ELSEVIER

Comput. Methods Appl. Mech. Engrg. 161 (1998) 365–387

Computer methods in applied mechanics and engineering

Numerical analysis of junctions between thin shells. Part 2: Approximation by finite element methods

Michel Bernadou[a,b,*], Annie Cubier[a,b]

[a] *Pôle Universitaire Léonard de Vinci, 92916 Paris La Défense Cedex, France*
[b] *INRIA, Domaine de Voluceau, B.P. 105, F-78153 Le Chesnay Cedex, France*

Received 23 June 1997

Abstract

The purpose of this work is to study the numerical analysis of junctions between thin shells. We describe the approximation by a 'pseudo-conforming' finite element method associated with the Argyris triangle, taking into account the numerical integration. Under suitable hypotheses on the integration schemes and on the data, we prove the convergence of this method, derive a priori error estimates and present some examples.

1. Introduction

The numerical analysis of junctions between thin plates was studied by Bernadou et al. [1] while Fayolle [2] thoroughly described the corresponding approximation by conforming finite element methods. Such an approximation is much more complicated in case of junctions between two general thin shells. Indeed, the transmission conditions which appear in the definition of the admissible spaces cannot be satisfied exactly in the associated discrete spaces. Thus, we consider pseudo-conforming finite element methods which are conforming everywhere except along the junction.

In the first part of this work [3], we analyze the continuous problems of junctions between two thin shells associated with an elastic or a rigid behaviour of the hinge. We start by giving the equilibrium equations of these problems and the corresponding variational formulations. We study the numerical properties of these equations and show the existence and uniqueness of the solution. We also prove that the solution of the elastic junction problem converges to the solution of the rigid junction problem when the coefficient of elastic stiffness of the hinge becomes very large.

In this second part of the work, we start in Section 2 by rewriting the continuous problems in terms of matrices and vectors, which are well adapted to the approximation by finite element methods. In Section 3, we build the discrete spaces which are associated with the Argyris triangle, and which contain the discrete junction conditions. The discretization of the transmission conditions is based on the results obtained by Zenisek [4] for nonhomogeneous boundary conditions. From Section 4, we restrict our attention to the elastic junction problem; we give the main results concerning the rigid junction problem in Section 6. Then, we state the first discrete problem taking into account the finite element approximation and the nonconformity of the method along the hinge. In this problem, we introduce an additional linear form which is void for continuous problem and which takes into account the nonconformity of the method. Then, we prove the existence and uniqueness of the

* Corresponding author.

0045-7825/98/$19.00
PII: S0045-7825(97)00327-7

solution. This result is based on the transmission of a clamped condition by the discrete junction conditions described in Section 3. Afterwards, we give an abstract error estimate which is reduced to the usual interpolation error; this is a direct consequence of the definition of the discrete problem by using an additional linear form as mentioned before. In Section 5, we study the second discrete problem which takes into account the additional effect of the numerical integration. We prove the uniform ellipticity of the bilinear forms and thus the existence and uniqueness of the solution of this problem. Next, we give criteria on the integration schemes and required regularity conditions on the data so that the finite element method converges. These conditions lead to asymptotic error estimates of the same order than the interpolation error. These results are based on local error estimates studied by Bernadou [5] and Cubier [6]. In Section 7, we illustrate previous results by some numerical tests on rigid and elastic junctions between a cylinder and a spherical end cap. In order to validate our results, we consider a corresponding test using the three-dimensional elasticity model instead of a shell model and which represents a good approximation of a rigid junction. The result of this is very close to ours obtained for rigid junction. Moreover, we again find numerically that the elastic junction becomes almost rigid when the coefficient of elastic stiffness of the hinge becomes very large.

Notations and references

In this second part, we make many references to the notations and to the results of Part 1, just by adding 'Part 1' before each reference.

2. Variational formulations in matrix form

In this section, we give new expressions in terms of vectors and matrices for the bilinear and linear forms which appear in the variational formulations of the elastic and rigid junction problems (Part 1, (3.10) and (3.24)). The bilinear form $a[.,.]$ which represents the addition of the strain energy of both shells, can be written

$$a[(\boldsymbol{u};\underset{\sim}{\boldsymbol{u}}),(\boldsymbol{v};\underset{\sim}{\boldsymbol{v}})]=\int_{\Omega}{}^{\mathrm{T}}[U][A][V]\,\mathrm{d}\xi^1\,\mathrm{d}\xi^2+\int_{\underset{\sim}{\Omega}}{}^{\mathrm{T}}[\underset{\sim}{U}][\underset{\sim}{A}][\underset{\sim}{V}]\,\mathrm{d}\underset{\sim}{\xi}^1\,\mathrm{d}\underset{\sim}{\xi}^2\,, \tag{2.1}$$

where the column matrix $[V]$ (and similarly $[U]$, $[\underset{\sim}{U}]$, $[\underset{\sim}{V}]$) is given by

$${}^{\mathrm{T}}[V]_{1\times 12}=[v_1\quad v_{1,1}\quad v_{1,2}\quad v_2\quad v_{2,1}\quad v_{2,2}\quad v_3\quad v_{3,1}\quad v_{3,2}\quad v_{3,11}\quad v_{3,12}\quad v_{3,22}]\,, \tag{2.2}$$

and where the symmetrical 12×12 matrix $[A]$ (respectively, $[\underset{\sim}{A}]$) depends only on the shell thickness e, on the mechanical characteristics of the shell and on the first, second and third partial derivatives of the application $\boldsymbol{\Phi}:\bar{\Omega}\to\bar{\mathcal{S}}$ (resp. $\underset{\sim}{\boldsymbol{\Phi}}:\bar{\underset{\sim}{\Omega}}\to\bar{\underset{\sim}{\mathcal{S}}}$) which maps $\bar{\Omega}$ (resp. $\underset{\sim}{\Omega}$) onto the middle surface $\mathcal{S}$ (resp. $\underset{\sim}{\mathcal{S}}$). Subsequently, we assume that $\boldsymbol{\Phi}\in(C^3(\bar{\Omega}))^3$ and $\underset{\sim}{\boldsymbol{\Phi}}\in(C^3(\bar{\underset{\sim}{\Omega}}))^3$.

The second bilinear form $b[.,.]$ which appears in the variational formulation of elastic junction between shells (Part 1, (3.10)) is associated with the strain energy of the hinge. We define a 24×24 matrix $[C]$ which only depends on the geometry of the hinge, and a column vector $[V\quad \underset{\sim}{V}]_{24\times 1}$ which collects the vectors $[V]$ and $[\underset{\sim}{V}]$ defined by relation (2.2). Thus, we have

$$b[(\boldsymbol{u};\underset{\sim}{\boldsymbol{u}}),(\boldsymbol{v};\underset{\sim}{\boldsymbol{v}})]=\int_{\Gamma}{}^{\mathrm{T}}[U\quad \underset{\sim}{U}][C][V\quad \underset{\sim}{V}]\,\mathrm{d}s\,. \tag{2.3}$$

Moreover, we introduce a new parameterization of the hinge Γ, as the image of a one-dimensional interval $\omega=]0,1[$ by a mapping $\underline{\boldsymbol{\Phi}}$, i.e. $\underline{\boldsymbol{\Phi}}:\bar{\omega}\to\bar{\Gamma}$. Now, we can substitute this application $\underline{\boldsymbol{\Phi}}$ into relation (2.3) to obtain

$$b[(\underline{\boldsymbol{u}};\underline{\underset{\sim}{\boldsymbol{u}}}),(\underline{\boldsymbol{v}};\underline{\underset{\sim}{\boldsymbol{v}}})]=\int_{\omega}{}^{\mathrm{T}}[\underline{U}\quad \underline{\underset{\sim}{U}}][\underline{C}][\underline{V}\quad \underline{\underset{\sim}{V}}]\,\mathrm{d}\omega\,, \tag{2.4}$$

where the underlined quantities are obtained by composition with the mapping $\underline{\boldsymbol{\Phi}}$ and are defined on the interval ω. The element $\mathrm{d}\omega$ is associated to the line element $\mathrm{d}s$ along the hinge Γ through the mapping $\underline{\boldsymbol{\Phi}}=\Sigma_{i=1}^{3}\underline{\Phi}_i(t)\boldsymbol{e}_i$:

$$\mathrm{d}s=[(\mathrm{d}x_1)^2+(\mathrm{d}x_2)^2+(\mathrm{d}x_3)^2]^{1/2}\,,$$

so that

$$\mathrm{d}\omega = [(\Phi_1'(t))^2 + (\underset{\sim}{\Phi}_2'(t))^2 + (\underset{\sim}{\Phi}_3'(t))^2]^{1/2}\, \mathrm{d}t\,.$$

The linear form $\ell[.]$ which represents the work of the external loads can be written (Part 1, (2.18) and (3.13))

$$\left.\begin{aligned} \ell[(\boldsymbol{v}; \underset{\sim}{\boldsymbol{v}})] = \int_{\Omega} {}^{\mathrm{T}}[\boldsymbol{P}][\boldsymbol{V}]\, \mathrm{d}\xi^1\, \mathrm{d}\xi^2 + \int_{\underset{\sim}{\Omega}} {}^{\mathrm{T}}[\underset{\sim}{\boldsymbol{P}}][\underset{\sim}{\boldsymbol{V}}]\, \mathrm{d}\underset{\sim}{\xi}^1\, \mathrm{d}\underset{\sim}{\xi}^2 \\ + \int_{\gamma_1} {}^{\mathrm{T}}[\boldsymbol{L}_s][\boldsymbol{V}]\, \mathrm{d}\gamma + \int_{\underset{\sim}{\gamma}_1} {}^{\mathrm{T}}[\underset{\sim}{\boldsymbol{L}}_s][\underset{\sim}{\boldsymbol{V}}]\, \mathrm{d}\underset{\sim}{\gamma}\,, \end{aligned}\right\} \tag{2.5}$$

where

$${}^{\mathrm{T}}[\boldsymbol{P}]_{1\times 12} = \sqrt{a}\,[p^1 \quad 0 \quad 0 \quad p^2 \quad 0 \quad 0 \quad p^3 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0] \tag{2.6}$$

and

$$\left.\begin{aligned} {}^{\mathrm{T}}[\boldsymbol{L}_s]_{1\times 12} = \sqrt{a_{\alpha\lambda}(g^{\alpha})'(g^{\lambda})'}\,[N^1 + b^1_{\beta}M^{\beta} \quad 0 \quad 0 \quad N^2 + b^2_{\beta}M^{\beta} \quad 0 \quad 0 \\ N^3 \quad M^1 \quad M^2 \quad 0 \quad 0 \quad 0]\,. \end{aligned}\right\} \tag{2.7}$$

In the above equations, p^i and N^i denote, respectively, the covariant components of the body force resultant, of the resultant and of the resultant moment of the surface loads while b^{β}_{α} and a are the second fundamental form and the determinant of the first fundamental form; all these quantities are referred to the middle surface $\mathcal{S}$ and, by definition, $\boldsymbol{M} = M^{\alpha}\boldsymbol{a}_{\alpha} \times \boldsymbol{a}_3$. The vectors $[\underset{\sim}{\boldsymbol{P}}]$, $[\underset{\sim}{\boldsymbol{L}}_s]$ are obtained by analogy.

3. Construction of the discrete admissible spaces

From now on, we shall assume that the domains Ω and $\underset{\sim}{\Omega}$ have polygonal boundaries. Then, we can exactly cover these domains by families of triangulations $\mathcal{T}_h$ and $\underset{\sim}{\mathcal{T}}_h$. Subsequently, we assume that these triangulations are compatible along the parts γ and $\underset{\sim}{\gamma}$ of the boundaries $\partial\Omega$ and $\partial\underset{\sim}{\Omega}$: in other words, their traces upon γ and $\underset{\sim}{\gamma}$ are the images of a one-dimensional triangulation $\underline{\mathcal{T}}_h$ of the interval ω through the mappings $\boldsymbol{F} = \boldsymbol{\Phi}_{|\Gamma}^{-1} \circ \underline{\boldsymbol{\Phi}}$ and $\underset{\sim}{\boldsymbol{F}} = \underset{\sim}{\boldsymbol{\Phi}}_{|\Gamma}^{-1} \circ \underline{\boldsymbol{\Phi}}$ (Fig. 1). From now on, for simplicity, we note $\boldsymbol{\Phi}_{|\Gamma}^{-1}$ and $\underset{\sim}{\boldsymbol{\Phi}}_{|\Gamma}^{-1}$ by $\boldsymbol{\Phi}^{-1}$ and $\underset{\sim}{\boldsymbol{\Phi}}^{-1}$. All these triangulations are assumed to be regular in the sense that

(i) There exists constants σ and $\underset{\sim}{\sigma}$ such that

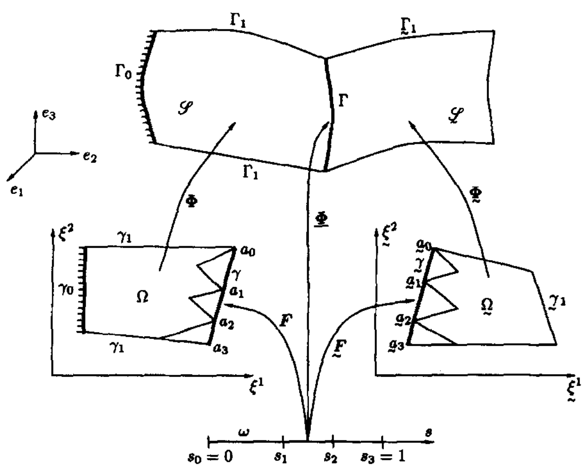
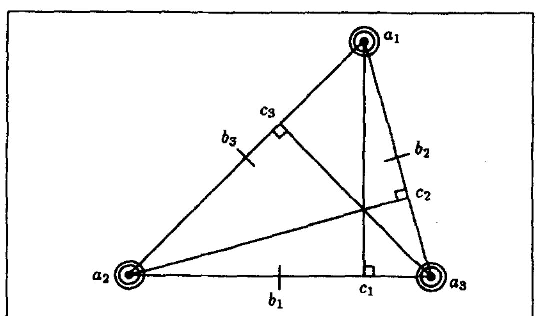


Fig. 1. Discretization of the hinge images.



$P_K = P_5(K)\,;\ \dim P_K = 21$

$\Sigma_K = \{p(a_i), Dp(a_i)(a_{i-1} - a_i),\ Dp(a_i)(a_{i+1} - a_i),\ 1 \le i \le 3,$
$D^2p(a_i)(a_{j+1} - a_{j-1})^2,\ 1 \le i, j \le 3,\ Dp(b_i)(a_i - c_i),\ 1 \le i \le 3\}$

Fig. 2. The Argyris triangle.

$$\forall K \in \mathcal{T}_h\,, \quad \frac{h_K}{\rho_K} \leqslant \sigma \quad \text{and} \quad \forall \underset{\sim}{K} \in \underset{\sim}{\mathcal{T}}_h\,, \quad \frac{h_{\underset{\sim}{K}}}{\rho_{\underset{\sim}{K}}} \leqslant \underset{\sim}{\sigma}\,, \tag{3.1}$$

where $h_K = \text{diam}(K)$, $h_{\underset{\sim}{K}} = \text{diam}(\underset{\sim}{K})$, $\rho_K = \sup\{\text{diam}(S),\ S \text{ is a ball contained in } K\}$ and $\rho_{\underset{\sim}{K}} = \sup\{\text{diam}(\underset{\sim}{S}),\ \underset{\sim}{S} \text{ is a ball contained in } \underset{\sim}{K}\}$.

(ii) Let h be a real number defined by

$$h = \sup\{\max_{K \in \mathcal{T}_h} h_K,\ \max_{\underset{\sim}{K} \in \underset{\sim}{\mathcal{T}}_h} h_{\underset{\sim}{K}},\ \max_{\underline{K} \in \underline{\mathcal{T}}_h} h_{\underline{K}}\}\,, \tag{3.2}$$

where $h_{\underline{K}} = \text{diam}(\underline{K})$. Then, we assume that

$$h \to 0\,. \tag{3.3}$$

With the triangulations $\mathcal{T}_h$ and $\underset{\sim}{\mathcal{T}}_h$, we associate the finite element spaces X_h and $\underset{\sim}{X}_h$ constructed from the Argyris triangle [7], whose definition is recalled in Fig. 2, and the spaces V_h and $\underset{\sim}{V}_h$:

$$V_h = V_{h1} \times V_{h1} \times V_{h2}\,, \qquad \underset{\sim}{V}_h = (\underset{\sim}{X}_h)^3$$

where

$$V_{h1} = \{v \in X_h;\ v = 0 \text{ along } \gamma_0\} \quad \text{and} \quad V_{h2} = \{v \in X_h;\ v = v_{,\nu} = 0 \text{ along } \gamma_0\}$$

and where ν is the outward unit normal vector to the boundary γ_0.

These definitions and those of (Part 1, (3.7) and (3.8)) lead to the inclusions

$$V_h \subset V \quad \text{and} \quad \underset{\sim}{V}_h \subset \underset{\sim}{V}\,. \tag{3.4}$$

In order to build the discrete admissible spaces, we have to discretize the junction conditions (equality of the displacements and equality of the tangential components of the rotations along the hinge), i.e. we have to express these conditions in terms of the degrees of freedom. This is a delicate step in the approximation of the continuous problems and it leads to the non-conformity of the method for the approximation of the transmission conditions along the hinge. The equality of the displacements along the hinge is a condition which appears for the elastic or the rigid junction problems as well. Thus, we begin by studying the discretization of this condition.

