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A WEAK GALERKIN FINITE ELEMENT SCHEME FOR THE CAHN-HILLIARD EQUATION

JUNPING WANG, QILONG ZHAI, RAN ZHANG, AND SHANGYOU ZHANG

ABSTRACT. This article presents a weak Galerkin (WG) finite element method for the Cahn-Hilliard equation. The WG method makes use of piecewise polynomials as approximating functions, with weakly defined partial derivatives (first and second order) computed locally by using the information in the interior and on the boundary of each element. A stabilizer is constructed and added to the numerical scheme for the purpose of providing certain weak continuities for the approximating function. A mathematical convergence theory is developed for the corresponding numerical solutions, and optimal order of error estimates are derived. Some numerical results are presented to illustrate the efficiency and accuracy of the method.

1. INTRODUCTION

This paper is concerned with the development of new numerical methods for the Cahn-Hilliard equation using weak Galerkin (WG) finite element techniques. The Cahn-Hilliard equation (1.1)–(1.4) was first introduced by John Cahn and John Hilliard in 1958 [6]. It describes the phenomenon of phase separation, or spinodal decomposition. The equation simulates the process that a two-phase alloy fluid separates into domains of pure of each component. For simplicity, we consider the model problem that seeks an unknown function $u = u(x,t)$ satisfying

$$u_t - \Delta w = g, \qquad \text{in } \Omega_T := \Omega \times (0,T), \tag{1.1}$$

$$w = f(u) - \gamma^2 \Delta u, \qquad \text{in } \Omega_T := \Omega \times (0,T), \tag{1.2}$$

$$\partial_n u = \partial_n \Delta u = 0, \qquad \text{on } \partial\Omega_T := \partial\Omega \times (0,T), \tag{1.3}$$

$$u(x,0) = u^0(x), \qquad \text{in } \Omega, \tag{1.4}$$

where Δ is the Laplacian operator, Ω is a bounded polygonal or polyhedral domain in $\mathbb{R}^d$ for $d = 2,3$, $\mathbf{n}$ denotes the outward unit normal vector along $\partial\Omega$, $f(s) = F'(s)$ with $F(s) = \frac{1}{4}(s^2-1)^2$, and $g = g(x,t)$ is given data on Ω_T. ∂_n denotes the

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directional derivative on $\partial\Omega$ in the normal direction $\mathbf{n}$. Without loss of generality, assume that the initial data satisfies $\int_\Omega u^0 = 0$.

In the Cahn-Hilliard equation (1.1), the function $u \in [-1, 1]$ is used to represent the concentration of each component, and γ is the interface parameter which governs the width of the transition region. The level set of the function u can be employed to identify the interface of the two components. Besides the two-phase fluid problem, the Cahn-Hilliard equation also has a wide range of applications in various areas of physics and industry, such as multiphase fluid flow, image processing, and planet formation. As to the detailed physical derivation and applications, readers are referred to [27] and the references therein.

The Cahn-Hilliard equation has no explicit formulation for its solution, and thus numerical methods are indispensable tools in practical simulation. In the last three decades, many numerical methods have been developed for the equation, including the finite difference method [9, 10, 21, 28], the finite element method [7, 13, 22], and the spectral method [3]. Due to the simplicity and robustness, the finite element method has been recognized as an efficient approach for the Cahn-Hilliard equation. However, since the Cahn-Hilliard equation is of fourth order, the usual conforming finite element method requires C^1 continuity for the approximation functions. It is well known that the construction for C^1-type elements is quite challenging in practical computation. There are mainly two ways to conquer this problem: one is to use mixed finite element methods, and the other is to use nonconforming or non-standard finite element methods. The central idea of mixed finite element methods is to reduce the fourth order equation into two second order equations, which then relax the smoothness requirement on finite element functions. The mixed finite element method is practical, and a lot of work has been dedicated to the study and development of this method; cf. [2, 11–13, 16, 19]. However, the mixed finite element method needs the solution of a saddle point problem for which a certain stability condition should be satisfied in the algorithmic development and analysis. The mixed finite element method also introduces new variables, which shall increase the size of the discrete linear system. Nonstandard finite element methods can solve the Cahn-Hilliard equation without introducing new variables, and do not usually require the approximating functions to be C^1 continuous. Among several of the nonstandard finite element methods developed for the Cahn-Hilliard equation are the nonconforming finite element method [14], the discontinuous Galerkin method [15, 16, 18], the local discontinuous Galerkin method [33], and the virtual element method [1].

Recently, the weak Galerkin (WG) finite element method has been developed for partial differential equations [8, 17, 29, 36]. The WG method employs piecewise polynomials in the finite element space, which can be discontinuous across elements. The key of WG method is to use locally defined weak derivative operators instead of the classical derivative operators, plus a stabilizer that ensures a certain weak continuity for the approximating function. The WG method has been applied to several classes of partial differential equations, including the biharmonic equation [23, 24, 35], the Stoke's equation [26, 31, 32, 34], and the Maxwell equation [25]. Many numerical techniques have also been applied to the WG method, including the a posteriori error estimators [8] and the two-level methods [20].

In this paper, we shall use the WG method to solve the Cahn-Hilliard equation. The main contributions of this paper are twofold. First, a WG finite element method is devised to show that the WG approach is applicable for polytopal meshes and can be extended to 3D polyhedral partitions without any difficulty. Second, a mathematical convergence theory is established for the corresponding WG finite element approximations. As a nonstandard finite element method, WG provides a numerical procedure for the Cahn-Hilliard equation without introducing auxiliary variables. On the other hand, the WG method usually consists of degrees of freedom on both interior and boundary of each element, and this results in a discrete system involving a large number of unknowns. To reduce the degrees of freedom, in this paper, we use a semi-implicit scheme and the Schur complement to eliminate the interior unknown which reduces the computational cost significantly.

The paper is organized as follows. In Section 2 we introduce a WG finite element method for the Cahn-Hilliard equation. In Section 3, we present a theoretical framework for a modified WG method for the linear biharmonic equation with boundary conditions that have not been dealt with in existing literature. In particular, a new weak Laplacian operator will be introduced here for an improved estimation of the Laplacian operator. In Section 4, we shall establish an optimal order of convergence for our WG numerical solutions for the Cahn-Hilliard equation. Finally, in Section 5, we report some computational results to demonstrate the stability and accuracy of the numerical approximations.

2. Weak Galerkin finite element scheme

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of polygons in 2D or polyhedra in 3D, such that $\mathcal{T}_h$ is shape regular in the sense as defined in [30]. Denote by $\mathcal{E}_h$ the set of all edges or flat faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ be the set of all interior edges or flat faces.

To devise the WG scheme for (1.1)–(1.4), we introduce the following finite element space

$$V_h = \{\{v_0, v_b, v_n\} : v_0 \in P_k(T) \quad \forall T \in \mathcal{T}_h, v_b \in P_k(e), \\ v_n \in P_{k-1}(e) \quad \forall e \in \mathcal{E}_h\}, \tag{2.1}$$

where $k \geq 2$ is an integer. Here, the component v_0 represents the interior value of v on each element, and the component v_b represents the value of v on each edge. Denote by V_h^0 the subspace of V_h with vanishing traces; i.e.,

$$V_h^0 = \{\{v_0, v_b, v_n\} \in V_h : v_n|_e = 0 \quad \forall e \in \partial T \cap \partial\Omega\}. \tag{2.2}$$

We also introduce the following finite element space

$$W_h = \{v \in V_h^0, \quad \int_\Omega v_0 = 0\}, \tag{2.3}$$

and a set of normal directions

$$\mathcal{N}_h = \{\mathbf{n}_e : \mathbf{n}_e \text{ is unit and normal to } e, \ e \in \mathcal{E}_h\}.$$

Then, on each edge, the component v_n represents the normal derivative of v in the direction $\mathbf{n}_e$. It should be pointed out that v_b and v_n are single-valued on each edge, and are irrelevant to the trace of v_0.

For any $v \in V_h$, define the weak Laplacian $\Delta_w v$ as follows.

Defintion 2.1 ([23])**.** For any $v \in V_h$, $\Delta_w v$ is defined as the unique polynomial in $P_k(T)$ on each element $T \in \mathcal{T}_h$ satisfying

$$(\Delta_w v, \varphi)_T = (v_0, \Delta\varphi)_T - \langle v_b, \nabla\varphi\cdot\mathbf{n}\rangle_{\partial T} + \langle v_n(\mathbf{n}_e\cdot\mathbf{n}), \varphi\rangle_{\partial T} \quad \forall\varphi \in P_k(T), \tag{2.4}$$

where $\mathbf{n}$ is the outward unit normal vector.

It should also be noticed that in the WG methods studied in [23, 35], the degree of $\Delta_w v$ was chosen to be $k-2$. But in this paper, the degree of $\Delta_w v$ is selected to be k in order to obtain a full order error estimate for the Cahn-Hillard equation. Thus, the properties of the weak Laplacian operator need to be derived again, which is to be accomplished in Section 3.

On each element $T \in \mathcal{T}_h$, denote by Q_0 the L^2 projection onto $P_k(T)$. For each edge/face $e \subset \partial T$, denote by Q_b the L^2 projection onto $P_k(e)$, and denote by Q_n the L^2 projection onto $P_{k-1}(e)$. Now for any $u \in H^2(\Omega)$, we shall combine these three projections together to define a projection into the finite element space V_h such that on the element T,

$$Q_h u = \{Q_0 u, Q_b u, Q_n(\nabla u\cdot\mathbf{n}_e)\}.$$

Next, we introduce the WG algorithm for the equations (1.1)–(1.4). To this end, we define three bilinear forms $s(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ as follows. For any $u_h = \{u_0, u_b, u_n\}$ and $v_h = \{v_0, v_b, v_n\}$ in V_h,

$$\begin{aligned} s(u_h, v_h) &= \sum_{T\in\mathcal{T}_h} h_T^{-1}\langle\nabla u_0\cdot\mathbf{n}_e - u_n, \nabla v_0\cdot\mathbf{n}_e - v_n\rangle_{\partial T} \\ &\quad + \sum_{T\in\mathcal{T}_h} h_T^{-3}\langle u_0 - u_b, v_0 - v_b\rangle_{\partial T}, \end{aligned} \tag{2.5}$$

$$b(u_h, v_h) = (\Delta_w u_h, \Delta_w v_h) + s(u_h, v_h), \tag{2.6}$$

$$c(u_h, v_h) = -(\Delta_w f(u_h), v_0), \tag{2.7}$$

where $f(u_h) = \{f(u_0), f(u_b), f'(u_b)u_n\}$.

