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A Zienkiewicz-type finite element applied to fourth-order problems☆

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ABSTRACT

This paper deals with convergence analysis and applications of a Zienkiewicz-type (Z-type) triangular element, applied to fourth-order partial differential equations. For the biharmonic problem we prove the order of convergence by comparison to a suitable modified Hermite triangular finite element. This method is more natural and it could be applied to the corresponding fourth-order eigenvalue problem. We also propose a simple postprocessing method which improves the order of convergence of finite element eigenpairs. Thus, an a posteriori analysis is presented by means of different triangular elements. Some computational aspects are discussed and numerical examples are given.

1. Introduction

In general, a finite element method (FEM) for treating fourth-order problems requires trial and test functions belonging to subspaces of the Sobolev space $H^2(\Omega)$, and this would require C^1-elements, i.e., piecewise polynomials which are C^1 across interelement boundaries. A motivation for avoiding the use of C^1 finite elements is their very high dimension. Also, in many cases the feasible C^0-elements for fourth-order problems give more simple and flexible computational schemes. However, the effective choice of a method is complex, depending on many aspects of the underlying problem. Herein, in order to avoid the C^1-requirement we will use a nonconforming Zienkiewicz-type (Z-type) triangle element [1,2] applied to some biharmonic problems. On the other hand, nonconforming finite elements are commonly used for approximation fourth-order eigenvalue problems (EVP) in linear plate theory.

The rest of the paper is organized as follows. In the next section we give a brief description of continuous fourth-order problems. In Section 3, a technique of finite element discretization is presented by means of nonconforming Z-type triangles. Section 4 is devoted to the main result. Here the error estimates are derived. In Section 5, a postprocessing technique for acceleration of the convergence for eigenpairs is presented. Finally, numerical examples are given to verify the validity of the analytic results.

2. Statement of the problems

Let Ω be a bounded polygonal domain in $\mathbf{R}^2$ with boundary $\partial\Omega$. Let also $H^m(\Omega)$ be the usual mth-order Sobolev space on Ω with a norm $\|\cdot\|_{m,\Omega}$ and a seminorm $|\cdot|_{m,\Omega}$. Throughout this paper, $(\cdot,\cdot)$ denotes the $L_2(\Omega)$-inner product.

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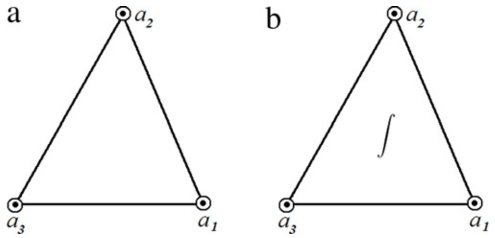


Fig. 1.

Consider the following fourth-order model problem for $f \in L_2(\Omega)$:

$$\Delta^2 u = f \quad \text{in } \Omega, \qquad u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $\nu = (\nu_1, \nu_2)$ is the unit outer normal vector of $\partial\Omega$ and Δ is the standard Laplacian operator.

Let us also consider a thin elastic plate corresponding to the domain Ω. If the material is homogeneous and isotropic, the question of the possible small vibrations of the plate leads to the basic eigenvalue problem:

$$\Delta^2 u = \lambda u \quad \text{in } \Omega, \qquad u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{2}$$

The weak formulation of the problem (1) is: find $u \in H_0^2(\Omega)$ such that

$$a(u, v) = (f, v), \quad \forall v \in H_0^2(\Omega), \tag{3}$$

where

$$a(u, v) = \int_\Omega \sum_{i,j=1}^{2} \partial_{ij}^2 u \, \partial_{ij}^2 v \mathrm{d}x, \quad \forall u, v \in H^2(\Omega).$$

By analogy with (3), the variational EVP corresponding to (2) is: find $(\lambda, u) \in \mathbf{R} \times H_0^2(\Omega)$ such that

$$a(u, v) = \lambda(u, v), \quad \forall v \in V = H_0^2(\Omega). \tag{4}$$

Obviously, the bilinear form $a(\cdot, \cdot)$ is symmetric and V-elliptic (see [3,4]). Moreover, the inclusion of V in $L_2(\Omega)$ is compact. Therefore, problem (4) has a countable infinite set of eigenvalues λ_j, all strictly positive and having finite multiplicity, without a finite accumulation point (see, e.g., [5]). The corresponding eigenfunctions u_j can be chosen to be orthonormal in $L_2(\Omega)$ and they constitute a Hilbert basis for V.

3. Finite element approximations

We shall approximate the solutions of (3) and (4) by the finite element method. Consider a family of triangulations $\tau_h = \cup_i K_i$ of $\overline{\Omega}$. Finite elements K_i fulfill standard assumptions (see [6, Chapter 3]). If h_i denotes the diameter of K_i, $h = \max_i h_i$ is the finite element parameter corresponding to any partition τ_h.

