A SIMPLIFIED ANALYSIS OF TWO PLATE BENDING ELEMENTS — THE MITC4 AND MITC9 ELEMENTS

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ABSTRACT

We consider the convergence behavior of two mixed-interpolated plate bending elements -- the MITC4 element which has already found wide use and a new element, the MITC9 element. A simplified analysis is given that renders valuable insight into the predictive capabilities of these elements.

1. INTRODUCTION

In this paper we analyze some finite element approximations of Mindlin-Reissner moderately thick plates. One element is the four-node element MITC4 [1,2] that was already analyzed from the mathematical point of view in [3]. The other element is a new element that we call MITC9, which is presently under testing and shows much promise.(*)

The type of analysis that we are carrying out here is in some sense a simplified one. In order to study the shear-locking phenomenon, we consider a sequence of plate bending problems $\{P_t\}$ with a thickness t going to zero, and the corresponding sequence $\{P_{th}\}$ of discretized problems. Now, instead of studying the convergence of P_{th} to P_t for positive t, we consider just the two limit problems P_{oh} and P_o, and we analyze convergence and error estimates only for this case. It is clear that, if P_{th} displays a "good behaviour" uniformly in t, then P_{oh} must also behave properly. Since the converse is not true, our analysis is not complete. However, we conjecture that the good behaviour for $t = 0$ is a very reasonable test that can be of great help in designing a new element. On the other hand, a comparison of the analysis of the MITC4 element in [3] (where the general case $t >$

(*)MITC4 denotes our element based on mixed-interpolated tensorial components using 4 nodes, and similarly for the abbreviations MITC8 and MITC9 [4].

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0 was considered) and the present analysis shows clearly that the study of the limit case alone is considerably simpler. In particular the relationship with the analysis of some incompressible fluid elements can be established much more clearly.

For the sake of simplicity we consider only uniform decompositions of a square plate. However, it will be clear from the analysis that, at least for the MITC9 element, the arguments also hold for the general case.

2. THE SEQUENCE OF PROBLEMS AND THE LIMIT PROBLEM

We consider the spaces: $\underline{\Theta} = (H_0^1(\Omega))^2$ and $W = H_0^1(\Omega)$ and a load function f given in $L^2(\Omega)$. The sequence of problems under consideration is:

$$P_t \qquad \operatorname*{Inf}_{\underline{\theta}\in\underline{\Theta},\, w\in W} \quad \frac{t^3}{2}\, a(\underline{\theta},\underline{\theta}) + \frac{\lambda t}{2}\, \|\underline{\theta}-\underline{\nabla} w\|_0^2 - t^3(f,w)$$

where $\frac{t^3}{2}\, a(\underline{\theta},\underline{\theta})$ is the bending internal energy, λ includes the shear correction factor and $\|\ \|_0$ and $(\ ,\ )$ represent respectively the norm and the inner product in $L^2(\Omega)$.

Assume now that we are given finite element subspaces $\underline{\Theta}_h \subset \underline{\Theta}$ and $W_h \subset W$. The corresponding discretized problem is described by

$$\tilde{P}_{th} \qquad \operatorname*{Inf}_{\underline{\theta}_h\in\underline{\Theta}_h,\, w_h\in W_h} \quad \frac{t^3}{2}\, a(\underline{\theta}_h,\underline{\theta}_h) + \frac{\lambda t}{2}\, \|\underline{\theta}_h-\underline{\nabla} w_h\|_0^2 - t^3(f,w_h).$$

