0.$$

Summing up, we have

$$w_{j,m}(r_k e^{i\theta_k}) = \sum_{\lambda_{k,l} > 0} \frac{c_{k,l}}{\sqrt{\omega_k}} \frac{r_k^{\lambda_{k,l}}}{\lambda_{k,l}} \cos(\lambda_{k,l}\theta_k + \Phi_{k+1})$$

near S_k and by (4,4,4,10) this shows that

$$w_{j,m} \in W_p^1(D_1)$$

for $\inf\{\lambda_{l,k} \mid \lambda_{l,k} > 0\} > 1 - 2/p$. This is true near each corner S_k and consequently we have

$$w_{j,m} \in W_p^1(\Omega)$$

for some $p > 2$. ■

We are now able to calculate the dimension of M_q in two particular cases 'adjoint' to the cases considered in Lemmas 4.4.4.3 and 4.4.4.5.

Theorem 4.4.4.8 *Assume that $\mathcal{D}$ is empty, that $(\Phi_j - \Phi_{j+1} + 2\omega_j/q)/\pi$ is not an integer for any j and that*

$$\tan \Phi_2 \leqslant \cdots \leqslant \tan \Phi_j \leqslant \tan \Phi_{j+1} \leqslant \cdots \leqslant \tan \Phi_N \leqslant \tan \Phi_1.$$

Then the dimension of N_q is less than or equal to

$$\mu(\Omega) = \sum_{j=1}^{N} \operatorname{card}\left\{m \in \mathbb{Z} \;\middle|\; -\frac{2\omega_j}{q} < \Phi_j - \Phi_{j+1} + m\pi < 0\right\} + 1.$$

Proof Let $v \in M_q$. Then $v \in L_q(\Omega)$ and is a solution of the problem (4,4,1,3). By Lemma 4.4.2.1, we know that v is smooth, far from the corners. Then near each corner S_j, v has an expansion given by Theorem 4.4.2.4. In other words, we have

$$v = \sum_{\lambda_{j,m} > -2/q} \frac{c_{j,m}}{\sqrt{\omega_j}} \frac{r_j^{\lambda_{j,m}}}{\lambda_{j,m}} \cos(\lambda_{j,m}\theta_j + \Phi_{j+1})$$

in $D = \Omega \cap \{r_j < \rho\}$ for some $\rho > 0$.

Again here, due to inequality (4,4,2,10), the series

$$\sum_{\lambda_{j,m} > 0} \frac{c_{j,m}}{\sqrt{\omega_j}} \frac{r_j^{\lambda_{j,m}}}{\lambda_{j,m}} \cos(\lambda_{j,m}\theta_j + \Phi_{j+1})$$

belongs to $W_p^1(D)$. This, together with Lemma 4.4.4.7, implies that

$$v - \sum_{-2/q < \lambda_{j,m} \leqslant 0} c_{j,m}\sigma_{j,m} \in W_p^1(D).$$

Since $\sigma_{j,m} \in W_p^1(\Omega \backslash \bar{D})$, it follows that

$$w = v - \sum_{\substack{j=1,2,\ldots,N \\ -2/q < \lambda_{j,m} \leqslant 0}} c_{j,m}\sigma_{j,m} \in W_p^1(\Omega)$$

for some $p > 2$.

Now w is solution of the homogeneous problem (4,4,1,3). Applying Lemma 4.4.4.3 we see that w must be a constant. Finally, the function $\sigma_{j,m}$ corresponding (possibly) to $\lambda_{j,m} = 0$ is eliminated from N_q by condition (4,4,1,7) of Theorem 4.4.1.8 as in the proof of Corollary 4.4.3.4. ∎

The same method of proof, with Lemma 4.4.4.3 replaced by Lemma 4.4.4.5, leads to the following statement.

Theorem 4.4.4.9 *Assume that $\mathcal{D} = \{3, 4, \ldots, N\}$, that*

$$(\Phi_j - \Phi_{j+1} + 2\omega_j/q)/\pi$$

is not an integer for any j and that

$$\tan \Phi_1 \leq \tan \Phi_2.$$

Then the dimension of N_q is less than or equal to

$$\mu(\Omega) = \sum_{j=1}^{N} \text{card}\left\{m \in \mathbb{Z} \,\middle|\, -\frac{2\omega_j}{q} < \Phi_j - \Phi_{j+1} + m\pi < 0\right\}.$$

Now, exactly as we did in Subsection 4.4.3, we shall derive existence results in the space spanned by $W_p^2(\Omega)$ and the functions $S_{j,m}$ corresponding to $-2/q < \lambda_{j,m} < 0$. Indeed, the result of Lemma 4.4.3.5 holds in the most general case. The analogue of Lemma 4.4.3.6 is the following.

