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OPTIMAL L^2 ERROR ESTIMATES OF UNCONDITIONALLY STABLE FINITE ELEMENT SCHEMES FOR THE CAHN–HILLIARD–NAVIER–STOKES SYSTEM*

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Abstract. The paper is concerned with the analysis of a popular convex-splitting finite element method (FEM) for the Cahn–Hilliard–Navier–Stokes system, which has been widely used in practice. Since the method is based on a combined approximation to multiple variables involved in the system, the approximation to one of the variables may seriously affect the accuracy for others. Optimal-order error analysis for such combined approximations is challenging. The previous works failed to present optimal error analysis in L^2-norm due to the weakness of the traditional approach. Here we first present an optimal error estimate in L^2-norm for the convex-splitting FEMs. We also show that optimal error estimates in the traditional (interpolation) sense may not always hold for all components in the coupled system due to the nature of the pollution/influence from lower-order approximations. Our analysis is based on two newly introduced elliptic quasi-projections and the superconvergence of negative norm estimates for the corresponding projection errors. Numerical examples are also presented to illustrate our theoretical results. More important is that our approach can be extended to many other FEMs and other strongly coupled phase field models to obtain optimal error estimates.

Key words. Cahn–Hilliard–Navier–Stokes, finite element methods, Ritz quasi-projection, optimal error estimates, unconditional stability

MSC codes. 65M12, 65N30, 65M60, 35K55

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1. Introduction.

We consider the following Cahn–Hilliard–Navier–Stokes (CHNS) phase field model

$$\partial_t \phi + \nabla\phi\cdot\mathbf{u} = \sigma\nabla\cdot(M\nabla\mu), \tag{1.1}$$
$$\mu = \sigma^{-1}\left(\phi^3-\phi\right) - \sigma\Delta\phi, \tag{1.2}$$
$$\partial_t\mathbf{u} - \eta\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \gamma\mu\nabla\phi, \tag{1.3}$$
$$\nabla\cdot\mathbf{u} = 0, \tag{1.4}$$

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in a bounded convex polygonal or polyhedral domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, for $t \in (0, T]$. In this model, $\mathbf{u}$ and p represent the velocity and pressure of the fluid, respectively, ϕ denotes the phase field variable, and μ denotes the chemical potential. Some positive physical parameters in this model include

M : mobility;

η : viscosity (inverse of the Reynolds number);

$\gamma : \gamma = \frac{1}{We^*}$, where We^* is the modified Weber number that measures relative strengths of the kinetic and surface energies;

σ : a dimensionless parameter that measures the interfacial width between the two phases.

Here, we consider (1.1)–(1.4) subject to the following no-flux/no-flow boundary conditions and initial conditions:

$$\partial_n \phi = \partial_n \mu = 0 \quad \text{and} \quad \mathbf{u} = \mathbf{0} \qquad \text{on } \partial\Omega \times (0, T], \tag{1.5}$$

$$\phi(x, 0) = \phi^0 \quad \text{and} \quad \mathbf{u}(x, 0) = \mathbf{u}^0 \qquad \text{in } \Omega. \tag{1.6}$$

The CHNS phase field model (1.1)–(1.6) can be used to describe the interface dynamics of two-phase, incompressible, and macroscopically immiscible Newtonian fluids with matched density [24, 28]. The mathematical theory of the problem (1.1)–(1.6) was investigated in [1] for the existence of weak solutions and in [17, 20] for the existence of strong solutions. For the CHNS model, the energy and mass are defined by

$$E(\phi, \mathbf{u}) = \int_\Omega \Big\{ \frac{\gamma}{4\sigma}(\phi^2 - 1)^2 + \frac{\gamma\sigma}{2}|\nabla\phi|^2 + \frac{1}{2}|\mathbf{u}|^2 \Big\} \mathrm{d}x, \quad M(\phi) = (\phi, 1), \tag{1.7}$$

respectively. It is well known that the system (1.1)–(1.6) is a mass-conservative gradient flow with respect to the energy (1.7). In other words, for any $t \in [0, T]$, the system (1.1)–(1.6) satisfies the following energy law,

$$E(\phi(t), \mathbf{u}(t)) + \int_0^t \Big\{ \gamma\sigma M \|\nabla\mu(s)\|_{L^2}^2 + \eta\|\nabla\mathbf{u}(s)\|_{L^2}^2 \Big\} \mathrm{d}s = E(\phi^0, \mathbf{u}^0), \tag{1.8}$$

and the mass conservation $(\phi(t), 1) = (\phi^0, 1)$ holds due to the fact $(\nabla\phi \cdot \mathbf{u}, 1) = 0$; see (4.18). Thus, it is desirable to construct numerical schemes that preserve such properties at a discrete level.

Numerical methods for the CHNS model for two-phase flow have been investigated extensively in the past several decades [10, 14, 17–19, 21–23, 34, 38]. A popular and effective scheme for discretizing the Cahn–Hilliard equation in time is based on the convex splitting technique, which was popularized by Eyre [16] and has been widely used by many researchers [4, 5, 11, 25, 35] for solving the phase field models in recent years. The advantage of such a scheme is that it generally inherits two important properties for the Cahn–Hilliard model: unconditional stability and unconditionally unique solvability, regardless of time step and spatial mesh size. Some other numerical methods, such as stabilization and scalar auxiliary variable (SAV) type methods, for solving the phase field models can be found in [2, 31, 33, 35].

Analysis of numerical methods for the CHNS model has been done extensively; see [8, 9, 12, 14, 15, 17, 21, 29]. In particular, Feng [17] proposed and analyzed a fully discrete finite element method, which satisfies a discrete energy law, for the CHNS model. Under the assumption of uniqueness of the solutions in continuous setting and by utilizing the discrete energy law, the author showed the convergence of the numerical solution to a weak solution of the CHNS phase field model as $\tau, h \to 0$, where τ and h denote the time step and spatial mesh size, respectively. Kay, Styles, and Welford [29] investigated both semidiscrete and fully discrete divergence-free type linear finite element schemes for the CHNS model. The authors obtained an optimal H^1-norm error estimate, i.e., order $\mathcal{O}(h)$, for the semidiscrete scheme, and proved an abstract convergence for the fully discrete scheme. In [21], Grün studied a fully discrete convex-splitting finite element scheme for a diffuse interface model, which describes two-phase flow of immiscible, incompressible viscous fluids with different mass densities. The convergence of the scheme was proved and no convergence rates were presented there. Recently, Diegel, Feng, and Wise [12] proposed a convex-splitting finite element method for the Cahn–Hilliard–Darcy–Stokes (CHDS) model, where the scheme was proved to be unconditionally energy stable and uniquely solvable. In this scheme, the standard Taylor–Hood finite element space $(\mathbf{u}_h^n, p_h^n) \in \mathbf{X}_h^{r+1} \times \mathring{S}_h^r$ is used for the velocity and pressure and $\phi_h^n, \mu_h^n \in S_h^r$ is used to ensure a mass conservation law at the discrete level. In [12], the error estimate

$$\|\nabla(\phi^n - \phi_h^n)\|_{L^2} + \|\mathbf{u}^n - \mathbf{u}_h^n\|_{L^2} \leq C(\tau + h^r) \tag{1.9}$$

and certain time-L^2-norm type estimates were established for the fully discrete scheme. The extension of the analysis to a second-order in time convex-splitting finite element scheme for the CHNS equations was made in [14], where the same spatial convergence result as (1.9) was obtained. More recently, Cai and Shen [8] carried out a rigorous error analysis for a fully discrete standard inf-sup stable finite element scheme for the CHNS system. The scheme was based on a stabilization technique and a projection method, where the latter was used for the decoupling of the velocity and pressure. For the Taylor–Hood finite element space $(\mathbf{u}_h^n, p_h^n) \in \mathbf{X}_h^{r+1} \times \mathring{S}_h^r$ and $\phi_h^n, \mu_h^n \in S_h^{r+1}$, a more precise estimate

$$\|\mathbf{u}^n - \mathbf{u}_h^n\|_{L^2} + \|\phi^n - \phi_h^n\|_{H^1} \leq C(\tau + h^{r+1}) \tag{1.10}$$

was obtained in [8]. Similar results were also presented for the Cahn–Hilliard equation and some other phase field models; see [6, 13, 32]. In the traditional sense (interpolation sense), the H^1-norm error estimate of the phase field variable ϕ in both (1.9) and (1.10) is optimal, while no optimal error estimate in the L^2-norm was presented. Moreover, for numerical velocity, the L^2-norm error estimate in (1.9) and (1.10) is two order and one order lower than the optimal one, respectively. This is mainly due to certain artificial pollution arising from the approximation for the phase field ϕ to the accuracy for numerical velocity in analysis. On the other hand, numerical methods for the coupled CHNS system are based on certain combined approximations to multiple variables involved. It is clear that the approximation to one may influence or pollute the accuracy for others. Optimal error estimates in the traditional sense may not always hold for all components in the coupled system due to the pollution.

In this paper, we present an optimal L^2-norm error estimate of convex-splitting finite element methods (FEMs) with a class of combined finite element spaces. In particular, for the commonly used (Taylor–Hood type) approximation $(\phi_h^n, \mu_h^n, \mathbf{u}_h^n, p_h^n) \in S_h^r \times S_h^r \times \mathbf{X}_h^{r+1} \times \mathring{S}_h^r$, we provide the error estimates

$$\left\{ \begin{array}{l} \|\mathbf{u}^n - \mathbf{u}_h^n\|_{L^2} \leq C(\tau + h^{k_r+2}), \\ \|\phi^n - \phi_h^n\|_{L^2} \leq C(\tau + h^{r+1}), \end{array} \right. \qquad k_r = \left\{ \begin{array}{ll} r & \text{for } r \geq 2, \\ r-1 & \text{for } r = 1\,. \end{array} \right. \tag{1.11}$$

Clearly, for $r \geq 2$, the above error estimates for both velocity and phase field are optimal in the sense of interpolation. For $r = 1$, the error estimate for the velocity is one order lower than for the interpolation, while our numerical results confirm that the estimate is also optimal. Thus our analysis is optimal for the coupled system.

The one-order reduction in this case is due to the nature of lacking the superconvergence of the H^{-1}-norm estimate for linear finite element approximation. As a result, the lower-order approximation, i.e., the linear finite element approximation, for the phase field ϕ will not achieve a superconvergence and this will generate a pollution to the accuracy of numerical velocity through the coupling term $\gamma\mu\nabla\phi$ in the Navier–Stokes equations. In other words, the numerical velocity in such a case possesses only the same convergence rate as that of the numerical phase field. For the uniform MINI type element $(\phi_h^n,\mu_h^n,\mathbf{u}_h^n,p_h^n)\in S_h^1\times S_h^1\times \mathbf{X}_h^{1b}\times \mathring{S}_h^1$, we prove the optimal error estimates

$$\|\mathbf{u}^n-\mathbf{u}_h^n\|_{L^2}+\|\phi^n-\phi_h^n\|_{L^2}\le C(\tau+h^2)\,. \tag{1.12}$$

The analysis presented in this paper is based on newly introduced Ritz and Stokes quasi-projections and the corresponding rigorous analysis of the projection errors in the H^{-1}-norm. In terms of these quasi-projections, the artificial pollution from the approximation to the phase field is avoided.

The rest of this paper is organized as follows. In section 2, we present our notations and our main theoretical results. In section 3, we introduce Ritz and Stokes quasi-projections, respectively, and provide the corresponding error estimates. In terms of these two quasi-projections, we prove optimal L^2-norm error estimates of the numerical scheme in section 4. Numerical examples are presented in section 5 to confirm our theoretical analysis.

2. Main results.

In this section, we introduce a fully discrete finite element method for the CHNS model (1.1)–(1.4), and present the main results.

2.1. Notations and variational formulation.

For any integer $k\ge 0$ and real number $1\le p\le\infty$, we denote by $W^{k,p}(\Omega)$ the conventional Sobolev space of functions defined on Ω with the abbreviations $H^k(\Omega)=W^{k,2}(\Omega)$ and $L^p(\Omega)=W^{0,p}(\Omega)$, and $L_0^p(\Omega)=\{q\in L^p(\Omega):\int_\Omega q\mathrm{d}x=0\}$. The closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ space is denoted by $W_0^{k,p}(\Omega)$, with the abbreviation $H_0^k(\Omega)=W_0^{k,2}(\Omega)$. We let $\mathring{H}^1(\Omega):=H^1(\Omega)\cap L_0^2(\Omega)$ and denote its the dual space by $H^{-1}(\Omega)$. Moreover, we denote by $\mathbf{W}^{k,p}(\Omega)=[W^{k,p}(\Omega)]^d$, $\mathbf{L}^p(\Omega)=[L^p(\Omega)]^d$, $\mathbf{H}_0^1(\Omega)=[H_0^1(\Omega)]^d$ the corresponding vector-valued Sobolev spaces with the norms $\|\cdot\|_{W^{k,p}}$, $\|\cdot\|_{H^k}$, and $\|\cdot\|_{L^p}$ for both scalar- and vector-valued functions. The inner products of both $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ are denoted by $(\cdot,\cdot)$. Here and thereafter, we set the parameters $M=1$, $\sigma=1$, $\eta=1$, and $\gamma=1$ in the CHNS model (1.1)–(1.4) to simplify the presentation. We note that the more general case $M,\sigma,\eta,\gamma>0$ can be considered in the analysis without essential additional difficulty.

With the above notations, it is easily seen that the exact solution $(\phi,\mu,\mathbf{u},p)$ of (1.1)–(1.4) satisfies

$$\big(\partial_t\phi,w\big)+(\nabla\mu,\nabla w)+b(\phi,\mathbf{u},w)=0, \tag{2.1}$$

$$(\mu,\varphi)=(\nabla\phi,\nabla\varphi)+\big(\phi^3-\phi,\varphi\big), \tag{2.2}$$

$$(\partial_t\mathbf{u},\mathbf{v})+(\nabla\mathbf{u},\nabla\mathbf{v})+b^*(\mathbf{u},\mathbf{u},\mathbf{v})-(\nabla\cdot\mathbf{v},p)-b(\phi,\mathbf{v},\mu)=0, \tag{2.3}$$

$$(\nabla\cdot\mathbf{u},q)=0, \tag{2.4}$$

for any test functions $(w,\varphi,\mathbf{v},q)\in(H^1(\Omega),H^1(\Omega),\mathbf{H}_0^1(\Omega),L_0^2(\Omega))$, where

$$b(\phi,\mathbf{u},w):=(\nabla\phi\cdot\mathbf{u},w), \tag{2.5}$$

$$b^*(\mathbf{u},\mathbf{v},\mathbf{w}):=\frac{1}{2}[(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w})-(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v})]. \tag{2.6}$$

2.2. Numerical scheme and main results. Let $\Im_h$ denote a quasi-uniform partition of Ω into triangles $\mathcal{K}_j$, $j=1,\ldots,M$, in $\mathbb{R}^2$ or tetrahedra in $\mathbb{R}^3$ with mesh size $h=\max_{1\le j\le M}\{\mathrm{diam}\mathcal{K}_j\}$. For any integer $r\ge 1$, we define the following finite element spaces,

$$\begin{aligned}
&S_h^r=\{v_h\in C(\Omega): v_h|_{\mathcal{K}_j}\in P_r(\mathcal{K}_j)\ \forall \mathcal{K}_j\in\Im_h\},\\
&\mathring{S}_h^r=S_h^r\cap L_0^2(\Omega),\\
&\mathbf{X}_h^{r+1}=\{\mathbf{v}_h\in \mathbf{H}_0^1(\Omega)^d: \mathbf{v}_h|_{\mathcal{K}_j}\in \mathbf{P}_{r+1}(\mathcal{K}_j)^d\ \forall \mathcal{K}_j\in\Im_h\},\\
&\mathbf{X}_h^{1b}=(S_h^1\oplus B_{d+1})^d\cap \mathbf{H}_0^1(\Omega),\qquad d=2,3,
\end{aligned}$$

where $P_r(\mathcal{K}_j)$ is the space of polynomials of degree r on $\mathcal{K}_j$, B_3 and B_4 are the spaces of cubic bubbles and quartic bubbles, respectively. Note that the spaces $\mathbf{X}_h^{r+1}\times\mathring{S}_h^r$ $(r\ge 1)$ and $\mathbf{X}_h^{1b}\times\mathring{S}_h^1$ are usually referred to as the generalized Taylor–Hood element and MINI element, respectively. It is well known that both the Taylor–Hood element and the MINI element satisfy the discrete inf-sup condition [3], i.e.,

$$\|q_h\|_{L^2}\le C\sup_{\mathbf{0}\ne\mathbf{v}_h\in\mathbf{X}_h^{r+1}/\mathbf{X}_h^{1b}}\frac{(q_h,\nabla\cdot\mathbf{v}_h)}{\|\mathbf{v}_h\|_{H^1}}\qquad \forall\, q_h\in\mathring{S}_h^r/\mathring{S}_h^1 \tag{2.7}$$

for some constant $C>0$. For the simplicity of notations, we define

$$\boldsymbol{\mathcal{X}}_h^r:=\begin{cases} S_h^r\times S_h^r\times\mathbf{X}_h^{r+1}\times\mathring{S}_h^r & \text{for } r\ge 2,\\ S_h^1\times S_h^1\times\mathbf{X}_h^{2}\times\mathring{S}_h^1 \text{ or } S_h^1\times S_h^1\times\mathbf{X}_h^{1b}\times\mathring{S}_h^1 & \text{for } r=1.\end{cases} \tag{2.8}$$

Let $\{t_n=n\tau\}_{n=0}^N$ denote a uniform partition of the time interval $[0,T]$ with a step size $\tau=T/N$. We denote $v^n=v(x,t_n)$ and $\mathbf{v}^n=\mathbf{v}(x,t_n)$. For any sequence $\{v^n\}_{n=1}^N$, we define

$$D_\tau v^n=\frac{v^n-v^{n-1}}{\tau}.$$

With the above notations, a fully discrete finite element convex-splitting scheme for the CHNS model (1.1)–(1.4) is to find $(\phi_h^n,\mu_h^n,\mathbf{u}_h^n,p_h^n)\in\boldsymbol{\mathcal{X}}_h^r$ such that

$$\Big(D_\tau\phi_h^n,w_h\Big)+\Big(\nabla\mu_h^n,\nabla w_h\Big)+b\Big(\phi_h^{n-1},\mathbf{u}_h^n,w_h\Big)=0, \tag{2.9}$$

$$\Big(\nabla\phi_h^n,\nabla\varphi_h\Big)-\Big(\mu_h^n,\varphi_h\Big)+\Big((\phi_h^n)^3-\phi_h^{n-1},\varphi_h\Big)=0, \tag{2.10}$$

$$\Big(D_\tau\mathbf{u}_h^n,\mathbf{v}_h\Big)+\Big(\nabla\mathbf{u}_h^n,\nabla\mathbf{v}_h\Big)+b^*\Big(\mathbf{u}_h^{n-1},\mathbf{u}_h^n,\mathbf{v}_h\Big)-\Big(\nabla\cdot\mathbf{v}_h,p_h^n\Big)-b\Big(\phi_h^{n-1},\mathbf{v}_h,\mu_h^n\Big)=0, \tag{2.11}$$

$$\Big(\nabla\cdot\mathbf{u}_h^n,q_h\Big)=0, \tag{2.12}$$

hold for all $(w_h,\varphi_h,\mathbf{v}_h,q_h)\in\boldsymbol{\mathcal{X}}_h^r$, and $n=1,2,\ldots,N$. At the initial time step, we choose $\phi_h^0=R_h\phi^0$ and $\mathbf{u}_h^0=\mathbf{I}_h\mathbf{u}^0$. Here, R_h and $\mathbf{I}_h$ denote the Ritz projection and L^2 projection operators (defined in section 3), respectively.

