V1: Mixed Formulation of Coupled Plates

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Abstract

Based on energy principle, the mathematical model of coupled plates has been established. However, the corresponding finite element method can't get the stress directly which is more concerned in practical engineering. In this article, a mixed formulation of coupled plates meeting at an arbitrary angle is established. And the well-poseness of the mixed formulation is provided.
Keywords: Coupled Plates, Mixed Formulation

1 Introduction

Many industrial constructions use, as basic components, elastic solids, bars, plates and shells. The numerical simulation of such assemblages needs a good approximation of each constitutive element as well as a good representation of their junctions. In engineering literature, there are many contributions on the best way to modelize and, particularly, to compute such constructions.

In Feng [3], the mechanical and mathematical models of composite structures have been proposed in general. Ciarlet [2] have dealt with the field by the asymptotic analysis. In recent paper [4], a composite elastic structure consisting of coupled plates meeting at a right angle and its finite element approximations are considered using the approach proposed by Feng. In this article, we consider an elastic structure consisting of coupled plates meeting at an arbitrary angle and its finite element method.

In this paper we restrict our attention to the numerical analysis of the junction between two plates. Our study is based on the following assumptions:(1) small deformations,(2) elastic, homogeneous, isotropic material,(3) deformation through the thickness obeys the Kirchhoff-Love hypothesis, (4) the junction can be assimilated to a rigid or an elastic hinge.

The contents of this paper can be outlined as follows. In the remaining part of this section, we will address the mechanical modelling of the junction of plates in terms of partial differential equations. We start by recalling the general plane stress model and Kirchhoff-Love plate model. Section 2 gives the variational formulations based on displacement only of the junctions and the corresponding existence results. In Section 3, we present the mixed variational form of the coupling model and propose the corresponding nonstandard finite element space. Furthermore, in Section 4, we provide a theoretical analysis of the well-posedness of the mixed formulation.

2 Preliminaries

2.1 Hypotheses and notations

In this section, we will give the notations of the coupled plates as figure 1. Denote the middle surface of the left plate S and the right plate $\underset{\sim}{S}$. Throughout this paper, we denote by $(\cdot)$ the quantities related to the plate S, while $\underset{\sim}{(\cdot)}$ denotes the quantities related to the plate $\underset{\sim}{S}$. So in the section 1 we will only show the symbols with the plate S, the symbols for the plate $\underset{\sim}{S}$ is totally same except needing to add the $\sim$ under it.

Let $\mathbb{R}^3$ be Euclidean space referred to the orthonormal system $(\hat{O};\hat{X},\hat{Y},\hat{Z})$ as the global coordinate. Denote $\Gamma = S \cap \underset{\sim}{S}$ be the interface with the direction unit vector $\boldsymbol{t}$. For the plate S, denote $\boldsymbol{n}$ as its unit outer normal vector on the boundary. Then the unit vector $\boldsymbol{l} = \boldsymbol{n} \times \boldsymbol{t}$ is perpendicular to the plane where the plate S is located. As for the plate $\underset{\sim}{S}$, take the opposite direction $\underset{\sim}{\boldsymbol{t}} = -\boldsymbol{t}$ on the Γ, and denote $\underset{\sim}{\boldsymbol{n}}$ as its unit outer normal vector on the boundary, then $\underset{\sim}{\boldsymbol{l}} = \underset{\sim}{\boldsymbol{n}} \times \underset{\sim}{\boldsymbol{t}}$ is located in plane where the plate $\underset{\sim}{S}$ is located. Therefore, the local coordinate on plate S is $(\boldsymbol{n},\boldsymbol{t},\boldsymbol{l})$ and on plate $\underset{\sim}{S}$ is $(\underset{\sim}{\boldsymbol{n}},\underset{\sim}{\boldsymbol{t}},\underset{\sim}{\boldsymbol{l}})$.

For the interface Γ, using the local coordinate $(\boldsymbol{n},\boldsymbol{t},\boldsymbol{l})$ same with plate S. Then we can define the angle $\theta = (\boldsymbol{n},\underset{\sim}{\boldsymbol{n}})_{\boldsymbol{t}}$ between two plates. When the plates are coplanar and placed side by side, the angle is equal to π, and when the plates coincide, the angle is equal to 0. The relation between two local coordinates is

$$\begin{aligned}
\boldsymbol{n} &= \underset{\sim}{\boldsymbol{n}}\cos\theta - \underset{\sim}{\boldsymbol{l}}\sin\theta, \quad \boldsymbol{t} = -\underset{\sim}{\boldsymbol{t}}, \quad \boldsymbol{l} = -\underset{\sim}{\boldsymbol{n}}\sin\theta - \underset{\sim}{\boldsymbol{l}}\cos\theta \\
\underset{\sim}{\boldsymbol{n}} &= \boldsymbol{n}\cos\theta - \boldsymbol{l}\sin\theta, \quad \underset{\sim}{\boldsymbol{t}} = -\boldsymbol{t}, \quad \underset{\sim}{\boldsymbol{l}} = -\boldsymbol{n}\sin\theta - \boldsymbol{l}\cos\theta
\end{aligned}$$

In this article, we assume that the external loads induce a distribution of forces whose resultant is $\boldsymbol{f}$ in plane and f_3 out plane on the middle surface S and whose resultant moment is $\boldsymbol{0}$ on S and $\underset{\sim}{\boldsymbol{f}}$ in plane and $\underset{\sim}{f_3}$ out plane on the middle surface $\underset{\sim}{S}$ and whose resultant moment is $\boldsymbol{0}$ on $\underset{\sim}{S}$.

We also assume that the plate S is clamped on a part ∂S_0 of its boundary, with measure $(\partial S_0 > 0)$. Then the boundary of the plate S can divided to $\partial S = \partial S_0 \cup \partial S_1 \cup \Gamma$, where ∂S_1 is the free boundary condition. As for the plate $\underset{\sim}{S}$, only the free boundary condition will be considered on $\partial \underset{\sim}{S}_1$ where $\partial\underset{\sim}{S} = \partial\underset{\sim}{S}_1 \cup \Gamma$.

For simplicity, we only consider the plates with the same property in this paper, it means these two plates have the same Young's modulus E, Poisson's ratio ν and thickness e.

In here and what follows, Latin indices take their values in $\{1,2,3\}$, Greek and primed Greek indices take their values in $\{1,2\}$ and $\{2,3\}$, respectively; the repeated index convention for summation is systematically used.