Lemma 4.4.4.10 *$\Delta S_{j,m}$ is not orthogonal to N_q for $-2/q \leq \lambda_{j,m} < 0$, $1 \leq j \leq N$, $\lambda_{j,m} \neq -1$.*

Proof Actually, we shall prove that

$$\int_\Omega \Delta S_{j,m} \sigma_{j,m} \, dx \, dy = \frac{1}{\lambda_{j,m}}. \tag{4,4,4,11}$$

Indeed we have $\sigma_{j,m} = \psi_{j,m} + w_{j,m}$, where

$$\psi_{j,m} = \frac{r_j^{\lambda_{j,m}}}{(\sqrt{\omega_j})\lambda_{j,m}} \cos(\lambda_{j,m}\theta_j + \Phi_{j+1})\eta_j$$

in the polar coordinates with origin at S_j. In addition, both $S_{j,m}$ and $w_{j,m}$ belong to $W_p^1(\Omega)$ for some $p > 2$. This allows us to apply the classical Green formula; thus we get

$$\begin{aligned}
&\int_\Omega \Delta S_{j,m} w_{j,m} \, dx \, dy - \int_\Omega S_{j,m} \Delta w_{j,m} \, dx \, dy \\
&= \sum_{l=1}^{N} \left\{ \left\langle \gamma_l \frac{\partial S_{j,m}}{\partial \nu_l} ; \gamma_l w_{j,m} \right\rangle - \left\langle \gamma_l S_{j,m} ; \gamma_l \frac{\partial w_{j,m}}{\partial \nu_l} \right\rangle \right\} \\
&= -\sum_{l=1}^{N} \tan \Phi_l (S_{j,m} w_{j,m}) \Big|_{S_{l-1}}^{S_l} = 0.
\end{aligned}$$

This is due to the boundary conditions

$$\gamma_l \frac{\partial S_{j,m}}{\partial \nu_l} + \tan \Phi_l \frac{\partial}{\partial \tau_l} \gamma_l S_{j,m} = 0 \qquad \text{on } \Gamma_l$$

$$\gamma_l \frac{\partial w_{j,m}}{\partial \nu_l} - \tan \Phi_l \frac{\partial}{\partial \tau_l} \gamma_l w_{j,m} = 0 \qquad \text{on } \Gamma_l,$$

to the properties of the support of $S_{j,m}$ and to the obvious fact that

$$S_{j,m}(S_j) = 0.$$

Since $\Delta\sigma_{j,m} = 0$ we have

$$\int_\Omega \Delta S_{j,m}\sigma_{j,m}\,dx\,dy = \int_\Omega \Delta S_{j,m}\psi_{j,m}\,dx\,dy - \int_\Omega S_{j,m}\Delta w_{j,m}\,dx\,dy. \tag{4,4,4,12}$$

Then, we cannot apply directly the Green formula to $S_{j,m}$ and $\psi_{j,m}$ because $\psi_{j,m}$ is singular at S_j. Thus, we are led to introduce

$$\Omega' = \Omega \cap \{r_j > \rho\},$$

where ρ is chosen such that $\eta_j(re^{i\theta}) = 1$ for $r < \rho$. Since the support of $\Delta S_{j,m}$ is contained in Ω', we have

$$\int_\Omega \Delta S_{j,m}\psi_{j,m}\,dx\,dy = \int_{\Omega'} \Delta S_{j,m}\psi_{j,m}\,dx\,dy.$$

We can now apply the classical Green formula in Ω', since both $\sigma_{j,m}$ and $\psi_{j,m}$ are smooth in $\bar{\Omega}'$.

