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A FACETED SHELL ELEMENT WITH LOOF NODES

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SUMMARY

A flat shell element which utilizes the linear strain triangle (LST) for membrane representation and a plate-bending element labelled DKL (for discrete Kirchhoff element with Loof nodes) previously derived by the authors in Reference 1, is presented. This element is free from the deficiencies of displacement incompatibility, singularity with coplanar elements, inability to model intersections and low order membrane strain representation, which are associated with existing faceted shell elements. The results of the numerical examples indicate that the convergence characteristics of this element are good.

INTRODUCTION

The first application of the finite element method to the analysis of shells consisted of replacing the curved shape with an assemblage of flat elements. By superposing the independent membrane and flexural behaviour with the appropriate spatial transformation, the desired shell model is obtained. This was soon followed by attempts to develop curved elements which give a better approximation to the actual shell curvature and hence provide the proper membrane–flexural coupling within each element. However, substantial difficulties exist in deriving the force–displacement relationships of curved shell elements. They are as described by Gallagher[2]: (a) choice of an appropriate shell theory which is consistent with the principles of energy and equilibrium, and invariant under rigid body displacements and transformation of the surface co-ordinates, (b) description of the element geometry, (c) representation of rigid body modes of behaviour by the assumed displacement field, and (d) retention of the required degree of inter-element continuity. Derivation of a completely general doubly curved element would require an analytic description of the shell geometry as well as high order representation of the displacements, as illustrated by the work of Argyris.[3]

However, problems relating to geometric representation are reduced somewhat by formulating the curved element in terms of a shallow shell theory, which allows all the necessary mathematical manipulations to be performed in a base reference plane. In addition, it is sufficient to assume constant geometric curvature over the element. This, however, introduces geometric discontinuities in the shell surface as adjacent elements which are portions of different parabolic surfaces will not match properly. For the shallow shell elements to converge to a deep shell solution, it is important that the shallowness assumption is employed relative to a local base plane rather than that of the global horizontal plane. However, regardless of whether the shallowness assumption is used or not, a proper description of the rigid body modes in elements based on curvilinear shell theory is only possible with the inclusion of transcendental functions into the displacement

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expression. This will violate interelement compatibility and it is more convenient to approximate the rigid body motions with higher order polynomials describing the displacements, as in the elements proposed by Cowper,[4] Yang[5] and Dawe.[6] This high order representation of the displacements leads to the introduction of additional degrees-of-freedom (d.o.f.), namely the first- and second-order derivatives, which complicates their use at shell intersections. Another possibility is to transform the curvilinear shell equations so that they are expressed in terms of a flat two-dimensional Euclidean reference surface. This formulation, as employed by Dupuis and Goel,[7] allows a proper representation of the rigid body motions but is deficient in representing 'sensitive' solutions.[8]

The isoparametric formulation, which has been used successfully for three-dimensional analysis,[9] appears to provide an alternative approach to analysing shells. It avoids the use of shell theories with their associated controversies. However, a straightforward application of the isoparametric concept results in a shell element which is ignorant of the kinematic constraints of thin shell problems. The retention of several nodal points through the thickness results in excessive d.o.f. which thus renders the element uneconomical. It also fails at a moderate length to thickness ratio due to displacement locking.

These difficulties associated with the three-dimensional solid elements are overcome with the degeneration concept introduced by Ahmad *et al.*[10] This directly discretizes the three-dimensional field equations in terms of mid-surface nodal variables from the outset. The constraint of straight 'normals' is introduced and the strain energy corresponding to stresses perpendicular to the middle surface is neglected. However, the normals do not remain normal after the deformation, which thus allows the degenerated shell to experience shear deformations. The resulting equilibrium equations are thus second-order differential equations and as such only C^0 continuous shape functions are required. It was soon noticed, however, that unsatisfactory results were obtained when the degenerated elements were applied in the thin shell regime. The model is too stiff, it exhibits shear and membrane locking and the rate of convergence is slow. An improvement to the basic model was achieved by Zienkiewicz *et al.*[11] through the use of the reduced integration technique and by Pawsey and Clough[12] through the use of selective reduced integration. These have the effect of reducing the number of constraints in the original model and thus prevent 'locking' of the element. Unfortunately this also reduces the rank of the elemental matrices which will lead to zero energy spurious modes in the global equations. This problem of rank deficiency is obviated in the 'Heterosis' element developed by Hughes and Cohen[13] and a shell element[14] based on this concept was found to behave well in thin shell applications.