For scalar functions v, vector-valued functions ψ, and matrix-valued functions $\boldsymbol{N}$ the first-order differential expressions

$$\nabla v, \nabla \psi, \operatorname{div} \psi, \operatorname{Div} \boldsymbol{N},$$

are defined in the weak sense on the corresponding domains of definition

$$H^1(\Omega), \left(H^1(\Omega)\right)^2,$$

In the case that all components are in $H^1(\Omega)$ they take on their classical form given as follows:

$$\begin{aligned}
\nabla v &= \begin{pmatrix} \partial_1 v \\ \partial_2 v \end{pmatrix}, \\
\nabla \psi &= \begin{pmatrix} \partial_1 \psi_1 & \partial_2 \psi_1 \\ \partial_1 \psi_2 & \partial_2 \psi_2 \end{pmatrix}, \\
\operatorname{div} \psi &= \partial_1 \psi_1 + \partial_2 \psi_2, \\
\operatorname{Div} \boldsymbol{N} &= \begin{pmatrix} \partial_1 \boldsymbol{N}_{11} + \partial_2 \boldsymbol{N}_{12} \\ \partial_1 \boldsymbol{N}_{21} + \partial_2 \boldsymbol{N}_{22} \end{pmatrix},
\end{aligned}$$

Moreover, the symmetric gradient and the symmetric Curl are introduced by

$$\varepsilon(\psi) = \frac{1}{2}\left(\nabla \psi + (\nabla \psi)^T\right),$$

2.2 Single flat shell model

The flat shell model can be decomposed to the plane stress model and plate bending model. Each of them is the simplify model form the classical linear elasticity theory. This section gives the equations of the single shell S with local coordinate and boundary condition as stated before. Assume that the condition on Γ is the same as it on ∂S_1.

The plane stress model can be derived by simplify the standard elasticity equations, we can derived the equation for plane stress model

$$\begin{aligned}
-\operatorname{Div} \boldsymbol{N} &= \boldsymbol{f}, && \text{on} \quad S \\
\boldsymbol{u} &= 0 && \text{on} \quad \partial S_0 \\
\boldsymbol{N}\boldsymbol{n} &= 0 && \text{on} \quad \partial S_1
\end{aligned} \tag{1}$$

where

$$\begin{aligned}
&\boldsymbol{N} = \tfrac{E}{1-\nu^2}((1-\nu)\boldsymbol{e} + \nu \operatorname{tr}(\boldsymbol{e})\mathbf{I}) = C_1 \boldsymbol{e}, \\
&\boldsymbol{e} = \tfrac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T).
\end{aligned} \tag{2}$$

Here $\boldsymbol{u} = (u_1, u_2), \boldsymbol{e}, \boldsymbol{N}$ are the displacement, strain, stress respectively.

For the plate bending model, we take the Kirchhoff-Love plate model can derive the following equation

$$\begin{aligned}
-\operatorname{divDiv} \boldsymbol{M} &= f_3, && && \text{on} \quad S \\
u_3 &= 0, && u_{3,n} = 0 && \text{on} \quad \partial S_0 \\
M_{\alpha\beta,\beta} n_\alpha + \left(M_{\alpha\beta} n_\beta t_\alpha\right)_{,t} &= 0, && M_{\alpha\beta} n_\alpha n_\beta = 0 && \text{on} \quad \partial S_1 \\
[\![M_{\alpha\beta} n_\alpha t_\beta]\!]_x &= 0, && && \text{at} \quad x \in \mathcal{V}_{\partial S_1}
\end{aligned} \tag{3}$$

where

$$
\begin{aligned}
&\boldsymbol{M} = \tfrac{Ee^3}{12(1-\nu^2)}((1-\nu)\boldsymbol{K} + \nu \operatorname{tr}(\boldsymbol{K})\mathbf{I}) = C_2\boldsymbol{K}. \\
&\boldsymbol{K} = -\nabla^2 u_3,
\end{aligned}
\tag{4}
$$

Here $u_3, \boldsymbol{K}, \boldsymbol{M}$ are the displacement or deflection in Z direction, curvature, moment, respectively.

3 Mathematical Model based displacement

Consider the equation of the junction plate as follows:

$$
\begin{aligned}
&N_{\alpha\beta,\beta} + p_\alpha = 0, \quad M_{\alpha\beta,\alpha\beta} + p_3 = 0 \quad \text{ in } S \\
&\utilde{N}_{\alpha\beta,\beta} + \utilde{p}_\alpha = 0, \quad \utilde{M}_{\alpha\beta,\alpha\beta} + \utilde{p}_3 = 0 \quad \text{ in } \utilde{S}
\end{aligned}
\tag{5}
$$

with boundary condition:

$$
\begin{array}{llll}
u_\alpha = 0, & u_3 = 0, & u_{3,n} = 0 & \text{on } \partial S_0 \\
N_{\alpha\beta} n_\beta = 0, & M_{\alpha\beta,\beta} n_\alpha + \left(M_{\alpha\beta} n_\beta t_\alpha\right)_{,t} = 0, & M_{\alpha\beta} n_\alpha n_\beta = 0 & \text{on } \partial S_1 \\
\utilde{N}_{\alpha\beta} \utilde{n}_\beta = 0, & \utilde{M}_{\alpha\beta,\beta} \utilde{n}_\alpha + \left(\utilde{M}_{\alpha\beta} \utilde{n}_\beta \utilde{t}_\alpha\right)_{,\utilde{t}} = 0, & \utilde{M}_{\alpha\beta} \utilde{n}_\alpha \utilde{n}_\beta = 0 & \text{on } \partial \utilde{S}_1 \\
\multicolumn{4}{l}{[\![\utilde{M}_{\alpha\beta} \utilde{n}_\alpha \utilde{t}_\beta]\!]_x = \utilde{M}_{\alpha\beta} \utilde{n}^1_\alpha \utilde{t}^1_\beta(x) - \utilde{M}_{\alpha\beta} \utilde{n}^2_\alpha \utilde{t}^2_\beta(x) = 0 \quad \text{for all} \quad x \in \mathcal{V}_{\partial S_1} \cup \mathcal{V}_{\partial \utilde{S}_1}}
\end{array}
\tag{6}
$$

where $\mathcal{V}_{\partial S_1}$ and $\mathcal{V}_{\partial \utilde{S}_1}$ denotes the set of corner points whose two adjacent edges do not meet at an angle of π (with corresponding normal and tangent vectors $\boldsymbol{n}_1, \boldsymbol{t}_1$ and $\boldsymbol{n}_2, \boldsymbol{t}_2$) belong to ∂S_1 and $\partial \utilde{S}_1$ respectively and conditions of junction upon Γ as follows:

$$
\begin{array}{lll}
u_{3,n} + \utilde{u}_{3,\utilde{n}} = 0, & M_{nn} + \utilde{M}_{\utilde{n}\utilde{n}} = 0 & \\
u_1 = \utilde{u}_1 \cos\theta - \utilde{u}_3 \sin\theta, & u_2 = -\utilde{u}_2, & u_3 = -\utilde{u}_1 \sin\theta - \utilde{u}_3 \cos\theta \\
N_{nn} = \utilde{N}_{\utilde{n}\utilde{n}} \cos\theta - \utilde{V}_{\utilde{l}} \sin\theta, & N_{nt} = -\utilde{N}_{\utilde{n}\utilde{t}}, & V_l = -\utilde{N}_{\utilde{n}\utilde{n}} \sin\theta - \utilde{V}_{\utilde{l}} \cos\theta
\end{array}
\tag{7}
$$

and the corner conditions on Γ:

$$
\utilde{M}_{\utilde{n}\utilde{t}} \sin\theta(x) = 0, \quad M_{tn}(x) - \utilde{M}_{\utilde{n}\utilde{t}} \cos\theta(x) = 0 \quad \text{for all } x \in \partial S_1 \cap \partial \utilde{S}_1 \cap \Gamma \tag{8}
$$

3.1 Classical variational form of clamped plates

We now introduce a space:

$$
\begin{aligned}
\mathcal{H} = \{\psi = (\boldsymbol{v}, v_3; \utilde{\boldsymbol{v}}, \utilde{v}_3)\,; \quad & (\boldsymbol{v}, v_3) \in (H^1(S))^2 \times H^2(S), \\
& (\utilde{\boldsymbol{v}}, \utilde{v}_3) \in (H^1(\utilde{S}))^2 \times H^2(\utilde{S})\},
\end{aligned}
$$

with norm

$$
\|\psi\|_{\mathcal{H}} := \left\{ \sum_\alpha \|v_\alpha\|^2_{1,S} + \|v_3\|^2_{2,S} + \sum_\alpha \|\utilde{v}_\alpha\|^2_{1,\utilde{S}} + \|\utilde{v}_3\|^2_{2,\utilde{S}} \right\},
$$

and a subspace

$$\begin{aligned}W = \{&\psi = (\boldsymbol{v}, v_3; \underset{\sim}{\boldsymbol{v}}, \underset{\sim}{v}_3) \in \mathcal{H}; v_\alpha = 0, v_3 = 0, v_{3,n} = 0 \text{ on } \partial S_0 \quad v_{3,n} + \underset{\sim}{v}_{3,\underset{\sim}{n}} = 0,\\ &\text{and } v_1 = \underset{\sim}{v}_1 \cos\theta - \underset{\sim}{v}_3 \sin\theta,\ v_2 = -\underset{\sim}{v}_2,\ v_3 = -\underset{\sim}{v}_1 \sin\theta - \underset{\sim}{v}_3 \cos\theta \text{ on } \Gamma\}\end{aligned}$$

Then in the weak form, the mathematical model of the elastic structure of coupled two plates with an angle θ is

$$\begin{cases} \text{to find } \phi = (\boldsymbol{u}, u_3; \underset{\sim}{\boldsymbol{u}}, \underset{\sim}{u}_3) \in W, \text{ such that} \\ D(\phi, \psi) = F(\psi) \quad \forall \psi \in W. \end{cases} \tag{9}$$

where

$$\begin{aligned} D(\phi, \psi) = &\int_S \frac{Ee}{1-\nu^2}\left((1-\nu) e_{\alpha\beta}(\boldsymbol{u}) e_{\alpha\beta}(\boldsymbol{v}) + \nu e_{\alpha\alpha}(\boldsymbol{u}) e_{\beta\beta}(\boldsymbol{v})\right) \mathrm{d}S \\ &+ \int_S \frac{Ee^3}{12\,(1-\nu^2)}\left((1-\nu) K_{\alpha\beta}(u_3) K_{\alpha\beta}(v_3) + \nu K_{\alpha\alpha}(u_3) K_{\beta\beta}(v_3)\right) \mathrm{d}S \\ &+ \int_{\underset{\sim}{S}} \frac{Ee}{1-\nu^2}\left((1-\nu) \underset{\sim}{e}_{\alpha\beta}(\underset{\sim}{\boldsymbol{u}}) \underset{\sim}{e}_{\alpha\beta}(\underset{\sim}{\boldsymbol{v}}) + \nu \underset{\sim}{e}_{\alpha\alpha}(\underset{\sim}{\boldsymbol{u}}) \underset{\sim}{e}_{\beta\beta}(\underset{\sim}{\boldsymbol{v}})\right) \mathrm{d}\underset{\sim}{S} \\ &+ \int_{\underset{\sim}{S}} \frac{Ee^3}{12\,(1-\nu^2)}\left((1-\nu) \underset{\sim}{K}_{\alpha\beta}(\underset{\sim}{u}_3) \underset{\sim}{K}_{\alpha\beta}(\underset{\sim}{v}_3) + \nu \underset{\sim}{K}_{\alpha\alpha}(\underset{\sim}{u}_3) \underset{\sim}{K}_{\beta\beta}(\underset{\sim}{v}_3)\right) \mathrm{d}\underset{\sim}{S} \end{aligned} \tag{10}$$

$$F(\psi) = \int_S \boldsymbol{f} \cdot \boldsymbol{u} \mathrm{d}S + \int_S f_3 u_3 \mathrm{d}S + \int_{\underset{\sim}{S}} \underset{\sim}{\boldsymbol{f}} \cdot \underset{\sim}{\boldsymbol{u}} \mathrm{d}\underset{\sim}{S} + \int_{\underset{\sim}{S}} \underset{\sim}{f}_3 \underset{\sim}{u}_3 \mathrm{d}\underset{\sim}{S} \tag{11}$$

By following lemma, the well-poseness of the variational formulation can be derived easily.

Lemma 1. *The subspace W is closed in $\mathcal{H}$.*

Lemma 2. *The bilinear form $D(\phi, \psi)$ is continuous and coercive on $W \times W$.*

4 Mixed Formulation

In the section2, the formulation of coupled plates with only the displacements as unkonwn are derived. In this section, we want to give the new mixed variational formulation. We introduce $\boldsymbol{N}$ and $\boldsymbol{M}$ for plane stress model and plate bending model for plate S and $\underset{\sim}{\boldsymbol{N}}$ and $\underset{\sim}{\boldsymbol{M}}$ for plate $\underset{\sim}{S}$.