As to the time direction, we use the backward Euler discretization and at the time step m, define

$$d_t u_h^m = \frac{u_h^m - u_h^{m-1}}{\tau_m},$$

where $\tau_m > 0$ is the time increment at the time-step m.

We are now in a position to introduce the WG algorithm for equations (1.1)–(1.4).

Weak Galerkin Algorithm 1. Set $u_h^0 = Q_h u^0 \in V_h^0$. For $m = 1, \ldots, M$, find $u_h^m \in W_h$, such that

$$(d_t u_h^m, v_0) + \gamma^2 b(u_h^m, v_h) + c(u_h^{m-1}, v_0) = (g, v_0) \quad \forall v_h \in W_h. \tag{2.8}$$

3. Error estimates for a linear problem

In order to study the convergence of the WG Algorithm 1, we need some error estimates for a linear biharmonic equation with boundary conditions derived from the Cahn-Hillard problem. To this end, consider the following boundary value problem for the biharmonic equation

$$\Delta^2 u = f, \quad \text{in } \Omega, \tag{3.1}$$

$$\partial_n u = \partial_n \Delta u = 0, \quad \text{on } \partial\Omega, \tag{3.2}$$

$$\int_\Omega u = 0, \tag{3.3}$$

where Ω is a polygonal or polyhedral domain. The corresponding WG scheme is given as follows.

Weak Galerkin Algorithm 2. Find $u_h \in W_h$, such that for any $v_h \in W_h$,

$$b(u_h, v_h) = (f, v_0), \tag{3.4}$$

where $b(\cdot,\cdot)$ is defined in (2.6).

The error estimate for the numerical scheme (3.4) can be derived by using the techniques developed in [23] and [35] with slight modification. First, we introduce a semi-norm $|\!|\!|\cdot|\!|\!|$ in V_h^0 by setting

$$|\!|\!|v|\!|\!|^2 = b(v, v). \tag{3.5}$$

Lemma 3.1. *$|\!|\!|\cdot|\!|\!|$ defines a norm on W_h.*

Proof. It suffices to verify the positivity property for this semi-norm. For any $v \in W_h$ with $|\!|\!|v|\!|\!| = 0$, we have $\Delta_w v = 0$ on each element T and $s(v, v) = 0$. From the definition of $s(v, v)$ we obtain $v_0 = v_b$ and $\nabla v_0 \cdot \mathbf{n}_e = v_n$ on each edge. Thus, v_0 and ∇v_0 are continuous in the domain Ω so that $v_0 \in H^2(\Omega)$.

From the definition of the weak Laplacian operator, we have that for any $\phi \in P_k(T)$,

$$\begin{aligned} 0 &= (\Delta_w v, \phi)_T \\ &= (v_0, \Delta\phi)_T - \langle v_b, \nabla\phi\cdot\mathbf{n}\rangle_{\partial T} + \langle v_n(\mathbf{n}\cdot\mathbf{n}_e), \phi\rangle_{\partial T} \\ &= (\Delta v_0, \phi)_T + \langle v_0 - v_b, \nabla\phi\cdot\mathbf{n}\rangle_{\partial T} - \langle \nabla v_0\cdot\mathbf{n} - v_n(\mathbf{n}\cdot\mathbf{n}_e), \phi\rangle_{\partial T} \\ &= (\Delta v_0, \phi), \end{aligned}$$

which implies $\Delta v_0 = 0$ on Ω. It follows from the conditions $\int_\Omega v_0 = 0$ and $\nabla v_0 \cdot \mathbf{n}_e = 0$ on $\partial\Omega$ that $v_0 = 0$ on Ω. Then $v_b = v_n = 0$ holds true, which completes the proof. $\square$

Lemma 3.2. *The Weak Galerkin Algorithm* 2 *has a unique solution.*

Proof. We only need to verify the uniqueness for the homogenous problem. Suppose $f = 0$, by setting $v_h = u_h$ in (3.4) we have

$$|\!|\!|u_h|\!|\!|^2 = b(u_h, u_h) = 0.$$

It follows from Lemma 3.1 that $u_h = 0$, which completes the proof. $\square$

The relationship between the operators Δ and Δ_w are revealed in the following lemma, which plays an essential role in the error analysis.

Lemma 3.3. *On each element $T \in \mathcal{T}_h$ and for any $u \in H^2(T)$ and $\phi \in P_k(T)$ there holds:*

$$(\Delta_w Q_h u, \phi)_T = (Q_0 \Delta u, \phi) + \langle Q_n(\nabla u\cdot\mathbf{n}) - \nabla u\cdot\mathbf{n}, \phi\rangle_{\partial T}. \tag{3.6}$$

Proof. For any $\phi \in P_k(T)$, from the definition of the discrete weak Laplacian (2.4) we obtain

$$\begin{aligned}(\Delta_w Q_h u, \phi)_T &= (Q_0 u, \Delta\phi)_T - \langle Q_b u, \nabla\phi\cdot\mathbf{n}\rangle_{\partial T} + \langle Q_n(\nabla u\cdot\mathbf{n}), \phi\rangle_{\partial T}\\ &= (u, \Delta\phi)_T - \langle u, \nabla\phi\cdot\mathbf{n}\rangle_{\partial T} + \langle \nabla u\cdot\mathbf{n}, \phi\rangle_{\partial T}\\ &\quad +\langle Q_n(\nabla u\cdot\mathbf{n}) - \nabla u\cdot\mathbf{n}, \phi\rangle_{\partial T}\\ &= (\Delta u, \phi)_T + \langle Q_n(\nabla u\cdot\mathbf{n}) - \nabla u\cdot\mathbf{n}, \phi\rangle_{\partial T}\\ &= (Q_0\Delta u, \phi) + \langle Q_n(\nabla u\cdot\mathbf{n}) - \nabla u\cdot\mathbf{n}, \phi\rangle_{\partial T},\end{aligned}$$

which completes the proof. □

The identity (3.6) indicates that the discrete weak Laplacian of the L^2 projection of u is an approximation of the Laplacian of u in the classical sense.

Let u be the exact solution of (3.1)–(3.3), and u_h be the numerical solution of (3.4). Denote by $e_h = Q_h u - u_h$ the error between the projection of u and the numerical approximation u_h. Then, we have the following error equation.

Lemma 3.4. *Let $u \in H^3(\Omega)$ be the exact solution of* (3.1)–(3.3)*, and let u_h be the numerical solution of* (3.4)*. The following equation holds true:*

$$b(e_h, v) = \ell(u, v) \qquad \forall v \in W_h,$$

where

$$\begin{aligned}\ell(u,v) &= \sum_{T\in\mathcal{T}_h} \langle Q_n(\nabla u\cdot\mathbf{n}) - \nabla u\cdot\mathbf{n}, \Delta_w v\rangle_{\partial T}\\ &\quad - \sum_{T\in\mathcal{T}_h} \langle Q_0\Delta u - \Delta u, \nabla v_0\cdot\mathbf{n} - v_n(\mathbf{n}\cdot\mathbf{n}_e)\rangle_{\partial T}\\ &\quad + \sum_{T\in\mathcal{T}_h} \langle \nabla(Q_0\Delta u - \Delta u)\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T} + s(Q_h u, v).\end{aligned} \tag{3.7}$$

Proof. From Lemma 3.3 and the definition of the weak Laplacian operator, we have

$$\begin{aligned}&(\Delta_w Q_h u, \Delta_w v)\\ =\ & (Q_0\Delta u, \Delta_w v) + \sum_{T\in\mathcal{T}_h} \langle Q_n(\nabla u\cdot\mathbf{n}) - \nabla u\cdot\mathbf{n}, \Delta_w v\rangle_{\partial T}\\ =\ & (\Delta Q_0\Delta u, v_0) + \sum_{T\in\mathcal{T}_h} \langle Q_n(\nabla u\cdot\mathbf{n}) - \nabla u\cdot\mathbf{n}, \Delta_w v\rangle_{\partial T}\\ & - \sum_{T\in\mathcal{T}_h} \langle \nabla Q_0\Delta u\cdot\mathbf{n}, v_b\rangle_{\partial T} + \sum_{T\in\mathcal{T}_h} \langle Q_0\Delta u, v_n(\mathbf{n}\cdot\mathbf{n}_e)\rangle_{\partial T}\\ =\ & (\Delta u, \Delta v_0) + \sum_{T\in\mathcal{T}_h} \langle Q_n(\nabla u\cdot\mathbf{n}) - \nabla u\cdot\mathbf{n}, \Delta_w v\rangle_{\partial T}\\ & + \sum_{T\in\mathcal{T}_h} \langle \nabla Q_0\Delta u\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T} - \sum_{T\in\mathcal{T}_h} \langle Q_0\Delta u, \nabla v_0\cdot\mathbf{n} - v_n(\mathbf{n}\cdot\mathbf{n}_e)\rangle_{\partial T}.\end{aligned}$$

By testing (3.1) against v_0 we obtain

$$\begin{aligned}(f, v_0) &= (\Delta^2 u, v_0)\\ &= (\Delta u, \Delta v_0) - \sum_{T\in\mathcal{T}_h} \langle \Delta u, \nabla v_0 \cdot \mathbf{n}\rangle_{\partial T} + \sum_{T\in\mathcal{T}_h} \langle \nabla\Delta u \cdot \mathbf{n}, v_0\rangle_{\partial T}\\ &= (\Delta u, \Delta v_0) - \sum_{T\in\mathcal{T}_h} \langle \Delta u, \nabla v_0 - v_n(\mathbf{n}\cdot\mathbf{n}_e)\cdot \mathbf{n}\rangle_{\partial T}\\ &\quad + \sum_{T\in\mathcal{T}_h} \langle \nabla\Delta u \cdot \mathbf{n}, v_0 - v_b\rangle_{\partial T}.\end{aligned}$$