Remark 2.1. The system (2.9)–(2.12) is unconditionally uniquely solvable and unconditionally energy stable and, also, mass conserved for the phase field if the same degree of finite element for the phase field function is chosen as for the pressure. The proof of the discrete mass conservation can be found in Lemma 4.1.

Remark 2.2. Analysis for the above scheme and some slightly different schemes based on the convex-splitting technique or stabilization has been done by many authors [8, 12, 14, 32]. However, no optimal L^2-norm estimates have been presented.

In this paper, we assume that the solution to the CHNS model (1.1)–(1.6) exists and satisfies the following regularities (for $r \geq 1$):

$$\begin{aligned}
&\phi \in H^2(0,T;L^2(\Omega)) \cap H^1(0,T;H^{r+1}(\Omega)) \cap C([0,T];W^{2,4}(\Omega)),\\
&\mu \in H^1(0,T;H^{r+1}(\Omega)),\\
&\mathbf{u} \in H^2(0,T;\mathbf{L}^2(\Omega)) \cap H^1(0,T;\mathbf{H}^{r+2}(\Omega)),\\
&p \in L^2(0,T;H^{r+1}(\Omega) \cap L_0^2(\Omega)).
\end{aligned} \tag{2.13}$$

We present our main result in the following theorem. The proof will be given in section 4.

THEOREM 2.1. *Assume that the system* (1.1)–(1.6) *has a unique solution* $(\phi, \mu, \mathbf{u}, p)$ *satisfying* (2.13)*. Then the fully discrete system* (2.9)–(2.12) *yields a unique solution* $(\phi_h^n, \mu_h^n, \mathbf{u}_h^n, p_h^n) \in \boldsymbol{\mathcal{X}}_h^r$*. Furthermore, there exists a positive constant* τ_0 *such that when* $\tau \leq \tau_0$*, the numerical solution satisfies the following error estimates:*

$$\max_{1\leq n\leq N} \|\mathbf{u}^n - \mathbf{u}_h^n\|_{L^2} \leq C_0(\tau + \mathcal{E}_h), \tag{2.14}$$

$$\left(\tau \sum_{n=1}^{N} \|\nabla(\mathbf{u}^n - \mathbf{u}_h^n)\|_{L^2}^2\right)^{\frac{1}{2}} \leq C_0(\tau + \mathcal{E}_h^\star), \tag{2.15}$$

$$\max_{1\leq n\leq N} \|\phi^n - \phi_h^n\|_{L^2} + \left(\tau \sum_{n=1}^{N} \|\mu^n - \mu_h^n\|_{L^2}^2\right)^{\frac{1}{2}} \leq C_0(\tau + h^{r+1}), \tag{2.16}$$

where

$$\mathcal{E}_h = \begin{cases} h^{r+2} & \text{for } r \geq 2, \\ h^{r+1} & \text{for } r = 1, \end{cases} \qquad \mathcal{E}_h^\star = \begin{cases} h^{r+1} & \text{for } r \geq 2, \\ h^{r+1} & \text{for } r = 1 \ ((\mathbf{u}_h^n, p_h^n) \in \mathbf{X}_h^2 \times \mathring{S}_h^1), \\ h^r & \text{for } r = 1 \ ((\mathbf{u}_h^n, p_h^n) \in \mathbf{X}_h^{1b} \times \mathring{S}_h^1), \end{cases} \tag{2.17}$$

and C_0 *is a positive constant independent of* τ *and* h.

Throughout this paper, we denote by C and C_ε (dependent on ε) generic constants, and ε a small generic constant, independent of τ, h, N, and C_0 in Theorem 2.1, which could be different at different occurrences.

2.3. Unconditional solvability and energy stability. In this section, we introduce two propositions for the unconditional solvability, stability property of scheme (2.9)–(2.12), and the boundedness of numerical solutions. The proofs follow those for the CHDS system given in [12, Theorem 2.10 and Lemmas 2.11–2.14]. Here, we omit the details.

PROPOSITION 2.2. *The fully discrete scheme* (2.9)–(2.12) *is uniquely solvable and unconditionally stable, and satisfies the following discrete energy law for all* $\tau, h > 0$,

$$\begin{aligned}
&\frac{1}{2}\|\nabla\phi_h^n\|_{L^2}^2 + \frac{1}{2}\|\mathbf{u}_h^n\|_{L^2}^2 + \frac{1}{4}\|(\phi_h^n)^2 - 1\|_{L^2}^2 + \tau\sum_{m=1}^{n}\|\nabla\mu_h^m\|_{L^2}^2 + \tau\sum_{m=1}^{n}\|\nabla\mathbf{u}_h^m\|_{L^2}^2 \\
&\leq \frac{1}{2}\|\nabla\phi_h^0\|_{L^2}^2 + \frac{1}{2}\|\mathbf{u}_h^0\|_{L^2}^2 + \frac{1}{4}\|(\phi_h^0)^2 - 1\|_{L^2}^2
\end{aligned} \tag{2.18}$$

for $n = 1, 2, \ldots, N$.

We denote by $\Delta_h : \mathring{S}_h^r \to \mathring{S}_h^r$ the discrete Laplacian operator, defined by

$$(-\Delta_h v_h, \xi_h) = (\nabla v_h, \nabla \xi_h) \quad \forall\, v_h, \xi_h \in \mathring{S}_h^r, \tag{2.19}$$

which is symmetric and positive definite on $\mathring{S}_h^r$. If v_h is a constant, we define $(-\Delta_h)^{\frac{1}{2}} v_h := 0$ and $(-\Delta_h)^{-\frac{1}{2}} v_h := 0$. Then we obtain the following boundedness of numerical solutions of scheme (2.9)–(2.12).

PROPOSITION 2.3. *Let $(\phi_h^n, \mu_h^n, \mathbf{u}_h^n, p_h^n) \in \boldsymbol{\mathcal{X}}_h^r$ be the unique solution of (2.9)–(2.12). The following estimates hold for all $\tau, h > 0$,*

$$\max_{1\le n\le N} \left(\|\mu_h^n\|_{L^2} + \|\phi_h^n\|_{L^\infty} + \|\Delta_h \phi_h^n\|_{L^2} + \|\nabla \phi_h^n\|_{L^6} + \|\mathbf{u}_h^n\|_{L^2} \right) \le C, \tag{2.20}$$

$$\left(\tau \sum_{n=1}^N \left(\|\nabla \mu_h^n\|_{L^2}^2 + \|\nabla \mathbf{u}_h^n\|_{L^2}^2 + \|D_\tau \phi_h^n\|_{L^2}^2 \right) \right)^{\frac{1}{2}} \le C, \tag{2.21}$$

where C is a positive constant independent of τ and h.

3. Ritz and Stokes quasi-projections.

Let $R_h : H^1(\Omega) \to S_h^r$ be a classic Ritz projection defined by [37],

$$(\nabla(\psi - R_h\psi), \nabla\varphi_h) = 0 \tag{3.1}$$

for all $\varphi_h \in S_h^r$ with $\int_\Omega (\psi - R_h\psi)dx = 0$. Following finite element theory [7], it holds that

$$\|\psi - R_h\psi\|_{L^s} + h\|\psi - R_h\psi\|_{W^{1,s}} \le Ch^{r+1}\|\psi\|_{W^{r+1,s}}, \tag{3.2}$$

$$\|\psi - R_h\psi\|_{H^{-1}} \le C\mathcal{E}_h\|\psi\|_{H^{r+1}}, \tag{3.3}$$

$$\|D_\tau(\psi^n - R_h\psi^n)\|_{L^2} + h\|D_\tau(\psi^n - R_h\psi^n)\|_{H^1} \le Ch^{r+1}\|D_\tau\psi^n\|_{H^{r+1}}, \tag{3.4}$$

$$\|D_\tau(\psi^n - R_h\psi^n)\|_{H^{-1}} \le C\mathcal{E}_h\|D_\tau\psi^n\|_{H^{r+1}} \tag{3.5}$$

for $s \in [2, \infty]$ and $n = 1, 2, \ldots, N$, where $\mathcal{E}_h$ is defined in (2.17).

Let $I_h : L^2(\Omega) \to S_h^r$ and $\mathbf{I}_h : \mathbf{L}^2(\Omega) \to \mathbf{X}_h^{r+1}/\mathbf{X}_h^{1b}$ denote the L^2 projection operators defined by

$$(v - I_h v, w_h) = 0 \qquad \forall w_h \in S_h^r, \tag{3.6}$$

$$(\mathbf{v} - \mathbf{I}_h\mathbf{v}, \mathbf{w}_h) = 0 \qquad \forall \mathbf{w}_h \in \mathbf{X}_h^{r+1}/\mathbf{X}_h^{1b}. \tag{3.7}$$

It is well known that the L^2 projection satisfies the following estimates:

$$\|v - I_h v\|_{L^2} + h\|\nabla(v - I_h v)\|_{L^2} \le Ch^{r+1}\|v\|_{H^{r+1}}, \tag{3.8}$$

$$\|\mathbf{v} - \mathbf{I}_h\mathbf{v}\|_{L^2} + h\|\nabla(\mathbf{v} - \mathbf{I}_h\mathbf{v})\|_{L^2} \le Ch^{r+2}\|\mathbf{v}\|_{H^{r+2}} \quad (\text{if } \mathbf{I}_h\mathbf{v} \in \mathbf{X}_h^{r+1}), \tag{3.9}$$

$$\|\mathbf{v} - \mathbf{I}_h\mathbf{v}\|_{L^2} + h\|\nabla(\mathbf{v} - \mathbf{I}_h\mathbf{v})\|_{L^2} \le Ch^2\|\mathbf{v}\|_{H^2} \quad (\text{if } \mathbf{I}_h\mathbf{v} \in \mathbf{X}_h^{1b}). \tag{3.10}$$

Based on the Ritz projection, we define a Ritz quasi-projection, $\Pi_h : H^1(\Omega) \to S_h^r$ by

$$(\nabla(\mu - \Pi_h\mu), \nabla w_h) + (\nabla(\phi - R_h\phi)\cdot \mathbf{u}, w_h) = 0 \tag{3.11}$$

for all $w_h \in S_h^r$ with $\int_\Omega(\mu - \Pi_h\mu)dx = 0$, and a Stokes quasi-projection $(\mathbf{P}_h, P_h) : \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \to \mathbf{X}_h^{r+1} \times \mathring{S}_h^r / \mathbf{X}_h^{1b} \times \mathring{S}_h^1$ by

$$(\nabla(\mathbf{u} - \mathbf{P}_h(\mathbf{u},p)), \nabla \mathbf{v}_h) - (p - P_h(\mathbf{u},p), \nabla\cdot\mathbf{v}_h) = (\mu\nabla(\phi - R_h\phi), \mathbf{v}_h), \tag{3.12}$$

$$(\nabla\cdot(\mathbf{u} - \mathbf{P}_h(\mathbf{u},p)), q_h) = 0 \tag{3.13}$$

for $(\mathbf{v}_h, q_h) \in \mathbf{X}_h^{r+1} \times \mathring{S}_h^r / \mathbf{X}_h^{1b} \times \mathring{S}_h^1$. For the simplicity of notations, we denote $\mathbf{P}_h\mathbf{u} := \mathbf{P}_h(\mathbf{u},p)$ and $P_hp := P_h(\mathbf{u},p)$. The corresponding estimates are presented in the following two lemmas, respectively.

Lemma 3.1. *For the Ritz quasi-projection* (3.11)*, it holds that*

$$\|\mu - \Pi_h\mu\|_{L^2} + h\|\nabla(\mu - \Pi_h\mu)\|_{L^2} \le Ch^{r+1}(\|\mathbf{u}\|_{L^\infty}\|\phi\|_{H^{r+1}} + \|\mu\|_{H^{r+1}}), \tag{3.14}$$

$$\|\mu - \Pi_h\mu\|_{H^{-1}} \le C\mathcal{E}_h(\|\mathbf{u}\|_{W^{1,4}}\|\phi\|_{H^{r+1}} + \|\mu\|_{H^{r+1}}), \tag{3.15}$$

and for $n = 1, 2, \ldots, N$*,*

$$\begin{aligned}&\|\nabla(D_\tau(\mu^n - \Pi_h\mu^n))\|_{L^2}\\ &\le Ch^r(\|\mathbf{u}^n\|_{L^\infty}\|D_\tau\phi^n\|_{H^r} + \|D_\tau\mathbf{u}^n\|_{L^\infty}\|\phi^{n-1}\|_{H^r} + \|D_\tau\mu^n\|_{H^{r+1}}),\end{aligned} \tag{3.16}$$

$$\begin{aligned}&\|D_\tau(\mu^n - \Pi_h\mu^n)\|_{H^{-1}}\\ &\le C\mathcal{E}_h(\|\mathbf{u}^n\|_{W^{1,4}}\|D_\tau\phi^n\|_{H^{r+1}} + \|D_\tau\mathbf{u}^n\|_{W^{1,4}}\|\phi^{n-1}\|_{H^{r+1}} + \|D_\tau\mu^n\|_{H^{r+1}}).\end{aligned} \tag{3.17}$$

Lemma 3.2. *For the Stokes quasi-projection defined in* (3.12)–(3.13)*, it holds that*

$$\|\mathbf{u} - \mathbf{P}_h\mathbf{u}\|_{L^2} \le \begin{cases} Ch^{r+2}(\|\mathbf{u}\|_{H^{r+2}} + \|p\|_{H^{r+1}} + \|\phi\|_{H^{r+1}}\|\mu\|_{H^2}) & \text{for } r \ge 2,\\ Ch^{r+1}(\|\mathbf{u}\|_{H^{r+1}} + \|p\|_{H^r} + \|\phi\|_{H^{r+1}}\|\mu\|_{H^2}) & \text{for } r = 1,\end{cases} \tag{3.18}$$

$$\begin{aligned}&\|\nabla(\mathbf{u} - \mathbf{P}_h\mathbf{u})\|_{L^2} + \|p - P_hp\|_{L^2}\\ &\le \begin{cases} Ch^{r+1}(\|\mathbf{u}\|_{H^{r+2}} + \|p\|_{H^{r+1}} + \|\phi\|_{H^{r+1}}\|\mu\|_{W^{1,4}}) & \text{for } r \ge 2,\\ Ch^{r+1}(\|\mathbf{u}\|_{H^{r+2}} + \|p\|_{H^{r+1}} + \|\phi\|_{H^{r+1}}\|\mu\|_{W^{1,4}}) & \text{for } r = 1 \ (\mathbf{X}_h^2 \times \mathring{S}_h^1),\\ Ch^{r}(\|\mathbf{u}\|_{H^{r+1}} + \|p\|_{H^{r}} + \|\phi\|_{H^{r}}\|\mu\|_{W^{1,4}}) & \text{for } r = 1 \ (\mathbf{X}_h^{1b} \times \mathring{S}_h^1),\end{cases}\end{aligned} \tag{3.19}$$

and

$$\|D_\tau(\mathbf{u}^n - \mathbf{P}_h\mathbf{u}^n)\|_{L^2} \le \begin{cases} Ch^{r+2}(\|D_\tau\mathbf{u}^n\|_{H^{r+2}} + \|D_\tau p^n\|_{H^{r+1}} + \|D_\tau\phi^n\|_{H^{r+1}}\|\mu^n\|_{H^2}\\ \qquad + \|\phi^{n-1}\|_{H^{r+1}}\|D_\tau\mu^n\|_{H^2}) \quad \text{for } r \ge 2,\\ Ch^{r+1}(\|D_\tau\mathbf{u}^n\|_{H^{r+1}} + \|D_\tau p^n\|_{H^{r}} + \|D_\tau\phi^n\|_{H^{r+1}}\|\mu^n\|_{H^2}\\ \qquad + \|\phi^{n-1}\|_{H^{r+1}}\|D_\tau\mu^n\|_{H^2}) \quad \text{for } r = 1.\end{cases} \tag{3.20}$$

From (3.18), we can see the boundedness

$$\|\mathbf{P}_h\mathbf{u}\|_{L^\infty} + \|\mathbf{P}_h\mathbf{u}\|_{W^{1,3}} \le C(\|\mathbf{u}\|_{H^2} + \|p\|_{H^2} + \|\phi\|_{H^2}\|\mu\|_{H^2}). \tag{3.21}$$

Lemma 3.3. *For the operators* $(-\Delta_h)^{\frac{1}{2}} : \mathring{S}_h^r \to \mathring{S}_h^r$ *and* $(-\Delta_h)^{-\frac{1}{2}} : \mathring{S}_h^r \to \mathring{S}_h^r$*, we have the estimates*

$$C^{-1}\|v_h\|_{H^1} \le \|(-\Delta_h)^{\frac{1}{2}}v_h\|_{L^2} \le C\|v_h\|_{H^1}, \tag{3.22}$$

$$C^{-1}\|v_h\|_{H^{-1}} \le \|(-\Delta_h)^{-\frac{1}{2}} v_h\|_{L^2} \le C\|v_h\|_{H^{-1}}, \tag{3.23}$$

and

$$\|v_h\|_{L^2} \le \left(\varepsilon\|\nabla v_h\|_{L^2}^2 + C\varepsilon^{-1}\|(-\Delta_h)^{-\frac{1}{2}} v_h\|_{L^2}^2\right)^{\frac{1}{2}} \tag{3.24}$$

for $v_h \in \mathring{S}_h^r$, where ε is an arbitrary positive constant independent of h.