A discussion on the relative merits of each of the above described formulations for analysing thin shells can be found in a survey paper by Gallagher.[15] It is fair comment, however, that no one particular approach is pre-eminent. The recent trend towards solving nonlinear shell problems, where computational cost effectiveness is paramount, has renewed interest in the flat shell elements, due to their simplicity and low cost of stiffness formulation. The advantages in using an assemblage of flat elements to model a shell are (a) simplicity and ease of formulation, (b) the rigid body motions are correctly represented, and (c) convergence to the deep shell solution is obtained. On the other hand, a geometrical approximation to the curved shell is introduced and flexural–membrane coupling is excluded within an element. Due to the physical idealization, 'parasitic' bending moments[16] may occur. This effect, as pointed out in Reference 17, is however frequently not present in translational shells.

Besides the above-mentioned deficiencies which are intrinsically related to a flat plate representation of the shell, currently available displacement based flat shell elements suffer from either one or more of the following difficulties. They are: singularity with coplanar elements, low order membrane strain representation, displacement incompatibility and inability to model intersecting shells. In modelling an arbitrary curved shell with flat elements, only triangular elements can be used. It is the objective of this paper to present an efficient and reliable triangular faceted shell element free from the previously mentioned problems associated with existing flat elements.

REVIEW OF TRIANGULAR FACETED SHELL ELEMENTS

The concept of using flat elements for analysing shells has been suggested by Greene *et al.*[18] as early as 1961. Initial attempts at formulating a flat triangular shell element by Clough and Tocher[19] and Argyris[20] did not yield satisfactory results due to the lack of suitable triangular bending stiffness matrices. Fortunately, satisfactory triangular plate bending elements were soon developed, which could then be incorporated into suitable faceted shell elements. This was accomplished by Zienkiewicz *et al.*[21] where the constant strain triangle (CST) was used in conjunction with the non-conforming bending element of Bazeley *et al.*[22] and by Clough and Johnson[23] with the compatible HCT bending element.[24]

This formulation of Clough and Johnson[23] was extended by Carr,[25] who improved the membrane approximation with the use of the quadratic strain triangle. The resulting element has 9 d.o.f. per node. Another higher order element, which utilizes the LST and a quartic plate bending element,[24] was proposed by Chu and Schnobrich.[17] This element has midside as well as corner nodes, with a total of 27 d.o.f. per element. Another element with midside nodes was presented by Razzaque,[27] where sub-cubic functions of the Zienkiewicz type[21] were used to describe the in-plane displacements. The derivative 'smoothed' bending element[28] was utilized to describe the bending action. The resultant element has 5 d.o.f. for each corner node and 3 d.o.f. for each midside node. However, the generation of the shape functions for the faceted element is somewhat tedious. In an attempt to incorporate the rotational stiffness about the normal to the shell, Olson and Bearden[29] employed a 9 d.o.f. plane stress element with the nodal in-plane rotation as a nodal parameter. This was combined with the BCIZ[22] plate-bending element to yield an 18 d.o.f. element. Convergence to the exact solution for the plane stress element used is not guaranteed as the displacement interpolation employed is incomplete. The simplest formulation of a faceted shell element is that due to Dawe,[30] where the CST is combined with the Morley[31] constant bending moment triangle to yield a 12 d.o.f. element. This element was also derived by Herrmann and Campbell[32] through a mixed variational approach.

