Nonlinearity

Lower semicontinuity of a non-hyperbolic attractor for the viscous Cahn - Hilliard equation

To cite this article: C M Elliott and I N Kostin 1996 *Nonlinearity* **9** 687

Nonlinearity **9** (1996) 687–702. Printed in the UK

Lower semicontinuity of a non-hyperbolic attractor for the viscous Cahn–Hilliard equation

C M Elliott† and I N Kostin‡§

† Centre for Mathematical Analysis and Its Applications, University of Sussex, Brighton BN1 9QH, UK

‡ Steklov Mathematical Institute, St.Petersburg Branch, 27 Fontanka, St.Petersburg 191011, Russia.

Received 1 June 1994
Recommended by P Constantin

Abstract. The unstable invariant set in the neighbourhood of a non-hyperbolic fixed point of a nonlinear operator is studied. Lower semicontinuity of the attractor for a gradient system having non-hyperbolic stationary points is proved. The result is applied to the semigroup generated by the viscous Cahn-Hilliard equation.

AMS classification scheme numbers: 35K35, 65N25, 65N35, 65M99

Introduction

The problem of lower semicontinuity of attractors for semigroups generated by evolution PDEs was first studied in [BV] and [HR]. The basic assumptions of the results proved in these papers, as well as of other similar results [Hu, KK, K1, S, see also further references in S] are (1) existence of the Lyapunov functional for the semigroup and (2) hyperbolicity of its stationary points.

The first result breaking this tradition was obtained in [K2], where the attractor for the Chafee–Infante problem in the case of the non-hyperbolic zero stationary point is proved to be lower-semicontinuous under a certain kind of perturbation. In the present paper we refine the technique used in [K2] and prove an abstract theorem applicable to a class of PDEs.

In section 1 we briefly recall the necessary definitions and basic facts of the theory of attractors for systems having a Lyapunov functional. In section 2 we prove a simple lemma reducing the question about lower-semicontinuity of the attractor of a semigroup to the question about lower-semicontinuity of unstable invariant sets in small neighbourhoods of the stationary points of this semigroup. Sections 3 and 4 are devoted to the study of the unstable invariant set in the neighbourhood of a non-hyperbolic fixed point of a nonlinear mapping. The main result of the first part of the paper—a theorem about lower semicontinuity of attractor having non-hyperbolic stationary points—is also proved in section 4.

§ Supported by the Royal Society and the Centre for Mathematical Analysis and Its Applications, University of Sussex.

0951-7715/96/030687+16$19.50

This result is then applied in sections 5 and 6 to the semigroup generated by the viscous Cahn–Hilliard equation in one space dimension

$$\alpha u_t + (1-\alpha)A^{-1}u_t = -Au - f(u), \qquad \alpha \in [0,1],$$
$$0 \leqslant x \leqslant \pi, \quad t \geqslant 0, \qquad u(0,\cdot) = v \in H_0^1(0,\pi), \tag{1}$$

where A is the unbounded operator in $L_2(0,\pi)$ defined on the domain $H^2(0,\pi) \cap H_0^1(0,\pi)$ by the formula $A = -d^2/dx^2$, and the function $f : \mathbb{R} \to \mathbb{R}$ belongs to a class of nonlinearities for which $f(u) = u^3 - bu$, $b \in \mathbb{R}$, is a typical representative (see section 5 for the exact description of this class). In the case $\alpha = 1$ problem (1) reduces to the Allen–Cahn model (Chafee–Infante problem), and in the case $\alpha = 0$ it gives the Cahn–Hilliard equation.

The main result of the second part of the paper (theorem 5.2) is the lower semicontinuity of the attractor $\mathcal{A}_\alpha$ of equation (1) in α.

1. Attractors of semigroups

Let X be a Banach space with norm $\|\cdot\|$, let $\mathcal{T}_+$ be either $\mathbb{R}_+ = [0,+\infty)$ or $t_0\mathbb{Z}_+ = \{kt_0, k \in \mathbb{Z}_+\}$, for some $t_0 > 0$, and, finally, let $S : \mathcal{T}_+ \times X \to X$ be a continuous in its second argument mapping, which satisfies the semigroup property

$$S(t_1, S(t_2, u)) = S(t_1 + t_2, u), \qquad \forall\, t_1, t_2 \in \mathcal{T}_+, \ \ \forall\, u \in X.$$

In the case $\mathcal{T}_+ = \mathbb{R}_+$ the family of maps $\{S(t,\cdot), t \in \mathcal{T}_+\}$ is called a continuous semigroup (semiflow) on X, while in the case $\mathcal{T}_+ = t_0\mathbb{Z}_+$ it is called a discrete semigroup on X.

For each $u \in X$ and nonempty bounded sets A, $B \subset X$ define

$$\mathrm{dist}(u, B) = \inf_{v \in B} \|u - v\|, \qquad \mathrm{dist}(A, B) = \sup_{u \in A} \mathrm{dist}(u, B).$$

A set A is called an *attracting set* of the semigroup $\{S(t,\cdot), t \in \mathcal{T}_+\}$, if

$$\mathrm{dist}(S(t,B), A) \to 0 \quad \text{as} \quad t \to \infty \quad \text{for all bounded } B \subset X.$$

The minimal of all closed attracting sets is called the *global attractor* (or simply attractor) of the semigroup. Extensive literature is devoted to the problem of existence of attractors, so we do not discuss this question here, rather assuming, whenever it necessary, that the global attractor does exist. We shall need the following result about elementary properties of the attractor (see [BV2, H, L1, L2, T] for the proofs, as well as for the results about existence of attractors).

Theorem 1.1. *Let a semigroup $\{S(t,\cdot), t \in \mathcal{T}_+\}$ in a Banach space X possess a compact global attractor $\mathcal{A}$. Then*

(1) *$\mathcal{A}$ is invariant, i.e. $S(t,\mathcal{A}) = \mathcal{A}$;*
(2) *$\mathcal{A}$ is the maximal of all bounded invariant sets;*
(3) *$\mathcal{A}$ is connected.*

The following lemma implies invariance of compact attractors with respect to reduction of the semigroup $\mathcal{T}_+$.

Lemma 1.2. *Let $\mathcal{A}$ be the compact global attractor of a semigroup $\{S(t,\cdot), t \in \mathcal{T}_+\}$, and let $t_0 \in \mathcal{T}_+\backslash\{0\}$. Then $\mathcal{A}$ is the global attractor of the semigroup $\{S(t,\cdot), t \in t_0\mathbb{Z}_+\}$.*

Proof. Clearly, $\mathcal{A}$ is a closed attracting set of the semigroup $\{S(t,\cdot), t \in t_0\mathbb{Z}_+\}$. It remains to show that it is the minimal of all such sets. Suppose it is not, i.e., there exists a smaller closed attracting set $A \subset \mathcal{A}$ of this semigroup. Then $\mathrm{dist}(S(t,\mathcal{A}), A) \to 0$ as $t \to \infty$,

$t \in t_0\mathbb{Z}_+$. On the other hand, by theorem 1.1(2), $\mathcal{A}$ is an invariant set of the semigroup $\{S(t,\cdot), t \in \mathcal{T}_+\}$, and therefore $\operatorname{dist}(S(t,\mathcal{A}), A) = \operatorname{dist}(\mathcal{A}, A) > 0$. This contradiction completes the proof. □

This result trivially reduces the question about upper and lower semicontinuity of attractors for arbitrary semigroups to the same question for discrete ones, which are in most situations easier to study. From now on we can, without loss of generality, consider only discrete semigroups, i.e., assume that $\mathcal{T}_+ = t_0\mathbb{Z}_+$, or, after obvious reparametrization, $\mathcal{T}_+ = \mathbb{Z}_+$. Denoting $S(\cdot) = S(1,\cdot)$, we have $S(t,\cdot) = S^t(\cdot)$, $t \in \mathbb{Z}_+$.

Let $\mathcal{O}$ be an open subset of X. By $\mathcal{W}(S,\mathcal{O})$ we denote the *unstable invariant set* of a mapping $S : \mathcal{O} \to X$ in $\mathcal{O}$, i.e., the set of all $u \in \mathcal{O}$ for which there exists a bounded sequence $u_k \in \mathcal{O}$, $k = 0, 1, \ldots$, such that

$$u_0 = u \qquad \text{and} \qquad u_k = S(u_{k+1}), \quad k = 0, 1, \ldots . \tag{1.1}$$

The following statement follows directly from theorem 1.1 and the above definition.

Lemma 1.3. *Let a discrete semigroup $\{S^t(\cdot), t \in \mathbb{Z}_+\}$ in a Banach space X possess a compact global attractor $\mathcal{A}$. Then $\mathcal{A} = \mathcal{W}(S, X)$.*

Sets $\mathcal{W}(S,\mathcal{O})$ and attractors for semigroups having a *Lyapunov functional* allow more detailed description. For a given set $B \subset X$ and a given mapping $S : B \to X$, a continuous function $V : B \to \mathbb{R}$ is called a Lyapunov functional for S on B, if

$$u \in B, \quad S(u) \in B, \quad \text{and} \quad S(u) \neq u \qquad \Rightarrow \qquad V(S(u)) < V(u).$$

Naturally, V is a Lyapunov functional of a semigroup $\{S^t(\cdot), t \in \mathbb{Z}_+\}$ on B, if it is a Lyapunov functional for the mapping S on B.

By $Z(S)$ we shall denote the set of all fixed points of a mapping S, i.e., $Z(S) = \{z \in X : S(z) = z\}$

Lemma 1.4. *Let a semigroup $\{S^t(\cdot), t \in \mathbb{Z}_+\}$ possessing a compact global attractor $\mathcal{A}$ have a Lyapunov functional on $\mathcal{A}$ and let the set $Z(S)$ of fixed points of $S(\cdot)$ be finite. Then every bounded sequence $u_k \in X$, $k = 0, 1, \ldots$, satisfying (1.1) converges to some $z \in Z(S)$.*

See [L3] for the proof.

