A CONSTRUCTIVE PROOF OF HELMHOLTZ'S THEOREM

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[Received 11 December 2018. Revise 28 July 2019. Accepted 05 August 2019]

Summary

It is a known result that any vector field $\boldsymbol{u}$ that is locally Hölder continuous on an arbitrary open set $\Omega \subset \mathbb{R}^3$ can be written on Ω as the sum of a gradient and a curl. Should Ω be unbounded, no conditions are required on the behaviour of $\boldsymbol{u}$ at infinity. We present a direct, self-contained proof of this theorem that only uses elementary techniques and has a constructive character. It consists in patching together local solutions given by the Newtonian potential that are then modified by harmonic approximations—based on solid spherical harmonics—to assure convergence near infinity for the resulting series.

1. Introduction

The classical Helmholtz theorem (**1**, **2**) on the decomposition of vector fields states that, under suitable conditions, a three-dimensional (3D) vector function $\boldsymbol{u}$ defined on a domain $D \subset \mathbb{R}^3$ can be written as the sum of a gradient and a curl, that is, that there exist a vector potential $\boldsymbol{A}$ and a scalar potential ψ such that

$$\boldsymbol{u} = \nabla\psi + \operatorname{curl}\boldsymbol{A}, \tag{1.1}$$

on D. The result is essential in elasticity and fluids mechanics. See the introduction of (**3**) for a detailed account of the history of the theorem and its applicability. We just mention that a decomposition of this kind is used in the incompressible Navier–Stokes equation to do away with the pressure (**4**, Section 1.3). The theory has evolved to establish weak analogues of (1.1) in L^p spaces, known as Helmholtz–Weyl decompositions (**5**, Section III.1), and it has been an active research topic for years; see, for instance, (**6**, **7**).

We wish to stress that the problem (1.1) is dealt with in this article without imposing any condition on ∂D. See (**8**, **9**) for classical references on the Helmholtz decomposition with boundary conditions.

We are concerned with the classical statement—the one involving C^k solutions—under the weakest possible conditions. The vector Poisson equation

$$\Delta\boldsymbol{v} = \boldsymbol{u}, \tag{1.2}$$

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Q. Jl Mech. Appl. Math, Vol. 72. No. 4
Advance Access publication 4 September 2019. doi:10.1093/qjmam/hbz016

together with the equality

$$\Delta \boldsymbol{v} = \nabla\,(\operatorname{div} \boldsymbol{v}) - \operatorname{curl}\,(\operatorname{curl} \boldsymbol{v}), \tag{1.3}$$

shows that the problem boils down to solving the scalar counterpart of (1.2). For the case where both the domain D and the scalar function f are regular enough, the Newtonian potential formula

$$v(\boldsymbol{r}) = -\frac{1}{4\pi}\int_D \frac{f(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,d\boldsymbol{r}' \tag{1.4}$$

provides a solution $v \in C^2(D)$ to the scalar Poisson equation

$$\Delta v = f \tag{1.5}$$

as long as f verifies

$$f(\boldsymbol{r}) = O\left(\frac{1}{|\boldsymbol{r}|^{2+\varepsilon}}\right), \qquad |\boldsymbol{r}| \to \infty, \tag{1.6}$$

for some $\varepsilon > 0$. Obviously, the condition (1.6) is necessary for the integral in (1.4) to converge only when the domain D is unbounded. Otherwise, the representation given by the Newtonian potential constitutes the final solution. So, we will consider henceforth that D is unbounded.

Since the growth condition (1.6) is strongly linked to the representation (1.4), the first question that arises is to what extent it can be weakened. Blumenthal (**10**) was the first to use the Green-type function

$$-\frac{1}{4\pi}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} - \frac{1}{|\boldsymbol{r}'|}\right)$$

instead of the Newtonian kernel to prove that, in order to guarantee a solution of the Poisson equation (1.5), it is sufficient for f to verify

$$f(\boldsymbol{r}) = O\left(\frac{1}{|\boldsymbol{r}|^{1+\varepsilon}}\right), \qquad |\boldsymbol{r}| \to \infty,$$

for some $\varepsilon > 0$. Subsequently, Gurtin (**11**), interested in the applications of Helmholtz's decomposition to elasticity, refined Blumenthal's method to obtain the weaker condition

$$\boldsymbol{u}(\boldsymbol{r}) = \boldsymbol{c} + O\left(\frac{1}{|\boldsymbol{r}|^{\varepsilon}}\right), \qquad |\boldsymbol{r}| \to \infty, \quad \boldsymbol{c} \in \mathbb{R}^3, \quad \varepsilon > 0,$$

for the Helmholtz decomposition (1.1) to take place.

In parallel, the distribution theory introduced by Sobolev and Schwartz allowed Malgrange (**12**) and Hörmander (**13**), among others, to prove very general results on the existence of weak solutions for linear partial differential equations with constant coefficients. From those results it follows that the Poisson equation (1.5) admits C^2 solutions regardless of the behaviour of f at ∞. To state this result accurately, we need to say a few words about the regularity required for the function f.

It turns out that the Poisson equation may not possess classical solutions if f is merely continuous, and it has been known since long (**14**, Chapter V) that the proper condition for f is the Hölder continuity.