3.1. The discrete admissible space for the elastic junction problem

First, let us recall the condition of continuity of the displacement along the hinge (Part 1, $(2.31)_1$):

$$\boldsymbol{u}(\eta) = \underset{\sim}{\boldsymbol{u}}(\underset{\sim}{\eta})\,, \quad \forall \eta \in \gamma\,, \quad \forall \underset{\sim}{\eta} \in \underset{\sim}{\gamma} \quad \text{such that } \boldsymbol{\Phi}(\eta) = \underset{\sim}{\boldsymbol{\Phi}}(\underset{\sim}{\eta})\,. \tag{3.5}$$

Eq. (3.5) is vectorial. For its discretization, we have to use components of displacements. The vectors $\boldsymbol{u}$ and $\underset{\sim}{\boldsymbol{u}}$ are expressed upon the contravariant bases $\{\boldsymbol{a}^1, \boldsymbol{a}^2, \boldsymbol{a}^3\}$ and $\{\underset{\sim}{\boldsymbol{a}}^1, \underset{\sim}{\boldsymbol{a}}^2, \underset{\sim}{\boldsymbol{a}}^3\}$ which differ along the hinge. Therefore, we have to write relation (3.5) upon one of these bases, for example $\{\underset{\sim}{\boldsymbol{a}}^i\}$:

$$\underset{\sim}{u}_i(\underset{\sim}{\eta}) = A_i^j(\eta, \underset{\sim}{\eta}) u_j(\eta)\,,$$

where $A_i^j(\eta, \underset{\sim}{\eta}) = \underset{\sim}{\boldsymbol{a}}_i(\underset{\sim}{\eta}) \cdot \boldsymbol{a}^j(\eta)$.

In addition, we introduce the two mappings $\boldsymbol{F} : \bar{\omega} = [0, 1] \to \gamma$ and $\underset{\sim}{\boldsymbol{F}} : \bar{\omega} \to \underset{\sim}{\gamma}$ which are assumed to be regular. The closed interval $\bar{\omega}$ is subdivided into $n + 1$ segments $[s_p, s_{p+1}]$, for $p = 0, \ldots, n$ with $s_0 = 0$ and $s_{n+1} = 1$. Thus, $\boldsymbol{F}(s_p) = a_p$ and $\underset{\sim}{\boldsymbol{F}}(s_p) = \underset{\sim}{a}_p$ where $\{a_p\}$, $\{\underset{\sim}{a}_p\}$ are the vertices of the triangles of $\mathcal{T}_h$ and $\underset{\sim}{\mathcal{T}}_h$ located along γ and $\underset{\sim}{\gamma}$.

By analogy with Zenisek [4] who considered the approximation of nonhomogeneous boundary conditions, the approximation through Argyris triangle leads naturally to impose the following conditions:

$$\left.\begin{aligned}
&\underset{\sim}{u}_{hi}\circ\underset{\sim}{F}(s_\ell)=\underset{\sim}{A}{}_i^j u_{hj}\circ F(s_\ell)\,,\\
&\frac{\mathrm{d}}{\mathrm{d}s}(\underset{\sim}{u}_{hi}\circ\underset{\sim}{F})(s_\ell)=\frac{\mathrm{d}}{\mathrm{d}s}(\underset{\sim}{A}{}_i^j u_{hj}\circ F)(s_\ell)\,,\\
&\frac{\mathrm{d}^2}{\mathrm{d}s^2}(\underset{\sim}{u}_{hi}\circ\underset{\sim}{F})(s_\ell)=\frac{\mathrm{d}^2}{\mathrm{d}s^2}(\underset{\sim}{A}{}_i^j u_{hj}\circ F)(s_\ell)\,,
\end{aligned}\right\}\tag{3.6}$$

for $\ell=0,\ldots,n+1$ and where we have set for clarity $\underset{\sim}{A}{}_i^j(s_\ell)=A_i^j(a_\ell,\underset{\sim}{a}_\ell)=A_i^j(F(s_\ell),\underset{\sim}{F}(s_\ell))$.

REMARK 3.1.1. Components u_{hi} and $\underset{\sim}{u}_{hj}$ are piecewise two-dimensional five degree polynomials. Since we have supposed that γ and $\underset{\sim}{\gamma}$ are rectilinear, the mappings F and $\underset{\sim}{F}$ are affine. Thus, the composed mappings $u_{hi}\circ F$ and $\underset{\sim}{u}_{hi}\circ\underset{\sim}{F}$ are piecewise one-dimensional five degree polynomials.

Now, we have to rewrite (3.6) in terms of degrees of freedom of Argyris triangle and thus to express these conditions on the reference domains Ω and $\underset{\sim}{\Omega}$. There is no problem for $(3.6)_1$ which can be directly written on the boundaries γ or $\underset{\sim}{\gamma}$. For $(3.6)_2$ we use the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}s}(u_{hi}\circ F)(s_\ell)=Du_{hi}(F(s_\ell))\cdot DF(s_\ell)$$

A unit tangent vector $\boldsymbol{\tau}$ to γ is given by: $|DF(s_\ell)|\boldsymbol{\tau}(a_\ell)=DF(s_\ell)$ so that by setting $u_{hi,\tau}(a_\ell)=Du_{hi}(a_\ell)\cdot\boldsymbol{\tau}(a_\ell)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}(u_{hi}\circ F)(s_\ell)=|DF(s_\ell)|u_{hi,\tau}(a_\ell)\,.\tag{3.7}$$

With similar arguments and since F is affine, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}(u_{hi}\circ F)(s_\ell)=|DF(s_\ell)|^2u_{hi,\tau\tau}(a_\ell)\,.\tag{3.8}$$

Thus, the discrete junction conditions for elastic problem are obtained for $\ell=0,\ldots,n+1$ by substituting (3.7) and (3.8) into relation (3.6):

$$\left.\begin{aligned}
&\underset{\sim}{u}_{hi}(\underset{\sim}{a}_\ell)=\underset{\sim}{A}{}_i^j(s_\ell)u_{hj}(a_\ell)\,,\\
&\underset{\sim}{u}_{hi,\underset{\sim}{\tau}}(\underset{\sim}{a}_\ell)=\left\{\left(\frac{\mathrm{d}}{\mathrm{d}s}\underset{\sim}{A}{}_i^j\right)(s_\ell)u_{hj}(a_\ell)+|DF(s_\ell)|\underset{\sim}{A}{}_i^j(s_\ell)u_{hj,\tau}(a_\ell)\right\}\Big/|D\underset{\sim}{F}(s_\ell)|\,,\\
&\underset{\sim}{u}_{hi,\underset{\sim}{\tau}\underset{\sim}{\tau}}(\underset{\sim}{a}_\ell)=\left\{\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2}\underset{\sim}{A}{}_i^j\right)(s_\ell)u_{hj}(a_\ell)+2|DF(s_\ell)|\left(\frac{\mathrm{d}}{\mathrm{d}s}\underset{\sim}{A}{}_i^j\right)(s_\ell)u_{hi,\tau}(a_\ell)\right.\\
&\qquad\left.+|DF(s_\ell)|^2\underset{\sim}{A}{}_i^j(s_\ell)u_{hj,\tau\tau}(a_\ell)\right\}\Big/|D\underset{\sim}{F}(s_\ell)|^2\,.
\end{aligned}\right\}\tag{3.9}$$

Thus, the admissible discrete displacement space for the elastic junction problem is defined by

$$\left.\begin{aligned}
W_{\mathrm{hel}}=\{&(\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h)\in V_h\times\underset{\sim}{V}_h,\text{ such that relations (3.9) are verified at the}\\
&\text{corresponding vertices }\{a_\ell\}\text{ and }\{\underset{\sim}{a}_\ell\}\text{ located on }\gamma\text{ and }\underset{\sim}{\gamma}\}
\end{aligned}\right\}\tag{3.10}$$

3.2. The discrete admissible space for the rigid junction problem

We proceed by similarity for the rigid junction problem which amounts to discretize the second condition (Part 1, $(2.30)_2$) related to the equality of the rotations. The approximation through Argyris triangle leads to impose the following conditions:

$$\left.\begin{aligned}
&[n^\beta(u_{h3,\beta}+b_\beta^\alpha u_{h\alpha})]\circ F(s_\ell)=[(\boldsymbol{t}\cdot\underset{\sim}{\boldsymbol{t}})\underset{\sim}{n}^\beta(\underset{\sim}{u}_{h3,\beta}+\underset{\sim}{b}{}_\beta^\alpha\underset{\sim}{u}_{h\alpha})]\circ\underset{\sim}{F}(s_\ell)\\
&[n^\beta(u_{h3,\beta}+b_\beta^\alpha u_{h\alpha})]\circ F(q_j)=[(\boldsymbol{t}\cdot\underset{\sim}{\boldsymbol{t}})\underset{\sim}{n}^\beta(\underset{\sim}{u}_{h3,\beta}+\underset{\sim}{b}{}_\beta^\alpha\underset{\sim}{u}_{h\alpha})]\circ\underset{\sim}{F}(q_j)\\
&\frac{\mathrm{d}}{\mathrm{d}s}\{[n^\beta(u_{h3,\beta}+b_\beta^\alpha u_{h\alpha})]\circ F\}(s_\ell)=\frac{\mathrm{d}}{\mathrm{d}s}\{[(\boldsymbol{t}\cdot\underset{\sim}{\boldsymbol{t}})\underset{\sim}{n}^\beta(\underset{\sim}{u}_{h3,\beta}+\underset{\sim}{b}{}_\beta^\alpha\underset{\sim}{u}_{h\alpha})]\circ\underset{\sim}{F}\}(s_\ell)
\end{aligned}\right\}\tag{3.11}$$

for $\ell = 0, \dots, n+1$ and $j = 0, \dots, n$, where $\boldsymbol{n} = n^\alpha \boldsymbol{a}_\alpha$ is the outward unit normal vector to the junction Γ in the tangent plane to $\mathcal{S}$ and q_j is the midpoint of $[s_\ell, s_{\ell+1}]$.

In order to obtain normal and tangential derivatives, we use the relation

$$v_{3,\beta} = \tau_\beta v_{3,\tau} + \nu_\beta v_{3,\nu}$$

where $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are, respectively, the unit tangent vector and the outward unit normal vector to the triangle on $\mathcal{T}_h$ which has a side on γ. Relations (3.11) give

$$\left.\begin{aligned}
&n^\beta(u_{h3,\beta} + b^\alpha_\beta u_{h\alpha})(a_\ell) = (\boldsymbol{t}\cdot\underset{\sim}{\boldsymbol{t}})\underset{\sim}{n}^\beta(\underset{\sim}{u}_{h3,\beta} + \underset{\sim}{b}{}^\alpha_\beta \underset{\sim}{u}_{h\alpha})(\underset{\sim}{a}_\ell)\\
&n^\beta(\tau_\beta u_{h3,\tau} + \nu_\beta u_{h3,\nu} + b^\alpha_\beta u_{h\alpha})(b_j) = (\boldsymbol{t}\cdot\underset{\sim}{\boldsymbol{t}})\underset{\sim}{n}^\beta(\underset{\sim}{\tau}_\beta \underset{\sim}{u}_{h3,\tau} + \underset{\sim}{\nu}_\beta \underset{\sim}{u}_{h3,\nu} + \underset{\sim}{b}{}^\alpha_\beta \underset{\sim}{u}_{h\alpha})(\underset{\sim}{b}_j)\\
&|DF(s_\ell)|\{n^\beta_{,\tau}(u_{h3,\beta} + b^\alpha_\beta u_{h\alpha}) + n^\beta(\tau_\beta u_{h3,\tau\tau} + \nu_\beta u_{h3,\nu\tau} + b^\alpha_\beta u_{h\alpha,\tau}\\
&\quad + b^\alpha_{\beta,\tau} u_{h\alpha})\}(a_\ell) = (\boldsymbol{t}\cdot\underset{\sim}{\boldsymbol{t}})|D\underset{\sim}{F}(s_\ell)|\{\underset{\sim}{n}^\beta_{,\underset{\sim}{\tau}}(\underset{\sim}{u}_{h3,\beta} + \underset{\sim}{b}{}^\alpha_\beta \underset{\sim}{u}_{h\alpha}) + \underset{\sim}{n}^\beta(\underset{\sim}{\tau}_\beta \underset{\sim}{u}_{h3,\underset{\sim}{\tau}\underset{\sim}{\tau}}\\
&\quad + \underset{\sim}{\nu}_\beta \underset{\sim}{u}_{h3,\underset{\sim}{\nu}\underset{\sim}{\tau}} + \underset{\sim}{b}{}^\alpha_\beta \underset{\sim}{u}_{h\alpha,\underset{\sim}{\tau}} + \underset{\sim}{b}{}^\alpha_{\beta,\underset{\sim}{\tau}} \underset{\sim}{u}_{h\alpha})\}(\underset{\sim}{a}_\ell)
\end{aligned}\right\} \tag{3.12}$$

for $\ell = 0, \dots, n+1$ and $j = 0, \dots, n$; τ_α are the components of the unit tangent vector used in (3.7) and b_j is the midpoint of $[a_\ell, a_{\ell+1}]$.

Thus, the admissible discrete displacement space for the rigid junction problem is defined by

$$\left.\begin{aligned}
W_{h\mathrm{rig}} = \{(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h) \in V_h \times \underset{\sim}{V}_h, &\text{ such that relations (3.9) and (3.12) are verified}\\
&\text{at the corresponding vertices } \{a_\ell\}, \{\underset{\sim}{a}_\ell\}, \{b_\ell\} \text{ and } \{\underset{\sim}{b}_\ell\} \text{ located on } \gamma \text{ and } \underset{\sim}{\gamma}\}
\end{aligned}\right\} \tag{3.13}$$

REMARK 3.2.1. This discretization of junction conditions involves the nonconformity of the approximation, i.e.

$$W_{h\mathrm{el}} \not\subset W_{\mathrm{el}} \quad \text{and} \quad W_{h\mathrm{rig}} \not\subset W_{\mathrm{rig}}\,.$$

Since the nonconformity just appears along the hinge while the method remains conform for all the other terms defined on Ω and $\underset{\sim}{\Omega}$, we say that the approximation method is pseudo-conforming.

REMARK 3.2.2. In relations (3.12), the quantities $u_{h3,\tau}(b_j)$, $u_{h\alpha}(b_j)$ and the associated quantities on $\underset{\sim}{\mathcal{S}}$, are not degrees of freedom of Argyris triangle, but they can be expressed from them through the definition of the interpolating function.

4. First discrete problem for elastic junction problem

From now on, we only consider the elastic junction problem. The rigid one could be considered similarly; we will give the corresponding main results in Section 6.

4.1. Definition of the first discrete problem

The following variational formulation takes only into account the finite element approximation; the effect of the numerical integration will be analyzed in Section 5.

$$\left.\begin{aligned}
&\text{Find } (\overset{*}{\boldsymbol{u}}{}^k_h; \overset{*}{\underset{\sim}{\boldsymbol{u}}}{}^k_h) \in W_{h\mathrm{el}} \text{ such that}\\
&a[(\overset{*}{\boldsymbol{u}}{}^k_h; \overset{*}{\underset{\sim}{\boldsymbol{u}}}{}^k_h), (\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)] + kb[(\overset{*}{\underline{\boldsymbol{u}}}{}^k_h; \overset{*}{\underset{\sim}{\underline{\boldsymbol{u}}}}{}^k_h), (\underline{\boldsymbol{v}}_h; \underset{\sim}{\underline{\boldsymbol{v}}}_h)] = \ell[(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)] + f[(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)]\,,\\
&\forall\, (\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h) \in W_{h\mathrm{el}}\,, \quad k \text{ constant} > 0\,,
\end{aligned}\right\} \tag{4.1}$$

where $\underline{\boldsymbol{v}}^k_h = \boldsymbol{v}^k_h \circ \boldsymbol{F}$, $\underset{\sim}{\underline{\boldsymbol{v}}}{}^k_h = \underset{\sim}{\boldsymbol{v}}{}^k_h \circ \underset{\sim}{\boldsymbol{F}}$ and where the space $W_{h\mathrm{el}}$ is defined by relation (3.10).