The following statement is a straightforward consequence of lemmas 1.3 and 1.4.

Corollary 1.5. *Let a semigroup $\{S^t(\cdot), t \in \mathbb{Z}_+\}$ possessing a compact global attractor $\mathcal{A}$ have a Lyapunov functional on $\mathcal{A}$ and let the set $Z(S)$ of fixed points of $S(\cdot)$ be finite. Then*

$$\mathcal{A} = \bigcup_{z \in Z(S)} \bigcup_{t \in \mathbb{Z}_+} S^t\big(\mathcal{W}(S, \mathcal{O}_z)\big),$$

the sets $\mathcal{O}_z$ being arbitrarily small neighbourhoods of the points z.

2. Upper and lower semicontinuity of attractors

Now we consider semigroups continuously depending on a parameter. Throughout sections 2–4 E stands for an arbitrary metric space having a non-isolated point, which we denote by α_0. Assume that for every $\alpha \in E$ a semigroup $\{S^t_\alpha(\cdot), t \in \mathbb{Z}_+\}$ possessing a compact attractor $\mathcal{A}_\alpha$ is defined.

Under rather general assumptions the set $\mathcal{A}_\alpha$ is *upper-semicontinuous* in α, i.e.,

$$\operatorname{dist}(\mathcal{A}_\alpha, \mathcal{A}_{\alpha_0}) \to 0, \qquad \alpha \to \alpha_0. \tag{2.1}$$

Relations of this sort were studied in [BV1, BV2, HLR, Hu, KK, S]. The result we cite here is proved in [KK].

Theorem 2.1. *Suppose that for every $\alpha \in E$ a semigroup $\{S^t_\alpha(\cdot), t \in \mathbb{Z}_+\}$ possesses a compact attractor $\mathcal{A}_\alpha$ and*

(1) *if $u_\alpha \in \mathcal{A}_\alpha$ and $u_\alpha \to u_{\alpha_0} \in X$ as $\alpha \to \alpha_0$, then $S_\alpha(u_\alpha) \to S_{\alpha_0}(u_{\alpha_0})$;*
(2) *the set $\bigcup_{\alpha\in E} \mathcal{A}_\alpha$ is precompact.*

Then $\operatorname{dist}(\mathcal{A}_\alpha, \mathcal{A}_{\alpha_0}) \to 0$ *as* $\alpha \to \alpha_0$.

Assumptions (1) and (2) hold in all reasonable situations arising from applications and are relatively easy to verify. Under these general assumptions the *lower semicontinuity*, i.e., the relation

$$\operatorname{dist}(\mathcal{A}_{\alpha_0}, \mathcal{A}_\alpha) \to 0, \qquad \alpha \to \alpha_0. \tag{2.2}$$

may not hold (see [KK, S] for simple counterexamples). However, for semigroups, having Lyapunov functionals, it is possible to infer the lower semicontinuity of attractors from a similar property of the unstable invariant sets in small neighbourhoods of fixed points. Given an open set $\mathcal{O} \subset X$ and a mapping $S_\alpha : \mathcal{O} \to X$ depending on a parameter $\alpha \in E$, we say that the unstable invariant set $\mathcal{W}(S_{\alpha_0}, \mathcal{O})$ has the *lower semicontinuity property* if for each $u \in \mathcal{W}(S_{\alpha_0}, \mathcal{O})$ and each sequence $\alpha_n \to \alpha_0$, $n = 1, 2, \ldots$, there exists a sequence $u_n \in \mathcal{W}(S_{\alpha_n}, \mathcal{O})$ such that $u_n \to u$ as $n \to \infty$.

Lemma 2.2. *Assume that for every $\alpha \in E$ a semigroup $\{S^t_\alpha(\cdot), t \in \mathbb{Z}_+\}$ possesses a compact attractor $\mathcal{A}_\alpha$ and*

(1) *if $u_\alpha \in \mathcal{A}_\alpha$ and $u_\alpha \to u_{\alpha_0} \in X$ as $\alpha \to \alpha_0$, then $S_\alpha(u_\alpha) \to S_{\alpha_0}(u_{\alpha_0})$;*
(2) *the semigroup $\{S^t_{\alpha_0}(\cdot), t \in \mathbb{Z}_+\}$ has a Lyapunov functional on $\mathcal{A}_{\alpha_0}$;*
(3) *the set $Z(S_{\alpha_0})$ of the fixed points of $S_{\alpha_0}(\cdot)$ is finite;*
(4) *each fixed point $z \in Z(S_{\alpha_0})$ has a neighbourhood $\mathcal{O}_z$ such that the unstable invariant set $\mathcal{W}(S_{\alpha_0}, \mathcal{O}_z)$ has the lower semicontinuity property.*

Then $\operatorname{dist}(\mathcal{A}_{\alpha_0}, \mathcal{A}_\alpha) \to 0$ *as* $\alpha \to \alpha_0$.

Proof. Assume the contrary: let $\operatorname{dist}(\mathcal{A}_{\alpha_0}, \mathcal{A}_{\alpha_n}) \geqslant \delta > 0$ for some sequence $\alpha_n \to \alpha_0$. Then $\operatorname{dist}(u_n, \mathcal{A}_{\alpha_n}) \geqslant \delta/2$ for some sequence $u_n \in \mathcal{A}_{\alpha_0}$. By the compactness of $\mathcal{A}_{\alpha_0}$ the sequence u_n contains a subsequence converging to some $u_0 \in \mathcal{A}_{\alpha_0}$. One then easily sees that

$$\operatorname{dist}(u_0, \mathcal{A}_{\alpha_n}) \geqslant \delta/4 \tag{2.3}$$

for all n sufficiently large for the inequality $\|u_n - u_0\| \leqslant \delta/4$ to hold. On the other hand, by corollary 1.5, there exists $z \in Z(S_{\alpha_0})$ such that $u_0 = S^t_{\alpha_0}(\bar{u}_0)$ for some $\bar{u}_0 \in \mathcal{W}(S_{\alpha_0}, \mathcal{O}_z)$ and some $t \in \mathbb{Z}_+$. By assumption (4), $\bar{u}_0 = \lim_{n\to\infty} \bar{u}_n$ for some sequence $\bar{u}_n \in \mathcal{W}(S_{\alpha_n}, \mathcal{O}_z)$. Denote $\tilde{u}_n = S^t_{\alpha_n}(\bar{u}_n)$. By corollary 1.5, $\tilde{u}_n \in \mathcal{A}_{\alpha_n}$. Besides, assumption (1) implies that $u_0 = \lim_{n\to\infty} \tilde{u}_n$, and therefore

$$\operatorname{dist}(u_0, \mathcal{A}_{\alpha_n}) \leqslant \|u_0 - \tilde{u}_n\| \to 0,$$

which is in contradiction with (2.3). □

Assumptions (1)–(3) of lemma 2.2 can be directly verified for a class of semigroups arising from applications. To verify assumption (4) we need some more abstract theory, which is presented in the following two sections.

3. Unstable invariant sets

In this section we study the unstable invariant set of a nonlinear mapping in the neighbourhood of its fixed point. We shall need the following notation. For Banach spaces

X, Y and an open set $\mathcal{O} \subset X$ by $C(\mathcal{O}, Y)$ and $C^1(\mathcal{O}, Y)$ we denote the spaces of continuous bounded mappings and continuously differentiable mappings $\phi : \mathcal{O} \to Y$ with the norms $\|\phi\|_C = \sup_{u\in\mathcal{O}} \|\phi(u)\|$ and $\|\phi\|_{C^1} = \sup_{u\in\mathcal{O}} \|\phi(u)\| + \sup_{u\in\mathcal{O}} \|\phi'(u)\|$, respectively. For every Lipschitz mapping $\phi : \mathcal{O} \to Y$ define its Lipschitz constant

$$\mathrm{Lip}(\phi) = \sup_{u,u'\in\mathcal{O},\ u\neq u'} \frac{\|\phi(u) - \phi(u')\|}{\|u - u'\|},$$

and denote by $\mathrm{Lip}(\mathcal{O}, Y)$ the space of Lipschitz mappings $\phi : \mathcal{O} \to Y$ with the norm $\|\phi\|_{\mathrm{Lip}} = \|\phi\|_C + \mathrm{Lip}(\phi)$.

Without loss of generality we shall assume in this section that the fixed point under consideration is zero. We now introduce a new sort of invariant sets, which play an essential role in the study of the sets $\mathcal{W}(S, \mathcal{O})$. Given a neighbourhood $\mathcal{O}$ of zero, a mapping $S : \mathcal{O} \to X$, $S(0) = 0$, and a number $\lambda \geqslant 1$, denote by $\mathcal{W}_\lambda(S, \mathcal{O})$ the *strongly unstable invariant set* of S, i.e., the set of all $u \in \mathcal{O}$ for which there exists a sequence $u_k \in \mathcal{O}$, $k = 0, 1, \ldots$, such that

$$\begin{aligned} &u_0 = u; \\ &u_k = S(u_{k+1}), \qquad k = 0, 1, \ldots; \\ &\|u_k\| = o(\lambda^{-k}). \end{aligned}$$

Theorem 3.1 follows from a result due to J C Wells [W] and provides sufficient conditions under which the set $\mathcal{W}_\lambda(S, \mathcal{O})$ for $\lambda > 1$ admits explicit description.