The proof of the above three lemmas will be given in the appendix.

For $v_h \in S_h^r$, by subtracting its mean value we obtain from (3.24) that

$$\begin{aligned}\|v_h\|_{L^2} &\le \left\|v_h - \tfrac{1}{|\Omega|}(v_h,1)\right\|_{L^2} + \left\|\tfrac{1}{|\Omega|}(v_h,1)\right\|_{L^2}\\ &\le \varepsilon\|\nabla v_h\|_{L^2} + C\varepsilon^{-1}\|(-\Delta_h)^{-\frac{1}{2}}\widetilde{v}_h\|_{L^2} + |(v_h,1)|,\end{aligned} \tag{3.25}$$

where $\widetilde{v}_h := v_h - \frac{1}{|\Omega|}(v_h,1)$. Several fundamental inequalities are presented below.

LEMMA 3.4 (see [7]). *The following Poincaré–Wirtinger inequality, embedding inequality, and inverse inequality hold:*

$$\|v\|_{L^2} \le C\left(\|\nabla v\|_{L^2}^2 + |(v,1)|^2\right)^{\frac{1}{2}}, \tag{3.26}$$

$$\|v\|_{L^\ell} \le C\|v\|_{H^1}, \tag{3.27}$$

$$\|v_h\|_{W^{m,s}} \le Ch^{n-m+\frac{d}{s}-\frac{d}{q}}\|v_h\|_{W^{n,q}}, \tag{3.28}$$

for $v \in H^1(\Omega)$, $v_h \in S_h^r, \mathring{S}_h^r, \mathbf{X}_h^{r+1}/\mathbf{X}_h^{1b}$, $1 \le \ell \le 6$, $0 \le n \le m \le 1$, and $1 \le q \le s \le \infty$.

LEMMA 3.5 (discrete Gronwall's inequality [27]). *Let τ, B, and a_k, b_k, c_k, γ_k, for integers $k \ge 0$, be nonnegative numbers such that*

$$a_n + \tau\sum_{k=0}^{n} b_k \le \tau\sum_{k=0}^{n}\gamma_k a_k + \tau\sum_{k=0}^{n} c_k + B \quad \textit{for} \quad n \ge 0,$$

suppose that $\tau\gamma_k < 1$, for all k, and set $\sigma_k = (1-\tau\gamma_k)^{-1}$. Then

$$a_n + \tau\sum_{k=0}^{n} b_k \le \exp\left(\tau\sum_{k=0}^{n}\gamma_k\sigma_k\right)\left(\tau\sum_{k=0}^{n} c_k + B\right) \quad \textit{for} \quad n \ge 0.$$

4. The proof of Theorem 2.1. The existence and uniqueness of solutions of scheme (2.9)–(2.12) have been given in Proposition 2.2. In this section, we focus on the proof of error estimates (2.14)–(2.16) in Theorem 2.1.

Let

$$e_\phi^n = R_h\phi^n - \phi_h^n, \quad e_\mu^n = \Pi_h\mu^n - \mu_h^n, \quad e_{\mathbf{u}}^n = \mathbf{P}_h\mathbf{u}^n - \mathbf{u}_h^n, \quad e_p^n = P_h p^n - p_h^n$$

for $n = 0, 1, \ldots, N$. In terms of Ritz quasi-projections defined in section 3, we subtract (2.1)–(2.4) from (2.9)–(2.12) to get the following error equations for $(e_\phi^n, e_\mu^n, e_{\mathbf{u}}^n, e_p^n)$,

(4.1)

$$\begin{aligned}&\Big(D_\tau e_\phi^n, w_h\Big) + \Big(\nabla e_\mu^n, \nabla w_h\Big)\\ &\quad = \Big(D_\tau(R_h\phi^n - \phi^n), w_h\Big) + \Big(b(\phi_h^{n-1}, \mathbf{u}_h^n, w_h) - b(R_h\phi^n, \mathbf{u}^n, w_h)\Big) + \Big(R_1^n, w_h\Big),\end{aligned}$$

(4.2)

$$\Big(\nabla e_\phi^n, \nabla\varphi_h\Big) - \Big(e_\mu^n, \varphi_h\Big) + \frac{1}{2}\Big(\mathcal{Z}^n(e_\phi^n - e_\phi^{n-1}), \varphi_h\Big)$$

$$= -\Big(\mathcal{Z}^n \overline{e}_\phi^{n-\frac{1}{2}}, \varphi_h\Big) + \Big(e_\phi^{n-1}, \varphi_h\Big) - \Big((\phi^n)^3 - (R_h\phi^n)^3, \varphi_h\Big) + \Big(\phi^{n-1} - R_h\phi^{n-1}, \varphi_h\Big) + \Big(\mu^n - \Pi_h\mu^n, \varphi_h\Big) + \big(R_2^n, \varphi_h\big),$$

(4.3)
$$\Big(D_\tau e_{\mathbf{u}}^n, \mathbf{v}_h\Big) + \Big(\nabla e_{\mathbf{u}}^n, \nabla \mathbf{v}_h\Big) - \Big(\nabla\cdot\mathbf{v}_h, e_p^n\Big) = \Big(b^*(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \mathbf{v}_h) - b^*(\mathbf{u}^{n-1}, \mathbf{u}^n, \mathbf{v}_h)\Big) - \Big(b(\phi_h^{n-1}, \mathbf{v}_h, \mu_h^n) - b(R_h\phi^n, \mathbf{v}_h, \mu^n)\Big) + \Big(D_\tau(\mathbf{P}_h\mathbf{u}^n - \mathbf{u}^n), \mathbf{v}_h\Big) + \Big(R_3^n, \mathbf{v_h}\Big),$$

(4.4)
$$\Big(\nabla\cdot e_{\mathbf{u}}^n, q_h\Big) = 0,$$

for $(w_h, \varphi_h, \mathbf{v}_h, q_h) \in \boldsymbol{\mathcal{X}}_h^r$ and $n = 1, \ldots, N$, where we have noted the identities $e_\phi^n = \overline{e}_\phi^{n-\frac{1}{2}} + \frac{1}{2}(e_\phi^n - e_\phi^{n-1})$, $\overline{e}_\phi^{n-\frac{1}{2}} := \frac{1}{2}(e_\phi^n + e_\phi^{n-1})$ and

(4.5)
$$\Big(R_h\phi^n\Big)^3 - \Big(\phi_h^n\Big)^3 = 3e_\phi^n \int_0^1 \Big((1-\theta)\phi_h^n + \theta R_h\phi^n\Big)^2 \mathrm{d}\theta = \frac{1}{2}(e_\phi^n - e_\phi^{n-1})\mathcal{Z}^n + \overline{e}_\phi^{n-\frac{1}{2}}\mathcal{Z}^n$$

with

$$\mathcal{Z}^n := 3\int_0^1 \Big((1-\theta)\phi_h^n + \theta R_h\phi^n\Big)^2 \mathrm{d}\theta. \tag{4.6}$$

In (4.1)–(4.4), R_1^n, R_2^n, R_3^n denote the truncation errors. By Taylor expansion, we have

$$\left(\tau\sum_{n=1}^N \big(\|R_1^n\|_{L^2}^2 + \|R_2^n\|_{L^2}^2 + \|R_3^n\|_{L^2}^2 + \|D_\tau R_2^n\|_{L^2}^2\big)\right)^{\frac{1}{2}} \le C\tau. \tag{4.7}$$

In the following subsections, we proceed to estimate the errors of numerical solutions.

4.1. Estimates for e_ϕ^n and e_μ^n. Taking $w_h = e_\mu^n$ and $\varphi_h = D_\tau e_\phi^n$ in (4.1)–(4.2), respectively, and adding the resulting equations together yield

$$\begin{aligned}&\frac{1}{2}D_\tau\|\nabla e_\phi^n\|_{L^2}^2 + \frac{1}{2\tau}\|\nabla(e_\phi^n - e_\phi^{n-1})\|_{L^2}^2 + \|\nabla e_\mu^n\|_{L^2}^2 + \frac{1}{2}\Big(\mathcal{Z}^n, \frac{|e_\phi^n - e_\phi^{n-1}|^2}{\tau}\Big)\\ &= \Big(D_\tau(R_h\phi^n - \phi^n), e_\mu^n\Big) + \Big(b(\phi_h^{n-1}, \mathbf{u}_h^n, e_\mu^n) - b(R_h\phi^n, \mathbf{u}^n, e_\mu^n)\Big) + \Big(R_1^n, e_\mu^n\Big)\\ &\quad + \Big(e_\phi^{n-1}, D_\tau e_\phi^n\Big) - \Big(\mathcal{Z}^n\overline{e}_\phi^{n-\frac{1}{2}}, D_\tau e_\phi^n\Big) - \Big((\phi^n)^3 - (R_h\phi^n)^3, D_\tau e_\phi^n\Big)\\ &\quad + \Big(\phi^{n-1} - R_h\phi^{n-1}, D_\tau e_\phi^n\Big) + \Big(\mu^n - \Pi_h\mu^n, D_\tau e_\phi^n\Big) + \Big(R_2^n, D_\tau e_\phi^n\Big) =: \sum_{j=1}^9 I_j^n.\end{aligned} \tag{4.8}$$

We first estimate I_i^n, $1\le i\le 4$, below. By (3.5), it is easy to see that

$$\begin{aligned}&I_1^n \le C\|D_\tau(R_h\phi^n - \phi^n)\|_{H^{-1}}\|e_\mu^n\|_{H^1} \le C\varepsilon^{-1}\mathcal{E}_h^2 + \varepsilon\|e_\mu^n\|_{H^1}^2,\\ &I_3^n \le C\|R_1^n\|_{L^2}\|e_\mu^n\|_{L^2} \le C\varepsilon^{-1}\|R_1^n\|_{L^2}^2 + \varepsilon\|e_\mu^n\|_{L^2}^2,\\ &I_4^n = -\frac{1}{2\tau}(\|e_\phi^{n-1}\|_{L^2}^2 - \|e_\phi^n\|_{L^2}^2 + \|e_\phi^{n-1} - e_\phi^n\|_{L^2}^2) \le \frac{1}{2}D_\tau\|e_\phi^n\|_{L^2}^2.\end{aligned}$$

By (2.5), the term I_2^n is bounded by

$$
\begin{aligned}
I_2^n &= \Big(\nabla\phi_h^{n-1}\cdot\mathbf{u}_h^n, e_\mu^n\Big) - \Big(\nabla R_h\phi^n\cdot\mathbf{u}^n, e_\mu^n\Big)\\
&= \Big(\nabla\phi_h^{n-1}\cdot(\mathbf{u}_h^n-\mathbf{u}^n), e_\mu^n\Big) + \Big(\nabla(\phi_h^{n-1}-R_h\phi^n)\cdot\mathbf{u}^n, e_\mu^n\Big)\\
&\le C\Big(\|e_{\mathbf{u}}^n+\mathbf{u}^n-\mathbf{P}_h\mathbf{u}^n\|_{L^2} + \|\nabla(e_\phi^{n-1}+R_h\phi^{n-1}-R_h\phi^n)\|_{L^2}\Big)\|e_\mu^n\|_{L^6}\\
&\le C\varepsilon^{-1}\Big(\|e_{\mathbf{u}}^n\|_{L^2}^2+\mathcal{E}_h^2+\|\nabla e_\phi^{n-1}\|_{L^2}^2+\tau^2\Big)+\varepsilon\|e_\mu^n\|_{H^1}^2, \qquad (4.9)
\end{aligned}
$$

where we used the uniform boundedness of $\|\nabla\phi_h^{n-1}\|_{L^3}$ given in Proposition 2.3, and error estimate (3.18) for the Stokes quasi-projection.

With the above estimates for I_i^n, $1\le i\le 4$, and a sufficiently small ε, (4.8) reduces to

$$
\begin{aligned}
&\frac{1}{2}D_\tau\|\nabla e_\phi^n\|_{L^2}^2+\frac{1}{2}\|\nabla e_\mu^n\|_{L^2}^2\\
&\le\varepsilon\|e_\mu^n\|_{L^2}^2+C_\varepsilon\Big(\mathcal{E}_h^2+\tau^2+\|e_{\mathbf{u}}^n\|_{L^2}^2+\|\nabla e_\phi^{n-1}\|_{L^2}^2+\|R_1^n\|_{L^2}^2\Big)+\frac{1}{2}D_\tau\|e_\phi^n\|_{L^2}^2+\sum_{j=5}^{9}I_j^n.
\end{aligned}
$$

Then we sum up the above estimate from time step t_1 to t_n to get

$$
\begin{aligned}
\|\nabla e_\phi^n\|_{L^2}^2+\tau\sum_{m=1}^n\|\nabla e_\mu^n\|_{L^2}^2 &\le C_\varepsilon\tau\sum_{m=1}^n\Big(\|e_{\mathbf{u}}^m\|_{L^2}^2+\|\nabla e_\phi^m\|_{L^2}^2\Big)+C_\varepsilon\Big(\mathcal{E}_h^2+\tau^2\Big)\\
&\quad+\varepsilon\tau\sum_{m=1}^n\|e_\mu^m\|_{L^2}^2+\|e_\phi^n\|_{L^2}^2+2\tau\sum_{m=1}^n\sum_{j=5}^9 I_j^m, \qquad (4.10)
\end{aligned}
$$

where we used $e_\phi^0=0$ and (4.7).

Next, we estimate the last term in (4.10). For sequences $\{f^m\}_{m=1}^n$ and $\{g^m\}_{m=1}^n$, we have the following formula for summation by parts:

$$
\tau\sum_{m=1}^n\Big(f^m,D_\tau g^m\Big)=-\tau\sum_{m=2}^n\Big(D_\tau f^m,g^{m-1}\Big)+\Big(f^n,g^n\Big)-\Big(f^1,g^0\Big). \qquad (4.11)
$$

Applying (4.11), we obtain the estimate

$$
\begin{aligned}
\tau\sum_{m=1}^n I_5^m=-\frac{\tau}{2}\sum_{m=1}^n\Big(\mathcal{Z}^m,D_\tau|e_\phi^m|^2\Big)&=\frac{\tau}{2}\sum_{m=1}^n\Big(D_\tau\mathcal{Z}^m,|e_\phi^{m-1}|^2\Big)-\Big(\mathcal{Z}^n,|e_\phi^n|^2\Big)\\
&\le C\tau\sum_{m=1}^n\|D_\tau\mathcal{Z}^m\|_{L^{\frac{3}{2}}}\|e_\phi^{m-1}\|_{L^6}^2,
\end{aligned}
$$

where we have noted $e_\phi^0=0$ and $\mathcal{Z}^n\ge 0$. By (4.11) and (3.5), we also have

$$
\begin{aligned}
\tau\sum_{m=1}^n I_7^m &= -\tau\sum_{m=2}^n\Big(D_\tau(\phi^{m-1}-R_h\phi^{m-1}),e_\phi^{m-1}\Big)+\Big(\phi^{n-1}-R_h\phi^{n-1},e_\phi^n\Big)\\
&\le C\tau\sum_{m=2}^n\|D_\tau(\phi^{m-1}-R_h\phi^{m-1})\|_{H^{-1}}\|e_\phi^{m-1}\|_{H^1}\\
&\quad+C\|\phi^{n-1}-R_h\phi^{n-1}\|_{H^{-1}}\|e_\phi^n\|_{H^1}\\
&\le C\tau\sum_{m=2}^n\|e_\phi^{m-1}\|_{H^1}^2+\varepsilon\|e_\phi^n\|_{H^1}^2+C\mathcal{E}_h^2.
\end{aligned}
$$

Using the same approach and the estimate (4.7) for truncation errors, we can easily get

$$\tau\sum_{m=1}^{n}\sum_{j=5}^{9} I_j^m \le C\tau\sum_{m=2}^{n}\|e_\phi^{m-1}\|_{H^1}^2+\varepsilon\|e_\phi^n\|_{H^1}^2+C\Big(\mathcal{E}_h^2+\tau^2\Big)$$
$$+C\tau\sum_{m=1}^{n}\|D_\tau\mathcal{Z}^m\|_{L^{\frac{3}{2}}}\|e_\phi^{m-1}\|_{H^1}^2.$$

Substituting the above results into (4.10) yields

(4.12)

$$\begin{aligned}
&\|\nabla e_\phi^n\|_{L^2}^2+\tau\sum_{m=1}^{n}\|\nabla e_\mu^n\|_{L^2}^2\\
&\le C_\varepsilon\tau\sum_{m=1}^{n}\Big(\|e_{\mathbf{u}}^m\|_{L^2}^2+\|\nabla e_\phi^m\|_{L^2}^2+\|e_\phi^{m-1}\|_{L^2}^2\Big)+C_\varepsilon\Big(\mathcal{E}_h^2+\tau^2\Big)+\varepsilon\tau\sum_{m=1}^{n}\|e_\mu^m\|_{L^2}^2\\
&\quad+C\tau\sum_{m=1}^{n}\|D_\tau\mathcal{Z}^m\|_{L^{\frac{3}{2}}}\|e_\phi^{m-1}\|_{H^1}^2+(2+\varepsilon)\|e_\phi^n\|_{L^2}^2.
\end{aligned}$$

4.2. Estimates for $e_{\mathbf{u}}^n$. Taking $\mathbf{v}_h=e_{\mathbf{u}}^n$ in (4.3) yields

(4.13)

$$\begin{aligned}
&\frac{1}{2}D_\tau\|e_{\mathbf{u}}^n\|_{L^2}^2+\frac{1}{2\tau}\|e_{\mathbf{u}}^n-e_{\mathbf{u}}^{n-1}\|_{L^2}^2+\|\nabla e_{\mathbf{u}}^n\|_{L^2}^2\\
&=\Big(b^*(\mathbf{u}_h^{n-1},\mathbf{u}_h^n,e_{\mathbf{u}}^n)-b^*(\mathbf{u}^{n-1},\mathbf{u}^n,e_{\mathbf{u}}^n)\Big)-\Big(b(\phi_h^{n-1},e_{\mathbf{u}}^n,\mu_h^n)-b(R_h\phi^n,e_{\mathbf{u}}^n,\mu^n)\Big)\\
&\quad+\Big(D_\tau(\mathbf{P}_h\mathbf{u}^n-\mathbf{u}^n),e_{\mathbf{u}}^n\Big)+\Big(R_3^n,e_{\mathbf{u}}^n\Big)=:\sum_{i=1}^{4}J_i,
\end{aligned}$$

where we used (4.4).