By comparison with Part 1, (3.10), we have introduced a new linear form $f[.]$ which takes into account the non-conformity of the approximation along the hinge, i.e.

$$f[(\boldsymbol{v}; \underset{\sim}{\boldsymbol{v}})] = \int_\Gamma \{\boldsymbol{N}\cdot\boldsymbol{v} - \underset{\sim}{\boldsymbol{N}}\cdot\underset{\sim}{\boldsymbol{v}}\}\,\mathrm{d}\Gamma = \int_\gamma \boldsymbol{N}\cdot\boldsymbol{v}\,\mathrm{d}\gamma + \int_{\underset{\sim}{\gamma}} \underset{\sim}{\boldsymbol{N}}\cdot\underset{\sim}{\boldsymbol{v}}\,\mathrm{d}\underset{\sim}{\gamma}\,,$$

where N and $\underset{\sim}{N}$ are the resultants of the surface load. This form is identically zero when $(\boldsymbol{v}; \underset{\sim}{\boldsymbol{v}}) \in W_{\text{el}}$ while it is generally different from zero when applied to elements $(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h) \in W_{\text{hel}}$. In that case, we rewrite in matrix form

$$f[(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)] = \int_\gamma {}^{\mathrm{T}}[N][V_h]\,\mathrm{d}\gamma + \int_{\underset{\sim}{\gamma}} {}^{\mathrm{T}}[\underset{\sim}{N}][\underset{\sim}{V}_h]\,\mathrm{d}\underset{\sim}{\gamma}\,, \tag{4.2}$$

where the column vectors $[V_h]$ and $[\underset{\sim}{V}_h]$ are defined in (2.2), and where we have set

$${}^{\mathrm{T}}[N]_{1\times 12} = [N^1 \quad 0 \quad 0 \quad N^2 \quad 0 \quad 0 \quad N^3 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

(and a similar expression for $[\underset{\sim}{N}]$). The introduction of this linear form $f[.]$ in (4.1) leads to a simplification in the abstract error estimate (see Section 4.3).

4.2. Uniform ellipticity

In this paragraph, we prove the existence and uniqueness of a solution for problem (4.1). That leads to show the uniform W_{hel}-ellipticity with respect to h of the bilinear form $a[.,.] + kb[.,.]$.

First, let us recall some definitions introduced in Part 1. Let space E be

$$E = (H^1(\Omega))^2 \times H^2(\Omega) \times (H^1(\underset{\sim}{\Omega}))^2 \times H^2(\underset{\sim}{\Omega})$$

equipped with the norm

$$\|(\boldsymbol{v}; \underset{\sim}{\boldsymbol{v}})\|_E = \{\|v_1\|^2_{1,\Omega} + \|v_2\|^2_{1,\Omega} + \|v_3\|^2_{2,\Omega} + \|\underset{\sim}{v}_1\|^2_{1,\underset{\sim}{\Omega}} + \|\underset{\sim}{v}_2\|^2_{1,\underset{\sim}{\Omega}} + \|\underset{\sim}{v}_3\|^2_{2,\underset{\sim}{\Omega}}\}^{1/2}\,.$$

The space W_{hel}, defined in (3.10), is a closed linear subspace of E and the above mapping is a norm on W_{hel}.

LEMMA 4.2.1. The application $(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h) \in W_{\text{hel}} \to \|(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)\|_{W_{\text{hel}}}$ is a norm on W_{hel} where

$$\|(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)\|_{W_{\text{hel}}} = \{a[(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h), (\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)] + kb[(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h), (\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)]\}^{1/2}\,.$$

In this expression, we consider the definition of the bilinear form $b[.,.]$ given in (2.3).

PROOF. This mapping is clearly a semi-norm. Thus, we just have to show that

$$\|(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)\|_{W_{\text{hel}}} = 0 \quad\Rightarrow\quad (\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h) = (\boldsymbol{0}, \boldsymbol{0}) \text{ in } \Omega \times \underset{\sim}{\Omega}\,.$$

The assumption $\|(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)\|_{W_{\text{hel}}} = 0$ immediately involves:

(i) $\int_\Omega eE^{\alpha\beta\lambda\mu}[\gamma_{\alpha\beta}(\boldsymbol{v}_h)\gamma_{\lambda\mu}(\boldsymbol{v}_h) + (e^2/12)\rho_{\alpha\beta}(\boldsymbol{v}_h)\rho_{\lambda\mu}(\boldsymbol{v}_h)]\sqrt{a}\,\mathrm{d}\xi^1\,\mathrm{d}\xi^2 = 0$ so that the boundary conditions on γ_0 give $\boldsymbol{v}_h = \boldsymbol{0}$ in Ω;

(ii) the discrete junction conditions (3.9) imply

$$\underset{\sim}{v}_{hi}(\underset{\sim}{a}_\ell) = \underset{\sim}{v}_{hi,\underset{\sim}{\tau}}(\underset{\sim}{a}_\ell) = \underset{\sim}{v}_{hi,\underset{\sim}{\tau}\underset{\sim}{\tau}}(\underset{\sim}{a}_\ell) = 0 \tag{4.3}$$

for $\ell = 0, \dots, n+1$, where $\{\underset{\sim}{a}_\ell\}$ is the set of vertices located on $\underset{\sim}{\gamma}$. Since $\underset{\sim}{v}_{hi}$ are five degree polynomials upon each triangle side located on $\underset{\sim}{\gamma}$, relations (4.3) imply tr $\underset{\sim}{v}_{hi} = 0$ on $\underset{\sim}{\gamma}$. Moreover, (Part 1, (2.16), (3.12)) and $b[(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h), (\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)] = 0$ leads to $\underset{\sim}{v}_{h3,\underset{\sim}{n}} = 0$ on Γ where $\underset{\sim}{\boldsymbol{n}} = \underset{\sim}{n}^\beta \underset{\sim}{\boldsymbol{a}}_\beta$ is the unit outward normal vector to Γ located in the tangent plane to $\underset{\sim}{\mathscr{S}}$.

Here, our purpose is to obtain clamped condition on $\underset{\sim}{\gamma}$, i.e. $\underset{\sim}{v}_{h3,\underset{\sim}{\nu}} = 0$ on $\underset{\sim}{\gamma}$. We point out that $\underset{\sim}{v}_{h3,\underset{\sim}{n}}$ and $\underset{\sim}{v}_{h3,\underset{\sim}{\nu}}$ are not the same quantities. Indeed, $\underset{\sim}{\boldsymbol{n}}$ is defined along Γ whereas $\underset{\sim}{\boldsymbol{\nu}}$ is defined along $\underset{\sim}{\gamma}$. The relation between these two quantities is

$$\underset{\sim}{v}_{h3,\underset{\sim}{n}} = \underset{\sim}{n}^\beta \underset{\sim}{v}_{h3,\beta} = \underset{\sim}{n}^\beta(\underset{\sim}{\tau}_\beta \underset{\sim}{v}_{h3,\underset{\sim}{\tau}} + \underset{\sim}{\nu}_\beta \underset{\sim}{v}_{h3,\underset{\sim}{\nu}}) \tag{4.4}$$

Moreover, note that $\underset{\sim}{v}_{h3} = 0$ on $\underset{\sim}{\gamma}$ implies $\underset{\sim}{v}_{h3,\underset{\sim}{\tau}} = 0$ on $\underset{\sim}{\gamma}$. Thus, by giving in addition relation (4.4) and the assumption $\underset{\sim}{v}_{h3,\underset{\sim}{n}} = 0$ on Γ, and noticing that the quantity $\underset{\sim}{n}^\beta \underset{\sim}{\nu}_\beta$ is different from zero, we obtain the required clamped condition $\underset{\sim}{v}_{h3,\underset{\sim}{\nu}} = 0$ on $\underset{\sim}{\gamma}$. Then, we obtain $\underset{\sim}{\boldsymbol{v}}_h = \boldsymbol{0}$ in $\underset{\sim}{\Omega}$. $\square$

LEMMA 4.2.2. Upon the space W_{hel}, *the norms* $\|(\boldsymbol{v}; \underset{\sim}{v})\|_E$ *and* $\|(\boldsymbol{v}; \underset{\sim}{v})\|_{W_{\text{hel}}}$ *are uniformly equivalent with respect to h.*

PROOF. The proof of this lemma is based on the same arguments to those of Part 1, Theorem 3.1.1. The uniform-ellipticity comes from the inclusions (3.4) which allow us to choose the same constants than for the continuous problem (Part 1, (3.10)). □

THEOREM 4.2.1. Problem (4.1) has one and only one solution.

PROOF. Since the bilinear form $a[.,.] + kb[.,.]$ is uniformly W_{hel}-elliptic and uniformly continuous with respect to h, and since the linear form $\ell[.] + f[.]$ is clearly uniformly continuous, we have just to apply the Lax–Milgram lemma to conclude. □

4.3. Abstract error estimate

The abstract error estimate is used in practice to obtain asymptotic error estimate. In the following theorem, the estimation is restricted to the usual approximation theory term $\inf_{(\boldsymbol{v}_h; \underset{\sim}{v}_h) \in W_{\text{hel}}} \|(\boldsymbol{u}^k; \underset{\sim}{u}^k) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)\|_E$ which is known as soon as a finite element approximation is chosen.

THEOREM 4.3.1. Let us consider the discrete problem (4.1) for which the bilinear form $a[.,.] + kb[.,.]$ *is uniformly* W_{hel}*-elliptic, i.e. there exists a constant* $\beta > 0$, *independent of h, such that*

$$a[(\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{v}_h; \underset{\sim}{v}_h)] + kb[(\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h), (\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h)] \geq \beta \|(\boldsymbol{v}_h; \underset{\sim}{v}_h)\|_E^2 , \quad \forall\, (\boldsymbol{v}_h; \underset{\sim}{v}_h) \in W_{\text{hel}} \tag{4.5}$$

We suppose, moreover, that there exists a constant $M > 0$, *independent of h, such that*

$$\left.\begin{array}{l} |a[(\boldsymbol{v}; \underset{\sim}{v}), (\boldsymbol{w}; \underset{\sim}{w})] + kb[(\underline{\boldsymbol{v}}; \underline{\underset{\sim}{v}}), (\underline{\boldsymbol{w}}; \underline{\underset{\sim}{w}})]| \leq M \|(\boldsymbol{v}; \underset{\sim}{v})\|_E \|(\boldsymbol{w}; \underset{\sim}{w})\|_E \\ \forall\, (\boldsymbol{v}; \underset{\sim}{v}) \in W_{\text{el}} + W_{\text{hel}} , \quad \forall\, (\boldsymbol{w}; \underset{\sim}{w}) \in W_{\text{el}} + W_{\text{hel}} . \end{array}\right\} \tag{4.6}$$

Then, there exists a constant C, independent of h, such that

$$\|(\boldsymbol{u}^k; \underset{\sim}{u}^k) - (\overset{*k}{\boldsymbol{u}}_h; \overset{*k}{\underset{\sim}{u}}_h)\|_E \leq C \inf_{(\boldsymbol{v}_h; \underset{\sim}{v}_h) \in W_{\text{hel}}} \|(\boldsymbol{u}^k; \underset{\sim}{u}^k) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)\|_E$$

where $(\boldsymbol{u}^k; \underset{\sim}{u}^k)$ *(resp.* $(\overset{*k}{\boldsymbol{u}}_h; \overset{*k}{\underset{\sim}{u}}_h)$*) denotes the solution of the continuous problem (Part 1, (3.10)) (resp. of the discrete problem (4.1)).*

PROOF. Lemma 4.2.2 involves that relation (4.5) is verified. Likewise, relation (4.6) is a consequence of continuity properties of the bilinear forms $a[.,.]$ and $b[.,.]$. Let $(\boldsymbol{v}_h; \underset{\sim}{v}_h)$ be any element of the space W_{hel}; we can write by using relations (4.1) and (4.5)

$$\begin{aligned} \beta \|(\overset{*k}{\boldsymbol{u}}_h; \overset{*k}{\underset{\sim}{u}}_h) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)\|_E^2 \leq{}& a[(\boldsymbol{u}^k; \underset{\sim}{u}^k) - (\boldsymbol{v}_h; \underset{\sim}{v}_h), (\overset{*k}{\boldsymbol{u}}_h; \overset{*k}{\underset{\sim}{u}}_h) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)] \\ &+ kb[(\underline{\boldsymbol{u}}^k; \underline{\underset{\sim}{u}}^k) - (\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h), (\underline{\overset{*k}{\boldsymbol{u}}}_h; \underline{\overset{*k}{\underset{\sim}{u}}}_h) - (\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h)] \\ &- a[(\boldsymbol{u}^k; \underset{\sim}{u}^k), (\overset{*k}{\boldsymbol{u}}_h; \overset{*k}{\underset{\sim}{u}}_h) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)] - kb[(\underline{\boldsymbol{u}}^k; \underline{\underset{\sim}{u}}^k), (\underline{\overset{*k}{\boldsymbol{u}}}_h; \underline{\overset{*k}{\underset{\sim}{u}}}_h) - (\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h)] \\ &+ \ell[(\overset{*k}{\boldsymbol{u}}_h; \overset{*k}{\underset{\sim}{u}}_h) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)] + f[(\overset{*k}{\boldsymbol{u}}_h; \overset{*k}{\underset{\sim}{u}}_h) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)] \end{aligned}$$

so that with the continuity property (4.6), we obtain

$$\begin{aligned} \beta \|(\overset{*k}{\boldsymbol{u}}_h; \overset{*k}{\underset{\sim}{u}}_h) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)\|_E \leq{}& M \|(\boldsymbol{u}^k; \underset{\sim}{u}^k) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)\|_E \\ + \sup_{(\boldsymbol{w}_h; \underset{\sim}{w}_h) \in W_{\text{hel}}} & \frac{|a[(\boldsymbol{u}^k; \underset{\sim}{u}^k), (\boldsymbol{w}_h; \underset{\sim}{w}_h)] + kb[(\underline{\boldsymbol{u}}^k; \underline{\underset{\sim}{u}}^k), (\underline{\boldsymbol{w}}_h; \underline{\underset{\sim}{w}}_h)] - \ell[(\boldsymbol{w}_h; \underset{\sim}{w}_h)] - f[(\boldsymbol{w}_h; \underset{\sim}{w}_h)]|}{\|(\boldsymbol{w}_h; \underset{\sim}{w}_h)\|_E} \end{aligned}$$

Now, let us obtain a new expression for $a[(\boldsymbol{u}^k; \underset{\sim}{u}^k), (\boldsymbol{w}_h; \underset{\sim}{w}_h)] + kb[(\underline{\boldsymbol{u}}^k; \underline{\underset{\sim}{u}}^k), (\underline{\boldsymbol{w}}_h; \underline{\underset{\sim}{w}}_h)]$. For that we come back to the equilibrium equations of the junction problem given in Part 1, (2.27)–(2.28). Making the product of these equations by test functions $(w_h, \underset{\sim}{w}_h) \in W_{\text{hel}}$ and using Green's formula, we finally get with notations introduced in (4.1)

$$\left.\begin{aligned} a[(u^k; \underset{\sim}{u}^k), (w_h; \underset{\sim}{w}_h)] + kb[(\underline{u}^k; \underset{\sim}{u}^k), (\underline{w}_h; \underline{\underset{\sim}{w}}_h)] &= \ell[(w_h; \underset{\sim}{w}_h)] + \int_\gamma N \cdot w_h \, d\gamma + \int_{\underset{\sim}{\gamma}} \underset{\sim}{N} \cdot \underset{\sim}{w}_h \, d\underset{\sim}{\gamma} \\ &= \ell[(w_h; \underset{\sim}{w}_h)] + f[(w_h; \underset{\sim}{w}_h)] , \end{aligned}\right\} \tag{4.7}$$

so that the second term in the above estimation disappears.

To conclude, it remains to use the triangular inequality and to take the minimum with respect to $(v_h; \underset{\sim}{v}_h) \in W_{\text{hel}}$. □

REMARK 4.3.1. The relation (4.7) allows us to cancel the consistency term that we usually find in the abstract error estimate associated with nonconforming finite element method. Here, the nonconformity only appears along the hinge Γ; its effect is circumvent by the introduction of the linear form $f[.]$ in (4.1).

Another discrete problem could also be considered by dropping term $f[.]$ in (4.1). It should be different from problem (4.1) and would lead to a more classical abstract error estimate including a consistency term based on the linear form $f[.]$. The solution of such a problem would be really close to that of problem (4.1).