Theorem 3.1. *Let P and $Q = I - P$ be bounded projections in a Banach space X. Let the norm in X satisfy $\|\cdot\| = \max\{\|P\cdot\|, \|Q\cdot\|\}$ (which can always be obtained by appropriate equivalent renormalization). Assume PX and QX are invariant subspaces of a bounded linear operator $L : X \to X$. Assume also that $\|L|_{QX}\| \leqslant 1$, while the operator $L|_{PX}$ has a bounded inverse and $\|L|_{PX}^{-1}\| = (2\lambda - 1)^{-1}$ for some $\lambda > 1$. Let $\mathcal{O} = \{u \in X : \|u\| < r\}$ for some $r > 0$ and let a mapping $\Phi \in \mathrm{Lip}(\mathcal{O}, X)$ satisfy $\Phi(0) = 0$ and $\mathrm{Lip}(\Phi) \equiv \sigma < (2\lambda - 1)^{-1}(\lambda - 1)$. Then the strongly unstable invariant set of the mapping $S(\cdot) = L + \Phi(\cdot)$ is given by*

$$\mathcal{W}_\lambda(S, \mathcal{O}) = \{y + g(y),\ \ y \in P\mathcal{O}\},$$

where the function $g \in \mathrm{Lip}(P\mathcal{O}, QX)$ satisfies $g(0) = 0$ and $\mathrm{Lip}(g) \leqslant \sigma(\lambda - 1)^{-1} < 1$. Besides, g continuously depends on $\Phi \in \mathrm{Lip}(\mathcal{O}, X)$ in the $C(P\mathcal{O}, QX)$-norm . If $\Phi \in C^1(\mathcal{O}, X)$, then $g \in C^1(P\mathcal{O}, QX)$. If $\Phi'(0) = 0$, then $g'(0) = 0$.

Remark. Assume X is a Hilbert space and L is a non-negative self-adjoint operator (which is often the case in applications). Then one can easily see that existence of P and Q satisfying the assumptions of theorem 3.1 is equivalent to the fact that the interval $(1, 2\lambda - 1)$ does not meet the spectrum of L for some $\lambda > 1$. Such λ always exists if L is compact. See also [HPS] for more general results of this kind for operators in a Banach space.

Clearly, $\mathcal{W}_\lambda(S, \mathcal{O}) \subset \mathcal{W}_1(S, \mathcal{O}) \subset \mathcal{W}(S, \mathcal{O})$ for all $\lambda \geqslant 1$. The equality $\mathcal{W}_1(S, \mathcal{O}) = \mathcal{W}(S, \mathcal{O})$ is easy to verify if the mapping S has a Lyapunov functional on $\mathcal{W}(S, \mathcal{O})$. In this section we study additional assumptions, under which the equality $\mathcal{W}_\lambda(S, \mathcal{O}) = \mathcal{W}_1(S, \mathcal{O})$ holds.

Lemma 3.2. *Let the assumptions of theorem 3.1 hold and let there exist a norm $\|\cdot\|^*$ on QX satisfying $\|u\|^* \leqslant C\|u\|$, $\forall u \in QX$, for some $C > 0$, such that*

$$\|QS(v + w) - QS(v)\|^* \leqslant \|w\|^* - \beta\|PS(v + w) - PS(v)\|, \tag{3.1}$$

where $\beta \geqslant C\sigma(\lambda-1)^{-1}$, for all $v \in \mathcal{W}_\lambda(S,\mathcal{O})$ and all $w \in QX$ satisfying $v+w \in \mathcal{O}$. Then $\mathcal{W}_\lambda(S,\mathcal{O}) = \mathcal{W}_1(S,\mathcal{O})$.

Proof. We have to show that $\mathcal{W}_\lambda(S,\mathcal{O}) \supset \mathcal{W}_1(S,\mathcal{O})$. Assume the contrary: let there exist a point $u \in \mathcal{O}\backslash\mathcal{W}_\lambda(S,\mathcal{O})$ such that $u = u_0$ for some sequence $u_k \in \mathcal{O}$, $k = 0, 1, \ldots$, satisfying $u_{k-1} = S(u_k)$ and converging to zero. Obviously, $u_k \in \mathcal{O}\backslash\mathcal{W}_\lambda(S,\mathcal{O})$ for all k (otherwise $u \in \mathcal{W}_\lambda(S,\mathcal{O})$). In the rest of the proof we show that such a sequence cannot exist. Denote $v_k = Pu_k + g(Pu_k)$, where the function $g \in C(P\mathcal{O}, QX)$ is that provided by theorem 3.1, i.e., $\mathcal{W}_\lambda(S,\mathcal{O}) = \{y + g(y),\ y \in P\mathcal{O}\}$. Define also $w_k = u_k - v_k = Qu_k - g(Pu_k) \in QX$, and $v'_{k-1} = S(v_k)$. For all k sufficiently large, $v'_{k-1} \in \mathcal{O}$ and therefore $v'_{k-1} \in \mathcal{W}_\lambda(S,\mathcal{O})$, since $v_k \in \mathcal{W}_\lambda(S,\mathcal{O})$. This implies that $Qv'_{k-1} = g(Pv'_{k-1})$. Using this fact, the triangle inequality, and assumption (3.1), we have

$$\begin{aligned}
\|w_{k-1}\|^* &= \|Qu_{k-1} - g(Pu_{k-1})\|^* \\
&\leqslant \|Qu_{k-1} - Qv'_{k-1}\|^* + \|g(Pu_{k-1}) - g(Pv'_{k-1})\|^* \\
&\leqslant \|QS(u_k) - QS(v_k)\|^* + C\mathrm{Lip}(g)\|Pu_{k-1} - Pv'_{k-1}\| \\
&\leqslant \|QS(u_k) - QS(v_k)\|^* + C\sigma(\lambda-1)^{-1}\|PS(u_k) - PS(v_k)\| \\
&\leqslant \|w_k\|^*,
\end{aligned}$$

for all k sufficiently large, and hence the sequence $\|w_k\|^*$ is bounded away from zero. On the other hand, both u_k and v_k converge to zero and therefore $\|w_k\|^* \leqslant C\|w_k\| \to 0$ as $k \to \infty$. This contradiction completes the proof. □

Remark. The proof of the lemma makes the geometric sense of condition (3.1) more or less transparent. This condition implies that the distance between a point $u \in \mathcal{O}$ and its projection onto $\mathcal{W}_\lambda(S,\mathcal{O})$, which is naturally defined by $v = Pu + g(Pu)$, does not increase along the trajectory of the operator S.

Another remark. The following example shows that assumption (3.1) involves a sort of sign condition on Φ. Let X be the space $\mathbb{R}^2$ of pairs (x, y) and let $L(x, y) = (2x, y)$, $P(x, y) = (x, 0)$, $Q(x, y) = (0, y)$, $\Phi((x, y)) = (\phi(x, y), -y^3)$, where $\phi(0, 0) = \phi_x(0, 0) = \phi_y(0, 0) = 0$. One can see that in this case the assumptions of theorem 3.1 hold for any $\sigma > 0$ and for the neighbourhood $\mathcal{O} = \{(x, y) \in X, |x| < r, |y| < r\}$, provided r is sufficiently small, and that $\mathcal{W}_\lambda(S,\mathcal{O}) = \{(x, 0), |x| < r\}$. For this situation condition (3.1) reads

$$|w - w^3| \leqslant |w| - \beta|\phi(x, w) - \phi(x, 0)|,$$

where β can be arbitrarily small. This is true if $|\phi(x, w) - \phi(x, 0)| \leqslant \beta^{-1}|w|^3$. On the other hand, for $\Phi((x, y)) = (\phi(x, y), y^3)$ condition (3.1) reads

$$|w + w^3| \leqslant |w| - \beta|\phi(x, w) - \phi(x, 0)|,$$

which is certainly never true.

One more remark. Suppose, in addition to the assumptions of theorem 3.1, the inequality $\|L|_{QX}\| < 1$ holds (for a non-negative self-adjoint operator L in a Hilbert space X this is true if and only if 1 is not a point of the spectrum of L). Then it is easy to show that the estimate (3.1) holds for $\|\cdot\|^* = \|\cdot\|$ provided σ is sufficiently small (see corollary 4.4 for details), and hence, by lemma 3.2, $\mathcal{W}_\lambda(S,\mathcal{O}) = \mathcal{W}_1(S,\mathcal{O})$. In the case $\|L|_{QX}\| < 1$ the zero fixed point of the mapping S is called *hyperbolic*. Our aim is to study the unstable invariant set in the neighbourhood of a *non-hyperbolic* fixed point.

The main result of this section is

Theorem 3.3. *Let $\mathcal{O}$ be a neighbourhood of* 0 *in a Banach space X, and let* 0 *be a fixed point of a mapping $S \in C^1(\mathcal{O}, X)$. Assume that P and $Q = I - P$ are bounded projections onto invariant subspaces of the operator $S'(0)$ and that $\|S'(0)|_{QX}\| \leqslant 1$, while the operator $S'(0)|_{PX}$ is invertible and $\|S'(0)|_{PX}^{-1}\| < 1$. Assume also that there exist a norm $\|\cdot\|^*$ on QX satisfying $\|u\|^* \leqslant C\|u\|$, $\forall u \in QX$, and positive numbers β and γ such that*

$$\|QS(v+w) - QS(v)\|^* \leqslant \|w\|^* - \beta\|PS(v+w) - PS(v)\|$$

for all $v \in \mathcal{O}$ and $w \in QX$ satisfying $v + w \in \mathcal{O}$ and $\|Qv\| \leqslant \gamma\|Pv\|$. Then there exists a neighbourhood $\mathcal{O}' \subset \mathcal{O}$ of zero, such that

$$\mathcal{W}_1(S, \mathcal{O}') = \{y + g(y),\ \ y \in P\mathcal{O}'\},$$

for some function $g \in C^1(P\mathcal{O}', QX)$ satisfying $g(0) = 0$ and $g'(0) = 0$.

The proof can be easily obtained by the proof of theorem 4.1, and so we omit it.

