ANALYSIS OF A LOCAL DISCONTINUOUS GALERKIN METHOD FOR FOURTH-ORDER TIME-DEPENDENT PROBLEMS

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Abstract. We analyze a local discontinuous Galerkin (LDG) method for fourth-order time-dependent problems. Optimal error estimates are obtained in one dimension and in multi-dimensions for Cartesian and triangular meshes. We extend the analysis to higher even-order equations and the linearized Cahn-Hilliard type equations. Numerical experiments are displayed to verify the theoretical results.

1. Introduction

In recent years, many finite element methods have been applied and analyzed for solving fourth-order problems, which have broad applications, such as the modeling of thin beams and plates, strain gradient elasticity, and phase separation in binary mixtures.

For fourth-order elliptic problems, mixed finite element methods introduced by Ciarlet and Raviart [6, 21, 22, 15], Hellan-Herrmann-Johnson (HHJ) mixed finite element method [1, 10, 23], and interior penalty methods [2, 19, 24, 20, 14, 4] have been studied, and optimal error estimates have been obtained.

Finite element methods have also been analyzed for fourth-order time-dependent problems. In [13], Elliott and Zheng applied a conforming finite element method to the Cahn-Hilliard equation, and optimal error estimates were obtained in L^2 and L^∞ norms under the assumptions that the approximate solution is bounded in L^∞ and the polynomial degree $k \geq 3$. In [11], Elliott and French applied a nonconforming finite element method based on the Morley nonconforming finite element to the Cahn-Hilliard equation, and they showed optimal error estimates in H^1 and L^∞ norms when piecewise quadratic polynomials are used. In [17], the lowest order Ciarlet-Raviart mixed finite element method was applied to the Cahn-Hilliard equation on quasi-uniform triangular meshes, and optimal error estimate was obtained. In [18], Li analyzed a mixed finite element methods for a fourth-order time-dependent equation on quasi-uniform rectangular meshes, and showed optimal estimates for the primary variable and its Laplacian. In [12], a semi-discrete splitting finite element method using piecewise linear polynomials was proposed, and second order accuracy was proved for triangular and rectangular meshes under the assumption that the approximate solution is bounded in L^∞. In [29], interior penalty methods was applied to the Cahn-Hilliard equation, and error estimate of order $k-1$ for piecewise polynomial of degree k was shown in the energy norm. In

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[16], a variant of Baker's discontinuous Galerkin (DG) method was applied to the Cahn-Hilliard equation and optimal error estimates were proved for $k \geq 3$.

The LDG method was introduced by Cockburn and Shu in [9] for time-dependent convection-diffusion systems. One of its features is to first rewrite a higher order differential equation into a system with first order equations, and then to apply the DG method to the first order system. Through the design of suitable numerical fluxes (traces) for the first order system, the LDG method has been successful in handling equations with high-order derivatives. In [28], Yan and Shu developed an LDG method for fourth-order time-dependent problems, and the L^2-stability was proved. The LDG method was also applied to the nonlinear Cahn-Hilliard type equations and systems, and energy stability was shown; see [26]. In [28] and [26], although no error estimates were obtained, numerical experiments show that the approximations converge with the optimal rate.

In this paper, we carry out the error analysis of the LDG method for fourth-order time-dependent problems in one-dimensional and multidimensional spaces. For the sake of simplicity and easy presentation, we restrict ourselves mainly to the model problem

$$u_t + \triangle^2 u = f \qquad \text{in } \Omega \times (0,T), \tag{1.1a}$$

with a periodic boundary condition and the initial condition

$$u|_{t=0} = u_0, \tag{1.1b}$$

where $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded domain. We discuss more general cases in Section 4.

We analyze the LDG method and obtain optimal error estimates in two different ways. In one dimension, what we do is *not* simply applying the energy argument used in [27] for KdV type equations, because this can only give us a sub-optimal error estimate. The novelty of our analysis is that we make the energy norm contain both u and its three derivatives. Then we eliminate many inter-element boundary terms by carefully choosing projections, which allows us to get an optimal error estimate for u. Similar arguments are applied to multidimensional Cartesian meshes and to higher even-order equations.

For multidimensional triangular meshes, the above argument gives us only a sub-optimal error estimate because we cannot eliminate the inter-element boundary terms that affect the convergence rate by using known projections. To get an optimal error estimate, we use a technique that is standard for parabolic problems, which makes use of the so-called elliptic projection; see [25, 13, 11, 17, 16]. We first prove an optimal convergence result of the LDG method for the corresponding elliptic problem, and then apply it to show the optimal error estimate of the time-dependent problem. Since the linearized Cahn-Hilliard equations are very similar to the model problem (1.1) except that there is a lower order term and the boundary condition is not periodic, we easily extend the analysis to the linearized Cahn-Hilliard equations and get optimal error estimates.

The paper is organized as follows. In Section 2, we define the LDG method for the fourth-order time-dependent problems, show the cell entropy inequality and the L^2-stability, and state the *a priori* error estimates. The error estimates are proved in Section 3. Then we extend the error estimates to higher even-order equations and the linearized Cahn-Hilliard type equations in Section 4. In Section 5, we display numerical experiments validating the theoretical results. We end with some concluding remarks in Section 6. In the Appendix, we prove the error estimates for the biharmonic problem, which we need to use in Section 3.

2. The method and the main results

2.1. **The LDG method for fourth-order time-dependent problems.** Before we introduce the LDG method, we rewrite the fourth-order equation (1.1) into a system of first-order equations

$$\boldsymbol{q} = \nabla u \quad \text{in } \Omega, \tag{2.2a}$$

$$z = \nabla \cdot \boldsymbol{q} \quad \text{in } \Omega, \tag{2.2b}$$

$$\boldsymbol{\sigma} = \nabla z \quad \text{in } \Omega, \tag{2.2c}$$

$$f = u_t + \nabla \cdot \boldsymbol{\sigma} \quad \text{in } \Omega. \tag{2.2d}$$

where $u, \boldsymbol{q}, z, \boldsymbol{\sigma}$ are periodic functions on Ω.

In order to define the LDG methods for the problem (2.2), let us introduce some notation. We denote by $\Omega_h = \{K\}$ a partition of the domain Ω and set $\partial\Omega_h := \{\partial K : K \in \Omega_h\}$. For example, in the one-dimensional case, K is a subinterval; in the two-dimensional case, K is a shape-regular triangle for triangular meshes, or a shape-regular rectangle for Cartesian meshes. The finite element spaces associated with the mesh Ω_h are of the form

$$\begin{aligned} \boldsymbol{V}_h :=& \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{v}|_K \in \boldsymbol{V}(K) \quad \forall K \in \Omega_h\}, \\ W_h :=& \{\omega \in L^2(\Omega) : \omega|_K \in W(K) \quad \forall K \in \Omega_h\}, \end{aligned}$$

where $\boldsymbol{V}(K)$ and $W(K)$ are local spaces on the element K. For one-dimensional meshes and multidimensional triangular meshes, we let $\boldsymbol{V}(K) = \boldsymbol{\mathcal{P}}^k(K)$ and $W(K) = \mathcal{P}^k(K)$, where $\mathcal{P}^k(K)$ is the space of polynomials of degree at most $k \geq 0$ defined on K, and $\boldsymbol{\mathcal{P}}^k(K) := [\mathcal{P}^k(K)]^d$. For multidimensional Cartesian meshes, we let $\boldsymbol{V}(K) = \boldsymbol{Q}^k(K)$ and $W(K) = Q^k(K)$, where $Q^k(K)$ is the space of tensor product of polynomials of degrees at most k in each variable and $\boldsymbol{Q}^k(K) := [Q^k(K)]^d$.

The LDG methods seek an approximation $(\boldsymbol{\sigma}_h, z_h, \boldsymbol{q}_h, u_h)$ to the exact solution $(\boldsymbol{\sigma}, z, \boldsymbol{q}, u)$, in a finite dimensional space $\boldsymbol{V}_h \times W_h \times \boldsymbol{V}_h \times W_h$ and determines it by requiring that on each $K \in \Omega_h$,

$$(\boldsymbol{q}_h, \boldsymbol{v})_K + (u_h, \nabla \cdot \boldsymbol{v})_K - \langle \widehat{u}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} = 0, \tag{2.3a}$$

$$(z_h, \omega)_K + (\boldsymbol{q}_h, \nabla \omega)_K - \langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, \omega \rangle_{\partial K} = 0, \tag{2.3b}$$

$$(\boldsymbol{\sigma}_h, \boldsymbol{\rho})_K + (z_h, \nabla \cdot \boldsymbol{\rho})_K - \langle \widehat{z}_h, \boldsymbol{\rho} \cdot \boldsymbol{n} \rangle_{\partial K} = 0, \tag{2.3c}$$

$$(u_{ht}, \eta)_K - (\boldsymbol{\sigma}_h, \nabla \eta)_K + \langle \widehat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}, \eta \rangle_{\partial K} = (f, \eta)_K, \tag{2.3d}$$

for all $(\boldsymbol{\rho}, \eta, \boldsymbol{v}, \omega) \in \boldsymbol{V}_h \times W_h \times \boldsymbol{V}_h \times W_h$. Here, the outward normal unit vector to ∂K is denoted by $\boldsymbol{n}$, and we have used the notation

$$\begin{aligned} (\boldsymbol{\rho}, \boldsymbol{v})_K &:= \int_K \boldsymbol{\rho}(x) \cdot \boldsymbol{v}(x) \, dx, \\ (\eta, \omega)_K &:= \int_K \eta(x) \, \omega(x) \, dx, \\ \langle \eta, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} &:= \int_{\partial K} \eta(\gamma) \, \boldsymbol{v}(\gamma) \cdot \boldsymbol{n} \, d\gamma, \end{aligned} \tag{2.4}$$

for any functions $\boldsymbol{\rho}, \boldsymbol{v}$ in $\boldsymbol{H}^1(\Omega_h) := [H^1(\Omega_h)]^d$ and η, ω in $H^1(\Omega_h)$, where the broken Sobolev space $H^1(\Omega_h)$ is the space of functions that are are elementwise in the H^1 Sobolev space. Note that in the integral in (2.4), if η or $\boldsymbol{v}$ is not single-valued on inter-element faces, we take its value on K and restrict on ∂K.

To complete the definition of the LDG methods, we need to define the numerical traces (also called numerical fluxes) $\widehat{u}_h, \widehat{\boldsymbol{q}}_h, \widehat{z}_h$ and $\widehat{\boldsymbol{\sigma}}_h$. To do that, we need to introduce some notation. We associate to this partition Ω_h the set of all faces $\mathscr{E}_h$. In the one-dimensional case, we define

$$\omega^{\pm}(x) = \lim_{\epsilon \downarrow 0} \omega(x \pm \epsilon) \quad \forall x \in \mathscr{E}_h.$$

In multidimensional case, let e be a face shared by elements K_1 and K_2, and define the unit normal vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ on e pointing exterior to K_1 and K_2, respectively. The *average* and the *jump* of a scalar-valued function ζ on $e \in \mathscr{E}_h$ are given by

$$\{\!\{\zeta\}\!\} := \frac{1}{2}(\zeta_1 + \zeta_2), \qquad [\![\zeta\, \boldsymbol{n}]\!] := \zeta_1 \boldsymbol{n}_1 + \zeta_2 \boldsymbol{n}_2,$$

where $\zeta_i := \zeta|_{\partial K_i}$. For a vector-valued function $\boldsymbol{\sigma}$, we define $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ analogously and set

$$\{\!\{\boldsymbol{\sigma}\}\!\} := \frac{1}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2), \qquad [\![\boldsymbol{\sigma} \cdot \boldsymbol{n}]\!] := \boldsymbol{\sigma}_1 \cdot \boldsymbol{n}_1 + \boldsymbol{\sigma}_2 \cdot \boldsymbol{n}_2 \qquad \text{on } e \in \mathscr{E}_h.$$

Let $\boldsymbol{v_0}$ be an arbitrarily chosen but fixed nonzero vector, and we define

$$\omega^{\pm} = \{\!\{\omega\}\!\} \pm \boldsymbol{\beta} \cdot [\![\omega\, \boldsymbol{n}]\!]$$

where $\boldsymbol{\beta}$ is any function on $\mathscr{E}_h$ such that, for $\boldsymbol{x} \in \partial K \cap \mathscr{E}_h$,

$$\boldsymbol{\beta} \cdot \boldsymbol{n}_K(\boldsymbol{x}) = \frac{1}{2}\mathrm{sign}\left(\boldsymbol{v_0}(\boldsymbol{x}) \cdot \boldsymbol{n}_K(\boldsymbol{x})\right).$$

The numerical traces $(\widehat{\boldsymbol{\sigma}}_h, \widehat{z}_h, \widehat{\boldsymbol{q}}_h, \widehat{u}_h)$ are defined on inter-element faces as the alternating fluxes

$$\widehat{u}_h = u_h^-, \quad \widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h^+, \quad \widehat{z}_h = z_h^-, \quad \widehat{\boldsymbol{\sigma}}_h = \boldsymbol{\sigma}_h^+. \tag{2.5}$$

Note that we can also choose

$$\widehat{u}_h = u_h^+, \quad \widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h^-, \quad \widehat{z}_h = z_h^+, \quad \widehat{\boldsymbol{\sigma}}_h = \boldsymbol{\sigma}_h^-.$$

On the boundary of Ω, the numerical traces are chosen to satisfy the periodic boundary condition. For example, in one dimension, we set $\Omega = (a, b)$. Then

$$\widehat{u}_h(a) = \widehat{u}_h(b) = u_h^-(b), \quad \widehat{\boldsymbol{q}}_h(a) = \widehat{\boldsymbol{q}}_h(b) = \boldsymbol{q}_h^+(a),$$
$$\widehat{z}_h(a) = \widehat{z}_h(b) = z_h^-(b), \quad \widehat{\boldsymbol{\sigma}}_h(a) = \widehat{\boldsymbol{\sigma}}_h(b) = \boldsymbol{\sigma}_h^+(a).$$

In multidimensional space, numerical traces are defined similarly.

2.2. Cell entropy inequality and L^2-stability.

In [28], the cell entropy inequality and the L^2-stability of the LDG method for the fourth-order time-dependent problem was proved for the one-dimensional case. Here we show the cell entropy inequality and the L^2-stability for multidimensional case in a similar way.

Theorem 2.1. *(Cell entropy inequality) Suppose $f = 0$. On each $K \in \Omega_h$, there exist numerical entropy fluxes $H_{\partial K}$ such that the solution given by the LDG method defined by* (2.3) *and* (2.5) *satisfies*

$$\frac{1}{2}\frac{d}{dt}\int_K u_h^2\, dx + H_{\partial K}(u_h, \boldsymbol{\sigma}_h) - H_{\partial K}(z_h, \boldsymbol{q}_h) \le 0,$$

where

$$H_{\partial K}(\omega, \boldsymbol{v}) = \langle \widehat{\omega}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} + \langle \widehat{\boldsymbol{v}}\cdot\boldsymbol{n}, \omega\rangle_{\partial K} - \langle \boldsymbol{v}\cdot\boldsymbol{n}, \omega\rangle_{\partial K} \quad \forall \omega\in W_h, \boldsymbol{v}\in \boldsymbol{V}_h.$$

Proof. Suppose $f = 0$. Adding equations (2.3b)-(2.3d) and then subtracting the equation (2.3a), we get the identity

$$(u_{ht}, \eta)_K + B_K(u_h, \boldsymbol{q}_h, z_h, \boldsymbol{\sigma}_h; \eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}) = 0, \tag{2.6}$$

for any $(\eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}) \in W_h\times \boldsymbol{V}_h\times W_h\times \boldsymbol{V}_h$, where

$$\begin{aligned}
&B_K(u_h, \boldsymbol{q}_h, z_h, \boldsymbol{\sigma}_h; \eta, \boldsymbol{v}, \omega, \boldsymbol{\rho})\\
=&(\boldsymbol{\sigma}_h, \boldsymbol{\rho})_K + (z_h, \omega)_K - (\boldsymbol{q}_h, \boldsymbol{v})_K\\
&-(\boldsymbol{\sigma}_h, \nabla\eta)_K + (z_h, \nabla\cdot\boldsymbol{\rho})_K + (\boldsymbol{q}_h, \nabla\omega)_K - (u_h, \nabla\cdot\boldsymbol{v})_K\\
&+\langle \widehat{\boldsymbol{\sigma}}_h\cdot\boldsymbol{n}, \eta\rangle_{\partial K} - \langle \widehat{z}_h, \boldsymbol{\rho}\cdot\boldsymbol{n}\rangle_{\partial K} - \langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, \omega\rangle_{\partial K} + \langle \widehat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K}.
\end{aligned}$$

Taking $(\eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}) = (u_h, \boldsymbol{\sigma}_h, z_h, \boldsymbol{q}_h)$ and using integration by parts, we get

$$B_K(u_h, \boldsymbol{q}_h, z_h, \boldsymbol{\sigma}_h; u_h, \boldsymbol{\sigma}_h, z_h, \boldsymbol{q}_h) = (z_h, z_h)_K + H_{\partial K}(u_h, \boldsymbol{\sigma}_h) - H_{\partial K}(z_h, \boldsymbol{q}_h).$$

Hence, from (2.6) we get

$$\frac{1}{2}\frac{d}{dt}\int_K u_h^2 dx + (z_h, z_h)_K + H_{\partial K}(u_h, \boldsymbol{\sigma}_h) - H_{\partial K}(z_h, \boldsymbol{q}_h) = 0, \tag{2.7}$$

which implies the cell entropy inequality. □

Using the definition of the numerical traces (2.5), we get the following property of the numerical entropy flux $H_{\partial K}(\omega, \boldsymbol{v})$.