With a partition τ_h we associate a finite dimensional space V_h by means of Z-type triangular elements. It is well-known that the Zienkiewicz triangle represents a reduced cubic Hermite finite element for which (see [1,7,2]):

- K is a triangle with vertices a_i, $1 \le i \le 3$;
- one possible set of degrees of freedom is (for any test function p)

$$p(a_i),\ 1 \le i \le 3 \quad \text{and} \quad Dp(a_i)(a_j - a_i),\ 1 \le i, j \le 3,\ i \ne j;$$

- $\mathcal{P}_K \subset \mathcal{P}_3(K)$ and $\dim \mathcal{P}_K = 9$ (Fig. 1(a)).

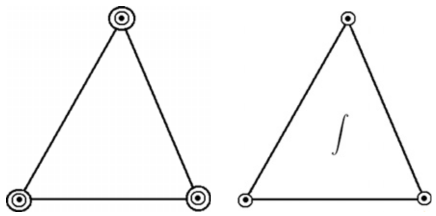
Using directional derivatives, there are a variety of ways to define a finite element. Some Z-type triangular elements having the same degrees of freedom can also be proposed in different ways [2].

So, for any triangle K we define

$$\mathcal{P}_K = \mathcal{P}_3'(K) = \mathcal{P}_2(K) + \operatorname{span}\left\{\lambda_i^2\lambda_j - \lambda_i\lambda_j^2,\ 1 \le i < j \le 3\right\},$$

where λ_i, $i = 1, 2, 3$, are the barycentric coordinates of K. Then we can define the shape function space by $\mathcal{P}_3'(K)$.

Lemma 3.1 ([2], Lemma 1)**.** *The set of degrees of freedom is $\mathcal{P}_K$-unisolvent.*



(a) dim $P_K = 18$. (b) dim $P_K = 10$.

Fig. 2.

Our approach requires the use of not just Z-type elements. In Fig. 1(b) the modified Hermite triangle is depicted, which uses the degrees of freedom of the Zienkiewicz element and the integral value of the corresponding element.

The Z-type finite element that we use (Fig. 1(a)) satisfies the following properties:

(i) It is an incomplete and nonconforming C^0-element for fourth-order problems.
(ii) It uses the degrees of freedom just like the Zienkiewicz triangle, but its polynomial space is obtained in a specific manner.
(iii) It takes values of functions and their derivatives at vertices as degrees of freedom and through this the global number of degrees of freedom is the smallest one.
(iv) It is convergent (applied to fourth-order problems) in contrast to the Zienkiewicz triangle, which is only convergent in parallel line conditions and is divergent in general grids.

The vertex point degrees of freedom used for the Zienkiewicz triangle are an advantage in finite element implementation. In addition the work is slightly reduced.

Then, the approximate variational problem of (3) is: find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \tag{5}$$

where

$$a_h(u_h, v_h) = \sum_{K\in\tau_h} \int_K \sum_{i,j=1}^{2} \frac{\partial^2 u_h}{\partial x_i \partial x_j} \frac{\partial^2 v_h}{\partial x_i \partial x_j} \mathrm{d}x.$$

By analogy, we determine the approximate eigenpairs (λ_h, u_h) using a nonconforming Z-type finite element. Then, the EVP corresponding to (4) is: find $(\lambda_h, u_h) \in \mathbf{R} \times V_h$ such that

$$a_h(u_h, v_h) = \lambda_h(u_h, v_h), \quad \forall v_h \in V_h. \tag{6}$$

4. Convergence analysis

First, we introduce the mesh-dependent norm and seminorm [7,2]. For any $v \in L_2(\Omega)$ with $v_{|_K} \in H^m(K),\ \forall K \in \tau_h$, we define

$$\|v\|_{m,h} = \left(\sum_{K\in\tau_h} \|v\|^2_{m,K}\right)^{1/2}, \qquad |v|_{m,h} = \left(\sum_{K\in\tau_h} |v|^2_{m,K}\right)^{1/2}.$$

In order to carry out convergence analysis of the Z-type element considered for fourth-order problems, we also consider the Hermite triangle with a suitably modified tenth degree of freedom. It takes an integral value on K instead of the value at the barycenter of K (see Fig. 2(b)).

Let Π_h denote the interpolation operator corresponding to the Z-type finite element partition τ_h and π_h be the interpolation operator related to the modified Hermite finite element. Our convergence analysis is based on the estimation of $\Pi_h v - \pi_h v$ for any $v \in H_0^2 \cap H^3(\Omega)$ on each element $K \in \tau_h$.

Theorem 4.1. *Let V_h be the FE space corresponding to the nonconforming Z-type element. Then there exists a constant $C = C(\Omega) > 0$, independent of h and such that*

$$\inf_{v_h\in V_h} \sum_{m=0}^{2} h^m |v - v_h|_{m,h} \le Ch^3 \|v\|_{3,\Omega}, \quad \forall v \in H_0^2 \cap H^3(\Omega).$$