Let us denote by Γ_l' the intersection of Γ_l with $\partial\Omega'$ and set

$$\gamma = \{r_j e^{i\theta_j} \mid r_j = \rho, \qquad 0 < \theta_j < \omega_j\}.$$

We have

$$\begin{aligned}
&\int_{\Omega'} \Delta S_{j,m}\psi_{j,m}\,dx\,dy - \int_{\Omega'} S_{j,m}\,\Delta\psi_{j,m}\,dx\,dy \\
&= \sum_{l=1}^{N} \int_{\Gamma_l'} \left[\frac{\partial S_{j,m}}{\partial \nu_l}\psi_{j,m} - S_{j,m}\frac{\partial \psi_{j,m}}{\partial \nu_l}\right] d\sigma - \int_\gamma \left[\frac{\partial S_{j,m}}{\partial r_j}\psi_{j,m} - S_{j,m}\frac{\partial \psi_{j,m}}{\partial r_j}\right] d\sigma \\
&= -\sum_{l=1}^{N} \tan \Phi_l \left(S_{j,m}\psi_{j,m}\Big|_{A_l}^{B_l}\right) - \mathcal{J}
\end{aligned}$$

due to the boundary conditions on $S_{j,m}$ and on $\psi_{j,m}$, i.e.

$$\gamma_l \frac{\partial \psi_{j,m}}{\partial \nu_l} - \tan \Phi_l \frac{\partial}{\partial \tau_l}\gamma_l\psi_{j,m} = 0 \qquad \text{on } \Gamma_l.$$

Here we denote by A_l the origin of Γ_l' and by B_l the endpoint of Γ_l', according to the positive orientation. We have also set

$$\mathcal{J} = \int_\gamma \left[\frac{\partial S_{j,m}}{\partial r_j}\psi_{j,m} - S_{j,m}\frac{\partial \psi_{j,m}}{\partial r_j}\right] d\sigma.$$

Due to the properties of the supports of $S_{j,m}$ and $\psi_{j,m}$, it turns out that

$$\int_{\Omega} \Delta S_{j,m}\psi_{j,m}\,dx\,dy - \int_{\Omega} S_{j,m}\,\Delta\psi_{j,m}\,dx\,dy$$

$$= \tan \Phi_{j+1}\frac{\cos^2 \Phi_{j+1}}{\omega_j\lambda_{j,m}^2} - \tan \Phi_j \frac{\cos^2 \Phi_j}{\omega_j\lambda_{j,m}^2} + \mathcal{J}. \qquad (4,4,4,13)$$

Finally, we calculate $\mathcal{J}$ explicitly. This is elementary, and we get

$$\mathcal{J} = \frac{1}{\lambda_{j,m}} + \frac{\sin \Phi_j \cos \Phi_j - \sin \Phi_{j+1} \cos \Phi_{j+1}}{\omega_j\lambda_{j,m}^2}. \qquad (4,4,4,14)$$

The identity (4,4,4,11) follows plainly from identities (4,4,4,12) to (4.4.4.14). ■

Corollary 4.4.4.11 *Assume that the hypotheses of Theorem* 4.4.4.8 *are fulfilled. Then for each* $f \in L_p(\Omega)$ *such that* $\int_\Omega f\,dx\,dy = 0$, *there exist real numbers* $c_{j,m}$ *and a function u such that*

$$u - \sum_{\substack{1 \le j \le N \\ -2/q < \lambda_{j,m} < 0 \\ \lambda_{j,m} \ne -1}} c_{j,m} S_{j,m} \in W_p^2(\Omega)$$

and u is a solution of problem (4,1,1).

Proof This is a simple consequence of the fact that $L_p(\Omega)$ is the space spanned by the annihilator of N_q, the constant functions and the functions $\Delta S_{j,m}$ corresponding to the eigenvalues such that

$$-\frac{2}{q} < \lambda_{j,m} < 0, \qquad \lambda_{j,m} \ne -1.$$

Clearly these functions are linearly independent (this follows from their explicit definition in identity (4,4,3,7)), do not belong to the annihilator of N_q by Lemma 4.4.4.10 and span a subspace of $L_p(\Omega)$ whose dimension is suitable by Theorem 4.4.4.8.

Replacing Theorem 4.4.4.8 by Theorem 4.4.4.9, we obtain the following statement.

Corollary 4.4.4.12 *Assume that the hypotheses of Theorem* 4.4.4.9 *are fulfilled. Then for each* $f \in L_p(\Omega)$ *there exist real numbers* $c_{j,m}$ *and a function u such that*

$$u - \sum_{\substack{1 \le j \le N \\ -2/q < \lambda_{j,m} < 0 \\ \lambda_{j,m} \ne -1}} c_{j,m} S_{j,m} \in W_p^2(\Omega)$$

and u is a solution of problem (4,1,1).