A faceted shell element was developed through the hybrid stress model by Dungar *et al.*[33] The equilibrium stress field employed had linearly varying in-plane forces, while the moments had a quadratic variation. Here, the in-plane boundary displacements are assumed to vary linearly parallel to an edge and cubically normal to it; while the out-of-plane displacements have a cubic variation. In another paper,[34] the membrane approximation was improved upon by using second-order polynomials for all interior stress fields. A similar type element but with linearly varying equilibrium stress fields was later presented by Yoshida.[35] In the hybrid stress elements described, the in-plane rotation is utilized as a nodal d.o.f. Sander and Becker[36] have suggested the use of a new family of triangular flat shell elements based on a combination of compatible displacement elements for membrane action and equilibrium stress elements for bending action.

Recent impetus towards the nonlinear analysis of shells has regenerated interest in the relatively simple and economical faceted formulation. The faceted 'TRUMP' shell element with 18 d.o.f. formulated by Argyris *et al.*,[37] through physical lumping ideas, was proposed towards solving nonlinear plate and shell problems. Another element formulated with nonlinear applications in mind is the element presented by Horrigmoe.[38] The CST was employed to represent the membrane action while a hybrid stress model utilizing a modified form of Reissner's variational principle was used to derive the bending stiffness. This resulted in a 15 d.o.f. faceted shell element. In a recent study of several three-node triangular thin plate elements with 3 d.o.f. per node, Batoz *et al.*[39] came to the conclusion that the element based on the discrete Kirchhoff model was most efficient. This particular element was utilized by Bathe and Ho[40] to represent the bending action in their flat shell element. The membrane action was modelled using the CST.

As the above review indicates, there exists an extensive literature on the faceted shell elements. However, most of the elements have only 2 rotational d.o.f. in the local element co-ordinate system. This is due to the fact that the in-plane rotation is missing in available compatible displacement based plane stress elements. Therefore if the elements meeting at a node are coplanar, a singularity will be present on transforming to the global co-ordinate system. This problem is overcomed by Zienkiewicz *et al.*[21] by applying a fictitious but consistent rotational stiffness to the in-plane rotation. The value added, however, depends on the precision of the computer, has no theoretical justification and should thus be kept to a minimum. Clough and Johnson's[23] solution was to define a tangent plane to the normal (ξ_3) of the shell at a vertex. The local rotational d.o.f. are then transformed to a set of surface co-ordinates, ξ_1, ξ_2 in the principal directions, which will then serve as the base co-ordinates for the rotations. However, difficulties occur with branches in a shell, as the 'normal' here cannot be easily specified. Also, the transformed local rotations will have a significant component about the normal (ξ_3) which cannot really be ignored, as is the case here. Olson and Bearden[29] and the hybrid elements[33–35] make use of a plane stress element, with the in-plane rotation as a nodal parameter, as another possible means of solution. There is, however, the added constraint that the included angle between the sides is constant throughout the deformation. Also, as pointed out by Irons,[41] serious errors are introduced if an extended patch of such coplanar elements are subjected to a horizontal tensile force.

The majority of the elements described above utilize the CST to represent the membrane action of the shell. This is due mainly to the fact that the nodal parameters of the CST will match up nicely with those of available compatible plate-bending elements to give three translational displacements at each corner. Experience in plane stress situations with the CST indicates, however, that it is a relatively poor performer. It is thus to be expected that a flat shell element using the CST will require a fine mesh subdivision to analyse a shell with significant membrane action, quite regardless of the bending approximation. It is our view that the relatively fine mesh required by most faceted shell elements to analyse the various shell examples in the literature, is not due so much to the inherent geometric approximation but rather to the poor membrane representation of the elements.

A further difficulty associated with displacement based flat shell elements is the incompatibility of the element displacements in the shell assemblage. Due to the different polynomial approximation orders of the displacement fields in available compatible plane stress and bending elements, displacement compatibility is not ensured when the elements are assembled together, as the adjacent elements are not all coplanar.