Proof. Let us estimate $\Pi_h v - \pi_h v$ on each finite element $K \in \tau_h$. For this purpose we transform any triangle K to the reference element

$$T = \{(t_1, t_2) : \ t_1, t_2 \ge 0,\ t_1 + t_2 \le 1\}.$$

The shape functions of the Z-type element on the reference element T are

$$\varphi_1(t_1,t_2) = -\frac{1}{2}t_1t_2 - t_1^2 + \frac{3}{2}t_1^2t_2 + \frac{1}{2}t_1t_2^2 + t_1^3; \qquad \left(\partial_{(1,0)}\varphi_1(1,0)=1\right)$$
$$\varphi_2(t_1,t_2) = 2t_1t_2 + 3t_1^2 - 2t_1^2t_2 - 2t_1t_2^2 - 2t_1^3; \qquad (\varphi_2(1,0)=1)$$
$$\varphi_3(t_1,t_2) = -\frac{1}{2}t_1t_2 - \frac{1}{2}t_1^2t_2 + \frac{1}{2}t_1t_2^2; \qquad \left(\partial_{(1,-1)}\varphi_3(1,0)=1\right)$$
$$\varphi_4(t_1,t_2) = -\frac{1}{2}t_1t_2 + \frac{1}{2}t_1^2t_2 - \frac{1}{2}t_1t_2^2; \qquad \left(\partial_{(-1,1)}\varphi_4(0,1)=1\right)$$
$$\varphi_5(t_1,t_2) = 2t_1t_2 + 3t_2^2 - 2t_1^2t_2 - 2t_1t_2^2 - 2t_2^3; \qquad (\varphi_5(0,1)=1)$$
$$\varphi_6(t_1,t_2) = -\frac{1}{2}t_1t_2 - t_2^2 + \frac{1}{2}t_1^2t_2 + \frac{3}{2}t_1t_2^2 + t_2^3; \qquad \left(\partial_{(0,1)}\varphi_6(0,1)=1\right)$$
$$\varphi_7(t_1,t_2) = -t_2 + \frac{3}{2}t_1t_2 + 2t_2^2 - \frac{1}{2}t_1^2t_2 - \frac{3}{2}t_1t_2^2 - t_2^3; \qquad \left(\partial_{(0,-1)}\varphi_7(0,0)=1\right)$$
$$\varphi_8(t_1,t_2) = 1 - 4t_1t_2 - 3t_1^2 - 3t_2^2 + 4t_1^2t_2 + 4t_1t_2^2 + 2t_1^3 + 2t_2^3; \qquad (\varphi_8(0,0)=1)$$
$$\varphi_9(t_1,t_2) = -t_1 + 2t_1^2 - t_1^3 + \frac{3}{2}t_1t_2 - \frac{3}{2}t_1^2t_2 - \frac{1}{2}t_1t_2^2. \qquad \left(\partial_{(-1,0)}\varphi_9(0,0)=1\right).$$

By analogy, we obtain the shape functions of the Hermite element on T consecutively from a_1 to a_3:

$$\psi_1(t_1,t_2) = 2t_1t_2 - t_1^2 - t_1^2t_2 - 2t_1t_2^2 + t_1^3; \qquad \left(\partial_{(1,0)}\psi_1(1,0)=1\right)$$
$$\psi_2(t_1,t_2) = -18t_1t_2 + 3t_1^2 + 18t_1^2t_2 + 18t_1t_2^2 - 2t_1^3; \qquad (\psi_2(1,0)=1)$$
$$\psi_3(t_1,t_2) = 2t_1t_2 - 3t_1^2t_2 - 2t_1x_2^2; \qquad \left(\partial_{(1,-1)}\psi_3(1,0)=1\right)$$
$$\psi_4(t_1,t_2) = 2t_1t_2 - 2t_1^2t_2 - 3t_1t_2^2; \qquad \left(\partial_{(-1,1)}\psi_4(0,1)=1\right)$$
$$\psi_5(t_1,t_2) = -18t_1t_2 + 3t_2^2 + 18t_1^2t_2 + 18t_1t_2^2 - 2t_2^3; \qquad (\psi_5(0,1)=1)$$
$$\psi_6(t_1,t_2) = 2t_1t_2 - t_2^2 - 2t_1^2t_2 - t_1t_2^2 + t_2^3; \qquad \left(\partial_{(0,1)}\psi_6(0,1)=1\right)$$
$$\psi_7(t_1,t_2) = -t_2 + 4t_1t_2 + 2t_2^2 - 3t_1^2t_2 - 4t_1t_2^2 - t_2^3; \qquad \left(\partial_{(0,-1)}\psi_7(0,0)=1\right)$$
$$\psi_8(t_1,t_2) = 1 - 24t_1t_2 - 3t_1^2 - 3t_2^2 + 24t_1^2t_2 + 24t_1t_2^2 + 2t_1^3 + 2t_2^3; \qquad (\psi_8(0,0)=1)$$
$$\psi_9(t_1,t_2) = -t_1 + 4t_1t_2 + 2t_1^2 - 4t_1^2t_2 - 3t_1t_2^2 - t_1^3; \qquad \left(\partial_{(-1,0)}\psi_9(0,0)=1\right)$$
$$\psi_{10}(t_1,t_2) = 60t_1t_2 - 60t_1^2t_2 - 60t_1t_2^2. \qquad \left(\frac{1}{\operatorname{meas} T}\int_T \psi_{10}(t)\mathrm{d}t = 1\right).$$