Definition 1.1. *Let $\Omega \subset \mathbb{R}^3$ be an arbitrary open set. We say that the function f is locally Hölder continuous on Ω with exponent $\alpha \in (0, 1]$ if, for each open ball B with $\overline{B} \subset \Omega$, there exists a constant $L_B > 0$ such that*

$$|f(\boldsymbol{r}) - f(\boldsymbol{s})| \leq L_B \, |\boldsymbol{r} - \boldsymbol{s}|^{\alpha} \tag{1.7}$$

for all $\boldsymbol{r}, \boldsymbol{s} \in \overline{B}$, and if so, we write $f \in C^{0,\alpha}(\Omega)$. If $f \in C^k(\Omega)$, $k \geq 1$, and $f^{(k)} \in C^{0,\alpha}(\Omega)$, then we write $f \in C^{k,\alpha}(\Omega)$. Obviously, we say that a vector function $\boldsymbol{u}$ belongs to the class $C^{k,\alpha}(\Omega)$ if each component function of $\boldsymbol{u}$ belongs to $C^{k,\alpha}(\Omega)$.

For instance, the function $f(\boldsymbol{r}) = 1/\log|\boldsymbol{r}|$ is not Hölder continuous on a neighbourhood of the origin for any $\alpha \in (0, 1]$. Thus, the Hölder continuity is stronger than continuity but weaker than differentiability.

Now, formula (1.3) and the Malgrange–Hörmander theory implies the following result, see (**13**, Corollaries 3.5.3 and 3.7.1). See also (**15**, Theorem 2.28), where is proven that a distributional solution of the Laplace equation is actually a regular function.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^3$ be an arbitrary open set and $\boldsymbol{u} \in C^{k,\alpha}(\Omega)$, $k \geq 0$. Then there exist $\boldsymbol{A}, \psi \in C^{k+1,\alpha}(\Omega)$ such that the decomposition (1.1) takes place on Ω.*

However, the proof of Theorem 1.2 given in (**13**) requires the use of the heavy machinery of functional analysis and distribution theory and Theorem 1.2 seems not to be so widely known in the community interested in Helmholtz's theorem as it would deserve to be. Except for (**3**) and (**16**), where using tools of classical analysis partial proofs of Theorem 1.2 are presented, we have found no references to Hörmander's work in the literature related to the Helmholtz decomposition.

The purpose of this article is to give an accessible, self-contained and relatively easy proof of Helmholtz's theorem in its most general version given by Theorem 1.2. This is done by building harmonic approximations based on solid spherical harmonics, for which we were partly inspired by Malgrange's work (**12**). The proof is constructive in the sense that all the objects that appear can be explicitly constructed and used to approximate the solution, but the proof does not give a closed representation of the decomposition, since it involves quite a few approximation processes. The main idea is simple and we outline it below. As mentioned, it is sufficient to find a regular solution to (1.5).

Suppose that f belongs to $C^{k,\alpha}(\Omega)$, $k \geq 0$, and Ω is an arbitrary open set of $\mathbb{R}^3$. By means of a *partition of unity* (**17**, Theorem 6.20) we decompose the function f so that we can write

$$f = \sum_{n=1}^{\infty} f_n,$$

where each of the functions f_n has compact support and is of the same regularity as f. Then there exist functions $v_n \in C^{k+2,\alpha}(\Omega)$, $n \in \mathbb{N}$, given by the formula (1.4), that verify

$$\Delta v_n = f_n$$

on Ω. In general, the series

$$\tilde{v} = \sum_{n=1}^{\infty} v_n$$

will not be convergent, so there is need to adjust it. This is achieved by approximating each of the functions v_n on compact sets conveniently chosen by means of a harmonic function h_n. The approximation involves several technically delicate steps.

As the functions h_n cannot be detected by the Laplacian operator, the function

$$v = \sum_{n=1}^{\infty} (v_n - h_n)$$

turns out to be convergent and is a solution to (1.5) on Ω that belongs to the class $C^{k+2,\alpha}(\Omega)$. Thus, the following result may be proven.

THEOREM 1.3. *Let $\Omega \subset \mathbb{R}^3$ be an arbitrary open set and $f \in C^{k,\alpha}(\Omega)$, $k \geq 0$. Then there exists $v \in C^{k+2,\alpha}(\Omega)$ such that $\Delta v = f$ on Ω.*

We wish to underline that Theorem 1.3 remains valid for arbitrary open subsets of $\mathbb{R}^n$, $n \geq 2$, and the proof runs along the same lines as those followed here. For the proof in the general case, the Newtonian potential must be replaced by the logarithmic potential if $n = 2$ or the corresponding Riesz potential when $n > 3$.

Theorem 1.2 straightforwardly follows from the formula (1.3) and Theorem 1.3. In the next section, we collect several known results about both the Newtonian potential and approximation theory as well as some technicalities that we will need to prove Theorem 1.3. The proof itself is given in section 3.

2. Preliminary results

The Hölder regularity of the Newtonian potential (1.4), very often expressed through the Schauder estimates, has been known since the beginning of the past century. We state below only what we need. For a reference, see (**18**, Chapter 2, Proposition 9.1).