Throughout the paper, the differential expression ∇v is only considered for functions $v \in H^1(\psi)$ and $\nabla^2 v$ is only considered for functions $v \subset H^2(\psi)$. Therefore, we define ∇v and $\nabla^2 v$ in the standard way as the vector consisting of all first-order partial derivatives and the matrix consisting of all second-order partial derivatives respectively.

Here we introduce the space:

$$\begin{aligned} V = \{\psi = (\boldsymbol{v}, v_3; \underset{\sim}{\boldsymbol{v}}, \underset{\sim}{v}_3)\,; \quad &(\boldsymbol{v}, v_3) \in (L^2(S))^2 \times L^2(S), \\ &(\underset{\sim}{\boldsymbol{v}}, \underset{\sim}{v}_3) \in (L^2(\underset{\sim}{S}))^2 \times L^2(\underset{\sim}{S})\}, \end{aligned}$$

with norm

$$\|\psi\|_V := \left\{ \sum_\alpha \|v_\alpha\|^2_{0,S} + \|v_3\|^2_{0,S} + \sum_\alpha \|\underset{\sim}{v}_\alpha\|^2_{0,\underset{\sim}{S}} + \|\underset{\sim}{v}_3\|^2_{0,\underset{\sim}{S}} \right\},$$

it easy to prove that the space V is Hilbert space with the above norm. And the space

$$\begin{aligned} V_{\text{sym}} = \{\Psi = (\boldsymbol{\tau}, \boldsymbol{\kappa}; \underset{\sim}{\boldsymbol{\tau}}, \underset{\sim}{\boldsymbol{\kappa}})\,; \quad & (\boldsymbol{\tau}, \boldsymbol{\kappa}) \in (L^2_{\text{sym}}(S))^2 \times L^2_{\text{sym}}(S), \\ & (\underset{\sim}{\boldsymbol{\tau}}, \underset{\sim}{\boldsymbol{\kappa}}) \in (L^2_{\text{sym}}(\underset{\sim}{S}))^2 \times L^2_{\text{sym}}(\underset{\sim}{S})\}, \end{aligned}$$

with norm

$$\|\Psi\|_{V_{\text{sym}}} := \left\{ \|\boldsymbol{\tau}\|^2_{0,S} + \|\boldsymbol{\kappa}\|^2_{0,S} + \|\underset{\sim}{\boldsymbol{\tau}}\|^2_{0,\underset{\sim}{S}} + \|\underset{\sim}{\boldsymbol{\kappa}}\|^2_{0,\underset{\sim}{S}} \right\},$$

Then define the operator $B := (-\nabla, \nabla^2; -\underset{\sim}{\nabla}, \underset{\sim}{\nabla}^2)$ takes value in W as follows:

$$B : D(B) = W \in V \longrightarrow V^*_{\text{sym}}$$

Obviously, $W \subset V$ and W is dense in V. So the operator B is densely defined linear operator (possibly unbounded). We can define the adjont operator $B^* := (\mathrm{Div}, \mathrm{divDiv}; \underset{\sim}{\mathrm{D}}\mathrm{iv}, \mathrm{di}\underset{\sim}{\mathrm{v}}\mathrm{Div})$ using the theory in functional analysis:

$$B^* : D(B^*) = \Sigma \in V_{\text{sym}} \longrightarrow V^*$$

The element $\Phi \in D(B^*)$ if and only if $\Phi \in V$ and there is a linear functional $G \in V^*$ such that

$$\langle B\psi, \Phi \rangle = \langle G, \psi \rangle \quad \text{for all} \quad \psi \in D(B) \tag{12}$$

where the notation means:

$$(\phi, \psi) := (\boldsymbol{u}, \boldsymbol{v})_{L^2} + (u_3, v_3)_{L^2} + (\underset{\sim}{\boldsymbol{u}}, \underset{\sim}{\boldsymbol{v}})_{L^2} + (\underset{\sim}{u}_3, \underset{\sim}{v}_3)_{L^2}$$

$$(\Phi, \Psi) := (\boldsymbol{N}, \boldsymbol{\tau})_{L^2_{\text{sym}}} + (\boldsymbol{M}, \boldsymbol{\kappa})_{L^2_{\text{sym}}} + (\underset{\sim}{\boldsymbol{N}}, \underset{\sim}{\boldsymbol{\tau}})_{L^2_{\text{sym}}} + (\underset{\sim}{\boldsymbol{M}}, \underset{\sim}{\boldsymbol{\kappa}})_{L^2_{\text{sym}}}$$

Here $\phi = (\boldsymbol{u}, u_3; \underset{\sim}{\boldsymbol{u}}, \underset{\sim}{u}_3)\,, \psi = (\boldsymbol{v}, v_3; \underset{\sim}{\boldsymbol{v}}, \underset{\sim}{v}_3) \in V$ and $\Phi = (\boldsymbol{N}, \boldsymbol{M}; \underset{\sim}{\boldsymbol{N}}, \underset{\sim}{\boldsymbol{M}})\,, \Psi = (\boldsymbol{\tau}, \boldsymbol{\kappa}; \underset{\sim}{\boldsymbol{\tau}}, \underset{\sim}{\boldsymbol{\kappa}}) \in V_{\text{sym}}$. In this case, we define $B^*\Phi = G$. Note that $\langle B^* y, x \rangle$ is well-defined for $x \in X$ and $y \in D(B^*)$ and we have in particular $\langle B^* y, x \rangle = \langle Bx, y \rangle \quad$ for all $x \in D(B), y \in D(B^*)$.

From the analysis before, the Hilbert space Σ is explicitly given by

$$\begin{aligned} \Sigma = \{\, \Phi \in V_{\text{sym}} :\ & \text{the functional } G : \psi \longrightarrow \\ & \int_S \nabla \boldsymbol{v} : \boldsymbol{N} dS + \int_S \nabla^2 v_3 : \boldsymbol{M} dS + \int_{\underset{\sim}{S}} \underset{\sim}{\nabla} \underset{\sim}{\boldsymbol{v}} : \underset{\sim}{\boldsymbol{N}} d\underset{\sim}{S} + \int_{\underset{\sim}{S}} \underset{\sim}{\nabla}^2 \underset{\sim}{v}_3 : \underset{\sim}{\boldsymbol{M}} d\underset{\sim}{S} \\ & \text{for all} \quad \psi \in W, \quad \text{is bounded w.r.t. the } V\text{-norm}\} \end{aligned}$$

equipped with the norm $\|\Phi\|_\Sigma = \left(\|\Phi\|^2_{V_{\text{sym}}} + \|B^*\Phi\|^2_{V^*} \right)^{1/2}$.