4. Lower semicontinuity property

We proceed with the study of the lower semicontinuity property of the unstable invariant set of a mapping S_α depending on a parameter $\alpha \in E$.

Theorem 4.1. *Let $\mathcal{O}$ be a neighbourhood of zero in a Banach space X, and let $S_\alpha : \mathcal{O} \to X$ be a continuous mapping depending on a parameter $\alpha \in E$. Suppose for every $\alpha \in E$ zero is a fixed point of S_α, and the following assumptions hold:*

(1) *the mappings S_α are Lipschitz and* $\mathrm{Lip}(S_\alpha - S_{\alpha_0}) \to 0$ *as $\alpha \to \alpha_0$;*
(2) *the mapping S_{α_0} has a Lyapunov functional on $\mathcal{W}(S_{\alpha_0}, \mathcal{O})$;*
(3) *zero is the only fixed point of S_{α_0} in $\mathcal{O}$;*
(4) *$S_{\alpha_0} \in C^1(\mathcal{O}, X)$ and $\|S'_{\alpha_0}(0)|_{QX}\| \leqslant 1$, $\|S'_{\alpha_0}(0)|_{PX}^{-1}\| < 1$, where P and $Q = I - P$ are bounded projections onto invariant subspaces of the operator $S'_{\alpha_0}(0)$;*
(5) *there exist a norm $\|\cdot\|^*$ on QX satisfying $\|u\|^* \leqslant C\|u\|$, $\forall u \in QX$, and positive numbers β and γ such that*
$$\|QS_{\alpha_0}(v+w) - QS_{\alpha_0}(v)\|^* \leqslant \|w\|^* - \beta\|PS_{\alpha_0}(v+w) - PS_{\alpha_0}(v)\|$$
for all $v \in \mathcal{O}$ and $w \in QX$ satisfying $v + w \in \mathcal{O}$ and $\|Qv\| \leqslant \gamma\|Pv\|$.

Then for some neighbourhood $\mathcal{O}' \subset \mathcal{O}$ of zero the unstable invariant set $\mathcal{W}(S_{\alpha_0}, \mathcal{O}')$ has the lower semicontinuity property.

Proof. The proof is based on theorem 3.1 and lemma 3.2. Denote $L = S'_{\alpha_0}(0)$ and $\Phi_\alpha(\cdot) = S_\alpha(\cdot) - L$. Clearly, $\Phi_{\alpha_0} \in C^1(\mathcal{O}, X)$ and $\Phi'_{\alpha_0}(0) = 0$. Define a new (equivalent to the original) norm on X by $\|\cdot\|_1 = \max\{\|P\cdot\|, \|Q\cdot\|\}$. Obviously, assumption (5) holds with $\|\cdot\|$ replaced by $\|\cdot\|_1$. Define $\lambda = (\|L|_{PX}^{-1}\|^{-1} + 1)/2$ and choose $r > 0$ sufficiently small for the set $\mathcal{O}' = \{u \in X, \|u\|_1 < r\}$ to be a subset of $\mathcal{O}$, and for $\sigma_{\alpha_0} = \mathrm{Lip}_1(\Phi_{\alpha_0}|_{\mathcal{O}'})$ to satisfy $\sigma_{\alpha_0} < (2\lambda - 1)^{-1}(\lambda - 1)$ and $\sigma_{\alpha_0}(\lambda - 1)^{-1} \leqslant \gamma$ (by $\mathrm{Lip}_1(\cdot)$ we denote the Lipschitz constant calculated using the norm $\|\cdot\|_1$). Assumption (1) implies that $\|\Phi_\alpha - \Phi_{\alpha_0}\|_{\mathrm{Lip}} \to 0$ as $\alpha \to \alpha_0$ and therefore $\sigma_\alpha \equiv \mathrm{Lip}_1(\Phi_\alpha|_{\mathcal{O}'}) < (2\lambda - 1)^{-1}(\lambda - 1)$ for all α sufficiently close to α_0. Then by theorem 3.1,

$$\mathcal{W}_\lambda(S_\alpha, \mathcal{O}') = \{y + g_\alpha(y),\ \ y \in P\mathcal{O}'\},$$

for some function $g_\alpha \in C(P\mathcal{O}', QX)$ depending continuously in this class on α. By the same theorem, $g_{\alpha_0}(0) = 0$, $\mathrm{Lip}_1(g_{\alpha_0}) \leqslant \gamma$, and hence $\|Qv\|_1 \leqslant \gamma\|Pv\|_1$ for all $v \in \mathcal{W}_\lambda(S_{\alpha_0}, \mathcal{O}')$. Therefore, all assumptions of lemma 3.2 (with $\|\cdot\|$ replaced by $\|\cdot\|_1$) are fulfilled, and so $\mathcal{W}_1(S_{\alpha_0}, \mathcal{O}') = \mathcal{W}_\lambda(S_{\alpha_0}, \mathcal{O}')$. Furthermore, conditions (2), (3) and lemma 1.4 yield $\mathcal{W}_1(S_{\alpha_0}, \mathcal{O}') = \mathcal{W}(S_{\alpha_0}, \mathcal{O}')$. The continuous dependence of g_α on α implies now that $u = \lim_{\alpha\to\alpha_0} u_\alpha$ for each $u \in \mathcal{W}_\lambda(S_{\alpha_0}, \mathcal{O}') = \mathcal{W}(S_{\alpha_0}, \mathcal{O}')$ and some $u_\alpha \in \mathcal{W}_\lambda(S_\alpha, \mathcal{O}') \subset \mathcal{W}(S_\alpha, \mathcal{O}')$, which completes the proof. □

The following statement generalizes theorem 4.1 on the case of the fixed point of S_α depending on α.

Corollary 4.2. *Let $\mathcal{O}$ be an open set in a Banach space X, and let $S_\alpha : \mathcal{O} \to X$ be a continuous mapping depending on a parameter $\alpha \in E$. Suppose for each $\alpha \in E$ there is a fixed point $z_\alpha \in \mathcal{O}$ of S_α. Assume also that*

(1) *the mappings S_α are Lipschitz and $\mathrm{Lip}(S_\alpha - S_{\alpha_0}) \to 0$ as $\alpha \to \alpha_0$;*
(2) *the mapping S_{α_0} has a Lyapunov functional on $\mathcal{W}(S_{\alpha_0}, \mathcal{O})$;*
(3) *z_{α_0} is the only fixed point of S_{α_0} in $\mathcal{O}$;*
(4) *$z_\alpha \to z_{\alpha_0}$ as $\alpha \to \alpha_0$;*
(5) *$S_{\alpha_0} \in C^1(\mathcal{O}, X)$ and $S'_{\alpha_0}(\,\cdot\,)$ is uniformly continuous on $\mathcal{O}$;*
(6) *$\|S'_{\alpha_0}(z_{\alpha_0})|_{QX}\| \leqslant 1$ and $\|S'_{\alpha_0}(z_{\alpha_0})|^{-1}_{PX}\| < 1$, where P and $Q = I - P$ are bounded projections onto invariant subspaces of the operator $S'_{\alpha_0}(z_{\alpha_0})$;*
(7) *there exist a norm $\|\cdot\|^*$ on QX satisfying $\|u\|^* \leqslant C\|u\|$, $\forall u \in QX$, and positive numbers β and γ such that*

$$\|QS_{\alpha_0}(v + w) - QS_{\alpha_0}(v)\|^* \leqslant \|w\|^* - \beta\|PS_{\alpha_0}(v + w) - PS_{\alpha_0}(v)\|$$

for all $v \in \mathcal{O}$ and $w \in QX$ satisfying $v + w \in \mathcal{O}$, $\|Q(v - z_{\alpha_0})\| \leqslant \gamma\|P(v - z_{\alpha_0})\|$.

Then for some neighbourhood $\mathcal{O}' \subset \mathcal{O}$ of z_{α_0} the unstable invariant set $\mathcal{W}(S_{\alpha_0}, \mathcal{O}')$ has the lower semicontinuity property.

Proof. Choose a neighbourhood $\hat{\mathcal{O}} \subset \mathcal{O}$ of z_{α_0} such that $\{u + z_\alpha - z_{\alpha_0},\ u \in \hat{\mathcal{O}}\} \subset \mathcal{O}$ for all α. Define $\tilde{S}_\alpha(\,\cdot\,) = S_\alpha(z_\alpha + \cdot\,) - z_\alpha$ and $\tilde{\mathcal{O}} = \{u - z_{\alpha_0}, u \in \hat{\mathcal{O}}\}$. One then easily verifies assumptions (2)–(5) of theorem 4.1 for the mappings $\tilde{S}_\alpha$ and the neighbourhood $\tilde{\mathcal{O}}$ of zero. To verify assumption (1) of theorem 4.1, fix some $u, v \in \tilde{\mathcal{O}}$. Using the triangle inequality and the differentiability of S_{α_0}, we have

$$\begin{aligned}
&\|\tilde{S}_\alpha(u) - \tilde{S}_{\alpha_0}(u) - \tilde{S}_\alpha(v) + \tilde{S}_{\alpha_0}(v)\| \\
&\quad = \|S_\alpha(z_\alpha + u) - S_{\alpha_0}(z_{\alpha_0} + u) - S_\alpha(z_\alpha + v) + S_{\alpha_0}(z_{\alpha_0} + v)\| \\
&\quad \leqslant \|S_\alpha(z_\alpha + u) - S_{\alpha_0}(z_\alpha + u) - S_\alpha(z_\alpha + v) + S_{\alpha_0}(z_\alpha + v)\| \\
&\quad + \|S_{\alpha_0}(z_\alpha + u) - S_{\alpha_0}(z_{\alpha_0} + u) - S_{\alpha_0}(z_\alpha + v) + S_{\alpha_0}(z_{\alpha_0} + v)\| \\
&\quad \leqslant \mathrm{Lip}(S_\alpha - S_{\alpha_0})\|u - v\| + \mathrm{Lip}\big(S_{\alpha_0}(z_\alpha + \cdot\,) - S_{\alpha_0}(z_{\alpha_0} + \cdot\,)\big)\|u - v\|.
\end{aligned}$$

Dividing through by $\|u - v\|$ and taking supremum in $u, v \in \tilde{\mathcal{O}}$, one obtains the estimate

$$\mathrm{Lip}(\tilde{S}_\alpha - \tilde{S}_{\alpha_0}) \leqslant \mathrm{Lip}(S_\alpha - S_{\alpha_0}) + \mathrm{Lip}\big(S_{\alpha_0}(z_\alpha + \cdot\,) - S_{\alpha_0}(z_{\alpha_0} + \cdot\,)\big),$$

the first term on the right-hand side of which is small, for α close to α_0, by assumption (1), and the second is small by assumptions (4) and (5). □

We are now in the position to state the main abstract result of the paper.

