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An existence result for a mixed variational problem arising from Contact Mechanics

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ABSTRACT

We consider a mixed variational problem involving a nonlinear, hemicontinuous, generalized monotone operator. The proposed problem consists of a variational equation in a real reflexive Banach space and a variational inequality in a subset of a second real reflexive Banach space. We investigate the existence of the solution using a fixed point theorem for set valued mapping. An example arising from Contact Mechanics illustrates the theory.

1. Introduction

The present paper focuses on the following mixed variational problem.

Problem 1. Given $f \in X'$, find $(u, \lambda) \in X \times \Lambda$ so that

$$(Au, v)_{X',X} + b(v, \lambda) = (f, v)_{X',X} \quad \text{for all } v \in X, \tag{1}$$

$$b(u, \mu - \lambda) \le 0 \quad \text{for all } \mu \in \Lambda. \tag{2}$$

Here and everywhere below X' denotes the dual of the space X and Λ is a subset of a space Y.

If X and Y are Hilbert spaces and $A : X \to X$ is a symmetric, continuous and strongly monotone operator, then we can write the following *saddle point problem*:

$$a(u, v) + b(v, \lambda) = (\tilde{f}, v)_X \quad \text{for all } v \in X, \tag{3}$$

$$b(u, \mu - \lambda) \le 0 \quad \text{for all } \mu \in \Lambda; \tag{4}$$

herein $a : X \times X \to \mathbb{R}$ is the bilinear, symmetric, continuous, X-elliptic form $a(u, v) = (Au, v)_X$ and $\tilde{f}$ is the unique element of X so that $(f, v)_{X',X} = (\tilde{f}, v)_X$ for all $v \in X$. If, in addition, $b(\cdot, \cdot) : X \times Y$ is a bilinear continuous form satisfying the "inf–sup property"

$$\exists\, \alpha > 0 : \inf_{\mu \in Y, \mu \neq 0_Y} \sup_{v \in X, v \neq 0_X} \frac{b(v, \mu)}{\|v\|_X \|\mu\|_Y} \ge \alpha$$

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and Λ is a closed convex subset of Y so that $0_Y \in \Lambda$, then the problem (3)–(4) has a unique solution $(u, \lambda) \in X \times \Lambda$ which is the unique saddle point of the following functional

$$\mathcal{L} : X \times \Lambda \to \mathbb{R} \quad \mathcal{L}(v, \mu) = \frac{1}{2}a(v, v) - (\tilde{f}, v)_X + b(v, \mu), \tag{5}$$

see e.g. [1,2]. The saddle point problem (3)–(4) can be related to the weak formulation of a class of unilateral frictionless or bilateral frictional contact problems, for linearly elastic materials, see for instance [2,3]. For a class of generalized saddle point problems related to the weak solvability of contact models involving a particular class of nonlinearly elastic materials we refer the reader to [4,5]; the weak solution of such a generalized saddle point problem is the unique fixed point of a single valued operator which is defined by means of the unique solution of an intermediate saddle point problem.

The current work focuses on a new theoretical result which will allow to explore contact models for another class of nonlinearly elastic materials; the key herein is not the saddle point theory; the key here is a fixed point theorem for set valued mapping. It is worth mentioning that mixed weak formulations in Contact Mechanics are appropriate approaches to efficiently approximate the weak solutions; see e.g. [6,7,3,8] for modern numerical techniques. The study on this direction is in progress. For a more complex view on mixed variational formulations in Mechanics we refer the reader also to [9–14].

In the present paper we shall study Problem 1 under the following assumptions.

Assumption 1. $(X, (\cdot, \cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot, \cdot)_Y, \|\cdot\|_Y)$ are two real reflexive Banach spaces.

Assumption 2. Λ is a closed convex bounded subset of Y so that $0_Y \in \Lambda$.

Assumption 3. There exists a functional $h : X \to \mathbb{R}$ so that:

- (i_1) $h(tw) = t^r h(w)$ for all $t > 0,\ w \in X$ and $r > 1$;
- (i_2) $(Av - Au, v - u)_{X',X} \geq h(v - u)$ for all $u, v \in X$;
- (i_3) If $(x_n)_n \subset X$ is a sequence so that $x_n \rightharpoonup x$ in X as $n \to \infty$, then $h(x) \leq \limsup_{n\to\infty} h(x_n)$.