In general, $\tilde{P}_{th}$ "locks" for small t. A common procedure is to reduce, somehow, the influence of the shear energy. We consider here the case in which the reduction is carried out in the following way: we assume that we are given a third finite element space, $\underline{\Gamma}_h$, and a linear operator R which takes values in $\underline{\Gamma}_h$. Then we use $\|R(\underline{\theta}_h-\underline{\nabla} w_h)\|_0^2$ instead of $\|\underline{\theta}_h-\underline{\nabla} w_h\|_0^2$ in the shear energy. For the sake of simplicity we shall assume that:

$$R\, \underline{\nabla} w_h = \underline{\nabla} w_h \text{ for all } w_h \in W_h \tag{1}$$

so that the discretized problem takes its final form

$$P_{th} \quad \operatorname*{Inf}_{\underline{\theta}_h \in \underline{\Theta}_h,\, w_h \in W_h} \frac{t^3}{2}\, a(\underline{\theta}_h,\underline{\theta}_h) + \frac{\lambda t}{2} \| R\underline{\theta}_h - \underline{\nabla} w_h \|_o^2 - t^3 (f, w_h).$$

Setting

$$\underline{\gamma} = \lambda t^{-2}(\underline{\theta} - \underline{\nabla} w) \text{ and } \underline{\gamma}_h = \lambda t^{-2}(R\underline{\theta}_h - \underline{\nabla} w_h) \tag{2}$$

the Euler equations of P_t and P_{th} are respectively

$$\begin{aligned} &a(\underline{\theta},\underline{\eta}) + (\underline{\gamma},\underline{\eta}-\underline{\nabla}\zeta) = (f,\zeta) \quad \forall\, \underline{\eta} \in \underline{\Theta},\ \forall\, \zeta \in W \\ &\underline{\gamma} = \lambda t^{-2} (\underline{\theta} - \underline{\nabla} w) \end{aligned} \tag{3}$$

and

$$\begin{aligned} &a(\underline{\theta}_h,\underline{\eta}) + (\underline{\gamma}_h, R\underline{\eta}-\underline{\nabla}\zeta) = (f,\zeta) \quad \forall\, \underline{\eta} \in \underline{\Theta}_h,\ \forall\, \zeta \in W_h \\ &\underline{\gamma}_h = \lambda t^{-2} (R\underline{\theta}_h - \underline{\nabla} w_h) \end{aligned} \tag{4}$$

From now on we will limit ourselves to the analysis of the limit problems

$$\begin{aligned} &a(\underline{\theta},\underline{\eta}) + (\underline{\gamma},\underline{\eta}-\underline{\nabla}\zeta) = (f,\zeta) \quad \forall\, \underline{\eta} \in \underline{\Theta},\ \forall\, \zeta \in W \\ &\underline{\theta} = \underline{\nabla} w \end{aligned} \tag{5}$$

and

$$\begin{aligned} &a(\underline{\theta}_h,\underline{\eta}) + (\underline{\gamma}_h, R\underline{\eta}-\underline{\nabla}\zeta) = (f,\zeta) \quad \forall\, \underline{\eta} \in \underline{\Theta}_h,\ \forall\, \zeta \in W_h \\ &R\underline{\theta}_h = \underline{\nabla} w_h\ . \end{aligned} \tag{6}$$

REMARK It is not difficult to show that (5) and (6) are the limit problems of (3) and (4) respectively, see for instance [3]. In particular the limit w will be the solution corresponding to the Kirchhoff model. Note also that the limit $\underline{\gamma}_h$ that appears in (6) will still belong to $R(\underline{\Theta}_h) - \underline{\nabla}(W_h)$. Although we are not studying here the convergence of $\underline{\gamma}_h$ to $\underline{\gamma}$, the results given in [5] with the discussion below give some insight into the behavior of $\underline{\gamma}_h$.

3. THE FINITE ELEMENT DISCRETIZATIONS

Following the discussion of the previous section, the finite element discretization is characterized by the choice of the finite element spaces $\underline{\Theta}_h, W_h, \underline{\Gamma}_h$ and by the choice of the linear operator R. Note that these choices are not independent from each other since we assumed (1) to be satisfied. We introduce now the two choices that we consider in this paper.