Theorem 4.3. *Let for every $\alpha \in E$ the semigroup $\{S^t_\alpha(\,\cdot\,), t \in \mathbb{Z}_+\}$ in a Banach space X possess a compact attractor $\mathcal{A}_\alpha$. Suppose for some neighbourhood $\mathcal{O} \subset X$ of the closure of the set $\bigcup_\alpha \mathcal{A}_\alpha$ the following assumptions hold:*

(L1) *the semigroup $\{S^t_{\alpha_0}(\,\cdot\,), t \in \mathbb{Z}_+\}$ has a Lyapunov functional on $\mathcal{A}_{\alpha_0}$;*
(L2) *$S_{\alpha_0}|_{\mathcal{O}} \in C^1(\mathcal{O}, X)$ and $S'_{\alpha_0}(\,\cdot\,)$ is uniformly continuous on $\mathcal{O}$;*
(L3) *the mappings $S_\alpha|_{\mathcal{O}}$ are Lipschitz and $\mathrm{Lip}(S_\alpha|_{\mathcal{O}} - S_{\alpha_0}|_{\mathcal{O}}) \to 0$ as $\alpha \to \alpha_0$;*

(L4) *the set* $Z(S_{\alpha_0})$ *of the fixed points of* $S_{\alpha_0}(\,\cdot\,)$ *is finite.*

Assume also that for each $z \in Z(S_{\alpha_0})$

(L5) *for every* $\alpha \in E$ *there exists a fixed point* z_α *of the mapping* S_α *such that* $z_\alpha \to z$ *as* $\alpha \to \alpha_0$*;*

(L6) *there exists a norm* $\|\cdot\|_{(z)}$ *on* X*, equivalent to the original one, such that the operator* $S'_{\alpha_0}(z)$ *satisfies*

$$\|S'_{\alpha_0}(z)u\|_{(z)} \geqslant (1+\delta)\|u\|_{(z)} \qquad \text{for} \quad u \in P_z X,$$
$$\|S'_{\alpha_0}(z)u\|_{(z)} \leqslant \|u\|_{(z)} \qquad \text{for} \quad u \in Q_z X,$$

for some $\delta > 0$*, where* P_z *and* $Q_z = I - P_z$ *are bounded projections onto invariant subspaces of* $S'_{\alpha_0}(z)$*;*

(L7) *there exist a neighbourhood* $\mathcal{O}_z \subset \mathcal{O}$ *of* z*, a norm* $\|\cdot\|^*_{(z)}$ *on* $Q_z X$ *satisfying* $\|u\|^*_{(z)} \leqslant C\|u\|_{(z)}$, $\forall u \in Q_z X$*, and positive numbers* β *and* γ *such that*
$$\|Q_z S_{\alpha_0}(v+w) - Q_z S_{\alpha_0}(v)\|^*_{(z)} \leqslant \|w\|^*_{(z)} - \beta\|P_z S_{\alpha_0}(v+w) - P_z S_{\alpha_0}(v)\|_{(z)}$$
for all $v \in \mathcal{O}_z$ *and* $w \in Q_z X$ *satisfying* $v+w \in \mathcal{O}_z$, $\|Q_z(v-z)\|_{(z)} \leqslant \gamma\|P_z(v-z)\|_{(z)}$.

Then $\operatorname{dist}(\mathcal{A}_{\alpha_0}, \mathcal{A}_\alpha) \to 0$ *as* $\alpha \to \alpha_0$.

Remark. See section 3 for remarks on the geometric sense of conditions (L6) and (L7) as well as of condition (L6′) of corollary 4.4.

Proof of theorem 4.3. To prove the theorem we apply lemma 2.2. Its conditions (2)–(4) are direct consequences of corollary 4.2, the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_{(z)}$, and the above assumptions. To verify condition (1) of lemma 2.2, fix some $z \in Z(S_{\alpha_0})$ and choose for each α some $z_\alpha \in Z(S_\alpha)$ such that $z_\alpha \to z$ as $\alpha \to \alpha_0$. Let $u_\alpha \in \mathcal{A}_\alpha$ and $u_\alpha \to u_{\alpha_0}$. Using the triangle inequality, we have

$$\begin{aligned}
\|S_\alpha(u_\alpha) - S_{\alpha_0}(u_{\alpha_0})\| &\leqslant \|S_\alpha(u_\alpha) - S_{\alpha_0}(u_\alpha)\| + \|S_{\alpha_0}(u_\alpha) - S_{\alpha_0}(u_{\alpha_0})\| \\
&\leqslant \|S_\alpha(u_\alpha) - S_{\alpha_0}(u_\alpha) - S_\alpha(z_\alpha) + S_{\alpha_0}(z_\alpha)\| \\
&\quad + \|S_\alpha(z_\alpha) - S_{\alpha_0}(z_\alpha)\| + \operatorname{Lip}(S_{\alpha_0}|_{\mathcal{O}})\|u_\alpha - u_{\alpha_0}\| \\
&\leqslant \operatorname{Lip}(S_\alpha|_{\mathcal{O}} - S_{\alpha_0}|_{\mathcal{O}})\|u_\alpha - z_\alpha\| + \|S_\alpha(z_\alpha) - S_{\alpha_0}(z)\| \\
&\quad + \|S_{\alpha_0}(z_\alpha) - S_{\alpha_0}(z)\| + \operatorname{Lip}(S_{\alpha_0}|_{\mathcal{O}})\|u_\alpha - u_{\alpha_0}\| \\
&\leqslant \operatorname{Lip}(S_\alpha|_{\mathcal{O}} - S_{\alpha_0}|_{\mathcal{O}})\|u_\alpha - z_\alpha\| + \|z_\alpha - z\| \\
&\quad + \operatorname{Lip}(S_{\alpha_0}|_{\mathcal{O}})\|z_\alpha - z\| + \operatorname{Lip}(S_{\alpha_0}|_{\mathcal{O}})\|u_\alpha - u_{\alpha_0}\|.
\end{aligned}$$

Assumption (L3) and the facts that $z_\alpha \to z$ and $u_\alpha \to u_{\alpha_0}$ as $\alpha \to \alpha_0$ imply that the right-hand side of this inequality tends to zero and hence condition (1) of lemma 2.2 also holds. □

The following version of the above theorem shows its connection with the known results about lower semicontinuity of attractors for semigroups having only hyperbolic stationary points.

Corollary 4.4. *Let for every* $\alpha \in E$ *the semigroup* $\{S^t_\alpha(\,\cdot\,), t \in \mathbb{Z}_+\}$ *in a Banach space* X *possess a compact attractor* $\mathcal{A}_\alpha$*. Suppose for some neighbourhood* $\mathcal{O} \subset X$ *of the closure of the set* $\bigcup_\alpha \mathcal{A}_\alpha$ *assumptions (L1)–(L4) hold. Assume also that for every* $z \in Z(S_{\alpha_0})$ *either assumptions (L5)–(L7) hold, or* z *is hyperbolic, i.e.,*

(L6′) *there exists a norm* $\|\cdot\|_{(z)}$ *on* X*, equivalent to the original one, such that the operator* $S'_{\alpha_0}(z)$ *satisfies*

$$\|S'_{\alpha_0}(z)u\|_{(z)} \geqslant (1+\delta)\|u\|_{(z)} \qquad \text{for} \quad u \in P_z X,$$
$$\|S'_{\alpha_0}(z)u\|_{(z)} \leqslant (1-\delta)\|u\|_{(z)} \qquad \text{for} \quad u \in Q_z X,$$

for some $\delta > 0$, *where* P_z *and* $Q_z = I - P_z$ *are bounded projections onto invariant subspaces of* $S'_{\alpha_0}(z)$.

Then $\mathrm{dist}(\mathcal{A}_{\alpha_0}, \mathcal{A}_\alpha) \to 0$ *as* $\alpha \to \alpha_0$.

Proof. Clearly, (L6′) implies (L6). The proof of (L5) is fairly standard. Apply the operator $Q_z - S'_{\alpha_0}(z)|^{-1}_{P_z X}$ to the equation $z_\alpha = S_\alpha(z_\alpha)$. This yields an equation of the form $z_\alpha = T_\alpha(z_\alpha)$, where T_α is a contraction in a small neighbourhood of z. Existence of the required fixed point z_α is then a consequence of the implicit function theorem.