Notice that (i_1) and (i_2) in Assumption 3 express a generalized monotonicity property for the operator $A : X \to X'$. According to the literature, the operator A is a relaxed h-monotone operator, see for example [15]; see also [16–20] for various generalizations of monotonicity such as pseudomonotonicity, quasimonotonicity, semimonotonicity, relaxed monotonicity.

Assumption 4. The operator $A : X \to X'$ is hemicontinuous, i.e., for all $u, v \in X$, the mapping $f : \mathbb{R} \to (-\infty, +\infty)$, $f(t) = (A(u + tv), v)_{X',X}$ is continuous at 0.

Assumption 5. $\frac{(Au,u)_{X',X}}{\|u\|_X} \to \infty$ as $\|u\|_X \to \infty$.

Assumption 6. The form $b : X \times Y \to \mathbb{R}$ is bilinear. In addition,

- for each sequence $(u_n)_n \subset X$ so that $u_n \rightharpoonup u$ in X as $n \to \infty$ we have $b(u_n, \mu) \to b(u, \mu)$ as $n \to \infty$, for all $\mu \in \Lambda$.
- for each sequence $(\lambda_n)_n \subset Y$ so that $\lambda_n \rightharpoonup \lambda$ in Y as $n \to \infty$, we have $b(v, \lambda_n) \to b(v, \lambda)$ as $n \to \infty$, for all $v \in X$.

In the present paper we shall prove that, under Assumptions 1–6, Problem 1 has at least one solution. Assumptions 1–6 impose a new technique in order to handle Problem 1, namely a fixed point technique involving a set valued mapping, instead of a saddle point technique. Let us recall here the main tool we use.

Theorem 1. *Let $\mathcal{K} \neq \emptyset$ be a convex subset of a Hausdorff topological vector space $\mathcal{E}$. Let $F : \mathcal{K} \to 2^{\mathcal{K}}$ be a set valued map so that*

(h_1) *for each $u \in \mathcal{K}$, $F(u)$ is a nonempty convex subset of $\mathcal{K}$;*
(h_2) *for each $v \in \mathcal{K}$, $F^{-1}(v) = \{u \in \mathcal{K} : v \in F(u)\}$ contains an open set O_v which may be empty;*
(h_3) $\bigcup_{v\in\mathcal{K}} \mathcal{O}_v = \mathcal{K}$;
(h_4) *there exists a nonempty set $\mathcal{V}_0$ contained in a compact convex subset $\mathcal{V}_1$ of $\mathcal{K}$ so that $\mathcal{D} = \bigcap_{v\in\mathcal{V}_0} \mathcal{O}_v^c$ is either empty or compact.*

Then, there exists $u_0 \in \mathcal{K}$ so that $u_0 \in F(u_0)$.

We note that $2^{\mathcal{K}}$ denotes the family of all subsets of $\mathcal{K}$, and $\mathcal{O}_v^c$ is the complement of $\mathcal{O}_v$ in $\mathcal{K}$. For a proof of this theorem we refer to [21].

We end this introductive part by specifying the structure of the rest of the paper. In Section 2 an existence result for an intermediate problem is given. In Section 3 we use the intermediate result to prove that, under Assumptions 1–6, Problem 1 has at least one solution. In Section 4 we give an example of functional spaces X and Y, operator A, bilinear form $b(\cdot, \cdot)$ and subset Λ so that Assumptions 1–6 are fulfilled. In the last section we discuss a contact model related to the example given in Section 4.

2. An auxiliary result

Let us construct a bounded convex closed nonempty subset of X as follows,

$$K_n = \{v \in X : \|v\|_X \leq n\}$$

where n is an arbitrarily fixed positive integer. We consider the following problem.