This motivates the new mixed formulation as follows: For $\boldsymbol{F} = (\boldsymbol{f}, f_3; \underset{\sim}{\boldsymbol{f}}, \underset{\sim}{f}_3) \in V^*$, find $\phi = (\boldsymbol{u}, u_3; \underset{\sim}{\boldsymbol{u}}, \underset{\sim}{u}_3) \in V$, $\Phi = (\boldsymbol{N}, \boldsymbol{M}; \underset{\sim}{\boldsymbol{N}}, \underset{\sim}{\boldsymbol{M}}) \in \Sigma$ such that

$$\begin{aligned} (\Phi, \Psi)_{C^{-1}} + \langle B^*\Psi, \phi\rangle &= 0 \qquad &&\text{for all} \quad \Psi = (\boldsymbol{\tau}, \boldsymbol{\kappa}; \underset{\sim}{\boldsymbol{\tau}}, \underset{\sim}{\boldsymbol{\kappa}}) \in \Sigma \\ \langle B^*\Phi, \psi\rangle &= -(\boldsymbol{F}, \psi) &&\text{for all} \quad \psi = (\boldsymbol{v}, v_3; \underset{\sim}{\boldsymbol{v}}, \underset{\sim}{v}_3) \in V \end{aligned} \tag{13}$$

where

$$(\Phi, \Psi)_{C^{-1}} = (C_1^{-1}\boldsymbol{N}, \boldsymbol{\tau})_{L^2_{\mathrm{sym}}(S)} + (C_2^{-1}\boldsymbol{M}, \boldsymbol{\kappa})_{L^2_{\mathrm{sym}}(S)} + (C_1^{-1}\underset{\sim}{\boldsymbol{N}}, \underset{\sim}{\boldsymbol{\tau}})_{L^2_{\mathrm{sym}}(\underset{\sim}{S})} + (C_2^{-1}\underset{\sim}{\boldsymbol{M}}, \underset{\sim}{\boldsymbol{\kappa}})_{L^2_{\mathrm{sym}}(\underset{\sim}{S})}$$

$$\langle B^*\Phi, \psi\rangle = (\mathrm{Div}\,\boldsymbol{N}, \boldsymbol{v})_{L^2(S)} + (\mathrm{divDiv}\,\boldsymbol{M}, v_3)_{L^2(S)} + (\underset{\sim}{\mathrm{Div}}\,\underset{\sim}{\boldsymbol{N}}, \underset{\sim}{\boldsymbol{v}})_{L^2(\underset{\sim}{S})} + (\underset{\sim}{\mathrm{div}}\mathrm{Div}\,\underset{\sim}{\boldsymbol{M}}, \underset{\sim}{v}_3)_{L^2(\underset{\sim}{S})}$$

$$(\boldsymbol{F}, \psi) = (\boldsymbol{f}, \boldsymbol{v})_{L^2(S)} + (f_3, v_3)_{L^2(S)} + (\underset{\sim}{\boldsymbol{f}}, \underset{\sim}{\boldsymbol{v}})_{L^2(\underset{\sim}{S})} + (\underset{\sim}{f}_3, \underset{\sim}{v}_3)_{L^2(\underset{\sim}{S})}$$

5 Wellposeness of the mixed formulation

Problem (13) is associated with a linear operator $\mathcal{A} : \Sigma \times V \longrightarrow (\Sigma \times V)^*$ given by

$$\langle \mathcal{A}(\Phi, \phi), (\Psi, \psi)\rangle = (\Phi, \Psi)_{C^{-1}} + \langle B^*\Psi, \phi\rangle + \langle B^*\Phi, \psi\rangle.$$

If the bilinear form a is symmetric, i.e., $a(\Phi, \Psi) = a(\Psi, \Phi)$, and nonnegative, i.e., $a(\Psi, \Psi) \geq 0$, which is fulfilled for (13), it is well-known that $\mathcal{A}$ is an isomorphism from $\Sigma \times V$ onto $(\Sigma \times V)^*$ if and only if the following conditions are satisfied (see, e.g., [1]) :

1. a is bounded: There is a constant $\|a\| > 0$ such that
$$|a(\Phi, \Psi)| \leq \|a\| \|\Phi\|_\Sigma \|\Psi\|_\Sigma \quad \text{for all } \Phi, \Psi \in \Sigma.$$

2. b is bounded: There is a constant $\|b\| > 0$ such that
$$|b(\Psi, \psi)| \leq \|b\| \|\Psi\|_\Sigma \|\psi\|_V \quad \text{for all } \Psi \in \Sigma, \psi \in V.$$

3. a is coercive on the kernel of b : There is a constant $\alpha > 0$ such that
$$a(\Psi, \Psi) \geq \alpha \|\Psi\|_\Sigma^2 \quad \text{for all } \Psi \in \mathrm{Ker}\, B$$
with $\mathrm{Ker}\, B = \{\Psi \in \Sigma : b(\Psi, \psi) = 0 \text{ for all } \psi \in V\}$.

4. b satisfies the inf-sup condition: There is a constant $\beta > 0$ such that
$$\inf_{0 \neq \psi \in V} \sup_{0 \neq \Psi \in \Sigma} \frac{b(\Psi, \psi)}{\|\Psi\|_\Sigma \|\psi\|_V} \geq \beta$$
we will refer to these conditions as Brezzi's conditions with constants $\|a\|, \|b\|, \alpha$, and β.

In order to verify Brezzi's conditions for (13), we need the following result on the relation between the primal problem (9) and the new mixed problem (13).