Lemma 2.2. *Suppose e is an inter-element face shared by the elements K_1 and K_2, then*

$$H_{\partial K_1\cap e}(\omega, \boldsymbol{v}) + H_{\partial K_2\cap e}(\omega, \boldsymbol{v}) = 0,$$

for any $\omega\in W_h$ and $\boldsymbol{v}\in\boldsymbol{V}_h$. Moreover, we have

$$\sum_{K\in\Omega_h} H_{\partial K}(\omega, \boldsymbol{v}) = 0.$$

Using Theorem 2.1 and Lemma 2.2, we immediately get the L^2-stability result.

Theorem 2.3. *(L^2-stability) Suppose that $f = 0$. The solution given by the LDG methods defined by (2.3) and (2.5) satisfies*

$$\frac{d}{dt}\int_{\Omega_h} u_h^2(x,t)\,dx \le 0,$$

2.3. **A priori error estimates.** In this subsection, we obtain a priori error estimates for the approximation $(\boldsymbol{\sigma}_h, z_h, \boldsymbol{q}_h, u_h) \in \boldsymbol{V}_h\times W_h\times \boldsymbol{V}_h\times W_h$ given by the LDG methods. To state them, we need to introduce some norms.

For any real-valued function η in $H^l(\Omega_h)$, we set

$$\|\,\eta\,\|_{H^l(\Omega_h)} := \Big(\sum_{K\in\Omega_h}\|\,\eta\,\|^2_{H^l(K)}\Big)^{\frac{1}{2}}.$$

For a vector-valued function $\boldsymbol{\rho} = (\sigma_1, \dots, \sigma_d) \in \boldsymbol{H}^l(\Omega_h)$ we set

$$\|\,\boldsymbol{\rho}\,\|_{\boldsymbol{H}^l(\Omega_h)} := \Big(\sum_{i=1}^{d}\|\,\sigma_i\,\|^2_{H^l(\Omega_h)}\Big)^{\frac{1}{2}}.$$

For each $K \in \Omega_h$, we denote h_K the diameter of K, and we set $h = \max_{K\in\Omega_h} h_K$.

The errors in the approximation of $u, \boldsymbol{q}, z$ and $\boldsymbol{\sigma}$ are given in the following theorem.

Theorem 2.4. *For one-dimensional meshes or multidimensional Cartesian meshes, if* $(u, \boldsymbol{q}, z, \boldsymbol{\sigma})$ *and* $(u_h, \boldsymbol{q}_h, z_h, \boldsymbol{\sigma}_h)$ *are solutions to* (2.2) *and* (2.3)*, respectively, then for* $T > 0$*, we have*

$$\begin{aligned}\|u(T) - u_h(T)\|_{L^2(\Omega_h)} + \|\boldsymbol{q}(T) - \boldsymbol{q}_h(T)\|_{\boldsymbol{L}^2(\Omega_h)} \leq& C\, h^{k+1},\\ \int_0^T (\|z - z_h\|_{L^2(\Omega_h)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\boldsymbol{L}^2(\Omega_h)})\, dt \leq& C\, h^{k+1},\end{aligned}$$

where C *is a constant independent of* h *and dependent on* $\|u\|_{H^{k+4}(\Omega_h)}$, $\|u(T)\|_{H^{k+3}(\Omega_h)}$, $\|u(0)\|_{H^{k+3}(\Omega_h)}$, $\|u_t\|_{H^{k+3}(\Omega_h)}$ *and* T.

If we use similar analysis for multi-dimensional triangular meshes instead of Cartesian meshes, we get the following sub-optimal convergence result.

Theorem 2.5. *For multi-dimensional triangular meshes, if* $(u, \boldsymbol{q}, z, \boldsymbol{\sigma})$ *and* $(u_h, \boldsymbol{q}_h, z_h, \boldsymbol{\sigma}_h)$ *are solutions to* (2.2) *and* (2.3)*, respectively, then for* $T > 0$*, we have*

$$\begin{aligned}\|u(T) - u_h(T)\|_{L^2(\Omega_h)} + \|\boldsymbol{q}(T) - \boldsymbol{q}_h(T)\|_{\boldsymbol{L}^2(\Omega_h)} \leq& C\, h^{k},\\ \int_0^T (\|z - z_h\|_{L^2(\Omega_h)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\boldsymbol{L}^2(\Omega_h)})\, dt \leq& C\, h^{k}.\end{aligned}$$

where C *is a constant independent of* h *and dependent on* $\|u\|_{H^{k+4}(\Omega_h)}$, $\|u(T)\|_{H^{k+2}(\Omega_h)}$, $\|u_t\|_{H^{k+2}(\Omega_h)}$ *and* T.

The error estimates in Theorem 2.5 are not optimal, because in the proof there are some boundary terms that we can not eliminate by choosing projections on triangular meshes. However, we can improve the estimates for $k \geq 1$ by imposing some regularity assumption.

To get optimal convergence result for fourth-order time-dependent problems on multi-dimensional triangular meshes, we consider the following biharmonic problem:

$$\boldsymbol{\psi} = \nabla\varphi \quad \text{in } \Omega, \tag{2.8a}$$

$$\xi = \nabla \cdot \boldsymbol{\psi} \quad \text{in } \Omega, \tag{2.8b}$$

$$\boldsymbol{\zeta} = \nabla\xi \quad \text{in } \Omega, \tag{2.8c}$$

$$\eta = \nabla \cdot \boldsymbol{\zeta} \quad \text{in } \Omega, \tag{2.8d}$$

with periodic boundary conditions. To make the problem well-defined, we assume that the average of φ on Ω is a given constant and that of η is zero.

If we assume the following elliptic regularity result (see e.g. [3])

$$\|\,\boldsymbol{\zeta}\,\|_{\boldsymbol{H}^1(\Omega_h)} + \|\xi\,\|_{\boldsymbol{H}^2(\Omega_h)} + \|\,\boldsymbol{\psi}\,\|_{\boldsymbol{H}^3(\Omega_h)} + \|\varphi\,\|_{\boldsymbol{H}^4(\Omega_h)} \leq \mathsf{C}_{er}\,\|\,\eta\,\|_{L^2(\Omega)}, \tag{2.9}$$

we can improve the error estimates in Theorem 2.5 to get an optimal convergence result.

Theorem 2.6. *Under the same assumption as in Theorem 2.5, if we further assume the regularity* (2.9) *for the elliptic problem* (2.8)*, then we have*

$$\|u(T) - u_h(T)\|_{L^2(\Omega_h)} \leq Ch^{k+1}.$$

for $k \geq 1$, where C is a constant independent of h and dependent on $\|u_0\|_{H^{k+4}(\Omega_h)}$, $\|u(T)\|_{H^{k+4}(\Omega_h)}$, $\|u_t\|_{H^{k+4}(\Omega_h)}$, C_{er} and T.

Remark: The proof of Theorem 2.6 does not work for the case $k = 0$. In the numerical experiments, we see that the method is not convergent for the piecewise constant case.

3. Proof of the error estimates

In this section, we prove the error estimates stated in the previous section. In subsection 3.1, we introduce some projections and inequalities that we are going to use in our analysis. In subsection 3.2, we prove Theorem 2.4 and Theorem 2.5, the optimal estimates for one-dimensional meshes and multidimensional Cartesian meshes, and the sub-optimal error estimates for multi-dimensional triangular meshes. To get the optimal convergence result for multi-dimensional triangular meshes in Theorem 2.6, we introduce a corresponding biharmonic problem in subsection 3.3. Then in subsection 3.4, we use the error estimate of the elliptic problem to get an optimal error estimate for the time-dependent problem.

3.1. **Projections and inequalities.** Before we introduce the projections that are defined on different types of meshes, first let us define the commonly used L^2-projection P^ℓ: Given a function $\eta \in L^2(\Omega_h)$ and an arbitrary simplex $K \in \Omega_h$, the restriction of $\mathsf{P}^\ell \eta$ to K is defined as the element of $\mathcal{P}^\ell(K)$ that satisfies

$$(\mathsf{P}^\ell \eta - \eta, \omega)_K = 0 \qquad \forall\, \omega \in \mathcal{P}^\ell(K). \tag{3.10}$$

To simplify the notation, we are going to denote P^k as P.

3.1.1. *Projections in one-dimensional space.* In one dimension, we use the special projections

$$\mathsf{P}^{\pm} : H^1(\Omega_h) \rightarrow W_h,$$

which are defined as the following. Given a function $\eta \in H^1(\Omega_h)$ and an arbitrary subinterval $K_j = (x_{j-1}, x_j)$, the restriction of $\mathsf{P}^{\pm}\eta$ to K_j are defined as the elements of $\mathcal{P}^k(K_j)$ that satisfy

$$\begin{aligned} &(\mathsf{P}^{+}\eta - \eta, \omega)_{K_j} = 0 \quad \forall\, \omega \in \mathcal{P}^{k-1}(K_j), \quad \text{and} \quad \mathsf{P}^{+}\eta(x_{j-1}) = \eta(x_{j-1}), \\ &(\mathsf{P}^{-}\eta - \eta, \omega)_{K_j} = 0 \quad \forall\, \omega \in \mathcal{P}^{k-1}(K_j), \quad \text{and} \quad \mathsf{P}^{-}\eta(x_j) = \eta(x_j). \end{aligned} \tag{3.11}$$

For $k = 0$, we use $\mathcal{P}^{-1}(K_j) = \{0\}$. For more details, see [5].

3.1.2. *Projection for Cartesian meshes.* For Cartesian meshes in multidimensional space, we use projections that are tensor products of the projections in one dimension; see [8]. For example, on a two-dimensional rectangle element $I \otimes J = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, we take $\boldsymbol{v}_0 = (1, 1)$. The projections P^- for scalar functions are defined as

$$\mathsf{P}^- = \mathsf{P}_x^- \otimes \mathsf{P}_y^-, \tag{3.12}$$

where the subscripts x and y indicate that the one-dimensional projections defined by (3.11) are applied with respect to the corresponding variable.

The projections Π^+ for vector-valued functions $\boldsymbol{\rho} = (\rho_1(x,y), \rho_2(x,y))$ are defined as

$$\Pi^+ \boldsymbol{\rho} = (\mathsf{P}_x^+ \otimes \mathsf{P}_y\, \rho_1, \mathsf{P}_x \otimes \mathsf{P}_y^+\, \rho_2), \tag{3.13}$$

where P_x^+ and P_y^+ are the one-dimensional projections defined by (3.11). It is easy to see that, for any $\boldsymbol{\rho} \in [H^1(\Omega_h)]^2$, the restriction of $\Pi^+\boldsymbol{\rho}$ to $I \otimes J$ are the elements of $[Q^k(I \otimes J)]^2$ that satisfy

$$\int_I \int_J (\Pi^+\boldsymbol{\rho} - \boldsymbol{\rho}) \nabla \omega \mathrm{dy}\mathrm{dx} = 0,$$

for any $\omega \in Q^k(I \otimes J)$, and

$$\int_J \big(\Pi^+\boldsymbol{\rho}(x_{i-1}, y) - \boldsymbol{\rho}(x_{i-1}, y)\big) \cdot \boldsymbol{n}\, \omega(x_{i-1}^+, y)\mathrm{dy} = 0 \quad \forall \omega \in Q^k(I \otimes J),$$
$$\int_I \big(\Pi^+\boldsymbol{\rho}(x, y_{j-1}) - \boldsymbol{\rho}(x, y_{j-1})\big) \cdot \boldsymbol{n}\, \omega(x, y_{j-1}^+)\mathrm{dx} = 0 \quad \forall \omega \in Q^k(I \otimes J).$$

3.1.3. *Projection for triangular meshes.* For triangular meshes in multidimensional space, we use the L^2-projection P^ℓ for scalar-valued functions and a projection Π^+ [7] for vector-valued functions.

The projection Π^+ is defined as: Given a function $\boldsymbol{\rho} \in \boldsymbol{H}^1(\Omega_h)$, an arbitrary simplex $K \in \Omega_h$, and an arbitrary edge $\tilde{e} \in \partial K$ that satisfies $\boldsymbol{v}_0 \cdot \boldsymbol{n}_{\tilde{e}} < 0$, the restriction of $\Pi^+\boldsymbol{\rho}$ to K is defined as the element of $\boldsymbol{\mathcal{P}}^k(K)$ that satisfies

$$(\Pi^+\boldsymbol{\rho} - \boldsymbol{\rho}, \boldsymbol{v})_K = 0 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{P}}^{k-1}(K), \text{ if } k \geq 1, \tag{3.14a}$$
$$\langle (\Pi^+\boldsymbol{\rho} - \boldsymbol{\rho}) \cdot \boldsymbol{n}, \omega \rangle_e = 0 \qquad \forall\, \omega \in \mathcal{P}^k(e) \text{ and } \forall e \in \partial K, e \neq \tilde{e}. \tag{3.14b}$$

3.1.4. *Approximation property of projections and some inequalities.* The projections defined in above subsections have the following approximation property; see [5, 8, 7].

Lemma 3.1. *Let $\mathbb{P}$ be the projection defined by* (3.10), (3.11) *or* (3.12). *Then for any $\eta \in H^{k+1}(\Omega)$,*

$$\|\mathbb{P}\eta - \eta\|_{L^2(\Omega_h)} + h^{1/2}\|\mathbb{P}\eta - \eta\|_{L^2(\mathscr{E}_h)} \leq Ch^{k+1}\|\eta\|_{H^{k+1}(\Omega_h)},$$

where C is independent of h.

Let $\boldsymbol{\Pi}$ be the projections defined by (3.13) *or* (3.14). *Then for any $\boldsymbol{\rho} \in [H^{k+1}(\Omega)]^d$,*

$$\|\boldsymbol{\Pi}\boldsymbol{\rho} - \boldsymbol{\rho}\|_{L^2(\Omega_h)} + h^{1/2}\|\boldsymbol{\Pi}\boldsymbol{\rho} - \boldsymbol{\rho}\|_{L^2(\mathscr{E}_h)} \leq Ch^{k+1}\|\boldsymbol{\rho}\|_{H^{k+1}(\Omega_h)},$$

where C is independent of h.

In the proofs of the error estimates, we are going to use the following inverse and trace inequalities.

Lemma 3.2. *For any $\boldsymbol{v} \in \boldsymbol{\mathcal{P}}^k(K)$ and $\omega \in \mathcal{P}^k(K)$ there exist positive constants C such that*

$$\|\boldsymbol{v} \cdot \boldsymbol{n}\|^2_{L^2(e)} \leq Ch_K^{-1}\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(K)},$$
$$\|\omega\|^2_{L^2(e)} \leq Ch_K^{-1}\|\omega\|^2_{L^2(K)},$$
$$\|\nabla\omega\|^2_{L^2(K)} \leq Ch_K^{-2}\|\omega\|^2_{L^2(K)};$$

where e is an face of K, and C is independent of the mesh size h.

Lemma 3.3. *For any $\boldsymbol{\rho} \in \boldsymbol{H}^1(K)$ and $\eta \in H^1(K)$ there exist positive constants C such that*

$$\|\boldsymbol{\rho}\cdot\boldsymbol{n}\|^2_{L^2(e)} \le C\|\boldsymbol{\rho}\|_{\boldsymbol{L}^2(K)}\|\boldsymbol{\rho}\|_{\boldsymbol{H}^1(K)},$$
$$\|\eta\|^2_{L^2(e)} \le C\|\eta\|_{L^2(K)}\|\eta\|_{H^1(K)},$$

where e is an face of K, and C is independent of the mesh size h.