In the following, we estimate J_i, $1\le i\le 4$, respectively. We first rewrite J_1 into

$$\begin{aligned}
J_1&=-b^*(\mathbf{u}_h^{n-1},e_{\mathbf{u}}^n,e_{\mathbf{u}}^n)-b^*(e_{\mathbf{u}}^{n-1},\mathbf{P}_h\mathbf{u}^n,e_{\mathbf{u}}^n)\\
&\quad-b^*(\mathbf{u}^{n-1}-\mathbf{P}_h\mathbf{u}^{n-1},\mathbf{u}^n,e_{\mathbf{u}}^n)-b^*(\mathbf{P}_h\mathbf{u}^{n-1},\mathbf{u}^n-\mathbf{P}_h\mathbf{u}^n,e_{\mathbf{u}}^n)=:\sum_{i=1}^{4}J_{1,i}\,.
\end{aligned}$$

By noting the definition (2.6) of $b^*(\cdot,\cdot,\cdot)$, it is easy to see that $J_{1,1}=0$, and

$$\begin{aligned}
J_{1,2}&=-\frac{1}{2}\Big(e_{\mathbf{u}}^{n-1}\cdot\nabla\mathbf{P}_h\mathbf{u}^n,e_{\mathbf{u}}^n\Big)+\frac{1}{2}\Big(e_{\mathbf{u}}^{n-1}\cdot\nabla e_{\mathbf{u}}^n,\mathbf{P}_h\mathbf{u}^n\Big)\\
&\le C\|e_{\mathbf{u}}^{n-1}\|_{L^2}\|\nabla\mathbf{P}_h\mathbf{u}^n\|_{L^3}\|e_{\mathbf{u}}^n\|_{L^6}+C\|e_{\mathbf{u}}^{n-1}\|_{L^2}\|\nabla e_{\mathbf{u}}^n\|_{L^2}\|\mathbf{P}_h\mathbf{u}^n\|_{L^\infty}\\
&\le\varepsilon\|\nabla e_{\mathbf{u}}^n\|_{L^2}^2+C\varepsilon^{-1}\|e_{\mathbf{u}}^{n-1}\|_{L^2}^2,\\
J_{1,3}&=-\frac{1}{2}\Big((\mathbf{u}^{n-1}-\mathbf{P}_h\mathbf{u}^{n-1})\cdot\nabla\mathbf{u}^n,e_{\mathbf{u}}^n\Big)+\frac{1}{2}\Big((\mathbf{u}^{n-1}-\mathbf{P}_h\mathbf{u}^{n-1})\cdot\nabla e_{\mathbf{u}}^n,\mathbf{u}^n\Big)\\
&\le C\|\mathbf{u}^{n-1}-\mathbf{P}_h\mathbf{u}^{n-1}\|_{L^2}\|\nabla\mathbf{u}^n\|_{L^3}\|e_{\mathbf{u}}^n\|_{L^6}\\
&\quad+C\|\mathbf{u}^{n-1}-\mathbf{P}_h\mathbf{u}^{n-1}\|_{L^2}\|\nabla e_{\mathbf{u}}^n\|_{L^2}\|\mathbf{u}^n\|_{L^\infty}\\
&\le\varepsilon\|\nabla e_{\mathbf{u}}^n\|_{L^2}^2+C\varepsilon^{-1}\mathcal{E}_h^2,
\end{aligned}$$

where we used (3.18), the projection error estimate (3.21), and (2.13). Using integration by parts, we further have

$$\begin{aligned}
J_{1,4} &= -\frac{1}{2}\Big(\mathbf{P}_h\mathbf{u}^{n-1}\cdot\nabla(\mathbf{u}^n-\mathbf{P}_h\mathbf{u}^n), e_{\mathbf{u}}^n\Big) + \frac{1}{2}\Big(\mathbf{P}_h\mathbf{u}^{n-1}\cdot\nabla e_{\mathbf{u}}^n, \mathbf{u}^n-\mathbf{P}_h\mathbf{u}^n\Big)\\
&= \frac{1}{2}\Big((\nabla\cdot\mathbf{P}_h\mathbf{u}^{n-1})(\mathbf{u}^n-\mathbf{P}_h\mathbf{u}^n), e_{\mathbf{u}}^n\Big) + \frac{1}{2}\Big(\mathbf{u}^n-\mathbf{P}_h\mathbf{u}^n, \mathbf{P}_h\mathbf{u}^{n-1}\cdot\nabla e_{\mathbf{u}}^n\Big)\\
&\quad + \frac{1}{2}\Big(\mathbf{P}_h\mathbf{u}^{n-1}\cdot\nabla e_{\mathbf{u}}^n, \mathbf{u}^n-\mathbf{P}_h\mathbf{u}^n\Big)\\
&\le C\|\nabla\cdot\mathbf{P}_h\mathbf{u}^{n-1}\|_{L^3}\|\mathbf{u}^n-\mathbf{P}_h\mathbf{u}^n\|_{L^2}\|e_{\mathbf{u}}^n\|_{L^6}\\
&\quad + C\|\mathbf{u}^n-\mathbf{P}_h\mathbf{u}^n\|_{L^2}\|\mathbf{P}_h\mathbf{u}^{n-1}\|_{L^\infty}\|\nabla e_{\mathbf{u}}^n\|_{L^2}\\
&\quad + C\|\mathbf{P}_h\mathbf{u}^{n-1}\|_{L^\infty}\|\nabla e_{\mathbf{u}}^n\|_{L^2}\|\mathbf{u}^n-\mathbf{P}_h\mathbf{u}^n\|_{L^2}\\
&\le \varepsilon\|\nabla e_{\mathbf{u}}^n\|_{L^2}^2 + C\varepsilon^{-1}\mathcal{E}_h^2.
\end{aligned}$$

Then we obtain

$$|J_1| \le \varepsilon\|\nabla e_{\mathbf{u}}^n\|_{L^2}^2 + C\varepsilon^{-1}\Big(\mathcal{E}_h^2 + \|e_{\mathbf{u}}^{n-1}\|_{L^2}^2\Big).$$

Second, by (2.5), the term J_2 can be estimated by

$$\begin{aligned}
J_2 &= -\Big\{\Big(\nabla\phi_h^{n-1}\cdot e_{\mathbf{u}}^n, \mu_h^n\Big) - \Big(\nabla R_h\phi^n\cdot e_{\mathbf{u}}^n, \mu^n\Big)\Big\}\\
&= \Big(\nabla\phi_h^{n-1}\cdot e_{\mathbf{u}}^n, e_\mu^n\Big) + \Big(\nabla\phi_h^{n-1}\cdot e_{\mathbf{u}}^n, \mu^n-\Pi_h\mu^n\Big)\\
&\quad + \Big(\nabla e_\phi^{n-1}\cdot e_{\mathbf{u}}^n, \mu^n\Big) + \Big(\nabla(R_h\phi^n - R_h\phi^{n-1})\cdot e_{\mathbf{u}}^n, \mu^n\Big)\\
&\le C\|\nabla\phi_h^{n-1}\|_{L^4}\|e_{\mathbf{u}}^n\|_{L^4}\|e_\mu^n\|_{L^2} + \Big(C\varepsilon^{-1}\|\nabla e_\phi^{n-1}\|_{L^2}^2 + \varepsilon\|\nabla e_{\mathbf{u}}^n\|_{L^2}^2 + C\varepsilon^{-1}\mathcal{E}_h^2\Big)\\
&\quad + C\|\nabla e_\phi^{n-1}\|_{L^2}\|e_{\mathbf{u}}^n\|_{L^2}\|\mu^n\|_{L^\infty} + C\|\nabla(R_h\phi^n - R_h\phi^{n-1})\|_{L^2}\|e_{\mathbf{u}}^n\|_{L^2}\|\mu^n\|_{L^\infty}\\
&\le C\|\nabla\phi_h^{n-1}\|_{L^4}\|e_{\mathbf{u}}^n\|_{L^2}^{\frac{1}{4}}\|\nabla e_{\mathbf{u}}^n\|_{L^2}^{\frac{3}{4}}\|e_\mu^n\|_{L^2} + C_\varepsilon\Big(\|\nabla e_\phi^{n-1}\|_{L^2}^2 + \|e_{\mathbf{u}}^n\|_{L^2}^2 + \mathcal{E}_h^2 + \tau^2\Big)\\
&\quad + \varepsilon\|\nabla e_{\mathbf{u}}^n\|_{L^2}^2\\
&\le C_\varepsilon\Big(\|e_{\mathbf{u}}^n\|_{L^2}^2 + \|\nabla e_\phi^{n-1}\|_{L^2}^2 + \mathcal{E}_h^2 + \tau^2\Big) + \varepsilon\Big(\|e_\mu^n\|_{L^2}^2 + \|\nabla e_{\mathbf{u}}^n\|_{L^2}^2\Big),
\end{aligned}$$

where we noted the uniform boundedness of $\|\nabla\phi_h^{n-1}\|_{L^4}$ given in Proposition 2.3, the error estimate (3.2) for the Ritz projection, and

$$\begin{aligned}
&\Big(\nabla\phi_h^{n-1}\cdot e_{\mathbf{u}}^n, \mu^n-\Pi_h\mu^n\Big)\\
&= -\Big(\nabla e_\phi^{n-1}\cdot e_{\mathbf{u}}^n, \mu^n-\Pi_h\mu^n\Big) + \Big(\nabla(R_h\phi^{n-1}-\phi^{n-1})\cdot e_{\mathbf{u}}^n, \mu^n-\Pi_h\mu^n\Big)\\
&\quad + \Big(\nabla\phi^{n-1}\cdot e_{\mathbf{u}}^n, \mu^n-\Pi_h\mu^n\Big)\\
&\le C\|\nabla e_\phi^{n-1}\|_{L^3}\|e_{\mathbf{u}}^n\|_{L^6}\|\mu^n-\Pi_h\mu^n\|_{L^2}\\
&\quad + C\|\nabla(R_h\phi^{n-1}-\phi^{n-1})\|_{L^3}\|e_{\mathbf{u}}^n\|_{L^6}\|\mu^n-\Pi_h\mu^n\|_{L^2}\\
&\quad + C\|\nabla\phi^{n-1}\cdot e_{\mathbf{u}}^n\|_{H^1}\|\mu^n-\Pi_h\mu^n\|_{H^{-1}}\\
&\le Ch^{-\frac{d}{6}}\|\nabla e_\phi^{n-1}\|_{L^2}\|e_{\mathbf{u}}^n\|_{L^6}h^2 + Ch\|e_{\mathbf{u}}^n\|_{L^6}\|\mu^n-\Pi_h\mu^n\|_{L^2}\\
&\qquad\qquad \text{(here use (3.28), (3.14), and (3.2))}\\
&\quad + C(\|\phi^{n-1}\|_{W^{2,3}}\|e_{\mathbf{u}}^n\|_{L^6} + \|\nabla\phi^{n-1}\|_{L^\infty}\|\nabla e_{\mathbf{u}}^n\|_{L^2})\|\mu^n-\Pi_h\mu^n\|_{H^{-1}}\\
&\le C\|\nabla e_\phi^{n-1}\|_{L^2}\|\nabla e_{\mathbf{u}}^n\|_{L^2} + Ch\|\nabla e_{\mathbf{u}}^n\|_{L^2}\|\mu^n-\Pi_h\mu^n\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+C(\|\phi^{n-1}\|_{W^{2,3}}\|\nabla e_{\mathbf{u}}^n\|_{L^2}+\|\nabla\phi^{n-1}\|_{L^\infty}\|\nabla e_{\mathbf{u}}^n\|_{L^2})\|\mu^n-\Pi_h\mu^n\|_{H^{-1}}\\
&\le C\varepsilon^{-1}\|\nabla e_\phi^{n-1}\|_{L^2}^2+\varepsilon\|\nabla e_{\mathbf{u}}^n\|_{L^2}^2+C\varepsilon^{-1}\mathcal{E}_h^2 \qquad \text{(here use (3.14) and (3.15))}.
\end{aligned}
$$

Finally, since

$$
\begin{aligned}
J_3+J_4&\le C\|D_\tau(\mathbf{P}_h\mathbf{u}^n-\mathbf{u}^n)\|_{L^2}\|e_{\mathbf{u}}^n\|_{L^2}+C\|R_3^n\|_{L^2}\|e_{\mathbf{u}}^n\|_{L^2}\\
&\le C\Big(\mathcal{E}_h^2+\|e_{\mathbf{u}}^n\|_{L^2}^2+\|R_3^n\|_{L^2}^2\Big)
\end{aligned}
$$

with the estimates of J_1 and J_2, and a sufficiently small ε, (4.13) reduces to

$$
\begin{aligned}
&D_\tau\|e_{\mathbf{u}}^n\|_{L^2}^2+\|\nabla e_{\mathbf{u}}^n\|_{L^2}^2\\
&\quad\le C_\varepsilon\Big(\|e_{\mathbf{u}}^n\|_{L^2}^2+\|e_{\mathbf{u}}^{n-1}\|_{L^2}^2+\|\nabla e_\phi^{n-1}\|_{L^2}^2+\mathcal{E}_h^2+\tau^2+\|R_3^n\|_{L^2}^2\Big)+\varepsilon\|e_\mu^n\|_{L^2}^2.
\end{aligned}
$$

We sum up the above estimate from time step t_1 to t_n to obtain

$$
\begin{aligned}
&\|e_{\mathbf{u}}^n\|_{L^2}^2+\tau\sum_{m=1}^n\|\nabla e_{\mathbf{u}}^m\|_{L^2}^2\\
&\le C_\varepsilon\tau\sum_{m=1}^n\Big(\|e_{\mathbf{u}}^m\|_{L^2}^2+\|\nabla e_\phi^m\|_{L^2}^2\Big)+C_\varepsilon\Big(\mathcal{E}_h^2+\tau^2\Big)+\varepsilon\tau\sum_{m=1}^n\|e_\mu^m\|_{L^2}^2, \qquad (4.14)
\end{aligned}
$$

where we have noted $\|e_{\mathbf{u}}^0\|_{L^2}\le C\mathcal{E}_h$ and (4.7).

4.3. L^2-norm estimates of e_ϕ^n and e_μ^n. In this subsection, we present some estimates for $\|e_\phi^n\|_{L^2}$ and $\|e_\mu^n\|_{L^2}$ in terms of Lemmas 3.3 and 3.4. For this purpose, we proceed to estimate $(e_\phi^n,1)^2$, $(e_\mu^n,1)^2$, and $\|(-\Delta_h)^{-\frac{1}{2}}e_\phi^n\|_{L^2}^2$, respectively, in the following two lemmas.

LEMMA 4.1. *For any $n=1,2,\dots,N$, and $\tau,h>0$, the following discrete mass conservation and estimate hold:*

$$(\phi_h^n,1)=(\phi_h^0,1), \qquad (4.15)$$

$$|(e_\phi^n,1)|\le C\mathcal{E}_h, \qquad (4.16)$$

$$|(e_\mu^n,1)|\le C\Big(\|\nabla e_\phi^n\|_{L^2}+\|\nabla e_\phi^{n-1}\|_{L^2}\Big)+C\Big(\tau+\mathcal{E}_h\Big), \qquad (4.17)$$

where C is a positive constant independent of n, h, and τ.