DESCRIPTION OF PROPOSED FACETED SHELL ELEMENT

When formulating a faceted shell element, it is important that the best balance between the displacement approximations for both the membrane and bending action, and the geometric approximation be achieved. The more successful plate bending elements have at least a linear stress variation. It appears appropriate therefore, that the membrane stress should be capable of a linear variation, thus suggesting the LST. The LST has been found to be generally superior to the CST in plane stress applications.[42,43] Choice of higher order stress distributions would make the resultant element too refined to justify as a flat element.

The authors in Reference 1 have developed an efficient and reliable plate-bending element to

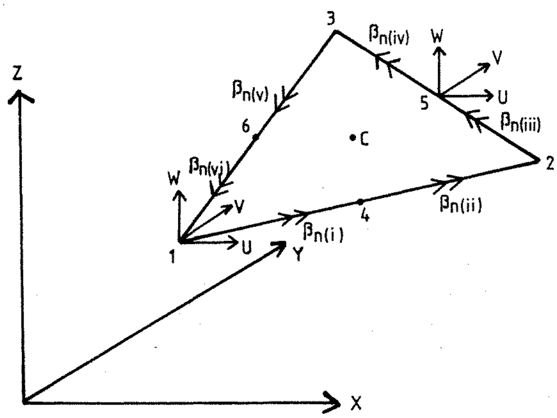


Figure 1. Nodal parameters of proposed shell element

combine with the LST to form a faceted shell element. This bending element has been shown in Reference 1 to perform well relative to available plate-bending elements. It has quadratic transverse displacements and so the problem of displacement incompatibility does not arise in the faceted shell model, as the in-plane displacements are also quadratic. The bending stiffness is formulated using Mindlin plate theory and so the normal rotations β_x and β_y are interpolated independently. A higher order term was added to the quadratic basis for the rotations so that shape functions can be found for the Loof nodes, which are the two Gauss points along each side of the triangle. In the absence of corner connections, it is impossible to attain even C^0 continuity. However, some form of generalized continuity is achieved for the rotations by interpolating at the Loof nodes and this plate-bending element was found to pass the patch test in Reference 1. The inclusion of shear into the bending model will lead to the element locking in thin plate situations and it will also exhibit very slow convergence. One commonly used solution for the degenerated elements is to employ either reduced or selective reduced integration. Although the convergence behaviour of the element is dramatically improved, it unfortunately very often results in the introduction of low energy spurious modes which destroy the solution. In the plate element proposed in Reference 1, the locking effect is alleviated by neglecting the strain energy due to shear and enforcing the Kirchhoff hypothesis at a number of discrete points as well as through the area of the triangle. This also has the effect of reducing the nodal parameters of the resultant plate element from an initial twenty to twelve. The stiffness matrix of the plate element is integrated exactly using a 7-point integration formula. The nodal parameters of the resultant shell element are depicted in Figure 1.

By choosing to interpolate the rotations at the Loof nodes, the problems associated with singularity when elements are coplanar and inability to model folds in the shell are avoided. In the plate examples studied in Reference 1, it was found that convergence under uniformly distributed loading is excellent but is less satisfactory under point loads. This, as pointed out by Irons,[44] could be due to the absence of corner slope connections. However, the resultant faceted shell nodal parameters permit it to model folded plates and shell branches without any difficulty and so this less gratifying response of the element to point loads will have to be tolerated.

FORMULATION OF PROPOSED ELEMENT

The local co-ordinate system and the positive direction of the kinematic variables are illustrated in Figure 2.