3.2. **Proofs of Theorem 2.4 and Theorem 2.5.** First, let us introduce some notation. We set

$$\begin{aligned}
(\boldsymbol{\rho},\boldsymbol{v})_{\Omega_h} &= \sum_{K\in\Omega_h}(\boldsymbol{\rho},\boldsymbol{v})_K,\\
(\eta,\omega)_{\Omega_h} &= \sum_{K\in\Omega_h}(\eta,\omega)_K,\\
\langle \eta,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} &= \sum_{K\in\Omega_h}\langle\eta,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K},\\
B(\phi,\boldsymbol{\psi},\xi,\boldsymbol{\chi};\eta,\boldsymbol{v},\omega,\boldsymbol{\rho}) &= \sum_{K\in\Omega_h}B_K(\phi,\boldsymbol{\psi},\xi,\boldsymbol{\chi};\eta,\boldsymbol{v},\omega,\boldsymbol{\rho}).
\end{aligned}$$

Note that the exact solution $(u,\boldsymbol{q},z,\boldsymbol{\sigma})$ of the problem (2.2) also satisfies the equation (2.6), so we have the Galerkin orthogonality

$$(\boldsymbol{e}_{ut},\eta)_{\Omega_h} + B(\boldsymbol{e}_u,\boldsymbol{e}_{\boldsymbol{q}},\boldsymbol{e}_z,\boldsymbol{e}_{\boldsymbol{\sigma}};\eta,\boldsymbol{v},\omega,\boldsymbol{\rho}) = 0, \tag{3.15}$$

for any $(\eta,\boldsymbol{v},\omega,\boldsymbol{\rho}) \in W_h\times\boldsymbol{V}_h\times W_h\times\boldsymbol{V}_h$, where we use the notation $e_p = p - p_h$, for $p = u,\boldsymbol{q},z$ and $\boldsymbol{\sigma}$.

Next, we use the Galerkin orthogonality (3.15) to obtain an intermediate result on the errors of u and z, and that on the errors of $\boldsymbol{q}$ and $\boldsymbol{\sigma}$. Then we combine the two intermediate results to get a priori error estimates, from which Theorem 2.4 and Theorem 2.5 follow.

In the following Lemmas, we let $\mathbb{P}$ and $\boldsymbol{\Pi}$ be some projections onto the finite element spaces W_h and $\boldsymbol{V}_h$, respectively, and we will specify the projections later.

Lemma 3.4.

$$(\mathbb{P}\boldsymbol{e}_{ut},\mathbb{P}\boldsymbol{e}_u)_{\Omega_h} + (\mathbb{P}\boldsymbol{e}_z,\mathbb{P}\boldsymbol{e}_z)_{\Omega_h} = T_1+T_2+T_3,$$

where

$$\begin{aligned}
T_1 =& -((u-\mathbb{P}u)_t,\mathbb{P}\boldsymbol{e}_u)_{\Omega_h} + (\boldsymbol{q}-\boldsymbol{\Pi}\boldsymbol{q},\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}})_{\Omega_h} - (z-\mathbb{P}z,\mathbb{P}\boldsymbol{e}_z)_{\Omega_h} - (\boldsymbol{\sigma}-\boldsymbol{\Pi}\boldsymbol{\sigma},\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{q}})_{\Omega_h},\\
T_2 =& (u-\mathbb{P}u,\nabla\cdot\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}})_{\Omega_h} - (\boldsymbol{q}-\boldsymbol{\Pi}\boldsymbol{q},\nabla\mathbb{P}\boldsymbol{e}_z)_{\Omega_h} - (z-\mathbb{P}z,\nabla\cdot\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{q}})_{\Omega_h} + (\boldsymbol{\sigma}-\boldsymbol{\Pi}\boldsymbol{\sigma},\nabla\mathbb{P}\boldsymbol{e}_u)_{\Omega_h},\\
T_3 =& \langle 1, -(u-\widehat{\mathbb{P}u})\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}\cdot\boldsymbol{n} + (\boldsymbol{q}-\widehat{\boldsymbol{\Pi}\boldsymbol{q}})\mathbb{P}\boldsymbol{e}_z\cdot\boldsymbol{n} + (z-\widehat{\mathbb{P}z})\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{n} - (\boldsymbol{\sigma}-\widehat{\boldsymbol{\Pi}\boldsymbol{\sigma}})\mathbb{P}\boldsymbol{e}_u\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}.
\end{aligned}$$

Proof. From the Galerkin orthogonality (3.15), we get

$$LHS = RHS,$$

where

$$\begin{aligned}
LHS =& (\mathbb{P}\boldsymbol{e}_{ut},\eta)_{\Omega_h} + B(\mathbb{P}\boldsymbol{e}_u,\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{q}},\mathbb{P}\boldsymbol{e}_z,\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}};\eta,\boldsymbol{v},\omega,\boldsymbol{\rho}),\\
RHS =& -((u-\mathbb{P}u)_t,\eta)_{\Omega_h} - B(u-\mathbb{P}u,\boldsymbol{q}-\boldsymbol{\Pi}\boldsymbol{q},z-\mathbb{P}z,\boldsymbol{\sigma}-\boldsymbol{\Pi}\boldsymbol{\sigma};\eta,\boldsymbol{v},\omega,\boldsymbol{\rho}).
\end{aligned}$$

Then we only need to take $(\eta,\boldsymbol{v},\omega,\boldsymbol{\rho}) = (\mathbb{P}\boldsymbol{e}_u,\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}},\mathbb{P}\boldsymbol{e}_z,\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{q}})$ and simplify LHS by integration by parts and Lemma 2.2. $\square$

Lemma 3.5.

$$(\mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{q}_t}, \mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{q}})_{\Omega_h} + (\mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}}, \mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}})_{\Omega_h} = S_1 + S_2 + S_3,$$

where

$$\begin{aligned}
S_1 =&((u-\mathbb{P}u)_t, \mathbb{P}\boldsymbol{e}_z)_{\Omega_h} - ((\boldsymbol{q}-\mathbf{\Pi}\boldsymbol{q})_t, \mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}})_{\Omega_h} - (z-\mathbb{P}z, (\mathbb{P}\boldsymbol{e}_u)_t)_{\Omega_h} - (\boldsymbol{\sigma}-\mathbf{\Pi}\boldsymbol{\sigma}, \mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}})_{\Omega_h},\\
S_2 =&-((u-\mathbb{P}u)_t, \nabla\cdot\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}})_{\Omega_h} - (\boldsymbol{q}-\mathbf{\Pi}\boldsymbol{q}, \nabla(\mathbb{P}\boldsymbol{e}_u)_t)_{\Omega_h}\\
&-(z-\mathbb{P}z, \nabla\cdot\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}})_{\Omega_h} - (\boldsymbol{\sigma}-\mathbf{\Pi}\boldsymbol{\sigma}, \nabla\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}})_{\Omega_h},\\
S_3 =&\langle 1, (u-\widehat{\mathbb{P}u})_t\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{n} + (\boldsymbol{q}-\widehat{\mathbf{\Pi}\boldsymbol{q}})(\mathbb{P}\boldsymbol{e}_u)_t\cdot\boldsymbol{n} + (z-\widehat{\mathbb{P}z})\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}\cdot\boldsymbol{n} + (\boldsymbol{\sigma}-\widehat{\mathbf{\Pi}\boldsymbol{\sigma}})\mathbb{P}\boldsymbol{e}_z\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}.
\end{aligned}$$

Proof. Differentiating the equation (2.3a) with respect to t, we have

$$((\boldsymbol{q}_h)_t, \boldsymbol{v})_K + ((u_h)_t, \nabla\cdot\boldsymbol{v})_K - \langle(\widehat{u}_h)_t, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} = 0. \tag{3.16}$$

We add equations (3.16), (2.3b) and (2.3c), subtract the equation (2.3d), and sum over all the element K. We get

$$(\boldsymbol{q}_{h\,t}, \boldsymbol{v})_{\Omega_h} + \tilde{B}(u_h, \boldsymbol{q}_h, z_h, \boldsymbol{\sigma}_h; \eta, \boldsymbol{\rho}, \omega, \boldsymbol{v}) = -(f,\eta)_{\Omega_h}, \tag{3.17}$$

where

$$\begin{aligned}
&\tilde{B}(u_h, \boldsymbol{q}_h, z_h, \boldsymbol{\sigma}_h; \eta, \boldsymbol{\rho}, \omega, \boldsymbol{v})\\
=&((\boldsymbol{q}_h)_t, \boldsymbol{v})_{\Omega_h} + (z_h, \omega)_{\Omega_h} + (\boldsymbol{\sigma}_h, \boldsymbol{\rho})_{\Omega_h} - (u_{h\,t}, \eta)_{\Omega_h}\\
&+((u_h)_t, \nabla\cdot\boldsymbol{v})_{\Omega_h} + (\boldsymbol{q}_h, \nabla\omega)_{\Omega_h} + (z_h, \nabla\cdot\boldsymbol{\rho})_{\Omega_h} + (\boldsymbol{\sigma}_h, \nabla\eta)_{\Omega_h}\\
&-\langle(\widehat{u}_h)_t, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} - \langle\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, \omega\rangle_{\partial\Omega_h} - \langle\widehat{z}_h, \boldsymbol{\rho}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} - \langle\widehat{\boldsymbol{\sigma}}_h\cdot\boldsymbol{n}, \eta\rangle_{\partial\Omega_h}
\end{aligned}$$

Since the exact solution $(u, \boldsymbol{q}, z, \boldsymbol{\sigma})$ of (2.2) also satisfies (3.17), we have the following new Galerkin orthogonality

$$(\boldsymbol{e}_{\boldsymbol{q}_t}, \boldsymbol{v})_{\Omega_h} + \tilde{B}(\boldsymbol{e}_u, \boldsymbol{e}_{\boldsymbol{q}}, \boldsymbol{e}_z, \boldsymbol{e}_{\boldsymbol{\sigma}}; \eta, \boldsymbol{\rho}, \omega, \boldsymbol{v}) = 0,$$

for any $(\eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}) \in W_h \times \boldsymbol{V}_h \times W_h \times \boldsymbol{V}_h$. This implies that

$$LHS = RHS,$$

where

$$\begin{aligned}
LHS =&(\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}_t}, \boldsymbol{v})_{\Omega_h} + \tilde{B}(\mathbb{P}\boldsymbol{e}_u, \mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}}, \mathbb{P}\boldsymbol{e}_z, \mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}; \eta, \boldsymbol{\rho}, \omega, \boldsymbol{v}),\\
RHS =&-((\boldsymbol{q}-\mathbf{\Pi}\boldsymbol{q})_t, \boldsymbol{v})_{\Omega_h} - \tilde{B}(u-\mathbb{P}u, \boldsymbol{q}-\mathbf{\Pi}\boldsymbol{q}, z-\mathbb{P}z, \boldsymbol{\sigma}-\mathbf{\Pi}\boldsymbol{\sigma}; \eta, \boldsymbol{\rho}, \omega, \boldsymbol{v}).
\end{aligned}$$

Then we only need to take $(\eta, \boldsymbol{\rho}, \omega, \boldsymbol{v}) = (\mathbb{P}\boldsymbol{e}_z, \mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}, (\mathbb{P}\boldsymbol{e}_u)_t, \mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}})$ in LHS and RHS, and simplify by using integration by parts and Lemma 2.2. This completes the proof. □

Lemma 3.6. *If we take the projections* $(\mathbb{P}, \mathbf{\Pi}) = (\mathsf{P}^-, \mathsf{P}^+)$ *defined by* (3.11) *for one-dimensional space, and take* $(\mathbb{P}, \mathbf{\Pi}) = (\mathsf{P}^-, \Pi^+)$ *defined by* (3.12) *and* (3.13) *for multidimensional Cartesian meshes, then we have*

$$\|\mathbb{P}\boldsymbol{e}_u(T)\|_{L^2(\Omega_h)} + \|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}}(T)\|_{L^2(\Omega_h)} + \int_0^T (\|\mathbb{P}\boldsymbol{e}_z\|_{L^2(\Omega_h)} + \|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}\|_{L^2(\Omega_h)})dt \le Ch^{k+1}, \tag{3.18}$$

for any $T > 0$, *where* C *is a constant independent of* h *and dependent on* $\|u\|_{H^{k+4}(\Omega_h)}$, $\|u(T)\|_{H^{k+3}(\Omega_h)}$, $\|u(0)\|_{H^{k+3}(\Omega_h)}$, $\|u_t\|_{H^{k+3}(\Omega_h)}$ *and* T.

Proof. From Lemma 3.4 and Lemma 3.5, we have

$$\frac{1}{2}\frac{d}{dt}\|\mathbb{P}\boldsymbol{e}_u\|^2_{L^2(\Omega_h)}+\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\Pi e_q}\|^2_{L^2(\Omega_h)}+\|\mathbb{P}\boldsymbol{e}_z\|^2_{L^2(\Omega_h)}+\|\boldsymbol{\Pi e_\sigma}\|^2_{L^2(\Omega_h)}=\sum_{i=1}^{3}(T_i+S_i).$$

Integrating both sides of above identity with respect to t over $(0,T)$, we get

$$\begin{aligned}&\frac{1}{2}\|\mathbb{P}\boldsymbol{e}_u(T)\|^2_{L^2(\Omega_h)}+\frac{1}{2}\|\boldsymbol{\Pi e_q}(T)\|^2_{L^2(\Omega_h)}+\int_0^T(\|\mathbb{P}\boldsymbol{e}_z\|^2_{L^2(\Omega_h)}+\|\boldsymbol{\Pi e_\sigma}\|^2_{L^2(\Omega_h)})dt\\ =&\frac{1}{2}\|\mathbb{P}\boldsymbol{e}_u(0)\|^2_{L^2(\Omega_h)}+\frac{1}{2}\|\boldsymbol{\Pi e_q}(0)\|^2_{L^2(\Omega_h)}+\sum_{i=1}^{3}\int_0^T(T_i+S_i)dt.\end{aligned}$$

It is easy to see that

$$\|\mathbb{P}\boldsymbol{e}_u(0)\|^2_{L^2(\Omega_h)}+\|\boldsymbol{\Pi e_q}(0)\|^2_{L^2(\Omega_h)}\le Ch^{2(k+1)},$$

where C is a constant depending on $\|u_0\|^2_{H^{k+2}(\Omega_h)}$. Next we estimate the terms $\int_0^T T_i\,dt$ and $\int_0^T S_i\,dt$ for $i=1,2,3$.

a. Estimate of $\int_0^T T_1 dt$. Using the approximation property of the projections, Lemma 3.1, we have

$$\begin{aligned}T_1\le&\|(u-\mathbb{P}u)_t\|_{L^2(\Omega_h)}\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)}+\|\boldsymbol{q}-\boldsymbol{\Pi q}\|_{L^2(\Omega_h)}\|\boldsymbol{\Pi e_\sigma}\|_{L^2(\Omega_h)}\\ &+\|z-\mathbb{P}z\|_{L^2(\Omega_h)}\|\mathbb{P}\boldsymbol{e}_z\|_{L^2(\Omega_h)}+\|\boldsymbol{\sigma}-\boldsymbol{\Pi\sigma}\|_{L^2(\Omega_h)}\|\boldsymbol{\Pi e_q}\|_{L^2(\Omega_h)}\\ \le&Ch^{k+1}(\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)}+\|\boldsymbol{\Pi e_q}\|_{L^2(\Omega_h)}+\|\mathbb{P}\boldsymbol{e}_z\|_{L^2(\Omega_h)}+\|\boldsymbol{\Pi e_\sigma}\|_{L^2(\Omega_h)}),\end{aligned}$$

where C depends on $\|u\|_{H^{k+4}(\Omega_h)}$. This implies that

$$\int_0^T T_1 dt\le Ch^{k+1}\int_0^T(\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)}+\|\boldsymbol{\Pi e_q}\|_{L^2(\Omega_h)}+\|\mathbb{P}\boldsymbol{e}_z\|_{L^2(\Omega_h)}+\|\boldsymbol{\Pi e_\sigma}\|_{L^2(\Omega_h)})dt.$$

b. Estimate of $\int_0^T S_1 dt$. Using the approximation property of the projections, Lemma 3.1, we have

$$\begin{aligned}S_1\le&\|(u-\mathbb{P}u)_t\|_{L^2(\Omega_h)}\|\mathbb{P}\boldsymbol{e}_z\|_{L^2(\Omega_h)}+\|(\boldsymbol{q}-\boldsymbol{\Pi q})_t\|_{L^2(\Omega_h)}\|\boldsymbol{\Pi e_q}\|_{L^2(\Omega_h)}\\ &+\|\boldsymbol{\sigma}-\boldsymbol{\Pi\sigma}\|_{L^2(\Omega_h)}\|\boldsymbol{\Pi e_\sigma}\|_{L^2(\Omega_h)}-(z-\mathbb{P}z,(\mathbb{P}\boldsymbol{e}_u)_t)_{\Omega_h}\\ \le&Ch^{k+1}(\|\boldsymbol{\Pi e_q}\|_{L^2(\Omega_h)}+\|\mathbb{P}\boldsymbol{e}_z\|_{L^2(\Omega_h)}+\|\boldsymbol{\Pi e_\sigma}\|_{L^2(\Omega_h)})-(z-\mathbb{P}z,(\mathbb{P}\boldsymbol{e}_u)_t)_{\Omega_h},\end{aligned}$$

where C depends on $\|u\|_{H^{k+4}(\Omega_h)}$ and $\|u_t\|_{H^{k+2}(\Omega_h)}$. Using integration by parts with respect to t, we get

$$\begin{aligned}\int_0^T S_1 dt\le&C\,h^{k+1}\int_0^T(\|\boldsymbol{\Pi e_q}\|_{L^2(\Omega_h)}+\|\mathbb{P}\boldsymbol{e}_z\|_{L^2(\Omega_h)}+\|\boldsymbol{\Pi e_\sigma}\|_{L^2(\Omega_h)})dt\\ &+\|(z-\mathbb{P}z)(T)\|_{L^2(\Omega_h)}\|\mathbb{P}\boldsymbol{e}_u(T)\|_{L^2(\Omega_h)}\\ &+\|(z-\mathbb{P}z)(0)\|_{L^2(\Omega_h)}\|\mathbb{P}\boldsymbol{e}_u(0)\|_{L^2(\Omega_h)}\\ &+\int_0^T\|(z-\mathbb{P}z)_t\|_{L^2(\Omega_h)}\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)}dt\\ \le&C\,h^{k+1}\int_0^T(\|\boldsymbol{\Pi e_q}\|_{L^2(\Omega_h)}+\|\mathbb{P}\boldsymbol{e}_z\|_{L^2(\Omega_h)}+\|\boldsymbol{\Pi e_\sigma}\|_{L^2(\Omega_h)})dt\\ &+C\,h^{2(k+1)}+\frac{1}{4}\|\mathbb{P}\boldsymbol{e}_u(T)\|^2_{L^2(\Omega_h)}+Ch^{k+1}\int_0^T\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)}dt,\end{aligned}$$

where C depends on $\|u\|_{H^{k+4}(\Omega_h)}$, $\|u(T)\|_{H^{k+3}(\Omega_h)}$, $\|u(0)\|_{H^{k+3}(\Omega_h)}$, $\|u_t\|_{H^{k+3}(\Omega_h)}$ and T.