Proof. Choosing $w=1$ in (2.1) yields $(\partial_t\phi,1)=0$ and therefore, we have

$$(\phi^n,1)=(\phi^0,1), \qquad (4.18)$$

which implies the mass conservation for the phase field function. Since $(\nabla\cdot\mathbf{u}_h^n,1)=\int_{\partial\Omega}\mathbf{u}_h^n\cdot\mathbf{n}\,\mathrm{d}\Gamma=0$, (2.12) implies that $(\nabla\cdot\mathbf{u}_h^n,q_h)=0$ for all $q_h\in S_h$. Then we choose $w_h=1$ in (2.9) to get $(D_\tau\phi_h^n,1)=-(\nabla\phi_h^{n-1}\cdot\mathbf{u}_h^n,1)=(\phi_h^{n-1},\nabla\cdot\mathbf{u}_h^n)=0$, which illustrates that ϕ_h^n satisfies the discrete mass conservation (4.15). By using (4.15) and (4.18), we obtain

$$
\begin{aligned}
|(e_\phi^n,1)|&=|(R_h\phi^n-\phi_h^n,1)|=|(R_h\phi^n-\phi^n+\phi^0-\phi_h^0,1)|\\
&\le C\|R_h\phi^n-\phi^n\|_{H^{-1}}+C\|\phi^0-\phi_h^0\|_{H^{-1}}\le C\mathcal{E}_h.
\end{aligned}
$$

Moreover, we take $\varphi_h = 1$ and $\varphi = 1$ in (2.10) and (2.2), respectively, to get

$$\begin{aligned}(\mu_h^n, 1) &= \Big((\phi_h^n)^3 - \phi_h^{n-1}, 1\Big),\\ (\mu^n, 1) &= \Big((\phi^n)^3 - \phi^n, 1\Big),\end{aligned}$$

which further shows

$$\begin{aligned}|(e_\mu^n, 1)| &= |(\Pi_h\mu^n - \mu^n + \mu^n - \mu_h^n, 1)|\\ &= \Big|(\Pi_h\mu^n - \mu^n, 1) + \big((\phi^n)^3 - (\phi_h^n)^3 - (\phi^{n-1} - \phi_h^{n-1}), 1\big) - (\phi^n - \phi^{n-1}, 1)\Big|\\ &\le C\|\Pi_h\mu^n - \mu^n\|_{H^{-1}} + C\|\phi^n - R_h\phi^n\|_{H^{-1}} + C\|e_\phi^n\|_{L^2}\\ &\quad + C\|\phi^{n-1} - R_h\phi^{n-1}\|_{H^{-1}} + C\|e_\phi^{n-1}\|_{L^2} + C\tau\\ &\le C(\tau + \mathcal{E}_h) + C(\|e_\phi^n\|_{L^2} + \|e_\phi^{n-1}\|_{L^2}).\end{aligned}$$

(4.17) follows from (3.26) and the above inequality. The proof of Lemma 4.1 is complete. ∎

Lemma 4.2. *For any $n = 1, 2, \ldots, N$, and $h > 0$, there exists a positive constant τ_1 such that when $\tau \le \tau_1$, the following estimate holds:*

$$\|(-\Delta_h)^{-\frac{1}{2}} e_\phi^n\|_{L^2}^2 \le C\tau \sum_{m=1}^{n} \|e_{\mathbf{u}}^m\|_{L^2}^2 + C\Big(\mathcal{E}_h^2 + \tau^2\Big). \tag{4.19}$$

Here, C is a positive constant independent of n, h, and τ.

Proof. We extend the operator $(-\Delta_h)^{-1} : \mathring{S}_h^r \to \mathring{S}_h^r$ to functions which are not mean-value zero by defining $A_h^{-1} : S_h^r \to S_h^r$ with

$$A_h^{-1} v_h := (-\Delta_h)^{-1}\Big(v_h - \frac{1}{|\Omega|}(v_h, 1)\Big) \quad \text{for } v_h \in S_h^r,$$

which satisfies $A_h^{-1} v_h = (-\Delta_h)^{-1} v_h$ for $v_h \in \mathring{S}_h^r$. Then, taking $w_h = A_h^{-1} e_\phi^n$ in (4.1) and $\varphi_h = \widetilde{e}_\phi^n := e_\phi^n - \frac{1}{|\Omega|}(e_\phi^n, 1)$ in (4.2), and adding the resulting equations together yield

$$\begin{aligned}&\frac{1}{2} D_\tau \|A_h^{-\frac{1}{2}} e_\phi^n\|_{L^2}^2 + \frac{1}{2\tau}\|A_h^{-\frac{1}{2}}(e_\phi^n - e_\phi^{n-1})\|_{L^2}^2 + \|\nabla e_\phi^n\|_{L^2}^2\\ &= \Big(A_h^{-\frac{1}{2}} D_\tau(R_h\phi^n - \phi^n), A_h^{-\frac{1}{2}} e_\phi^n\Big) + \Big(b(\phi_h^{n-1}, \mathbf{u}_h^n, A_h^{-1} e_\phi^n) - b(R_h\phi^n, \mathbf{u}^n, A_h^{-1} e_\phi^n)\Big)\\ &\quad + \Big(A_h^{-\frac{1}{2}} R_1^n, A_h^{-\frac{1}{2}} e_\phi^n\Big) - \frac{1}{2}\Big(\mathcal{Z}^n(e_\phi^n - e_\phi^{n-1}), \widetilde{e}_\phi^n\Big) - \Big(\mathcal{Z}^n \overline{e}_\phi^{n-\frac{1}{2}}, \widetilde{e}_\phi^n\Big) + (e_\phi^{n-1}, \widetilde{e}_\phi^n)\\ &\quad - \Big((\phi^n)^3 - (R_h\phi^n)^3, \widetilde{e}_\phi^n\Big) + \Big(\phi^{n-1} - R_h\phi^{n-1}, \widetilde{e}_\phi^n\Big) + \Big(\mu^n - \Pi_h\mu^n, \widetilde{e}_\phi^n\Big) + \Big(R_2^n, \widetilde{e}_\phi^n\Big)\\ &\le C\|A_h^{-\frac{1}{2}} e_\phi^n\|_{L^2}^2 + C_\varepsilon\Big(\mathcal{E}_h^2 + \|R_1^n\|_{L^2}^2 + \|e_\phi^n\|_{L^2}^2 + \|e_\phi^{n-1}\|_{L^2}^2 + \|R_2^n\|_{L^2}^2\Big) + \varepsilon\|\widetilde{e}_\phi^n\|_{H^1}^2\\ &\quad + \Big(b(\phi_h^{n-1}, \mathbf{u}_h^n, A_h^{-1} e_\phi^n) - b(R_h\phi^n, \mathbf{u}^n, A_h^{-1} e_\phi^n)\Big),\end{aligned} \tag{4.20}$$

where we noted $(\nabla e_\mu^n, \nabla A_h^{-1} e_\phi^n) = (e_\mu^n, \widetilde{e}_\phi^n)$ and used (3.2), $\|\mathcal{Z}^n\|_{L^\infty} \le C$, and Lemma 4.1.

To estimate the last term in (4.20), we rewrite it into

$$
\begin{aligned}
&b(\phi_h^{n-1}, \mathbf{u}_h^n, A_h^{-1}e_\phi^n) - b(R_h\phi^n, \mathbf{u}^n, A_h^{-1}e_\phi^n) \\
&= \Big(\nabla\phi_h^{n-1}\cdot\mathbf{u}_h^n, A_h^{-1}e_\phi^n\Big) - \Big(\nabla R_h\phi^n\cdot\mathbf{u}^n, A_h^{-1}e_\phi^n\Big) \\
&= -\Big(\nabla\phi_h^{n-1}\cdot e_{\mathbf{u}}^n, A_h^{-1}e_\phi^n\Big) - \Big(\nabla\phi_h^{n-1}\cdot(\mathbf{u}^n - \mathbf{P}_h\mathbf{u}^n), A_h^{-1}e_\phi^n\Big) \\
&\quad - \Big(\nabla e_\phi^{n-1}\cdot\mathbf{u}^n, A_h^{-1}e_\phi^n\Big) - \Big(\nabla(R_h\phi^n - R_h\phi^{n-1})\cdot\mathbf{u}^n, A_h^{-1}e_\phi^n\Big) =: \sum_{i=1}^4 K_i.
\end{aligned}
$$

Then we have the bound for K_1 as follows,

$$
\begin{aligned}
K_1 = -\Big(A_h^{-\frac12}(\nabla\phi_h^{n-1}\cdot e_{\mathbf{u}}^n), A_h^{-\frac12}e_\phi^n\Big) &\le C\|\nabla\phi_h^{n-1}\cdot e_{\mathbf{u}}^n\|_{H^{-1}}\|A_h^{-\frac12}e_\phi^n\|_{L^2} \\
&\le C\|\nabla\phi_h^{n-1}\cdot e_{\mathbf{u}}^n\|_{L^{\frac65}}\|A_h^{-\frac12}e_\phi^n\|_{L^2} \\
&\le C\Big(\|e_{\mathbf{u}}^n\|_{L^2}^2 + \|A_h^{-\frac12}e_\phi^n\|_{L^2}^2\Big),
\end{aligned}
$$

where we used (3.23) in the first inequality. Similarly, we have

$$
\begin{aligned}
K_2 = -\Big(A_h^{-\frac12}(\nabla\phi_h^{n-1}\cdot(\mathbf{u}^n - \mathbf{P}_h\mathbf{u}^n)), A_h^{-\frac12}e_\phi^n\Big) &\le C\Big(\|\mathbf{u}^n - \mathbf{P}_h\mathbf{u}^n\|_{L^2}^2 + \|A_h^{-\frac12}e_\phi^n\|_{L^2}^2\Big) \\
&\le C\Big(\mathcal{E}_h^2 + \|A_h^{-\frac12}e_\phi^n\|_{L^2}^2\Big).
\end{aligned}
$$

By using integration by parts and $\nabla\cdot\mathbf{u}^n = 0$, we further have

$$
\begin{aligned}
K_3 = \Big(\widetilde{e}_\phi^{n-1}, \mathbf{u}^n\cdot\nabla A_h^{-1}e_\phi^n\Big) &\le C\|\widetilde{e}_\phi^{n-1}\|_{L^2}\|\mathbf{u}^n\|_{L^\infty}\|A_h^{-\frac12}e_\phi^n\|_{L^2} \\
&\le \Big(\varepsilon\|\nabla e_\phi^{n-1}\|_{L^2} + C\varepsilon^{-1}\|A_h^{-\frac12}e_\phi^{n-1}\|_{L^2}\Big)\|A_h^{-\frac12}e_\phi^n\|_{L^2} \\
&\le \varepsilon\|\nabla e_\phi^{n-1}\|_{L^2}^2 + C\varepsilon^{-1}\Big(\|A_h^{-\frac12}e_\phi^{n-1}\|_{L^2}^2 + \|A_h^{-\frac12}e_\phi^n\|_{L^2}^2\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
K_4 = \Big(R_h\phi^n - R_h\phi^{n-1}, \mathbf{u}^n\cdot\nabla A_h^{-1}e_\phi^n\Big) &\le C\|R_h\phi^n - R_h\phi^{n-1}\|_{L^2}\|\mathbf{u}^n\|_{L^\infty}\|A_h^{-\frac12}e_\phi^n\|_{L^2} \\
&\le C\Big(\tau^2 + \|A_h^{-\frac12}e_\phi^n\|_{L^2}^2\Big).
\end{aligned}
$$

The above results yield

$$
\begin{aligned}
&b(\phi_h^{n-1}, \mathbf{u}_h^n, A_h^{-1}e_\phi^n) - b(R_h\phi^n, \mathbf{u}^n, A_h^{-1}e_\phi^n) \\
&\le C_\varepsilon\Big(\|e_{\mathbf{u}}^n\|_{L^2}^2 + \|A_h^{-\frac12}e_\phi^n\|_{L^2}^2 + \|A_h^{-\frac12}e_\phi^{n-1}\|_{L^2}^2 + \mathcal{E}_h^2 + \tau^2\Big) + \varepsilon\|\nabla e_\phi^{n-1}\|_{L^2}^2.
\end{aligned}
$$

Hence, substituting the last equation into (4.20), we obtain

$$
\begin{aligned}
&D_\tau\|A_h^{-\frac12}e_\phi^n\|_{L^2}^2 + \|\nabla e_\phi^n\|_{L^2}^2 \\
&\quad\le \varepsilon(\|\nabla e_\phi^n\|_{L^2}^2 + \|\nabla e_\phi^{n-1}\|_{L^2}^2) + C_\varepsilon\Big(\|A_h^{-\frac12}e_\phi^n\|_{L^2}^2 + \|A_h^{-\frac12}e_\phi^{n-1}\|_{L^2}^2 \\
&\qquad + \mathcal{E}_h^2 + \tau^2 + \|R_1^n\|_{L^2}^2 + \|R_2^n\|_{L^2}^2 + \|e_{\mathbf{u}}^n\|_{L^2}^2\Big),
\end{aligned}
$$

where we used (3.24)–(3.25). Summing up the above estimate and using (4.7) yield

$$
\|A_h^{-\frac12}e_\phi^n\|_{L^2}^2 + \tau\sum_{m=1}^n\|\nabla e_\phi^m\|_{L^2}^2 \le C\tau\sum_{m=1}^n\Big(\|A_h^{-\frac12}e_\phi^m\|_{L^2}^2 + \|e_{\mathbf{u}}^m\|_{L^2}^2\Big) + C\Big(\mathcal{E}_h^2 + \tau^2\Big).
$$

By Gronwall's inequality, there exists a positive constant τ_1 such that (4.19) holds when $\tau \le \tau_1$. The proof of Lemma 4.2 is complete. ∎

From (3.26) and Lemma 4.1, we can see that

$$\begin{aligned}\|e_\mu^n\|_{L^2} &\le C\Big(\|\nabla e_\mu^n\|_{L^2} + |(e_\mu^n,1)|\Big)\\ &\le C\Big(\|\nabla e_\mu^n\|_{L^2} + \|\nabla e_\phi^n\|_{L^2} + \|\nabla e_\phi^{n-1}\|_{L^2} + \mathcal{E}_h + \tau\Big).\end{aligned} \tag{4.21}$$

Furthermore, by using (3.25) and Lemmas 4.1–4.2, we have

$$\begin{aligned}\|e_\phi^n\|_{L^2}^2 &\le \varepsilon\|\nabla e_\phi^n\|_{L^2}^2 + C\varepsilon^{-1}\|(-\Delta_h)^{-\frac{1}{2}}e_\phi^n\|_{L^2}^2 + C(e_\phi^n,1)^2\\ &\le \varepsilon\|\nabla e_\phi^n\|_{L^2}^2 + C_\varepsilon\tau\sum_{m=1}^n\|e_{\mathbf{u}}^m\|_{L^2}^2 + C_\varepsilon\Big(\mathcal{E}_h^2+\tau^2\Big).\end{aligned} \tag{4.22}$$

Now we turn back to the proof of Theorem 2.1. Substituting the above L^2-norm estimates into (4.12) and (4.14) and adding them together give

(4.23)

$$\begin{aligned}&\|\nabla e_\phi^n\|_{L^2}^2 + \|e_{\mathbf{u}}^n\|_{L^2}^2 + \tau\sum_{m=1}^n\Big(\|\nabla e_\mu^m\|_{L^2}^2 + \|\nabla e_{\mathbf{u}}^m\|_{L^2}^2\Big)\\ &\le C_\varepsilon\tau\sum_{m=1}^n\Big(\|e_{\mathbf{u}}^m\|_{L^2}^2 + \|\nabla e_\phi^m\|_{L^2}^2 + \|e_\phi^{m-1}\|_{L^2}^2\Big) + C_\varepsilon\Big(\mathcal{E}_h^2+\tau^2\Big) + \varepsilon\tau\sum_{m=1}^n\|e_\mu^m\|_{L^2}^2\\ &\quad + C\tau\sum_{m=1}^n\|D_\tau\mathcal{Z}^m\|_{L^{\frac{3}{2}}}\|e_\phi^{m-1}\|_{H^1}^2 + (2+\varepsilon)\|e_\phi^n\|_{L^2}^2\\ &\le C_\varepsilon\tau\sum_{m=1}^n\Big(\|e_{\mathbf{u}}^m\|_{L^2}^2 + \|\nabla e_\phi^m\|_{L^2}^2\Big) + C_\varepsilon\Big(\mathcal{E}_h^2+\tau^2\Big) + \varepsilon\tau\sum_{m=1}^n\|\nabla e_\mu^m\|_{L^2}^2\\ &\quad + C\tau\sum_{m=1}^n\|D_\tau\mathcal{Z}^m\|_{L^{\frac{3}{2}}}\left(\|\nabla e_\phi^{m-1}\|_{L^2}^2 + \tau\sum_{k=1}^{m-1}\|e_{\mathbf{u}}^k\|_{L^2}^2 + \mathcal{E}_h^2+\tau^2\right) + \varepsilon\|\nabla e_\phi^n\|_{L^2}^2.\end{aligned}$$