5. Second discrete problem: additional effect of numerical integration

5.1. Definition of the second discrete problem

The integrals defined over the domains Ω and $\underset{\sim}{\Omega}$ have to be evaluated over all the triangles $K \in \mathcal{T}_h$ and $\underset{\sim}{K} \in \underset{\sim}{\mathcal{T}}_h$ and they are seldom exactly computed in practice. One rather uses numerical integration schemes. Then, let us consider a numerical integration scheme defined over a reference triangle $\hat{K}$ (for more details see Ciarlet [8]):

$$\int_{\hat{K}} \hat{\phi}(\hat{x}) \, d\hat{x} \sim \sum_{\ell=1}^{L} \omega_\ell \hat{\phi}(\hat{b}_\ell) .$$

All the integrals appearing in the expressions of $a[.,.]$ and $\ell[.]$ are of the form $\int_K \phi(x) \, dx$. We use the usual correspondence between ϕ and $\hat{\phi}$ through the affine invertible mapping

$$F_K : \hat{x} \in \hat{K} \to F_K(\hat{x}) = B_K \hat{x} + b_K \in K ,$$

where B_K is an invertible matrix, b_K is a vector of $\mathbb{R}^2$ such that $F_K(\hat{a}_i) = a_i$, $i = 1, 2, 3$, where $\hat{a}_i$, a_i are the vertices of the triangles $\hat{K}$ and K. Then, the numerical integration scheme over the triangle $\hat{K}$ automatically induces a numerical integration scheme over K, namely

$$\int_K \phi(x) \, dx \sim \sum_{\ell=1}^{L} \omega_{\ell,K} \phi(b_{\ell,K}) ,$$

where $\omega_{\ell,K} = \det(B_K)\hat{\omega}_\ell$ and $b_{\ell,K} = F_K(\hat{b}_\ell)$, $1 \leq \ell \leq L$. Moreover, we define the error functionals:

$$\hat{E}(\hat{\phi}) = \int_{\hat{K}} \hat{\phi}(\hat{\xi}) \, d\hat{\xi} - \sum_{\ell=1}^{L} \hat{\omega}_\ell \hat{\phi}(\hat{b}_\ell) , \qquad E_K(\phi) = \int_K \phi(\xi) \, d\xi - \sum_{\ell=1}^{L} \omega_{\ell,K} \phi(b_{\ell,K}) , \tag{5.1}$$

so that

$$E_K(\phi) = \det B_K \hat{E}(\hat{\phi}) .$$

We define similar functions for the domain $\underset{\sim}{\Omega}$ and for the triangles $\underset{\sim}{K} \in \underset{\sim}{\mathcal{T}}_h$. Finally, we introduce a numerical integration scheme over each triangle side $\hat{K}'$ located upon the boundaries γ, $\underset{\sim}{\gamma}$, γ_1, $\underset{\sim}{\gamma}_1$, ω:

$$\int_{\hat{K}'} \hat{\phi}'(\hat{s}) \, d\hat{s} \sim \sum_{\ell=1}^{L'} \hat{\omega}'_\ell \hat{\phi}'(\hat{b}'_\ell) .$$

Thus, we use the correspondence between $\hat{\phi}'$ and ϕ' through the affine invertible mapping

$$G_{K'}: \hat{s} \in \hat{K}' \to G_{K'}(\hat{s}) = \alpha\hat{s} + \beta \in K' ,$$

where $\alpha \neq 0$. We have $G_{K'}(\hat{s}_\alpha) = s_\alpha$, $\alpha = 1, 2$ where $\hat{s}_\alpha, s_\alpha$ are the corresponding vertices of the sides $\hat{K}'$ and K'. Then, we have a numerical integration scheme over the segment K'

$$\int_{K'} \phi'(s)\,\mathrm{d}s \sim \sum_{\ell=1}^{L'} \omega_{\ell,K'}\phi'(b_{\ell,K'}) ,$$

with $\omega_{\ell,K'} = \alpha\hat{\omega}'_\ell$ and $b_{\ell,K'} = G_{K'}(\hat{b}'_\ell)$, $1 \leq \ell \leq L'$ and the associated error functional

$$\hat{E}'(\hat{\phi}') = \int_{\hat{K}'} \hat{\phi}'(\hat{s})\,\mathrm{d}\hat{s} - \sum_{\ell=1}^{L'} \hat{\omega}'_\ell\hat{\phi}'(\hat{b}'_\ell) , \qquad E'_{K'}(\phi') = \int_{K'} \phi'(s)\,\mathrm{d}s - \sum_{\ell=1}^{L'} \omega_{\ell,K'}\phi'(b_{\ell,K'}) , \tag{5.2}$$

so that

$$E'_{K'}(\phi') = \alpha\hat{E}'(\hat{\phi}') .$$

Now, we can give the expression of the second discrete problem which takes into account the additional effect of the numerical integration.

$$\left.\begin{array}{l} \text{Find } (\boldsymbol{u}^k_h, \underset{\sim}{\boldsymbol{u}}^k_h) \in W_{\mathrm{hel}} \text{ such that} \\ a_h[(\boldsymbol{u}^k_h; \underset{\sim}{\boldsymbol{u}}^k_h), (\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)] + kb_h[(\underline{\boldsymbol{u}}^k_h; \underline{\underset{\sim}{\boldsymbol{u}}}^k_h), (\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{\boldsymbol{v}}}_h)] = \ell_h[(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)] + f_h[(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)] \\ \forall\, (\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h) \in W_{\mathrm{hel}} , \quad k \text{ constant} > 0 , \end{array}\right\} \tag{5.3}$$

where we have set (compare with (2.1) and (2.4)):

$$\left.\begin{array}{rl} a_h[(\boldsymbol{u}_h; \underset{\sim}{\boldsymbol{u}}_h), (\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)] = & \displaystyle\sum_{K\in\mathcal{T}_h} \sum_{\ell=1}^{L} \omega_{\ell,K}\{{}^{\mathrm{T}}[\boldsymbol{U}_h][\boldsymbol{A}][\boldsymbol{V}_h]\}(b_{\ell,K}) \\ & + \displaystyle\sum_{\underset{\sim}{K}\in\mathcal{T}_h} \sum_{\ell=1}^{L} \omega_{\ell,\underset{\sim}{K}}\{{}^{\mathrm{T}}[\underset{\sim}{\boldsymbol{U}}_h][\underset{\sim}{\boldsymbol{A}}][\underset{\sim}{\boldsymbol{V}}_h]\}(b_{\ell,\underset{\sim}{K}}) \end{array}\right\} \tag{5.4}$$

$$b_h[(\underline{\boldsymbol{u}}_h; \underline{\underset{\sim}{\boldsymbol{u}}}_h), (\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{\boldsymbol{v}}}_h)] = \sum_{\underline{K}\in\mathcal{T}_h} \sum_{\ell=1}^{L'} \omega_{\ell,K'}\{{}^{\mathrm{T}}[\underline{\boldsymbol{U}}_h \quad \underline{\underset{\sim}{\boldsymbol{U}}}_h][\underline{\boldsymbol{C}}][\underline{\boldsymbol{V}}_h \quad \underline{\underset{\sim}{\boldsymbol{V}}}_h]\}(b_{\ell,\underline{K}}) . \tag{5.5}$$

The linear forms are defined by (compare with (2.5)):

$$\left.\begin{array}{rl} \ell_h[(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)] = & \displaystyle\sum_{K\in\mathcal{T}_h} \sum_{\ell=1}^{L} \omega_{\ell,K}\{{}^{\mathrm{T}}[\boldsymbol{P}][\boldsymbol{V}_h]\}(b_{\ell,K}) + \sum_{\underset{\sim}{K}\in\mathcal{T}_h} \sum_{\ell=1}^{L} \omega_{\ell,\underset{\sim}{K}}\{{}^{\mathrm{T}}[\underset{\sim}{\boldsymbol{P}}][\underset{\sim}{\boldsymbol{V}}_h]\}(b_{\ell,\underset{\sim}{K}}) \\ & + \displaystyle\sum_{K'\in G_1} \sum_{\ell=1}^{L'} \omega_{\ell,K'}\{{}^{\mathrm{T}}[\boldsymbol{L}_s][\boldsymbol{V}_h]\}(b_{\ell,K'}) + \sum_{\underset{\sim}{K}'\in \underset{\sim}{G}_1} \sum_{\ell=1}^{L'} \omega_{\ell,\underset{\sim}{K}'}\{{}^{\mathrm{T}}[\underset{\sim}{\boldsymbol{L}}_s][\underset{\sim}{\boldsymbol{V}}_h]\}(b_{\ell,\underset{\sim}{K}'}) \end{array}\right\} \tag{5.6}$$

where G_1 and $\underset{\sim}{G}_1$ denote the sets of the sides of triangles which are located upon γ_1 and $\underset{\sim}{\gamma}_1$. Moreover, (compare with (4.2)):

$$f_h[(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)] = \sum_{K'\in G} \sum_{\ell=1}^{L'} \omega_{\ell,K'}\{{}^{\mathrm{T}}[\boldsymbol{N}][\boldsymbol{V}_h]\}(b_{\ell,K'}) + \sum_{\underset{\sim}{K}'\in \underset{\sim}{G}} \sum_{\ell=1}^{L'} \omega_{\ell,\underset{\sim}{K}'}\{{}^{\mathrm{T}}[\underset{\sim}{\boldsymbol{N}}][\underset{\sim}{\boldsymbol{V}}_h]\}(b_{\ell,\underset{\sim}{K}'}) \tag{5.7}$$

where G and $\underset{\sim}{G}$ denote again the sets of the sides of triangles located on γ and $\underset{\sim}{\gamma}$.

5.2. Abstract error estimate

THEOREM 5.2.1. Let us consider a family of discrete problems (5.3) for which the bilinear forms $a_h[.,.] + kb_h[.,.]$ are W_{hel}-elliptic, uniformly with respect to h, i.e. there exists a constant $\beta > 0$, independent of h, such that

$$a_h[(\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{v}_h; \underset{\sim}{v}_h)] + kb_h[(\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h), (\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h)] \geq \beta \|(\boldsymbol{v}_h; \underset{\sim}{v}_h)\|_E^2, \quad \forall (\boldsymbol{v}_h; \underset{\sim}{v}_h) \in W_{\text{hel}}.$$

Then, there exists a constant C, independent of h, such that

$$\begin{aligned}
\|(\boldsymbol{u}^k; \underset{\sim}{u}^k) - (\boldsymbol{u}_h^k; \underset{\sim}{u}_h^k)\|_E \leq C \inf_{(\boldsymbol{v}_h; \underset{\sim}{v}_h) \in W_{\text{hel}}} &\Bigg\{ \|(\boldsymbol{u}^k; \underset{\sim}{u}^k) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)\|_E \\
&+ \sup_{(\boldsymbol{w}_h; \underset{\sim}{w}_h) \in W_{\text{hel}}} \Bigg\{ \frac{|a[(\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{w}_h; \underset{\sim}{w}_h)] - a_h[(\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{w}_h; \underset{\sim}{w}_h)]|}{\|(\boldsymbol{w}_h; \underset{\sim}{w}_h)\|_E} \\
&+ k \frac{|b[(\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h), (\underline{\boldsymbol{w}}_h; \underline{\underset{\sim}{w}}_h)] - b_h[(\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h), (\underline{\boldsymbol{w}}_h; \underline{\underset{\sim}{w}}_h)]|}{\|(\boldsymbol{w}_h; \underset{\sim}{w}_h)\|_E} \Bigg\} \Bigg\} \\
&+ C \sup_{(\boldsymbol{w}_h; \underset{\sim}{w}_h) \in W_{\text{hel}}} \frac{|a[(\boldsymbol{u}^k; \underset{\sim}{u}^k), (\boldsymbol{w}_h; \underset{\sim}{w}_h)] + kb[(\underline{\boldsymbol{u}}^k; \underline{\underset{\sim}{u}}^k), (\underline{\boldsymbol{w}}_h; \underline{\underset{\sim}{w}}_h)] - \ell_h[(\boldsymbol{w}_h; \underset{\sim}{w}_h)] - f_h[(\boldsymbol{w}_h; \underset{\sim}{w}_h)]|}{\|(\boldsymbol{w}_h; \underset{\sim}{w}_h)\|_E}
\end{aligned} \tag{5.8}$$

where $(\boldsymbol{u}^k; \underset{\sim}{u}^k)$ (resp. $(\boldsymbol{u}_h^k; \underset{\sim}{u}_h^k)$) denotes the solution of the continuous problem (Part 1, (3.10)) (resp. of the discrete problem (5.3)).

PROOF. The assumption of W_{hel}-ellipticity involves the existence and uniqueness of a solution $(\boldsymbol{u}_h^k; \underset{\sim}{u}_h^k)$ for the discrete problem (5.3). Then, let $(\boldsymbol{v}_h; \underset{\sim}{v}_h)$ be any element of the space W_{hel}; we can write

$$\begin{aligned}
\beta \|(\boldsymbol{u}_h^k; \underset{\sim}{u}_h^k) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)\|_E^2 &\leq a_h[(\boldsymbol{u}_h^k; \underset{\sim}{u}_h^k) - (\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{u}_h^k; \underset{\sim}{u}_h^k) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)] \\
&\quad + kb_h[(\underline{\boldsymbol{u}}_h^k; \underline{\underset{\sim}{u}}_h^k) - (\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h), (\underline{\boldsymbol{u}}_h^k; \underline{\underset{\sim}{u}}_h^k) - (\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h)] \\
&= a[(\boldsymbol{u}^k; \underset{\sim}{u}^k) - (\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{u}_h^k; \underset{\sim}{u}_h^k) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)] \\
&\quad + kb[(\underline{\boldsymbol{u}}^k; \underline{\underset{\sim}{u}}^k) - (\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h), (\underline{\boldsymbol{u}}_h^k; \underline{\underset{\sim}{u}}_h^k) - (\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h)] \\
&\quad + a[(\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{u}_h^k; \underset{\sim}{u}_h^k) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)] - a_h[(\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{u}_h^k; \underset{\sim}{u}_h^k) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)] \\
&\quad + kb[(\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h), (\underline{\boldsymbol{u}}_h^k; \underline{\underset{\sim}{u}}_h^k) - (\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h)] - kb_h[(\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h), (\underline{\boldsymbol{u}}_h^k; \underline{\underset{\sim}{u}}_h^k) - (\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h)] \\
&\quad - a[(\boldsymbol{u}^k; \underset{\sim}{u}^k), (\boldsymbol{u}_h^k; \underset{\sim}{u}_h^k) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)] - kb[(\underline{\boldsymbol{u}}^k; \underline{\underset{\sim}{u}}^k), (\underline{\boldsymbol{u}}_h^k; \underline{\underset{\sim}{u}}_h^k) - (\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h)] \\
&\quad + \ell_h[(\boldsymbol{u}_h^k, \underset{\sim}{u}_h^k) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)] + f_h[(\boldsymbol{u}_h^k, \underset{\sim}{u}_h^k) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)]
\end{aligned}$$

so that with the continuity property (4.6) we obtain

$$\begin{aligned}
\beta \|(\boldsymbol{u}_h^k; \underset{\sim}{u}_h^k) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)\|_E &\leq M \|(\boldsymbol{u}^k; \underset{\sim}{u}^k) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)\|_E \\
&+ \sup_{(\boldsymbol{w}_h; \underset{\sim}{w}_h) \in W_{\text{hel}}} \Bigg\{ \frac{|a[(\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{w}_h; \underset{\sim}{w}_h)] - a_h[(\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{w}_h; \underset{\sim}{w}_h)]|}{\|(\boldsymbol{w}_h; \underset{\sim}{w}_h)\|_E} \\
&+ k \frac{|b[(\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h), (\underline{\boldsymbol{w}}_h; \underline{\underset{\sim}{w}}_h)] - b_h[(\underline{\boldsymbol{v}}_h; \underline{\underset{\sim}{v}}_h), (\underline{\boldsymbol{w}}_h; \underline{\underset{\sim}{w}}_h)]|}{\|(\boldsymbol{w}_h; \underset{\sim}{w}_h)\|_E} \Bigg\} \\
&+ \sup_{(\boldsymbol{w}_h; \underset{\sim}{w}_h) \in W_{\text{hel}}} \frac{|a[(\boldsymbol{u}^k; \underset{\sim}{u}^k), (\boldsymbol{w}_h; \underset{\sim}{w}_h)] + kb[(\underline{\boldsymbol{u}}^k; \underline{\underset{\sim}{u}}^k), (\underline{\boldsymbol{w}}_h; \underline{\underset{\sim}{w}}_h)] - \ell_h[(\boldsymbol{w}_h; \underset{\sim}{w}_h)] - f_h[(\boldsymbol{w}_h; \underset{\sim}{w}_h)]|}{\|(\boldsymbol{w}_h; \underset{\sim}{w}_h)\|_E}
\end{aligned}$$

To conclude, it remains to use the triangular inequality and to take the minimum with respect to $(\boldsymbol{v}_h; \underset{\sim}{v}_h) \in W_{\text{hel}}$. $\square$

In the estimate (5.8), in addition to the usual approximation theory term $\inf \|(\boldsymbol{u}^k; \underset{\sim}{u}^k) - (\boldsymbol{v}_h; \underset{\sim}{v}_h)\|$, we find two additional terms which measure the consistency error between the bilinear forms $a[.,.]$ and $a_h[.,.]$, $b[.,.]$ and $b_h[.,.]$; they take into account the error due to the numerical integration. Finally, the last term combines the error due to both approximations, i.e. nonconforming approximation along the hinge and use of the numerical integration techniques.

5.3. Uniform ellipticity

The uniform W_{hel}-ellipticity is based on the local error estimate theorems given by Bernadou [5, pp. 53–61] for a triangle K and by Cubier [6, pp. 76–87] for a triangle side K'. These theorems give a general result of error estimate; they specify criteria on the choice of numerical integration schemes in order to obtain the same order of asymptotic error estimate than for exact integration.

THEOREM 5.3.1. Let $\mathcal{T}_h$ and $\underset{\sim}{\mathcal{T}}_h$ be regular families of triangulations of the domains Ω and $\underset{\sim}{\Omega}$ satisfying properties (3.1) to (3.3). Let (K, P_K, Σ_K) and $(\underset{\sim}{K}, P_{\underset{\sim}{K}}, \Sigma_{\underset{\sim}{K}})$ be two almost affine families of finite elements associated with the Argyris triangle. Thus, we have

$$P_K = P_5(K), \quad \forall K \in \mathcal{T}_h \quad \text{and} \quad P_{\underset{\sim}{K}} = P_5(\underset{\sim}{K}), \quad \forall \underset{\sim}{K} \in \underset{\sim}{\mathcal{T}}_h .$$

Moreover, assume that the integration scheme on the reference triangle $\hat{K}$ satisfies the following conditions:

(i) the integration nodes $\hat{b}_\ell \in \overline{\hat{K}}$, $\forall \ell = 1, \ldots, L$;
(ii) $\hat{E}(\hat{\varphi}) = 0$, $\forall \hat{\varphi} \in P_8(\hat{K})$.