It follows from scheme (3.4) that

$$\begin{aligned}b(e_h, v) &= (\Delta_w(Q_h u - u_h), \Delta_w v) + s(Q_h u - u_h, v)\\ &= (\Delta_w Q_h u, \Delta_w v) - (f, v_0) + s(Q_h u, v)\\ &= \ell(u, v),\end{aligned} \tag{3.8}$$

which completes the proof. □

On the regular polytopal partition, the following trace inequality and inverse inequality hold true. The proof can be found in [30]

Lemma 3.5 (Trace inequality)**.** *For any $\varphi \in H^1(T)$, the following inequality holds true on each element $T \in \mathcal{T}_h$:*

$$\|\varphi\|_{\partial T}^2 \le C(h_T^{-1}\|\varphi\|_T^2 + h_T\|\varphi\|_{1,T}^2).$$

Lemma 3.6 (Inverse inequality)**.** *There exists a constant C such that on each element $T \in \mathcal{T}_h$, one has*

$$\|\nabla\psi\|_T \le C h_T^{-1}\|\psi\|_T \qquad \forall \psi \in P_k(T).$$

The following lemma is devoted to an estimate of the three terms in $\ell(u, v)$.

Lemma 3.7. *Let $u_h \in V_h$ be the weak Galerkin finite element solution arising from (3.4) with finite element functions of order $k \ge 2$. Assume that the exact solution of the biharmonic equation (3.1)–(3.3) satisfies $u \in H^{k+1}(\Omega)$ and $\Delta u \in H^2(\Omega)$. Then, there exists a constant C such that*

$$\left|\sum_{T\in\mathcal{T}_h} \langle Q_n(\nabla u\cdot\mathbf{n}) - \nabla u\cdot\mathbf{n}, \Delta_w v\rangle_{\partial T}\right| \le Ch^{k-1}\|u\|_{k+1}|\!|\!| v|\!|\!|, \tag{3.9}$$

$$\left|\sum_{T\in\mathcal{T}_h} \langle Q_0\Delta u - \Delta u, \nabla v_0\cdot\mathbf{n} - v_n(\mathbf{n}\cdot\mathbf{n}_e)\rangle_{\partial T}\right| \le Ch^{k-1}\|u\|_{k+1}|\!|\!| v|\!|\!|, \tag{3.10}$$

$$\left|\sum_{T\in\mathcal{T}_h} \langle \nabla(Q_0\Delta u - \Delta u)\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T}\right| \le Ch^{k-1}(\|u\|_{k+1} + h\delta_{k},2\|\Delta u\|_2)|\!|\!| v|\!|\!|, \tag{3.11}$$

$$|s(Q_h u, v)| \le Ch^{k-1}\|u\|_{k+1}|\!|\!| v|\!|\!|. \tag{3.12}$$

Proof. The proof of (3.10) and (3.12) can be found in Theorem 6.2 in [23]. Thus, we shall only focus on the treatment of (3.9) and (3.11).

Denote by $\tilde{Q}_0$ the L^2 projection onto $[P_{k-1}(T)]^d$. Then, from the trace inequality and the inverse inequality we have

$$\begin{aligned}
&\left|\sum_{T\in\mathcal{T}_h}\langle Q_n(\nabla u\cdot\mathbf{n})-\nabla u\cdot\mathbf{n},\Delta_w v\rangle_{\partial T}\right|\\
\le\ &\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|\tilde{Q}_0\nabla u-\nabla u\|_{\partial T}^2\right)^{\frac12}\left(\sum_{T\in\mathcal{T}_h}h_T\|\Delta_w v\|_{\partial T}^2\right)^{\frac12}\\
\le\ &C\left(\sum_{T\in\mathcal{T}_h}h_T^{-2}\|\tilde{Q}_0\nabla u-\nabla u\|_T^2+\|\tilde{Q}_0\nabla u-\nabla u\|_{1,T}^2\right)^{\frac12}\|\Delta_w v\|\\
\le\ &Ch^{k-1}\|u\|_{k+1}|||v|||.
\end{aligned}$$

Similarly, it follows from the trace inequality and the inverse inequality that

$$\begin{aligned}
&\left|\sum_{T\in\mathcal{T}_h}\langle\nabla(Q_0\Delta u-\Delta u)\cdot\mathbf{n},v_0-v_b\rangle_{\partial T}\right|\\
\le\ &\left(\sum_{T\in\mathcal{T}_h}h_T^3\|\nabla(Q_0\Delta u-\Delta u)\|_{\partial T}^2\right)^{\frac12}\left(\sum_{T\in\mathcal{T}_h}h_T^{-3}\|v_0-v_b\|_{\partial T}^2\right)^{\frac12}\\
\le\ &C\left(\sum_{T\in\mathcal{T}_h}h_T^2\|Q_0\Delta u-\Delta u\|_{1,T}^2+h_T^4\|Q_0\Delta u-\Delta u\|_{2,T}^2\right)^{\frac12}s(v,v)^{\frac12}\\
\le\ &Ch^{k-1}(\|u\|_{k+1}+h\delta_{k,2}\|\Delta u\|_2)|||v|||,
\end{aligned}$$

which completes the proof. □

Corollary 3.8. *Let $u_h\in V_h$ be the weak Galerkin finite element solution arising from (3.4) with finite element functions of order $k\ge 2$. Assume that the exact solution of the biharmonic equation (3.1)–(3.3) satisfies $u\in H^{k+1}(\Omega)$ and $\Delta u\in H^2(\Omega)$. Then, we have, for $\ell(u,v)$ defined in (3.7),*

$$|\ell(u,v)|\le Ch^{k-1}(\|u\|_{k+1}+h\delta_{k,2}\|\Delta u\|_2)|||v|||.$$

Remark 3.1. The problem (3.1)–(3.3) can be written in a mixed form as follows:

$$\begin{aligned}
-\Delta w &= f, && \text{in } \Omega,\\
-\Delta u &= w, && \text{in } \Omega,\\
\frac{\partial w}{\partial n}=\frac{\partial u}{\partial n} &= 0, && \text{on } \partial\Omega,\\
\int_\Omega u &= 0.
\end{aligned}$$

Thus, the variable $w=-\Delta u$ satisfies the Poisson equation with homogeneous Neumann boundary condition. It follows that $w=-\Delta u\in H^2(\Omega)$ when Ω is convex polygonal.

With these preparations, we are ready to establish an optimal order error estimate for the error function e_h in the trip-bar norm.

Theorem 3.9. *Let $u_h \in V_h$ be the weak Galerkin finite element solution arising from* (3.4) *with finite element functions of order $k \geq 2$. Assume that the exact solution of biharmonic equation* (3.1)–(3.3) *satisfies $u \in H^{k+1}(\Omega)$ and $\Delta u \in H^2(\Omega)$. Then, there exists a constant C such that*

$$\|\!|u_h - Q_h u|\!\| \leq Ch^{k-1}(\|u\|_{k+1} + h\delta_{k,2}\|\Delta u\|_2). \tag{3.13}$$

Proof. By setting $v = e_h$ in Lemma 3.4, we arrive at

$$\|\!|e_h|\!\|^2 = \ell(u, e_h).$$

From Lemma 3.7, we have

$$\begin{aligned} \|\!|e_h|\!\|^2 &= \ell(u, e_h) \\ &\leq Ch^{k-1}(\|u\|_{k+1} + h\delta_{k,2}\|\Delta u\|_2)\|\!|e_h|\!\|, \end{aligned}$$

which completes the proof. □

To obtain an error estimate in the L^2 norm, we consider the following dual problem

$$\Delta^2\psi = e_0, \quad \text{in } \Omega, \tag{3.14}$$

$$\frac{\partial\psi}{\partial\mathbf{n}} = \frac{\partial\Delta\psi}{\partial\mathbf{n}} = 0, \quad \text{on } \partial\Omega, \tag{3.15}$$

$$\int_\Omega \psi = 0. \tag{3.16}$$

Assume that the solution of the dual problem (3.14)–(3.16) has the following regularity estimate:

$$\|\psi\|_3 + \|\Delta\psi\|_2 \leq C\|e_0\|. \tag{3.17}$$

For a high order error estimate in L^2, the following assumption is required:

$$\|\psi\|_4 \leq C\|e_0\|. \tag{3.18}$$

Remark 3.2. For the clamp boundary condition, when the polygonal domain Ω is convex, the solution u belongs to $H^3(\Omega)$. When all the corners of Ω are less than 126.2839°, the solution u belongs to $H^4(\Omega)$; cf. Theorem 2 in [4].

For the Cahn-Hilliard type boundary condition, when all the corners w in Ω satisfy $w \leq \frac{\pi}{2}$, the solution u belongs to $H^3(\Omega)$. When all the corners w in Ω satisfy $\frac{\pi}{3} < w < \frac{\pi}{2}$ or $w = \frac{\pi}{2}$, the solution u belongs to $H^4(\Omega)$. Especially, $u \in H^4(\Omega)$ when Ω is a rectangular domain; cf. p. 2107, Appendix A in [5].

Lemma 3.10. *Assume that the solution of* (3.1)–(3.2) *satisfies $u \in H^3(\Omega)$, and the regularity assumption* (3.17) *holds true for the dual problem* (3.14)–(3.15). *Then we have the following identity,*

$$\|e_0\|^2 = \ell(u, Q_h\psi) - \ell(\psi, e_h),$$

where $\ell(\cdot,\cdot)$ is defined in (3.7).