PROPOSITION 2.1. *Let $\Omega \subset \mathbb{R}^3$ be an arbitrary open set. Let f be a function with compact support contained in Ω. Suppose that $f \in C^{k,\alpha}(\Omega)$, with $k \geq 0$. Then the Newtonian potential*

$$v_f(\boldsymbol{r}) = -\frac{1}{4\pi} \int_{\Omega} \frac{f(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, d\boldsymbol{r}'$$

belongs to the class $C^{k+2,\alpha}(\Omega)$.

The next ingredient in our proof is the use of partitions of unity. This is a standard tool often used in analysis to obtain global results from local properties. The proof of the next lemma may be found in (**17**, Theorem 6.20).

LEMMA 2.2. *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary open set and Γ a family of open sets whose union is Ω. Then there exists a sequence of function $\{\phi_n\}$ with the following properties:*

(i) All the functions ϕ_n are non-negative and belong to $C^{\infty}(\Omega)$.
(ii) The support of each ϕ_n is a compact set contained in some element of Γ.
(iii) For all $\boldsymbol{r} \in \Omega$, it holds that

$$\sum_{n=1}^{\infty} \phi_n(\boldsymbol{r}) = 1.$$

(iv) For each compact subset K of Ω, there exist a natural number N and an open set $W \supset K$ such that

$$\sum_{n=1}^{N} \phi_n(\boldsymbol{r}) = 1, \quad \boldsymbol{r} \in W.$$

DEFINITION 2.3. *A sequence of functions $\{\phi_n\}$ as it appears in the statement of Lemma 2.2 is called a partition of unity in Ω subordinated to the open cover Γ of Ω.*

For our purposes, one of the most interesting features of Lemma 2.2 is that the sequence of functions $\{\phi_n\}$ can be explicitly constructed. Indeed, a careful reading of the proof in (**17**) reveals that everything relies on the existence of a C^∞ plateau function, that is, a function ψ such that $\psi(\boldsymbol{r}) = 1$ for all $\boldsymbol{r} \in B_1$ and $\psi(\boldsymbol{r}) = 0$ for all $\boldsymbol{r} \notin B_2$, where B_1 and B_2 are arbitrary concentric balls with $\overline{B}_1 \subset B_2$. There are many ways to carry out such a construction. Perhaps one of the simplest is to consider the function

$$\psi(\boldsymbol{r}) = \frac{g(r_2^2 - |\boldsymbol{r}|^2)}{g(r_2^2 - |\boldsymbol{r}|^2) + g(|\boldsymbol{r}|^2 - r_1^2)}, \quad \boldsymbol{r} \in \mathbb{R}^n,$$

where r_1 and r_2 are the radii of B_1 and B_2, respectively, the function g is given by

$$g(x) = \begin{cases} 0, & \text{if } x < 0, \\ e^{-1/x}, & \text{if } x \geq 0, \end{cases}$$

and we have supposed that the two balls are centred at the origin.

Note that it follows from the properties (iii) and (iv) in Lemma 2.2 that every point of Ω has a neighbourhood which intersects the supports of only finitely many ϕ_n. Hence a partition of unity is often called locally finite.

The last and most important ingredient in our approach to Theorem 1.3 is that of approximation by harmonic functions. Most of the results we present on this subject matter may be found in (**19**, Chapter 1) or (**20**, Sections 2.6 and 7.9). We have opted for sketching the proofs in order to make the paper as self-contained as possible, to check that the approximations are constructive, and due to the fact that the main result we use is given in (**19**, Theorem 1.10) with much more generality than what is needed here. So, we have decided to provide an adapted version in Proposition 2.7. We limit ourselves to 3D objects, although the results are valid in $\mathbb{R}^n$ with slight modifications here and there.

For reasons that will become apparent later, we specifically need to approximate a function that is harmonic on a compact set K by functions that are harmonic on a larger open set $\Omega \supset K$. In doing so, an essential principle is that the approximations must have enough singularities to mimic the singularities of the function to be approximated. This translates into a topological condition regarding K and Ω that we will specify below. For the time being, we deal with a particular but important case that provides the basics to building the harmonic approximation.

It is well known that a harmonic function on an annulus can be developed in terms of homogeneous harmonic polynomials, the so-called solid spherical harmonics. To be more precise, let H_k denotes the vector space of all homogeneous harmonic polynomials of degree k on $\mathbb{R}^3$, $k \geq 0$, and set

$$A = \left\{ \boldsymbol{r} \in \mathbb{R}^3 \,:\, 0 \leq r_1 < |\boldsymbol{r} - \boldsymbol{a}| < r_2 \leq +\infty \right\}.$$

Then any function v harmonic on the annulus A can be written as

$$v(\boldsymbol{r}) = \sum_{k=0}^{\infty} Y_k(\boldsymbol{r}-\boldsymbol{a}) + \sum_{k=0}^{\infty} \frac{Z_k(\boldsymbol{r}-\boldsymbol{a})}{|\boldsymbol{r}-\boldsymbol{a}|^{2k+1}}, \quad \boldsymbol{r} \in A, \tag{2.1}$$

where $Y_k, Z_k \in H_k$, $k \geq 0$. The functions Y_k, Z_k are obtained integrating v on the boundary of an annulus $A' \subset A$, see (**21**, Theorem 2.42). The convergence of the first series of (2.1) is absolute and uniform of compact subsets of $\{\boldsymbol{r} \in \mathbb{R}^3 : |\boldsymbol{r}-\boldsymbol{a}| < r_2\}$, whereas that of the second one is absolute and uniform on compact subsets of $\{\boldsymbol{r} \in \mathbb{R}^3 : r_1 < |\boldsymbol{r}-\boldsymbol{a}|\}$; see (**20**, Theorem 2.5.3) for details. The Laurent-type expansion (2.1) allows us to prove the following lemma on harmonic approximation.