By noting the definition (4.6) of $\mathcal{Z}^n$, we obtain

$$\begin{aligned}\|D_\tau\mathcal{Z}^n\|_{L^2} &= \left\|\frac{3}{\tau}\int_0^1\Big[\Big((1-\theta)\phi_h^n+\theta R_h\phi^n\Big)^2 - \Big((1-\theta)\phi_h^{n-1}+\theta R_h\phi^{n-1}\Big)^2\Big]d\theta\right\|_{L^2}\\ &= 3\bigg\|\int_0^1\Big((1-\theta)(\phi_h^n+\phi_h^{n-1}) + \theta(R_h\phi^n+R_h\phi^{n-1})\Big)\\ &\qquad\times\Big((1-\theta)D_\tau\phi_h^n + \theta R_hD_\tau\phi^n\Big)\mathrm{d}\theta\bigg\|_{L^2}\\ &\le 3\int_0^1\Big\|\Big((1-\theta)(\phi_h^n+\phi_h^{n-1}) + \theta(R_h\phi^n+R_h\phi^{n-1})\Big)\\ &\qquad\times\Big((1-\theta)D_\tau\phi_h^n + \theta R_hD_\tau\phi^n\Big)\Big\|_{L^2}\mathrm{d}\theta\\ &\le 3\int_0^1\Big\|(1-\theta)(\phi_h^n+\phi_h^{n-1}) + \theta(R_h\phi^n+R_h\phi^{n-1})\Big\|_{L^\infty}\\ &\qquad\times\Big\|(1-\theta)D_\tau\phi_h^n + \theta R_hD_\tau\phi^n\Big\|_{L^2}\mathrm{d}\theta\\ &\le C\int_0^1\max_{1\le n\le N}\Big(\|\phi_h^n\|_{L^\infty} + \|R_h\phi^n\|_{L^\infty}\Big)\Big\|(1-\theta)D_\tau\phi_h^n + \theta R_hD_\tau\phi^n\Big\|_{L^2}\mathrm{d}\theta\end{aligned}$$

$$\leq C\int_0^1 \Big((1-\theta)\|D_\tau\phi_h^n\|_{L^2}+\theta\|R_hD_\tau\phi^n\|_{L^2}\Big)\mathrm{d}\theta$$
$$\leq C\Big(\|D_\tau\phi_h^n\|_{L^2}+\|D_\tau\phi^n\|_{H^1}\Big),$$

where we used Proposition 2.3 and $\|R_h\phi\|_{L^\infty}\leq C\|\phi\|_{H^2}$. Then we have

$$\tau\sum_{n=1}^N\|D_\tau\mathcal{Z}^n\|_{L^2}^2\leq C\tau\sum_{n=1}^N\Big(\|D_\tau\phi_h^n\|_{L^2}+\|D_\tau\phi^n\|_{H^1}\Big)^2\leq C,$$

which implies $\tau^{\frac{1}{2}}\|D_\tau\mathcal{Z}^n\|_{L^{\frac{3}{2}}}\leq C$ and

$$\tau\sum_{n=1}^N\|D_\tau\mathcal{Z}^n\|_{L^{\frac{3}{2}}}\leq C\left(\tau\sum_{n=1}^N\|D_\tau\mathcal{Z}^n\|_{L^{\frac{3}{2}}}^2\right)^{\frac{1}{2}}\leq C\left(\tau\sum_{n=1}^N\|D_\tau\mathcal{Z}^n\|_{L^2}^2\right)^{\frac{1}{2}}\leq C.$$

With the above results and choosing a sufficiently small ε, the inequality (4.23) reduces to

(4.24)
$$\|\nabla e_\phi^n\|_{L^2}^2+\|e_{\mathbf{u}}^n\|_{L^2}^2+\tau\sum_{m=1}^n\Big(\|\nabla e_\mu^m\|_{L^2}^2+\|\nabla e_{\mathbf{u}}^m\|_{L^2}^2\Big)$$
$$\leq C_\varepsilon\tau\sum_{m=1}^n\Big(\|e_{\mathbf{u}}^m\|_{L^2}^2+\|\nabla e_\phi^m\|_{L^2}^2\Big)+C_\varepsilon\Big(\mathcal{E}_h^2+\tau^2\Big)+C\tau\sum_{m=1}^n\|D_\tau\mathcal{Z}^m\|_{L^{\frac{3}{2}}}\|\nabla e_\phi^{m-1}\|_{L^2}^2.$$

By using the discrete Gronwall's inequality, i.e., Lemma 3.5, there exists a positive constant τ_2 such that

$$\|\nabla e_\phi^n\|_{L^2}^2+\|e_{\mathbf{u}}^n\|_{L^2}^2+\tau\sum_{m=1}^n\Big(\|\nabla e_\mu^m\|_{L^2}^2+\|\nabla e_{\mathbf{u}}^m\|_{L^2}^2\Big)\leq C\Big(\mathcal{E}_h^2+\tau^2\Big)\tag{4.25}$$

when $\tau\leq\tau_0:=\min\{\tau_1,\tau_2\}$. Combining the above result and (4.21)–(4.22), and the estimates (3.2) and (3.18)–(3.19), the error estimates (2.14)–(2.16) in Theorem 2.1 follow immediately for $n=1,2,\ldots,N$.

Remark 4.3. This paper focuses on the new optimal error estimates of commonly used finite element approximations for the CHNS phase field model under the regularity assumption (2.13). The global existence of the solution of the required regularity for the current model in three dimensions is generally unknown. For smooth initial data satisfying sufficient compatibility conditions, the local-in-time existence of smooth solutions to the Navier–Stokes (NS) equations in three dimensions satisfying $\partial_t^m u\in L^\infty(0,T;L^2(\Omega))$ and $u\in L^\infty(0,T;H^{2m}(\Omega))$ was proved in [36, Theorems 3.2 and 6.1] (for an arbitrary large m). For nonsmooth initial data, the global-in-time existence in two dimensions and local-in-time existence in three dimensions of smooth solutions to the NS equations away from $\mathrm{t}=0$ was proved in [30, Lemma 3.2] and [36, Proposition 7.1], respectively. The local-in-time existence of solutions to the CHNS system in two dimensions with temporal H^2 regularity away from $t=0$ was proved in [26].

5. Numerical results. In this section, we present numerical results to illustrate the convergence and stability of scheme (2.9)–(2.12), as shown in Theorem 2.1 and Proposition 2.2.

Example 5.1. We first consider the system (1.1)–(1.4) in a unit square domain $\Omega = (0,1) \times (0,1)$ with the initial conditions

$$\begin{cases} \phi^0 = 0.24\cos(2\pi x)\cos(2\pi y) + 0.4\cos(\pi x)\cos(3\pi y), \\ \mathbf{u}^0 = (0,0), \end{cases} \tag{5.1}$$

where the parameters of the problem are chosen as $M = 0.1, \eta = 0.01, \gamma = 0.04, \sigma = 0.04$. We take $T = 0.25$.

To investigate the convergence rate, we solve the problem (1.1)–(1.4) by the method in (2.9)–(2.12) with three different approximations:

Case 1 $(r=1)$*:* P1 for ϕ_h^n and μ_h^n and MINI element for $\mathbf{u}_h^n$ and p_h^n;

Case 2 $(r=1)$*:* P1 for ϕ_h^n and μ_h^n and Taylor–Hood element (P2–P1) for $\mathbf{u}_h^n$ and p_h^n;

Case 3 $(r=2)$*:* P2 for ϕ_h^n and μ_h^n, Taylor–Hood element (P3–P2) for $\mathbf{u}_h^n$ and p_h^n. Since the exact solutions are unknown, we define the errors by

$$\|\theta_v\|_{\ell^\infty(L^2)} := \max_{1\le n\le N} \|v_{\mathrm{ref}}^n - v_h^n\|_{L^2}, \quad \|\theta_v\|_{\ell^2(X)} := \left(\tau \sum_{n=1}^{N} \|v_{\mathrm{ref}}^n - v_h^n\|_X^2\right)^{\frac{1}{2}}, \tag{5.2}$$

where v_h^n denotes a numerical solution and v_{ref}^n denotes a reference solution computed by using the same finite element approximation as for v_h^n with the sufficiently small mesh size $h = \sqrt{2}/512$ and the time step size $\tau = 1/8196$ for Table 5.1 and $h = \sqrt{2}/128$, $\tau = 1/16384$ for Table 5.2. Numerical results given in Tables 5.1–5.2 indicate that the convergence rates of the scheme both in space and time are consistent with the theoretical results presented in Theorem 2.1. In particular, for Case 2 (Table 5.1), the

TABLE 5.1
Spatial convergence rates with $\tau = 1/8196$.

	$h/\sqrt{2}$	$\|\theta_\phi\|_{\ell^\infty(L^2)}$	Rate	$\|\theta_\mu\|_{\ell^2(L^2)}$	Rate	$\|\theta_{\mathbf{u}}\|_{\ell^\infty(L^2)}$	Rate	$\|\theta_{\mathbf{u}}\|_{\ell^2(H^1)}$	Rate
	1/32	8.09e-03	-	2.42e-02	-	5.82e-04	-	1.34e-02	-
Case 1	1/64	2.07e-03	1.97	6.12e-03	1.99	1.48e-04	1.98	5.21e-03	1.36
	1/128	5.00e-04	2.05	1.47e-03	2.06	3.55e-05	2.06	2.35e-03	1.15
	1/32	8.11e-03	-	2.42e-02	-	5.13e-04	-	6.26e-03	–
Case 2	1/64	2.07e-03	1.97	6.12e-03	1.99	1.30e-04	1.98	1.06e-03	2.56
	1/128	5.02e-04	2.05	1.47e-03	2.06	3.12e-05	2.06	2.12e-04	2.32
	1/64	4.99e-05	-	5.47e-05	-	1.15e-06	-	2.24e-04	-
Case 3	1/128	6.25e-06	3.00	6.67e-06	3.03	7.65e-08	3.91	2.92e-05	2.94
	1/256	7.76e-07	3.01	8.23e-07	3.02	4.82e-09	3.99	3.53e-06	3.05

TABLE 5.2
Temporal convergence rates with $h = \sqrt{2}/128$.

	τ	$\|\theta_\phi\|_{\ell^\infty(L^2)}$	Rate	$\|\theta_\mu\|_{\ell^2(L^2)}$	Rate	$\|\theta_{\mathbf{u}}\|_{\ell^\infty(L^2)}$	Rate	$\|\theta_{\mathbf{u}}\|_{\ell^2(H^1)}$	Rate
	1/256	1.69e-02	-	9.90e-02	-	1.40e-03	-	6.37e-03	-
Case 1	1/512	8.48e-03	1.00	5.01e-02	0.98	7.34e-04	0.94	3.33e-03	0.94
	1/1024	4.14e-03	1.03	2.46e-02	1.03	3.67e-04	1.00	1.66e-03	1.00
	1/256	1.69e-02	-	9.90e-02	-	1.40e-03	-	6.37e-03	–
Case 2	1/512	8.48e-03	1.00	5.01e-02	0.98	7.34e-04	0.94	3.33e-03	0.94
	1/1024	4.14e-03	1.03	2.46e-02	1.03	3.67e-04	1.00	1.66e-03	1.00
	1/256	1.69e-02	-	9.90e-02	-	1.40e-03	-	6.37e-03	-
Case 3	1/512	8.48e-03	1.00	5.01e-02	0.98	7.34e-04	0.94	3.33e-03	0.94
	1/1024	4.14e-03	1.03	2.46e-02	1.03	3.67e-04	1.00	1.66e-03	1.00

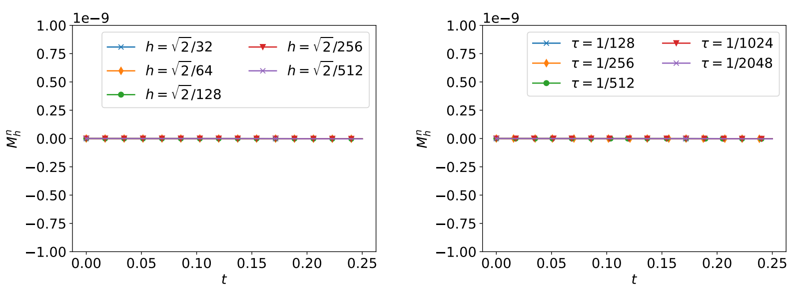
L^2-norm error for the velocity $\mathbf{u}_h^n$ is on the order of $O(h^2)$ in spatial direction, which is one order lower than for interpolation and is confirmed numerically to be optimal. It shows that in this case, the lower-order approximation to ϕ and μ does pollute the accuracy of the numerical velocity.

Also we evaluate the mass of ϕ_h^n and energy of the system in case 3 ($r=2$) with different mesh sizes and time steps. Here we define the numerical mass and energy by

$$M_h^n := (\phi_h^n, 1), \quad E_h^n := \frac{1}{2}\|\mathbf{u}_h^n\|_{L^2}^2 + \frac{\gamma}{4\sigma}\|(\phi_h^n)^2 - 1\|_{L^2}^2 + \frac{\gamma\sigma}{2}\|\nabla\phi_h^n\|_{L^2}^2.$$

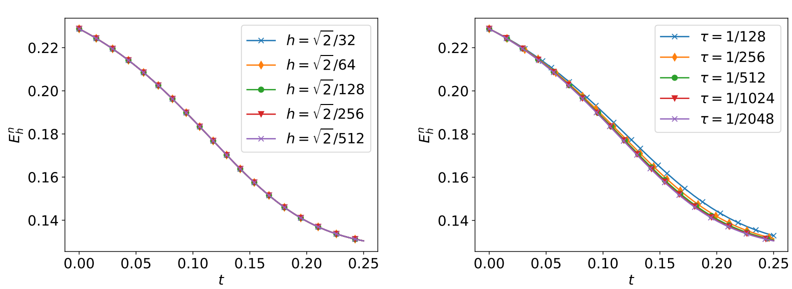
For this example, it is noted that $\int_\Omega \phi dx = \int_\Omega \phi_0 dx = 0$. We plot the mass M_h^n and the energy E_h^n in Figures 5.1 and 5.2, respectively, with different mesh sizes and time steps. We can observe clearly that the mass M_h^n is conserved and the energy E_h^n is decreasing as the time increases, which are consistent with theoretical analysis.

Example 5.2. Second, we consider a shape relaxation example in the domain $\Omega = (0,1) \times (0,1)$ with a rotational boundary condition $\mathbf{u} = (y - 0.5, -x + 0.5)$ on $\partial\Omega$. In this example, we investigate the performance of scheme (2.9)–(2.12) with critical phase field initial data, i.e., $\phi^0 = 1$ in a polygonal domain with reentrant corners and $\phi^0 = -1$ in the remaining part of Ω. The initial shape and velocity ($\mathbf{u}^0 = (y - 0.5, -x + 0.5)$) are shown in Figure 5.3. The problem was studied numerically in [29]. Here, we set $\sigma = 0.02$, $M = 0.005$, $\gamma = 0.002$, $\eta = 0.01$, and $T = 5$. We solve the problem by using scheme (2.9)–(2.12) with the $P1$ element for ϕ_h^n and μ_h^n and the MINI element for $\mathbf{u}_h^n$ and p_h^n. As the exact solutions are unknown, we compute the errors by (5.2). Here, we also compute the relative errors defined by



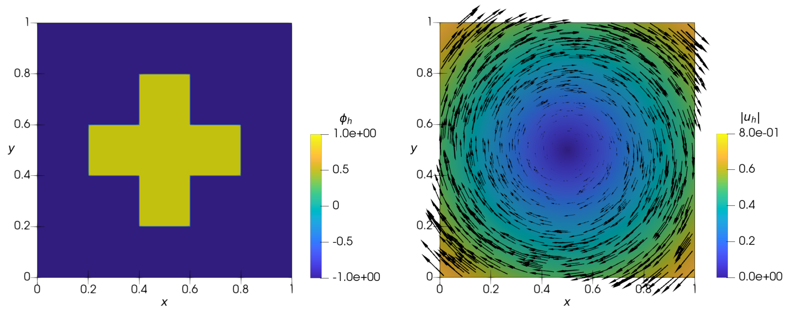
(a) Mass with different h (b) Mass with different τ

FIG. 5.1. *Mass of the phase field* ϕ.



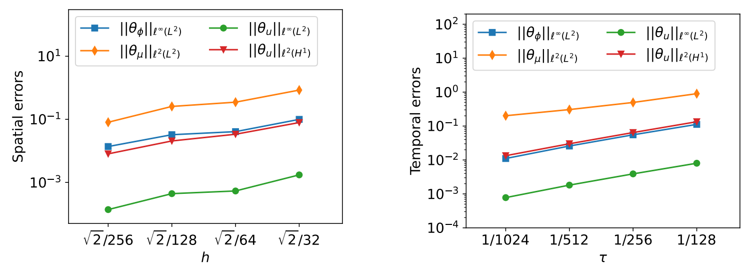
(a) Energy with different h (b) Energy with different τ

FIG. 5.2. *Energy of the system.*



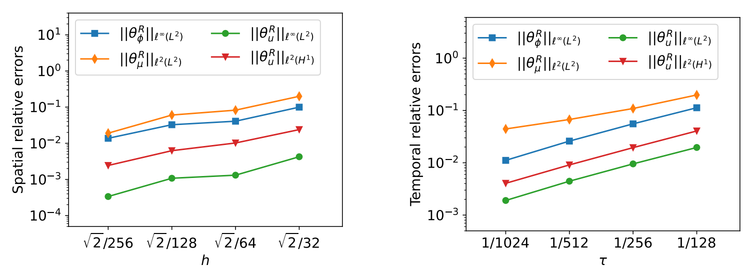
(a) Initial phase field ϕ^0 (b) Initial velocity $\mathbf{u}^0$

FIG. 5.3. *Initial conditions for Example* 5.2.