Likewise, the integration scheme on the reference segment $\hat{K}'$ verifies

(iii) the integration nodes $\hat{b}'_\ell \in \overline{\hat{K}'}$, $\forall \ell = 1, \ldots, L'$;
(iv) $\hat{E}'(\hat{\varphi}) = 0$, $\forall \hat{\varphi} \in P_8(\hat{K}')$.

Then, for any given $A_{IJ} \in W^{1,\infty}(\Omega)$, $\underset{\sim}{A}_{IJ} \in W^{1,\infty}(\underset{\sim}{\Omega})$, $I, J = 1, \ldots, 12$ and $\underline{C}_{IJ} \in W^{1,\infty}(\omega)$, $I, J = 1, \ldots, 24$, there exist constants $\beta > 0$ and $h_1 > 0$, independent of h, such that for all $h \in]0, h_1]$, we have

$$a_h[(\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{v}_h; \underset{\sim}{v}_h)] + kb_h[(\underline{v}_h; \underset{\sim}{\underline{v}}_h), (\underline{v}_h; \underset{\sim}{\underline{v}}_h)] \geq \beta \|(\boldsymbol{v}_h; \underset{\sim}{v}_h)\|_E^2 , \quad \forall (\boldsymbol{v}_h; \underset{\sim}{v}_h) \in W_{\mathrm{hel}} . \tag{5.9}$$

PROOF. For any $(\boldsymbol{v}_h; \underset{\sim}{v}_h) \in W_{\mathrm{hel}} \subset V_h \times \underset{\sim}{V}_h$, the inclusions (3.4) allow us to write

$$\left.\begin{aligned} a_h[(\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{v}_h; \underset{\sim}{v}_h)] + kb_h[(\underline{v}_h; \underset{\sim}{\underline{v}}_h), (\underline{v}_h; \underset{\sim}{\underline{v}}_h)] &= a[(\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{v}_h; \underset{\sim}{v}_h)] + kb[(\underline{v}_h; \underset{\sim}{\underline{v}}_h), (\underline{v}_h; \underset{\sim}{\underline{v}}_h)] \\ &\quad + a_h[(\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{v}_h; \underset{\sim}{v}_h)] - a[(\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{v}_h; \underset{\sim}{v}_h)] \\ &\quad + kb_h[(\underline{v}_h; \underset{\sim}{\underline{v}}_h), (\underline{v}_h; \underset{\sim}{\underline{v}}_h)] - kb[(\underline{v}_h; \underset{\sim}{\underline{v}}_h), (\underline{v}_h; \underset{\sim}{\underline{v}}_h)] , \end{aligned}\right\} \tag{5.10}$$

where $a[.,.]$ and $b[.,.]$ are defined by relations (2.1) and (2.4). According to the proof of Lemma 4.2.2, there exists a constant $\alpha > 0$, independent of h, such that for any $(\boldsymbol{v}_h; \underset{\sim}{v}_h) \in W_{\mathrm{hel}}$

$$a[(\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{v}_h; \underset{\sim}{v}_h)] + kb[(\underline{v}_h; \underset{\sim}{\underline{v}}_h), (\underline{v}_h; \underset{\sim}{\underline{v}}_h)] \geq \alpha \|(\boldsymbol{v}_h; \underset{\sim}{v}_h)\|_E^2 . \tag{5.11}$$

This result and the assumptions of Theorem 5.3.1 allow us to apply Theorem 1.3.3 of Bernadou [5, p. 57] to the different types of terms which occur in the second-hand member of the following inequality

$$|a[(\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{v}_h; \underset{\sim}{v}_h)] - a_h[(\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{v}_h; \underset{\sim}{v}_h)]| \leq \sum_{K \in \mathcal{T}_h} \sum_{I,J=1}^{12} |E_K(A_{IJ} V_{hI} V_{hJ})| + \sum_{\underset{\sim}{K} \in \underset{\sim}{\mathcal{T}}_h} \sum_{I,J=1}^{12} |E_{\underset{\sim}{K}}(\underset{\sim}{A}_{IJ} \underset{\sim}{V}_{hI} \underset{\sim}{V}_{hJ})| .$$

Thus, there exists a constant $C > 0$, independent of h, which can change from an inequality to the next and such that

$$\left.\begin{aligned} &|a[(\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{v}_h; \underset{\sim}{v}_h)] - a_h[(\boldsymbol{v}_h; \underset{\sim}{v}_h), (\boldsymbol{v}_h; \underset{\sim}{v}_h)]| \\ &\quad \leq C\left(\sum_{K \in \mathcal{T}_h} h_K \left(\sum_{I,J=1}^{12} \|A_{IJ}\|_{1,\infty,\Omega} \right) \|\boldsymbol{v}_h\|_{V(K)}^2 + \sum_{\underset{\sim}{K} \in \underset{\sim}{\mathcal{T}}_h} h_{\underset{\sim}{K}} \left(\sum_{I,J=1}^{12} \|\underset{\sim}{A}_{IJ}\|_{1,\infty,\underset{\sim}{\Omega}} \right) \|\underset{\sim}{v}_h\|_{\underset{\sim}{V}(\underset{\sim}{K})}^2 \right) \\ &\quad \leq Ch \sup\left\{ \sum_{I,J=1}^{12} \|A_{IJ}\|_{1,\infty,\Omega} , \sum_{I,J=1}^{12} \|\underset{\sim}{A}_{IJ}\|_{1,\infty,\underset{\sim}{\Omega}} \right\} \|(\boldsymbol{v}_h; \underset{\sim}{v}_h)\|_E^2 \\ &\quad \leq Ch \|(\boldsymbol{v}_h; \underset{\sim}{v}_h)\|_E^2 . \end{aligned}\right\} \tag{5.12}$$

Now, we have to estimate the term $b[(\underline{v}_h; \underset{\sim}{\underline{v}}_h), (\underline{v}_h; \underset{\sim}{\underline{v}}_h)] - b_h[(\underline{v}_h; \underset{\sim}{\underline{v}}_h), (\underline{v}_h; \underset{\sim}{\underline{v}}_h)]$. By using definition (5.2) of the error functional, we obtain

$$
|b[(\underline{\boldsymbol{v}}_h;\underset{\sim}{\underline{\boldsymbol{v}}}_h),(\underline{\boldsymbol{v}}_h;\underset{\sim}{\underline{\boldsymbol{v}}}_h)]-b_h[(\underline{\boldsymbol{v}}_h;\underset{\sim}{\underline{\boldsymbol{v}}}_h),(\underline{\boldsymbol{v}}_h;\underset{\sim}{\underline{\boldsymbol{v}}}_h)]|\leqslant\sum_{\underline{K}\in\underline{\mathcal{T}}_h}\sum_{I,J=1}^{24}|E'_{\underline{K}}(\underline{\boldsymbol{C}}_{IJ}[\underline{\boldsymbol{V}}_h\quad\underset{\sim}{\underline{\boldsymbol{V}}}_h]_I[\underline{\boldsymbol{V}}_h\quad\underset{\sim}{\underline{\boldsymbol{V}}}_h]_J)|\,.
$$

If we denote

$$
I_1=\{1,\ldots,12\}\times\{1,\ldots,12\}\,;\qquad I_2=\{13,\ldots,24\}\times\{13,\ldots,24\}\,;
$$
$$
I_3=\{1,\ldots,12\}\times\{13,\ldots,24\}\,;\qquad I_4=\{13,\ldots,24\}\times\{1,\ldots,12\}\,,
$$

then we have the relation

$$
\left.\begin{aligned}
&\sum_{\underline{K}\in\underline{\mathcal{T}}_h}\sum_{I,J=1}^{24}|E'_{\underline{K}}(\underline{\boldsymbol{C}}_{IJ}[\underline{\boldsymbol{V}}_h\quad\underset{\sim}{\underline{\boldsymbol{V}}}_h]_I[\underline{\boldsymbol{V}}_h\quad\underset{\sim}{\underline{\boldsymbol{V}}}_h]_J|=\sum_{\underline{K}\in\underline{\mathcal{T}}_h}\sum_{I,J\in I_1}|E'_{\underline{K}}(\underline{\boldsymbol{C}}_{IJ}[\underline{\boldsymbol{V}}_h]_I[\underline{\boldsymbol{V}}_h]_J)|\\
&\quad+\sum_{\underline{K}\in\underline{\mathcal{T}}_h}\sum_{I,J\in I_2}|E'_{\underline{K}}(\underline{\boldsymbol{C}}_{IJ}[\underset{\sim}{\underline{\boldsymbol{V}}}_h]_I[\underset{\sim}{\underline{\boldsymbol{V}}}_h]_J)|+\sum_{\underline{K}\in\underline{\mathcal{T}}_h}\sum_{I,J\in I_3\cup I_4}|E'_{\underline{K}}(\underline{\boldsymbol{C}}_{IJ}[\underline{\boldsymbol{V}}_h]_I[\underset{\sim}{\underline{\boldsymbol{V}}}_h]_J)|
\end{aligned}\right\}\tag{5.13}
$$

With the hypotheses made in the statement of the theorem, we can apply to these one-dimensional integration terms of the relation (5.13) the same kind of technique and we obtain (see Cubier [6, p. 83] for details)

$$
\sum_{\underline{K}\in\underline{\mathcal{T}}_h}\sum_{I,J\in I_1}|E'_{\underline{K}}(\underline{\boldsymbol{C}}_{IJ}[\underline{\boldsymbol{V}}_h]_I[\underline{\boldsymbol{V}}_h]_J)|\leqslant Ch\sum_{\underline{K}\in\underline{\mathcal{T}}_h}\Big(\sum_{I,J\in I_1}\|\underline{\boldsymbol{C}}_{IJ}\|_{1,\infty,\underline{K}}\Big)\|\boldsymbol{v}_h\|^2_{V(K)}\,,
$$
$$
\sum_{\underline{K}\in\underline{\mathcal{T}}_h}\sum_{I,J\in I_2}|E'_{\underline{K}}(\underline{\boldsymbol{C}}_{IJ}[\underset{\sim}{\underline{\boldsymbol{V}}}_h]_I[\underset{\sim}{\underline{\boldsymbol{V}}}_h]_J)|\leqslant Ch\sum_{\underline{K}\in\underline{\mathcal{T}}_h}\Big(\sum_{I,J\in I_2}\|\underline{\boldsymbol{C}}_{IJ}\|_{1,\infty,\underline{K}}\Big)\|\underset{\sim}{\boldsymbol{v}}_h\|^2_{\underset{\sim}{V}(\underline{K})}\,,
$$
$$
\sum_{\underline{K}\in\underline{\mathcal{T}}_h}\sum_{I,J\in I_3\cup I_4}|E'_{\underline{K}}(\underline{\boldsymbol{C}}_{IJ}[\underline{\boldsymbol{V}}_h]_I[\underset{\sim}{\underline{\boldsymbol{V}}}_h]_J)|\leqslant Ch\sum_{\underline{K}\in\underline{\mathcal{T}}_h}\Big(\sum_{I,J\in I_3\cup I_4}\|\underline{\boldsymbol{C}}_{IJ}\|_{1,\infty,\underline{K}}\Big)\|\boldsymbol{v}_h\|_{V(K)}\|\underset{\sim}{\boldsymbol{v}}_h\|_{\underset{\sim}{V}(\underline{K})}\,,
$$

so that the substitution of these inequalities into relation (5.13) proves the existence of a constant C, independent of h, such that

$$
\left.\begin{aligned}
&|b[(\underline{\boldsymbol{v}}_h;\underset{\sim}{\underline{\boldsymbol{v}}}_h),(\underline{\boldsymbol{v}}_h;\underset{\sim}{\underline{\boldsymbol{v}}}_h)]-b_h[(\underline{\boldsymbol{v}}_h;\underset{\sim}{\underline{\boldsymbol{v}}}_h),(\underline{\boldsymbol{v}}_h;\underset{\sim}{\underline{\boldsymbol{v}}}_h)]|\\
&\quad\leqslant Ch\Big(\sum_{I,J=1}^{24}\|\underline{\boldsymbol{C}}_{IJ}\|_{1,\infty,\omega}\Big)\|(\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h)\|^2_E\leqslant Ch\|(\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h)\|^2_E\,.
\end{aligned}\right\}\tag{5.14}
$$

By substituting relations (5.11), (5.12), (5.14) into relation (5.10), we get

$$
a_h[(\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h),(\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h)]+kb_h[(\underline{\boldsymbol{v}}_h;\underset{\sim}{\underline{\boldsymbol{v}}}_h),(\underline{\boldsymbol{v}}_h;\underset{\sim}{\underline{\boldsymbol{v}}}_h)]\geqslant(\alpha-Ch)\|(\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h)\|^2_E
$$

and it suffices to take $\beta=\alpha/2$ and $h_1=\alpha/(2C)$ to obtain the expected estimate (5.9). □

REMARK 5.3.1. Numerical integration schemes exact for polynomial of degree eight can be found in Dunavant [9, p. 1140] for the triangle and in Zienkiewicz and Taylor [10, p. 173] for an interval.

5.4. Asymptotic error estimate

Now, we are able to evaluate the different terms of the abstract error estimate (5.8) and to derive an asymptotic error estimate $\|(\boldsymbol{u};\underset{\sim}{\boldsymbol{u}})-(\boldsymbol{u}_h;\underset{\sim}{\boldsymbol{u}}_h)\|$ between the solution $(\boldsymbol{u};\underset{\sim}{\boldsymbol{u}})$ of the continuous problem (Part 1, (3.10)) and the solutions of the discrete problems (5.3). Before giving the asymptotic error estimate theorem, let us specify some notations:

$$
\|(\boldsymbol{p};\underset{\sim}{\boldsymbol{p}})\|_{(W^{4,q}(\Omega))^3\times(W^{4,q}(\Omega))^3}=\Big\{\sum_{i=1}^{3}\|p^i\|^q_{4,q,\Omega}+\sum_{i=1}^{3}\|\underset{\sim}{p}^i\|^q_{4,q,\Omega}\Big\}^{1/q}\,,
$$
$$
\|(\boldsymbol{L}_s,\underset{\sim}{\boldsymbol{L}}_s)\|_{(W^{5,s}(\gamma_1))^{12}\times(W^{5,s}(\underset{\sim}{\gamma}_1))^{12}}=\Big\{\sum_{I=1}^{12}\|\boldsymbol{L}_{sI}\|^s_{5,s,\gamma_1}+\sum_{I=1}^{12}\|\underset{\sim}{\boldsymbol{L}}_{sI}\|^s_{5,s,\underset{\sim}{\gamma}_1}\Big\}^{1/s}\,,
$$

where q,s are two integer numbers $\geqslant 1$.

THEOREM 5.4.1. Let $\mathcal{T}_h$ and $\underset{\sim}{\mathcal{T}}_h$ be two regular families of triangulations of the domains Ω and $\underset{\sim}{\Omega}$ satisfying the properties (3.1) to (3.3). Let (K, P_K, Σ_K) and $(\underset{\sim}{K}, P_{\underset{\sim}{K}}, \Sigma_{\underset{\sim}{K}})$ be two almost affine families of finite elements associated with the Argyris triangle. Moreover, we assume that the numerical integration scheme on the reference triangle $\hat{K}$ satisfies the following conditions:

(i) the integration nodes $\hat{b}_\ell \in \overline{\hat{K}}$, $\forall\, \ell = 1, \dots, L$;

(ii) $\hat{E}(\hat{\varphi}) = 0$, $\forall\, \hat{\varphi} \in P_8(\hat{K})$.

Likewise, the numerical integration scheme on the reference interval $\hat{K}'$ verifies

(iii) the integration nodes $\hat{b}'_\ell \in \overline{\hat{K}'}$, $\forall\, \ell = 1, \dots, L'$;

(iv) $\hat{E}'(\hat{\varphi}) = 0$, $\forall\, \hat{\varphi} \in P_8(\hat{K}')$.

Assume that

(v) the solution $(\boldsymbol{u}^k; \underset{\sim}{\boldsymbol{u}}^k) \in W_{\text{hel}}$ of the continuous problem (Part 1, (3.10)) belongs to the space $K(\Omega) \times \underset{\sim}{K}(\underset{\sim}{\Omega}) = (H^5(\Omega))^2 \times H^6(\Omega) \times (H^5(\underset{\sim}{\Omega}))^2 \times H^6(\underset{\sim}{\Omega})$,

(vi) $\boldsymbol{A}_{IJ} \in W^{4,\infty}(\Omega)$, $\underset{\sim}{\boldsymbol{A}}_{IJ} \in W^{4,\infty}(\underset{\sim}{\Omega})$ for $I, J = 1, \dots, 12$;

(vii) $\underset{\sim}{C}_{IJ} \in W^{4,\infty}(\omega)$ for $I, J = 1, \dots, 24$;

(viii) $p^i \in W^{4,q}(\Omega)$, $\underset{\sim}{p}^i \in W^{4,q}(\underset{\sim}{\Omega})$ for $i = 1, \dots, 3$;

(ix) $N^i \in W^{5,s}(\gamma)$, $\underset{\sim}{N}^i \in W^{5,s}(\underset{\sim}{\gamma})$ for $i = 1, \dots, 3$;

(x) $\boldsymbol{L}_{sI} \in W^{5,s}(\gamma_1)$, $\underset{\sim}{\boldsymbol{L}}_{sI} \in W^{5,s}(\underset{\sim}{\gamma}_1)$ for $I = 1, \dots, 12$,

where q, s are integer numbers ≥ 2.