LEMMA 2.4. *Let $B_1, B_2 \subset \mathbb{R}^3$ be balls centred at $\boldsymbol{a}$ with radius r_1 and r_2, respectively, and $r_1 < r_2$. Let v a harmonic function on $\mathbb{R}^3 \setminus \overline{B}_1$. Then, for each $\varepsilon > 0$, there exists a harmonic function h on $\mathbb{R}^3 \setminus \{\boldsymbol{a}\}$ such that $|v-h| < \varepsilon$ on $\mathbb{R}^3 \setminus B_2$.*

PROOF. It follows from the Laurent expansion (2.1) that the function v has the representation

$$v(\boldsymbol{r}) = v_1(\boldsymbol{r}) + \sum_{k=0}^{\infty} \frac{Z_k(\boldsymbol{r}-\boldsymbol{a})}{|\boldsymbol{r}-\boldsymbol{a}|^{2k+1}}, \quad \boldsymbol{r} \in \mathbb{R}^3 \setminus \overline{B}_1, \tag{2.2}$$

where v_1 is harmonic in $\mathbb{R}^3$. We set $r = (r_1 + r_2)/2$. It is clear that the series (2.2) is uniformly and absolutely convergent on the sphere $S = \left\{\boldsymbol{r} \in \mathbb{R}^3 : |\boldsymbol{r}-\boldsymbol{a}| = r\right\}$. So, there exists $k_0 \geq 0$ such that

$$\sum_{k=k_0+1}^{\infty} \frac{|Z_k(\boldsymbol{r}-\boldsymbol{a})|}{|\boldsymbol{r}-\boldsymbol{a}|^{2k+1}} = \sum_{k=k_0+1}^{\infty} \frac{|Z_k(\boldsymbol{r}-\boldsymbol{a})|}{r^{2k+1}} < \varepsilon, \quad \boldsymbol{r} \in S. \tag{2.3}$$

Then, we consider the function

$$h(\boldsymbol{r}) = v_1(\boldsymbol{r}) + \sum_{k=0}^{k_0} \frac{Z_k(\boldsymbol{r}-\boldsymbol{a})}{|\boldsymbol{r}-\boldsymbol{a}|^{2k+1}},$$

which is harmonic in $\mathbb{R}^3 \setminus \{\boldsymbol{a}\}$. Besides, if $|\boldsymbol{r}-\boldsymbol{a}| > r_2$ we have

$$|v(\boldsymbol{r}) - h(\boldsymbol{r})| = \left|\sum_{k=k_0+1}^{\infty} \frac{Z_k(\boldsymbol{r}-\boldsymbol{a})}{|\boldsymbol{r}-\boldsymbol{a}|^{2k+1}}\right| = \left|\sum_{k=k_0+1}^{\infty} Z_k\left(\frac{r(\boldsymbol{r}-\boldsymbol{a})}{|\boldsymbol{r}-\boldsymbol{a}|}\right) \frac{1}{r^{2k+1}} \frac{r^{k+1}}{|\boldsymbol{r}-\boldsymbol{a}|^{k+1}}\right|$$

$$< \sum_{k=k_0+1}^{\infty} \left|Z_k\left(\frac{r(\boldsymbol{r}-\boldsymbol{a})}{|\boldsymbol{r}-\boldsymbol{a}|}\right)\right| \frac{1}{r^{2k+1}} \left(\frac{r}{r_2}\right)^{k+1} < \varepsilon,$$

where we have used (2.3) in the last step. ■

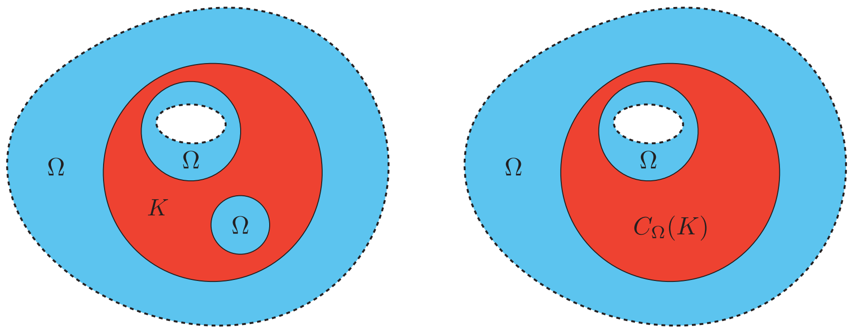

Fig. 1 Sets K and Ω (left), and $C_\Omega(K)$ (right)

In order to deal with the general case we need to delve into the topological structure of K and Ω.