Problem 2. Given $f \in X'$, find $(u_n, \lambda_n) \in K_n \times \Lambda$ so that

$$(Au_n, v - u_n)_{X',X} + b(v, \lambda_n) - b(u_n, \mu) \geq (f,\ v - u_n)_{X',X} \quad \text{for all } (v, \mu) \in K_n \times \Lambda. \tag{6}$$

Lemma 1. *A pair $(u_n, \lambda_n) \in K_n \times \Lambda$ is a solution of Problem 2 if and only if it verifies*

$$(Av, v - u_n)_{X',X} + b(v, \lambda_n) - b(u_n, \mu) \geq (f, v - u_n)_{X',X} + h(v - u_n) \quad \text{for all } (v, \mu) \in K_n \times \Lambda. \tag{7}$$

Proof. Let (u_n, λ_n) be a solution of Problem 2. By Assumption 3 we have

$$(Av, v - u_n)_{X',X} \geq (Au_n, v - u_n)_{X',X} + h(v - u_n)$$

and from this

$$(Au_n, v - u_n)_{X',X} \leq (Av, v - u_n)_{X',X} - h(v - u_n).$$

Combining the previous inequality with (6) we obtain (7).

Conversely, assume that $(u_n, \lambda_n) \in K_n \times \Lambda$ verifies (7). We shall prove that this pair (u_n, λ_n) is a solution of Problem 2. To start, let us take (w, ζ) an arbitrary pair in $K_n \times \Lambda$. Setting in (7) $v = u_n + t(w - u_n)$ and $\mu = \lambda_n + t(\zeta - \lambda_n)$ with $t \in (0, 1)$, then

$$\begin{aligned} &t(A(u_n + t(w - u_n)), w - u_n)_{X',X} + b(u_n, \lambda_n) + tb(w - u_n, \lambda_n) - b(u_n, \lambda_n) - tb(u_n, \zeta - \lambda_n) \\ &\quad \geq t(f, w - u_n)_{X',X} + t^r h(w - u_n). \end{aligned}$$

After dividing by $t > 0$ we obtain

$$(A(u_n + t(w - u_n)), w - u_n)_{X',X} + b(w - u_n, \lambda_n) - b(u_n, \zeta - \lambda_n) \geq (f, w - u_n)_{X',X} + t^{r-1}h(w - u_n).$$

Passing to the limit when $t \to 0$ and using the hemicontinuity of the operator A, we obtain (6). $\square$

In the study of Problem 2 we have the following existence result.

Theorem 2. *If Assumptions 1–4 and 6 hold true, then Problem 2 has at least one solution $(u_n, \lambda_n) \in K_n \times \Lambda$.*

Proof. Arguing by contradiction, for each $(u, \lambda) \in K_n \times \Lambda$ there exists $(v, \mu) \in K_n \times \Lambda$ so that

$$(Au, v - u)_{X',X} + b(v, \lambda) - b(u, \mu) < (f,\ v - u)_{X',X}.$$

Let us define a set valued map $F : K_n \times \Lambda \to 2^{K_n \times \Lambda}$ as follows:

$$F(u, \lambda) = \left\{(v, \mu) \in K_n \times \Lambda : (Au, v - u)_{X',X} + b(v, \lambda) - b(u, \mu) < (f,\ v - u)_{X',X}\right\}.$$

We shall prove that this map verifies (h_1)–(h_4) in Theorem 1 with $\mathcal{K} = K_n \times \Lambda$ and $\mathcal{E} = X \times Y$.

Let $(u, \lambda) \in K_n \times \Lambda$. Since Problem 2 has no solution, then $F(u, \lambda) \neq \emptyset$. Besides, $F(u, \lambda)$ is a convex set. Indeed, let $(v_1, \mu_1),\ (v_2, \mu_2) \in K_n \times \Lambda$ and $t \in [0, 1]$. The following two inequalities hold true:

$$\begin{aligned} &(Au, tv_1 - tu)_{X',X} + b(tv_1, \lambda) - b(u, t\mu_1) < (f,\ tv_1 - tu)_{X',X}, \\ &(Au, (1 - t)v_2 - (1 - t)u)_{X',X} + b((1 - t)v_2, \lambda) - b(u, (1 - t)\mu_2) < (f,\ (1 - t)v_2 - (1 - t)u)_{X',X}. \end{aligned}$$

By summing this two last inequalities we get

$$(Au, tv_1 + (1 - t)v_2 - u)_{X',X} + b(tv_1 + (1 - t)v_2, \lambda) - b(u, t\mu_1 + (1 - t)\mu_2) < (f,\ tv_1 + (1 - t)v_2 - u)_{X',X}.$$

Hence, $(tv_1 + (1 - t)v_2, t\mu_1 + (1 - t)\mu_2) \in F(u, \lambda)$. Therefore, (h_1) in Theorem 1 is fulfilled.