We finally show that (L6′) implies (L7). Denote $\Phi(\cdot) = S_{\alpha_0}(\cdot) - S'_{\alpha_0}(z)$ and choose the neighbourhood $\mathcal{O}_z \subset \mathcal{O}$ of z sufficiently small for $\sigma = \mathrm{Lip}(\Phi|_{\mathcal{O}_z})$ to satisfy $2p\sigma \leqslant \delta$, where $p = \max\{\|P_z\|, \|Q_z\|\}$ (the Lipschitz constant and the norms of the projections are calculated using the norm $\|\cdot\|_{(z)}$). Then we have

$$\|Q_z S_{\alpha_0}(v+w) - Q_z S_{\alpha_0}(v)\|_{(z)} \leqslant (1 - \delta + p\sigma)\|w\|_{(z)},$$
$$\|P_z S_{\alpha_0}(v+w) - P_z S_{\alpha_0}(v)\|_{(z)} \leqslant p\sigma\|w\|_{(z)},$$

and therefore assumption (L7) of theorem 4.3 holds for $\|\cdot\|^*_{(z)} = \|\cdot\|_{(z)}$ and $\beta = 1$. □

By considering the proof of corollary 4.2, one notices that the assumption about the uniform continuity of the derivatives $S'_{\alpha_0}(\cdot)$ is not necessary provided the stationary point z_α of S_α is independent of α. Thus we have the following version of our lower semicontinuity result.

Corollary 4.5. *Let for every* $\alpha \in E$ *the semigroup* $\{S^t_\alpha(\cdot), t \in \mathbb{Z}_+\}$ *in a Banach space* X *possess a compact attractor* $\mathcal{A}_\alpha$. *Suppose for some neighbourhood* $\mathcal{O} \subset X$ *of the closure of the set* $\bigcup_\alpha \mathcal{A}_\alpha$ *assumptions (L1), (L3), and (L4) hold. Let also*

(L2′) $S_{\alpha_0}|_{\mathcal{O}} \in C^1(\mathcal{O}, X)$;

(L5′) $Z(S_{\alpha_0}) \subset Z(S_\alpha)$ for every $\alpha \in E$.

Finally assume that every $z \in Z(S_{\alpha_0})$ *satisfies either assumptions (L6) and (L7), or assumption (L6′). Then* $\mathrm{dist}(\mathcal{A}_{\alpha_0}, \mathcal{A}_\alpha) \to 0$ *as* $\alpha \to \alpha_0$.

5. Viscous Cahn–Hilliard equation

We now proceed with the study of lower semicontinuity in $\alpha \in [0, 1]$ of the attractor for the semigroup generated in $H^1_0 \equiv H^1_0(0, \pi)$ by the evolutionary problem

$$B_\alpha u_t = -Au - f(u), \qquad u(0, \cdot) = v \in H^1_0 \tag{5.1}$$

Here A is the unbounded operator in $L_2(0, \pi)$ defined by the formula $A = -\mathrm{d}^2/\mathrm{d}x^2$ on the domain $H^2(0, \pi) \cap H^1_0$, and $B_\alpha = \alpha I + (1 - \alpha)A^{-1}$. The function $f : \mathbb{R} \to \mathbb{R}$ is locally C^1 and satisfies

$$f(0) = 0; \tag{5.2}$$
$$s^{-1} f(s) < f'(s), \qquad \forall s \in \mathbb{R}\backslash\{0\}; \tag{5.3}$$
$$s^{-1} f(s) \to +\infty, \qquad \text{as} \quad |s| \to \infty; \tag{5.4}$$
$$G^{-1}|s|^q \leqslant f'(s) - f'(0) \leqslant G|s|^q, \qquad |s| \leqslant s_0 \tag{5.5}$$

for some positive G, q, and s_0.

For every $\alpha \in [0, 1]$ and every $v \in H^1_0$ problem (5.1) has a unique global solution $u \in C(\mathbb{R}_+, H^1_0)$, depending on v continuously in the norm of $C([0, t]; H^1_0)$ for all positive t, and therefore we can define the semigroup of continuous solution operators $S_\alpha : \mathbb{R}_+ \times H^1_0 \to H^1_0$ by $S_\alpha(t, v) = u(t, \cdot)$.

Theorem 5.1. *Under the above assumptions the mappings $S_\alpha(t,\cdot)$ are continuously differentiable. For every $t \in \mathbb{R}_+$ and every bounded set $\mathcal{O} \in H_0^1$ the mapping $S_\alpha(t,\cdot)$ depends continuously in the norm of the class $C^1(\mathcal{O}, H_0^1)$ on α. For every $\alpha \in [0,1]$ the semigroup $S_\alpha(t,\cdot)$ has a global attractor $\mathcal{A}_\alpha \subset H_0^1$. These attractors are uniformly bounded in $H^2(0,\pi)$. They depend upper-semicontinuously on α, i.e.,* $\operatorname{dist}(\mathcal{A}_\alpha, \mathcal{A}_{\alpha_0}) \to 0$ *as $\alpha \to \alpha_0$ for every $\alpha_0 \in [0,1]$.*

Existence of the semigroups and the statements listed in theorem 5.1 are proved in [BEGSS] and [ES]. In [ES] it is also proved the lower-semicontinuity of $\mathcal{A}_\alpha$ under the assumption that all stationary solutions are hyperbolic. Here we prove lower-semicontinuity without this additional assumption.

Theorem 5.2. *Under the above assumptions the attractor $\mathcal{A}_\alpha$ lower-semicontinuously depends on α, i.e.,* $\operatorname{dist}(\mathcal{A}_{\alpha_0}, \mathcal{A}_\alpha) \to 0$ *as $\alpha \to \alpha_0$.*

By lemma 1.2, for every $\alpha \in [0,1]$ the set $\mathcal{A}_\alpha$ is also the attractor of the discrete semigroup $\{S_\alpha(t,\cdot), t \in \mathbb{Z}_+\}$. In the rest of the paper we verify assumptions of corollary 4.5 for the family of mappings $S_\alpha(\cdot) \equiv S_\alpha(1,\cdot)$ and thus prove theorem 5.2.

We first introduce the necessary notation. Throughout sections 5 and 6, as well as in the appendix, L_2 and H_0^1 denote the functional spaces $L_2(0,\pi)$ and $H_0^1(0,\pi)$ with the norms $|u|_0$ and $|u|_1 = |A^{1/2}u|_0 = |u_x|_0$, respectively. Scalar product in L_2 is denoted by $(\cdot,\cdot)$. By H^{-1} we denote the dual of H_0^1 with the norm $|u|_{-1} = |A^{-1/2}u|_0$. We shall also use the norm $|u|_{(\alpha)} = |B_\alpha^{1/2}u|_0$. Clearly,

$$|u|_{-1} \leqslant |u|_{(\alpha)} \leqslant |u|_0 \leqslant |u|_1. \tag{5.6}$$

Assumptions (L2′) and (L3) are consequences of theorem 5.1. Direct calculation shows that the continuous function $V : H_0^1 \to \mathbb{R}$ defined by $V(u) = \frac{1}{2}|u|_1^2 + (F(u), 1)$, where F is a primitive of f, satisfies $\mathrm{d}V(u)/\mathrm{d}t = -|u_t|_{(\alpha)}^2$ for every solution $u = u(t,x)$ of (5.1), and therefore is a Lyapunov functional on H_0^1 for the semigroup $S_\alpha(t,\cdot)$.

To verify the remaining assumptions of corollary 4.5 we consider the stationary problem

$$Az + f(z) = 0, \qquad z \in H_0^1. \tag{5.7}$$

Existence of a Lyapunov functional implies that the set of solutions of this equation coincides with the set $Z(S_\alpha)$ of fixed points of the mapping $S_\alpha(\cdot)$. This, in particular, means that $Z(S_\alpha)$ does not depend on α, and therefore assumption (L5′) is satisfied. Clearly, zero is always a solution to (5.7). The structure of the set of non-zero solutions essentially depends only on the parameter $b = -f'(0)$.

Theorem 5.3. *Under assumptions (5.2)–(5.4) for every $b \leqslant 1$ equation (5.7) has no non-zero solutions. For $b \in (k^2, (k+1)^2]$, $k = 1, 2, \ldots$, it has exactly $2k$ non-zero solutions. For every non-zero solution z of (5.7) zero is not an eigenvalue of the operator $A + f'(z)$ considered as an unbounded operator in L_2 with the domain H_0^1.*

The proof of the first claim of the theorem is given in [CI] (see also [He1]). The second claim is proved in [He2] (see also [K2] for a different proof).

Fix some $z \in Z(S_\alpha) \setminus \{0\}$. The technique developed in [He1] shows that $S'_\alpha(z) = \exp\bigl(-B_\alpha^{-1}(A + f'(z))\bigr)$ (see [ES] for details). The operator $S'_\alpha(z)$ is a bounded positive definite operator in L_2, and, by theorem 5.3, its spectrum does not contain 1. Choose ν sufficiently small for the interval $(1-\nu, 1+\nu)$ not to meet the spectrum of $S'_\alpha(z)$ and denote by P_z and Q_z the projections onto the invariant subspaces of $S'_\alpha(z)$ corresponding to the subsets $[1+\nu, +\infty)$ and $(0, 1-\nu]$ of the spectral axis, respectively. Define a new (equivalent to the original one) norm $\|\cdot\|_{(z)}$ on H_0^1 by $\|v\|_{(z)}^2 = (Av + f'(z)v + Cv, v)$, where the constant C is large enough for the operator $A + f'(z) + C$ to be positive definite. One then can see that assumption (L6′) holds, i.e., all non-zero fixed points of S_α are hyperbolic.

Obviously, 0 is an eigenvalue of the operator $A + f'(0) = A$ if and only if $b = k^2$, $k = 1, 2, \ldots$. Repeating the above argument we have that for the zero fixed point of the mapping $S_\alpha(\cdot)$ assumption (L6′) holds provided $b \neq k^2$. To complete the proof of theorem 5.2 it therefore remains to verify assumptions (L6) and (L7) for the zero fixed point of the mapping $S_\alpha(\cdot)$ in the case $b = k^2$, $k = 1, 2, \ldots$, which is the subject of the following section.