c. Estimate of $\int_0^T (T_2 + T_3 + S_2 + S_3)\, dt$ in one dimension. Using the definition of the numerical traces, (2.5), and the definitions of the projections $\mathsf{P}^{\pm}$, (3.11), we get

$$T_2 = S_2 = T_3 = S_3 = 0.$$

So

$$\int_0^T (T_2 + T_3 + S_2 + S_3)\, dt = 0.$$

d. Estimate of $\int_0^T (T_2 + T_3 + S_2 + S_3)\, dt$ on multidimensional Cartesian meshes. Using the definition of the numerical traces, (2.5), and the definition of the projection Π, (3.13), we have

$$\begin{aligned}
T_2 &= (u - \mathbb{P}u, \nabla \cdot \boldsymbol{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}})_{\Omega_h} - (z - \mathbb{P}z, \nabla \cdot \boldsymbol{\Pi} \boldsymbol{e}_{\boldsymbol{q}})_{\Omega_h},\\
T_3 &= -\langle u - \widehat{\mathbb{P}u}, \boldsymbol{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}} \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} + \langle z - \widehat{\mathbb{P}z}, \boldsymbol{\Pi} \boldsymbol{e}_{\boldsymbol{q}} \cdot \boldsymbol{n} \rangle_{\partial\Omega_h},\\
S_2 &= -\left((u - \mathbb{P}u)_t, \nabla \cdot \boldsymbol{\Pi} \boldsymbol{e}_{\boldsymbol{q}}\right)_{\Omega_h} - (z - \mathbb{P}z, \nabla \cdot \boldsymbol{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}})_{\Omega_h},\\
S_3 &= \langle (u - \widehat{\mathbb{P}u})_t, \boldsymbol{\Pi} \boldsymbol{e}_{\boldsymbol{q}} \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} + \langle z - \widehat{\mathbb{P}z}, \boldsymbol{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}} \cdot \boldsymbol{n} \rangle_{\partial\Omega_h}.
\end{aligned}$$

The projection $\mathbb{P} = \mathsf{P}^-$ defined by (3.12) on Cartesian meshes has the following superconvergence property (see [8] Lemma 3.6)

Lemma 3.7. *Suppose $(\eta, \boldsymbol{\rho}) \in H^{k+2}(\Omega_h) \times \boldsymbol{V}_h$ and the projection P^- is defined by* (3.12)*, then we have*

$$|(\eta - \mathsf{P}^-\eta, \nabla \cdot \boldsymbol{\rho})_{\Omega_h} - \langle \eta - \widehat{\mathsf{P}^-\eta}, \boldsymbol{\rho} \cdot \boldsymbol{n}_K \rangle_{\partial\Omega_h}| \le C h^{k+1} \|\eta\|_{H^{k+2}(\Omega_h)} \|\boldsymbol{\rho}\|_{L^2(\Omega_h)},$$

where C only depends on k and the shape regular constant.

Using Lemma 3.7, we have

$$T_2 + T_3 + S_2 + S_3 \le C h^{k+1} (\|\boldsymbol{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}}\|_{L^2(\Omega_h)} + \|\boldsymbol{\Pi} \boldsymbol{e}_{\boldsymbol{q}}\|_{L^2(\Omega_h)}),$$

where C depends on $\|u\|_{H^{k+4}(\Omega_h)}$ and $\|u_t\| H^{k+2}(\Omega_h)$. Hence

$$\int_0^T (T_2 + T_3 + S_2 + S_3)\, dt \le C h^{k+1} \int_0^T (\|\boldsymbol{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}}\|_{L^2(\Omega_h)} + \|\boldsymbol{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}}\|_{L^2(\Omega_h)})\, dt,$$

where C depends on $\|u\|_{H^{k+4}(\Omega_h)}$, and $\|u_t\| H^{k+2}(\Omega_h)$.

e. Conclusion. Gathering the estimates for $\int_0^T T_i\, dt$ and $\int_0^T S_i\, dt$, $i = 1, 2, 3$, we get

$$\begin{aligned}
&\frac{1}{4}\|\mathbb{P} \boldsymbol{e}_u(T)\|^2_{L^2(\Omega_h)} + \frac{1}{2}\|\boldsymbol{\Pi} \boldsymbol{e}_{\boldsymbol{q}}(T)\|^2_{L^2(\Omega_h)} + \int_0^T (\|\mathbb{P} \boldsymbol{e}_z\|^2_{L^2(\Omega_h)} + \|\boldsymbol{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}}\|^2_{L^2(\Omega_h)}) dt\\
\le & C h^{k+1} \int_0^T (\|\mathbb{P} \boldsymbol{e}_u\|_{L^2(\Omega_h)} + \|\boldsymbol{\Pi} \boldsymbol{e}_{\boldsymbol{q}}\|_{L^2(\Omega_h)} + \|\mathbb{P} \boldsymbol{e}_z\|_{L^2(\Omega_h)} + \|\boldsymbol{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}}\|_{L^2(\Omega_h)}) dt + C h^{2(k+1)},
\end{aligned}$$

where C depends on $\|u\|_{H^{k+4}(\Omega_h)}$, $\|u(T)\|_{H^{k+3}(\Omega_h)}$, $\|u(0)\|_{H^{k+3}(\Omega_h)}$, $\|u_t\|_{H^{k+3}(\Omega_h)}$ and T. Finally, (3.18) follows by using Gronwall's inequality. $\square$

Lemma 3.8. *For multidimensional triangular meshes, if we take the projection $\mathbb{P}$ to be the L^2 projection P^k defined by (3.10), and $\mathbf{\Pi}$ to be the projection Π^+ defined by (3.14), then we have*

$$\|\mathbb{P}\boldsymbol{e}_u(T)\|_{L^2(\Omega_h)} + \|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}}(T)\|_{L^2(\Omega_h)} + \int_0^T (\|\mathbb{P}\boldsymbol{e}_z\|_{L^2(\Omega_h)} + \|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}\|_{L^2(\Omega_h)})dt \le Ch^k,$$

where C depends on $\|u_t\|_{H^{k+1}(\Omega_h)}$, $\|u\|_{H^{k+4}(\Omega_h)}$ and T.

Proof. From Lemma 3.4 and Lemma 3.5, we have

$$\frac{1}{2}\frac{d}{dt}\|\mathbb{P}\boldsymbol{e}_u\|^2_{L^2(\Omega_h)} + \frac{1}{2}\frac{d}{dt}\|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}}\|^2_{L^2(\Omega_h)} + \|\mathbb{P}\boldsymbol{e}_z\|^2_{L^2(\Omega_h)} + \|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}\|^2_{L^2(\Omega_h)} = \sum_{i=1}^{3}(T_i + S_i).$$

Using the definition of the projections $\mathbb{P}$ and $\mathbf{\Pi}$, we get that

$$T_2 = S_2 = 0.$$

By the definition of the projection $\mathbb{P}$, (3.10), we have

$$\begin{aligned} T_1 =&(\boldsymbol{q} - \mathbf{\Pi}\boldsymbol{q}, \mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}})_{\Omega_h} - (\boldsymbol{\sigma} - \mathbf{\Pi}\boldsymbol{\sigma}, \mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}})_{\Omega_h} \\ \le&\|\boldsymbol{q} - \mathbf{\Pi}\boldsymbol{q}\|_{L^2(\Omega_h)}\|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}\|_{L^2(\Omega_h)} + \|\boldsymbol{\sigma} - \mathbf{\Pi}\boldsymbol{\sigma}\|_{L^2(\Omega_h)}\|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}}\|_{L^2(\Omega_h)} \end{aligned}$$

and

$$\begin{aligned} S_1 =&-((\boldsymbol{q} - \mathbf{\Pi}\boldsymbol{q})_t, \mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}})_{\Omega_h} - (\boldsymbol{\sigma} - \mathbf{\Pi}\boldsymbol{\sigma}, \mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}})_{\Omega_h} \\ \le&\|(\boldsymbol{q} - \mathbf{\Pi}\boldsymbol{q})_t\|_{L^2(\Omega_h)}\|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}}\|_{L^2(\Omega_h)} + \|\boldsymbol{\sigma} - \mathbf{\Pi}\boldsymbol{\sigma}\|_{L^2(\Omega_h)}\|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}\|_{L^2(\Omega_h)}. \end{aligned}$$

Hence

$$T_1 + S_1 \le Ch^{k+1}(\|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}}\|_{L^2(\Omega_h)} + \|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}\|_{L^2(\Omega_h)}),$$

Where the constant C depends only on $\|u\|_{H^{k+4}(\Omega_h)}$ and $\|u_t\|_{H^{k+2}(\Omega_h)}$. By the definition of the numerical traces, (2.5), and the definition of the projection $\mathbf{\Pi}$, (3.14b), we have

$$\begin{aligned} T_3 =&\langle 1, -(u - \widehat{\mathbb{P}u})\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}} \cdot \boldsymbol{n} + (z - \widehat{\mathbb{P}z})\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}} \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} \\ \le&\|u - \widehat{\mathbb{P}u}\|_{L^2(\partial\Omega_h)}\|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}} \cdot \boldsymbol{n}\|_{L^2(\partial\Omega_h)} + \|z - \widehat{\mathbb{P}z}\|_{L^2(\partial\Omega_h)}\|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}} \cdot \boldsymbol{n}\|_{L^2(\partial\Omega_h)}, \end{aligned}$$

and

$$\begin{aligned} S_3 =&\langle 1, -(u - \widehat{\mathbb{P}u})_t\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}} \cdot \boldsymbol{n} + (z - \widehat{\mathbb{P}z})\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}} \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} \\ \le&\|(u - \widehat{\mathbb{P}u})_t\|_{L^2(\partial\Omega_h)}\|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}} \cdot \boldsymbol{n}\|_{L^2(\partial\Omega_h)} + \|z - \widehat{\mathbb{P}z}\|_{L^2(\partial\Omega_h)}\|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}} \cdot \boldsymbol{n}\|_{L^2(\partial\Omega_h)}. \end{aligned}$$

Using inverse and trace inequalities, Lemma 3.2 and Lemma 3.3, we get

$$T_3 + S_3 \le Ch^k(\|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}}\|_{L^2(\Omega_h)} + \|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}\|_{L^2(\Omega_h)}),$$

where the constant C depends on $\|u_t\|_{H^{k+1}(\Omega_h)}$ and $\|u\|_{H^{k+3}(\Omega_h)}$. Hence

$$\sum_{i=0}^{3}(T_i + S_i) \le Ch^{2k} + \frac{1}{2}(\|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}}\|^2_{L^2(\Omega_h)} + \|\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}\|^2_{L^2(\Omega_h)}),$$

where C depends on $\|u_t\|_{H^{k+2}(\Omega_h)}$ and $\|u\|_{H^{k+4}(\Omega_h)}$. We complete the proof of the lemma by using Gronwall's inequality. $\square$

Proofs of Theorem 2.4 and Theorem 2.5: We only need to use triangle inequality, and apply Lemma 3.6 and Lemma 3.8.

Notice that in Lemma 3.8, we are not able to estimate the terms T_3 and S_3 in an optimal order, and we lose an order h due to the use of inverse inequality and trace inequality. This makes the error estimate in Theorem 2.5 sub-optimal.

3.3. **Fourth-order elliptic problems.** Suppose $(u, \boldsymbol{q}, z, \boldsymbol{\sigma})$ is the exact solution to the fourth-order time-dependent problem (2.2). At any time t, we can consider it as a solution to the following fourth-order elliptic problem

$$\boldsymbol{q} = \nabla u \quad \text{in } \Omega, \tag{3.19a}$$

$$z = \nabla \cdot \boldsymbol{q} \quad \text{in } \Omega, \tag{3.19b}$$

$$\boldsymbol{\sigma} = \nabla z \quad \text{in } \Omega, \tag{3.19c}$$

$$\tilde{f} = \nabla \cdot \boldsymbol{\sigma} \quad \text{in } \Omega. \tag{3.19d}$$

where $u, \boldsymbol{q}, z, \boldsymbol{\sigma}$ are periodic functions, and $\tilde{f} = f - u_t$.

The LDG methods give an approximation $(\boldsymbol{\sigma}_h, z_h, \boldsymbol{q}_h, u_h) \in \boldsymbol{V}_h \times W_h \times \boldsymbol{V}_h \times W_h$ to the exact solution $(\boldsymbol{\sigma}, z, \boldsymbol{q}, u)$, by requiring that

$$(\boldsymbol{q}_h, \boldsymbol{v})_{\Omega_h} + (u_h, \nabla \cdot \boldsymbol{v})_{\Omega_h} - \langle \widehat{u}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial \Omega_h} = 0, \tag{3.20a}$$

$$(z_h, \omega)_{\Omega_h} + (\boldsymbol{q}_h, \nabla \omega)_{\Omega_h} - \langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, \omega \rangle_{\partial \Omega_h} = 0, \tag{3.20b}$$

$$(\boldsymbol{\sigma}_h, \boldsymbol{\rho})_{\Omega_h} + (z_h, \nabla \cdot \boldsymbol{\rho})_{\Omega_h} - \langle \widehat{z}_h, \boldsymbol{\rho} \cdot \boldsymbol{n} \rangle_{\partial \Omega_h} = 0, \tag{3.20c}$$

$$-(\boldsymbol{\sigma}_h, \nabla \eta)_{\Omega_h} + \langle \widehat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}, \eta \rangle_{\partial \Omega_h} = (\tilde{f}, \eta)_{\Omega_h}, \tag{3.20d}$$

for all $(\boldsymbol{\rho}, \eta, \boldsymbol{v}, \omega) \in \boldsymbol{V}_h \times W_h \times \boldsymbol{V}_h \times W_h$, and

$$\int_{\Omega_h} (u - u_h) = 0. \tag{3.20e}$$

Remark: We make the assumption $\int_{\Omega_h} (u - u_h) = 0$ so that the approximation given by (3.20) is well-defined. In the elliptic problems with periodic boundary conditions, u_h is determined up to additive constants, so we can make such assumptions. The proof of the existence and uniqueness of the approximate solution is standard, and we skip it for simplicity.

Theorem 3.9. *Suppose u and u_h are the solutions of* (3.19) *and* (3.20)*, respectively. For $k \geq 1$, we have*

$$\|u - u_h\|_{L^2(\Omega_h)} \leq C h^{k+1}.$$

where C depends on $\|u\|_{H^{k+4}(\Omega_h)}$ and the elliptic regularity constant C_{er}.

We prove Theorem 3.9 by duality argument in the Appendix.

3.4. **Proof of Theorem 2.6.** Here, we apply the elliptic projection to prove the optimal convergence result of the fourth-order time-dependent problem, using the error estimate of the fourth-order elliptic problem shown in the previous subsection.

Suppose $(u, \boldsymbol{q}, z, \boldsymbol{\sigma})$ is the exact solution of the fourth-order time-dependent problem (2.2), $(u_h, \boldsymbol{q}_h, z_h, \boldsymbol{\sigma}_h)$ is the approximate solution of the time-dependent problem defined by the LDG method in (2.3) and $(R_h u, R_h \boldsymbol{q}, R_h z, R_h \boldsymbol{\sigma})$ is the approximate solution of the corresponding fourth-order elliptic problem given by (3.20).