The MITC4 Element [1,2]

We set

$$\underline{\Theta}_h = \{\underline{\eta} \mid \underline{\eta} \in (H_0^1(\Omega))^2, \underline{\eta}|_K \in (Q_1)^2 \ \forall\ K\} \tag{7}$$

$$W_h = \{\zeta \mid \zeta \in H_0^1(\Omega), \zeta|_K \in Q_1 \ \forall\ K\} \tag{8}$$

where, here and in the following, Q_1 is the set of polynomials of degree ≤ 1 in each variable and K is the current element in the discretization (we recall that we assumed a uniform decomposition of a square Ω). The space $\underline{\Gamma}_h$ is given by

$$\underline{\Gamma}_h = \{\underline{\delta} \mid \underline{\delta}|_K \in TR(K) \ \forall\ K,\ \underline{\delta}\cdot\underline{\tau} \text{ continuous at the interelement boundaries}\} \tag{9}$$

where $\underline{\tau}$ is the tangential unit vector to each edge of each element and

$$TR(K) = \{\underline{\delta} \mid \delta_1 = a_1 + b_1 y,\ \delta_2 = a_2 + b_2 x\} \tag{10}$$

is a sort of "rotated Raviart-Thomas" space. We have finally to introduce the reduction operator R. We describe its action on the current element: for $\underline{\eta}$ smooth in K, $R\underline{\eta}|_K$ is the unique element in TR(K) that satisfies

$$\int_e (\underline{\eta} - R\underline{\eta})\cdot\underline{\tau}\, ds = 0 \text{ for all edges e of } K. \tag{11}$$

Note that if $\underline{\eta} \in \underline{Q}_1$ then (11) is satisfied if and only if $\underline{\eta}\cdot\underline{\tau} = R(\underline{\eta})\cdot\underline{\tau}$ at the midpoints of each edge. Hence clearly (1) holds.

The MITC9 Element

We introduce now a new element. We set

$$\underline{\Theta}_h = \{\underline{\eta} \mid \underline{\eta} \in (H_0^1(\Omega))^2, \underline{\eta}|_K \in (Q_2)^2 \ \forall\ K\} \tag{12}$$

$$W_h = \{\zeta \mid \zeta \in H_0^1(\Omega),\ \zeta|_K \in Q_2^r \ \forall\ K\} \tag{13}$$

where Q_2 is the space of polynomials of degree ≤ 2 in each variable (corresponding to a 9 node element) and Q_2^r is its usual serendipity reduction (corresponding to an 8 node element). In order to introduce the space $\underline{\Gamma}_h$ we define first the space of polynomials

$$G = \{\underline{\delta} \mid \delta_1 = a_1 + b_1 x + c_1 y + d_1\, x\, y + e_1 y^2, \tag{14}$$

$$\delta_2 = a_2 + b_2 x + c_2 y + d_2\, x\, y + e_2 x^2\}$$

which is some kind of rotated Brezzi-Douglas-Fortin-Marini space. Note that if $\zeta \in Q_2^r$ then $\underline{\nabla}\zeta \in G$. This is the main reason why W_h has been discretized with the interpolations of 8-node elements instead of 9-node elements. (But could we use a larger space G ? We shall deal with this question briefly later on.) We introduce now the space $\underline{\Gamma}_h$:

$$\underline{\Gamma}_h = \{\underline{\delta} \mid \underline{\delta}|_K \in G \ \forall\ K,\ \underline{\delta}\cdot\underline{\tau} \text{ continuous at the interelement boundaries}\}. \tag{15}$$

Further, we define the action of the reduction operator R on the current element K in the following way: for $\underline{\eta}$ smooth in K, $R\underline{\eta}|_K$ is the unique element in G that satisfies

$$\int_e (\underline{\eta}-R\underline{\eta})\cdot\underline{\tau}\ p_1(s)\, ds = 0 \quad \forall\ e \text{ an edge of } K,\ \forall\ p_1(s) \text{ polynomial of degree} \leq 1 \text{ on } e \tag{16}$$

$$\int_K (\underline{\eta}-R\underline{\eta})\,dx\, dy = 0. \tag{17}$$

Here again (1) is satisfied. Note also that if $\underline{\eta} \in \underline{Q}_2$ then (16) holds if and only if $\underline{\eta}\cdot\underline{\tau} = (R\underline{\eta})\cdot\underline{\tau}$ at the two Gauss points of each edge.