DEFINITION 2.5. *Let Ω be an arbitrary open set of $\mathbb{R}^3$. We say that $E \subset \Omega$ is Ω-bounded if $\overline{E}$ is a compact subset of Ω.*

Let K be an arbitrary compact subset of Ω. We define the polynomial convex hull of K with respect to Ω as the union of K with all the Ω-bounded components of $\Omega \setminus K$. Such a set is denoted by $C_\Omega(K)$.

Figure 1 displays the planar section of an example showing K, Ω and $C_\Omega(K)$. Roughly speaking, the polynomial convex hull of a compact set with respect to Ω is the very compact set together with all of its 'holes' that do not contain 'holes' of Ω. The following lemma makes this statement precise.

LEMMA 2.6. *Let K be a compact subset of an arbitrary open set $\Omega \subset \mathbb{R}^3$. For each component W of $\Omega \setminus K$, let H be the component of $\mathbb{R}^3 \setminus K$ that contains W. Then, the following assertions are equivalent.*

(i) W is Ω-bounded.
(ii) W is bounded and $\partial W \subset K$.
(iii) W is bounded and $W = H$.

PROOF. Given any set $A \subset \mathbb{R}^3$, it follows from the definition of a connected component that the only non-empty connected sets that are at once open and closed sets of A are its connected components. Besides, if A is open, then the components of A are also open subsets of $\mathbb{R}^3$, see (22, Chapter V, Theorem 4.2).

Obviously, (ii) implies (i), since then $\overline{W}$ is a compact set contained in Ω. Let us prove that (i) implies (ii). Let W be a Ω-bounded component of $\Omega \setminus K$. In particular W is bounded. Besides, as mentioned, W is an open set. Therefore, if $\boldsymbol{r} \in \partial W$ then $\boldsymbol{r} \notin W$. We know from hypotheses that $\partial W \subset \Omega$, that is, $\boldsymbol{r} \in \Omega$. We claim that $\boldsymbol{r} \in K$. Otherwise, $\boldsymbol{r} \in \Omega \setminus K$, but as $\boldsymbol{r}$ does not belong to W, necessarily there exists another component W' of $\Omega \setminus K$ that is also an open set and that contains $\boldsymbol{r}$. As $\boldsymbol{r}$ is in the interior of W', there exists a ball B centred at $\boldsymbol{r}$ with radius sufficiently small so as to be completely contained in W'. But B also contains points of W, since $\boldsymbol{r} \in \partial W$. Thus $W \cap W' \neq \emptyset$, which is absurd. Therefore, we have proven that $\partial W \subset K$.

Next, we prove that (ii) implies (iii). If $\partial W \subset K$, we can write

$$W = \overline{W} \setminus K = \overline{W} \cap \left(\mathbb{R}^3 \setminus K\right).$$

Thus, W is a connected open set contained in $\mathbb{R}^3 \setminus K$, that, at the same time, is closed in the topology of $\mathbb{R}^3 \setminus K$. Necessarily, W is a connected component of $\mathbb{R}^3 \setminus K$, and, therefore, coincides with H.

Finally, it is easy to see that (iii) implies (ii). Indeed, suppose that $W = H$ and consider $\boldsymbol{r} \in \partial W$. As before, $\boldsymbol{r} \notin W$ and, therefore, $\boldsymbol{r} \notin H$. If $\boldsymbol{r} \notin K$, then $\boldsymbol{r}$ would have to belong to another component of $\mathbb{R}^3 \setminus K$, say H'. As $\boldsymbol{r}$ is in the interior of H', there exists a ball B centred at $\boldsymbol{r}$ with radius sufficiently small so as to be completely contained in H'. But B also contains points of $W = H$, since $\boldsymbol{r} \in \partial W$. Thus $H \cap H' \neq \emptyset$, which is again absurd. Then $\partial W \subset K$. ■

It is clear that $C_{\mathbb{R}^3}(K) = \mathbb{R}^3 \setminus U$, where U is the unbounded connected component of $\mathbb{R}^3 \setminus K$. Hence, the set $C_{\mathbb{R}^3}(K)$ is compact. On the other hand, note that the relation between the components of $\Omega \setminus K$ and those of $\mathbb{R}^3 \setminus K$ is onto, since, if H is a connected component of $\mathbb{R}^3 \setminus K$, then, following the same arguments as in the proof of Lemma 2.6, we obtain $\partial H \subset K \subset \Omega$ and $H \cap \Omega \neq \emptyset$. Therefore, as a consequence of Lemma 2.6, we have $C_\Omega(K) = C_{\mathbb{R}^3}(K) \setminus V$, where V is the union of all the components of $\Omega \setminus K$ that are not Ω-bounded. In particular, $C_\Omega(K)$ is always a compact set.

Now, suppose that v is a harmonic function on a neighbourhood of the compact set K shown in Fig. 1. Should there exist a sequence $\{h_n\}$ of harmonic functions on Ω converging uniformly to v on K, the sequence $\{h_n\}$ would also converge, because of the maximum principle, on the Ω-bounded components of $\Omega \setminus K$, defining thus a harmonic extension of v, which may not be possible depending on the particular function v considered. One way to ensure the existence of the harmonic approximation is requiring that there do not exist Ω-bounded components of $\Omega \setminus K$. That is the idea behind the following result to which Lemma 2.4 serves as a stepping stone.