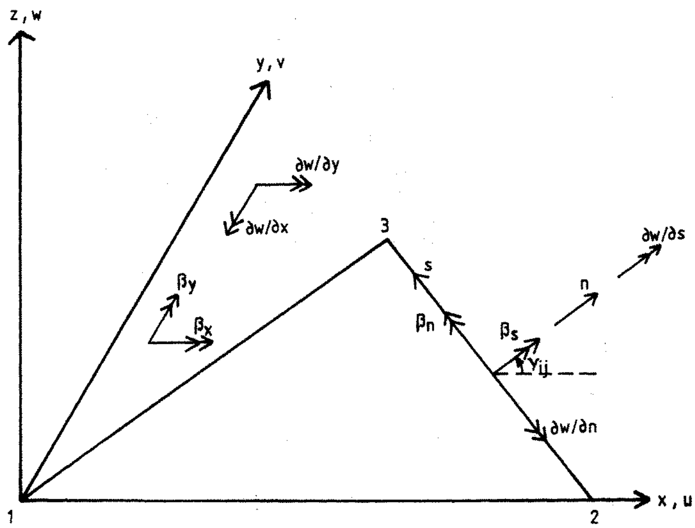


Figure 2. Local axes of shell element

Membrane stiffness matrix

The membrane strains are

$$\{\varepsilon^m\} = \begin{bmatrix} \varepsilon^m_{xx} \\ \varepsilon^m_{yy} \\ \varepsilon^m_{xy} \end{bmatrix} = \begin{bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{bmatrix} \tag{1}$$

and the corresponding stress–strain relationship is

$$\{\sigma^m\} = [D]\{\varepsilon^m\} \tag{2}$$

where $[D]$ is the constitutive matrix and u, v are the in-plane displacements. The membrane strain energy for an element of thickness, t and area, A is

$$U_m = \frac{t}{2}\int_A \{\varepsilon^m\}^{\mathrm{T}}[D]\{\varepsilon^m\}\,\mathrm{d}A \tag{3}$$

The linear strain triangle is employed here and hence the interpolation functions used are

$$\begin{aligned} u &= \{\phi_u\}^{\mathrm{T}}\{u^n\} \\ v &= \{\phi_v\}^{\mathrm{T}}\{v^n\} \end{aligned} \tag{4}$$

where $\{\phi_u\}^{\mathrm{T}}$ and $\{\phi_v\}^{\mathrm{T}}$ are equal to

$$[\zeta_1(2\zeta_1 - 1)\ \ \zeta_2(2\zeta_2 - 1)\ \ \zeta_3(2\zeta_3 - 1)\ \ 4\zeta_1\zeta_2\ \ 4\zeta_2\zeta_3\ \ 4\zeta_1\zeta_3] \tag{5}$$

and

$$\begin{aligned} \{u^n\}^{\mathrm{T}} &= [u_1 \ldots u_6] \\ \{v^n\}^{\mathrm{T}} &= [v_1 \ldots v_6] \end{aligned} \tag{6}$$

The membrane strain–displacement relationship is thus

$$\{\varepsilon^m\} = \begin{bmatrix} \{\phi_{u_{,x}}\}^{\mathrm{T}} & \{0\} \\ \{0\} & \{\phi_{v_{,y}}\}^{\mathrm{T}} \\ \{\phi_{u_{,y}}\}^{\mathrm{T}} & \{\phi_{v_{,x}}\}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \{u^n\} \\ \{v^n\} \end{bmatrix} = [B^m]\begin{bmatrix} \{u^n\} \\ \{v^n\} \end{bmatrix} \tag{7}$$

with the element membrane stiffness matrix being

$$[k^m] = t\int_A [B^m]^{\mathrm{T}}[D][B^m]\,\mathrm{d}A \tag{8}$$

Bending stiffness matrix

From Mindlin plate theory, the bending strains are expressed in terms of the normal rotations β_x and β_y as

$$\{\varepsilon^b\} = \begin{bmatrix} \varepsilon^b_{xx} \\ \varepsilon^b_{yy} \\ \varepsilon^b_{xy} \end{bmatrix} = z\begin{bmatrix} \beta_{x,x} \\ -\beta_{y,y} \\ \beta_{x,y} - \beta_{y,x} \end{bmatrix} = z\{\kappa\} \tag{9}$$

and the transverse shear strains are

$$\{\varepsilon^s\} = \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} w_{,x} + \beta_x \\ w_{,y} - \beta_y \end{bmatrix} \tag{10}$$