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We could think of using $\underline{\eta} = R\underline{\eta}$ at the center of the element instead of (17). However, our proof is then not applicable, although numerical experiments may show good element behavior even in this case.

It is clear that for a general decomposition R should be defined by covariant interpolations (see [1,2,4]).

4. THE ERROR ANALYSIS

It is convenient to recall the definition of the differential operators

$$\varphi \rightarrow \underline{\mathrm{rot}}(\varphi) = (-\partial\varphi/\partial y,\ \partial\varphi/\partial x)$$

and

$$\underline{\varphi} = (\varphi_1, \varphi_2) \rightarrow \mathrm{rot}\underline{\varphi} = (\partial\varphi_1/\partial y - \partial\varphi_2/\partial x).$$

We now look for a "pressure space" Q_h made of discontinuous finite element functions(**) such that, for all $\underline{\eta} \in \underline{\Theta}$, we have

$$(\mathrm{rot}\ \underline{\eta}, q_h) = (\mathrm{rot}(R\underline{\eta}), q_h) \quad \forall\ q_h \in Q_h \tag{18}$$

and

$$\mathrm{rot}(\underline{\Gamma}_h) \subseteq Q_h. \tag{19}$$

Conditions (18), (19) are strictly related to the so-called "commuting diagram property" of Douglas and Roberts [6] that is used in the study of mixed methods for elliptic equations. It is easy to check that (18) and (19) hold if we take for the MITC4 element

$$Q_h = \{q \,|\, q|_K \in P_0\ \forall\ K\} \tag{20}$$

and for the MITC9 element

$$Q_h = \{q \,|\, q|_K \in P_1\ \forall\ K\}. \tag{21}$$

(**) This space corresponds to the pressure space in incompressible solutions.

In both cases P_k denotes the set of polynomials of total degree $\leq k$: hence Q_h has local dimension 1 in the MITC4 case and 3 in the MITC9 case.

In order to analyze the error between $\underline{\theta}$ and $\underline{\theta}_h$ in (5) - (6) (and as a consequence the error between w and w_h) we want to build a pair $\hat{\underline{\theta}}, \hat{w}$ in $\underline{\Theta}_h \times W_h$ such that $\|\underline{\theta}-\hat{\underline{\theta}}\|_1$ is optimally small and

$$R\hat{\underline{\theta}} = \underline{\nabla}\hat{w}. \tag{22}$$

Condition (22) implies

$$\text{rot } R\hat{\underline{\theta}} = 0 \tag{23}$$

which, in its turn, using (18), (19) is equivalent to

$$(\text{rot } \hat{\underline{\theta}}, q_h) = 0 \quad \forall\, q_h \in Q_h. \tag{24}$$

A possible way of constructing $\hat{\underline{\theta}}$ is the following. For $\underline{\theta}$ given in $(H_0^1(\Omega))^2$ and satisfying $\text{rot}\underline{\theta} = 0$, consider the problem:

$$\text{Find } \underline{\beta},\ p \in \underline{\Theta} \times L^2(\Omega) \text{ such that}$$

$$a(\underline{\beta},\underline{\eta}) + (p,\ \text{rot}\underline{\eta}) = a(\underline{\theta},\underline{\eta}) \qquad \forall\ \underline{\eta} \in \underline{\Theta} \tag{25}$$

$$(q, \text{rot}\underline{\beta}) = 0 \qquad \forall\ q \in L^2(\Omega)$$

and its approximation

$$\text{Find } \hat{\underline{\theta}},\ p_h \in \underline{\Theta}_h \times Q_h \text{ such that}$$

$$a(\hat{\underline{\theta}},\underline{\eta}) + (p_h, \text{rot}\underline{\eta}) = a(\underline{\theta},\eta) \qquad \forall\ \underline{\eta} \in \underline{\Theta}_h \tag{26}$$