Proposition 2.7. *Let Ω be an arbitrary open set of $\mathbb{R}^3$ and v a function harmonic on a neighbourhood of the compact set $K \subset \Omega$. Suppose that $K = C_\Omega(K)$. Then, for each $\varepsilon > 0$, there exists a function h harmonic on Ω such that $|f - h| < \varepsilon$ on K.*

Proof. Let $U \supset K$ be the open set in which v is harmonic. We may suppose that $U \subset \Omega$. As K is a compact set, it is possible to find a finite number of balls whose union W satisfies $K \subset W \subset \overline{W} \subset U$. We can apply the divergence theorem or any of its allied results to v and W, since the boundary set ∂W is regular enough to do that, see (**14**, Chapter IV) or (**23**, Theorem 7.8.5). Then, using Green's third identity (**14**, Section VIII.4), we have

$$v(\boldsymbol{r}) = \frac{1}{4\pi} \iint_{\partial W} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \frac{\partial v}{\partial \boldsymbol{n}}(\boldsymbol{r}')\, d\sigma(\boldsymbol{r}') - \frac{1}{4\pi} \iint_{\partial W} v(\boldsymbol{r}') \frac{\partial}{\partial \boldsymbol{n}} \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\right) d\sigma(\boldsymbol{r}'), \quad \boldsymbol{r} \in K,$$

where $d\sigma$ stands for surface area measure and $\boldsymbol{n}$ denotes the outer unit normal to ∂W. The above integral can be approximated uniformly on K by a Riemann sum, where the function

$$f(\boldsymbol{r}) = \frac{\partial}{\partial \boldsymbol{n}} \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\right)$$

can be in turn uniformly approximated on K by a difference quotient for fixed $\boldsymbol{r}'$. In conclusion, given $\varepsilon > 0$, there exists a function h of the form

$$h(\boldsymbol{r}) = \sum_{m=1}^{M} \frac{a_m}{|\boldsymbol{r} - \boldsymbol{r}_m|}, \qquad \boldsymbol{r}_m \in U \setminus K,\ a_m \in \mathbb{R},\ m = 1, \ldots, M, \tag{2.4}$$

such that $|v - h| < \varepsilon$ on K. Then it is sufficient to prove that any function of the form (2.4) can be uniformly approximated on K by functions that are harmonic in Ω, and we can limit ourselves to the approximation of the function

$$h_0(\boldsymbol{r}) = \frac{a_0}{|\boldsymbol{r} - \boldsymbol{r}_0|}, \qquad \boldsymbol{r}_0 \in \Omega \setminus K,\ a_0 \in \mathbb{R}.$$

The idea is to push the pole $\boldsymbol{r}_0$ away from Ω with the aid of Lemma 2.4. Indeed, the point $\boldsymbol{r}_0$ necessarily belongs to some connected component W of $\Omega \setminus K$. So, we distinguish two cases according to whether W is bounded or not. Suppose first that W is bounded. Due to Lemma 2.6, we know that there is a point $\boldsymbol{a} \notin \Omega$ in the component H of $\mathbb{R}^3 \setminus K$ that contains W, since, by hypotheses, there do not exist Ω-bounded components of $\Omega \setminus K$. Then there exist a finite number of balls $B_k \subset H$, $k = 1, \ldots, m$, connecting $\boldsymbol{r}_0$ and $\boldsymbol{a}$. That is, if we call $\boldsymbol{a}_k$ the centre of B_k, $k = 1, \ldots, m$, then it holds that $\boldsymbol{r}_0 \in B_1$, $\boldsymbol{a}_k \in B_{k+1}$, $k = 1, \ldots, m-1$, and $\boldsymbol{a}_m = \boldsymbol{a}$. By repeated application of Lemma 2.4, we obtain functions h_k harmonic in $\mathbb{R}^3 \setminus \{\boldsymbol{a}_k\}$, $k = 1, \ldots, m$, such that

$$|h_{k-1}(\boldsymbol{r}) - h_k(\boldsymbol{r})| < \frac{\varepsilon}{m}, \quad \boldsymbol{r} \in \mathbb{R}^3 \setminus B_k, \quad k = 1, \ldots, m.$$

As all the sets $\mathbb{R}^3 \setminus B_k$ contain K, we have proven

$$|h_0(\boldsymbol{r}) - h_m(\boldsymbol{r})| < \varepsilon, \quad \boldsymbol{r} \in K,$$

where h_m is harmonic on $\mathbb{R}^3 \setminus \{\boldsymbol{a}\}$, and, therefore, on Ω.