Let us check (h_2). For every $(v, \mu) \in K_n \times \Lambda$ we introduce $F^{-1}(v, \mu)$ as follows,

$$\begin{aligned} F^{-1}(v, \mu) &= \{(u, \lambda) \in K_n \times \Lambda : (v, \mu) \in F(u, \lambda)\} \\ &= \{(u, \lambda) \in K_n \times \Lambda : (Au, v - u)_{X',X} + b(v, \lambda) - b(u, \mu) < (f,\ v - u)_{X',X}\}. \end{aligned}$$

Besides, for every $(v, \mu) \in K_n \times \Lambda$ we define

$$\mathcal{O}_{(v,\mu)} = \{(u, \lambda) \in K_n \times \Lambda : (Av, v - u)_{X',X} + b(v, \lambda) - b(u, \mu) < (f,\ v - u)_{X',X} + h(v - u)\}.$$

The following inclusion holds true:

$$[F^{-1}(v, \mu)]^c \subseteq \mathcal{O}^c_{(v,\mu)}. \tag{8}$$

Indeed, if $(u, \lambda) \in [F^{-1}(v, \mu)]^c$, then

$$(Au, v - u)_{X',X} + b(v, \lambda) - b(u, \mu) \geq (f, v - u)_{X',X}.$$

Using Assumption 3, we have

$$(Au, v - u)_{X',X} \leq (Av, v - u)_{X',X} - h(v - u).$$

By combining these last two inequalities we are led to

$$(Av, v - u)_{X',X} + b(v, \lambda) - b(u, \mu) \geq (f, v - u)_{X',X} + h(v - u).$$

Hence,

$$(u, \lambda) \in \mathcal{O}^c_{(v,\mu)}$$

which concludes (8). Now, we deduce that

$$\mathcal{O}_{(v,\mu)} \subseteq F^{-1}(v, \mu).$$

Let us prove that $\mathcal{O}^c_{(v,\mu)}$ is weakly closed. To that end, let $(u_m, \lambda_m)_m \subset \mathcal{O}^c_{(v,\mu)}$ be a sequence so that $(u_m, \lambda_m) \rightharpoonup (u, \lambda)$ in $X \times Y$ as $m \to \infty$. Thus, $u_m \rightharpoonup u$ in X as $m \to \infty$ and $\lambda_m \rightharpoonup \lambda$ in Y as $m \to \infty$.

Since, for all $m \geq 1$, we have

$$(Av, v - u_m)_{X',X} + b(v, \lambda_m) - b(u_m, \mu) \geq (f, v - u_m)_{X',X} + h(v - u_m),$$

then, by (i_3) in Assumptions 3 and 6, passing to the superior limit as $m \to \infty$ we deduce that $(u, \lambda) \in \mathcal{O}^c_{(v,\mu)}$. As $\mathcal{O}^c_{(v,\mu)}$ is weakly closed then $\mathcal{O}_{(v,\mu)}$ is weakly open.

Let us verify now that

$$K_n \times \Lambda = \bigcup_{(v,\mu)\in K_n\times\Lambda} \mathcal{O}_{(v,\mu)}.$$

Clearly,

$$\bigcup_{(v,\mu)\in K_n\times\Lambda} \mathcal{O}_{(v,\mu)} \subseteq K_n \times \Lambda.$$

It remains to prove the following inclusion,

$$K_n \times \Lambda \subseteq \bigcup_{(v,\mu)\in K_n\times\Lambda} \mathcal{O}_{(v,\mu)}.$$

Indeed, let $(u, \lambda) \in K_n \times \Lambda$. As Problem 2 has no solution, based on Lemma 1 it follows that there exists $(v, \mu) \in K_n \times \Lambda$ so that $(u, \lambda) \in \mathcal{O}_{(v,\mu)}$. Thus, (h_3) holds true.