It is easy to see that

$$B(R_h u, R_h \boldsymbol{q}, R_h z, R_h \boldsymbol{\sigma}; \eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}) = B(u, \boldsymbol{q}, z, \boldsymbol{\sigma}; \eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}),$$

for any $(\eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}) \in W_h \times \boldsymbol{V}_h \times W_h \times \boldsymbol{V}_h$. Note that

$$(u_{h\,t}, \eta)_{\Omega_h} + B(u_h, \boldsymbol{q}_h, z_h, \boldsymbol{\sigma}_h; \eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}) = (u_t, \eta)_{\Omega_h} + B(u, \boldsymbol{q}, z, \boldsymbol{\sigma}; \eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}),$$

so we have

$$\begin{aligned}
&((u_h - R_h u)_t, \eta)_{\Omega_h} + B(u_h - R_h u, \boldsymbol{q}_h - R_h \boldsymbol{q}, z_h - R_h z, \boldsymbol{\sigma}_h - R_h \boldsymbol{\sigma}; \eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}) \\
=&(u_{h\,t}, \eta)_{\Omega_h} + B(u_h, \boldsymbol{q}_h, z_h, \boldsymbol{\sigma}_h; \eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}) \\
&- (R_h u_t, \eta)_{\Omega_h} - B(R_h u, R_h \boldsymbol{q}, R_h z, R_h \boldsymbol{\sigma}; \eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}) \\
=&(u_t, \eta)_{\Omega_h} + B(u, \boldsymbol{q}, z, \boldsymbol{\sigma}; \eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}) \\
&- (R_h u_t, \eta)_{\Omega_h} - B(R_h u, R_h \boldsymbol{q}, R_h z, R_h \boldsymbol{\sigma}; \eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}) \\
=&(u_t - R_h u_t, \eta)_{\Omega_h}.
\end{aligned}$$

Taking $(\eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}) = (u_h - R_h u, \boldsymbol{\sigma}_h - R_h \boldsymbol{\sigma}, z_h - R_h z, \boldsymbol{q}_h - R_h \boldsymbol{q})$, we have

$$\frac{1}{2}\frac{\partial}{\partial t}\|u_h - R_h u\|^2_{L^2(\Omega_h)} + \|z_h - R_h z\|^2_{L^2(\Omega_h)} = (u_t - R_h u_t, u_h - R_h u)_{\Omega_h}, \tag{3.21}$$

which implies that

$$\frac{\partial}{\partial t}\|u_h - R_h u\|_{L^2(\Omega_h)} \leq \|u_t - R_h u_t\|_{L^2(\Omega_h)}.$$

Integrating with respect to t over $(0, T)$,

$$\|u_h(T) - R_h u(T)\|_{L^2(\Omega_h)} \leq \|u_h(0) - R_h u(0)\|_{L^2(\Omega_h)} + \int_0^T \|u_t - R_h u_t\|_{L^2(\Omega_h)}.$$

Note that $u_h(0) = \mathsf{P} u_0$ and $R_h u(0) = R_h u_0$, so by triangle inequality and Theorem 3.9, we have

$$\|u_h(0) - R_h u(0)\|_{L^2(\Omega_h)} \leq C h^{k+1},$$

where C depends on $\|u_0\|_{H^{k+4}(\Omega_h)}$ and C_{er}. From Theorem 3.9, we get

$$\|u_t - R_h u_t\|_{L^2(\Omega_h)} \leq C h^{k+1},$$

where C depends on $\|u_t\|_{H^{k+4}(\Omega_h)}$ and C_{er}. Hence

$$\|u_h(T) - R_h u(T)\|_{L^2(\Omega_h)} \leq C h^{k+1},$$

where C depends on $\|u_0\|_{H^{k+4}(\Omega_h)}$, $\|u_t\|_{H^{k+4}(\Omega_h)}$, C_{er} and T. By triangle inequality and Theorem 3.9,

$$\|u_h(T) - u(T)\|_{L^2(\Omega_h)} \leq \|u_h(T) - R_h u(T)\|_{L^2(\Omega_h)} + \|R_h u(T) - u(T)\|_{L^2(\Omega_h)} \leq C h^{k+1},$$

where C depends on $\|u_0\|_{H^{k+4}(\Omega_h)}$, $\|u(T)\|_{H^{k+4}(\Omega_h)}$, $\|u_t\|_{H^{k+4}(\Omega_h)}$, C_{er} and T. This completes the proof of Theorem 2.6.

4. Extensions

The techniques that we use for the fourth-order time-dependent problem (1.1) can also be applied to prove stability and error estimates for other types of problems. In this section, we extend Theorem 2.3–2.6 to partial differential equations whose orders are even, and to the linearized Cahn-Hilliard equations.

4.1. **Even-order equations.** We extend the L^2-stability and the error estimates to even-order equations, for example, sixth-order equations, eighth-order equations, etc.

To illustrate how we extend the analysis to higher even-order equations, we briefly sketch the corresponding results for the following sixth-order problem:

$$u_t - \triangle^3 u = f \quad \text{in } \Omega \times (0, T), \tag{4.22}$$

with periodic boundary condition and the initial condition

$$u|_{t=0} = u_0.$$

First we rewrite the sixth-order problem (4.22) into a system of first order equations

$$\boldsymbol{q} = \nabla u, \;\; z = \nabla \cdot \boldsymbol{q}, \;\; \boldsymbol{\sigma} = \nabla z, \;\; \xi = \nabla \cdot \boldsymbol{\sigma}, \;\; \boldsymbol{\zeta} = \nabla \xi, \;\; f = u_t - \nabla \cdot \boldsymbol{\zeta} \tag{4.23}$$

Then we can define the approximations $(u_h, \boldsymbol{q}_h, z_h, \boldsymbol{\sigma}_h, \xi_h, \boldsymbol{\zeta}_h) \in \boldsymbol{V}_h \times W_h \times \boldsymbol{V}_h \times W_h \times \boldsymbol{v}_h \times W_h$ to the exact solution $(u, \boldsymbol{q}, z, \boldsymbol{\sigma}, \xi, \boldsymbol{\zeta})$ by requiring that on each $K \in \Omega_h$,

$$\begin{aligned}
(\boldsymbol{q}_h, \boldsymbol{v})_K + (u_h, \nabla \cdot \boldsymbol{v})_K - \langle \widehat{u}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} =& 0, \\
(z_h, \omega)_K + (\boldsymbol{q}_h, \nabla \omega)_K - \langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, \omega \rangle_{\partial K} =& 0, \\
(\boldsymbol{\sigma}_h, \boldsymbol{\rho})_K + (z_h, \nabla \cdot \boldsymbol{\rho})_K - \langle \widehat{z}_h, \boldsymbol{\rho} \cdot \boldsymbol{n} \rangle_{\partial K} =& 0, \\
(\xi_h, \phi)_K + (\boldsymbol{\sigma}_h, \nabla \phi)_K - \langle \widehat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}, \phi \rangle_{\partial K} =& 0, \\
(\boldsymbol{\zeta}_h, \boldsymbol{\psi})_K + (\xi_h, \nabla \cdot \boldsymbol{\psi})_K - \langle \widehat{\xi}_h, \boldsymbol{\psi} \cdot \boldsymbol{n} \rangle_{\partial K} =& 0, \\
(u_{ht}, \eta)_K - (\boldsymbol{\zeta}_h, \nabla \eta)_K + \langle \widehat{\boldsymbol{\zeta}}_h \cdot \boldsymbol{n}, \eta \rangle_{\partial K} =& (f, \eta)_K,
\end{aligned}$$

for any $(\boldsymbol{v}, \omega, \boldsymbol{\rho}, \phi, \boldsymbol{\psi}, \eta) \in \boldsymbol{V}_h \times W_h \times \boldsymbol{V}_h \times W_h \times \boldsymbol{v}_h \times W_h$, where the alternative numerical fluxes are defined as

$$\widehat{u}_h = u_h^-, \quad \widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h^+, \quad \widehat{z}_h = z_h^-, \quad \widehat{\boldsymbol{\sigma}}_h = \boldsymbol{\sigma}_h^+, \quad \widehat{\xi}_h = \xi_h^-, \quad \widehat{\boldsymbol{\zeta}}_h = \boldsymbol{\zeta}_h^+,$$

or

$$\widehat{u}_h = u_h^+, \quad \widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h^-, \quad \widehat{z}_h = z_h^+, \quad \widehat{\boldsymbol{\sigma}}_h = \boldsymbol{\sigma}_h^-, \quad \widehat{\xi}_h = \xi_h^+, \quad \widehat{\boldsymbol{\zeta}}_h = \boldsymbol{\zeta}_h^-.$$

Similar to getting the identity (2.7), we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_h} u_h^2(x,t)dx + (\boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h)_{\Omega_h} + \mathcal{H} = (f, u_h)_{\Omega_h},$$

where

$$\mathcal{H} = \sum_{K \in \Omega_h} (-H_{\partial K}(u_h, \boldsymbol{\zeta}_h) + H_{\partial K}(\xi_h, \boldsymbol{q}_h) - H_{\partial K}(z_h, \boldsymbol{\sigma}_h)).$$

Then using the property of the numerical flux, Lemma 2.2, we get the following stability result.

Theorem 4.1. *(L^2-stability) Suppose that $f = 0$. The approximate solution of the sixth-order equation given by the LDG method* (2.3) *satisfies*

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_h} u_h^2(x,t)\,dx \le 0,$$

Similar to the a priori error estimates for fourth-order equations, Theorem 2.4 and Theorem 2.5, we have the following a priori error estimates for sixth-order equations.

Theorem 4.2. *Suppose* $(u, \boldsymbol{q}, z, \boldsymbol{\sigma}, \xi, \boldsymbol{\zeta})$ *is the solution to* (4.23) *and* $(u_h, \boldsymbol{q}_h, z_h, \boldsymbol{\sigma}_h, \xi_h, \boldsymbol{\zeta}_h)$ *is the approximation given by the LDG method. For one-dimensional meshes or multidimensional Cartesian meshes, we have*

$$\|u(T) - u_h(T)\|_{L^2(\Omega_h)} + \|\boldsymbol{q}(T) - \boldsymbol{q}_h(T)\|_{L^2(\Omega_h)} + \|z(T) - z_h(T)\|_{L^2(\Omega_h)} \leq C\, h^{k+1},$$

and

$$\int_0^T (\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\boldsymbol{L}^2(\Omega_h)} + \|\xi - \xi_h\|_{L^2(\Omega_h)} + \|\boldsymbol{\zeta} - \boldsymbol{\zeta}_h\|_{\boldsymbol{L}^2(\Omega_h)})dt \leq C\, h^{k+1}.$$

For multidimensional triangular meshes, we have

$$\|u(T) - u_h(T)\|_{L^2(\Omega_h)} + \|\boldsymbol{q}(T) - \boldsymbol{q}_h(T)\|_{\boldsymbol{L}^2(\Omega_h)} + \|z(T) - z_h(T)\|_{L^2(\Omega_h)} \leq C\, h^k,$$

and

$$\int_0^T (\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\boldsymbol{L}^2(\Omega_h)} + \|\xi - \xi_h\|_{L^2(\Omega_h)} + \|\boldsymbol{\zeta} - \boldsymbol{\zeta}_h\|_{\boldsymbol{L}^2(\Omega_h)})\, dt \leq C\, h^k,$$

where C *is a constant independent of* h.

4.2. **Linearized Cahn-Hilliard equations.** We extend the L^2-stability and the error estimates, Theorem 2.3–2.6, to the linearized Cahn-Hilliard equations.

The following problem can be considered as a simple version of the Cahn-Hilliard problem linearized in the neighborhood of some point:

$$u_t - \triangle u + \triangle^2 u = 0 \qquad \text{in } \Omega \times (0, T), \tag{4.24a}$$

with the boundary condition

$$\frac{\partial}{\partial \boldsymbol{n}} u = \frac{\partial}{\partial \boldsymbol{n}} \triangle u = 0 \qquad \text{on } \partial\Omega, \tag{4.24b}$$

and the initial condition

$$u = u_0 \quad \text{at } t = 0. \tag{4.24c}$$

The fourth-order equation (4.24a) can be written as a system of first order equations

$$\boldsymbol{q} = \nabla u, \quad z = \nabla \cdot \boldsymbol{q}, \quad \boldsymbol{\sigma} = \nabla(z + u), \quad f = u_t + \nabla \cdot \boldsymbol{\sigma}. \tag{4.25}$$

The approximation $(u_h, \boldsymbol{q}_h, z_h, \boldsymbol{\sigma}_h) \in \boldsymbol{V}_h \times W_h \times \boldsymbol{V}_h \times W_h$ given by the LDG method is defined by

$$(\boldsymbol{q}_h, \boldsymbol{v})_K + (u_h, \nabla \cdot \boldsymbol{v})_K - \langle \widehat{u}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} = 0, \tag{4.26}$$

$$(z_h, \omega)_K + (\boldsymbol{q}_h, \nabla \omega)_K - \langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, \omega \rangle_{\partial K} = 0, \tag{4.27}$$

$$(\boldsymbol{\sigma}_h, \boldsymbol{\rho})_K + (z_h + u_h, \nabla \cdot \boldsymbol{\rho})_K - \langle \widehat{z}_h + \widehat{u}_h, \boldsymbol{\rho} \cdot \boldsymbol{n} \rangle_{\partial K} = 0, \tag{4.28}$$

$$(u_{ht}, \eta)_K - (\boldsymbol{\sigma}_h, \nabla \eta)_K + \langle \widehat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}, \eta \rangle_{\partial K} = (f, \eta)_K, \tag{4.29}$$

for any $(\boldsymbol{v}, \omega, \boldsymbol{\rho}, \eta) \in \boldsymbol{V}_h \times W_h \times \boldsymbol{V}_h \times W_h$. The alternative numerical fluxes are defined as on inter-element faces

$$\widehat{u}_h = u_h^-, \quad \widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h^+, \quad \widehat{z}_h = z_h^-, \quad \widehat{\boldsymbol{\sigma}}_h = \boldsymbol{\sigma}_h^+,$$

or

$$\widehat{u}_h = u_h^+, \quad \widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h^-, \quad \widehat{z}_h = z_h^+, \quad \widehat{\boldsymbol{\sigma}}_h = \boldsymbol{\sigma}_h^-,$$

and on the boundary of Ω

$$\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} = \widehat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n} = 0, \quad \widehat{u}_h = u_h, \quad \widehat{z}_h = z_h.$$

Suppose $f = 0$. Similar to the identities (2.6), from above weak formulation we get

$$(u_{ht}, \eta)_K + \tilde{\tilde{B}}_K(u_h, \boldsymbol{q}_h, z_h, \boldsymbol{\sigma}_h; \eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}) = 0,$$

where

$$\tilde{\tilde{B}}_K(u_h, \boldsymbol{q}_h, z_h, \boldsymbol{\sigma}_h; \eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}) = B_K(u_h, \boldsymbol{q}_h, z_h, \boldsymbol{\sigma}_h; \eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}) + (u_h, \boldsymbol{\rho})_K - \langle \widehat{u}_h, \boldsymbol{\rho}\cdot\boldsymbol{n}\rangle_{\partial K},$$

where B_K is defined in (2.6). Taking $(\boldsymbol{v}, \omega, \boldsymbol{\rho}, \eta) = (u_h, \boldsymbol{\sigma}_h, z_h, \boldsymbol{q}_h)$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_h} u_h^2(x,t)dx + (z_h, z_h)_{\Omega_h} + \mathcal{H} = -(u_h, z_h)_{\Omega_h},$$

where

$$\mathcal{H} = \sum_{K\in\Omega_h} (H_{\partial K}(u_h, \boldsymbol{\sigma}_h) - H_{\partial K}(z_h + u_h, \boldsymbol{q}_h)).$$

Note that $\mathcal{H} = 0$ by the property of the entropy flux, and that

$$-(u_h, z_h) \le \frac{1}{2}(u_h, u_h)_{\Omega_h} + \frac{1}{2}(z_h, z_h)_{\Omega_h}.$$

Using Gronwall's inequality, we get the following stability result.

Theorem 4.3. *Suppose that $f = 0$. The approximate solution of the linearized Cahn-Hilliard equation given by the LDG method* (4.26) *satisfies*

$$\int_{\Omega_h} u_h^2(x,t)\,dx \le e^t \int_{\Omega_h} u_0^2(x)\,dx.$$

Similar to Lemma 3.4 and Lemma 3.5, we get the following lemma.