Using these shape functions we calculate

$$\begin{aligned}(\Pi_h v - \pi_h v)_{|T} &= (60t_1t_2 - 60t_1^2t_2 - 60t_1t_2^2)\left[\frac{v(a_1)+v(a_2)+v(a_3)}{3}\right. \\ &\quad - \frac{\partial_{(a_1-a_2)}v(a_1) + \partial_{(a_1-a_3)}v(a_1)}{24} - \frac{\partial_{(a_2-a_1)}v(a_2) + \partial_{(a_2-a_3)}v(a_2)}{24} \\ &\quad \left. - \frac{\partial_{(a_3-a_1)}v(a_3) + \partial_{(a_3-a_2)}v(a_3)}{24} - \frac{1}{\operatorname{meas} T}\int_T v(t)\mathrm{d}t\right] \\ &= 60t_1t_2(1-t_1-t_2)E_T(v) \le \frac{20}{9}|E_T(v)|,\end{aligned} \tag{7}$$

where

$$E_T(v) = \left[\frac{1}{3}\sum_{i=1}^{3} v(a_i) - \frac{1}{24}\sum_{\substack{i,j=1\\ i\neq j}}^{3} \partial_{(a_i-a_j)}v(a_i) - \frac{1}{\operatorname{meas} T}\int_T v(t)\,\mathrm{d}t\right]$$

is the error functional of the quadrature formula.

It is easy to see that

$$E_T(v) = 0 \quad \text{for any } v \in \mathcal{P}_2(T).$$

Therefore, from the Bramble–Hilbert lemma [6] there exists a constant $C > 0$ such that

$$|E_T(v)| \le C|v|_{3,T}.$$

Let F_K be the invertible affine mapping which maps the reference finite element T onto the finite element K:

$$F_K : T \to K, \qquad t \to x = F_K(t) = B_K(t) + b_K$$

with $B_K \in \mathbf{R}^{2\times 2},\ b_K \in \mathbf{R}^{2\times 1}$, and where $\det B_K = \mathcal{O}(h^2)$.

So, we obtain

$$|E_T(v)| \le Ch^3 \,(\det B_K)^{-1/2}\, |v|_{3,K}. \tag{8}$$

Thus, using that

$$\|\Pi_h v - \pi_h v\|_{0,K} = (\det B_K)^{1/2}\, \|\Pi_h v - \pi_h v\|_{0,T},$$

from (7) and (8) it follows that

$$\begin{aligned}\|\Pi_h v - \pi_h v\|_{0,h} &= \left(\sum_{K\in\tau_h} \|\Pi_h v - \pi_h v\|_{0,K}^2\right)^{1/2} \\ &\le Ch^3 \|v\|_{3,\Omega}.\end{aligned} \tag{9}$$

By explicit calculations, we also obtain

$$\partial_i\, (\Pi_h v - \pi_h v)_{|T} = t_j(1-2t_i-t_j)E_T(v), \quad i,j\in\{1,2\},\ i\ne j,$$

where $_{|T}$ denotes the restriction to the reference element T.

Since $(\Pi_h v - \pi_h v) \in \mathcal{P}_3(K)$, for the last equality in the conjecture with the inverse inequality [6, pp. 133–137] we obtain

$$|\partial_i\, (\Pi_h v - \pi_h v)\,|_{|T} \le Ch^2 |v|_{3,K},$$

and, in the same manner,

$$|\partial_{ij}\, (\Pi_h v - \pi_h v)\,|_{|T} \le Ch |v|_{3,K}, \quad i,j = 1,2.$$

These inequalities and (9) give

$$\|\Pi_h v - \pi_h v\|_{m,h} \le Ch^{3-m}\|v\|_{3,\Omega}, \quad m = 0,1,2. \tag{10}$$

Finally, we use the fact that the order of convergence of $\|v-\pi_h v\|_{m,h},\ m = 0,1,2$, is an optimal one for the cubic Hermite polynomials (see [7,6]). From this and (10) we apply the inequality

$$\|v - \Pi_h v\|_{m,h} \le \|v - \pi_h v\|_{m,h} + \|\Pi_h v - \pi_h v\|_{m,h}.$$

Then the result of the theorem follows from the FE interpolation theory. □

Remark 1. Observe that for $m = 0, 1$ we have

$$\|\Pi_h v - \pi_h v\|_{m,h} = \|\Pi_h v - \pi_h v\|_{m,\Omega}.$$

Now, we shall prove the main estimate:

Theorem 4.2. *Let $u \in H^3(\Omega)\cap H_0^2(\Omega)$ be the solution of (3) and $u_h \in V_h$ be the solution of the problem (5) using a Z-type finite element. Then there exists a constant $C = C(\Omega) > 0$, independent of h and such that*

$$\|u - u_h\|_{2,h} \le Ch\|u\|_{3,\Omega}. \tag{11}$$

Proof. By Theorem 4.1, it is readily seen that for any $v \in H_0^2(\Omega)$

$$\lim_{h\to 0}\ \inf_{v_h\in V_h} \|v - v_h\|_{2,h} = 0.$$

Therefore

$$\lim_{h\to 0} \|u - u_h\|_{2,h} = 0.$$

Having in mind that V_h is constructed of Z-type nonconforming finite elements, we have (see also [2, Lemma 3])

$$|a_h(v, v_h) - (\Delta^2 v, v_h)| \le Ch|v|_{3,\Omega}|v_h|_{2,h}, \quad \forall v\in H^3(\Omega),\ \forall v_h \in V_h. \tag{12}$$