Finally, we verify (h_4). Let us set $\mathcal{V}_0 = \mathcal{V}_1 = K_n \times \Lambda$. The set $\mathcal{D} = \bigcap_{(v,\mu)\in K_n\times\Lambda} \mathcal{O}^c_{(v,\mu)}$ is empty or weakly closed as it is the intersection of weakly closed sets $\mathcal{O}^c_{(v,\mu)}$. As $K_n \times \Lambda$ is a nonempty closed convex bounded subset of the reflexive space $X \times Y$, it follows that $K_n \times \Lambda$ is weakly compact. Therefore, $\mathcal{D}$ is either empty or weakly compact.

Hence, all hypotheses of Theorem 1 hold true in the weak topology. We deduce that there exists $(u_0, \lambda_0) \in F(u_0, \lambda_0)$. Henceforth,

$$(Au_0, u_0 - u_0)_{X',X} + b(u_0, \lambda_0) - b(u_0, \lambda_0) < (f, u_0 - u_0)_{X',X}$$

which is impossible. □

3. The main result

In this section we use Theorem 2 in order to prove the following existence result.

Theorem 3. *Assumptions 1–6 hold true. Then Problem 1 has at least one solution.*

Proof. Due to Theorem 2, for each positive integer n there exists $(u_n, \lambda_n) \in K_n \times \Lambda$ so that (6) is fulfilled for all $(v, \mu) \in K_n \times \Lambda$.

We claim that there exists n_0 a positive integer so that $\|u_{n_0}\|_X < n_0$, where (u_{n_0}, λ_{n_0}) is a solution of Problem 2 corresponding to the sets K_{n_0} and Λ.

Arguing by contradiction, we suppose that $\|u_n\|_X = n$ for all positive integers n. Setting $v = 0_X$ and $\mu = 0_Y$ in (6) we are led to

$$(Au_n, u_n)_{X',X} \leq (f, u_n)_{X',X} \leq \|f\|_{X'} \|u_n\|_X$$

and from this,

$$\frac{(Au_n, u_n)_{X',X}}{\|u_n\|_X} \leq \|f\|_{X'}.$$

Passing to the limit as $n \to \infty$ and using Assumption 5 we get a contradiction.

Let us prove now that the pair (u_{n_0}, λ_{n_0}) is a solution of Problem 1. Setting $n = n_0$ in (6) we deduce that, for all pairs $(w, \mu) \in K_{n_0} \times \Lambda$, the following inequality holds true:

$$(Au_{n_0}, w - u_{n_0})_{X',X} + b(w - u_{n_0}, \lambda_{n_0}) + b(u_{n_0}, \lambda_{n_0} - \mu) \geq (f,\ w - u_{n_0})_{X',X}. \tag{9}$$

Let $\varepsilon > 0$. We define $w \in K_{n_0}$ as follows

$$w = u_{n_0} + \varepsilon(z - u_{n_0}) \tag{10}$$

where $z \in X$. If $z = u_{n_0}$ we can take $\varepsilon = 1$, else $\varepsilon = \frac{|n_0 - \|u_{n_0}\|_X|}{\|z - u_{n_0}\|_X}$.

Let us take $\mu = \lambda_{n_0}$ in (9) and use (10). Dividing by ε it follows that

$$(Au_{n_0}, z - u_{n_0})_{X',X} + b(z - u_{n_0}, \lambda_{n_0}) \geq (f,\ z - u_{n_0})_{X',X} \quad \text{for all } z \in X.$$

Setting now in this last inequality $z = u_{n_0} \pm v$ where $v \in X$, we get

$$(Au_{n_0}, v)_{X',X} + b(v, \lambda_{n_0}) = (f, v)_{X',X} \quad \text{for all } v \in X.$$

Thus, the pair $(u_{n_0}, \lambda_{n_0}) \in K_{n_0} \times \Lambda$ verifies the first line of Problem 1. Setting now $w = u_{n_0}$ in (9) we obtain

$$b(u_{n_0}, \mu - \lambda_{n_0}) \leq 0 \quad \text{for all } \mu \in \Lambda.$$

Therefore, the second line of Problem 1 is also verified. It follows that the pair (u_{n_0}, λ_{n_0}) is a solution of Problem 1. □

4. An example

In this section we shall present an example of spaces X, Y, subset Λ, operator A and form $b(\cdot, \cdot)$ which verify Assumptions 1–6.