Theorem 1. *Let ϕ be the solution of the primal problem* (9) *for $\boldsymbol{F} \in V^*$. Then we have $\Phi = -\boldsymbol{C}B\phi$ and (Φ, ϕ) solves the mixed problem* (13). *Here and thereafter, the $\boldsymbol{C}B = (C_1\nabla, C_2\nabla^2; C_1\utilde{\nabla}, C_2\utilde{\nabla}^2)$.*

Proof. Since $\phi \in W$ solves (9), it follows that $\Phi \in \Sigma$ and

$$\begin{aligned}
\langle B\phi, \Phi\rangle &= \int_S \nabla\boldsymbol{u} : \boldsymbol{N} dS + \int_S \nabla^2 u_3 : \boldsymbol{M} dS + \int_{\utilde{S}} \utilde{\nabla}\utilde{\boldsymbol{u}} : \utilde{\boldsymbol{N}} d\utilde{S} + \int_{\utilde{S}} \utilde{\nabla}^2 \utilde{u}_3 : \utilde{\boldsymbol{M}} d\utilde{S} \\
&= -\left(\int_S \boldsymbol{e} : \boldsymbol{N} dS + \int_S \boldsymbol{K} : \boldsymbol{M} dS + \int_{\utilde{S}} \utilde{\boldsymbol{e}} : \utilde{\boldsymbol{N}} d\utilde{S} + \int_{\utilde{S}} \utilde{\boldsymbol{K}} : \utilde{\boldsymbol{M}} d\utilde{S} \right) \\
&= -(\boldsymbol{F}, \phi) \quad \text{for all} \quad \phi \in W
\end{aligned}$$

From the definition of the domain of the adjoint operator in (12) we obtain that $B^*\Phi = -F \in V^*$, which shows that $\boldsymbol{M} \in \boldsymbol{V}$ and that the second equation of (13) is satisfied. Using (8), we receive

$$\begin{aligned}
(B^*\Psi, \phi) &= \langle B\phi, \Psi\rangle \\
&= \int_S \nabla\boldsymbol{u} : \boldsymbol{\tau} dS + \int_S \nabla^2 u_3 : \boldsymbol{\kappa} dS + \int_{\utilde{S}} \utilde{\nabla}\utilde{\boldsymbol{u}} : \utilde{\boldsymbol{\tau}} d\utilde{S} + \int_{\utilde{S}} \utilde{\nabla}^2 \utilde{u}_3 : \utilde{\boldsymbol{\kappa}} d\utilde{S} \\
&= -\int_S C_1^{-1}\boldsymbol{N} : \boldsymbol{\tau} dS - \int_S C_2^{-1}\boldsymbol{M} : \boldsymbol{\kappa} dS \\
&\qquad - \int_{\utilde{S}} C_1^{-1}\utilde{\boldsymbol{N}} : \utilde{\boldsymbol{\tau}} d\utilde{S} - \int_{\utilde{S}} C_2^{-1}\utilde{\boldsymbol{M}} : \utilde{\boldsymbol{\kappa}} d\utilde{S}
\end{aligned} \tag{14}$$

for all $\Psi \in \Sigma$, which proves the first equation. $\square$

Theorem 2. *The mixed problem defined by* (13) *satisfies Brezzi's cenditions with the constants $\|a\| = 1/\lambda_{\min}(\mathcal{C}), \|b\| = 1, \alpha = 1/\lambda_{\max}(\mathcal{C})$, and $\beta = (1 + c)$, where $c = c'\lambda_{\max}(\mathcal{C})/\lambda_{\min}(\mathcal{C})$ and c' as in* (6).

Proof. The verification of the first three parts of Brezzi's conditions is simple and, therefore, omitted. For showing the inf-sup condition, let ϕ^ψ be the solution of the primal problem (9) with the right-hand side $\boldsymbol{F}^\psi = -(\psi, .)_V \in V^*$ for a fixed but arbitrary $\psi \in V$. From Theorem 1 it follows that $\Phi^\psi = -\boldsymbol{C}B\phi^\psi \in V_{\text{sym}}$, and (Φ^ψ, ϕ^ψ) is a solution of the corresponding mixed problem (13). From the second line of the mixed formulation (13) we obtain

$$\left(B^*\Phi^\psi, \psi\right) = (\psi, \psi)_V = \|\psi\|_V^2$$

and

$$\left\|B^*\Phi^\psi\right\|_{V^*} = \sup_{q\in V} \frac{\left(B^*\Phi^\psi, q\right)}{\|q\|_V} = \sup_{q\in V} \frac{(\psi, q)_V}{\|q\|_V} = \|\psi\|_V.$$

Using the stability estimate (6) we obtain

$$\begin{aligned}
\left\|\Phi^\psi\right\|_{V_{\text{sym}}}^2 &= \left\|C_1\nabla\boldsymbol{u}^\psi\right\|_{L^2(S)}^2 + \left\|C_2\nabla^2 u_3^\psi\right\|_{L^2(S)}^2 + \left\|C_1\utilde{\nabla}\utilde{\boldsymbol{u}}^\psi\right\|_{L^2(\utilde{S})}^2 + \left\|C_2\utilde{\nabla}^2 \utilde{u}_3^\psi\right\|_{L^2(\utilde{S})}^2 \\
&\le \lambda_{\max}(\boldsymbol{C}) D(\phi^\psi, \phi^\psi) = \lambda_{\max}(\boldsymbol{C}) \left(\boldsymbol{F}^\psi, \phi^\psi\right) \\
&\le \lambda_{\max}(\boldsymbol{C}) \left\|\boldsymbol{F}^\psi\right\|_{W^*} \left\|\phi^\psi\right\|_W \le c\left\|\boldsymbol{F}^\psi\right\|_{W^*}^2 \le c\left\|\boldsymbol{F}^\psi\right\|_{V^*}^2 = c\|\psi\|_V^2
\end{aligned}$$

with $c = c'\lambda_{\max}(\boldsymbol{C})/\lambda_{\min}(\boldsymbol{C})$. Hence,

$$\left\|\Phi^{\psi}\right\|_{\Sigma}^{2} = \left\|\Phi^{\psi}\right\|_{V_{\rm sym}}^{2} + \left\|B^{*}\Phi^{\psi}\right\|_{V^{*}}^{2} \leq (1+c)\|\psi\|_{V}^{2}.$$

Therefore,

$$\sup_{0\neq\Psi\in\Sigma} \frac{(B^{*}\Psi,\psi)}{\|\Psi\|_{\Sigma}} \geq \frac{\left(B^{*}\Phi^{\psi},\psi\right)}{\|\Phi^{\psi}\|_{\Sigma}} \geq (1+c)^{-1/2}\|\psi\|_{V},$$

which completes the proof. □

Theorem 3. *For $\boldsymbol{F} \in V^{*}$, the primal problem (9) and the mixed problem (13) are equivalent in the following sense: If ϕ solves (9), then $\Phi = -\boldsymbol{C}B\phi \in \Sigma$ and (Φ,ϕ) solves (13). Conversely, if (Φ,ϕ) solves (13), then $\phi \in W$ and solves (9).*

Proof. The first part has already been shown in Theorem 1. Since both problems are uniquely solvable the reverse direction is true as well. □