Proof. From Lemma 3.3 and the definition of the weak Laplacian operator, we have

$$
\begin{aligned}
& (\Delta_w Q_h\psi, \Delta_w e_h) \\
= \;& (Q_0\Delta\psi, \Delta_w e_h) + \sum_{T\in\mathcal{T}_h} \langle Q_n(\nabla\psi\cdot\mathbf{n}) - \nabla\psi\cdot\mathbf{n}, \Delta_w e_h\rangle_{\partial T} \\
= \;& (\Delta Q_0\Delta\psi, e_0) + \sum_{T\in\mathcal{T}_h} \langle Q_n(\nabla\psi\cdot\mathbf{n}) - \nabla\psi\cdot\mathbf{n}, \Delta_w e_h\rangle_{\partial T} \\
& - \sum_{T\in\mathcal{T}_h} \langle \nabla Q_0\Delta\psi\cdot\mathbf{n}, e_b\rangle_{\partial T} + \sum_{T\in\mathcal{T}_h} \langle Q_0\Delta\psi, e_n(\mathbf{n}\cdot\mathbf{n}_e)\rangle_{\partial T} \\
= \;& (\Delta\psi, \Delta e_0) + \sum_{T\in\mathcal{T}_h} \langle Q_n(\nabla\psi\cdot\mathbf{n}) - \nabla\psi\cdot\mathbf{n}, \Delta_w e_h\rangle_{\partial T} \\
& + \sum_{T\in\mathcal{T}_h} \langle \nabla Q_0\Delta\psi\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial T} - \sum_{T\in\mathcal{T}_h} \langle Q_0\Delta\psi, \nabla e_0\cdot\mathbf{n} - e_n(\mathbf{n}\cdot\mathbf{n}_e)\rangle_{\partial T}.
\end{aligned}
$$

Testing (3.14) by e_0, it follows from Lemma (3.4) that

$$
\begin{aligned}
\|e_0\|^2 = \;& (\Delta^2\psi, e_0) \\
= \;& (\Delta\psi, \Delta e_0) - \sum_{T\in\mathcal{T}_h} \langle \Delta\psi, \nabla e_0\cdot\mathbf{n}\rangle_{\partial T} + \sum_{T\in\mathcal{T}_h} \langle \nabla\Delta\psi\cdot\mathbf{n}, e_0\rangle_{\partial T} \\
= \;& (\Delta\psi, \Delta e_0) - \sum_{T\in\mathcal{T}_h} \langle \Delta\psi, \nabla e_0 - e_n(\mathbf{n}\cdot\mathbf{n}_e)\cdot\mathbf{n}\rangle_{\partial T} \\
& + \sum_{T\in\mathcal{T}_h} \langle \nabla\Delta\psi\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial T} \\
= \;& b(Q_h\psi, e_h) - \ell(\psi, e_h) \\
= \;& \ell(u, Q_h\psi) - \ell(\psi, e_h).
\end{aligned}
$$

□

For the term $\ell(u, Q_h\psi)$, we have the following estimate.

Lemma 3.11. *Assume that $u \in H^{k+1}(\Omega)$ is the solution of (3.1)–(3.2) satisfying $\Delta u \in H^2(\Omega)$. Let $\psi \in H^{3+s}(\Omega)$ be the solution of (3.14)–(3.15) with $s \in [0,1]$. Then we have the following estimate:*

$$
\ell(u, Q_h\psi) \le Ch^{k+s}(\|u\|_{k+1} + h\delta_{k,2}\|\Delta u\|_2)\|\psi\|_{3+s}.
$$

Proof. We estimate $\ell(u, Q_h\psi)$ term by term. From the trace inequality and the Cauchy-Schwarz inequality, it follows that for the first term, we have

$$
\begin{aligned}
& \left| \sum_{T\in\mathcal{T}_h} \langle Q_n(\nabla u\cdot\mathbf{n}) - \nabla u\cdot\mathbf{n}, \Delta_w Q_h\psi\rangle_{\partial T} \right| \\
\le \;& \left(\sum_{T\in\mathcal{T}_h} \|Q_n(\nabla u\cdot\mathbf{n}) - \nabla u\cdot\mathbf{n}\|_{\partial T}^2 \right)^{\frac12} \left(\sum_{T\in\mathcal{T}_h} \|Q_0(\Delta\psi) - Q_n(\Delta\psi)\|_{\partial T}^2 \right)^{\frac12} \\
\le \;& Ch^{k+s}\|u\|_{k+1}\|\psi\|_{3+s}.
\end{aligned}
$$

For the second term, we have

$$\begin{aligned}
&\left|\sum_{T\in\mathcal{T}_h}\langle Q_0\Delta u-\Delta u, \nabla Q_0\psi\cdot\mathbf{n}-Q_n(\nabla\psi\cdot\mathbf{n})\rangle_{\partial T}\right|\\
\le\ &\left(\sum_{T\in\mathcal{T}_h}\|Q_0\Delta u-\Delta u\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h}\|\nabla Q_0\psi\cdot\mathbf{n}-Q_n(\nabla\psi\cdot\mathbf{n})\|_{\partial T}^2\right)^{\frac{1}{2}}\\
\le\ &Ch^{k+s}\|u\|_{k+1}\|\psi\|_{3+s}.
\end{aligned}$$

For the third term, it can be seen that

$$\begin{aligned}
&\left|\sum_{T\in\mathcal{T}_h}\langle \nabla(Q_0\Delta u-\Delta u)\cdot\mathbf{n}, Q_0\psi-Q_b\psi\rangle_{\partial T}\right|\\
\le\ &\left(\sum_{T\in\mathcal{T}_h}\|\nabla(Q_0\Delta u-\Delta u)\cdot\mathbf{n}\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h}\|Q_0\psi-Q_b\psi\|_{\partial T}^2\right)^{\frac{1}{2}}\\
\le\ &Ch^{k+s}(\|u\|_{k+1}+h\delta_{k,2}\|\Delta u\|_2)\|\psi\|_{3+s}.
\end{aligned}$$

As to the last term, we have

$$\begin{aligned}
s(Q_hu,Q_h\psi) =\ &\sum_{T\in\mathcal{T}_h}h_T^{-1}\langle\nabla Q_0u\cdot\mathbf{n}-Q_n(\nabla u\cdot\mathbf{n}),\nabla Q_0\psi\cdot\mathbf{n}-Q_n(\nabla\psi\cdot\mathbf{n})\rangle_{\partial T}\\
&+\sum_{T\in\mathcal{T}_h}h_T^{-3}\langle Q_0u-Q_bu,Q_0\psi-Q_b\psi\rangle_{\partial T}\\
\le\ &\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|\nabla Q_0u\cdot\mathbf{n}-Q_n(\nabla u\cdot\mathbf{n})\|_{\partial T}^2\right)^{\frac{1}{2}}\\
&\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|\nabla Q_0\psi\cdot\mathbf{n}-Q_n(\nabla\psi\cdot\mathbf{n})\|_{\partial T}^2\right)^{\frac{1}{2}}\\
&+\left(\sum_{T\in\mathcal{T}_h}h_T^{-3}\|Q_0u-Q_bu\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h}h_T^{-3}\|Q_0\psi-Q_b\psi\|_{\partial T}^2\right)^{\frac{1}{2}}\\
\le\ &Ch^{k+s}\|u\|_{k+1}\|\psi\|_{3+s}. \qquad \square
\end{aligned}$$

Using the Nitsche's technique, we obtain the following error estimate in L^2.

Theorem 3.12. *Let $u_h\in V_h$ be the weak Galerkin finite element solution arising from (3.4) with finite element functions of order $k\ge 2$. Assume that the exact solution of the biharmonic equation (3.1)–(3.3) satisfies $u\in H^{k+1}(\Omega)$, $\Delta u\in H^2(\Omega)$ and the dual problem (3.14)–(3.16) satisfies the assumption (3.17). Then, there exists a constant C such that*

$$\|u_0-Q_0u\|\le Ch^k(\|u\|_{k+1}+h\delta_{k,2}\|\Delta u\|_2). \tag{3.19}$$

Furthermore, if $k\ge 3$ and the dual problem (3.14)–(3.16) satisfies the H^4-regularity assumption (3.18), then the following optimal order of error estimate holds true:

$$\|u_0-Q_0u\|\le Ch^{k+1}\|u\|_{k+1}. \tag{3.20}$$

Proof. From Lemma 3.10, we have

$$\|e_0\|^2 \quad = \quad \ell(u, Q_h\psi) - \ell(\psi, e_h).$$

It follows from (3.17) that $\psi \in H^3(\Omega)$ is the solution of (3.14)–(3.16) satisfying $\Delta\psi \in H^2(\Omega)$. From Lemma 3.7 we have

$$\ell(\psi, e_h) \le Ch^{k-1}(\|\psi\|_{k+1} + h\delta_{k,2}\|\Delta\psi\|_2)|\!|\!|e_h|\!|\!|.$$

Since $\|\psi\| \in H^3(\Omega)$ and $\Delta\psi \in H^2(\Omega)$, by taking $k = 2$ in Lemma 3.7 and using Theorem 3.9 we obtain

$$\begin{aligned} \ell(\psi, e_h) &\le Ch(\|\psi\|_3 + h\|\Delta\psi\|_2)|\!|\!|e_h|\!|\!| \\ &\le Ch^k(\|u\|_{k+1} + h\delta_{k,2}\|\Delta u\|_2)(\|\psi\|_3 + h\|\Delta\psi\|_2). \end{aligned}$$

Taking $s = 0$ in Lemma 3.11 we have

$$\ell(u, Q_h\psi) \le Ch^k(\|u\|_{k+1} + h\delta_{k,2}\|\Delta u\|_2)\|\psi\|_3.$$

Thus, from assumption (3.17) we conclude that

$$\begin{aligned} \|e_0\|^2 &\le |\ell(u, Q_h\psi)| + |\ell(\psi, e_h)| \\ &\le Ch^k(\|\psi\|_3 + h\|\Delta\psi\|_2)(\|u\|_{k+1} + h\delta_{k,2}\|\Delta u\|_2) \\ &\le Ch^k(\|u\|_{k+1} + h\delta_{k,2}\|\Delta u\|_2)\|e_0\|, \end{aligned}$$

which implies

$$\|e_0\| \quad \le \quad Ch^k(\|u\|_{k+1} + h\delta_{k,2}\|\Delta u\|_2).$$

Similarly, under the assumption (3.18) and $k \ge 3$. By taking $k = 3$ in Lemma 3.7 and using Theorem 3.9 we obtain

$$\begin{aligned} \ell(\psi, e_h) &\le Ch^2(\|\psi\|_4)|\!|\!|e_h|\!|\!| \\ &\le Ch^{k+1}\|u\|_{k+1}\|\psi\|_4. \end{aligned}$$