The corresponding stress–strain relation for the bending stress is

$$\{\sigma^b\} = [D]\{\varepsilon^b\} \tag{11}$$

Integrating the stress through the thickness results in the bending moments, $\{M\}$ per unit length. They are

$$\{M\} = \begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \frac{t^3}{12}[D]\{\kappa\} \tag{12}$$

with the bending strain energy of the shell being

$$U_b = \frac{1}{2}\frac{t^3}{12}\int_A \{\kappa\}^{\mathrm{T}}[D]\{\kappa\}\,\mathrm{d}A \tag{13}$$

The interpolation function for the transverse displacement, w is

$$w = \{\phi_w\}^{\mathrm{T}}\{w^n\} \tag{14}$$

where

$$\{w^n\}^{\mathrm{T}} = [w_1 \ldots w_6]$$

and $\{\phi_w\}^{\mathrm{T}}$ is as given in equation (5). In evaluating the shape functions for the normal rotations β_x and β_y, the following basis is used:

$$\beta_x = \{L\}^{\mathrm{T}}\{a\} \tag{15}$$

where

$$\{L\}^{\mathrm{T}} = [1 \;\; \zeta_1 \;\; \zeta_2 \;\; \zeta_1^2 \;\; \zeta_1\zeta_2 \;\; \zeta_2^2 \;\; (2\zeta_1^3 + 3\zeta_1^2\zeta_2 - 3\zeta_1\zeta_2^2 - 2\zeta_2^3)]$$

$\{a\}$ is the vector of generalized displacements. Substituting the area co-ordinates of the Loof and central nodes into equation (15) results in

$$\{\beta^n_x\} = [A]\{a\}$$

where

$$\{\beta^n_x\}^{\mathrm{T}} = [\beta_{x(i)} \ldots \beta_{x(c)}] \tag{16}$$

Thus

$$\begin{aligned} \beta_x &= \{L\}^{\mathrm{T}}[A^{-1}]\{\beta^n_x\} \\ &= \{\phi\}^{\mathrm{T}}\{\beta^n_x\} \end{aligned} \tag{17}$$

Similarly,

$$\beta_y = \{\phi\}^{\mathrm{T}}\{\beta_y^n\} \tag{18}$$

In order that the element is capable of modelling thin plate behaviour, the following shear constraints are imposed:

$$\gamma_{sz} = 0$$

i.e.

$$w_{,s(j)} - \beta_{s(j)} = 0 \quad \text{at } j = (i)\ldots(vi) \tag{19}$$

and

$$\int_A \gamma_{xz}\,\mathrm{d}A = \int_A \gamma_{yz}\,\mathrm{d}A = 0$$

i.e.

$$\int_A (\beta_x + w_{,x})\,\mathrm{d}A = \int_A (-\beta_y + w_{,y})\,\mathrm{d}A = 0 \tag{20}$$

On application of equations (19) and (20), the original 20 d.o.f. for the bending element are reduced to 12, i.e.

$$\{\beta_x^n\} = [C_x]\{\delta^n\}$$

and

$$\{\beta_y^n\} = [C_y]\{\delta^n\} \tag{21}$$

where

$$\{\delta^n\}^{\mathrm{T}} = [w_1 \ldots w_6 \;\; \beta_{n(i)} \ldots \beta_{n(vi)}]$$