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary Γ. Let p be a real number so that $\infty > p \geq 4$. We define a subspace of $W^{1,p}(\Omega)$ as follows,

$$X = \{v : v \in W^{1,p}(\Omega),\ \gamma v = 0 \text{ a.e. on } \Gamma_D\} \tag{11}$$

where Γ_D is a part of Γ with positive Lebesgue measure and $\gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ is the Sobolev trace operator. Recall that γ is a linear continuous operator. It is known that the space X is a Banach space endowed with the norm

$$\|u\|_X = \|\nabla u\|_{L^p(\Omega)^N}.$$

Let p' be the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. We now consider Γ_C a part of Γ so that $meas(\Gamma_C) > 0$ and $\Gamma_C \cap \Gamma_D = \emptyset$. Then, we can take

$$Y = L^{p'}(\Gamma_C). \tag{12}$$

The spaces X and Y fulfill Assumption 1.

Next, we define a subset of Y as follows:

$$\Lambda = \left\{\mu \in Y : \langle \mu, \gamma v_{|\Gamma_C}\rangle \leq \int_{\Gamma_C} g|\gamma v(\boldsymbol{x})|\, d\Gamma \text{ for all } v \in X\right\}, \tag{13}$$

where g is a positive real number. This subset fulfills Assumption 2.

Denoting by $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^2$, we can define $A : X \to X'$ as follows: for each $u \in X,\ Au \in X'$ so that

$$(Au, v)_{X',X} = \int_\Omega \mu\|\nabla u(\boldsymbol{x})\|^{p-2}\nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\, dx \quad \text{for all } v \in X \tag{14}$$

where μ is a positive real number. The operator A is a Lipschitz continuous, monotone operator. Therefore, the operator A is hemicontinuous, relaxed h-monotone with $h \equiv 0$. We deduce that Assumptions 3–4 hold true. Besides, for each $u \in X,\ u \neq 0_X$, we have

$$\frac{(Au, u)_{X',X}}{\|u\|_X} = \mu\|u\|_X^{p-1}.$$

Therefore, Assumption 5 is also fulfilled.

Finally, we define $b : X \times L^{p'}(\Gamma_C) \to \mathbb{R}$ as follows

$$b(v, \mu) = \langle \mu, \gamma v_{|\Gamma_C} \rangle, \tag{15}$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $L^{p'}(\Gamma_C)$ and $L^p(\Gamma_C)$. Taking into account the properties of the trace operator we can see that the bilinear form b verifies Assumption 6.

To simplify the presentation, an easy to follow example arising from Contact Mechanics was presented. In the next section we shall discuss a simplified model in elasticity which can be related to this example.

5. A frictional contact problem

Let us consider the following boundary value problem.

Problem 3. Find $u : \bar{\Omega} \to \mathbb{R}$ so that

$$\operatorname{div}(\mu \|\nabla u(\boldsymbol{x})\|^{p-2} \nabla u(\boldsymbol{x})) + f_0(\boldsymbol{x}) = 0 \qquad \text{in } \Omega, \tag{16}$$

$$u(\boldsymbol{x}) = 0 \qquad \text{on } \Gamma_D, \tag{17}$$

$$\mu \,\|\nabla u(\boldsymbol{x})\|^{p-2}\, \partial_\nu u(\boldsymbol{x}) = f_2(\boldsymbol{x}) \qquad \text{on } \Gamma_N, \tag{18}$$

$$\left.\begin{aligned} &|\mu \,\|\nabla u(\boldsymbol{x})\|^{p-2}\, \partial_\nu\, u(\boldsymbol{x})| \le g, \\ &\mu \,\|\nabla u(\boldsymbol{x})\|^{p-2}\, \partial_\nu u(\boldsymbol{x}) = -g\, \frac{u(\boldsymbol{x})}{|u(\boldsymbol{x})|} \quad \text{if } u(\boldsymbol{x}) \neq 0 \end{aligned}\right\} \qquad \text{on } \Gamma_C. \tag{19}$$

This problem models the antiplane shear deformation of a nonlinearly elastic cylindrical body, in frictional contact on Γ_C with a rigid foundation. See [22] for details on the antiplane contact models. We also refer to the works [23–26] which treat antiplane contact problems in a general setting of hemivariational inequalities.