Lemma 4.4.

$$(\mathbb{P}\boldsymbol{e}_{ut}, \mathbb{P}\boldsymbol{e}_u)_{\Omega_h} + (\mathbb{P}\boldsymbol{e}_z, \mathbb{P}\boldsymbol{e}_z)_{\Omega_h} = T_1 + T_2 + T_3 + T_4,$$
$$(\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{q}_t}, \boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{q}})_{\Omega_h} + (\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}})_{\Omega_h} = S_1 + S_2 + S_3 + S_4,$$

where T_i and S_i, $i = 1, 2, 3$, are the same as in Lemma 3.4 and Lemma 3.5, and

$$T_4 = -(\boldsymbol{e}_u, \nabla\cdot\boldsymbol{\Pi}\boldsymbol{q}) + \langle \widehat{\boldsymbol{e}_u}, \boldsymbol{\Pi}\boldsymbol{q}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h},$$
$$S_4 = -(\boldsymbol{e}_u, \nabla\cdot\boldsymbol{\Pi}\boldsymbol{\sigma}) + \langle \widehat{\boldsymbol{e}_u}, \boldsymbol{\Pi}\boldsymbol{\sigma}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}.$$

It is easy to check that

$$\begin{aligned} T_4 \le &-(z - \mathbb{P}z, \mathbb{P}\boldsymbol{e}_u)_{\Omega_h} + \frac{1}{2}(\mathbb{P}\boldsymbol{e}_u, \mathbb{P}\boldsymbol{e}_u)_{\Omega_h} + \frac{1}{2}(\mathbb{P}\boldsymbol{e}_z, \mathbb{P}\boldsymbol{e}_z)_{\Omega_h} \\ &-(u - \mathbb{P}u, \nabla\cdot\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{q}})_{\Omega_h} + \langle u - \widehat{\mathbb{P}u}, \boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}, \\ S_4 \le &(\boldsymbol{q} - \boldsymbol{\Pi}\boldsymbol{q}, \boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}})_{\Omega_h} + \frac{1}{2}(\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{q}}, \boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{q}})_{\Omega_h} + \frac{1}{2}(\mathbb{P}\boldsymbol{e}_{\boldsymbol{\sigma}}, \mathbb{P}\boldsymbol{e}_{\boldsymbol{\sigma}})_{\Omega_h}. \end{aligned}$$

The estimates of T_4 and S_4 are similar to those of T_i and S_i, $i = 1, 2, 3$, in the proofs of Lemma 3.6 and Lemma 3.8. So, we get that Lemma 3.6 and Lemma 3.8 also hold for the linearized Cahn-Hilliard equations. Therefore, similar to Theorem 2.4 and Theorem 2.5, we have the following a priori error estimates.

Theorem 4.5. *Suppose* $(u, \boldsymbol{q}, z, \boldsymbol{\sigma})$ *is the solution to* (4.25) *and* $(u_h, \boldsymbol{q}_h, z_h, \boldsymbol{\sigma}_h)$ *is the approximation given by the LDG method* (4.26). *For one-dimensional meshes or multidimensional Cartesian meshes, we have*

$$\|u(T) - u_h(T)\|_{L^2(\Omega_h)} + \|\boldsymbol{q}(T) - \boldsymbol{q}_h(T)\|_{L^2(\Omega_h)} \leq C\,h^{k+1},$$

and

$$\int_0^T (\|z - z_h\|_{L^2(\Omega_h)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\boldsymbol{L}^2(\Omega_h)})dt \leq C\,h^{k+1}.$$

For multidimensional triangular meshes, we have

$$\|u(T) - u_h(T)\|_{L^2(\Omega_h)} + \|\boldsymbol{q}(T) - \boldsymbol{q}_h(T)\|_{\boldsymbol{L}^2(\Omega_h)} \leq C\,h^{k},$$

and

$$\int_0^T (\|z - z_h\|_{L^2(\Omega_h)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\boldsymbol{L}^2(\Omega_h)})\,dt \leq C\,h^{k},$$

where C *is a constant independent of* h.

To get an optimal error estimate for triangular meshes, we consider the corresponding elliptic problem of (4.25):

$$\boldsymbol{q} = \nabla u, \quad z = \nabla\cdot\boldsymbol{q}, \quad \boldsymbol{\sigma} = \nabla(z+u), \quad f = u_t + \nabla\cdot\boldsymbol{\sigma}. \tag{4.30}$$

Similar to Theorem 3.9, we assume the elliptic regularity (2.9) and get the optimal convergence of the elliptic problem (4.30) by duality argument. Then we set $\tilde{\tilde{B}} := \sum_{K\in\Omega_h} \tilde{\tilde{B}}$ and define the elliptic projection by

$$\tilde{\tilde{B}}(u_h, \boldsymbol{q}_h, z_h, \boldsymbol{\sigma}_h; \eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}) = \tilde{\tilde{B}}(u, \boldsymbol{q}, z, \boldsymbol{\sigma}; \eta, \boldsymbol{v}, \omega, \boldsymbol{\rho}),$$

where $(u, \boldsymbol{q}, z, \boldsymbol{\sigma})$ is the exact solution to the linearized Cahn-Hilliard equation (4.25). Similar to (3.21), we get

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\|u_h - R_h u\|^2_{L^2(\Omega_h)} + \|z_h - R_h z\|^2_{L^2(\Omega_h)}\\ =&(u_t - R_h u_t, u_h - R_h u)_{\Omega_h} - (u_h - R_h u, z_h - R_h z),\end{aligned}$$

which implies that

$$\frac{\partial}{\partial t}\|u_h - R_h u\|^2_{L^2(\Omega_h)} \leq \int_0^T \|u_t - R_h u_t\|^2_{L^2(\Omega_h)} + 2\int_0^T \|u_h - R_h u\|^2_{L^2(\Omega_h)}.$$

Using the optimal convergence of the elliptic projection and Gronwall's inequality, we get the following optimal error estimate.

Theorem 4.6. *Suppose* u *is the exact solution to* (4.24), u_h *is the approximate solution given by the LDG method, and the elliptic regularity* (2.9) *holds for the adjoint problem of* (4.30). *Then for* $k \geq 1$, *we have*

$$\|u(T) - u_h(T)\|_{L^2(\Omega_h)} \leq Ch^{k+1}.$$

where C *is a constant independent of* h.

5. Numerical results

In this section, we use numerical experiments to validate the theoretical convergence properties of the LDG method for the fourth-order time-dependent problem.

in [28], numerical results were displayed for the LDG method on the one-dimensional equation $u_t + u_{xxxx} = 0$ with periodic boundary condition. The L^∞ error on uniform meshes at $T = 1$ is of order $k+1$ for $k = 0, 1, 2, 3$.

Here we implement the LDG method for the fourth-order time-dependent problem on two-dimensional triangular meshes. The uniform meshes that we use is obtained by discretizing $\Omega = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})$ with squares of side 2^{-l} which are then divided into two triangles as indicated in Fig. 1; the resulting mesh is denoted by "mesh=l".

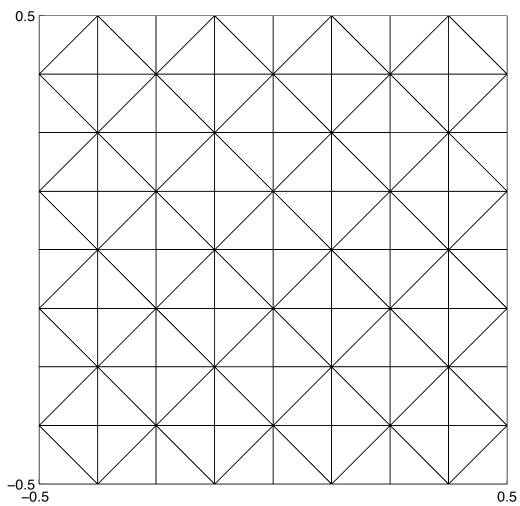

FIGURE 1. Example of a mesh with $h = 1/2^3$.

The test problem is obtained by choosing f so that the exact solution of (1.1) is $u(x, y, t) = \sin(2\pi x)\sin(2\pi y)\exp(t)$ on the domain $\Omega \times (0, 1)$. The history of convergence of the LDG method with a fourth-order implicit Runge-Kutta time discretization is displayed in Table 1 for polynomials of degree $k = 1, 2$.

In Table 1, we observe that for k=1,2, the quantity $\|u - u_h\|_{L^2(\Omega)}$ has optimal convergence rates, which is consistent with Theorem 2.6. For the case of $k = 0$, numerical results show that the approximation to u does not converge.

6. Conclusions

In this paper we apply the LDG method to fourth-order time-dependent problems, and we obtain optimal approximations for one-dimensional and multidimensional meshes by using different techniques. Then we extend our optimal convergence results to higher even-order equations and the linearized Cahn-Hilliard equation.

TABLE 1. History of convergence of the LDG method for $u_t + \triangle^2 u = f$.

	mesh	$\|u-u_h\|_{L^2(\Omega)}$			mesh	$\|u-u_h\|_{L^2(\Omega)}$	
k	ℓ	error	order	k	ℓ	error	order
	1	.86e-0	-		1	.23e-0	-
	2	.40e-0	1.12		2	.22e-1	3.40
1	3	.80e-1	2.30	2	3	.72e-2	1.59
	4	.22e-1	1.86		4	.91e-3	2.99
	5	.62e-2	1.82		5	.11e-3	3.00
	6	.16e-2	1.95		6	.14e-4	2.98

7. Appendix: Proof of Theorem 3.9

Here we prove Theorem 3.9, the optimal error estimate of the elliptic problem, in three steps. First, we introduce error equations and get intermediate error estimates for $\boldsymbol{q}, z$ and $\boldsymbol{\sigma}$ that depend on the error of u. Second, we use duality argument to get an intermediate result about error of u that depends on errors of $\boldsymbol{q}, z$ and $\boldsymbol{\sigma}$. Finally, we combine all the intermediate results to obtain the a priori error estimate of u.

7.1. First, let us introduce the error equations. Because the exact solution of the elliptic problem (3.19) satisfies the weak formulation (3.20), we have the following error equations.

$$(\boldsymbol{e_q}, \boldsymbol{v})_K + (\boldsymbol{e}_u, \nabla\cdot \boldsymbol{v})_K - \langle \widehat{\boldsymbol{e}_u}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} = 0, \tag{7.31a}$$

$$(\boldsymbol{e}_z, \omega)_K + (\boldsymbol{e_q}, \nabla\omega)_K - \langle \widehat{\boldsymbol{e_q}}\cdot\boldsymbol{n}, \omega\rangle_{\partial K} = 0, \tag{7.31b}$$

$$(\boldsymbol{e_\sigma}, \boldsymbol{\rho})_K + (\boldsymbol{e}_z, \nabla\cdot \boldsymbol{\rho})_K - \langle \widehat{\boldsymbol{e}_z}, \boldsymbol{\rho}\cdot\boldsymbol{n}\rangle_{\partial K} = 0, \tag{7.31c}$$

$$-(\boldsymbol{e_\sigma}, \nabla\eta)_K + \langle \widehat{\boldsymbol{e_\sigma}}\cdot\boldsymbol{n}, \eta\rangle_{\partial K} = 0, \tag{7.31d}$$

Now we use above error equations to prove the following intermediate results.

Lemma 7.1. *Suppose* $\boldsymbol{\Pi}$ *and* $\mathbb{P}$ *are projections defined by* (3.14) *and* (3.10)*. We have*

$$(i)\ \ \|\boldsymbol{\Pi}\boldsymbol{e_\sigma}\|_{L^2(\Omega_h)} \le Ch^k, \tag{7.32a}$$

$$(ii)\ \ \|\mathbb{P}\boldsymbol{e}_z\|_{L^2(\Omega_h)} \le Ch^k + Ch^{k/2}\|\mathbb{P}\boldsymbol{e_q}\|^{1/2}_{L^2(\Omega_h)}, \tag{7.32b}$$

$$(iii)\ \ \|\boldsymbol{\Pi}\boldsymbol{e_q}\|_{L^2(\Omega_h)} \le Ch^k + C\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)}, \tag{7.32c}$$

where C *depends on* $\|u\|_{H^{k+4}(\Omega_h)}$.

Proof. (i) Taking $\boldsymbol{\rho} = \boldsymbol{\Pi}\boldsymbol{e_\sigma}$ in the error equation (7.31c), we get

$$\begin{aligned}(\boldsymbol{e_\sigma}, \boldsymbol{\Pi}\boldsymbol{e_\sigma})_{\Omega_h} &= -(\boldsymbol{e}_z, \nabla\cdot\boldsymbol{\Pi}\boldsymbol{e_\sigma})_{\Omega_h} + \langle \widehat{\boldsymbol{e}_z}, \boldsymbol{\Pi}\boldsymbol{e_\sigma}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\\ &= -(z-\mathbb{P}z, \nabla\cdot\boldsymbol{\Pi}\boldsymbol{e_\sigma})_{\Omega_h} - (\mathbb{P}\boldsymbol{e}_z, \nabla\cdot\boldsymbol{\Pi}\boldsymbol{e_\sigma})_{\Omega_h} + \langle \widehat{\boldsymbol{e}_z}, \boldsymbol{\Pi}\boldsymbol{e_\sigma}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}.\end{aligned}$$

By the property of the projection $\mathbb{P}$, we get that the first term on the right hand side is zero. Hence, by integration by parts, we have

$$\begin{aligned}(\boldsymbol{e_\sigma},\boldsymbol{\Pi e_\sigma})_{\Omega_h} =&(\nabla\mathbb{P}\boldsymbol{e}_z,\boldsymbol{\Pi e_\sigma})_{\Omega_h}-\langle\mathbb{P}\boldsymbol{e}_z,\boldsymbol{\Pi e_\sigma}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}+\langle\widehat{\boldsymbol{e}_z},\boldsymbol{\Pi e_\sigma}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h},\\ =&(\nabla\mathbb{P}\boldsymbol{e}_z,\boldsymbol{e_\sigma})_{\Omega_h}-\langle\mathbb{P}\boldsymbol{e}_z,\boldsymbol{\Pi e_\sigma}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}+\langle\widehat{\boldsymbol{e}_z},\boldsymbol{\Pi e_\sigma}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h},\end{aligned}$$

by the property of $\boldsymbol{\Pi}$. Then taking $\eta=\mathbb{P}\boldsymbol{e}_z$ in the error equation (7.31d),

$$\begin{aligned}(\boldsymbol{e_\sigma},\boldsymbol{\Pi e_\sigma})_{\Omega_h} =&\langle\widehat{\boldsymbol{e_\sigma}}\cdot\boldsymbol{n},\mathbb{P}\boldsymbol{e}_z\rangle_{\partial\Omega_h}-\langle\mathbb{P}\boldsymbol{e}_z,\boldsymbol{\Pi e_\sigma}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}+\langle\widehat{\boldsymbol{e}_z},\boldsymbol{\Pi e_\sigma}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h},\\ =&\langle(\boldsymbol{\sigma}-\widehat{\boldsymbol{\Pi\sigma}})\cdot\boldsymbol{n},\mathbb{P}\boldsymbol{e}_z\rangle_{\partial\Omega_h}+\langle z-\widehat{\mathbb{P}z},\boldsymbol{\Pi e_\sigma}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}+\sum_{K\in\Omega_h}H_{\partial K}(\mathbb{P}\boldsymbol{e}_z,\boldsymbol{\Pi e_\sigma}).\end{aligned}$$

Using the property of $\boldsymbol{\Pi}$ and Lemma 2.2, we have that the first and the third terms on the right hand are zeros. Hence

$$(\boldsymbol{e_\sigma},\boldsymbol{\Pi e_\sigma})_{\Omega_h}=\langle z-\widehat{\mathbb{P}z},\boldsymbol{\Pi e_\sigma}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h},$$

which implies that

$$\begin{aligned}\|\boldsymbol{\Pi e_\sigma}\|^2_{L^2(\Omega_h)}=&-(\boldsymbol{\sigma}-\boldsymbol{\Pi\sigma},\boldsymbol{\Pi e_\sigma})_{\Omega_h}+\langle z-\widehat{\mathbb{P}z},\boldsymbol{\Pi e_\sigma}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\\ \le&\|\boldsymbol{\sigma}-\boldsymbol{\Pi\sigma}\|_{L^2(\Omega_h)}\|\boldsymbol{\Pi e_\sigma}\|_{L^2(\Omega_h)}+Ch^{-1/2}\|z-\mathbb{P}z\|_{L^2(\partial\Omega_h)}\|\boldsymbol{\Pi e_\sigma}\|_{L^2(\Omega_h)},\end{aligned}$$

So by the properties of $\boldsymbol{\Pi}$ and $\mathbb{P}$, and trace inequality, we have

$$\|\boldsymbol{\Pi e_\sigma}\|_{L^2(\Omega_h)}\le\|\boldsymbol{\sigma}-\boldsymbol{\Pi\sigma}\|_{L^2(\Omega_h)}+Ch^{-1/2}\|z-\mathbb{P}z\|_{L^2(\partial\Omega_h)}\le Ch^k,$$

where C depends on $\|u\|_{H^{k+4}(\Omega_h)}$.