Suppose now that W is unbounded. Then there exists a sequence of balls $\{B_k\}$ with radii less than 1 and all of them contained in W connecting $\boldsymbol{r}_0$ and ∞. That is, if we call $\boldsymbol{a}_k$ the centre of B_k, $k \in \mathbb{N}$, then it holds that $\boldsymbol{r}_0 \in B_1$, $\boldsymbol{a}_k \in B_{k+1}$, $k \in \mathbb{N}$, and $\lim_{k\to\infty} |\boldsymbol{a}_k| = \infty$. Again, by repeated application of Lemma 2.4, we obtain functions h_k harmonic in $\mathbb{R}^3 \setminus \{\boldsymbol{a}_k\}$, $k = 1, \ldots, m$, such that

$$|h_{k-1}(\boldsymbol{r}) - h_k(\boldsymbol{r})| < \frac{\varepsilon}{2^k}, \quad \boldsymbol{r} \in \mathbb{R}^3 \setminus B_k, \quad k \in \mathbb{N}.$$

Therefore, the sequence $\{h_k\}$ is a Cauchy sequence uniformly on compact subsets of $\mathbb{R}^3$ and then converges to a function h that is harmonic on $\mathbb{R}^3$ (**20**, Theorem 1.5.1). Note that

$$|h_0(\boldsymbol{r}) - h_k(\boldsymbol{r})| < \varepsilon \left(\frac{1}{2} + \cdots + \frac{1}{2^k}\right), \quad \boldsymbol{r} \in K, \quad k \in \mathbb{N},$$

whence the result follows by taking limits as k tends to ∞. ■

Finally, let us see that, given an arbitrary open set $\Omega \subset \mathbb{R}^3$, it is always possible to find an increasing sequence of compact sets whose union is Ω and whose properties allow us both to construct a partition of unity in Ω and to apply Proposition 2.7 on harmonic approximation.

LEMMA 2.8. *Let Ω be an arbitrary open set of $\mathbb{R}^3$. Then there exists a sequence of compact sets $\{K_n\}_{n\in\mathbb{N}}$ whose union is Ω verifying the following properties:*

(i) For each $n \in \mathbb{N}$, the set K_n is contained in the interior of K_{n+1}.
(ii) Every compact set of Ω is contained in some K_n.
(iii) $K_n = C_\Omega(K_n)$ for each $n \in \mathbb{N}$.

PROOF. We may suppose that $\Omega \neq \mathbb{R}^3$. Otherwise, the result is trivial. Then, we have $\partial\Omega \neq \emptyset$ and, for each natural number n, we define

$$K_n = \overline{B(\mathbf{0}, n)} \cap \left\{\boldsymbol{r} \in \Omega \, : \, \mathrm{dist}(\boldsymbol{r}, \partial\Omega) \geq 1/n\right\},$$

where $B(\boldsymbol{s}, r)$ denotes the open ball with centre $\boldsymbol{s}$ and radius r. Consider the first natural number N for which K_N has non-empty interior. We will reason from N on; at the end of the proof we can rename the whole sequence so that it begins at $n = 1$.

For each $n \geq N$, the set K_n is a compact set. It is very easy to see that

$$\mathbb{R}^3 \setminus \left\{\boldsymbol{r} \in \Omega \, : \, \mathrm{dist}(\boldsymbol{r}, \partial\Omega) \geq 1/n\right\} = \bigcup_{\boldsymbol{a}\notin\Omega} B(\boldsymbol{a}; 1/n).$$

Then

$$V_n = \mathbb{R}^3 \setminus K_n = B(\infty, n) \cup \bigcup_{\boldsymbol{a}\notin\Omega} B(\boldsymbol{a}; 1/n), \tag{2.5}$$

where $B(\infty, n) = \left\{\boldsymbol{r} \in \mathbb{R}^3 \, : \, |\boldsymbol{r}| > n\right\}$. Consequently, we have the equality

$$\Omega = \bigcup_{n=N}^{\infty} K_n.$$

Denote by U_n the interior of K_n, $n \geq N$. Basic standard arguments provide that $K_n \subset U_{n+1}$ for all $n \geq N$, which proves the part (i) and, besides, it holds that

$$\Omega = \bigcup_{n=N}^{\infty} U_n.$$

As a consequence, if K is a compact subset of Ω, then there exists $N_1 > N$ such that

$$K \subset U_N \cup \cdots \cup U_{N_1} \subset K_{N_1},$$

which proves (ii).

Finally, let us prove (iii). This last property plays a key role in the proof of Theorem 1.3. It is enough to prove that there do not exist connected components Ω-bounded of $\Omega \setminus K_n$. Let W be a connected component of $\Omega \setminus K_n$ and let H be the connected component of $V_n = \mathbb{R}^3 \setminus K_n$ that contains it. Should W be Ω-bounded, it follows from Lemma 2.6 that $W = H$. This will lead us to a contradiction. In the first place, as $\overline{W}$ is a compact subset of Ω, necessarily $W = H$ is bounded. Then, see (2.5), H cannot contain $B(\infty, n)$ and it has to contain some ball of the type $B(\boldsymbol{a}, 1/n)$, with $\boldsymbol{a} \notin \Omega$. That is, $H = W$ contains a point that does not belong to Ω, which is absurd, since $W \subset \Omega$. ■

3. Proof

We are now in the position to prove Theorem 1.3 for which most of the work has already been done.

First, given Ω, take a sequence of compact sets $\{K_n\}_{n\in\mathbb{N}}$ as in Lemma 2.8. For each $n \in \mathbb{N}$, we denote by U_n the interior of K_n. We may additionally suppose, without loss of generality, that $U_1 \neq \emptyset$, as is done in the proof of Lemma 2.8.