Herein $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary Γ partitioned into three measurable parts Γ_D, Γ_N, Γ_C with positive Lebesgue measures. Referring the body to a Cartesian coordinate system $Ox_1x_2x_3$ so that the generators of the cylinder are parallel with the axis Ox_3, the domain $\Omega \subset Ox_1x_2$ denotes the cross section of the cylinder. The functions $f_0 = f_0(x_1, x_2) : \Omega \to \mathbb{R}$, $f_2 = f_2(x_1, x_2) : \Gamma_N \to \mathbb{R}$ are related to the density of the volume forces and the density of the surface traction, respectively, and $g > 0$ is the friction bound. The vector $\boldsymbol{\nu} = (\nu_1, \nu_2)$, $\nu_i = \nu_i(x_1, x_2)$, for each $i \in \{1, 2\}$, represents the outward unit normal vector to the boundary of Ω and $\partial_\nu u = \nabla u \cdot \boldsymbol{\nu}$. The behavior of the nonlinearly elastic material is described by the following constitutive law:

$$\boldsymbol{\sigma}(\boldsymbol{x}) = k \operatorname{tr} \boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))\, \boldsymbol{I}_3 + \mu \|\boldsymbol{\varepsilon}^D(\boldsymbol{u}(x))\|^{p-2} \boldsymbol{\varepsilon}^D(\boldsymbol{u}(\boldsymbol{x})) \tag{20}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, tr is the trace of a Cartesian tensor of second order, $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor, $\boldsymbol{u}$ is the displacement vector, $\boldsymbol{I}_3$ is the identity tensor, $k, \mu > 0$ are material parameters and p is a constant so that $4 \le p < \infty$. We recall that $\boldsymbol{\tau}^D$ denotes the *deviator* of a tensor $\boldsymbol{\tau}$, defined by $\boldsymbol{\tau}^D = \boldsymbol{\tau} - \frac{1}{3} (\operatorname{tr} \boldsymbol{\tau})\, \boldsymbol{I}_3$. The constitutive law (20) is a Hencky-type constitutive law; see for instance [27] and the references therein.

The unknown of the problem is the function $u = u(x_1, x_2) : \bar{\Omega} \to \mathbb{R}$ that represents the third component of the displacement vector $\boldsymbol{u}$. We recall that, in the antiplane physical setting, the displacement vectorial field has the particular form $\boldsymbol{u} = (0, 0, u(x_1, x_2))$. Once the field u is determined, the stress tensor $\boldsymbol{\sigma}$ can be computed:

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & \mu \dfrac{\partial u}{\partial x_1} \\ 0 & 0 & \mu \dfrac{\partial u}{\partial x_2} \\ \mu \dfrac{\partial u}{\partial x_1} & \mu \dfrac{\partial u}{\partial x_2} & 0 \end{pmatrix}.$$

The mechanical problem has the following structure: (16) represents the equilibrium equation, (17) is the displacement boundary condition, (18) is the traction boundary condition and (19) is Tresca's law of dry friction; see e.g. [22,27] for more details on frictional laws.

We shall study Problem 3 assuming that

$$f_0 \in L^{p'}(\Omega), \qquad f_2 \in L^{p'}(\Gamma_N). \tag{21}$$

In order to write a weak formulation we start assuming that u is a smooth enough function which verifies (16)–(19). Let us multiply the first line of Problem 3 by $v \in C^\infty(\overline{\Omega})$. Using the integration by parts formula in $\mathbb{R}^2$, for all $v \in C^\infty(\bar{\Omega})$, we obtain

$$\int_\Omega \mu \|\nabla u(\boldsymbol{x})\|^{p-2} \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\, dx = \int_\Omega f_0(\boldsymbol{x})\, v(\boldsymbol{x})\, dx + \int_\Gamma \mu\, \|\nabla u(\boldsymbol{x})\|^{p-2} \partial_\nu u(\boldsymbol{x})\, v(\boldsymbol{x})\, d\Gamma.$$

As $\overline{C^\infty(\overline{\Omega})} = W^{1,p}(\Omega)$ we deduce that, for all $v \in W^{1,p}(\Omega)$,

$$\int_\Omega \mu \|\nabla u(\boldsymbol{x})\|^{p-2} \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\, dx = \int_\Omega f_0(\boldsymbol{x})\, v(\boldsymbol{x})\, dx + \int_\Gamma \mu\, \|\nabla u(\boldsymbol{x})\|^{p-2} \partial_\nu u(\boldsymbol{x})\, \gamma v(\boldsymbol{x})\, d\Gamma.$$