Now with the help of Theorems 4.4.3.7 and Corollaries 4.4.4.11 and 4.4.4.12, we reach our final goal.

Theorem 4.4.4.13 *We assume that $(\Phi_j - \Phi_{j+1} + 2\omega_j/q)/\pi$ is not an integer for any j. Then for each $f \in L_p(\Omega)$ (such that $\int_\Omega f\,dx\,dy = 0$ when $\mathscr{D}$ is empty) there exist real numbers $c_{j,m}$ and a function u (possibly non-unique) such that*

$$u - \sum_{\substack{1 \le j \le N \\ -2/q < \lambda_{j,m} < 0 \\ \lambda_{j,m} \neq -1}} c_{j,m} S_{j,m} \in W_p^2(\Omega)$$

and u is the solution of problem (4,1,1).

Proof We start from one solution $u \in H^1(\Omega)$ to problem (4,1,1); such a solution exists by Lemma 4.4.4.2. Then we study locally the behaviour of u. Since we have

$$\Delta u \in L_p(\Omega),$$

it follows plainly that $\eta u \in W_p^2(\Omega)$ for every $\eta \in \mathscr{D}(\Omega)$. This describes the smoothness of u inside Ω.

Then let us look at the behaviour of u near the regular points of the boundary. For that purpose we let $\eta \in \mathscr{D}(\bar{\Omega})$ have a support which does not meet $\bar{\Gamma}_l$ for $l \neq j$. Since $u \in H^1(\Omega)$, we have

$$\Delta \eta u \in L_p(\Omega) + L_2(\Omega)$$

and

$$\left(\gamma_j \frac{\partial}{\partial \nu_j} + \beta_j \frac{\partial}{\partial \tau_j} \gamma_j\right) \eta u = \left(\frac{\partial \eta}{\partial \nu_j} + \beta_j \frac{\partial \eta}{\partial \tau_j}\right) \gamma_j u \in H^{1/2}(\Gamma_j) + W_p^{1-1/p}(\Gamma_j).$$

Choosing a plane open subset Ω' with a $C^{1,1}$ boundary such that $\partial \Omega' \supseteq \Gamma_j$ and such that $\bar{\Omega}'$ contains the support of η, we see that

$$\begin{cases} \Delta \widetilde{\eta u} \in L_p(\Omega') + L_2(\Omega') \\ \left(\gamma \dfrac{\partial}{\partial \nu} + \beta_j \dfrac{\partial}{\partial \tau} \gamma\right) \eta u \in H^{1/2}(\partial \Omega') + W_p^{1-1/p}(\partial \Omega') \end{cases}$$

Therefore, we have $\eta u \in H^2(\Omega) + W_p^2(\Omega)$ by Theorem 2.4.1.3. Varying η this shows that $u \in H^2(\Omega \setminus V) + W_p^2(\Omega \setminus V)$, where V is any closed neighbourhood of the corners of Ω. Accordingly, it follows that

$$\left(\gamma \frac{\partial}{\partial \nu} + \beta_j \frac{\partial}{\partial \tau} \gamma\right) \widetilde{\eta u} \in H^{3/2}(\partial \Omega') \subset W_p^{1-1/p}(\partial \Omega').$$

Applying again Theorem 2.4.1.3, we see that $\eta u \in W_p^2(\Omega)$. Varying η this shows that

$$u \in W_p^2(\Omega \setminus V).$$

Finally, let us study the behaviour of u near one of the corners, say S_1. We shall use one of the model problems studied before. For that purpose we introduce new boundary conditions on Γ_j, $3 \leqslant j \leqslant N$, as follows:

First case $1 \in \mathcal{D}$ or $2 \in \mathcal{D}$: we set

$$L_j = I, \qquad j = 3, \ldots, N.$$

Second case 1 and $2 \in \mathcal{N}$, $\tan \Phi_1 > \tan \Phi_2$: we set

$$L_j = \frac{\partial}{\partial \nu_j} + \tan \Phi_j' \frac{\partial}{\partial \tau_j}, \qquad j = 3, \ldots, N$$

with $\tan \Phi_2 \leqslant \tan \Phi_j' \leqslant \tan \Phi_{j+1}' \leqslant \cdots \leqslant \tan \Phi_1$, $j = 3, \ldots, N$.