Now, for the solutions u and u_h we apply the second Strang lemma (see [6, Theorem 31.1]):

$$\|u - u_h\|_{2,h} \le C\left(\inf_{v_h\in V_h} \|u - v_h\|_{2,h} + \sup_{\substack{v_h\in V_h\\ v_h\ne 0}} \frac{|a_h(u,v_h) - (f, v_h)|}{\|v_h\|_{2,h}}\right).$$

Then, the main estimate (11) follows from (12) and the result of Theorem 4.1. □

We can apply the results of the last two theorems to the corresponding fourth-order elliptic EVP (2) (see [3]).

It is to be noted here that the sesquilinear form a_h is uniformly elliptic, i.e. ($\alpha > 0$),

$$\alpha \|\Delta v\|_{0,\Omega}^2 \le a_h(v, v), \quad \forall v \in H_0^2(\Omega).$$

Let us also define the elliptic projection $\mathcal{R}_h \in \mathcal{L}(V, V_h)$ by

$$a_h(\mathcal{R}_h u - u, v_h) = 0, \quad \forall v_h \in V_h. \tag{13}$$

Thus, $\mathcal{R}_h$ is an operator of orthogonal projection from V over V_h with respect to the scalar product $a_h(\cdot, \cdot)$. The next theorem gives the error estimate of the eigenvalues using a nonconforming Z-type finite element.

Theorem 4.3. *Let (λ, u) and (λ_h, u_h) be eigensolutions of (4) and (6), respectively. Then for any simple eigenvalue λ_m ($m \ge 1$), $\lambda_{m,h} \longrightarrow \lambda_m$ ($h \to 0$). Moreover, if the corresponding eigenfunction u_m belongs to $H_0^2(\Omega) \cap H^3(\Omega)$, then*

$$|\lambda_m - \lambda_{m,h}| \le Ch^2 \|u_m\|_{3,\Omega}^2. \tag{14}$$

Proof. First, let us notice that from Theorem 4.1 and (13), it follows that for any $u \in V$

$$\|u - \mathcal{R}_h u\|_{2,h} \le C \inf_{v_h \in V_h} \|u - v_h\|_{2,h}. \tag{15}$$

We shall estimate the difference $\lambda_m - \lambda_{m,h}$ for any integer $m \ge 1$. For this purpose we introduce the space V_m which is generated by the first m (exact) eigenfunctions $\{u_i\}$, $1 \le i \le m$. The approximate eigenvalue $\lambda_{m,h}$ can be characterized as various extrema of the Rayleigh quotient [3]. Then (dim $(\mathcal{R}_h V_m) = m$)

$$\lambda_{m,h} \le \max_{0 \ne v_h \in \mathcal{R}_h V_m} \frac{a_h(v_h, v_h)}{\|v_h\|_{0,\Omega}^2} = \max_{\substack{v \in V_m \\ \|v\|_{0,\Omega}=1}} \frac{a_h(\mathcal{R}_h v, \mathcal{R}_h v)}{\|\mathcal{R}_h v\|_{0,\Omega}^2}.$$

Since $\mathcal{R}_h v$ is an orthogonal projection over V_h with respect to $a_h(\cdot, \cdot)$, we have

$$a_h(\mathcal{R}_h v, \mathcal{R}_h v) \le a_h(v, v),$$

and therefore

$$\lambda_{m,h} \le \sup_{\substack{v \in V_m \\ \|v\|_{0,\Omega}=1}} \frac{a_h(v, v)}{\|\mathcal{R}_h v\|_{0,\Omega}^2} \le \lambda_m \sup_{\substack{v \in V_m \\ \|v\|_{0,\Omega}=1}} \frac{1}{\|\mathcal{R}_h v\|_{0,\Omega}^2}. \tag{16}$$

In the last inequality we suppose that $a_h(v, v) = a(v, v)$. This is true if for example $V_m \subset H^2(\Omega)$. We also emphasize that $\|v_h\|_{s,h} = \|v_h\|_{s,\Omega}$, $s = 0, 1$, for all $v_h \in V_h$. For this, let us consider a function $v \in V_m$ such that $\|v\|_{0,\Omega} = 1$. Then

$$v = \sum_{i=1}^m \alpha_i u_i \quad \text{with} \quad \sum_{i=1}^m \alpha_i^2 = 1.$$

Using that v is normalized with respect to $L_2(\Omega)$ we obtain

$$1 - \|\mathcal{R}_h v\|_{0,\Omega}^2 = (v - \mathcal{R}_h v, v + \mathcal{R}_h v) = 2(v - \mathcal{R}_h v, v) - \|v - \mathcal{R}_h v\|_{0,\Omega}^2,$$

or

$$\|\mathcal{R}_h v\|_{0,\Omega}^2 \ge 1 - 2(v - \mathcal{R}_h v, v). \tag{17}$$

On the other hand, from (4) we derive

$$(v - \mathcal{R}_h v, v) = \sum_{i=1}^m \alpha_i (v - \mathcal{R}_h v, u_i) = \sum_{i=1}^m \frac{\alpha_i}{\lambda_i} a_h(v - \mathcal{R}_h v, u_i).$$