6. Non-hyperbolic zero equilibrium

In this section we shall use the notation $\hat{f}(s) = f(s) + bs$, where $b = -f'(0)$. Since $\hat{f}'(0) = 0$, the local Lipschitz constant of $\hat{f}$ defined by

$$\phi(r) = \sup_{|u|_1, |v|_1 \leqslant r} \frac{|\hat{f}(u) - \hat{f}(v)|_0}{|u - v|_1}$$

tends to zero as $r \to 0$.

Define P_0, Q_{01}, and Q_{02} to be the orthogonal projections onto the invariant subspaces of the operator $A - b$ corresponding to the subsets $(-\infty, 0)$, $\{0\}$, and $(0, +\infty)$ of the spectral axis, respectively, and denote $Q_0 = Q_{01} + Q_{02}$. Taking into account that $S'_\alpha(0) = \exp(-B_\alpha^{-1}(A - b))$, one can see that for some $\delta > 0$ the zero fixed point of $S_\alpha(\cdot)$ satisfies assumption (L6) for $\|\cdot\|_{(0)} = |\cdot|_1$.

Lemma 6.1. *For every $\rho > 0$ and $\gamma \in (0, 1]$ there exists $\rho_0 > 0$ such that*

$$|S_\alpha(t, v)|_1 \leqslant \rho, \tag{6.1}$$

$$|Q_0 S_\alpha(t, v)|_1 \leqslant \gamma |P_0 S_\alpha(t, v)|_1 \tag{6.2}$$

for all $t \in [0, 1]$ provided $|v|_1 \leqslant \rho_0$ and $|Q_0 v|_1 \leqslant \gamma |P_0 v|_1$.

Proof. Existence of ρ_0 sufficiently small for (6.1) to hold is a trivial consequence of the fact that the operators $S_\alpha(t, \cdot)$ are uniformly in $t \in [0, 1]$ bounded in $C^1(H_0^1)$ (see [ES]). The idea of the proof of (6.2) is borrowed from [MS]. Denote $u(t, \cdot) = S_\alpha(t, v)$, $u_1 = Q_{01} u$, $u_2 = Q_{02} u$, $u_3 = P_0 u$ and multiply the equation

$$B_\alpha u_t = -Au + bu - \hat{f}(u) \tag{6.3}$$

by $B_\alpha^{-1} A u_2$. Taking into account that the operator $(A - b)|_{Q_{02} H_0^1}$ is positive definite, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_2|_1^2 &\leqslant -\nu |B_\alpha^{-1/2} A u_2|_0^2 + |\hat{f}(u)|_1 |B_\alpha^{-1} A u_2|_{-1} \\ &\leqslant -\nu |B_\alpha^{-1/2} A u_2|_0^2 + \phi(\rho)|u|_1 |B_\alpha^{-1/2} A u_2|_0 \\ &\leqslant \frac{1}{4\nu}\phi^2(\rho)|u|_1^2\end{aligned} \tag{6.4}$$

for some positive ν. Multiplication of 6.3 by $B_\alpha^{-1} A u_1$ and $B_\alpha^{-1} A u_3$ after even simpler manipulations yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_1|_1^2 &\leqslant C\phi(\rho)|u|_1^2, \\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_3|_1^2 &\geqslant \nu|u_3|_1^2 - C\phi(\rho)|u|_1^2\end{aligned} \tag{6.5}$$

for some $C > 0$ (here we also used the fact that the subspaces $Q_{01}H_0^1$ and $P_0H_0^1$ are finite-dimensional and therefore all norms on these subspaces are equivalent). Consider the function $W : [0, 1] \to \mathbb{R}$ defined by $W(t) = |u_1|_1^2 + |u_2|_1^2 - \gamma |P_0 u|_1^2$. Clearly, $W(0) \leqslant 0$. With the help of (6.4) and (6.5) one then easily verifies that $W(t) = 0$ implies $W'(t) < 0$ for all $t \in [0, 1]$ provided ρ_0 is sufficiently small, and therefore $W(t) \leqslant 0$ for all $t \in [0, 1]$. □

To verify assumption (L7) we shall also need the following technical statement.

Lemma 6.2. *There exist positive numbers ρ, δ, and γ such that*

$$(\hat{f}(u + y) - \hat{f}(u), y) \geqslant \delta |\hat{f}(u + y) - \hat{f}(u)|_0 \, |y|_0,$$

for all $u \in H_0^1$ and $y \in Q_{01}H_0^1$ satisfying

$$|u|_1 \leqslant \rho, \qquad |u + y|_1 \leqslant \rho, \qquad |Q_0 u|_1 \leqslant \gamma |P_0 u|_1.$$

See the appendix for the proof.

Fix some $v \in H_0^1$ and $w \in Q_0H_0^1$ and denote $u(t, \cdot) = S_\alpha(t, v)$, $y(t, \cdot) = S_\alpha(t, v + w) - S_\alpha(t, v)$, $y_1 = Q_{01}y$, $y_2 = Q_{02}y$, and $y_3 = P_0 y$. Using lemmas 6.1 and 6.2, one then shows that for any $\rho > 0$ there exist positive numbers ρ_0, δ, and γ such that

$$|u|_1 \leqslant \rho, \qquad |u + y|_1 \leqslant \rho, \qquad |u + y_1|_1 \leqslant \rho, \qquad |u + y_1 + y_2|_1 \leqslant \rho, \tag{6.6}$$

$$(\hat{f}(u + y_1) - \hat{f}(u), y_1) \geqslant \delta |\hat{f}(u + y_1) - \hat{f}(u)|_0 \, |y_1|_0, \tag{6.7}$$

for all $t \in [0, 1]$, provided

$$|v|_1 < \rho_0, \qquad |v + w|_1 < \rho_0, \qquad |Q_0 v|_1 \leqslant \gamma |P_0 v|_1. \tag{6.8}$$

Assuming that v and w satisfy (6.8), multiply the equation

$$B_\alpha y_t = -Ay + by - \left(\hat{f}(u + y) - \hat{f}(u)\right) \tag{6.9}$$

by $|y_1|_0^{-1} y_1$ and take into account the relations $(-A + b)|_{Q_{01}H_0^1} = 0$, $B_\alpha|_{Q_{01}H_0^1} = \alpha + (1 - \alpha)b^{-1}$ and inequality (6.7) to obtain

$$\begin{aligned}
(\alpha + (1 - \alpha)b^{-1}) \frac{\mathrm{d}}{\mathrm{d}t} |y_1|_0 = &- |y_1|_0^{-1} (\hat{f}(u + y_1) - \hat{f}(u), y_1) \\
&- |y_1|_0^{-1} (\hat{f}(u + y_1 + y_2) - \hat{f}(u + y_1), y_1) \\
&- |y_1|_0^{-1} (\hat{f}(u + y) - \hat{f}(u + y_1 + y_2), y_1) \\
\leqslant &- \delta |\hat{f}(u + y_1) - \hat{f}(u)|_0 + \phi(\rho)|y_2|_0 + \phi(\rho)|y_3|_0.
\end{aligned} \tag{6.10}$$

Now multiply (6.9) by $B_\alpha y_2$ and notice that the operator $B_\alpha(A - b)|_{Q_{02}H_0^1}$ is positive definite. This yields

$$\begin{aligned}
|B_\alpha y_2|_0 \frac{\mathrm{d}}{\mathrm{d}t} |B_\alpha y_2|_0 = &-(B_\alpha(A - b)y_2, y_2) - (\hat{f}(u + y_1) - \hat{f}(u), B_\alpha y_2) \\
&- (\hat{f}(u + y_1 + y_2) - \hat{f}(u + y_1), B_\alpha y_2) \\
&- (\hat{f}(u + y) - \hat{f}(u + y_1 + y_2), B_\alpha y_2) \\
\leqslant &-2\nu |y_2|_0^2 + |\hat{f}(u + y_1) - \hat{f}(u)|_0 |B_\alpha y_2|_0 \\
&+ \phi(\rho)|y_2|_0 |B_\alpha y_2|_0 + \phi(\rho)|y_3|_0 |B_\alpha y_2|_0
\end{aligned}$$

for some $\nu > 0$. Dividing through by $|B_\alpha y_2|_0$ and using inequalities (5.6), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} |B_\alpha y_2|_0 \leqslant -\nu |y_2|_0 + |\hat{f}(u + y_1) - \hat{f}(u)|_0 + \phi(\rho)|y_3|_0 \tag{6.11}$$

provided ρ_0 is sufficiently small for ρ to satisfy $\phi(\rho) \leqslant \nu$. Multiplication by $B_\alpha y_3$ after similar manipulations yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|B_\alpha y_3|_0 &\leqslant -\nu|y_3|_0 + b|y_3|_0 + |\hat f(u+y_1) - \hat f(u)|_0 + \phi(\rho)|y_2|_0 \\ &\leqslant -\nu|y_3|_0 + b^2|B_\alpha y_3|_0 + |\hat f(u+y_1) - \hat f(u)|_0 + \phi(\rho)|y_2|_0\end{aligned}$$

for all ρ_0 sufficiently small (here we also used the fact that $B_\alpha^{-1} \leqslant A \leqslant b$ on $P_0H_0^1$). The last inequality implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-b^2t}|B_\alpha y_3|_0 \right] \leqslant -\nu|y_3|_0 + |\hat f(u+y_1) - \hat f(u)|_0 + \phi(\rho)|y_2|_0. \qquad (6.12)$$

Define a new norm $\|\cdot\|_{(0)}^*$ on $Q_0H_0^1$ by

$$\|\cdot\|_{(0)}^* = 2(\alpha + (1-\alpha)b^{-1})|Q_{01}\cdot|_0 + \delta|B_\alpha Q_{02}\cdot|_0$$

and choose ρ_0 even smaller, if necessary, so that $(2+\delta)\phi(\rho) \leqslant \nu\delta$. It then follows from (6.10), (6.11), and (6.12) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|Q_0y\|_{(0)}^* + \delta e^{-b^2t}|B_\alpha P_0 y|_0 \right] \leqslant 0,$$

which, after integration in t yields

$$\begin{aligned}\|Q_0S_\alpha(v+w) - Q_0S_\alpha(v)\|_{(0)}^* &\leqslant \|w\|_{(0)}^* - \delta e^{-b^2}\left|B_\alpha\big(P_0S_\alpha(v+w) - P_0S_\alpha(v)\big)\right|_0 \\ &\leqslant \|w\|_{(0)}^* - \delta b^{-3/2}e^{-b^2}|P_0S_\alpha(v+w) - P_0S_\alpha(v)|_1.\end{aligned}$$

This completes verification of assumption (L7) and thus proves theorem 5.2.