$$(q, \text{rot}\hat{\underline{\theta}}) = 0 \qquad \forall\ q \in Q_h$$

Note that (25) is a kind of Stokes problem and its solution is given by $\underline{\beta} = \underline{\theta}$, $p = 0$. If the pair $\underline{\Theta}_h$, Q_h is a suitable finite element discretization for the Stokes problem one might expect to have optimal error bounds for $\hat{\underline{\theta}}-\underline{\theta}$. For instance, in the case of the MITC4 element the pair $\underline{\Theta}_h$, Q_h is the classical bilinear velocities-constant pressure (or Q_1-P_0) element, and we know that with minor assumptions on the decomposition that are surely satisfied in the present case:

$$\|\underline{\theta}-\hat{\underline{\theta}}\|_1 \leq c\ h\ \|\underline{\theta}\|_2 \quad \text{for } Q_1 - P_0 \text{ element.} \tag{27}$$

On the other hand in the case of the MITC9 element the pair $\underline{\Theta}_h$, Q_h is the biquadratic velocities and linear pressure (the Q_2-P_1) element and for a general decomposition:

$$\|\underline{\theta}-\hat{\underline{\theta}}\|_1 \leq c\ h^2\ \|\underline{\theta}\|_3 \quad \text{for } Q_2 - P_1 \text{ element.} \tag{28}$$

Note on the other hand that once $\hat{\underline{\theta}}$ satisfying (23) has been found, then one can uniquely determine the $\hat{w} \in W$ that satisfies (22). It is easy to check that in our two cases such a $\hat{w}$ is an element of W_h.

We are now ready for proving error estimates. We set

$$\underline{\delta} = \underline{\theta}_h - \hat{\underline{\theta}}\ ; \quad \xi = w_h - \hat{w} \tag{29}$$

and we note that

$$R\underline{\delta} = \underline{\nabla}\xi. \tag{30}$$

Now we have

$$\begin{aligned}
\alpha\|\underline{\delta}\|_1^2 \leq a(\underline{\delta},\underline{\delta}) &= a(\underline{\theta}_h-\underline{\theta},\underline{\delta}) + a(\underline{\theta}-\hat{\underline{\theta}},\underline{\delta}) = \\
&= -(\underline{\gamma}_h,R\underline{\delta}) + (\underline{\gamma},\underline{\delta}) + a(\underline{\theta}-\hat{\underline{\theta}},\underline{\delta}) = \\
&= (\underline{\gamma},\underline{\delta}-R\underline{\delta}) - (\underline{\gamma}_h-\underline{\gamma},R\underline{\delta}) + a(\underline{\theta}-\hat{\underline{\theta}},\underline{\delta}) = \\
&= (\underline{\gamma},\underline{\delta}-R\underline{\delta}) - (\underline{\gamma}_h-\underline{\gamma},\underline{\nabla}\xi) + a(\underline{\theta}-\hat{\underline{\theta}},\underline{\delta}) = \\
&= (\underline{\gamma},\underline{\delta}-R\underline{\delta}) + a(\underline{\theta}-\hat{\underline{\theta}},\underline{\delta}) \leq \\
&\leq \left[\sup_{\underline{\beta} \in \underline{\Theta}_h} \frac{(\underline{\gamma},\underline{\beta}-R\underline{\beta})}{\|\underline{\beta}\|_1} + c\ \|\underline{\theta}-\hat{\underline{\theta}}\|_1 \right] \|\underline{\delta}\|_1
\end{aligned} \tag{31}$$

which implies

$$\|\underline{\delta}\|_1 \leq c \left\{ \|\underline{\theta}-\hat{\underline{\theta}}\|_1 + \sup_{\underline{\beta} \in \underline{\Theta}_h} (\underline{\gamma},\underline{\beta}-R\underline{\beta})/\|\underline{\beta}\|_1 \right\} \tag{32}$$