Applying equality (13) we get

$$(v - \mathcal{R}_h v, v) = \sum_{i=1}^m \frac{\alpha_i}{\lambda_i} a_h(v - \mathcal{R}_h v, u_i - \mathcal{R}_h u_i).$$

Next, as $a_h(\cdot, \cdot)$ is continuous, i.e. ($M = const > 0$),

$$a_h(u, v) \le M \|u\|_{2,h} \|v\|_{2,h},$$

and we obtain

$$(v - \mathcal{R}_h v, v) \le M \|v - \mathcal{R}_h v\|_{2,h} \left\| \sum_{i=1}^m \frac{\alpha_i}{\lambda_i} (u_i - \mathcal{R}_h u_i) \right\|_{2,h}.$$

Having in mind that λ_1 is the smallest eigenvalue, by the Cauchy–Schwarz inequality we estimate

$$\begin{aligned}\left\|\sum_{i=1}^{m}\frac{\alpha_i}{\lambda_i}(u_i-\mathcal{R}_h u_i)\right\|_{2,h} &\le \left(\sum_{i=1}^{m}\frac{\alpha_i^2}{\lambda_i^2}\right)^{1/2}\left(\sum_{i=1}^{m}\|u_i-\mathcal{R}_h u_i\|_{2,h}^2\right)^{1/2}\\ &\le \frac{\sqrt{m}}{\lambda_1}\sup_{\substack{v\in V_m\\ \|v\|_{0,\Omega}=1}}\|v-\mathcal{R}_h v\|_{2,h}. \end{aligned}\tag{18}$$

Combining (16)–(18), we obtain

$$\lambda_{m,h}\le\left(1+C\sup_{\substack{v\in V_m\\ \|v\|_{0,\Omega}=1}}\|v-\mathcal{R}_h v\|_{2,h}^2\right)\lambda_m.$$

We rewrite the last result as

$$|\lambda_{m,h}-\lambda_m|\le C(\lambda)\sup_{\substack{v\in V_m\\ \|v\|_{0,\Omega}=1}}\|v-\mathcal{R}_h v\|_{2,h}^2\le C\sum_{i=1}^{m}\|u_i-\mathcal{R}_h u_i\|_{2,h}^2.$$

Now, let us suppose that $V_m\subset H^3(\Omega)\cap H_0^2(\Omega)$ and $u_i\in H^3(\Omega),\ i=1,\dots,m$. Applying (15) and the approximation property of the Z-type finite element proved in Theorem 4.1 to the last inequality we prove the estimate (14).

As a corollary of the considerations above, if λ_m is a simple eigenvalue, for the corresponding eigenfunctions we have

$$\|u_{m,h}-u_m\|_{2,h}\le C\sup_{\substack{v\in V_m\\ \|v\|_{0,\Omega}=1}}\|v-\mathcal{R}_h v\|_{2,h}\le C\left(\sum_{i=1}^{m}\|u_i-\mathcal{R}_h u_i\|_{2,h}^2\right)^{1/2}.$$

From (15) under $V_m\subset H^3(\Omega)\cap H_0^2(\Omega)$ we get

$$\|u_{m,h}-u_m\|_{2,h}\le Ch\left(\sum_{i=1}^{m}\|u_i\|_{3,h}^2\right)^{1/2}.\quad\Box\tag{19}$$

5. The superconvergent postprocessing technique

At present, modern engineering and scientific computing use intensively superconvergence postprocessing methods. Procedures for accelerating the convergence of FE approximations of the eigenpairs are developed by authors for different problems (see, e.g., [8,9,4]). Herein we prove that these ideas could be applied to biharmonic EVP approximated by nonconforming finite elements. We present a relatively simple postprocessing method that gives better accuracy for eigenvalues. It is based on a postprocessing technique whereby an additional solving of a source problem on augmented FE space is involved. This method is illustrated with a numerical example in the next section.

Let u_h be any approximate eigenfunction of (6) with $\|u_h\|_{0,\Omega}=1$. Since the FE solution u_h obtained with the nonconforming Z-type element is already known, we consider the following variational elliptic problem:

$$a(\widetilde{u},v)=(u_h,v),\quad\forall v\in V.\tag{20}$$

Theorem 5.1. *Let the FE space V_h be constructed of Z-type nonconforming triangular elements. If (λ,u) is an eigenpair of problem (4), $u\in H^3(\Omega)$ and (λ_h,u_h) is the corresponding solution of (6). We also suppose that the eigenfunctions are normalized: $\|u\|_{0,\Omega}=\|u_h\|_{0,\Omega}=1$. Then*

$$|\lambda-\widetilde{\lambda}|=\mathcal{O}(\|u-u_h\|_{0,\Omega}^2),\tag{21}$$

where $\widetilde{\lambda}$ is defined by

$$\widetilde{\lambda}=\frac{1}{(\widetilde{u},u_h)}.$$

Proof. For any function $w\in H^r(\Omega),\ r\ge 0$, consider the following elliptic problem:

$$a(u,v)=(w,v)\quad\forall v\in V.$$

Then the operator $\mathcal{T}:H^r(\Omega)\to V$ defined by $u=\mathcal{T}w,\ u\in V$, is the solution operator for the boundary value (source) problem. Since $a(\cdot,\cdot)$ is a symmetric form and $a(u,v)$ is an inner product on V, then the operator $\mathcal{T}$ is symmetric and positive (see also [3]). Thus we have

$$a(\mathcal{T}u,v)=(u,v)\quad\forall v\in V.$$

On the other hand if u is a solution of the EVP (3), then

$$a(u, \mathcal{T}u) = \lambda(u, \mathcal{T}u).$$

From the symmetry of $\mathcal{T}$ we easily get

$$\frac{1}{\lambda} - \frac{1}{\widetilde{\lambda}} = (\mathcal{T}u, u) - (\mathcal{T}u_h, u_h) = (\mathcal{T}u, u) - (\mathcal{T}u_h, u_h) + (\mathcal{T}(u-u_h), u-u_h) - (\mathcal{T}(u-u_h), u-u_h).$$

Consequently

$$\frac{1}{\lambda} - \frac{1}{\widetilde{\lambda}} = 2(\mathcal{T}u, u-u_h) - (\mathcal{T}(u-u_h), u-u_h). \tag{22}$$

Now we will estimate the two terms in the right-hand side of (22):

$$\begin{aligned} 2(\mathcal{T}u, u-u_h) &= \frac{2}{\lambda}(1-(u,u_h)) = \frac{1}{\lambda}((u,u) - 2(u,u_h) + (u_h,u_h)) \\ &= \frac{1}{\lambda}(u-u_h, u-u_h) \le C\|u-u_h\|^2_{0,\Omega}. \end{aligned}$$

Having in mind that the operator $\mathcal{T}$ is bounded, we have

$$|\,(\mathcal{T}(u-u_h), u-u_h)\,| \le C\|u-u_h\|^2_{0,\Omega}.$$

Finally, from (22) and the last two inequalities we derive the estimate (21). □

As a corollary of the above result, if the FE partitions are regular, then the following superconvergent estimate holds:

$$|\lambda - \widetilde{\lambda}| \le Ch^6\|u\|^2_{3,\Omega}.$$

Nevertheless, this estimate is not very practical since the exact solution of the source problem (20) is hardly ever available. To make it useful for computational practice we need to approximate appropriately $\widetilde{\lambda}$. So, the FE solution, which corresponds to (20) is

$$a_h(\widetilde{u}_h, v_h) = (u_h, v_h), \quad \forall v_h \in \widetilde{V}_h, \tag{23}$$

where $\widetilde{V}_h$ will be made precise. Namely,

(i) $\widetilde{V}_h^{(1)}$ is constructed using a modified Hermite element (Fig. 2(b)). Recall that this element is also a nonconforming one.
(ii) $\widetilde{V}_h^{(2)}$ is a finite element space obtained by using the conforming Bell's triangle shown in Fig. 2(a) (see also [7,6]). We point out here that Bell's triangle is optimal among triangular polygonal finite elements of class C^1 (see [6, Theorem 9.3]), since dim $\mathcal{P}_K \ge 18$ for such kinds of finite elements.

It is to be noted here that the set of degrees of freedom of Bell's triangle could be chosen as follows (for any test function p):

$$p(a_i),\ 1 \le i \le 3; \quad Dp(a_i)(a_j - a_i),\ 1 \le i, j \le 3,\ i \ne j;$$

$$D^2p(a_i)(a_j - a_i)(a_k - a_i),\ 1 \le i, j, k \le 3,\ i \ne j, i \ne k.$$

In Fig. 2 • denotes evaluation at the point, ◯ denotes evaluation of first derivatives at the center of the circle and the outer circle denotes evaluation of second-order derivatives at the center of the circle.

We define

$$\widetilde{\lambda}_h = \frac{1}{(\widetilde{u}_h, u_h)},$$

where u_h and $\widetilde{u}_h$ are the solutions of (6) and (23), respectively.

Theorem 5.2. *Let the assumptions of Theorem 5.1 be fulfilled and let us use the finite element subspaces $\widetilde{V}_h^{(s)}$, $s = 1, 2$. Then the following estimate holds:*

$$|\lambda - \widetilde{\lambda}_h| \le Ch^{2(s+1)}, \quad s = 1, 2. \tag{24}$$

Proof. Using the definition of λ and $\widetilde{\lambda}_h$ we have

$$\begin{aligned}\frac{1}{\widetilde{\lambda}}-\frac{1}{\widetilde{\lambda}_h} &= (\widetilde{u},u_h)-(\widetilde{u}_h,u_h)=a(\widetilde{u},\widetilde{u})-a_h(\widetilde{u}_h,\widetilde{u}_h)\\ &= a_h(\widetilde{u}-\widetilde{u}_h,\widetilde{u})+a_h(\widetilde{u}_h,\widetilde{u})-a_h(\widetilde{u}_h,\widetilde{u}_h)\\ &= a_h(\widetilde{u}-\widetilde{u}_h,\widetilde{u})-a_h(\widetilde{u}-\widetilde{u}_h,\widetilde{u}_h),\end{aligned}$$

and consequently

$$\frac{1}{\widetilde{\lambda}}-\frac{1}{\widetilde{\lambda}_h}=a_h(\widetilde{u}-\widetilde{u}_h,\widetilde{u}-\widetilde{u}_h).$$