Appendix A.

Here we prove an analytical result including lemma 6.2 as a particular case.

Lemma A.1. *Let a C^1 function $\psi : \mathbb{R} \to \mathbb{R}$, $\psi(0) = \psi'(0) = 0$, satisfy*

$$G^{-1}|s|^q \leqslant \psi'(s) \leqslant G|s|^q, \qquad \forall s \in (-s_0, s_0) \qquad (A.1)$$

for some positive G, s_0, and q. Assume Π is the orthogonal projection onto a finite dimensional subspace of H_0^1 such that every function $v \in \Pi H_0^1 \setminus \{0\}$ is non-zero almost everywhere. Then there exist positive numbers ρ, δ, and γ such that inequality

$$(\psi(u') - \psi(u), u' - u) \geqslant \delta|\psi(u') - \psi(u)|_0\, |u' - u|_0$$

holds provided

$$|u'|_1 < \rho, \qquad |u|_1 < \rho, \qquad (A.2)$$

$$u' - u \in \Pi H_0^1, \qquad (A.3)$$

$$|(I-\Pi)u|_1 \leqslant \gamma|\Pi u|_1, \qquad |(I-\Pi)u'|_1 \leqslant \gamma|\Pi u'|_1. \qquad (A.4)$$

Proof. We agree to denote by C_i, $i = 1, 2, \ldots$, different positive constants depending only on ψ. Integration of assumption (A.1) in s yields

$$C_1(|s'| + |s|)^q(s'-s)^2 \leqslant (\psi(s') - \psi(s))(s'-s) \leqslant C_2(|s'| + |s|)^q(s'-s)^2 \qquad (A.5)$$

for all s' and s sufficiently close to zero. ¿From (A.5), taking into account the continuity of the embedding $H_0^1 \subset C(0,\pi)$ and the equivalence of all norms on ΠH_0^1, we have inequalities

$$C_3(\psi(u') - \psi(u), u' - u) \geqslant \int (|u'| + |u|)^q (u' - u)^2 \, dx,$$

$$|\psi(u') - \psi(u)|_0 \leqslant C_4\left(\int (|u'| + |u|)^{2q} (u' - u)^2 \, dx\right)^{\frac{1}{2}} \leqslant C_5\left(|u'|_1^q + |u|_1^q\right) |u' - u|_0$$

for all $u', u \in H_0^1$ sufficiently small. To prove the lemma it therefore remains to show that

$$C_6 \int (|u'| + |u|)^q (u' - u)^2 \, dx \geqslant \left(|u'|_1^q + |u|_1^q\right)|u' - u|_0^2. \tag{A.6}$$

Instead we prove the inequality

$$C_6 \int (|u'| + |u|)^q v^2 \, dx \geqslant \left(|u'|_1^q + |u|_1^q\right)|v|_0^2 \tag{A.7}$$

for all $v \in \Pi H_0^1$ and all $u', u \in H_0^1$ satisfying (A.4). Assume first that $u', u \in \Pi H_0^1$. Since inequality (7.5) is homogeneous in v and in pairs (u', u), it is true if and only if

$$C_6 \int (|u'| + |u|)^q v^2 \, dx \geqslant 1 \tag{A.8}$$

for all $v, u', u \in \Pi H_0^1$ satisfying $|v|_0 = 1$, $|u'|_1^q + |u|_1^q = 1$. These conditions define a compact set of triples $(v, u', u) \in \Pi H_0^1 \times \Pi H_0^1 \times \Pi H_0^1$, which we denote by Ω. The integral on left-hand side of (A.8) is a continuous functional on Ω. It is everywhere positive in Ω and therefore bounded away from zero. This proves (A.7) for $u', u \in \Pi H_0^1$.

Now consider general u', u satisfying (A.4). The triangle inequality, the Hölder inequality, and the result just proved imply

$$\begin{aligned}
\int (|u'| + |u|)^q v^2 \, dx &\geqslant C_7 \int (|\Pi u'| + |\Pi u|)^q v^2 \, dx - \int (|u' - \Pi u'| + |u - \Pi u|)^q v^2 \, dx \\
&\geqslant C_7 C_6^{-1}\left(|\Pi u'|_1^q + |\Pi u|_1^q\right) |v|_0^2 - C_8\left(|u' - \Pi u'|_1^q + |u - \Pi u|_1^q\right) |v|_0^2 \\
&\geqslant C_9\left(|u'|_1^q + |u|_1^q\right)|v|_0^2 + C_9\left(|\Pi u'|_1^q + |\Pi u|_1^q\right)|v|_0^2 \\
&\quad - C_{10}\left(|u' - \Pi u'|_1^q + |u - \Pi u|_1^q\right)|v|_0^2
\end{aligned}$$

(here we also used the fact that all norms on ΠH_0^1 are equivalent). Finally, taking into account conditions (A.4), we have

$$\int (|u'| + |u|)^q v^2 \, dx \geqslant C_9\left(|u'|_1^q + |u|_1^q\right)|v|_0^2$$

provided γ is sufficiently small. □

Proposition 6.2 now follows from lemma A.1 with $\Pi = P_0 + Q_{01}$.

References

[BV1] Babin A V and Vishik M I 1986 Unstable invariant manifolds of semigroups of nonlinear operators and their perturbations *Usp. Mat. Nauk/* **41** (Engl. transl. *Russian Math. Surveys* **41**(4) (1986) 1–41)

[BV2] Babin A V and Vishik M I 1992 *Attractors of Evolution Equations* (Amsterdam: North Holland)

[BEGSS] Bai F, Elliott C M, Gardiner A, Spence A and Stuart A M 1995 The Viscous Cahn–Hilliard equation. Part 1: Computations *Nonlinearity* **8** 131–60

[CI] Chafee N and Infante E F 1974 A bifurcation problem for a nonlinear parabolic equation *J. Appl. Anal.* **4** 17–37

[ES] Elliott C M and Stuart A M 1996 The Viscous Cahn–Hilliard equation. Part 2 *J. Diff. Eq.* in press

[H] Hale J K 1988 Asymptotic behavior of dissipative systems *Math. Surveys Monographs* vol 25 (Providence, RI: Am. Math. Soc.)

[HLR] Hale J K, Lin X B and Raugel G 1988 Upper-semicontinuity of attractors for approximations of semigroups and partial differential equations *Math. Comput.* **5** 89–123

[HR] Hale J K and Raugel G 1989 Lower-semicontinuity of attractors of gradient systems and applications *Ann. Math. Pura Appl.* **154** 281–326

[He1] Henry D 1981 Geometric theory of semilinear parabolic equations *Lecture Notes in Math.* vol 840 (Berlin: Springer)

[He2] Henry D 1985 Some infinite dimensional Morse–Smale systems defined by parabolic differential equations *J. Diff. Eq.* **59** 165–205

[HPS] Hirsch M, Pugh C and Shub M 1977 Invariant manifolds *Lecture Notes in Math.* vol 583 (Berlin: Springer)

[Hu] Humphries A R 1993 Numerical analysis of dynamical systems *PhD thesis* University of Bath

[KK] Kapitanskiĭ L V and Kostin I N 1990 Attractors of nonlinear evolution equations and their approximations *Algebra i Analiz* **2** (Engl. transl. *Leningrad Math. J.* **2**(1) (1991) 97–117)

[K1] Kostin I N 1990 A regular approach to a problem on the attractors of singularly perturbed equations *Zap. Nauchn. Semin. LOMI* **181** (Engl. transl. *J. Sov. Math.* **62** (1992) 2664–89)

[K2] Kostin I N 1995 Lower semicontinuity of a non-hyperbolic attractor *J. London Math. Soc.* **52** 568–82

[L1] Ladyzhenskaya O A 1987 On determination of minimal global attractors for the Navier–Stokes equations and other partial differential equations *Usp. Mat. Nauk* **42** (Engl. transl. *Russian Math. Surveys* **42**(6) 27–73)

[L2] Ladyzhenskaya O A 1991 *Attractors for Semigroups and Evolution Equations* (Cambridge: Cambridge University Press)

[L3] Ladyzhenskaya O A 1992 Some complements and corrections to my papers on the theory of attractors for abstract semigroups *J. Sov. Math.* **62** 2789–94

[MS] Mallet–Paret J and Sell G R 1988 Inertial manifolds for reaction-diffusion equations in higher space dimensions *J. Am. Math. Soc.* **1** 805–64

[N] Novick–Cohen A 1988 On the viscous Cahn–Hilliard equation *Material Instabilities in continuum and related mathematical problems* ed J M Ball pp 329–42

[S] Stuart A 1994 Numerical analysis of dynamical systems *Acta Numerica* 1994, 467–572 (Cambridge: Cambridge University Press)

[T] Temam R 1988 Infinite-dimensional dynamical systems in mechanics and physics *Appl. Math. Sci.* vol 68 (Berlin: Springer)

[W] Wells J C 1976 Invariant manifolds of nonlinear operators *Pacific J. Math.* **62** 285–93