(ii) Adding error equations (7.31b)-(7.31d), subtracting (7.31a), and summing over all element $K\in\Omega_h$, we get

$$B(\boldsymbol{e}_u,\boldsymbol{e_q},\boldsymbol{e}_z,\boldsymbol{e_\sigma};\eta,\boldsymbol{v},\omega,\boldsymbol{\rho})=0.$$

Taking $(\eta,\boldsymbol{v},\omega,\boldsymbol{\rho})=(\mathbb{P}\boldsymbol{e}_u,\boldsymbol{\Pi e_\sigma},\mathbb{P}\boldsymbol{e}_z,\boldsymbol{\Pi e_q})$, we get

$$(\mathbb{P}\boldsymbol{e}_z,\mathbb{P}\boldsymbol{e}_z)_{\Omega_h}=\sum_{i=1}^5 T_i,$$

where

$$\begin{aligned}T_1:=&-(z-\mathbb{P}z,\mathbb{P}\boldsymbol{e}_z)_{\Omega_h}+(\boldsymbol{\sigma}-\boldsymbol{\Pi\sigma},\nabla\mathbb{P}\boldsymbol{e}_u)_{\Omega_h}-(z-\mathbb{P}z,\nabla\cdot\boldsymbol{\Pi e_q})_{\Omega_h}\\ &-(\boldsymbol{q}-\boldsymbol{\Pi q},\nabla\mathbb{P}\boldsymbol{e}_z)_{\Omega_h}+(u-\mathbb{P}u,\nabla\cdot\boldsymbol{\Pi e_\sigma})_{\Omega_h},\\ T_2:=&-(\boldsymbol{\sigma}-\boldsymbol{\Pi\sigma},\boldsymbol{\Pi e_q})_{\Omega_h}+(\boldsymbol{q}-\boldsymbol{\Pi q},\boldsymbol{\Pi e_\sigma})_{\Omega_h},\\ T_3:=&-\langle(\boldsymbol{\sigma}-\widehat{\boldsymbol{\Pi\sigma}})\cdot\boldsymbol{n},\mathbb{P}\boldsymbol{e}_u\rangle_{\partial\Omega_h}+\langle(\boldsymbol{q}-\widehat{\boldsymbol{\Pi q}})\cdot\boldsymbol{n},\mathbb{P}\boldsymbol{e}_z\rangle_{\partial\Omega_h},\\ T_4:=&\langle z-\widehat{\mathbb{P}z},\boldsymbol{\Pi e_q}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}-\langle u-\widehat{\mathbb{P}u},\boldsymbol{\Pi e_\sigma}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h},\\ T_5:=&-\sum_{K\in\Omega_h}(H_{\partial K}(\mathbb{P}\boldsymbol{e}_u,\boldsymbol{\Pi e_\sigma})-H_{\partial K}(\mathbb{P}\boldsymbol{e}_z,\boldsymbol{\Pi e_q})).\end{aligned}$$

By the properties of $\mathbb{P}$ and $\boldsymbol{\Pi}$, we have

$$T_1=T_3=0.$$

By Lemma 2.2,

$$T_5 = 0.$$

Using inverse inequality and trace inequality, we have

$$T_2 \le Ch^{k+1}(\|\boldsymbol{\Pi e_q}\|_{L^2(\Omega_h)} + \|\boldsymbol{\Pi e_\sigma}\|_{L^2(\Omega_h)}),$$

and

$$T_4 \le Ch^{k}(\|\boldsymbol{\Pi e_q}\|_{L^2(\Omega_h)} + \|\boldsymbol{\Pi e_\sigma}\|_{L^2(\Omega_h)}),$$

where C depends on $\|u\|_{H^{k+4}(\Omega_h)}$. So

$$\begin{aligned}\|\mathbb{P}\boldsymbol{e}_z\|^2_{L^2(\Omega_h)} \le& Ch^k(\|\mathbb{P}\boldsymbol{e_q}\|_{L^2(\Omega_h)} + \|\boldsymbol{\Pi e_\sigma}\|_{L^2(\Omega_h)})\\ \le& Ch^k\|\mathbb{P}\boldsymbol{e_q}\|_{L^2(\Omega_h)} + Ch^{2k},\end{aligned}$$

by the estimate of error of $\boldsymbol{\sigma}$, (7.32a).

(iii): Taking $\boldsymbol{v} = \boldsymbol{\Pi e_q}$ in the error equation (7.31a), we get

$$\begin{aligned}(\boldsymbol{e_q}, \boldsymbol{\Pi e_q})_{\Omega_h} =& -(\boldsymbol{e}_u, \nabla\cdot\boldsymbol{\Pi e_q})_{\Omega_h} + \langle\widehat{\boldsymbol{e}_u}, \boldsymbol{\Pi e_q}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\\ =& -(u - \mathbb{P}u, \nabla\cdot\boldsymbol{\Pi e_q})_{\Omega_h} - (\mathbb{P}\boldsymbol{e}_u, \nabla\cdot\boldsymbol{\Pi e_q})_{\Omega_h} + \langle\widehat{\boldsymbol{e}_u}, \boldsymbol{\Pi e_q}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}.\end{aligned}$$

By the property of the L^2-projection $\mathbb{P}$, (3.10), we get that the first term on the right hand side is zero. Hence, using integration by parts, we get

$$\begin{aligned}(\boldsymbol{e_q}, \boldsymbol{\Pi e_q})_{\Omega_h} =&(\nabla\mathbb{P}\boldsymbol{e}_u, \boldsymbol{\Pi e_q})_{\Omega_h} - \langle\mathbb{P}\boldsymbol{e}_u, \boldsymbol{\Pi e_q}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + \langle\widehat{\boldsymbol{e}_u}, \boldsymbol{\Pi e_q}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h},\\ =&(\nabla\mathbb{P}\boldsymbol{e}_u, \boldsymbol{e_q})_{\Omega_h} - \langle\mathbb{P}\boldsymbol{e}_u, \boldsymbol{\Pi e_q}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + \langle\widehat{\boldsymbol{e}_u}, \boldsymbol{\Pi e_q}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h},\end{aligned}$$

by the property of $\boldsymbol{\Pi}$. Then using the error equation (7.31b) by taking $\omega = \mathbb{P}\boldsymbol{e}_u$, we have

$$\begin{aligned}(\boldsymbol{e_q}, \boldsymbol{\Pi e_q})_{\Omega_h} =& -(\boldsymbol{e}_z, \mathbb{P}\boldsymbol{e}_u)_{\Omega_h} + \langle\widehat{\boldsymbol{e_q}}\cdot\boldsymbol{n}, \mathbb{P}\boldsymbol{e}_u\rangle_{\partial\Omega_h}\\ &- \langle\mathbb{P}\boldsymbol{e}_u, \boldsymbol{\Pi e_q}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + \langle\widehat{\boldsymbol{e}_u}, \boldsymbol{\Pi e_q}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h},\end{aligned}$$

which implies that

$$\begin{aligned}\|\boldsymbol{\Pi e_q}\|^2_{L^2(\Omega_h)} =& -(\boldsymbol{q} - \boldsymbol{\Pi q}, \boldsymbol{\Pi e_q})_{\Omega_h} - (\boldsymbol{e}_z, \mathbb{P}\boldsymbol{e}_u)_{\Omega_h}\\ &+ \langle\widehat{\boldsymbol{e_q}}\cdot\boldsymbol{n}, \mathbb{P}\boldsymbol{e}_u\rangle_{\partial\Omega_h} - \langle\mathbb{P}\boldsymbol{e}_u, \boldsymbol{\Pi e_q}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + \langle\widehat{\boldsymbol{e}_u}, \boldsymbol{\Pi e_q}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}.\end{aligned}$$

We can rewrite above identity as

$$\begin{aligned}\|\boldsymbol{\Pi e_q}\|^2_{L^2(\Omega_h)} =& -(\boldsymbol{q} - \boldsymbol{\Pi q}, \boldsymbol{\Pi e_q})_{\Omega_h} - (\boldsymbol{e}_z, \mathbb{P}\boldsymbol{e}_u)_{\Omega_h} + \langle(\widehat{\boldsymbol{q} - \boldsymbol{\Pi q}})\cdot\boldsymbol{n}, \mathbb{P}\boldsymbol{e}_u\rangle_{\partial\Omega_h}\\ &+ \langle u - \widehat{\mathbb{P}u}, \boldsymbol{\Pi e_q}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + \sum_{K\in\Omega_h} H_{\partial K}(\mathbb{P}\boldsymbol{e}_u, \boldsymbol{\Pi e_q}).\end{aligned}$$

By Lemma 2.2 and the definition of the projection $\boldsymbol{\Pi}$, we get

$$\begin{aligned}\|\boldsymbol{\Pi e_q}\|^2_{L^2(\Omega_h)} =& -(\boldsymbol{q} - \boldsymbol{\Pi q}, \boldsymbol{\Pi e_q})_{\Omega_h} - (\boldsymbol{e}_z, \mathbb{P}\boldsymbol{e}_u)_{\Omega_h} + \langle u - \widehat{\mathbb{P}u}, \boldsymbol{\Pi e_q}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\\ \le&\|\boldsymbol{q} - \boldsymbol{\Pi q}\|_{L^2(\Omega_h)}\|\boldsymbol{\Pi e_q}\|_{L^2(\Omega_h)} + \|z - \mathbb{P}z\|_{L^2(\Omega_h)}\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)}\\ &+ \|\mathbb{P}\boldsymbol{e}_z\|_{L^2(\Omega_h)}\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)} + Ch^{-1/2}\|u - \mathbb{P}u\|_{L^2(\partial\Omega_h)}\|\boldsymbol{\Pi e_q}\|_{L^2(\Omega_h)}.\end{aligned}$$

This implies that

$$\|\mathbf{\Pi} \boldsymbol{e_q}\|_{L^2(\Omega_h)} \leq Ch^k + C\|\mathbb{P}\boldsymbol{e}_z\|_{L^2(\Omega_h)} + C\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)},$$

where C depends on $\|u\|_{H^{k+4}(\Omega_h)}$. Using the error estimate of z, (7.32b), we get

$$\|\mathbf{\Pi} \boldsymbol{e_q}\|_{L^2(\Omega_h)} \leq Ch^k + C\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)}.$$

□

7.2. In this subsection, we consider the adjoint problem (2.8) and use the duality argument to prove an intermediate result about error of u.

Before state and prove the intermediate result, let us introduce an L^2-projection P_∂ defined as follows. Given any function $\eta \in L^2(\mathscr{E}_h)$ and an arbitrary face $e \in \mathscr{E}_h$, the restriction of $\mathsf{P}_\partial \eta$ to e is defined as the element of $\mathcal{P}^k(e)$ that satisfies

$$\langle \mathsf{P}_\partial \eta - \eta, \omega\rangle_e = 0, \qquad \forall\, \omega \in \mathcal{P}^k(e).$$

Next, we state the intermediate result of error of u and prove it in four steps.

Lemma 7.2. *We have*

$$(\mathbb{P}\boldsymbol{e}_u, \eta)_{\Omega_h} = \sum_{i=1}^{4} E_i,$$

where

$$\begin{aligned}
E_1 =&(\boldsymbol{e_q}, \boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta})_{\Omega_h} + (z - \mathbb{P}z, \mathbb{P}\xi - \xi)_{\Omega_h} - (\boldsymbol{e_\sigma}, \mathbf{\Pi}\boldsymbol{\psi} - \boldsymbol{\psi})_{\Omega_h},\\
E_2 =& -(\nabla\cdot(\boldsymbol{q} - \mathbf{\Pi}\boldsymbol{q}), \mathbb{P}\xi - \xi)_{\Omega_h} - (\nabla(z - \mathbb{P}z), \boldsymbol{\psi} - \mathbf{\Pi}\boldsymbol{\psi})_{\Omega_h} - (\nabla(\boldsymbol{\sigma} - \mathbf{\Pi}\boldsymbol{\sigma}), \mathbb{P}\varphi - \varphi)_{\Omega_h},\\
E_3 =&\langle(\widehat{\boldsymbol{q}}_h - \boldsymbol{q}_h, (\mathbb{P}\xi - \xi)\boldsymbol{n}\rangle_{\partial\Omega_h} + \langle(\widehat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h)\cdot\boldsymbol{n}, \mathbb{P}\varphi - \varphi\rangle_{\partial\Omega_h}.
\end{aligned}$$

Proof. **Step 1:** By the adjoint equation (2.8d), we have

$$\begin{aligned}
(\mathbb{P}\boldsymbol{e}_u, \eta)_{\Omega_h} =&(\mathbb{P}\boldsymbol{e}_u, \nabla\cdot\boldsymbol{\zeta})_{\Omega_h}\\
=&(\mathbb{P}\boldsymbol{e}_u, \nabla\cdot(\boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta}))_{\Omega_h} + (\mathbb{P}\boldsymbol{e}_u, \nabla\cdot\mathbf{\Pi}\boldsymbol{\zeta})_{\Omega_h}
\end{aligned}$$

Using integration by parts,

$$\begin{aligned}
(\mathbb{P}\boldsymbol{e}_u, \eta)_{\Omega_h} =&\langle\mathbb{P}\boldsymbol{e}_u, (\boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} - (\nabla\mathbb{P}\boldsymbol{e}_u, \boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta})_{\Omega_h} + (\mathbb{P}\boldsymbol{e}_u, \nabla\cdot\mathbf{\Pi}\boldsymbol{\zeta})_{\Omega_h}\\
=&\langle\mathbb{P}\boldsymbol{e}_u, (\boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} - (\nabla\mathbb{P}\boldsymbol{e}_u, \boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta})_{\Omega_h}\\
&+ (\boldsymbol{e}_u, \nabla\cdot\mathbf{\Pi}\boldsymbol{\zeta})_{\Omega_h} - (u - \mathbb{P}u, \nabla\cdot\mathbf{\Pi}\boldsymbol{\zeta})_{\Omega_h}.
\end{aligned}$$

Using the property of the projection $\mathbf{\Pi}$ and $\mathbb{P}$, we have that the second and the fourth terms on the right hand side of the last equality are zeros. So

$$\begin{aligned}
(\mathbb{P}\boldsymbol{e}_u, \eta)_{\Omega_h} =&\langle\mathbb{P}\boldsymbol{e}_u, (\boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + (\boldsymbol{e}_u, \nabla\cdot\mathbf{\Pi}\boldsymbol{\zeta})_{\Omega_h}\\
=&\langle\mathbb{P}\boldsymbol{e}_u, (\boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} - (\boldsymbol{e_q}, \mathbf{\Pi}\boldsymbol{\zeta}) + \langle\widehat{\boldsymbol{e}_u}, \mathbf{\Pi}\boldsymbol{\zeta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}
\end{aligned}$$

by the error equation (7.31a). Because u, $\widehat{u}_h$ and $\boldsymbol{\zeta}$ are single-valued on interior faces and periodic on $\partial\Omega$, we have

$$\langle\widehat{\boldsymbol{e}_u}, \boldsymbol{\zeta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} = 0.$$

Then

$$\begin{aligned}
(\mathbb{P}\boldsymbol{e}_u, \eta)_{\Omega_h} =&\langle\mathbb{P}\boldsymbol{e}_u, (\boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} - (\boldsymbol{e_q}, \mathbf{\Pi}\boldsymbol{\zeta}) - \langle\widehat{\boldsymbol{e}_u}, (\boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\\
=&\langle\widehat{u}_h - u_h, (\boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + \langle\mathbb{P}u - u, (\boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} - (\boldsymbol{e_q}, \mathbf{\Pi}\boldsymbol{\zeta}).
\end{aligned}$$

By the properties of the projection $\mathbf{\Pi}$ and the definition of the numerical trace $\widehat{u}_h$, we get that the first term on the right hand side is zero. Hence, by the adjoint equation (2.8c), we get

$$(\mathbb{P}\boldsymbol{e}_u, \eta)_{\Omega_h} = \langle \mathbb{P}u - u, (\boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta}) \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} + (\boldsymbol{e}_{\boldsymbol{q}}, \boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta})_{\Omega_h} + A_1, \tag{7.33}$$

where $A_1 := -(\boldsymbol{e}_{\boldsymbol{q}}, \nabla\xi)_{\Omega_h}$.