Taking $s = 1$ in Lemma 3.11 we have

$$\ell(u, Q_h\psi) \le Ch^{k+1}(\|u\|_{k+1})\|\psi\|_4.$$

Thus, from assumption (3.18) we conclude that

$$\begin{aligned} \|e_0\|^2 &\le |\ell(u, Q_h\psi)| + |\ell(\psi, e_h)| \\ &\le Ch^{k+1}(\|u\|_{k+1})\|\psi\|_4 \\ &\le Ch^{k+1}\|u\|_{k+1}\|e_0\|, \end{aligned}$$

which implies

$$\|e_0\| \quad \le \quad Ch^{k+1}\|u\|_{k+1},$$

which completes the proof. □

Remark 3.3. In Theorem 3.12, we have a sub-optimal order of convergence for $k = 2$ and optimal order of convergence for $k \ge 3$. For the quadratic element, the convergence order is $O(h^2)$, and for the higher order element of P_k, $k \ge 3$, the convergence order is $O(h^{k+1})$.

For any $v \in V_h$, define the edge-based norms as follows:

$$\|v_b\|_{\mathcal{E}_h}^2 = \sum_{e\in\mathcal{E}_h} h_e \|v_b\|_{L^2(e)}^2,$$
$$\|v_n\|_{\mathcal{E}_h}^2 = \sum_{e\in\mathcal{E}_h} h_e \|v_n\|_{L^2(e)}^2.$$

Similar to Theorem 5.3 in [35], we have the following estimates.

Theorem 3.13. *Let $u_h \in V_h$ be the weak Galerkin finite element solution arising from* (3.4) *with finite element functions of order $k \geq 2$. Assume that the exact solution of the biharmonic equation* (3.1)–(3.3) *satisfies $u \in H^{k+1}(\Omega)$, $\Delta u \in H^2(\Omega)$ and the dual problem* (3.14)–(3.16) *satisfies the assumption* (3.17)*. Then, there exists a constant C such that*

$$\|u_b - Q_b u\|_{\mathcal{E}_h} \leq Ch^k(\|u\|_{k+1} + h\delta_{k,2}\|\Delta u\|_2), \tag{3.21}$$
$$\|u_n - Q_n(\nabla u \cdot \mathbf{n}_e)\|_{\mathcal{E}_h} \leq Ch^{k-1}(\|u\|_{k+1} + h\delta_{k,2}\|\Delta u\|_2). \tag{3.22}$$

Furthermore, if $k \geq 3$ and the dual problem (3.14)–(3.16) *satisfies the H^4-regularity assumption* (3.18)*, the following estimate holds true:*

$$\|u_b - Q_b u\|_{\mathcal{E}_h} \leq Ch^{k+1}\|u\|_{k+1}, \tag{3.23}$$
$$\|u_n - Q_n(\nabla u \cdot \mathbf{n}_e)\|_{\mathcal{E}_h} \leq Ch^k\|u\|_{k+1}. \tag{3.24}$$

Proof. Denote $e_h = Q_h u - u_h$. From Theorem 3.9 and Theorem 3.12, we have

$$\begin{aligned} & \|u_b - Q_b u\|_{\mathcal{E}_h} \\ \leq\ & \|u_0 - Q_0 u\|_{\mathcal{E}_h} + \|e_0 - e_b\|_{\mathcal{E}_h} \\ \leq\ & C(\|Q_0 u - u_0\| + h^2 s(e_h, e_h)) \\ \leq\ & Ch^k(\|u\|_{k+1} + h\delta_{k,2}\|\Delta u\|_2). \end{aligned}$$

From Theorem 3.9, Theorem 3.12, and the inverse inequality, we obtain

$$\begin{aligned} & \|u_n - Q_n(\nabla u \cdot \mathbf{n}_e)\|_{\mathcal{E}_h} \\ \leq\ & \|\nabla(Q_0 u - u_0)\cdot \mathbf{n}_e\|_{\mathcal{E}_h} + \|\nabla e_0 \cdot \mathbf{n}_e - e_n\|_{\mathcal{E}_h} \\ \leq\ & C(h^{-1}\|Q_0 u - u_0\| + hs(e_h, e_h)) \\ \leq\ & Ch^{k-1}(\|u\|_{k+1} + h\delta_{k,2}\|\Delta u\|_2). \end{aligned}$$

When $k \geq 3$ and the dual problem (3.14)–(3.16) satisfies the H^4-regularity assumption (3.18), the proof is similar and thus omitted. □

4. Error estimates for the Cahn-Hilliard equation

Based on the results established in Section 3, we shall establish an error estimate for the WG-FEM solution u_h arising from (2.8).

Since the nonlinear term $c(\cdot,\cdot)$ is explicit in (2.8), the solvability of (2.8) is a straightforward application of Lemma 3.2.

Denote by $E_h u$ the elliptic projection of u satisfying $\int_\Omega E_h u = \int_\Omega u$ and

$$b(E_h u, v_h) = (\Delta^2 u, v_0) \quad \forall v_h \in W_h. \tag{4.1}$$

Let u_h^m be the numerical solution of the Cahh-Hilliard problem at the m-th step of the WG algorithm (2.8). Define error functions η and ξ as follows:

$$\eta^m = u^m - E_h u^m, \quad \xi^m = E_h u^m - u_h^m. \tag{4.2}$$

The estimation of η^m has been derived in Theorems 3.9–3.13 in various Sobolev norms. It remains to estimate the second term ξ^m. To this end, we first derive the error equation for the WG finite element solution obtained from (2.8).

Lemma 4.1. *Let u^m and $u_h^m \in V_h$ be the solution of* (1.1)–(1.4) *and* (2.8)*, respectively. Then the error functions η^m and ξ^m satisfy the following equation:*

$$\begin{aligned}(d_t\xi^m, v_0) + \gamma^2 b(\xi^m, v_h)& \\ = (R^m, v_0) - (d_t\eta^m, v_0) &+ (\Delta f(u^m) - \Delta_w f(u_h^{m-1}), v_0)\end{aligned} \tag{4.3}$$

for all $v \in V_h$, where

$$R^m = d_t u^m - u_t^m = -\tau_m^{-1}\int_{t_{m-1}}^{t_m}(t - t_{m-1})u_{tt}(t)dt.$$

Proof. For any $v_h \in V_h$, we have

$$\begin{aligned}&(d_t\xi^m, v_0) + \gamma^2 b(\xi^m, v_h)\\ =\ & (d_t E_h u^m, v_0) + \gamma^2 b(E_h u^m, v_h) - (d_t u_h^m, v_0) - \gamma^2 b(u_h^m, v_h)\\ =\ & (d_t E_h u^m, v_0) + \gamma^2 b(\Delta^2 u^m, v_0) - (g, v_0) - (\Delta_w f(u_h^{m-1}), v_0)\\ =\ & (d_t E_h u^m, v_0) - (u_t^m, v_0) + (\Delta f(u^m), v_0) - (\Delta_w f(u_h^{m-1}), v_0),\end{aligned}$$

which leads to the error equation (4.3). □

Next, we shall establish some estimates for all three terms on the right-hand side of the error equation (4.3).

The nonlinear function $f(u)$ is not globally Lipschitz on $\mathbf{R}$. However, on any given interval $[-L, L]$ the function f is Lipschitz continuous. It is also trivial to see that f, f', and f'' are continuous and uniformly bounded on $[-L, L]$ and

$$|f(x) - f(y)| \le C_L|x - y| \quad \forall x, y \in [-L, L].$$

In the application to the Cahn-Hilliard equation, the size of the interval will be taken to be $L = 2\max_{0\le t\le T}\|u(t)\|_\infty + C_0$ in Theorem 4.2. Note that L is a constant depending upon the exact solution u. In particular, it will be shown that for sufficiently small meshsize h and time step $\tau \le h^{\frac{1}{2}}$, the value of the numerical solution u_h lies in $[-L, L]$. Thus, the effective domain for the nonlinear function will be bounded by the interval $[-L, L]$ on which the function f is smooth and Lipschitz continuous.

Assume that the solution u of (1.1)–(1.4) satisfies the following regularity assumptions:

$$\begin{aligned} &u \in C(0,T;L^2(\Omega)),\\ &u, u_t \in L^2(0,T;H^3(\Omega)\cap H^{k+1}(\Omega)),\\ &u_{tt} \in L^2(0,T;L^2(\Omega)). \end{aligned}$$

Theorem 4.2. *Assume that the solution $u \in H^{k+1}$ is the exact solution of* (1.1)–(1.4) *satisfying $\Delta u \in H^2(\Omega)$ and u_h^m is the WG solution arising from Algorithm* 1 *with $k \geq 2$. Assume $\tau = \max_{1\leq m\leq M}\tau_m$ and $h = \max_{T\in\mathcal{T}_h} h_T$ are sufficiently small and $\tau \leq \min_{T\in\mathcal{T}_h} h_T^{\frac{1}{2}}$, then the following estimates hold true for the error function $\xi^m = E_h u^m - u_h^m$:*

$$\|\xi_0^m\|^2 \leq Ce^{C_2T}(\tau^2 + h^{2k}), \tag{4.4}$$

$$\gamma^2\sum_{n=1}^{m}\tau_n|||\xi^n|||^2 \leq Ce^{C_2T}(\tau^2 + h^{2k}), \tag{4.5}$$

where the constant C_2 is proportional to C_L^2.