(a) $\tau = 1/256$ (b) $h = \sqrt{2}/128$

FIG. 5.4. *Errors in the spatial and temporal directions.*

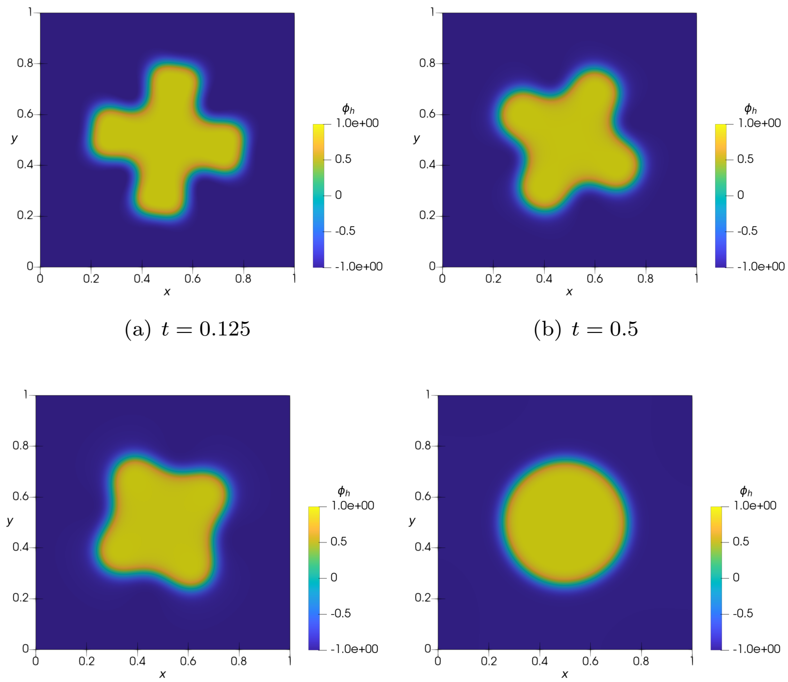


(a) $\tau = 1/256$ (b) $h = \sqrt{2}/128$

FIG. 5.5. *Relative errors in the spatial and temporal directions.*

$$\|\theta_v^R\|_{\ell^\infty(L^2)} := \frac{\|\theta_v\|_{\ell^\infty(L^2)}}{\max_{1\le n\le N}\|v_h^n\|_{L^2}}, \quad \|\theta_v^R\|_{\ell^2(X)} := \frac{\|\theta_v\|_{\ell^2(X)}}{\left(\tau\sum_{n=1}^N \|v_h^n\|_X^2\right)^{\frac{1}{2}}}. \tag{5.3}$$

The errors and relative errors in the spatial direction (temporal direction) are computed by fixing the temporal step size (spatial mesh size) with $h_{\mathrm{ref}} = \sqrt{2}/256$ ($\tau_{\mathrm{ref}} = 1/4096$). The numerical results are given in Figures 5.4 and 5.5, which show that the scheme (2.9)–(2.12) is convergent in such a case. In addition, the relaxation process of the shape under the flow and the snapshot of velocity and phase field at different times are shown in Figures 5.6–5.7. Clearly, as the flow runs clockwise, the shape relaxes and changes to a circle finally.



(a) $t = 0.125$ (b) $t = 0.5$

(c) $t = 1$ (d) $t = 5$

FIG. 5.6. *Phase field at different times.*

6. Appendix. In this appendix, we present the proof for Lemmas 3.1–3.3.

6.1. Proof of Lemma 3.1.

Proof. By choosing $w_h = \Pi_h \mu - I_h \mu$ in (3.11) and using (3.2) and (3.8), it is easy to get the estimate for $\|\nabla(\mu - \Pi_h \mu)\|_{L^2}$ in (3.14).

To prove the estimate for $\|\mu - \Pi_h \mu\|_{L^2}$ in (3.14), for any $w \in H^1(\Omega)$, we introduce the following dual problem,

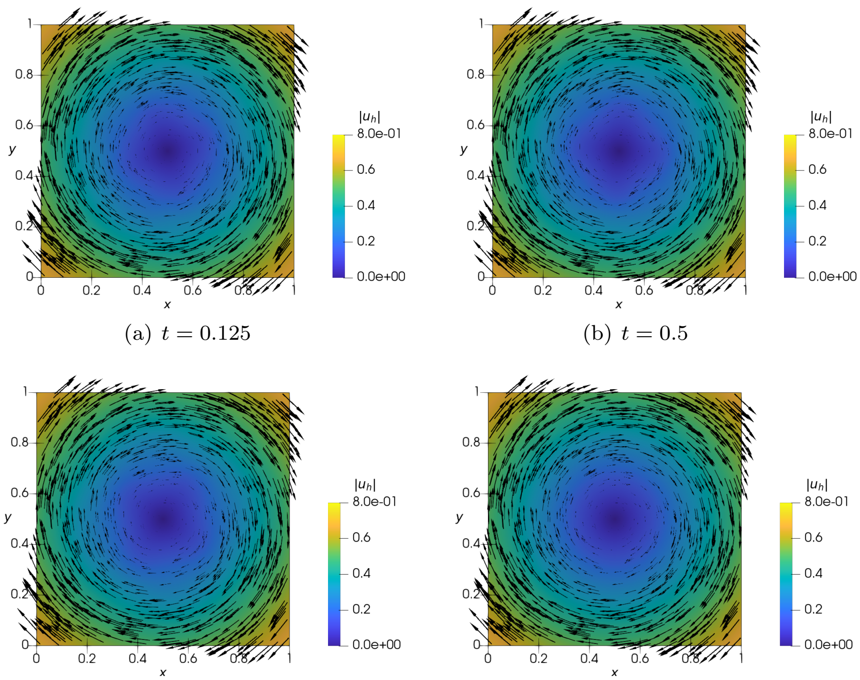
$$-\Delta v = w - \frac{1}{|\Omega|}\int_\Omega w \mathrm{d}x, \qquad x \in \Omega, \tag{6.1}$$

with $\partial_n v = 0$ on $\partial\Omega$. We denote $\xi_\mu := \mu - \Pi_h \mu$. By (3.11), we have

$$\begin{aligned}
(\xi_\mu, w) &= \left(\xi_\mu, w - \frac{1}{|\Omega|}\int_\Omega w \mathrm{d}x\right)\\
&= (\nabla \xi_\mu, \nabla v)\\
&= (\nabla \xi_\mu, \nabla(v - v_h)) + (\nabla \xi_\mu, \nabla v_h)\\
&= (\nabla \xi_\mu, \nabla(v - v_h)) - (\nabla(\phi - R_h\phi)\cdot \mathbf{u}, v_h)\\
&= (\nabla \xi_\mu, \nabla(v - v_h)) + (\phi - R_h\phi, \mathbf{u}\cdot\nabla v_h) \quad \text{(here use } \nabla\cdot\mathbf{u} = 0)\\
&= (\nabla \xi_\mu, \nabla(v - v_h)) + (\phi - R_h\phi, \mathbf{u}\cdot\nabla(v_h - v)) + (\phi - R_h\phi, \mathbf{u}\cdot\nabla v)
\end{aligned} \tag{6.2}$$

for all $v_h \in S_h^r$. It is noted that $\|v\|_{H^s} \le C\|w\|_{H^{s-2}}$ for $s = 2, 3$. Together with the estimate (3.2), we have

$$(\xi_\mu, w) \le \|\nabla\xi_\mu\|_{L^2}\|\nabla(v - v_h)\|_{L^2} + \|\phi - R_h\phi\|_{L^2}\|\mathbf{u}\|_{L^\infty}\|\nabla(v_h - v)\|_{L^2}$$



(a) $t = 0.125$ (b) $t = 0.5$

(c) $t = 1$ (d) $t = 5$

FIG. 5.7. *Velocity at different times.*

$$
\begin{aligned}
&\quad + \|\phi - R_h\phi\|_{L^2}\|\mathbf{u}\|_{L^\infty}\|\nabla v\|_{L^2}\\
&\le Ch^r(\|\mathbf{u}\|_{L^\infty}\|\phi\|_{H^{r+1}} + \|\mu\|_{H^{r+1}})h\|v\|_{H^2} + Ch^{r+1}\|\phi\|_{H^{r+1}}\|\mathbf{u}\|_{L^\infty}h\|v\|_{H^2}\\
&\quad + Ch^{r+1}\|\phi\|_{H^{r+1}}\|\mathbf{u}\|_{L^\infty}\|v\|_{H^1}\\
&\le Ch^{r+1}(\|\mathbf{u}\|_{L^\infty}\|\phi\|_{H^{r+1}} + \|\mu\|_{H^{r+1}})\|w\|_{L^2}. \qquad (6.3)
\end{aligned}
$$

Then, choosing $w = \xi_\mu$ completes the proof of (3.14).

Furthermore, we can also estimate (6.2) as follows:

$$
\begin{aligned}
|(\xi_\mu, w)| &\le \|\nabla\xi_\mu\|_{L^2}\|\nabla(v - v_h)\|_{L^2} + \|\phi - R_h\phi\|_{L^2}\|\mathbf{u}\|_{L^\infty}\|\nabla(v_h - v)\|_{L^2}\\
&\quad + \|\phi - R_h\phi\|_{H^{-1}}\|\mathbf{u}\cdot\nabla v\|_{H^1}\\
&\le Ch^r(\|\mathbf{u}\|_{L^\infty}\|\phi\|_{H^{r+1}} + \|\mu\|_{H^{r+1}})h^2\|v\|_{H^3} + Ch^{r+1}\|\phi\|_{H^{r+1}}\|\mathbf{u}\|_{L^\infty}h\|v\|_{H^2}\\
&\quad + Ch^{r+2}\|\phi\|_{H^{r+1}}(\|\mathbf{u}\|_{W^{1,3}}\|v\|_{W^{1,6}} + \|\mathbf{u}\|_{L^\infty}\|v\|_{H^2})\\
&\le Ch^{r+2}(\|\mathbf{u}\|_{W^{1,4}}\|\phi\|_{H^{r+1}} + \|\mu\|_{H^{r+1}})\|w\|_{H^1}
\end{aligned}
$$

for $v_h \in S_h^r$ and $r \ge 2$, where we used (3.3) in the second to last inequality. Clearly, the above result yields (3.15) for $r \ge 2$. For $r = 1$, since $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, the estimate (3.15) follows from (3.14) immediately.

Now, it remains to prove (3.16)–(3.17). From (3.11), we can see that

$$
\begin{aligned}
&(\nabla D_\tau(\mu^n - \Pi_h\mu^n), \nabla w_h) + (\nabla D_\tau(\phi^n - R_h\phi^n)\cdot\mathbf{u}^n, w_h)\\
&\qquad + (\nabla(\phi^{n-1} - R_h\phi^{n-1})\cdot D_\tau\mathbf{u}^n, w_h) = 0 \qquad (6.4)
\end{aligned}
$$

for all $w_h \in S_h^r$. By choosing $w_h = D_\tau(\Pi_h\mu^n - I_h\mu^n)$ in (6.4) and using (3.2) and (3.4), it is easy to get the estimate (3.16).

Similarly, to prove (3.17), we denote $\zeta_\mu^n := D_\tau(\mu^n - \Pi_h \mu^n)$ and get

$$
\begin{aligned}
(\zeta_\mu^n, w) &= \left(\zeta_\mu^n, w - \frac{1}{|\Omega|}\int_\Omega w \mathrm{d}x\right)\\
&= (\nabla\zeta_\mu^n, \nabla v)\\
&= (\nabla\zeta_\mu^n, \nabla(v - v_h)) + (\nabla\zeta_\mu^n, \nabla v_h)\\
&= (\nabla\zeta_\mu^n, \nabla(v - v_h)) - (\nabla D_\tau(\phi^n - R_h\phi^n)\cdot \mathbf{u}^n, v_h)\\
&\quad - (\nabla(\phi^{n-1} - R_h\phi^{n-1})\cdot D_\tau\mathbf{u}^n, v_h)\\
&= (\nabla\zeta_\mu^n, \nabla(v - v_h)) + (D_\tau(\phi^n - R_h\phi^n), \mathbf{u}^n\cdot\nabla v_h)\\
&\quad + (\phi^{n-1} - R_h\phi^{n-1}, D_\tau\mathbf{u}^n\cdot\nabla v_h)\\
&= (\nabla\zeta_\mu^n, \nabla(v - v_h)) + (D_\tau(\phi^n - R_h\phi^n), \mathbf{u}^n\cdot\nabla(v_h - v))\\
&\quad + (D_\tau(\phi^n - R_h\phi^n), \mathbf{u}^n\cdot\nabla v))\\
&\quad + (\phi^{n-1} - R_h\phi^{n-1}, D_\tau\mathbf{u}^n\cdot\nabla(v_h - v)) + (\phi^{n-1} - R_h\phi^{n-1}, D_\tau\mathbf{u}^n\cdot\nabla v) \qquad (6.5)
\end{aligned}
$$

for all $v_h \in S_h^r$, where v is the solution of (6.1) and $\nabla\cdot\mathbf{u} = 0$ is used. It follows that

$$
\begin{aligned}
|(\zeta_\mu^n, w)| &\le \|\nabla\zeta_\mu^n\|_{L^2}\|\nabla(v - v_h)\|_{L^2} + \|D_\tau(\phi^n - R_h\phi^n)\|_{L^2}\|\mathbf{u}^n\|_{L^\infty}\|\nabla(v_h - v)\|_{L^2}\\
&\quad + \|D_\tau(\phi^n - R_h\phi^n)\|_{H^{-1}}\|\mathbf{u}^n\cdot\nabla v\|_{H^1}\\
&\quad + \|\phi^{n-1} - R_h\phi^{n-1}\|_{L^2}\|D_\tau\mathbf{u}^n\|_{L^\infty}\|\nabla(v_h - v)\|_{L^2}\\
&\quad + \|\phi^{n-1} - R_h\phi^{n-1}\|_{H^{-1}}\|D_\tau\mathbf{u}^n\cdot\nabla v\|_{H^1}\\
&\le Ch^r(\|\mathbf{u}^n\|_{L^\infty}\|D_\tau\phi^n\|_{H^r} + \|D_\tau\mathbf{u}^n\|_{L^\infty}\|\phi^{n-1}\|_{H^r} + \|D_\tau\mu^n\|_{H^{r+1}})h^2\|v\|_{H^3}\\
&\quad + Ch^{r+1}\|D_\tau\phi^n\|_{H^{r+1}}\|\mathbf{u}^n\|_{L^\infty}h\|v\|_{H^2}\\
&\quad + Ch^{r+2}\|D_\tau\phi^n\|_{H^{r+1}}(\|\mathbf{u}^n\|_{W^{1,3}}\|v\|_{W^{1,6}} + \|\mathbf{u}^n\|_{L^\infty}\|v\|_{H^2})\\
&\quad + Ch^{r+1}\|\phi^{n-1}\|_{H^{r+1}}\|D_\tau\mathbf{u}^n\|_{L^\infty}h\|v\|_{H^2}\\
&\quad + Ch^{r+2}\|\phi^{n-1}\|_{H^{r+1}}(\|D_\tau\mathbf{u}^n\|_{W^{1,3}}\|v\|_{W^{1,6}} + \|D_\tau\mathbf{u}^n\|_{L^\infty}\|v\|_{H^2})\\
&\le Ch^{r+2}(\|\mathbf{u}^n\|_{W^{1,4}}\|D_\tau\phi^n\|_{H^{r+1}} + \|D_\tau\mathbf{u}^n\|_{W^{1,4}}\|\phi^{n-1}\|_{H^{r+1}}\\
&\quad + \|D_\tau\mu^n\|_{H^{r+1}})\|w\|_{H^1}
\end{aligned}
$$

for $v_h \in S_h^r$ and $r \ge 2$, where we used (3.16) and (3.4)–(3.5) in the second to last inequality, and noted $\|v\|_{H^s} \le C\|w\|_{H^{s-2}}$ for $s = 2, 3$ in the last inequality. Clearly, the above result yields (3.17) for $r \ge 2$. For $r = 1$, by noting $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ and choosing $w = \zeta_\mu^n$ in (6.5), the estimate (3.17) follows similarly as that for (6.3). The proof of Lemma 3.1 is complete. ∎

6.2. Proof of Lemma 3.2.

Proof. Let $\xi_{\mathbf{u}} := \mathbf{u} - \mathbf{P}_h\mathbf{u}$ and $\xi_p := p - P_h p$. Taking $\mathbf{v}_h = \mathbf{P}_h\mathbf{u} - \mathbf{I}_h\mathbf{u}$ into (3.12) yields

$$
\begin{aligned}
\|\nabla\xi_{\mathbf{u}}\|_{L^2}^2 &= (\nabla\xi_{\mathbf{u}}, \nabla(\mathbf{u} - \mathbf{I}_h\mathbf{u})) - (\xi_p, \nabla\cdot(\mathbf{P}_h\mathbf{u} - \mathbf{I}_h\mathbf{u})) - (\mu\nabla(\phi - R_h\phi), \mathbf{P}_h\mathbf{u} - \mathbf{I}_h\mathbf{u})\\
&= (\nabla\xi_{\mathbf{u}}, \nabla(\mathbf{u} - \mathbf{I}_h\mathbf{u})) - (\xi_p, \nabla\cdot(\mathbf{P}_h\mathbf{u} - \mathbf{u})) - (\xi_p, \nabla\cdot(\mathbf{u} - \mathbf{I}_h\mathbf{u}))\\
&\quad + (\phi - R_h\phi, \nabla\mu\cdot(\mathbf{P}_h\mathbf{u} - \mathbf{I}_h\mathbf{u})) + (\mu(\phi - R_h\phi), \nabla\cdot(\mathbf{P}_h\mathbf{u} - \mathbf{I}_h\mathbf{u}))\\
&= (\nabla\xi_{\mathbf{u}}, \nabla(\mathbf{u} - \mathbf{I}_h\mathbf{u})) - (p - I_h p, \nabla\cdot(\mathbf{P}_h\mathbf{u} - \mathbf{u})) - (\xi_p, \nabla\cdot(\mathbf{u} - \mathbf{I}_h\mathbf{u}))\\
&\quad + (\phi - R_h\phi, \nabla\mu\cdot(\mathbf{P}_h\mathbf{u} - \mathbf{I}_h\mathbf{u})) + (\mu(\phi - R_h\phi), \nabla\cdot(\mathbf{P}_h\mathbf{u} - \mathbf{I}_h\mathbf{u}))\\
&\le \|\nabla\xi_{\mathbf{u}}\|_{L^2}\|\nabla(\mathbf{u} - \mathbf{I}_h\mathbf{u})\|_{L^2} + \|p - I_h p\|_{L^2}\|\nabla\cdot\xi_{\mathbf{u}}\|_{L^2}\\
&\quad + \|\xi_p\|_{L^2}\|\nabla\cdot(\mathbf{u} - \mathbf{I}_h\mathbf{u})\|_{L^2} + \|(\phi - R_h\phi)\nabla\mu\|_{L^{\frac{6}{5}}}\|\mathbf{P}_h\mathbf{u} - \mathbf{I}_h\mathbf{u}\|_{L^6}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \|\mu(\phi - R_h\phi)\|_{L^2}\|\nabla\cdot(\mathbf{P}_h\mathbf{u} - \mathbf{I}_h\mathbf{u})\|_{L^2} \\
&\le \varepsilon\Big(\|\nabla\xi_{\mathbf{u}}\|_{L^2}^2 + \|\xi_p\|_{L^2}^2\Big) \\
&\quad + C_\varepsilon\Big(\|\nabla(\mathbf{u} - \mathbf{I}_h\mathbf{u})\|_{L^2}^2 + \|p - I_hp\|_{L^2}^2 + \|\phi - R_h\phi\|_{L^2}^2\|\mu\|_{W^{1,4}}^2\Big),
\end{aligned}
\tag{6.6}
$$

where we used (3.13) in the third equality. To derive the estimate of $\|\xi_p\|_{L^2}$, we use the discrete inf-sup condition (2.7) and obtain

$$
\begin{aligned}
\|I_hp - P_hp\|_{L^2} &\le C\frac{(I_hp - P_hp, \nabla\cdot\mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|_{L^2}} \\
&\le C\|I_hp - p\|_{L^2} + C\|\nabla\xi_{\mathbf{u}}\|_{L^2} - C\frac{(\phi - R_h\phi, \nabla\mu\cdot\mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|_{L^2}} \\
&\quad - C\frac{(\mu(\phi - R_h\phi), \nabla\cdot\mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|_{L^2}} \quad \text{(use (3.12))} \\
&\le C\|I_hp - p\|_{L^2} + C\|\nabla\xi_{\mathbf{u}}\|_{L^2} + C\|(\phi - R_h\phi)\nabla\mu\|_{L^{\frac{6}{5}}} \\
&\quad + C\|\mu(\phi - R_h\phi)\|_{L^2}
\end{aligned}
\tag{6.7}
$$

for $\mathbf{v}_h \in \mathbf{X}_h^{r+1}/\mathbf{X}_h^{1b}$. Substituting the above estimate into (6.6) and using (3.2) lead to (3.19).