Substituting into equations (17) and (18) yields

$$\beta_x = \{\phi\}^{\mathrm{T}}[C_x]\{\delta^n\}$$

and

$$\beta_y = \{\phi\}^{\mathrm{T}}[C_y]\{\delta^n\} \tag{22}$$

From equations (9) and (22), the bending strain–displacement relationship is thus

$$\{\varepsilon^b\} = z\begin{bmatrix} \{\phi_{,x}\}^{\mathrm{T}}[C_x] \\ -\{\phi_{,y}\}^{\mathrm{T}}[C_y] \\ \{\phi_{,y}\}^{\mathrm{T}}[C_x] - \{\phi_{,x}\}^{\mathrm{T}}[C_y] \end{bmatrix}\{\delta^n\} = z[B^b]\{\delta^n\} \tag{23}$$

and the element bending stiffness matrix is

$$[K^b] = \frac{t^2}{12}\int_A [B^b]^{\mathrm{T}}[D][B^b]\,\mathrm{d}A \tag{24}$$

The matrices $[A^{-1}]$, $[C_x]$, $[C_y]$, $\{\phi_{,x}\}^{\mathrm{T}}$ and $\{\phi_{,y}\}^{\mathrm{T}}$ are given explicitly by the authors in Reference 1 and will not be repeated here.

Transformation to global co-ordinates

The 12×12 membrane and 12×12 bending stiffness matrices are now combined to form the stiffness matrix of the faceted shell. The nodal displacement vector corresponding to the shell stiffness, $[k^e]$ is

$$\{\delta^e\}^{\mathrm{T}} = [u_1 v_1 w_1 \ldots w_6 \;\; \beta_{n(i)} \ldots \beta_{n(vi)}]$$

As stated earlier, the proper membrane–flexural coupling is attained by effecting the proper spatial transformation. $[T]$ is the 3×3 transformation matrix relating the local co-ordinate system to the global co-ordinates. The local element stiffness matrix expressed in the global co-ordinates is

$$[k^G] = [A]^T[k^e][A]$$

where

$$[A] = [[T]\ldots[T] \pm 1 \cdots \pm 1] \tag{25}$$

To elaborate on the transformation in equation (25) affecting the normal rotations at the Loof nodes, the local numbering 1, 2, 3 of the vertices of the triangle is compared with the global one. If for an edge, both counts are of ascending order, then the rotations have the same sense as shown in Figure 1. If this is not so, the rows and columns in $[k^e]$ associated with $\beta_{n(i)}$ and $\beta_{n(ii)}$ along side 1–2 for example, are interchanged and their signs reversed before assembly into the global stiffness matrix.

The consistent load vector corresponding to the element stiffness matrix, $[K^e]$ for a uniformly distributed loading of p_x, p_y and p_z is

$$\{U\}^T = \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ p_x\frac{A}{3}\ \ p_y\frac{A}{3}\ \ p_z\frac{A}{3}\ldots 0\ 0\ 0\ 0\ 0\ 0\right]$$

since u, v and w are specified as quadratic polynomials.

NUMERICAL EXAMPLES

The accuracy and convergence characteristics of the proposed faceted shell element are demonstrated on several shell examples. Its ability to analyse intersections and branches in a shell is illustrated with a folded plate example. Where applicable, the results are compared with other faceted, curved or degenerated shell elements available in the literature. A list of the shell elements utilized for comparison is given in Table I.

Cylindrical shell roof

The cylindrical shell roof shown in Figure 3 has been frequently analysed by proposers of various shell elements and has become somewhat of a standard for comparison of shell elements.

Table I. Shell elements

Element	Ref.	D.o.f./ element	Symbol
Present element (flat)		24	●
Carr (flat)	25	27	*
Clough and Johnson (flat)	23	15	×
Olson and Bearden (flat)	29	18	■
Bonnes and Dhatt (curved)	45	36	▲
Brebbia and Connor (curved)	46	20	⊡
Cowper *et al.* (curved)	4	36	○
Dawe (curved)	6	54	△
MacNeal (4-node, degenerated)	47	20	◆
Nukulchai (4-node, degenerated)	48	24	□
Lagrange 9-NODE (full integration)	49	45	▼
Lagrange 9-NODE (reduced integration)	49	45	▽