The continuity of the a_h-form gives

$$|\widetilde{\lambda}-\widetilde{\lambda}_h|\le C\|\widetilde{u}-\widetilde{u}_h\|^2_{2,h}.$$

By arguments similar to those in Theorem 4.3 and by standard assumptions for the smoothness of $\widetilde{u}$ we can derive the estimate as (19) (see also [6]):

$$\|\widetilde{u}-\widetilde{u}_h\|_{2,h}\le Ch^{s+1}\|\widetilde{u}\|_{2s+1,\Omega},\quad s=1,2.$$

The superconvergent estimate (24) follows from the last inequality, (21) and

$$|\lambda-\widetilde{\lambda}_h|\le|\lambda-\widetilde{\lambda}|+|\widetilde{\lambda}-\widetilde{\lambda}_h|.\quad\square$$

Now we can present a postprocessing algorithm which will give improved approximations of the eigenvalues:

1. Solve the eigenvalue problem (6) using a Z-type finite element and as a result obtain $\lambda_h\in\mathbf{R}$ and $u_h\in V_h$.
2. Solve the source problem (23) and find $\widetilde{u}_h\in\widetilde{V}_h^{(s)},\ s=1$ or $s=2$.
3. Compute $\widetilde{\lambda}=\frac{1}{(\widetilde{u}_h,u_h)}$.

The value $\widetilde{\lambda}_h$ represents a new (and better) approximation of λ.

6. Numerical results

The theoretical results are illustrated by reporting this example of a related two-dimensional biharmonic eigenvalue problem. Let Ω be a square domain:

$$\Omega:-\frac{\pi}{2}<x_i<\frac{\pi}{2},\quad i=1,2.$$

The model problem considered is

$$\Delta^2u=\lambda u\quad\text{in }\Omega,$$

$$u=\frac{\partial u}{\partial\nu}=0\quad\text{on }\partial\Omega.$$

For this problem the exact eigenvalues are not known. We use their lower and upper bounds obtained in [10] (see also [11]).

In Table 1 the results from our numerical experiments for the first four eigenvalues are given. They illustrate the computation by means of Z-type finite elements. The domain is divided into a uniform mesh with $2n^2$ isosceles triangular Z-type elements and the mesh parameter is $h=\pi/n,\ n=5,6,7,8$. As can be seen, the numerical implementation confirms the convergence asserted by Theorems 4.1–4.3. The proposed Z-type elements are appropriate, especially for computing eigenvalues. And for the approximate eigenvalues and eigenfunctions obtained by means of the proposed elements some postprocessing technique could easily be applied to improve both the rate of convergence to the exact solution and the properties of the approximate eigenfunctions.

In Table 2 the implementation of the proposed postprocessing algorithm, putting to use modified Hermite elements ($s=1$), is shown. Here we give the computational results obtained after division of the domain into $2n^2$ isosceles triangular elements, where $n=4,6,8$. Regardless of the fact that the second and the third eigenvalue are equal, the proposed postprocessing technique is put into effect for both of them. As can be seen, the postprocessing takes effect especially well on a coarse grid.

It is to be noted here that if one is mainly interested in the computing of eigenvalues, one can successfully apply Hermite elements, because they completely ensure an improvement of the approximate values. However, as regards eigenfunctions, the use of Bell's finite elements for the postprocessing procedure not only gives a better improvement compared to the use of Hermite elements, but also improves the smoothness of the approximate eigenfunctions.

Table 1
Eigenvalues computed by means of Z-type finite elements.

Number of FEs	λ_1	λ_2	λ_3	λ_4
50	14.8023	64.1267	70.4634	162.3236
72	14.0465	60.3152	65.3925	156.5718
98	13.6470	58.6170	62.5207	153.7372
128	13.4721	57.7318	61.3971	152.4870
Bounds:				
Lower	13.2820	55.2400	55.2400	120.0070
Upper	13.3842	56.5610	56.5610	124.0740

Table 2
Eigenvalues computed by means of Z-type finite elements (FEM) and their improvements as a result of the postprocessing procedure (PP) achieved by means of Hermite finite elements.

Number of FEs		λ_1	λ_2	λ_3	λ_4
32	FEM	15.0780	67.1107	75.2068	168.1998
	PP	14.8992	63.2144	71.7732	153.0853
72	FEM	14.0465	60.3152	65.3925	156.5718
	PP	13.9459	58.7350	63.3821	149.8718
128	FEM	13.4721	57.7318	61.3971	152.4870
	PP	13.4152	56.1393	58.9924	148.6162
Bounds:	Lower	13.2820	55.2400	55.2400	120.0070
	Upper	13.3842	56.5610	56.5610	124.0740

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