To prove (3.18), we introduce the solution $\mathbf{v}$ of

$$
\begin{aligned}
-\Delta\mathbf{v} + \nabla q &= \xi_{\mathbf{u}}, \\
\nabla\cdot\mathbf{v} &= 0
\end{aligned}
$$

in Ω, with $\mathbf{v} = \mathbf{0}$ on $\partial\Omega$. Then we have

$$
\begin{aligned}
\|\xi_{\mathbf{u}}\|_{L^2}^2 &= (\nabla\mathbf{v}, \nabla\xi_{\mathbf{u}}) - (q, \nabla\cdot\xi_{\mathbf{u}}) \\
&= (\nabla(\mathbf{v} - \mathbf{I}_h\mathbf{v}), \nabla\xi_{\mathbf{u}}) + (\nabla\mathbf{I}_h\mathbf{v}, \nabla\xi_{\mathbf{u}}) - (q - I_hq, \nabla\cdot\xi_{\mathbf{u}}) \\
&= (\nabla(\mathbf{v} - \mathbf{I}_h\mathbf{v}), \nabla\xi_{\mathbf{u}}) + (\xi_p, \nabla\cdot\mathbf{I}_h\mathbf{v}) - (\phi - R_h\phi, \nabla\cdot(\mu\mathbf{I}_h\mathbf{v})) - (q - I_hq, \nabla\cdot\xi_{\mathbf{u}}) \\
&= (\nabla(\mathbf{v} - \mathbf{I}_h\mathbf{v}), \nabla\xi_{\mathbf{u}}) + (\xi_p, \nabla\cdot(\mathbf{I}_h\mathbf{v} - \mathbf{v})) - (\phi - R_h\phi, \mu\nabla\cdot(\mathbf{I}_h\mathbf{v} - \mathbf{v})) \\
&\quad - (\phi - R_h\phi, \nabla\mu\cdot(\mathbf{I}_h\mathbf{v} - \mathbf{v} + \mathbf{v})) - (q - I_hq, \nabla\cdot\xi_{\mathbf{u}}) \quad \text{(here use } \nabla\cdot\mathbf{v} = 0) \\
&\le Ch\|\mathbf{v}\|_{H^2}\Big(\|\nabla\xi_{\mathbf{u}}\|_{L^2} + \|\xi_p\|_{L^2} + \|\phi - R_h\phi\|_{L^2}\|\mu\|_{W^{1,4}}\Big) \\
&\quad + C\|\phi - R_h\phi\|_{H^{-1}}\|\nabla\mu\cdot\mathbf{v}\|_{H^1} + Ch\|q\|_{H^1}\|\nabla\cdot\xi_{\mathbf{u}}\|_{L^2} \\
&\le Ch\|\xi_{\mathbf{u}}\|_{L^2}\Big(\|\nabla\xi_{\mathbf{u}}\|_{L^2} + \|\xi_p\|_{L^2} + \|\phi - R_h\phi\|_{L^2}\|\mu\|_{W^{1,4}}\Big) \\
&\quad + C\|\phi - R_h\phi\|_{H^{-1}}\|\mu\|_{H^2}\|\xi_{\mathbf{u}}\|_{L^2} + Ch\|\xi_{\mathbf{u}}\|_{L^2}\|\nabla\cdot\xi_{\mathbf{u}}\|_{L^2},
\end{aligned}
$$

where we used $\|\mathbf{v}\|_{H^2} + \|q\|_{H^1} \le C\|\xi_{\mathbf{u}}\|_{L^2}$ in the last inequality. Thus (3.18) follows from the above estimate and (6.6)–(6.7) immediately.

From (3.12)–(3.13), we can also see that

$$
\begin{aligned}
&(\nabla D_\tau(\mathbf{u}^n - \mathbf{P}_h\mathbf{u}^n), \nabla\mathbf{v}_h) - (D_\tau(p^n - P_hp^n), \nabla\cdot\mathbf{v}_h) \\
&\quad = (\mu^n\nabla D_\tau(\phi^n - R_h\phi^n), \mathbf{v}_h) + (D_\tau\mu^n\nabla(\phi^{n-1} - R_h\phi^{n-1}), \mathbf{v}_h),
\end{aligned}
\tag{6.8}
$$

$$
(\nabla\cdot D_\tau(\mathbf{u}^n - \mathbf{P}_h\mathbf{u}^n), q_h) = 0 \tag{6.9}
$$

for $(\mathbf{v}_h, q_h) \in \mathbf{X}_h^{r+1}\times\mathring{S}_h^r/\mathbf{X}_h^{1b}\times\mathring{S}_h^1$. By a similar approach as used in (3.18)–(3.19), we can easily obtain the estimate (3.20). The proof of Lemma 3.2 is complete. ∎

6.3. Proof of Lemma 3.3.

Proof. First, by the definition (2.19) of Δ_h, it is easy to see that for $v_h \in \mathring{S}_h^r$

$$\|(-\Delta_h)^{\frac{1}{2}} v_h\|_{L^2}^2 = (-\Delta_h v_h, v_h) = \|\nabla v_h\|_{L^2}^2 \sim \|v_h\|_{H^1}^2,$$

where the equivalence relation holds due to $\int_\Omega v_h \mathrm{d}x = 0$. The above result yields (3.22).

Second, on the one hand, for given $v_h \in \mathring{S}_h^r$ and any $\phi \in \mathring{H}^1(\Omega)$, we have

$$\begin{aligned}(v_h, \phi) = (v_h, I_h\phi) = ((-\Delta_h)^{-\frac{1}{2}} v_h, (-\Delta_h)^{\frac{1}{2}} I_h\phi) &\le \|(-\Delta_h)^{-\frac{1}{2}} v_h\|_{L^2} \|(-\Delta_h)^{\frac{1}{2}} I_h\phi\|_{L^2} \\ &= \|(-\Delta_h)^{-\frac{1}{2}} v_h\|_{L^2} \|\nabla I_h\phi\|_{L^2} \\ &\le C\|(-\Delta_h)^{-\frac{1}{2}} v_h\|_{L^2} \|\phi\|_{H^1},\end{aligned}$$

which implies $\|v_h\|_{H^{-1}} \le C\|(-\Delta_h)^{-\frac{1}{2}} v_h\|_{L^2}$. On the other hand, by choosing $\phi = (-\Delta_h)^{-1} v_h$ we also obtain

$$\begin{aligned}\sup_{\phi\in \mathring{H}^1(\Omega),\phi\neq 0} \frac{(v_h,\phi)}{\|\nabla\phi\|_{L^2}} \ge \sup_{\phi\in \mathring{S}_h^r,\phi\neq 0} \frac{(v_h,\phi)}{\|\nabla\phi\|_{L^2}} &\ge \frac{(v_h, (-\Delta_h)^{-1} v_h)}{\|\nabla(-\Delta_h)^{-1} v_h\|_{L^2}} \\ &= \frac{\|(-\Delta_h)^{-\frac{1}{2}} v_h\|_{L^2}^2}{\|(-\Delta_h)^{-\frac{1}{2}} v_h\|_{L^2}} = \|(-\Delta_h)^{-\frac{1}{2}} v_h\|_{L^2}.\end{aligned}$$

Combining the above estimates leads to (3.23).

Finally, it is straightforward to get

$$\|v_h\|_{L^2}^2 = ((-\Delta_h)^{\frac{1}{2}} v_h, (-\Delta_h)^{-\frac{1}{2}} v_h) \le \|(-\Delta_h)^{\frac{1}{2}} v_h\|_{L^2} \|(-\Delta_h)^{-\frac{1}{2}} v_h\|_{L^2},$$

and with (3.22), (3.24) follows immediately. The proof of Lemma 3.3 is complete. ∎

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REFERENCES

[1] H. Abels, *On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities*, Arch. Ration. Mech. Anal., 194 (2009), pp. 463–506.

[2] G. Akrivis, B. Li, and D. Li, *Energy-decaying extrapolated RK-SAV methods for the Allen–Cahn and Cahn–Hilliard equations*, SIAM J. Sci. Comput., 41 (2019), pp. A3703–A3727.

[3] D. N. Arnold, F. Brezzi, and M. Fortin, *A stable finite element for the Stokes equations*, Calcolo, 21 (1984), pp. 337–344.

[4] J. W. Barrett and J. F. Blowey, *Finite element approximation of a model for phase separation of a multi-component alloy with non-smooth free energy*, Numer. Math., 77 (1997), pp. 1–34.

[5] J. W. Barrett and J. F. Blowey, *Finite element approximation of a model for phase separation of a multi-component alloy with a concentration-dependent mobility matrix*, IMA J. Numer. Anal., 18 (1998), pp. 287–328.

[6] S. C. Brenner, A. E. Diegel, and L.-Y. Sung, *A robust solver for a mixed finite element method for the Cahn–Hilliard equation*, J. Sci. Comput., 77 (2018), pp. 1234–1249.

[7] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods,* Texts Appl. Math. 15, 3rd ed., Springer, New York, 2008.

[8] Y. Cai and J. Shen, *Error estimates for a fully discretized scheme to a Cahn–Hilliard phase-field model for two-phase incompressible flows*, Math. Comp., 87 (2018), pp. 2057–2090.

[9] W. Chen, Y. Liu, C. Wang, and S. M. Wise, *Convergence analysis of a fully discrete finite difference scheme for the Cahn–Hilliard–Hele–Shaw equation*, Math. Comp., 85 (2016), pp. 2231–2257.

[10] Y. Chen and J. Shen, *Efficient, adaptive energy stable schemes for the incompressible Cahn–Hilliard Navier–Stokes phase-field models*, J. Comput. Phys., 308 (2016), pp. 40–56.
[11] C. Collins, J. Shen, and S. M. Wise, *Unconditionally stable finite difference multigrid schemes for the Cahn–Hilliard–Brinkman equation*, Commun. Comput. Phys., 13 (2013), pp. 929–957.
[12] A. E. Diegel, X. H. Feng, and S. M. Wise, *Analysis of a mixed finite element method for a Cahn–Hilliard–Darcy–Stokes system*, SIAM J. Numer. Anal., 53 (2015), pp. 127–152.
[13] A. E. Diegel, C. Wang, and S. M. Wise, *Stability and convergence of a second-order mixed finite element method for the Cahn–Hilliard equation*, IMA J. Numer. Anal., 36 (2016), pp. 1867–1897.
[14] A. E. Diegel, C. Wang, X. Wang, and S. M. Wise, *Convergence analysis and error estimates for a second order accurate finite element method for the Cahn–Hilliard–Navier–Stokes system*, Numer. Math., 137 (2017), pp. 495–534.
[15] C. M. Elliott and S. Larsson, *Error estimates with smooth and nonsmooth data for a finite element method for the Cahn–Hilliard equation*, Math. Comp., 58 (1992), pp. 603–630.
[16] D. J. Eyre, *Unconditionally gradient stable time marching the Cahn–Hilliard equation*, in Computational and Mathematical Models of Microstructural Evolution, J. W. Bullard, R. Kalia, M. Stoneham, and L. Q. Chen, eds., Materials Research Society, Warrendale, PA, 1998, pp. 1686–1712.
[17] X. Feng, *Fully discrete finite element approximations of the Navier–Stokes–Cahn–Hilliard diffuse interface model for two-phase fluid flows*, SIAM J. Numer. Anal., 44 (2006), pp. 1049–1072.
[18] X. Feng, Y. He, and C. Liu, *Analysis of finite element approximations of a phase field model for two-phase fluids*, Math. Comp., 76 (2007), pp. 539–571.
[19] H. Garcke, K. Lam, E. Sitka, and V. Styles, *A Cahn–Hilliard–Darcy model for tumour growth with chemotaxis and active transport*, Math. Models Methods Appl. Sci., 26 (2016), pp. 1095–1148.
[20] A. Giorgini, A. Miranville, and R. Temam, *Uniqueness and regularity for the Navier-Stokes-Cahn–Hilliard system*, SIAM J. Math. Anal., 51 (2019), pp. 2535–2574.
[21] G. Grün, *On convergent schemes for diffuse interface models for two-phase flow of incompressible fluids with general mass densities*, SIAM J. Numer. Anal., 51 (2013), pp. 3036–3061.
[22] G. Grün and F. Klingbeil, *Two-phase flow with mass density contrast: Stable schemes for a thermodynamic consistent and frame indifferent diffuse-interface model*, J. Comput. Phys., 257 (2014), pp. 708–725.
[23] Z. Guo, P. Lin, and J. S. Lowengrub, *A numerical method for the quasi-incompressible Cahn–Hilliard–Navier-Stokes equations for variable density flows with a discrete energy law*, J. Comput. Phys., 276 (2014), pp. 486–507.
[24] M. E. Gurtin, D. Polignone, and J. Viñals, *Two-phase binary fluids and immiscible fluids described by an order parameter*, Math. Models Methods Appl. Sci., 6 (1996), pp. 815–831.
[25] D. Han and X. Wang, *A second order in time, uniquely solvable, unconditionally stable numerical scheme for Cahn–Hilliard–Navier–Stokes equation*, J. Comput. Phys., 290 (2015), pp. 139–156.
[26] Y. He and X. Feng, *Uniform H^2-regularity of solution for the 2D Navier–Stokes/Cahn–Hilliard phase field model*, J. Math. Anal. Appl., 441 (2016), pp. 815–829.
[27] J. G. Heywood and R. Rannacher, *Finite element approximation of the nonstationary Navier–Stokes problem* IV*: Error analysis for second-order time discretization*, SIAM J. Numer. Anal., 27 (1990), pp. 353–384.
[28] P. C. Hohenberg and B. I. Halperin, *Theory of dynamic critical phenomena*, Rev. Mod. Phys., 49 (1977).
[29] D. Kay, V. Styles, and R. Welford, *Finite element approximation of a Cahn–Hilliard–Navier–Stokes system*, Interfaces Free Bound., 10 (2008), pp. 15–43.
[30] B. Li, S. Ma, and Y. Ueda, *Analysis of fully discrete finite element methods for 2D Navier–Stokes equations with critical initial data*, ESAIM Math. Model. Numer. Anal., 56 (2022), pp. 2105–2139.
[31] X. Li and J. Shen, *On a SAV-MAC scheme for the Cahn–Hilliard–Navier–Stokes phase-field model and its error analysis for the corresponding Cahn–Hilliard–Stokes case*, Math. Models Methods Appl. Sci., 30 (2020), pp. 2263–2297.
[32] Y. Liu, W. Chen, C. Wang, and S. M. Wise, *Error analysis of a mixed finite element method for a Cahn–Hilliard–Hele-Shaw system*, Numer. Math., 135 (2017), pp. 679–709.
[33] J. Shen, J. Xu, and J. Yang, *The scalar auxiliary variable (SAV) approach for gradient flows*, J. Comput. Phys., 353 (2018), pp. 407–416.

[34] J. Shen and X. Yang, *A phase-field model and its numerical approximation for two-phase incompressible flows with different densities and viscosities*, SIAM J. Sci. Comput., 32 (2010), pp. 1159–1179.

[35] J. Shen and X. Yang, *Decoupled, energy stable schemes for phase-field models of two-phase incompressible flows*, SIAM J. Numer. Anal., 53 (2015), pp. 279–296.

[36] R. Temam, *Navier–Stokes Equations and Nonlinear Functional Analysis*, CBMS-NSF Regional Conf. Ser. in Appl. Math. 66, 2nd ed., SIAM, Philadelphia, 1995.

[37] M. F. Wheeler, *A priori L^2 error estimates for Galerkin approximations to parabolic partial differential equations*, SIAM J. Numer. Anal., 10 (1973), pp. 723–759.

[38] J. Zhao and D. Han, *Second-order decoupled energy-stable schemes for Cahn–Hilliard–Navier–Stokes equations*, J. Comput. Phys., 443 (2021), 110536.