On the basis of the increasing sequences of compact and open sets $\{K_n\}$ and $\{U_n\}$, we build another one in the following way. Let $V_1 = U_2$ and $V_n = U_{n+1} \setminus K_{n-1}$, $n \geq 2$. By using the fact that $K_{n-1} \subset U_n$, $n \geq 2$, it is easy to prove by induction the equality

$$U_{n+1} = \bigcup_{k=1}^{n} V_k, \quad n \in \mathbb{N}.$$

Then

$$\Omega = \bigcup_{n=1}^{\infty} V_n, \tag{3.1}$$

where each V_n is a bounded open set. We then consider, according to Lemma 2.2, a partition of unity $\{\phi_k\}$ in Ω subordinated to the open cover (3.1). Note that, as none of the open sets V_n is redundant, there exists, for each V_n, at least one function ϕ_k whose support is contained in V_n, whereas, as the partition is locally finite and all the V_n are relatively compact, there are at most finitely many ϕ_k for each V_n. If we add the functions of each of such groups and consider each sum as a one single function, we may suppose, renaming the whole sequence if need be, that the support of each ϕ_n is contained in V_n for each $n \in \mathbb{N}$.

Now, set $f_n = \phi_n f$, $n \in \mathbb{N}$. It holds that

$$f(\boldsymbol{r}) = \sum_{n=1}^{\infty} f_n(\boldsymbol{r}), \quad \boldsymbol{r} \in \Omega. \tag{3.2}$$

For each $n \in \mathbb{N}$, the function f_n belongs to the class $C^{k,\alpha}(\Omega)$, $k \geq 0$, and has compact support contained in V_n. It follows from Proposition 2.1 that there exist functions $v_n \in C^{k+2,\alpha}(\Omega)$ such that

$$\Delta v_n(\boldsymbol{r}) = f_n(\boldsymbol{r}), \quad \boldsymbol{r} \in \Omega, \quad n \in \mathbb{N}.$$

Therefore, if $n \geq 2$, each one of the functions v_n is harmonic on a neighbourhood of K_{n-1}. It follows from Proposition 2.7 and part (iii) of Lemma 2.8 that there exist functions h_n harmonic on Ω such that

$$\|v_n - h_n\|_{K_{n-1}} \leq \frac{1}{2^{n-1}}, \quad n \geq 2. \tag{3.3}$$

Let us define on Ω the function

$$v = v_1 + \sum_{n=2}^{\infty} (v_n - h_n). \tag{3.4}$$

We need to check that the series defining v is convergent. Consider a fixed compact set $K \subset \Omega$. It follows from properties (i) and (ii) in Lemma 2.8 that there exists $N \in \mathbb{N}$ such that $K \subset K_n$ for all

$n \geq N$. If we write

$$\sum_{n=2}^{\infty}(v_n - h_n) = \sum_{n=2}^{N}(v_n - h_n) + \sum_{n=N+1}^{\infty}(v_n - h_n), \tag{3.5}$$

it is clearly seen that the convergence of the series is equivalent to that of the second series in the right-hand side of (3.5). Now, if we use the inequalities (3.3), we arrive at the estimate

$$\sum_{n=N+1}^{\infty}\|v_n - h_n\|_K \leq \sum_{n=N+1}^{\infty}\|v_n - h_n\|_{K_{n-1}} = \sum_{n=N+1}^{\infty}\frac{1}{2^{n-1}} = \frac{1}{2^{N-1}},$$

which implies, due to the Weierstrass M-test, that the series in (3.4) converges uniformly on K and then on compact subsets of Ω. Consequently, the function v is well defined and continuous on Ω. Actually, much more can be said about v. Note that, fixed $N \in \mathbb{N}$, we have

$$v = v_1 + \sum_{n=2}^{N}(v_n - h_n) + \sum_{n=N+1}^{\infty}(v_n - h_n),$$

where the functions $v_n - h_n$, $n \geq N+1$, are harmonic on U_N. As we have proven that the above series converges uniformly on compact subsets of Ω, we may write

$$v = v_1 + \sum_{n=2}^{N}(v_n - h_n) + H_N,$$

where H_N is harmonic on U_N (**20**, Theorem 1.5.1). Therefore, v belongs to the class $C^{k+2,\alpha}(U_N)$ and it holds that

$$\Delta v = \sum_{n=1}^{N}\Delta v_n = \sum_{n=1}^{N} f_n.$$

on U_N. Hence, if $\boldsymbol{r} \in U_N$, we obtain

$$\Delta v(\boldsymbol{r}) = \sum_{n=1}^{N} f_n(\boldsymbol{r}) = f(\boldsymbol{r}),$$

due to (3.2) and the fact that the rest of functions f_n, $n > N$, are supported on compact subsets of $\Omega \setminus K_N$. As $N \in \mathbb{N}$ is arbitrary, this reasoning proves the rest of the properties required for v, and Theorem 1.3 is proven.

REMARK 3.1. The technique used in the proof of Theorem 1.3, which basically consists in carrying out an approximation on an increasing sequence of open sets retaining at the same time certain important properties outside each open set, is well known in analysis and was first used by Mittag–Leffler. Cf. (**24**, Theorem 13.10).

Acknowledgements

The research of B. de la Calle was supported by Dirección General de Investigación Científica y Técnica under grant MTM2014-54053-P and by Universidad Politécnica de Madrid under grant GI-1505440113. B. de la Calle wishes to thank N. Eldredge for his help with Lemma 2.6.

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