Third case 1 and $2 \in \mathcal{N}$, $\tan \Phi_1 \leqslant \tan \Phi_2$: we set

$$L_j = I, \qquad j = 3, \ldots, N.$$

In all cases, we have $\eta_1 u \in H^1(\Omega)$ and

$$\begin{cases} \Delta \eta_1 u \in L_p(\Omega) & \\ \gamma_j L_j \eta_1 u = 0 & \text{on } \Gamma_j, \quad 3 \leqslant j \leqslant N \\ \gamma_j \eta_1 u = 0 & \text{on } \Gamma_j, \quad j \in \mathcal{D} \cap \{1, 2\} \\ \gamma_j \dfrac{\partial \eta_1 u}{\partial \nu_j} + \tan \Phi_j \dfrac{\partial}{\partial \tau_j} \gamma_j \eta_1 u = 0 & \text{on } \Gamma_j, \quad j \in \mathcal{N} \cap \{1, 2\}. \end{cases}$$

This is a problem that we have already solved in Theorem 4.4.3.7 in the first case, Corollary 4.4.4.11 in the second case and in Corollary 4.4.4.12 in the third case. Accordingly, there exists $v \in H^1(\Omega)$ and constants $c_{1,m}$ such that

$$v - \sum_{\substack{-2/q < \lambda_{1,m} < 0 \\ \lambda_{1,m} \neq -1}} c_{1,m} S_{1,m} \in W_p^2(\Omega \cap V) \tag{4,4,4,15}$$

where V is a neighbourhood of S_1 and v is the solution of the same problem as $\eta_1 u$. In other words, we have

$$\begin{cases} \eta_1 u - v \in H^1(\Omega) & \\ \Delta(\eta_1 u - v) = 0 & \text{in } \Omega \\ \gamma_j(\eta_1 u - v) = 0 & \text{on } \Gamma_j, \quad j \in \mathcal{D} \cap \{1, 2\} \\ \left(\gamma_j \dfrac{\partial}{\partial \nu_j} + \tan \Phi_j \dfrac{\partial}{\partial \tau_j} \gamma_j\right)(\eta_1 u - v) = 0 & \text{on } \Gamma_j, \quad j \in \mathcal{N} \cap \{1, 2\}. \end{cases}$$

The last step is to apply Theorem 4.4.2.4 to $\eta_1 u - v$. This shows that $\eta_1 u - v$ can be expanded as follows near zero:

$$\eta_1 u - v = \sum_{\lambda_{1,m} < 0} \frac{c_m}{\sqrt{\omega_1}} \frac{r_1^{-\lambda_{1,m}}}{\lambda_{1,m}} \cos(\lambda_{1,m} \theta_1 + \Phi_2) + \mathcal{K},$$

where

$$|c_m| \leq L\,|m|^{1/q}\rho^{\lambda_{1,m}}$$

for some L and ρ. Consequently, we have

$$\eta_1 u - v - \sum_{\substack{-2/q<\lambda_{1,m}<0\\ \lambda_{1,m}\neq -1}} c_m S_{1,m} \in W_p^2(\Omega\cap V). \tag{4,4,4,16}$$

Adding (4,4,4,15) and (4,4,4,16) we see that

$$\eta_1 u - \sum_{\substack{2/q<\lambda_{1,m}<1\\ \lambda_{1,m}\neq -1}} (c_m + c_{1,m}) S_{1,m} \in W_p^2(\Omega\cap V).$$

A similar result holds for $\eta_j u$ near S_j for each j and this completes the proof of Theorem 4.4.4.13. ■

We conclude with a statement concerning the nonhomogeneous boundary value problem.