Then, there exist constants $C > 0$ and $h_1 > 0$, independent of h, such that for any $h \in]0, h_1]$, we have

$$\left.\begin{aligned} \|(\boldsymbol{u}^k; \underset{\sim}{\boldsymbol{u}}^k) - (\boldsymbol{u}_h^k; \underset{\sim}{\boldsymbol{u}}_h^k)\|_E \leq Ch^4\{&\|(\boldsymbol{u}^k; \underset{\sim}{\boldsymbol{u}}^k)\|_{K(\Omega)\times \underset{\sim}{K}(\underset{\sim}{\Omega})} + \|(\boldsymbol{p}; \underset{\sim}{\boldsymbol{p}})\|_{(W^{4,q}(\Omega))^3 \times (W^{4,q}(\underset{\sim}{\Omega}))^3} \\ &+ h^{1/2}[\|(\boldsymbol{N}; \underset{\sim}{\boldsymbol{N}})\|_{(W^{5,s}(\gamma))^3 \times (W^{5,s}(\underset{\sim}{\gamma}))^3} + \|(\boldsymbol{L}_s, \underset{\sim}{\boldsymbol{L}}_s)\|_{(W^{5,s}(\gamma_1))^{12} \times (W^{5,s}(\underset{\sim}{\gamma}_1))^{12}}]\} \end{aligned}\right\} \tag{5.15}$$

where $(\boldsymbol{u}_h^k; \underset{\sim}{\boldsymbol{u}}_h^k)$ is the solution of the discrete problem (5.3).

PROOF. The conditions for applying Theorem 5.3.1 are satisfied. Hence, the condition of uniform W_{hel}-ellipticity is verified and it is possible to apply Theorem 5.2.1. Therefore, we are going to evaluate the different terms of the second hand member of the inequality (5.8). The proof takes five steps.

Step 1: Estimate of $\inf_{(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h) \in W_{\text{hel}}} \|(\boldsymbol{u}^k; \underset{\sim}{\boldsymbol{u}}^k) - (\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)\|_E$

Let Π_h be the W_{hel} interpolation operator on the space W_{el}. We define

$$\Pi_h(\boldsymbol{v}; \underset{\sim}{\boldsymbol{v}}) = (\pi_h \boldsymbol{v}, \underset{\sim}{\pi}_h \underset{\sim}{\boldsymbol{v}})$$

where π_h and $\underset{\sim}{\pi}_h$ are the associated interpolation operators on V_h and $\underset{\sim}{V}_h$. Then, with Ciarlet ([8, p. 124])

$$\left.\begin{aligned} \inf_{(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h) \in W_{\text{hel}}} \|(\boldsymbol{u}^k; \underset{\sim}{\boldsymbol{u}}^k) - (\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)\|_E &\leq \|(\boldsymbol{u}^k; \underset{\sim}{\boldsymbol{u}}^k) - \Pi_h(\boldsymbol{u}^k; \underset{\sim}{\boldsymbol{u}}^k)\|_E \\ &\leq Ch^4 \|(\boldsymbol{u}^k; \underset{\sim}{\boldsymbol{u}}^k)\|_{K(\Omega)\times \underset{\sim}{K}(\underset{\sim}{\Omega})}\,. \end{aligned}\right\} \tag{5.16}$$

Step 2: Estimate of

$$\sup_{(\boldsymbol{w}_h; \underset{\sim}{\boldsymbol{w}}_h) \in W_{\text{hel}}} \frac{|a[\Pi_h(\boldsymbol{u}^k; \underset{\sim}{\boldsymbol{u}}^k), (\boldsymbol{w}_h; \underset{\sim}{\boldsymbol{w}}_h)] - a_h[\Pi_h(\boldsymbol{u}^k; \underset{\sim}{\boldsymbol{u}}^k), (\boldsymbol{w}_h; \underset{\sim}{\boldsymbol{w}}_h)]|}{\|(\boldsymbol{w}_h; \underset{\sim}{\boldsymbol{w}}_h)\|_E}$$

By using relations (2.1) and (5.1):

$$\left.\begin{aligned} &|a[\Pi_h(\boldsymbol{u}^k; \underset{\sim}{\boldsymbol{u}}^k), (\boldsymbol{w}_h; \underset{\sim}{\boldsymbol{w}}_h)] - a_h[\Pi_h(\boldsymbol{u}^k; \underset{\sim}{\boldsymbol{u}}^k), (\boldsymbol{w}_h; \underset{\sim}{\boldsymbol{w}}_h)| \\ &\quad \leq \sum_{K \in \mathcal{T}_h} \sum_{I,J=1}^{12} |E_K(\boldsymbol{A}_{IJ}[\pi_h \boldsymbol{U}^k]_I \boldsymbol{W}_J)| + \sum_{\underset{\sim}{K} \in \underset{\sim}{\mathcal{T}}_h} \sum_{I,J=1}^{12} |E_{\underset{\sim}{K}}(\underset{\sim}{\boldsymbol{A}}_{IJ}[\underset{\sim}{\pi}_h \underset{\sim}{\boldsymbol{U}}^k]_I \underset{\sim}{\boldsymbol{W}}_J)|\,. \end{aligned}\right\} \tag{5.17}$$

We restrict our attention to the first term of the second-hand number of inequality (5.17). The hypotheses of Bernadou [5, Theorem 1.3.3, p. 57] are verified. Thus, we obtain the existence of a constant $C > 0$, independent of h, such that

$$\sum_{K\in\mathcal{T}_h}\sum_{I,J=1}^{12}|E_K(A_{IJ}[\pi_h U^k]_I W_J)| \leq C\sum_{K\in\mathcal{T}_h} h_K^4\left(\sum_{I,J=1}^{12}\|A_{IJ}\|_{4,\infty,K}\right)(\|\pi_h u_1^k\|_{5,K}^2+\|\pi_h u_2^k\|_{5,K}^2+\|\pi_h u_3^k\|_{6,K}^2)^{1/2}\|w_h\|_{V(K)}$$

where

$$\pi_h u = [\pi_h u_1, (\pi_h u_1)_{,1}, (\pi_h u_1)_{,2}, \pi_h u_2, (\pi_h u_2)_{,1}, (\pi_h u_2)_{,2}, \pi_h u_3, (\pi_h u_3)_{,1}, (\pi_h u_3)_{,2}, (\pi_h u_3)_{,11}, (\pi_h u_3)_{,12}, (\pi_h u_3)_{,22}]$$

The interpolation operator π_h leaves the space $P_5(K)$ invariant, and we obtain

$$\|\pi_h u_i\|_{5,K} \leq \|u_i\|_{5,K} + \|u_i - \pi_h u_i\|_{5,K} \leq C\|u_i\|_{5,K}, \quad i=1,2,3.$$

Thus,

$$\sum_{K\in\mathcal{T}_h}\sum_{I,J=1}^{12}|E_K(A_{IJ}[\pi_h U^k]_I[W]_{hJ})| \leq C\max_{K\in\mathcal{T}_h}(h_K)^4\left(\sum_{I,J=1}^{12}\|A_{IJ}\|_{4,\infty,\Omega}\right)\|u^k\|_{K(\Omega)}\|w_h\|_{V(\Omega)} \tag{5.18}$$

Similarly, for the shell $\underset{\sim}{\mathcal{S}}$, we could prove

$$\sum_{\underset{\sim}{K}\in\underset{\sim}{\mathcal{T}}_h}\sum_{I,J=1}^{12}|E_{\underset{\sim}{K}}(\underset{\sim}{A}_{IJ}[\underset{\sim}{\pi}_h\underset{\sim}{U}^k]_I[\underset{\sim}{W}]_{hJ})| \leq C\max_{\underset{\sim}{K}\in\underset{\sim}{\mathcal{T}}_h}(h_{\underset{\sim}{K}})^4\left(\sum_{I,J=1}^{12}\|\underset{\sim}{A}_{IJ}\|_{4,\infty,\underset{\sim}{\Omega}}\right)\|\underset{\sim}{u}^k\|_{\underset{\sim}{K}(\underset{\sim}{\Omega})}\|\underset{\sim}{w}_h\|_{\underset{\sim}{V}(\underset{\sim}{\Omega})} \tag{5.19}$$

By combining inequalities (5.18), (5.19) and definition (3.2), we obtain

$$\begin{aligned}&|a[\Pi_h(u^k;\underset{\sim}{u}^k),(w_h;\underset{\sim}{w}_h)] - a_h[\Pi_h(u^k;\underset{\sim}{u}^k),(w_h;\underset{\sim}{w}_h)]|\\ &\leq Ch^4\sup\left\{\sum_{I,J=1}^{12}\|A_{IJ}\|_{4,\infty,\Omega}, \sum_{I,J=1}^{12}\|\underset{\sim}{A}_{IJ}\|_{4,\infty,\underset{\sim}{\Omega}}\right\}\{\|u^k\|_{K(\Omega)}\|w_h\|_{V(\Omega)} + \|\underset{\sim}{u}^k\|_{\underset{\sim}{K}(\underset{\sim}{\Omega})}\|\underset{\sim}{w}_h\|_{\underset{\sim}{V}(\underset{\sim}{\Omega})}\}\end{aligned}$$

so that

$$\left.\begin{aligned}&\sup_{(w_h;\underset{\sim}{w}_h)\in W_{hel}}\frac{|a[\Pi_h(u^k;\underset{\sim}{u}^k),(w_h;\underset{\sim}{w}_h)] - a_h[\Pi_h(u^k;\underset{\sim}{u}^k),(w_h;\underset{\sim}{w}_h)]|}{\|(w_h;\underset{\sim}{w}_h)\|_E}\\ &\leq Ch^4\sup\left\{\sum_{I,J=1}^{12}\|A_{IJ}\|_{4,\infty,\Omega}, \sum_{I,J=1}^{12}\|\underset{\sim}{A}_{IJ}\|_{4,\infty,\underset{\sim}{\Omega}}\right\}\|(u^k;\underset{\sim}{u}^k)\|_{K(\Omega)\times\underset{\sim}{K}(\underset{\sim}{\Omega})}\end{aligned}\right\} \tag{5.20}$$

Step 3: Estimate of

$$\sup_{(w_h;\underset{\sim}{w}_h)\in W_{hel}}\frac{|b[\Pi_h(\underline{u}^k;\underset{\sim}{\underline{u}}^k),(\underline{w}_h;\underset{\sim}{\underline{w}}_h)] - b_h[\Pi_h(\underline{u}^k;\underset{\sim}{\underline{u}}^k),(\underline{w}_h;\underset{\sim}{\underline{w}}_h)]|}{\|(w_h;\underset{\sim}{w}_h)\|_E}$$

The restriction of an Argyris triangle to one of its side is a P_5-one-dimensional finite element, so that we define the interpolating function:

$$\Pi_h(\underline{u}^k;\underset{\sim}{\underline{u}}^k) = (\pi_h u^k|_\gamma \circ F, \underset{\sim}{\pi}_h\underset{\sim}{u}^k|_{\underset{\sim}{\gamma}}\circ\underset{\sim}{F}).$$

By using the matrix expressions (2.4) and (5.5) of the bilinear forms $b[.,.]$ and $b_h[.,.]$, a similar one-dimensional study to the previous ones gives (for details, see Cubier [6, Theorem 5.2.4, p. 83])

$$\begin{aligned}&|b[\Pi_h(\underline{u}^k;\underset{\sim}{\underline{u}}^k),(\underline{w}_h;\underset{\sim}{\underline{w}}_h)] - b_h[\Pi_h(\underline{u}^k;\underset{\sim}{\underline{u}}^k),(\underline{w}_h;\underset{\sim}{\underline{w}}_h)]|\\ &\leq Ch^4\left(\sum_{I,J=1}^{24}\|\underline{C}_{IJ}\|_{4,\infty,\underline{K}}\right)[\|u^k\|_{K(\Omega)}\|w_h\|_{V(\Omega)} + \|\underset{\sim}{u}^k\|_{\underset{\sim}{K}(\underset{\sim}{\Omega})}\|\underset{\sim}{w}_h\|_{\underset{\sim}{V}(\underset{\sim}{\Omega})}]\end{aligned}$$

so that, we obtain

$$\left.\begin{aligned}&\frac{|b[\Pi_h(\underline{u}^k;\underset{\sim}{u}^k),(\underline{w}_h;\underset{\sim}{\underline{w}}_h)]-b_h[\Pi_h(\underline{u}^k;\underset{\sim}{u}^k),(\underline{w}_h;\underset{\sim}{\underline{w}}_h)]|}{\|(w_h;\underset{\sim}{w}_h)\|_E}\\&\leqslant Ch^4\Bigg(\sum_{I,J=1}^{24}\|\underset{\sim}{C}_{IJ}\|_{4,\infty,\omega}\Bigg)\|(u^k;\underset{\sim}{u}^k)\|_{K(\Omega)\times\underset{\sim}{K}(\underset{\sim}{\Omega})}\,.\end{aligned}\right\}\tag{5.21}$$

Step 4: Estimate of

$$\sup_{(w_h;\underset{\sim}{w}_h)\in W_{\mathrm{hel}}}\frac{|a[(u^k;\underset{\sim}{u}^k),(w_h;\underset{\sim}{w}_h)]+kb[(\underline{u}^k;\underset{\sim}{\underline{u}}^k),(\underline{w}_h;\underset{\sim}{\underline{w}}_h)]-\ell_h[(w_h;\underset{\sim}{w}_h)]-f_h[(w_h;\underset{\sim}{w}_h)]|}{\|(w_h;\underset{\sim}{w}_h)\|_E}$$

By using relation (4.7), we obtain the new estimate:

$$\begin{aligned}&\sup_{(w_h;\underset{\sim}{w}_h)\in W_{\mathrm{hel}}}\frac{|a[(u^k;\underset{\sim}{u}^k),(w_h;\underset{\sim}{w}_h)]+kb[(\underline{u}^k;\underset{\sim}{\underline{u}}^k),(\underline{w}_h;\underset{\sim}{\underline{w}}_h)]-\ell_h[(w_h;\underset{\sim}{w}_h)]-f_h[(w_h;\underset{\sim}{w}_h)]|}{\|(w_h;\underset{\sim}{w}_h)\|_E}\\&\leqslant\sup_{(w_h;\underset{\sim}{w}_h)\in W_{\mathrm{hel}}}\frac{|\ell[(w_h;\underset{\sim}{w}_h)]-\ell_h[(w_h;\underset{\sim}{w}_h)]|}{\|(w_h;\underset{\sim}{w}_h)\|_E}+\sup_{(w_h;\underset{\sim}{w}_h)\in W_{\mathrm{hel}}}\frac{|f[(w_h;\underset{\sim}{w}_h)]-f_h[(w_h;\underset{\sim}{w}_h)]|}{\|(w_h;\underset{\sim}{w}_h)\|_E}\end{aligned}$$

In this inequality, we find the term $|f[(w_h;\underset{\sim}{w}_h)]-f_h[(w_h;\underset{\sim}{w}_h)]|$ which combines the errors due to the nonconformity of the finite element method and to the use of numerical integration. Thus, it remains to study the following estimates

- Estimate of

$$\sup_{(w_h;\underset{\sim}{w}_h)\in W_{\mathrm{hel}}}\frac{|\ell[(w_h;\underset{\sim}{w}_h)]-\ell_h[(w_h;\underset{\sim}{w}_h)]|}{\|(w_h;\underset{\sim}{w}_h)\|_E}$$

We can rewrite the linear forms $\ell[.]$ and $\ell_h[.]$ by using the matrix decompositions introduced in Section 2; thus we obtain

$$\left.\begin{aligned}|\ell[(w_h;\underset{\sim}{w}_h)]-\ell_h[(w_h;\underset{\sim}{w}_h)]|\leqslant{}&\sum_{K\in\mathcal{T}_h}\sum_{I=1}^{12}|E_K([P]_I[W_h]_I)|+\sum_{\underset{\sim}{K}\in\underset{\sim}{\mathcal{T}}_h}\sum_{I=1}^{12}|E_{\underset{\sim}{K}}([\underset{\sim}{P}]_I[\underset{\sim}{W}_h]_I)|\\&+\sum_{K'\in G_1}\sum_{I=1}^{12}|E'_{K'}([L_s]_I[W_h]_I)|+\sum_{\underset{\sim}{K}'\in\underset{\sim}{G}_1}\sum_{I=1}^{12}|E'_{\underset{\sim}{K}'}([\underset{\sim}{L}_s]_I[\underset{\sim}{W}_h]_I)|\end{aligned}\right\}\tag{5.22}$$

where the sets G_1 and $\underset{\sim}{G}_1$ are defined in (5.6). The first two terms of the second-hand member of (5.22) correspond to body force resultants; the last two terms correspond to surface load resultants. In order to estimate these terms, we can use Bernadou [5, Theorem 1.3.2, p. 54] and their one-dimensional counterparts by Cubier [6, Theorem 5.2.3, p. 79]. That leads to

$$\begin{aligned}|\ell[(w_h;\underset{\sim}{w}_h)]-\ell_h[(w_h;\underset{\sim}{w}_h)]|\leqslant{}&C\sum_{K\in\mathcal{T}_h}h_K^4(\mathrm{mes}(K))^{1/2-1/q}\Bigg(\sum_{I=1}^{12}\|P_I\|_{4,q,K}^q\Bigg)^{1/q}\|w_h\|_{V(K)}\\&+C\sum_{\underset{\sim}{K}\in\underset{\sim}{\mathcal{T}}_h}h_{\underset{\sim}{K}}^4(\mathrm{mes}(\underset{\sim}{K}))^{1/2-1/q}\Bigg(\sum_{I=1}^{12}\|\underset{\sim}{P}_I\|_{4,q,\underset{\sim}{K}}^q\Bigg)^{1/q}\|\underset{\sim}{w}_h\|_{\underset{\sim}{V}(\underset{\sim}{K})}\\&+C\sum_{K\in G_1}h_{K^*}^{9/2}(\mathrm{mes}(K'))^{1/2-1/s}\Bigg(\sum_{I=1}^{12}\|L_{sI}\|_{5,s,K'}^s\Bigg)^{1/s}\|w_h\|_{V(K^*)}\\&+C\sum_{\underset{\sim}{K}\in\underset{\sim}{G}_1}h_{\underset{\sim}{K}^*}^{9/2}(\mathrm{mes}(\underset{\sim}{K}'))^{1/2-1/s}\Bigg(\sum_{I=1}^{12}\|\underset{\sim}{L}_{sI}\|_{5,s,\underset{\sim}{K}'}^s\Bigg)^{1/s}\|\underset{\sim}{w}_h\|_{\underset{\sim}{V}(\underset{\sim}{K}^*)}\,,\end{aligned}$$

where K^* (resp. $\underset{\sim}{K}^*$) denote the triangles belonging to $\mathcal{T}_h$ (resp. $\underset{\sim}{\mathcal{T}}_h$) which count the sides K' (resp. $\underset{\sim}{K}'$) located on γ_1 (resp. $\underset{\sim}{\gamma}_1$) among their own sides. Fig. 3 summarizes these notations.