Theorem 4. *Let $\Phi = (\boldsymbol{N},\boldsymbol{M};\utilde{\boldsymbol{N}},\utilde{\boldsymbol{M}}) \in L^{2}_{\rm sym}\cap C^{1}_{\rm sym}$. Then $\Phi \in \Sigma$ if and only if*

$$\begin{array}{lll}
\operatorname{Div}\boldsymbol{N} \in (L^{2}(S))^{2} \quad \operatorname{divDiv}\boldsymbol{M} \in L^{2}(S) & & on \quad S\\
\utilde{\operatorname{Div}}\,\utilde{\boldsymbol{N}} \in (L^{2}(\utilde{S}))^{2} \quad \utilde{\operatorname{div}}\utilde{\operatorname{Div}}\,\utilde{\boldsymbol{M}} \in L^{2}(\utilde{S}) & & on \quad \utilde{S}\\
\boldsymbol{N}\boldsymbol{n} = 0 \quad M_{nn} = 0 \quad V = 0 & & on \quad \partial S_{1}\\
\utilde{\boldsymbol{N}}\utilde{\boldsymbol{n}} = 0 \quad \utilde{M}_{\utilde{n}\utilde{n}} = 0 \quad \utilde{V} = 0 & & on \quad \partial\utilde{S}_{1}\\
[\![M_{nt}]\!]_{x} = 0 \quad for\ all \quad x \in \mathcal{V}^{0}_{\partial S_{1}} \quad [\![\utilde{M}_{\utilde{n}}]\!]_{x} = 0 & & for\ all \quad x \in \mathcal{V}^{0}_{\partial\utilde{S}_{1}}\\
\utilde{N}_{nn} = N_{nn}\cos\theta - V\sin\theta \quad \utilde{V} = -N_{nn}\sin\theta - V\cos\theta & & \utilde{N}_{\utilde{n}\utilde{t}} = -N_{nt}\\
M_{nn} + \utilde{M}_{\utilde{n}\utilde{n}} = 0 & & on \quad \Gamma\\
[\![M_{nt}]\!]_{x}\sin\theta = 0 \quad [\![\utilde{M}_{\utilde{n}\utilde{t}}]\!]_{x}\sin\theta - [\![M_{nt}]\!]_{x}\cos\theta = 0 & & for\ all \quad x \in \mathcal{V}_{\Gamma}
\end{array} \tag{15}$$

Here we assume that the interface Γ is a straight line with two endpoints denoted as $\mathcal{V}_{\Gamma}$. Furthermore, if the angle $\theta \neq \pi$, then the condition can be simplified to $[\![M_{nt}]\!]_{x} = 0$ and $[\![\utilde{M}_{\utilde{n}\utilde{t}}]\!]_{x} = 0$ for all $x \in \mathcal{V}_{\Gamma}$.

Proof. For $\phi \in W$, define the functional G act on ϕ as follows:

$$\begin{aligned}
\langle G,\phi\rangle &= \langle B\phi,\Phi\rangle\\
&= -\int_{S}\boldsymbol{N} : \nabla\boldsymbol{u}dS + \int_{S}\boldsymbol{M} : \nabla^{2}u_{3}dS - \int_{\utilde{S}}\utilde{\boldsymbol{N}} : \utilde{\nabla}\utilde{\boldsymbol{u}}d\utilde{S} + \int_{\utilde{S}}\utilde{\boldsymbol{M}}\cdot\utilde{\nabla}\utilde{u}_{3}d\utilde{S}\\
&= -\left(\int_{S}\boldsymbol{N} : \boldsymbol{e}dS + \int_{S}\boldsymbol{M} : \boldsymbol{K}dS + \int_{\utilde{S}}\utilde{\boldsymbol{N}} : \utilde{\boldsymbol{e}}d\utilde{S} + \int_{\utilde{S}}\utilde{\boldsymbol{M}}\cdot\utilde{\boldsymbol{K}}d\utilde{S}\right)
\end{aligned} \tag{16}$$

Using the integration by parts for the terms related to plate S and assume that Φ satisfy some smooth condition can obtain:

$$\begin{aligned}
\int_S \boldsymbol{N} : \boldsymbol{e} dS &= -\int_S \operatorname{Div} \boldsymbol{N} \cdot \boldsymbol{u} dS + \int_{\partial S} \boldsymbol{N}\boldsymbol{n} \cdot \boldsymbol{u} ds \\
\int_S \boldsymbol{M} : \boldsymbol{K} dS &= \int_S \operatorname{Div} \boldsymbol{M} \cdot \nabla u_3 dS - \int_{\partial S} \boldsymbol{M}\boldsymbol{n} \cdot \nabla u_3 ds \\
&= -\int_S \operatorname{divDiv} \boldsymbol{M} u_3 dS + \int_{\partial S} \operatorname{Div} \boldsymbol{M} \cdot \boldsymbol{n} u_3 ds - \int_{\partial S} \boldsymbol{M}\boldsymbol{n} \cdot \nabla u_3 ds \\
&= -\int_S \operatorname{divDiv} \boldsymbol{M} u_3 dS + \int_{\partial S} \operatorname{Div} \boldsymbol{M} \cdot \boldsymbol{n} u_3 ds \\
&\quad - \int_{\partial S} M_{nn} \partial_n u_3 ds - \int_{\partial S} M_{nt} \partial_t u_3 ds \\
&= -\int_S \operatorname{divDiv} \boldsymbol{M} u_3 dS + \int_{\partial S} \operatorname{Div} \boldsymbol{M} \cdot \boldsymbol{n} u_3 ds \\
&\quad - \int_{\partial S} M_{nn} \partial_n u_3 ds - \int_{\partial S} \partial_t M_{nt} u_3 ds + [\![M_{nt} u_3]\!]_x \text{ for all } x \in \mathcal{V}_{\partial S}
\end{aligned} \tag{17}$$