Furthermore, if $k \geq 3$ and $u \in H^4(\Omega)$, then the following estimate holds true:

$$\begin{aligned} \|\xi_0^m\|^2 &\leq Ce^{C_2T}(\tau^2 + h^{2k+2}),\\ \gamma^2\sum_{n=1}^{m}\tau_n|||\xi^n|||^2 &\leq Ce^{C_2T}(\tau^2 + h^{2k+2}). \end{aligned}$$

Proof. By letting $v_h = \xi^m$ in the error equation (4.3), we have

$$\begin{aligned} &\frac{1}{2}d_t\|\xi_0^m\|^2 + \frac{\tau_m}{2}\|d_t\xi_0^m\|^2 + \gamma^2|||\xi^m|||^2\\ = \ &(R^m, \xi_0^m) - (d_t\eta^m, \xi_0^m) + (\Delta f(u^m) - \Delta_w f(u_h^{m-1}), \xi_0^m). \end{aligned} \tag{4.6}$$

The rest of the proof is devoted to the estimation of the terms on the right-hand side of (4.6). For the first term, we use the Cauchy-Schwarz inequality to obtain

$$(R^m, \xi_0^m) = \sum_{T\in\mathcal{T}_h}(R^m, \xi_0^m)_T \leq \|R^m\|\|\xi_0^m\|. \tag{4.7}$$

From the Cauchy-Schwarz inequality we have

$$\begin{aligned} \|R^m\|^2 &= \int_\Omega\left(\tau_m^{-1}\int_{t_{m-1}}^{t_m}(t-t_{m-1})u_{tt}dt\right)^2 dx\\ &\leq \int_\Omega \tau_m\int_{t_{m-1}}^{t_m}u_{tt}^2 dtdx\\ &= \tau_m\int_{t_{m-1}}^{t_m}\int_\Omega u_{tt}^2 dxdt\\ &= \tau_m\int_{t_{m-1}}^{t_m}\|u_{tt}\|^2 dt. \end{aligned}$$

Substituting the above inequality into (4.7) gives

$$(R^m, \xi_0^m) \quad \leq \quad C\tau_m^{\frac{1}{2}}(A_1^m)^{\frac{1}{2}}\|\xi_0^m\|, \tag{4.8}$$

where $A_1^m = \displaystyle\int_{t_{m-1}}^{t_m} \|u_{tt}\|^2 dt$.

From the definition of E_h (4.1), for any time interval (t_{m-1}, t_m) and $v_h \in V_h$ we have

$$\begin{aligned}
&\left(E_h \int_{t_{m-1}}^{t_m} u_t dt, v_h\right) \\
= \quad & \int_\Omega \Delta^2 \left(\int_{t_{m-1}}^{t_m} u_t dt\right) v_h dx \\
= \quad & \int_{t_{m-1}}^{t_m} \int_\Omega \Delta^2 u_t v_h dx dt \\
= \quad & \int_\Omega \int_{t_{m-1}}^{t_m} E_h u_t dt v_h dx \\
= \quad & \left(\int_{t_{m-1}}^{t_m} E_h u_t dt, v_h\right),
\end{aligned}$$

which leads to

$$E_h \int_{t_{m-1}}^{t_m} u_t dt = \int_{t_{m-1}}^{t_m} E_h u_t dt.$$

For the second term, from Theorem 3.12 we have

$$\begin{aligned}
(d_t\eta^m, \xi_0^m) \quad &= \quad \tau_m^{-1}(u^m - u^{m-1} - E_h(u^m - u^{m-1}), \xi_0^m) \\
&= \quad \tau_m^{-1}\left(\int_{t_{m-1}}^{t_m} u_t - E_h u_t dt, \xi_0^m\right) \\
&\leq \quad \tau_m^{-1}\left(\int_\Omega \left(\int_{t_{m-1}}^{t_m} u_t - E_h u_t dt\right)^2 dx\right)^{\frac{1}{2}} \|\xi_0^m\| \\
&\leq \quad C\tau_m^{-\frac{1}{2}}\left(\int_{t_{m-1}}^{t_m} \|u_t - E_h u_t\|^2 dt\right)^{\frac{1}{2}} \|\xi_0^m\| \\
&\leq \quad C\tau_m^{-\frac{1}{2}} h^k (A_2^m)^{\frac{1}{2}}\|\xi_0^m\|, \qquad (4.9)
\end{aligned}$$

where $A_2^m = \displaystyle\int_{t_{m-1}}^{t_m} \|u_t\|_k^2 dt$.

The third term can be estimated by using the Cauchy-Schwarz inequality as follows:

$$\begin{aligned}
& (\Delta f(u^m) - \Delta_w f(u_h^{m-1}), \xi_0^m) \\
= \;& (f(u^m) - f(u_0^{m-1}), \Delta\xi_0^m) - \sum_{T\in\mathcal{T}_h} \left\langle f(u^m) - f(u_b^{m-1}), \frac{\partial \xi_0^m}{\partial \mathbf{n}} \right\rangle_{\partial T} \\
& + \sum_{T\in\mathcal{T}_h} \left\langle f'(u^m)\frac{\partial u^m}{\partial \mathbf{n}} - f'(u_b^{m-1})u_n^{m-1}(\mathbf{n}\cdot\mathbf{n}_e), \xi_0^m \right\rangle_{\partial T} \\
= \;& (f(u^m) - f(u_0^{m-1}), \Delta\xi_0^m) - \sum_{T\in\mathcal{T}_h} \left\langle f(u^m) - f(u_b^{m-1}), \left(\frac{\partial \xi_0^m}{\partial \mathbf{n}} - \xi_n^m\right)(\mathbf{n}\cdot\mathbf{n}_e) \right\rangle_{\partial T} \\
& + \sum_{T\in\mathcal{T}_h} \left\langle (f'(u^m) - f'(u_b^{m-1}))\frac{\partial u^m}{\partial \mathbf{n}} \right. \\
& \qquad\qquad \left. -f'(u_b^{m-1})\left(\frac{\partial u^m}{\partial \mathbf{n}} - u_n^{m-1}(\mathbf{n}\cdot\mathbf{n}_e)\right), \xi_0^m - \xi_b^m \right\rangle_{\partial T} \\
\leq \;& C\|f(u^m) - f(u_0^{m-1})\|\|\Delta\xi_0^m\| \\
& + C\left(\sum_{T\in\mathcal{T}_h} \|f(u^m) - f(u_b^{m-1})\|_{\partial T}^2\right)^{\frac{1}{2}} \left(\sum_{T\in\mathcal{T}_h} \|\frac{\partial \xi_0^m}{\partial \mathbf{n}_e} - \xi_n^m\|_{\partial T}\right)^{\frac{1}{2}} \\
& + C\left(\sum_{T\in\mathcal{T}_h} \|f'(u^m) - f'(u_b^{m-1})\|_{\partial T}^2 \right. \\
& \qquad\qquad \left. + \|\frac{\partial u^m}{\partial \mathbf{n}_e} - u_n^{m-1}\|_{\partial T}^2\right)^{\frac{1}{2}} \left(\sum_{T\in\mathcal{T}_h} \|\xi_0^m - \xi_b^m\|_{\partial T}\right)^{\frac{1}{2}} \\
\leq \;& C\|u^m - u_0^{m-1}\|\|\!|\xi^m|\!\| + Ch^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h} \|u^m - u_b^{m-1}\|_{\partial T}^2\right)^{\frac{1}{2}} |\!|\!|\xi^m|\!|\!| \\
& + Ch^{\frac{3}{2}}\left(\sum_{T\in\mathcal{T}_h} \left\|\frac{\partial u^m}{\partial \mathbf{n}_e} - u_n^{m-1}\right\|_{\partial T}^2\right)^{\frac{1}{2}} |\!|\!|\xi^m|\!|\!|.
\end{aligned}$$

Notice that

$$\begin{aligned}
& \|u^m - u_0^{m-1}\| \\
\leq \;& \|u^m - u^{m-1}\| + \|u^{m-1} - E_h u_0^{m-1}\| + \|E_h u_0^{m-1} - u_0^{m-1}\| \\
= \;& \|u^m - u^{m-1}\| + \|\eta_0^{m-1}\| + \|\xi_0^{m-1}\|,
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
& \|u^m - u_b^{m-1}\|_{\partial T} \\
\leq \;& \|u^m - u^{m-1}\|_{\partial T} + \|u^{m-1} - (E_h u^{m-1})_b\|_{\partial T} \\
& + \|\xi_0^{m-1}\|_{\partial T} + \|\xi_0^{m-1} - \xi_b^{m-1}\|_{\partial T} \\
\leq \;& C(h^{-\frac{1}{2}}\|u^m - u^{m-1}\|_T + h^{\frac{1}{2}}\|u^m - u^{m-1}\|_{1,T}) \\
& + C(\|\eta_b^{m-1}\|_{\partial T} + h^{-\frac{1}{2}}\|\xi_0^{m-1}\|_T + \|\xi_0^{m-1} - \xi_b^{m-1}\|_{\partial T})
\end{aligned} \tag{4.11}$$

and

$$
\begin{aligned}
(4.12)\qquad & \left\| \frac{\partial u^m}{\partial \mathbf{n}_e} - u_n^{m-1} \right\|_{\partial T} \\
\le\ & \left\| \frac{\partial u^m}{\partial \mathbf{n}_e} - \frac{\partial u^{m-1}}{\partial \mathbf{n}_e} \right\|_{\partial T} + \left\| \frac{\partial u^{m-1}}{\partial \mathbf{n}_e} - (E_h u^{m-1})_n \right\|_{\partial T} \\
& + \left\| \frac{\partial \xi_0^{m-1}}{\partial \mathbf{n}_e} \right\|_{\partial T} + \left\| \frac{\partial \xi_0^{m-1}}{\partial \mathbf{n}_e} - \xi_n^{m-1} \right\|_{\partial T} \\
\le\ & C\left(h^{-\frac{1}{2}} \|u^m - u^{m-1}\|_{1,T} + h^{\frac{1}{2}} \|u^m - u^{m-1}\|_{2,T} \right) \\
& + C\left(\|\eta_n^{m-1}\|_{\partial T} + h^{-\frac{3}{2}} \|\xi_0^{m-1}\|_T + \left\| \frac{\partial \xi_0^{m-1}}{\partial \mathbf{n}_e} - \xi_n^{m-1} \right\|_{\partial T} \right).
\end{aligned}
$$