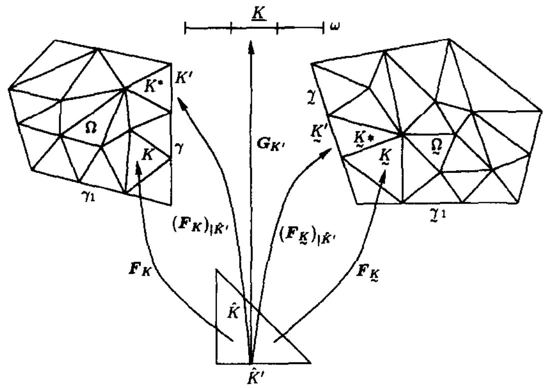


Fig. 3. Some notations.

From definitions (2.6) of vectors $\boldsymbol{P}$ and $\underset{\sim}{\boldsymbol{P}}$, we have

$$\sum_{I=1}^{12} \|\boldsymbol{P}_I\|^q_{4,q,K} = \sum_{i=1}^{3} \|p^i\|^q_{4,q,K} \quad \text{and} \quad \sum_{I=1}^{12} \|\underset{\sim}{\boldsymbol{P}}_I\|^q_{4,q,\underset{\sim}{K}} = \sum_{i=1}^{3} \|\underset{\sim}{p}^i\|^q_{4,q,\underset{\sim}{K}} \,.$$

Then, we use the inequality

$$\sum_K |a_K b_K c_K| \leqslant \Big(\sum_K |a_K|^\alpha\Big)^{1/\alpha}\Big(\sum_K |b_K|^\beta\Big)^{1/\beta}\Big(\sum_K |c_K|^\gamma\Big)^{1/\gamma}$$

which is valid for any real numbers α, β, $\gamma \geqslant 1$ satisfying $1/\alpha + 1/\beta + 1/\gamma = 1$. Here, we take $1/\alpha = 1/2 - 1/q$, $\beta = q$ and $\gamma = 2$ for body force terms, and $1/\alpha = 1/2 - 1/s$, $\beta = s$ and $\gamma = 2$ for surface load terms, for any $q \geqslant 2$ and $s \geqslant 2$. Hence

$$\left.\begin{aligned} \sup_{(\boldsymbol{w}_h;\, \underset{\sim}{\boldsymbol{w}}_h) \in W_{\text{hel}}} \frac{|\ell[(\boldsymbol{w}_h;\underset{\sim}{\boldsymbol{w}}_h)] - \ell_h[(\boldsymbol{w}_h;\underset{\sim}{\boldsymbol{w}}_h)]|}{\|(\boldsymbol{w}_h;\underset{\sim}{\boldsymbol{w}}_h)\|_E} &\leqslant Ch^4\{\|(\boldsymbol{p};\underset{\sim}{\boldsymbol{p}})\|_{(W^{4,q}(\Omega))^3\times(W^{4,q}(\underset{\sim}{\Omega}))^3} \\ &\quad + h^{1/2}\|(\boldsymbol{L}_s, \underset{\sim}{\boldsymbol{L}}_s)\|_{(W^{5,s}(\gamma_1))^{12}\times(W^{5,s}(\underset{\sim}{\gamma}_1))^{12}}\} \end{aligned}\right\} \tag{5.23}$$

• Estimate of

$$\sup_{(\boldsymbol{w}_h;\, \underset{\sim}{\boldsymbol{w}}_h) \in W_{\text{hel}}} \frac{|f[(\boldsymbol{w}_h;\underset{\sim}{\boldsymbol{w}}_h)] - f_h[(\boldsymbol{w}_h;\underset{\sim}{\boldsymbol{w}}_h)]|}{\|(\boldsymbol{w}_h;\underset{\sim}{\boldsymbol{w}}_h)\|_E}$$

The final result is obtained by taking into account the matrix expression of $f[.]$ and $f_h[.]$ defined by relation (5.7); a proof similar to the previous ones (see Cubier [6, Theorem 5.2.3, p. 79]) would lead to

$$\sup_{(\boldsymbol{w}_h;\, \underset{\sim}{\boldsymbol{w}}_h) \in W_{\text{hel}}} \frac{|f[(\boldsymbol{w}_h;\underset{\sim}{\boldsymbol{w}}_h)] - f_h[(\boldsymbol{w}_h;\underset{\sim}{\boldsymbol{w}}_h)]|}{\|(\boldsymbol{w}_h;\underset{\sim}{\boldsymbol{w}}_h)\|_E} \leqslant Ch^{9/2}\|(\boldsymbol{N};\underset{\sim}{\boldsymbol{N}})\|_{(W^{5,s}(\gamma))^3\times(W^{5,s}(\underset{\sim}{\gamma}))^3} \tag{5.24}$$

Step 5: Final estimate (5.15).

We can apply Theorem 5.2.1 and we get the estimate (5.15) by substitution of the inequalities (5.16), (5.20), (5.21), (5.23) and (5.24) into the abstract error estimate (5.8) written for $(\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h) = \Pi_h(\boldsymbol{u}^k;\underset{\sim}{\boldsymbol{u}}^k)$. □

6. Extension to the rigid junction problem

The rigid junction problem can be stated as follows:

$$\left.\begin{array}{l}\text{Find } (\boldsymbol{u}_{h\mathrm{rig}};\underset{\sim}{\boldsymbol{u}}_{h\mathrm{rig}})\in W_{h\mathrm{rig}} \text{ such that for any } (\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h)\in W_{h\mathrm{rig}}\\ a_h[(\boldsymbol{u}_{h\mathrm{rig}};\underset{\sim}{\boldsymbol{u}}_{h\mathrm{rig}}),(\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h)]=\ell_h[(\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h)]+f_h^*[(\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h)]\end{array}\right\}\tag{6.1}$$

where the space $W_{h\mathrm{rig}}$, the bilinear form $a_h[.,.]$ and the form $\ell_h[.]$ are, respectively, given by relations (3.13), (5.4), (5.6) and, concerning the approximation along the hinge:

$$f_h^*[(\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h)]=\sum_{K'\in G}\sum_{\ell=1}^{L'}\omega_{\ell,K'}\{{}^{\mathrm{T}}[\boldsymbol{L}_s][\boldsymbol{V}_h]\}(b_{\ell,K'})+\sum_{\underset{\sim}{K}'\in \underset{\sim}{G}}\sum_{\ell=1}^{L'}\omega_{\ell,\underset{\sim}{K}'}\{{}^{\mathrm{T}}[\underset{\sim}{\boldsymbol{L}}_s][\underset{\sim}{\boldsymbol{V}}_h]\}(b_{\ell,\underset{\sim}{K}'})$$

where $\boldsymbol{L}_s$ and $\underset{\sim}{\boldsymbol{L}}_s$ are defined in (2.7), and sets G and $\underset{\sim}{G}$ are introduced in relation (5.7).

THEOREM 6.1 (Abstract error estimate). Let us consider a family of discrete problems (6.1). We suppose that the bilinear forms $a_h[.,.]$ are $W_{h\mathrm{rig}}$-elliptic, uniformly with respect to h, i.e. there exists a constant $\beta>0$, independent of h, such that

$$a_h[(\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h),(\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h)]\geqslant\beta\|(\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h)\|_E^2\,,\quad\forall\,(\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h)\in W_{h\mathrm{rig}}\,.$$

Then, there exists a constant $C>0$, independent of h, such that

$$\begin{aligned}\|(\boldsymbol{u}_{\mathrm{rig}};\underset{\sim}{\boldsymbol{u}}_{\mathrm{rig}})-(\boldsymbol{u}_{h\mathrm{rig}};\underset{\sim}{\boldsymbol{u}}_{h\mathrm{rig}})\|_E\leqslant C\inf_{(\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h)\in W_{h\mathrm{rig}}}&\left\{\|(\boldsymbol{u}_{\mathrm{rig}};\underset{\sim}{\boldsymbol{u}}_{\mathrm{rig}})-(\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h)\|_E\right.\\ &\left.+\sup_{(\boldsymbol{w}_h;\underset{\sim}{\boldsymbol{w}}_h)\in W_{h\mathrm{rig}}}\frac{|a[(\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h),(\boldsymbol{w}_h;\underset{\sim}{\boldsymbol{w}}_h)]-a_h[(\boldsymbol{v}_h;\underset{\sim}{\boldsymbol{v}}_h),(\boldsymbol{w}_h;\underset{\sim}{\boldsymbol{w}}_h)]|}{\|(\boldsymbol{w}_h;\underset{\sim}{\boldsymbol{w}}_h)\|_E}\right\}\\ &+C\sup_{(\boldsymbol{w}_h;\underset{\sim}{\boldsymbol{w}}_h)\in W_{h\mathrm{rig}}}\frac{|a[(\boldsymbol{u}_{\mathrm{rig}};\underset{\sim}{\boldsymbol{u}}_{\mathrm{rig}}),(\boldsymbol{w}_h;\underset{\sim}{\boldsymbol{w}}_h)]-\ell_h[(\boldsymbol{w}_h;\underset{\sim}{\boldsymbol{w}}_h)]-f_h^*[(\boldsymbol{w}_h;\underset{\sim}{\boldsymbol{w}}_h)]|}{\|(\boldsymbol{w}_h;\underset{\sim}{\boldsymbol{w}}_h)\|_E}\end{aligned}$$

where $(\boldsymbol{u}_{\mathrm{rig}};\underset{\sim}{\boldsymbol{u}}_{\mathrm{rig}})$ and $(\boldsymbol{u}_{h\mathrm{rig}};\underset{\sim}{\boldsymbol{u}}_{h\mathrm{rig}})$ denote the solutions of the continuous and discrete problems (Part 1, (3.24)) and (6.1).

PROOF. The proof is similar to that of Theorem 5.2.1 for the elastic hinge problem; it suffices to note that $b[.,.]=b_h[.,.]=0$. □

THEOREM 6.2 (Asymptotic error estimate). Let $\mathcal{T}_h$ and $\underset{\sim}{\mathcal{T}}_h$ be two regular families of triangulations of the domains Ω and $\underset{\sim}{\Omega}$ satisfying the properties (3.1) to (3.3). Let (K, P_K, Σ_K) and $(\underset{\sim}{K}, P_{\underset{\sim}{K}}, \Sigma_{\underset{\sim}{K}})$ be two almost affine families of finite elements associated with the Argyris triangle. Moreover, we assume that the numerical integration scheme on the reference triangle $\hat{K}$ satisfies the following conditions:

(i) the integration nodes $\hat{b}_\ell\in\hat{K}$, $\forall\,\ell=1,\dots,L$;
(ii) $\hat{E}(\hat{\varphi})=0$, $\forall\,\hat{\varphi}\in P_8(\hat{K})$.
Likewise, the numerical integration scheme on the reference interval $\hat{K}'$ verifies
(iii) the integration nodes $\hat{b}'_\ell\in\hat{K}'$, $\forall\,\ell=1,\dots,L'$;
(iv) $\hat{E}'(\hat{\varphi})=0$, $\forall\,\hat{\varphi}\in P_8(\hat{K}')$.

Assume that
(v) the solution $(\boldsymbol{u}_{\mathrm{rig}};\underset{\sim}{\boldsymbol{u}}_{\mathrm{rig}})\in W_{\mathrm{rig}}$ of the continuous problem (Part 1, (3.24)) belongs to the space $K(\Omega)\times\underset{\sim}{K}(\underset{\sim}{\Omega})=(H^5(\Omega))^2\times H^6(\Omega)\times(H^5(\underset{\sim}{\Omega}))^2\times H^6(\underset{\sim}{\Omega})$,
(vi) $A_{IJ}\in W^{4,\infty}(\Omega)$, $\underset{\sim}{A}_{IJ}\in W^{4,\infty}(\underset{\sim}{\Omega})$ for $I,J=1,\dots,12$;
(vii) $p^i\in W^{4,q}(\Omega)$, $\underset{\sim}{p}^i\in W^{4,q}(\underset{\sim}{\Omega})$ for $i=1,\dots,3$;
(viii) $\boldsymbol{L}_{sI}\in W^{5,s}(\gamma_1\cup\gamma)$, $\underset{\sim}{\boldsymbol{L}}_{sI}\in W^{5,s}(\underset{\sim}{\gamma}_1\cup\gamma)$ for $I=1,\dots,12$,
where q, s are integer numbers $\geqslant 2$. Then, there exist constants $C>0$ and $h_1>0$, independent of h, such that for any $h\in\,]0,h_1]$, we have

$$\|(\boldsymbol{u}_{\rm rig}; \underset{\sim}{\boldsymbol{u}}_{\rm rig}) - (\boldsymbol{u}_{h\rm rig}; \underset{\sim}{\boldsymbol{u}}_{h\rm rig})\|_E \leq Ch^4\{\|(\boldsymbol{u}_{\rm rig}; \underset{\sim}{\boldsymbol{u}}_{\rm rig})\|_{K(\Omega)\times \underset{\sim}{K}(\underset{\sim}{\Omega})}$$
$$+ \|(\boldsymbol{p}; \underset{\sim}{\boldsymbol{p}})\|_{(W^{4,q}(\Omega))^3\times(W^{4,q}(\underset{\sim}{\Omega}))^3} + h^{1/2}\|(\boldsymbol{L}_s, \underset{\sim}{\boldsymbol{L}}_s)\|_{(W^{5,s}(\gamma_1\cup\gamma))^{12}\times(W^{5,s}(\underset{\sim}{\gamma}_1\cup\underset{\sim}{\gamma}))^{12}}\}$$

where $(\boldsymbol{u}_{h\rm rig}; \underset{\sim}{\boldsymbol{u}}_{h\rm rig})$ *is the solution of the discrete problem (6.1).*

PROOF. The uniform ellipticity arises from the equivalence between the norm $\|.\|_E$ and the norm on $W_{h\rm rig}$ defined by $\|(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)\|_{W_{h\rm rig}} = \{a_h[(\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h), (\boldsymbol{v}_h; \underset{\sim}{\boldsymbol{v}}_h)]\}^{1/2}$ and from Bernadou [5, Theorem 1.3.3]. Then, we can obtain the asymptotic error estimate exactly in the same way than for the elastic junction problem. □

7. Numerical results

7.1. Description of the example

We consider the example given in (Part 1, Section 2.4), i.e. a circular cylinder with a spherical end cap subjected to a vertical body force. The cylinder is clamped along its foundations. The notations and the data used for this benchmark are given in Fig. 4 and Table 1. The axisymmetry of this structure leads to simplifications.