Then the terms related to the plate S can be summarized as follows

$$\int_S \operatorname{Div} \boldsymbol{N} \cdot \boldsymbol{u} dS + \int_S \operatorname{divDiv} \boldsymbol{M} u_3 dS + bd1 + bd2 + bd3 + bd4 \tag{18}$$

where the simplified symbol means using $\partial S = \partial S_0 \cup \partial S_1 \cup \Gamma$

$$\begin{aligned}
bd1 &= -\int_{\partial S} \boldsymbol{N}\boldsymbol{n} \cdot \boldsymbol{u} ds \\
&= -\int_{\partial S_0} \boldsymbol{N}\boldsymbol{n} \cdot \boldsymbol{u} ds - \int_{\partial S_1} \boldsymbol{N}\boldsymbol{n} \cdot \boldsymbol{u} ds - \int_{\Gamma} \boldsymbol{N}\boldsymbol{n} \cdot \boldsymbol{u} ds \\
bd2 &= -\int_{\partial S} M_{nn} \partial_n u_3 ds \\
&= -\int_{\partial S_0} M_{nn} \partial_n u_3 ds - \int_{\partial_n S_1} M_{nn} \partial_n u_3 ds - \int_{\Gamma} M_{nn} \partial_n u_3 ds \\
bd3 &= -\int_{\partial S} V u_3 ds \\
&= -\int_{\partial S_0} V u_3 ds - \int_{\partial S_1} V u_3 ds - \int_{\Gamma} V u_3 ds \\
bd4 &= [\![M_{nt} u_3]\!]_x = \begin{cases} 0, & x \in \partial S_0 \cap \mathcal{V}_{\partial S} \\ [\![M_{nt}]\!]_x u_3, & otherwise \end{cases}
\end{aligned} \tag{19}$$

The same procedure can be used for the plate $\underset{\sim}{S}$ easily to get

$$\int_{\underset{\sim}{S}} \underset{\sim}{\operatorname{Div}} \underset{\sim}{\boldsymbol{N}} \cdot \underset{\sim}{\boldsymbol{u}} d\underset{\sim}{S} + \int_{\underset{\sim}{S}} \underset{\sim}{\operatorname{div}}\operatorname{Div} \underset{\sim}{\boldsymbol{M}} \underset{\sim}{u}_3 d\underset{\sim}{S} + b\underset{\sim}{d}1 + b\underset{\sim}{d}2 + b\underset{\sim}{d}3 + b\underset{\sim}{d}4 \tag{20}$$

where the simplified symbol means using $\partial \utilde{S} = \partial \utilde{S}_1 \cup \Gamma$

$$\begin{aligned}
b\utilde{d}1 &= -\int_{\partial \utilde{S}_1} \utilde{\boldsymbol{N}}\utilde{\boldsymbol{n}} \cdot \utilde{\boldsymbol{u}} d\utilde{s} - \int_{\Gamma} \utilde{\boldsymbol{N}}\utilde{\boldsymbol{n}} \cdot \utilde{\boldsymbol{u}} d\utilde{s} \\
b\utilde{d}2 &= -\int_{\partial \utilde{S}_1} \utilde{M}_{\utilde{n}\utilde{n}} \partial \utilde{u}_3 d\utilde{s} - \int_{\Gamma} \utilde{M}_{\utilde{n}\utilde{n}} \partial \utilde{u}_3 d\utilde{s} \\
b\utilde{d}3 &= -\int_{\partial \utilde{S}_1} \utilde{V} \utilde{u}_3 d\utilde{s} - \int_{\Gamma} \utilde{V} \utilde{u}_3 d\utilde{s} \\
b\utilde{d}4 &= \llbracket \utilde{M}_{\utilde{n}\utilde{t}} \utilde{u}_3 \rrbracket_x \quad \text{for all} \quad x \in \mathcal{V}_{\partial \utilde{S}}
\end{aligned} \tag{21}$$

Combine the terms with S and $\utilde{S}$ together, and note that the integral in Γ in the contrary direction because the different local coordinate $\boldsymbol{t} = -\utilde{\boldsymbol{t}}$, then

$$\begin{aligned}
bd1 + bd3 + b\utilde{d}1 + b\utilde{d}3 = &-\int_{\partial S_1} \boldsymbol{N}\boldsymbol{n} \cdot \boldsymbol{u} ds - \int_{\partial S_1} V u_3 ds \\
&- \int_{\partial \utilde{S}_1} \utilde{\boldsymbol{N}}\utilde{\boldsymbol{n}} \cdot \utilde{\boldsymbol{u}} d\utilde{s} - \int_{\partial \utilde{S}_1} \utilde{V} \utilde{u}_3 d\utilde{s} \\
&- \int_{\Gamma} \boldsymbol{N}\boldsymbol{n} \cdot \boldsymbol{u} + V u_3 ds - \int_{\Gamma} \utilde{\boldsymbol{N}}\utilde{\boldsymbol{n}} \cdot \utilde{\boldsymbol{u}} + \utilde{V} \utilde{u}_3 d\utilde{s}
\end{aligned} \tag{22}$$

Because the relationship between the two local coordinate, we have

$$u_1 = \utilde{u}_1 \cos\theta - \utilde{u}_3 \sin\theta \quad u_2 = -\utilde{u}_2 \quad u_3 = -\utilde{u}_1 \sin\theta - \utilde{u}_3 \cos\theta$$

then if the boundary condition (15), we have:

$$\langle G, \phi \rangle = \int_S \mathrm{Div}\, \boldsymbol{N} \cdot \boldsymbol{u} dS + \int_S \mathrm{divDiv}\, \boldsymbol{M} u_3 dS + \int_{\utilde{S}} \utilde{\mathrm{D}}\mathrm{iv}\, \utilde{\boldsymbol{N}} \cdot \utilde{\boldsymbol{u}} d\utilde{S} + \int_{\utilde{S}} \mathrm{di}\utilde{\mathrm{v}}\mathrm{Div}\, \utilde{\boldsymbol{M}} \utilde{u}_3 d\utilde{S}$$

which is obviously bounded w.r.t. the V-norm. Hence $\Phi \in \Sigma$.

On the other hand, if $\Phi \in \Sigma$, then the functional G given by (16) is bounded w.r.t. the V-norm. For $\phi \in W$ we obtain

$$\langle G, \phi \rangle = \int_S \mathrm{Div}\, \boldsymbol{N} \cdot \boldsymbol{u} dS + \int_S \mathrm{divDiv}\, \boldsymbol{M} u_3 dS + \int_{\utilde{S}} \utilde{\mathrm{D}}\mathrm{iv}\, \utilde{\boldsymbol{N}} \cdot \utilde{\boldsymbol{u}} d\utilde{S} + \int_{\utilde{S}} \mathrm{di}\utilde{\mathrm{v}}\mathrm{Div}\, \utilde{\boldsymbol{M}} \utilde{u}_3 d\utilde{S} \tag{23}$$

Note that all expressions in (23) are continuous in v w.r.t. the V-norm. One can show that W is dense in V w.r.t. the V-norm. Then it follows that (23) is valid for all $\phi \in W$. This implies together with (16) that all the boundary integral must be zero. From this the boundary conditions in (15) follow by standard arguments. $\square$

6 Conclusion

7 Appendix

7.1 Equations

References

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