Then we conclude from (4.10)–(4.12) that

$$
\begin{aligned}
& \left(\Delta f(u^m) - \Delta_w f(u_h^{m-1}), \xi_0^m \right) \\
& \le C(\|u^m - u^{m-1}\| + h\|u^m - u^{m-1}\|_1 + h^2\|u^m - u^{m-1}\|_2) |\!|\!| \xi^m |\!|\!| \\
& \quad + C(\|\eta_0^{m-1}\| + \|\eta_b^{m-1}\|_{\mathcal{E}_h} + h\|\eta_n^{m-1}\|_{\mathcal{E}_h} + h^2 |\!|\!| \xi^{m-1} |\!|\!| + \|\xi_0^{m-1}\|) |\!|\!| \xi^m |\!|\!| \\
(4.13)\qquad & \le C(\tau_m^{\frac{1}{2}} (A_3^m)^{\frac{1}{2}} + h^k \|u^{m-1}\|_{k+1} + \|\xi_0^{m-1}\|) |\!|\!| \xi^m |\!|\!| + Ch^2 |\!|\!| \xi^{m-1} |\!|\!| |\!|\!| \xi^m |\!|\!|,
\end{aligned}
$$

where $A_3^m = \displaystyle\int_{t_{m-1}}^{t_m} \|u_t\|_2^2 \, dt$.

Combining the above estimates with the error equation (4.3) we arrive at:

$$
\begin{aligned}
& \frac{1}{2} d_t \|\xi_0^m\|^2 + \frac{\tau_m}{2} \|d_t \xi_0^m\|^2 + \gamma^2 |\!|\!| \xi^m |\!|\!|^2 \\
\le\ & C\left(\tau_m^{\frac{1}{2}} (A_3^m)^{\frac{1}{2}} + h^k \|u^{m-1}\|_{k+1} + h^k \|\xi_0^{m-1}\| \right) |\!|\!| \xi^m |\!|\!| + C\tau_m^{\frac{1}{2}} (A_1^m)^{\frac{1}{2}} \|\xi_0^m\| \\
& + Ch^2 |\!|\!| \xi^{m-1} |\!|\!| |\!|\!| \xi^m |\!|\!| + C\tau_m^{-\frac{1}{2}} h^k (A_2^m)^{\frac{1}{2}} \|\xi_0^m\| \\
\le\ & \frac{\gamma^2}{2} |\!|\!| \xi_0^m |\!|\!|^2 + \frac{1}{2} \|\xi_0^m\|^2 + \frac{C}{\gamma^2} \|\xi_0^{m-1}\|^2 + \frac{C}{\gamma^2} h^2 |\!|\!| \xi^{m-1} |\!|\!|^2 + C\tau_m A_1^m \\
& + \frac{C}{\gamma^2} \tau_m A_3^m + \frac{C}{\gamma^2} h^{2k} \|u^{m-1}\|_{k+1}^2 + C\tau_m^{-1} h^{2k} A_2^m.
\end{aligned}
$$

Multiplying both sides of the above inequality by $2\tau_m$, we obtain

$$
\begin{aligned}
(4.14)\qquad & (1 - \tau_m) \|\xi_0^m\|^2 + \gamma^2 \tau_m |\!|\!| \xi^m |\!|\!|^2 \\
\le\ & (1 + \frac{C}{\gamma^2} \tau_m) \|\xi_0^{m-1}\|^2 + \frac{C}{\gamma^2} \tau_m h^2 |\!|\!| \xi^{m-1} |\!|\!|^2 + \Gamma^m,
\end{aligned}
$$

where

$$
\Gamma^m = \frac{C}{\gamma^2} \left(\tau_m^2 A_3^m + \tau_m h^{2k} \|u^{m-1}\|_{k+1}^2 \right) + C\tau_m^2 A_1^m + Ch^{2k} A_2^m.
$$

Denote $\alpha_m = \frac{1 + \frac{C}{\gamma^2} \tau_m}{1 - \tau_m}$, $C_1 = 2(\frac{C}{\gamma^2} + 1)$, and $C_2 = \frac{C}{2\gamma^2}$. Then

$$
e^{C_1 \tau_m} < \alpha_m < e^{C_2 \tau_m} \quad \text{for } 0 < \tau_m < \frac{1}{4}.
$$

From the equation (4.14), we get

$$
\begin{aligned}
\|\xi_0^m\|^2 &\le \alpha_m\|\xi_0^{m-1}\|^2 + 2\Gamma^m + \frac{C}{\gamma^2}\tau_m h^2 |||\xi^{m-1}|||^2 \\
&\le \alpha_m(\alpha_{m-1}\|\xi_0^{m-2}\|^2 + 2\Gamma^{m-1} \\
&\quad + \frac{C}{\gamma^2}\tau_{m-1}h^2|||\xi^{m-2}|||^2) + 2\Gamma^m + \frac{C}{\gamma^2}\tau_m h^2|||\xi^{m-1}|||^2 \\
&\le \prod_{n=1}^m \alpha_n \|\xi_0^0\|^2 + 2\sum_{n=1}^m \prod_{j=n}^m \alpha_j \left(\Gamma^n + \frac{C}{\gamma^2}\tau_n h^2 |||\xi^{n-1}|||^2\right) \\
&\le e^{C_2T}\left(\|\xi_0^0\|^2 + 2\sum_{n=1}^m\left(\Gamma^n + \frac{C}{\gamma^2}\tau_n h^2|||\xi^{n-1}|||^2\right)\right) \\
&\le Ce^{C_2T}\left(\tau^2 + h^{2k}\right) + \frac{C}{\gamma^2}h^2 e^{C_2T}\sum_{n=1}^m \tau_n|||\xi^n|||^2.
\end{aligned}
\tag{4.15}
$$

From the regularity assumptions, $\sum_{l=1}^m A_1^l$, $\sum_{l=1}^m A_2^l$, and $\sum_{l=1}^m A_3^l$ are all bounded. Summing the equation (4.14) from 1 to m, we obtain

$$
\begin{aligned}
\gamma^2\sum_{n=1}^m \tau_n|||\xi^n|||^2 &\le -\|\xi_0^m\|^2 + \|\xi_0^0\|^2 + \frac{C}{\gamma^2}\sum_{n=1}^m \tau_n\|\xi_0^n\|^2 \\
&\quad + \frac{C}{\gamma^2}h^2e^{C_2T}\sum_{n=1}^m\tau_n|||\xi^n|||^2 + \sum_{n=1}^m \Gamma^n \\
&\le Ce^{C_2T}(\tau^2 + h^{2k}) + \frac{C}{\gamma^4}h^2e^{C_2T}\sum_{n=1}^m \tau_n|||\xi^n|||^2.
\end{aligned}
\tag{4.16}
$$

Combining (4.15) and (4.16) we have

$$
\|\xi_0^m\|^2 + \gamma^2\sum_{n=1}^m \tau_n|||\xi^n|||^2 \le Ce^{C_2T}(\tau^2 + h^{2k}),
\tag{4.17}
$$

provided that the meshsize h is sufficiently small.

By setting $h_{\min} = \min_{T\in\mathcal{T}_h} h_T$, from the inverse inequality and the error estimate in L^2 we arrive at

$$
\begin{aligned}
\|u_h^m\|_{L^\infty} &\le \|u_h^m - E_hu^m\|_{L^\infty} + \|u^m - E_hu^m\|_{L^\infty} + \|u^m\|_{L^\infty} \\
&\le Ch_{\min}^{-\frac12}(\|\xi_0^m\| + \|\eta_0^m\|) + \|u^m\|_{L^\infty} \\
&\le Ch_{\min}^{-\frac12}(\tau + h^k) + \|u^m\|_{L^\infty},
\end{aligned}
$$

when $\tau \le h_{\min}^{\frac12}$ and h is sufficiently small, we have $\|u_h^m\|_{L^\infty} \le 2\|u^m\|_{L^\infty} + C_0$, where C_0 is a constant independent of h and τ. It follows that the range of the numerical solution u_h lies in $[-L, L]$ when $\tau \le h^{\frac12}$ and h is sufficiently small. On the interval $[-L, L]$, there exists a constant C_L such that

$$
|f(x) - f(y)| \le C_L|x - y| \quad \forall x, y \in [-L, L],
$$

i.e., f is Lipschitz continuous on $[-L, L]$.

For $k \ge 3$ and $u \in H^4(\Omega)$, the corresponding error estimate can be derived analogously by following a similar procedure. Details are thus omitted. $\square$

5. Numerical results

In this section, we use four numerical examples to verify the theoretical results derived in previous sections.

5.1. P_2-WG for the biharmonic equation.

We solve the linear biharmonic problem (3.1)–(3.2) with the constraint (3.3) being replaced by

$$\int_\Omega u = \frac{4}{\pi^2}.$$

The exact solution is chosen as

$$u(x,y) = \cos(\pi x)\cos(\pi y), \tag{5.1}$$

defined in the unit square domain $\Omega = (0,1) \times (0,1)$. The right-hand side function f in equation (3.1) is calculated by matching the exact solution u.

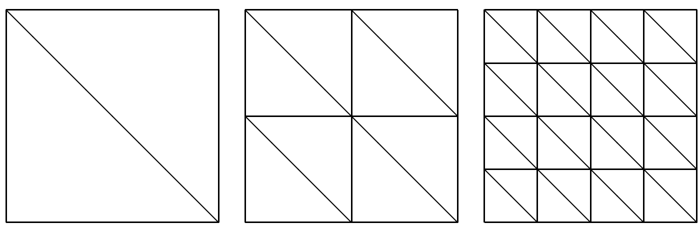

Figure 5.1. The level 1, 2, and 3 uniform grids.