Let X be the space defined in (11). Using (18), for all $v \in X$ we have

$$\int_\Omega \mu \|\nabla u(\boldsymbol{x})\|^{p-2} \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\, dx = \int_\Omega f_0(\boldsymbol{x})\, v(\boldsymbol{x})\, dx + \int_{\Gamma_N} f_2(\boldsymbol{x})\, \gamma v(\boldsymbol{x})\, d\Gamma + \int_{\Gamma_C} \mu\, \|\nabla u(\boldsymbol{x})\|^{p-2} \partial_\nu u(\boldsymbol{x})\, \gamma v(\boldsymbol{x})\, d\Gamma.$$

Taking into account (21), we can define $f \in X'$ as follows

$$(f, v)_{X',X} = \int_\Omega f_0(\boldsymbol{x})\, v(\boldsymbol{x})\, dx + \int_{\Gamma_D} f_2(\boldsymbol{x})\, \gamma v(\boldsymbol{x})\, d\Gamma \quad \text{for all } v \in X. \tag{22}$$

Using now the definition (14) we get

$$(Au,\ v)_{X',X} = (f, v)_{X',X} + \int_{\Gamma_C} \mu\, \|\nabla u(\boldsymbol{x})\|^{p-2} \partial_\nu u(\boldsymbol{x})\, \gamma v(\boldsymbol{x})\, d\Gamma \quad \text{for all } v \in X. \tag{23}$$

Next, we define a Lagrange multiplier $\lambda \in Y$ as follows:

$$\langle \lambda, z \rangle = -\int_{\Gamma_C} \mu\, \|\nabla u(\boldsymbol{x})\|^{p-2} \partial_\nu u(\boldsymbol{x})\, z(\boldsymbol{x})\, d\Gamma \quad \text{for all } z \in L^p(\Gamma_C), \tag{24}$$

where Y is the space defined in (12). Notice that, due to (19), we have $\lambda \in \Lambda$.

Let us rewrite (23) as

$$(Au,\ v)_{X',X} = (f, v)_{X',X} - \langle \lambda, \gamma v|_{\Gamma_C} \rangle \quad \text{for all } v \in X.$$

By the definition of the Lagrange multiplier λ, (24), and the definition of the form $b(\cdot,\cdot)$, (15), we obtain

$$(Au,\ v)_{X',X} + b(v, \lambda) = (f, v)_{X',X} \quad \text{for all } v \in X. \tag{25}$$

The friction law (19) leads us to the identity

$$\int_{\Gamma_C} \mu\, \|\nabla u(\boldsymbol{x})\|^{p-2} \partial_\nu u(\boldsymbol{x})\, u(\boldsymbol{x})\, d\Gamma = -\int_{\Gamma_C} g|u(\boldsymbol{x})|\, d\Gamma.$$

Thus,

$$b(u, \lambda) = \int_{\Gamma_C} g|u(\boldsymbol{x})|\, d\Gamma. \tag{26}$$

By the definition (13) we are led to

$$b(u, \zeta) \le \int_{\Gamma_C} g|u(\boldsymbol{x})|\, d\Gamma \quad \text{for all } \zeta \in \Lambda. \tag{27}$$

Subtract now (26) from (27) to obtain the inequality

$$b(u, \zeta - \lambda) \le 0 \quad \text{for all } \zeta \in \Lambda. \tag{28}$$

Therefore, Problem 3 has the following weak formulation.

Problem 4. Find $u \in X$ and $\lambda \in \Lambda \subset Y$ so that (25) and (28) hold true.

Theorem 4. *If* $4 \le p < \infty,\ k, \mu, g > 0,\ f_0 \in L^{p'}(\Omega)$, *and* $f_2 \in L^{p'}(\Gamma_N)$, *then* Problem 4 *has at least one solution.*

Proof. We apply Theorem 3. $\square$

As each solution of Problem 4 is called *weak solution* of Problem 3, Theorem 4 ensures us that Problem 3 has at least one weak solution.

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