Corollary 4.4.4.14 *Under the assumptions of Theorem* 4.4.4.13, *let* $f\in L_p(\Omega)$ *and* $g_j \in W_p^{2-1/p}(\Gamma_j)$, $j\in\mathscr{D}$ *and* $g_j\in W_p^{1-1/p}(\Gamma_j)$, $j\in\mathscr{N}$ *be given such that*

$$g_j(S_j) = \frac{\partial g_{j+1}}{\partial \mu_j}(S_j) \qquad \text{if } j\in\mathscr{N} \quad \text{and} \quad j+1\in\mathscr{D}$$

or

$$g_{j+1}(S_j) = \frac{\partial g_j}{\partial \mu_{j+1}}(S_j) \qquad \text{if } j\in\mathscr{D} \quad \text{and} \quad j+1\in\mathscr{N}$$

or

$$g_{j+1}(S_j) = \frac{\mu_j\cdot\mu_{j+1}}{|\mu_{j+1}|^2}\, g_j(S_j) \quad \text{if } j\in\mathscr{N} \quad \text{and} \quad j+1\in\mathscr{N}$$

whenever $\boldsymbol{\mu}_j$ *is parallel to* $\boldsymbol{\mu}_{j+1}$ $p>2$, *and* $g_j(S_j)=g_{j+1}(S_j)$ *if* $j\in\mathscr{D}$ *and* $j+1\in\mathscr{D}$. *Then there exists a function u and numbers* $c_{j,m}$ *(possibly non-unique) such that*

$$u - \sum_{\substack{1\leq j\leq N\\ -2/q<\lambda_{j,m}<0\\ \lambda_{j,m}\neq -1}} c_{j,m} S_{j,m} \in W_p^2(\Omega)$$

and u is solution of problem (4,1,1).

Remark 4.4.4.15 In this whole chapter we have excluded the domains with cuts (i.e. $\omega_j = 2\pi$ for some j) for simplicity. However, if we allow cuts, the basic *a priori* inequality of Section 4.3 remains valid (see Remarks 4.3.1.7 and 4.3.2.7). The main tool in Section 4.4 has been the

Green formula in a Lipschitz domain. To handle domains with cuts requires derivation of the corresponding Green formula. This can be achieved by using the trick, described at the beginning of Section 1.7, of considering separately the restrictions of the functions to Ω_+ and Ω_-. Accordingly the results of Theorem 4.4.4.13 and Corollary 4.4.4.14 hold for domains with cuts (i.e. if we allow $\omega_j = 2\pi$ in the statements).

Remark 4.4.4.16 In the particular case of self-adjoint boundary conditions (i.e. either Dirichlet or Neumann) along a cut the results mentioned above may be easily deduced from the Theorem 4.4.4.13. Indeed, to make an example, let us assume that $\omega_j = 2\pi$ and $j, j+1 \in \mathcal{D}$. Then by looking at the problem locally one can assume that Ω is symmetric with respect to Γ_j. A rotation and a translation reduce the problem to the particular case when $S_j = 0$ and Γ_j together with Γ_{j+1} lie on the positive x-axis.

Now let us write u as the sum of an even function u_e and an odd function u_0 with respect to y:

$$u_e(x, y) = \frac{u(x, y) + u(x, -y)}{2}, \qquad u_0(x, y) = \frac{u(x, y) - u(x, -y)}{2}.$$

Assuming that $f \in L_p(\Omega)$, $g_j = 0$ and $g_{j+1} = 0$ imply that u_0 fulfils a homogeneous Dirichlet on the axis $y = 0$ in a neighbourhood of O. Therefore we have

$$u_0 \in W_p^2(V \cap \Omega), \tag{4,4,4,17}$$

where V is a suitable neighbourhood of O. In the same way u_e fulfils a homogeneous mixed boundary condition near O:

$$\begin{cases} u_e(x, 0) = 0, & 0 < x < \delta \\ D_y u_e(x, 0) = 0, & -\delta < x < 0 \end{cases}$$

for some $\delta > 0$. Accordingly, by Theorem 4.4.4.13 there exist constants $c_{j,m}$ such that

$$u_e - \sum_{1/2 - 2/q < m < 1/2} c_{j,m} r_j^{-m+1/2} \sin\left(m - \tfrac{1}{2}\right)\theta_j \in W_p^2(V \cap \Omega). \tag{4,4,4,18}$$

By adding (4,4,4,17) and (4,4,4,18) we obtain the behaviour of u near 0. This is an alternative proof of the results stated in Remark 4.4.4.15. The same method allows one to handle the following cases:

$$\begin{cases} \omega_j = 2\pi, & j \in \mathcal{D}, \quad j+1 \in \mathcal{N} \quad \text{or} \quad j \in \mathcal{N}, \quad j+1 \in \mathcal{D} \\ \omega_j = 2\pi, & j \in \mathcal{N}, \quad j+1 \in \mathcal{N}. \end{cases}$$