In the case of the MITC4 element we have

$$|(\underline{\gamma},\underline{\beta}-R\underline{\beta})| \leq \|\underline{\gamma}\|_0 \; \|\underline{\beta}-R\underline{\beta}\|_0 \leq c \; h \; \|\underline{\gamma}\|_0 \; \|\underline{\beta}\|_1 \tag{33}$$

and using (32), (33) and (27) we obtain

$$\|\underline{\delta}\|_1 \leq c \; h \; (\|\underline{\theta}\|_2 + \|\underline{\gamma}\|_0) \tag{34}$$

and finally from (34), (29), (27) and the triangle inequality:

$$\|\underline{\theta}-\underline{\theta}_h\|_1 \leq c \; h(\|\underline{\theta}\|_2+\|\underline{\gamma}\|_0) \qquad \text{for MITC4} \tag{35}$$

Let us consider now the case of the MITC9 element, and set $\hat{\underline{\gamma}}$ = mean value of $\underline{\gamma}$ in each K. Using (17) we have

$$(\underline{\gamma},\underline{\beta}-R\underline{\beta}) = (\underline{\gamma}-\hat{\underline{\gamma}},\underline{\beta}-R\underline{\beta}) \tag{36}$$

so that

$$\begin{aligned} |(\underline{\gamma},\underline{\beta}-R\underline{\beta})| &\leq \|\underline{\gamma}-\hat{\underline{\gamma}}\|_0 \; \|\underline{\beta}-R\underline{\beta}\|_0 \leq \\ &\leq c \; h^2\|\underline{\gamma}\|_1 \; \|\underline{\beta}\|_1 \end{aligned} \tag{37}$$

and from (32), (37) and (28)

$$\|\underline{\delta}\|_1 < c \; h^2(\|\underline{\theta}\|_3 + \|\underline{\gamma}\|_1) \tag{38}$$

so that from (38), (29), (28) and the triangle inequality

$$\|\underline{\theta}-\underline{\theta}_h\|_1 \leq c \; h^2(\|\underline{\theta}\|_3 + \|\underline{\gamma}\|_1) \qquad \text{for MITC9} \tag{39}$$

Finally we want to estimate $w-w_h$. Since $\underline{\nabla} w_h = R\underline{\theta}_h$ we have

$$\underline{\nabla}(w-w_h) = \underline{\theta} - R\underline{\theta}_h = (\underline{\theta}-R\underline{\theta}) + R(\underline{\theta}-\underline{\theta}_h) \tag{40}$$

It is easy to check that

$$\|\underline{\theta}-R\underline{\theta}\|_0 \leq c \; h\|\underline{\theta}\|_1 \qquad \text{for MITC4} \tag{41}$$

$$\|\underline{\theta}-R\underline{\theta}\|_0 \leq c \; h^2\|\underline{\theta}\|_2 \qquad \text{for MITC9} \tag{42}$$

while in both cases

$$\|R(\underline{\theta}-\underline{\theta}_h)\|_0 \leq c\ \|\underline{\theta}-\underline{\theta}_h\|_1. \tag{43}$$

Therefore from (40), (41), (43) and (35):

$$\|\underline{\nabla}w - \underline{\nabla}w_h\|_0 \leq c\ h\ (\|\underline{\theta}\|_2 + \|\underline{\gamma}\|_0) \text{ for MITC4} \tag{44}$$

and from (40), (42), (43) and (39)

$$\|\underline{\nabla}w-\underline{\nabla}w_h\|_0 \leq c\ h^2\ (\|\underline{\theta}\|_3 + \|\underline{\gamma}\|_1) \quad \text{for MITC9.} \tag{45}$$

REMARKS

Hinton and Huang [7] suggested other constructions of mixed-interpolated elements and gave interesting numerical results.

The use of 9 nodes to describe W_h, considering our theory, would require the $\underline{\Gamma}_h$ to be of the form

$$(a_1 + b_1x + c_1y + d_1\ x\ y + e_1y^2 + f_1\ x\ y^2,$$

$$a_2 + b_2x + c_2y + d_2\ x\ y + e_2x^2 + f_2\ x^2y)$$

in each K. Then, since we need (19), we would have Q_h made of bilinear (instead of linear) functions in each element. Hence the pair $\underline{\theta}_h$, Q_h will be of the type (Q_2-Q_1) which is not as good as the (Q_2-P_1) pair [8].

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