We apply the P_2 weak Galerkin algorithm, i.e., $k = 2$ in the finite element space V_h^0 (2.1) and (2.2), and in the computation of the weak-Laplacian (2.4). The first three levels of grid are depicted in Figure 5.1. The optimal order of convergence of the method is confirmed by the numerical results shown as in Table 5.1 where the discrete norms $|\cdot|_{s_1}$ and $|\cdot|_{s_2}$ are induced by the two bilinear forms in $s(\cdot,\cdot)$, (2.5). It can be seen that the weak Galerkin finite element method does achieve the optimal order of convergence in the H^2-like discrete norm. It is interesting to note that the numerical solutions seem to be super-convergent in the L^2 norm. A theoretical investigation for such a superconvergence could be taken as a future research topic.

Table 5.1. The errors, where $e_h = Q_h u - u_h$, and the orders of convergence (n in the $O(h^n)$ error), by the P_2 element (2.1) on the uniform grids (Figure 5.1) for (5.1).

level	$\|Q_0u - u_0\|$	n	$\|\!\|e_h\|\!\|$	n	$\|e_h\|_{s_1}$	n	$\|e_h\|_{s_2}$	n
1	2.17481		7.48116		9.99672		12.64885	
2	0.23105	3.2	9.96075	0.0	5.07280	1.0	2.92707	2.1
3	0.01054	4.5	5.37375	0.9	1.02005	2.3	0.30465	3.3
4	0.00061	4.1	2.73367	1.0	0.14057	2.9	0.06932	2.1
5	0.00006	3.4	1.37275	1.0	0.02645	2.4	0.01859	1.9

5.2. **P_2-WG for the Cahn-Hilliard equation.** In the present test, we solve the nonlinear Cahn-Hilliard equation (1.1) by using the P_2 weak Galerkin finite element method, where the exact solution is given by

$$u(x, y) = e^{-4\pi^4 t} \cos(\pi x)\cos(\pi y). \tag{5.2}$$

The function g in (1.1) is computed by matching the exact solution u with $\gamma = 1$. Table 5.1 illustrates the numerical errors on the fourth grid computed by using three time-step sizes:

$$\tau = 0.002,\ 0.001 \ \text{ and } \ 0.0005.$$

From Table 5.1, we see a first order of convergence in the time discretization:

$$\|Q_0 u - u_0\| = O(\tau), \quad \|\!|Q_h u - u_h|\!\| = O(\tau).$$

TABLE 5.2. The errors, where $e_h = (Q_h u - u_h)(0.01)$, and the orders of convergence (n in the $O(\tau^n)$ error), by the P_2 element (2.1) on the uniform grids (Figure 5.1) for (5.2).

τ	$\|Q_0 u - u_0\|$	order n	$\|\!|e_h|\!\|$	order n
0.0020	0.02728		0.55173	
0.0010	0.01365	1.00	0.29534	0.90
0.0005	0.00679	1.01	0.18049	0.71

5.3. **P_2-WG for the two-phase Cahn-Hilliard equation.** In this numerical experiment, we solve the nonlinear Cahn-Hilliard equation (1.1) by using the P_2 weak Galerkin finite element method. The exact solution for this test case is not known. This example is taken from [18] for which we have $g = 0$ in (1.1) and $\gamma = 0.01$. The initial condition u^0 is plotted in Figure 5.2. The time step in our computational is given by $\tau = 0.0001$.

The WG solution on the grid level 5 (cf. Fig. 5.1) is plotted in the right figure of Figure 5.2. On the fifth grid, the number of unknowns is

$$2n^2(6) + (2n(n+1) + n^2)(3+2) = 7072, \quad \text{where } n = 2^{5-1}.$$

At each time level, we use the conjugate gradient iterative algorithm to solve the implicit finite element equation. We also use some adaptive grids with hanging nodes (local refinement and coarsening after 100 time-steps) to compute the solution. Figure 5.3 illustrates the final grid (at time $t = 0.3$) after the local refinement. We note that the computation on adaptive grids is much more efficient than on uniform grids, due to the sharp change of the solution near the interface.

5.4. **P_4-WG for the two-phase Cahn-Hilliard equation.** In our final numerical test, we solve the nonlinear Cahn-Hilliard equation (1.1) by using the P_4 weak Galerkin finite element method. In this setting, the initial condition is shown in the left diagram of Figure 5.4. This example is also taken from [18] with $g = 0$ in (1.1) and

$$\gamma = 0.01, \quad \text{and time-step } \ \tau = 0.0001.$$

As the initial condition is a smooth function, the process is expected to take a longer time to reach the steady-state of the solution. We first try to solve this

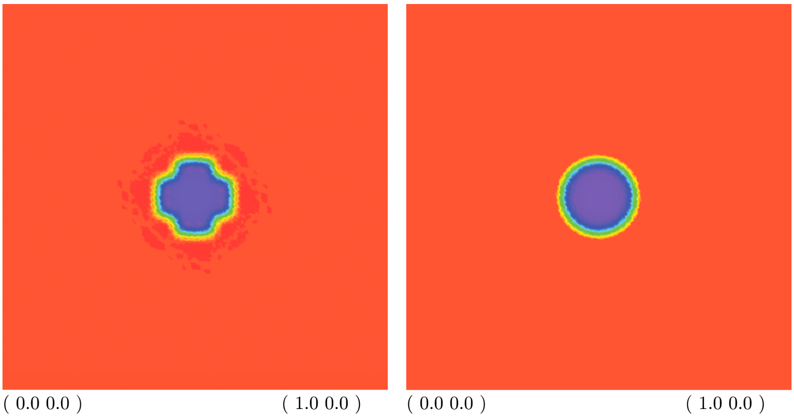

FIGURE 5.2. The WG initial solution $u_0(0)$ and the WG steady-state solution $u_0(0.3)$.

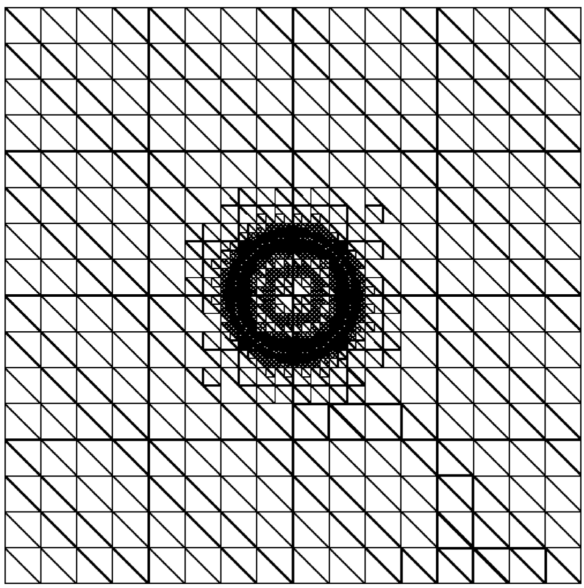

FIGURE 5.3. The final adaptive grid (at $t = 0.3$) for WG method solving problem of Figure 5.2.

problem by using P_2-WG finite element method. But the computational results are not satisfactory; the iteration often stalls at some discrete steady-state solutions in which the discrete interface appears to be rough and the discrete concentration force $f(u)$ seems to balance the smoothing force of the discrete biharmonic operator. We then moved to P_4-WG finite element method on grid level 6; i.e., $k = 4$ in (2.1) and (2.4). On the grid level 6, the number of unknowns for P_4-WG is

$$2n^2(15) + (2n(n+1) + n^2)(5+4) = 14880, \quad \text{where } n = 2^{6-1}.$$

At each time level, we apply the conjugate gradient iterative scheme to solve the implicit finite element equations. Figure 5.4 illustrates the numerical solutions at three times levels. It shows that the numerical solutions take a longer time (than the last testing case) to reach its steady-state. Figure 5.5 demonstrates more computational results at other time levels till the steady-state solution is well approached. This test shows the accuracy and stability of high order WG finite element methods in practical scientific computing.

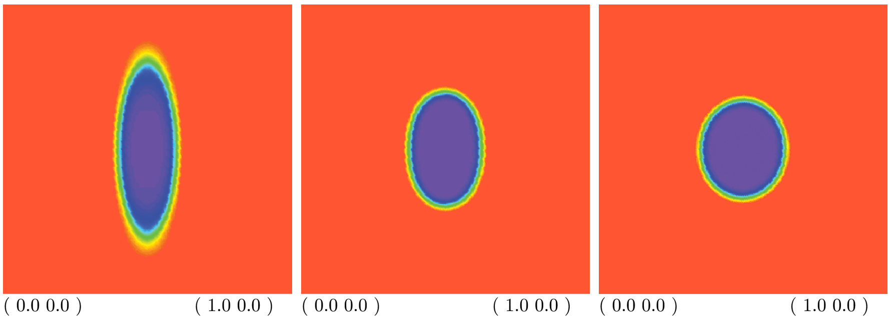

FIGURE 5.4. The WG solution $u_0(0)$, and solutions $u_0(0.15)$, $u_0(0.3)$.

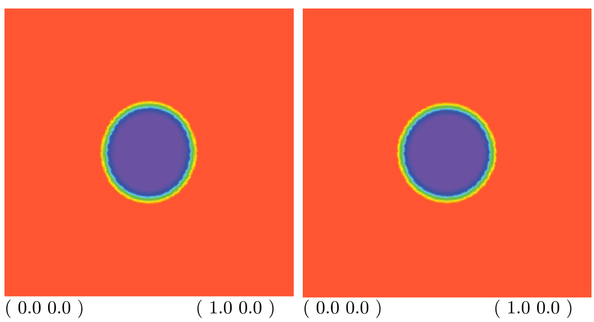

FIGURE 5.5. The WG solution $u_0(0.4)$ and $u_0(0.5)$.

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Division of Mathematical Sciences, National Science Foundation, Arlington, Virginia 22230
Email address: jwang@nsf.gov

Department of Mathematics, Jilin University, Changchun, China
Email address: diql15@mails.jlu.edu.cn

Department of Mathematics, Jilin University, Changchun, China
Email address: zhangran@mail.jlu.edu.cn

Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716
Email address: szhang@udel.edu