Step 2: Now we estimate the term A_1. Using integration by parts, we have

$$\begin{aligned} A_1 =& (\boldsymbol{e}_{\boldsymbol{q}}, \nabla(\mathbb{P}\xi - \xi))_{\Omega_h} - (\boldsymbol{e}_{\boldsymbol{q}}, \nabla\mathbb{P}\xi)_{\Omega_h} \\ =& \langle \boldsymbol{e}_{\boldsymbol{q}}, (\mathbb{P}\xi - \xi)\boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla \cdot \boldsymbol{e}_{\boldsymbol{q}}, \mathbb{P}\xi - \xi)_{\Omega_h} - (\boldsymbol{e}_{\boldsymbol{q}}, \nabla\mathbb{P}\xi)_{\Omega_h}. \end{aligned}$$

By the property of the projection $\mathbb{P}$, we have

$$\begin{aligned} A_1 =& \langle \boldsymbol{e}_{\boldsymbol{q}}, (\mathbb{P}\xi - \xi)\boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla \cdot \mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}}, \mathbb{P}\xi - \xi)_{\Omega_h} - (\nabla \cdot (\boldsymbol{q} - \mathbf{\Pi}\boldsymbol{q}), \mathbb{P}\xi - \xi)_{\Omega_h} \\ & - (\boldsymbol{e}_{\boldsymbol{q}}, \nabla\mathbb{P}\xi)_{\Omega_h} \\ =& \langle \boldsymbol{e}_{\boldsymbol{q}}, (\mathbb{P}\xi - \xi)\boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla \cdot (\boldsymbol{q} - \mathbf{\Pi}\boldsymbol{q}), \mathbb{P}\xi - \xi)_{\Omega_h} - (\boldsymbol{e}_{\boldsymbol{q}}, \nabla\mathbb{P}\xi)_{\Omega_h}. \end{aligned}$$

Taking $\omega = \mathbb{P}\xi$ in the error equation (7.31b), we get

$$A_1 = \langle \boldsymbol{e}_{\boldsymbol{q}}, (\mathbb{P}\xi - \xi)\boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla \cdot (\boldsymbol{q} - \mathbf{\Pi}\boldsymbol{q}), \mathbb{P}\xi - \xi)_{\Omega_h} + (\boldsymbol{e}_z, \mathbb{P}\xi)_{\Omega_h} - \langle \widehat{\boldsymbol{e}_{\boldsymbol{q}}}, \mathbb{P}\xi\boldsymbol{n} \rangle_{\partial\Omega_h}.$$

Note that $\boldsymbol{q}$, $\widehat{\boldsymbol{q}}_h$ and ξ are single-valued on interior faces and periodic on $\partial\Omega$, we have

$$\langle \widehat{\boldsymbol{e}_{\boldsymbol{q}}}, \xi\boldsymbol{n} \rangle_{\partial\Omega_h} = 0,$$

which implies that

$$\begin{aligned} A_1 =& \langle (\boldsymbol{e}_{\boldsymbol{q}}, (\mathbb{P}\xi - \xi)\boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla \cdot (\boldsymbol{q} - \mathbf{\Pi}\boldsymbol{q}), \mathbb{P}\xi - \xi)_{\Omega_h} + (\boldsymbol{e}_z, \mathbb{P}\xi)_{\Omega_h} - \langle \widehat{\boldsymbol{e}_{\boldsymbol{q}}}, (\mathbb{P}\xi - \xi)\boldsymbol{n} \rangle_{\partial\Omega_h} \\ =& \langle (\widehat{\boldsymbol{q}}_h - \boldsymbol{q}_h, (\mathbb{P}\xi - \xi)\boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla \cdot (\boldsymbol{q} - \mathbf{\Pi}\boldsymbol{q}), \mathbb{P}\xi - \xi)_{\Omega_h} + (\boldsymbol{e}_z, \mathbb{P}\xi)_{\Omega_h}. \end{aligned}$$

Using the adjoint equation (2.8b), we have

$$A_1 = \langle \widehat{\boldsymbol{q}}_h - \boldsymbol{q}_h, (\mathbb{P}\xi - \xi)\boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla \cdot (\boldsymbol{q} - \mathbf{\Pi}\boldsymbol{q}), \mathbb{P}\xi - \xi)_{\Omega_h} + (\boldsymbol{e}_z, \mathbb{P}\xi - \xi)_{\Omega_h} + (\boldsymbol{e}_z, \nabla \cdot \boldsymbol{\psi})_{\Omega_h}.$$

Since $\mathbb{P}$ is the L^2-projection, we have

$$A_1 = \langle \widehat{\boldsymbol{q}}_h - \boldsymbol{q}_h, (\mathbb{P}\xi - \xi)\boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla \cdot (\boldsymbol{q} - \mathbf{\Pi}\boldsymbol{q}), \mathbb{P}\xi - \xi)_{\Omega_h} + (z - \mathbb{P}z, \mathbb{P}\xi - \xi)_{\Omega_h} + A_2, \tag{7.34}$$

where $A_2 := (\boldsymbol{e}_z, \nabla \cdot \boldsymbol{\psi})_{\Omega_h}$.

Step 3: Next we estimate the term A_2 in a way similar to that of terms A_1. Using integration by parts, we have

$$\begin{aligned} A_2 =& (\boldsymbol{e}_z, \nabla \cdot (\boldsymbol{\psi} - \mathbf{\Pi}\boldsymbol{\psi}))_{\Omega_h} + (\boldsymbol{e}_z, \nabla \cdot \mathbf{\Pi}\boldsymbol{\psi})_{\Omega_h} \\ =& \langle \boldsymbol{e}_z, (\boldsymbol{\psi} - \mathbf{\Pi}\boldsymbol{\psi}) \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla\boldsymbol{e}_z, \boldsymbol{\psi} - \mathbf{\Pi}\boldsymbol{\psi})_{\Omega_h} + (\boldsymbol{e}_z, \nabla \cdot \mathbf{\Pi}\boldsymbol{\psi})_{\Omega_h}. \end{aligned}$$

By the property of the projection $\mathbf{\Pi}$, we have that

$$\begin{aligned} A_2 =& \langle \boldsymbol{e}_z, (\boldsymbol{\psi} - \mathbf{\Pi}\boldsymbol{\psi}) \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla(z - \mathbb{P}z), \boldsymbol{\psi} - \mathbf{\Pi}\boldsymbol{\psi})_{\Omega_h} - (\nabla\mathbb{P}\boldsymbol{e}_z, \boldsymbol{\psi} - \mathbf{\Pi}\boldsymbol{\psi})_{\Omega_h} \\ & + (\boldsymbol{e}_z, \nabla \cdot \mathbf{\Pi}\boldsymbol{\psi})_{\Omega_h} \\ =& \langle \boldsymbol{e}_z, (\boldsymbol{\psi} - \mathbf{\Pi}\boldsymbol{\psi}) \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla(z - \mathbb{P}z), \boldsymbol{\psi} - \mathbf{\Pi}\boldsymbol{\psi})_{\Omega_h} + (\boldsymbol{e}_z, \nabla \cdot \mathbf{\Pi}\boldsymbol{\psi})_{\Omega_h}. \end{aligned}$$

Using the error equation (7.31c), we get

$$A_2 = \langle \boldsymbol{e}_z, (\boldsymbol{\psi} - \boldsymbol{\Pi\psi}) \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla(z - \mathbb{P}z), \boldsymbol{\psi} - \boldsymbol{\Pi\psi})_{\Omega_h} - (\boldsymbol{e}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi\psi})_{\Omega_h} + \langle \widehat{\boldsymbol{e}_z}, \boldsymbol{\Pi\psi} \cdot \boldsymbol{n} \rangle_{\partial\Omega_h}.$$

Note that z, $\widehat{z}_h$ and $\boldsymbol{\psi}$ and single-valued on interior faces and periodic on $\partial\Omega$, so

$$\langle \widehat{\boldsymbol{e}_z}, \boldsymbol{\psi} \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} = 0.$$

Hence we have

$$\begin{aligned} A_2 =& \langle \boldsymbol{e}_z, (\boldsymbol{\psi} - \boldsymbol{\Pi\psi}) \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla(z - \mathbb{P}z), \boldsymbol{\psi} - \boldsymbol{\Pi\psi})_{\Omega_h} - (\boldsymbol{e}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi\psi})_{\Omega_h} \\ & - \langle \widehat{\boldsymbol{e}_z}, (\boldsymbol{\psi} - \boldsymbol{\Pi\psi}) \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} \\ =& \langle \widehat{z}_h - z_h, (\boldsymbol{\psi} - \boldsymbol{\Pi\psi}) \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla(z - \mathbb{P}z), \boldsymbol{\psi} - \boldsymbol{\Pi\psi})_{\Omega_h} - (\boldsymbol{e}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi\psi})_{\Omega_h}. \end{aligned}$$

Since on any edge e, either $\widehat{z}_h = z_h$ or $\langle \boldsymbol{\psi} - \boldsymbol{\Pi\psi}, \boldsymbol{v} \rangle_e = 0$ for any $\boldsymbol{v} \in \boldsymbol{\mathcal{P}}^k(e)$, we get that the first term on the right hand side of A_2 is zero. So

$$A_2 = -(\nabla(z - \mathbb{P}z), \boldsymbol{\psi} - \boldsymbol{\Pi\psi})_{\Omega_h} - (\boldsymbol{e}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi\psi})_{\Omega_h}.$$

By the adjoint equation (2.8a), we have

$$A_2 = -(\nabla(z - \mathbb{P}z), \boldsymbol{\psi} - \boldsymbol{\Pi\psi})_{\Omega_h} - (\boldsymbol{e}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi\psi} - \boldsymbol{\psi})_{\Omega_h} + A_3, \tag{7.35}$$

where $A_3 := -(\boldsymbol{e}_{\boldsymbol{\sigma}}, \nabla\varphi)_{\Omega_h}$.

Step 4: We estimate the term A_3. Using integration by parts,

$$\begin{aligned} A_3 =& (\boldsymbol{e}_{\boldsymbol{\sigma}}, \nabla(\mathbb{P}\varphi - \varphi))_{\Omega_h} - (\boldsymbol{e}_{\boldsymbol{\sigma}}, \nabla\mathbb{P}\varphi)_{\Omega_h} \\ =& \langle \boldsymbol{e}_{\boldsymbol{\sigma}}, (\mathbb{P}\varphi - \varphi)\boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla \boldsymbol{e}_{\boldsymbol{\sigma}}, \mathbb{P}\varphi - \varphi)_{\Omega_h} - (\boldsymbol{e}_{\boldsymbol{\sigma}}, \nabla\mathbb{P}\varphi)_{\Omega_h}. \end{aligned}$$

Using the error equation (7.31d), we get that

$$A_3 = \langle \boldsymbol{e}_{\boldsymbol{\sigma}}, (\mathbb{P}\varphi - \varphi)\boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla \boldsymbol{e}_{\boldsymbol{\sigma}}, \mathbb{P}\varphi - \varphi)_{\Omega_h} - \langle \widehat{\boldsymbol{e}_{\boldsymbol{\sigma}}} \cdot \boldsymbol{n}, \mathbb{P}\varphi \rangle_{\partial\Omega_h}$$

By the property of the L^2-projection $\mathbb{P}$, we have that

$$\begin{aligned} A_3 =& \langle \boldsymbol{e}_{\boldsymbol{\sigma}}, (\mathbb{P}\varphi - \varphi)\boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla(\boldsymbol{\sigma} - \boldsymbol{\Pi\sigma}), \mathbb{P}\varphi - \varphi)_{\Omega_h} \\ & - (\nabla \boldsymbol{\Pi e}_{\boldsymbol{\sigma}}, \mathbb{P}\varphi - \varphi)_{\Omega_h} - \langle \widehat{\boldsymbol{e}_{\boldsymbol{\sigma}}} \cdot \boldsymbol{n}, \mathbb{P}\varphi \rangle_{\partial\Omega_h} \\ =& \langle \boldsymbol{e}_{\boldsymbol{\sigma}}, (\mathbb{P}\varphi - \varphi)\boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla(\boldsymbol{\sigma} - \boldsymbol{\Pi\sigma}), \mathbb{P}\varphi - \varphi)_{\Omega_h} - \langle \widehat{\boldsymbol{e}_{\boldsymbol{\sigma}}} \cdot \boldsymbol{n}, \mathbb{P}\varphi \rangle_{\partial\Omega_h}. \end{aligned}$$

Note that $\boldsymbol{\sigma}$, $\widehat{\boldsymbol{\sigma}}_h$ and φ are single-valued on interior faces and periodic on $\partial\Omega$, we have that

$$\langle \widehat{\boldsymbol{e}_{\boldsymbol{\sigma}}} \cdot \boldsymbol{n}, \varphi \rangle_{\partial\Omega_h} = 0.$$

Hence

$$\begin{aligned} A_3 =& \langle \boldsymbol{e}_{\boldsymbol{\sigma}}, (\mathbb{P}\varphi - \varphi)\boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla(\boldsymbol{\sigma} - \boldsymbol{\Pi\sigma}), \mathbb{P}\varphi - \varphi)_{\Omega_h} - \langle \widehat{\boldsymbol{e}_{\boldsymbol{\sigma}}}, (\mathbb{P}\varphi - \varphi)\boldsymbol{n} \rangle_{\partial\Omega_h} \\ =& \langle \widehat{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_h, (\mathbb{P}\varphi - \varphi)\boldsymbol{n} \rangle_{\partial\Omega_h} - (\nabla(\boldsymbol{\sigma} - \boldsymbol{\Pi\sigma}), \mathbb{P}\varphi - \varphi)_{\Omega_h} \end{aligned} \tag{7.36}$$

To complete the proof of Lemma 7.2, we only need to combine (7.33) and the estimates of the terms A_1, A_2, and A_3, (7.34)-(7.36). $\square$

7.3. **Proof of Theorem 3.9:** Taking $\eta = \mathbb{P}\boldsymbol{e}_u$ in Lemma 7.2, we get

$$\|\mathbb{P}\boldsymbol{e}_u\|^2_{L^2(\Omega_h)} = \sum_{i=1}^{3} E_i.$$

Then we estimate E_i for $i = 1, \cdots, 3$. By the property of the projections $\mathbb{P}$ and $\boldsymbol{\Pi}$, we have

$$\begin{aligned} E_1 \le C(&h\|\boldsymbol{e_q}\|_{L^2(\Omega_h)}\|\boldsymbol{\zeta}\|_{H^1(\Omega_h)} + h^{\min\{2,k+1\}}\|z - \mathbb{P}z\|_{L^2(\Omega_h)}\|\xi\|_{H^2(\Omega_h)} \\ &+ h^{\min\{3,k+1\}}\|\boldsymbol{e_\sigma}\|_{L^2(\Omega_h)}\|\boldsymbol{\psi}\|_{H^3(\Omega_h)}). \end{aligned}$$

Using the regularity (2.9), we have

$$E_1 \le C(h^{k+2} + h^{\min\{2,k+1\}}\|\boldsymbol{\Pi}\boldsymbol{e_q}\|_{L^2(\Omega_h)} + h^{\min\{3,k+1\}}\|\boldsymbol{\Pi}\boldsymbol{e_\sigma}\|_{L^2(\Omega_h)})\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)},$$

where C depends on $\|u\|_{H^{k+4}(\Omega_h)}$ and the elliptic regularity constant C_{er}. Using the properties of the projection $\mathbb{P}$ and $\boldsymbol{\Pi}$, we get

$$\begin{aligned} E_2 \le C(&h^{k+\min\{2,k+1\}}\|\boldsymbol{q}\|_{H^{k+1}(\Omega_h)}\|\xi\|_{H^2(\Omega_h)} \\ &+ h^{k+\min\{3,k+1\}}\|z\|_{H^{k+1}(\Omega_h)}\|\boldsymbol{\psi}\|_{H^3(\Omega_h)} \\ &+ h^{k+\min\{4,k+1\}}\|\boldsymbol{\sigma}\|_{H^{k+1}(\Omega_h)}\|\varphi\|_{H^4(\Omega_h)}). \end{aligned}$$

Using the regularity (2.9), we have

$$E_2 \le Ch^{k+1}\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)},$$

where C depends on $\|u\|_{H^{k+4}(\Omega_h)}$ and C_{er}. By inverse inequality in Lemma 3.2, we have

$$E_3 \le C(h^{\min\{k,1\}}\|\boldsymbol{e_q}\|_{L^2(\Omega_h)}\|\xi\|_{H^2(\Omega_h)} + h^{\min\{k,3\}}\|\boldsymbol{e_\sigma}\|_{L^2(\Omega_h)}\|\varphi\|_{H^4(\Omega_h)}).$$

Using the regularity (2.9),

$$E_3 \le C(h^{k+1} + h^{\min\{k,1\}}\|\boldsymbol{\Pi}\boldsymbol{e_q}\|_{L^2(\Omega_h)} + h^{\min\{k,3\}}\|\boldsymbol{\Pi}\boldsymbol{e_\sigma}\|_{L^2(\Omega_h)})\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)},$$

where C depends on $\|u\|_{H^{k+4}(\Omega_h)}$ and C_{er}. Hence, we have

$$\sum_{i=1}^{3} E_i \le C(h^{k+1} + h^{\min\{k,1\}}\|\boldsymbol{\Pi}\boldsymbol{e_q}\|_{L^2(\Omega_h)} + h^{\min\{k,3\}}\|\boldsymbol{\Pi}\boldsymbol{e_\sigma}\|_{L^2(\Omega_h)})\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)}.$$

which implies that

$$\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)} \le C(h^{k+1} + h^{\min\{k,1\}}\|\boldsymbol{\Pi}\boldsymbol{e_q}\|_{L^2(\Omega_h)} + h^{\min\{k,3\}}\|\boldsymbol{\Pi}\boldsymbol{e_\sigma}\|_{L^2(\Omega_h)}),$$

where C depends on $\|u\|_{H^{k+4}(\Omega_h)}$ and C_{er}. To complete the proof, we only need to apply the error estimates of $\boldsymbol{\Pi}\boldsymbol{e_q}$ and $\boldsymbol{\Pi}\boldsymbol{e_\sigma}$ in Lemma (7.1) for $k \ge 1$.

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