The mappings are given by

$$\boldsymbol{\Phi}: \begin{cases} x = r\cos\xi^2 \\ y = r\sin\xi^2 \\ z = \xi^1 \end{cases} \qquad \underset{\sim}{\boldsymbol{\Phi}}: \begin{cases} x = R\cos\left(\delta + \dfrac{\xi^1 - L}{R}\right)\cos\xi^2 \\ y = R\cos\left(\delta + \dfrac{\xi^1 - L}{R}\right)\sin\xi^2 \\ z = R\sin\left(\delta + \dfrac{\xi^1 - L}{R}\right) + L - R\sin\delta \end{cases}$$

Here, we consider the junction between the sphere and the cylinder; the above mappings are such that $\boldsymbol{\Phi}^{-1}(\Gamma) = \underset{\sim}{\boldsymbol{\Phi}}^{-1}(\Gamma) = \gamma$. Thus, we do not need to introduce the third mapping $\underline{\boldsymbol{\Phi}}$ and the interval ω (see (2.3) and (2.4)). In this benchmark the bilinear form $b[.,.]$ is given by

$$b[(\boldsymbol{u}; \underset{\sim}{\boldsymbol{u}}), (\boldsymbol{v}; \underset{\sim}{\boldsymbol{v}})] = \int_\gamma \left(u_{3,1} - \underset{\sim}{u}_{3,1} - \frac{1}{R}\underset{\sim}{u}_1\right)\left(v_{3,1} - \underset{\sim}{v}_{3,1} - \frac{1}{R}\underset{\sim}{v}_1\right) \mathrm{d}\gamma$$

where $\mathrm{d}\gamma = r^2\,\mathrm{d}\xi^2$ (Part 1, (2.15) and (3.12)).

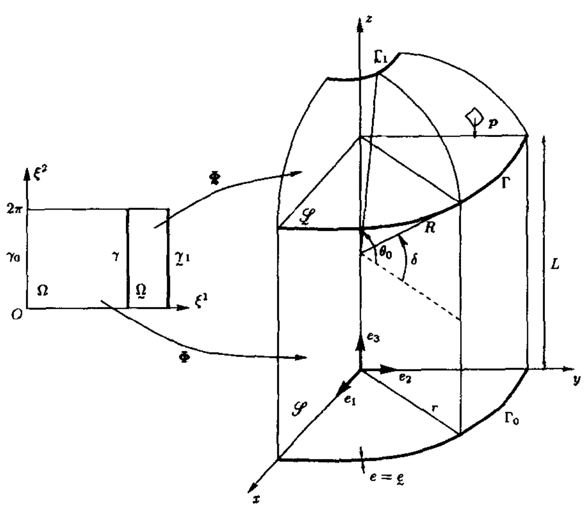


Fig. 4. Geometry of the problem.

Table 1
Data of the example

Dimensions	Material constants	Forces
$L = 130.9$ m $r = 95.1$ m; $R = 100$ m $e = \underset{\sim}{e} = 1$ m $\theta_0 = \frac{2\pi}{5}$; $\delta = \frac{\pi}{10}$	$E = \underset{\sim}{E} = 10^5\ \mathrm{N/m^2}$ $\nu = \underset{\sim}{\nu} = 0.33$	$p = \underset{\sim}{p} = -be_3$ $b = 10\ \mathrm{N/m^2}$

7.2. Reference solution

As a reference deformed configuration of the structure we use the approximated solution of the three-dimensional linearized elasticity model. The data allow us to restrict the problem to a two-dimensional axisymmetrical elasticity problem set on a radial section of the structure. We approximate this problem by a finite element method using P_1-Lagrange finite elements. In order to have a good approximation of the deformed configuration, we have used a mesh including 17 408 triangles. Fig. 5 gives the result that we have obtained with the Modulef code (see Bernadou et al. [11]).

7.3. Rigid junction using Argyris triangle

For this test, we use the Argyris triangle described in Fig. 2 and the mesh given in Fig. 6.

Because of the axisymmetry of the problem, we restrict the reference domains to an axial strip of width $\ell_b = \pi/16$ and we impose symmetry conditions on the boundaries δ_1 and δ_2 (Fig. 6). The rigid junction

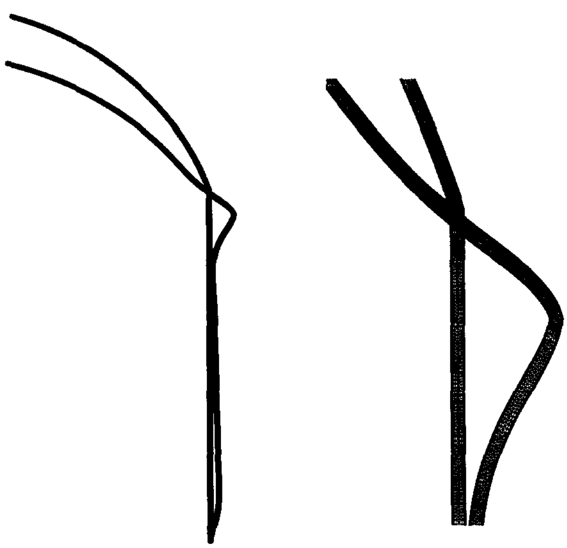


Fig. 5. Reference deformed configuration for rigid junction obtained from a 2d accurate approximation of the axisymmetrical problem and zoom of the junction.

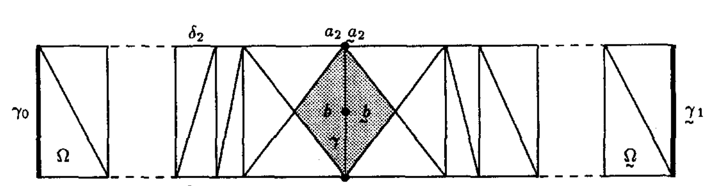


Fig. 6. Mesh.

conditions along γ are obtained from Eqs. (3.9) and (3.12). They consist of linear relations between the degrees of freedom located on γ. The equality of the displacements along the junction is given by $\underset{\sim}{u}_i = A_i^j u_j$ with $A_i^j = \underset{\sim}{a}_i \cdot a^j$. In our example, we have

$$A_1^2 = A_2^1 = A_2^3 = A_3^2 = 0\,, \qquad A_1^1 = A_3^3 = \cos\delta\,, \qquad A_2^2 = \frac{R}{r}\cos\delta = 1\,,$$
$$A_1^3 = \sin\delta\,, \qquad A_3^1 = -\sin\delta$$

where δ is defined in Table 1 and Fig. 4. Conditions (3.9) can be written as

$$\left\{\begin{aligned} &\underset{\sim}{u}_{h1}(\underset{\sim}{a}_\alpha) = \cos\delta\, u_{h1}(a_\alpha) + \sin\delta\, u_{h3}(a_\alpha) \\ &\underset{\sim}{u}_{h2}(\underset{\sim}{a}_\alpha) = u_{h2}(a_\alpha) \\ &\underset{\sim}{u}_{h3}(\underset{\sim}{a}_\alpha) = -\sin\delta\, u_{h1}(a_\alpha) + \cos\delta\, u_{h3}(a_\alpha) \end{aligned}\right\} \tag{7.1}$$

$$\left\{\begin{aligned} &\underset{\sim}{u}_{h1,2}(\underset{\sim}{a}_\alpha) = \cos\delta\, u_{h1,2}(a_\alpha) + \sin\delta\, u_{h3,2}(a_\alpha) \\ &\underset{\sim}{u}_{h2,2}(\underset{\sim}{a}_\alpha) = u_{h2,2}(a_\alpha) \\ &\underset{\sim}{u}_{h3,2}(\underset{\sim}{a}_\alpha) = -\sin\delta\, u_{h1,2}(a_\alpha) + \cos\delta\, u_{h3,2}(a_\alpha) \end{aligned}\right\} \tag{7.2}$$

$$\left\{\begin{aligned} &\underset{\sim}{u}_{h1,22}(\underset{\sim}{a}_\alpha) = \cos\delta\, u_{h1,22}(a_\alpha) + \sin\delta\, u_{h3,22}(a_\alpha) \\ &\underset{\sim}{u}_{h2,22}(\underset{\sim}{a}_\alpha) = u_{h2,22}(a_\alpha) \\ &\underset{\sim}{u}_{h3,22}(\underset{\sim}{a}_\alpha) = -\sin\delta\, u_{h1,22}(a_\alpha) + \cos\delta\, u_{h3,22}(a_\alpha) \end{aligned}\right\} \tag{7.3}$$

where a_α, $\underset{\sim}{a}_\alpha$ are displayed in Fig. 6, $\alpha = 1, 2$.

The equality of rotations is given by the conditions (see (3.12)):

$$u_{h3,1}(a_\alpha) - \underset{\sim}{u}_{h3,1}(\underset{\sim}{a}_\alpha) - \frac{1}{R}\underset{\sim}{u}_{h1}(\underset{\sim}{a}_\alpha) = 0$$

$$u_{h3,12}(a_\alpha) - \underset{\sim}{u}_{h3,12}(\underset{\sim}{a}_\alpha) - \frac{1}{R}\underset{\sim}{u}_{h1,2}(\underset{\sim}{a}_\alpha) = 0$$

$$u_{h31}(b) - \underset{\sim}{u}_{h3,1}(\underset{\sim}{b}) - \frac{1}{R}\underset{\sim}{u}_{h1}(\underset{\sim}{b}) = 0$$

where b, $\underset{\sim}{b}$ are the midpoints of $[a_1, a_2]$ and $[\underset{\sim}{a}_1, \underset{\sim}{a}_2]$. In the above relation, term $\underset{\sim}{u}_{h1}(\underset{\sim}{b})$ is not a degree of freedom of Argyris triangle, but the expression of the interpolating function for Argyris triangle (see for example [5, (1.5.1), p. 99]) gives

$$\begin{aligned} \underset{\sim}{u}_{h1}(\underset{\sim}{b}) &= \frac{1}{2}[\underset{\sim}{u}_{h1}(\underset{\sim}{a}_1) + \underset{\sim}{u}_{h1}(\underset{\sim}{a}_2)] + \frac{5}{32}[D\underset{\sim}{u}_{h1}(\underset{\sim}{a}_1)(\underset{\sim}{a}_2 - \underset{\sim}{a}_1) + D\underset{\sim}{u}_{h1}(\underset{\sim}{a}_2)(\underset{\sim}{a}_1 - \underset{\sim}{a}_2)] \\ &\quad + \frac{1}{64}[D^2\underset{\sim}{u}_{h1}(\underset{\sim}{a}_1)(\underset{\sim}{a}_2 - \underset{\sim}{a}_1)^2 + D^2\underset{\sim}{u}_{h1}(\underset{\sim}{a}_2)(\underset{\sim}{a}_2 - \underset{\sim}{a}_1)^2] \\ &= \frac{1}{2}[\underset{\sim}{u}_{h1}(\underset{\sim}{a}_1) + \underset{\sim}{u}_{h1}(\underset{\sim}{a}_2)] + \frac{5}{32}\ell_b[\underset{\sim}{u}_{h1,2}(\underset{\sim}{a}_1) - \underset{\sim}{u}_{h1,2}(\underset{\sim}{a}_2)] + \frac{1}{64}\ell_b^2[\underset{\sim}{u}_{h1,22}(\underset{\sim}{a}_1) + \underset{\sim}{u}_{h1,22}(\underset{\sim}{a}_2)] \end{aligned}$$

where $\ell_b = \pi/16$ is the width of the radial strip into consideration in this test. The numerical results are given in Fig. 7. They are very close to the reference solution considered in Section 7.2. This is not surprising since the computation of the reference solution through 2D elasticity equations does not introduce effects of elastic hinge including a jump of rotation. Thus, this reference solution is really very close to the rigid hinge solution.

7.4. Elastic junction

In order to simulate an elastic junction, we have implemented in the Modulef code a junction finite element which is built from two Argyris triangles which joint along one of their sides (see Fig. 8) and whose rigidity matrix includes the bilinear form $b_h[.,.]$. We use again the mesh given in Fig. 6, the junction element is represented in grey. We have seen in Part 1 that the elasticity of the junction depends on a coefficient k. When k is small (about 10), the junction is elastic and is very distorted. When k is very large (about 10^7), the junction

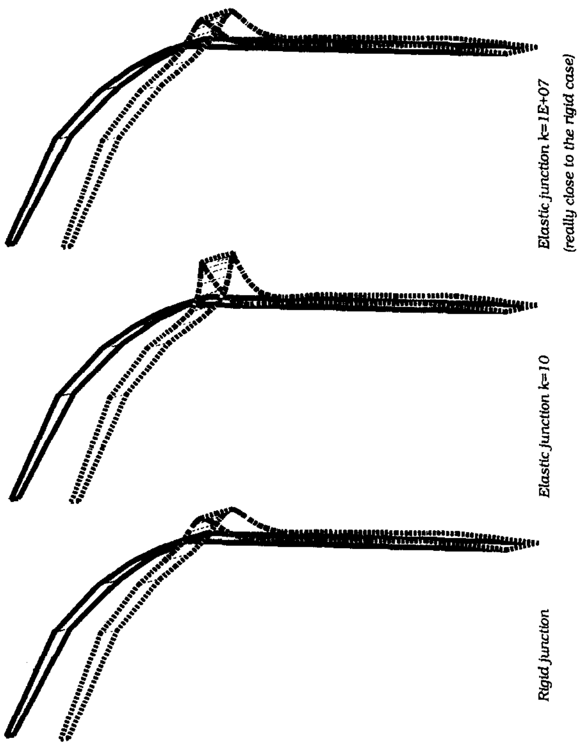
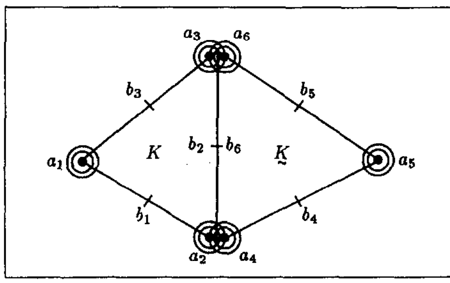


Fig. 7. Numerical results.



$$P_K = (P_5)^6 \cap \{relations(7.1)\ to\ (7.3)\}$$

$$dim\, P_K = 108$$

Fig. 8. Shell junction finite element (interpolation of the unknowns u_i on K and $\underset{\sim}{u}_i$ on $\underset{\sim}{K}$ for $i = 1, 2, 3$).

becomes almost rigid and the angle between the two shells remains unchanged during the deformation (see Fig. 7).

8. Final remarks

In this study we have approximated the thin shell junction problem by using the high accuracy finite element method based on the Argyris element. Another possibility is to use the delinquent finite element introduced in Sander [12]. This element seems promising since it combines high accuracy (an error estimate in $O(h^2)$ in energy norm would seem reasonable) with a choice of degrees of freedom well adapted to the junction problem. These elements are schematically shown in Fig. 9. A study along these lines is in progress by Bernadou et al. [13].

Another interesting study would be the study of the junction between two shells using Reissner–Mindlin model. In this way, the transverse shear effects and the boundary layers phenomena have to be considered.

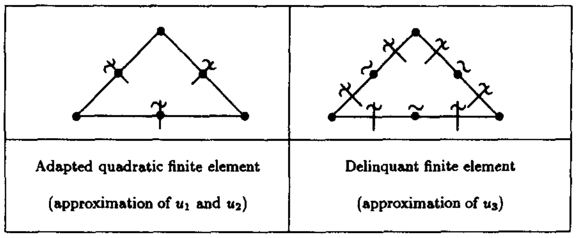


Fig. 9. An interesting combination of finite elements for the thin shell junction problem.

Acknowledgments

This work is a part of the Human Capital and Mobility Program 'Shells: Mathematical Modeling and Analysis, Scientific Computing' of the Commission of the European Communities (Contract No. ERBCHRXCT940536).

References

[1] M. Bernadou, S. Fayolle and F. Léné, Numerical analysis of junctions between plates, Comput. Methods Appl. Mech. Engrg. 74 (1989) 307–326.
[2] S. Fayolle, Sur l'analyse numérique de raccords de poutres et de plaques, Thèse de l'Université P. et M. Curie, 1987.
[3] M. Bernadou and A. Cubier, Numerical analysis of junctions between thin shells. Part 1: Continuous problems, Comput. Methods Appl. Mech. Engrg. 161 (1998) 349–363.
[4] A. Zenisek, Nonhomogeneous boundary conditions and curved triangular finite element, Apl. Mat. 26 (1981) 121–140.
[5] M. Bernadou, Finite Element Methods for Thin Shell Problems (J. Wiley & Sons, Chichester, 1996).
[6] A. Cubier, Analyse et simulation numériques de jonctions de coques minces, Thèse de l'Université P. et M. Curie, 1994.
[7] J.H. Argyris, I. Fried and D.W. Scharpf, The TUBA family of plate elements for the matrix displacement method, Aero. J. Roy. Aeronaut. Soc. 72 (1968) 701–709.
[8] P.G. Ciarlet, The Finite Element Method for Elliptic Problems (North-Holland, Amsterdam, 1978).
[9] D.A. Dunavant, High degree efficient symmetrical gaussian quadrature rules for the triangle, Int. J. Numer. 13 (1985) 1129–1148.
[10] O.C. Zienkiewicz and R.L. Taylor, The Finite Element Method, 4th edition, Vol. 1, Basic Formulation and Linear Problems (McGraw-Hill, New York, 1989).
[11] M. Bernadou, P.L. George, A. Hassim, P. Joly, P. Laug, B. Muller, A. Perronet, E. Saltel, D. Steer, G. Vanderbork and M. Vidrascu, Modulef: A Modular Library of Finite Elements, I.N.R.I.A, 1986.
[12] G. Sander, Modèles d'éléments finis délinquants, Groupe de travail 'Méthodes Numériques pour les Problèmes de Coques', 17 Janvier 1985, Clamart.
[13] M. Bernadou, F. Palma, P. Trouvé and M.A. Vilariño, to appear.