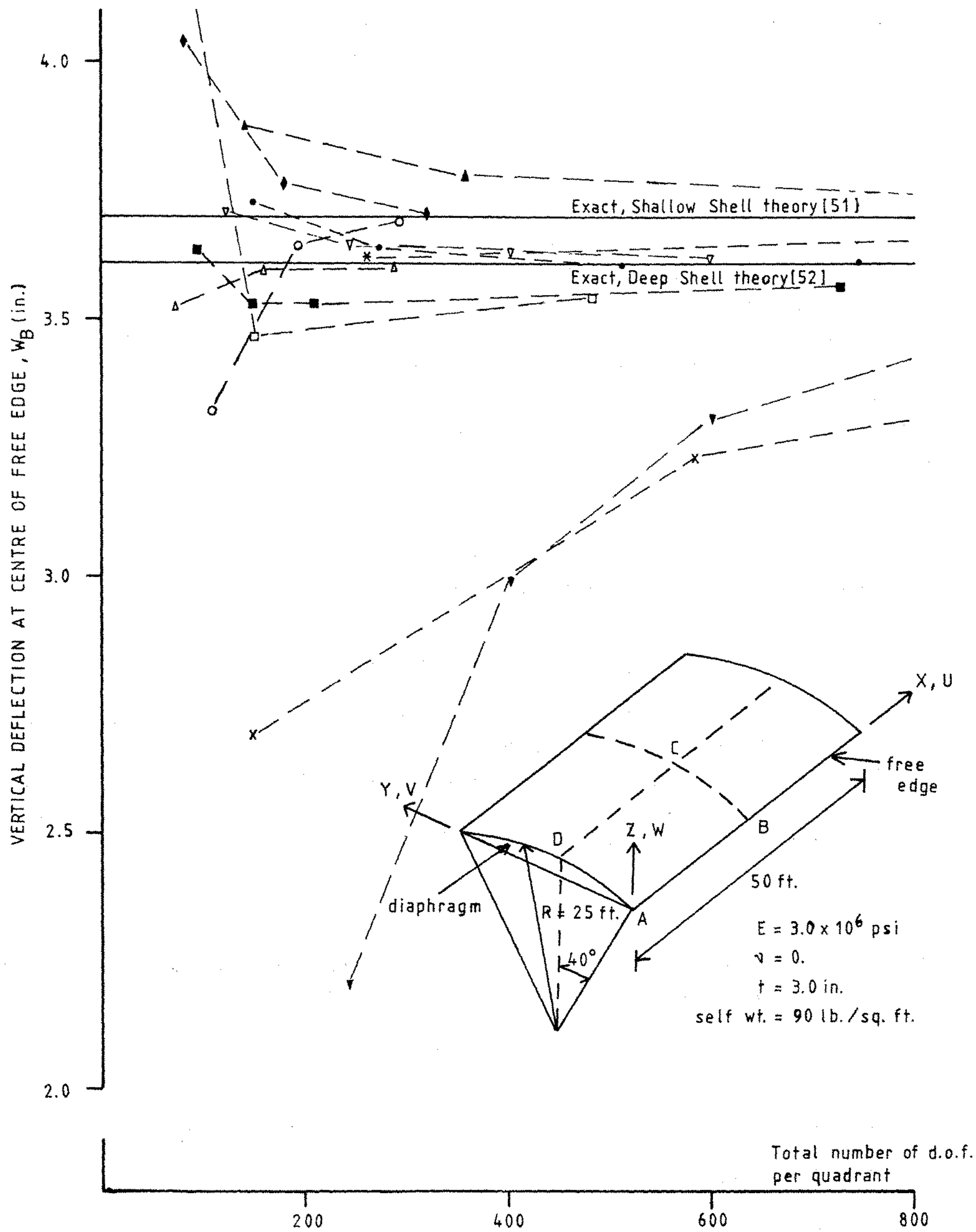
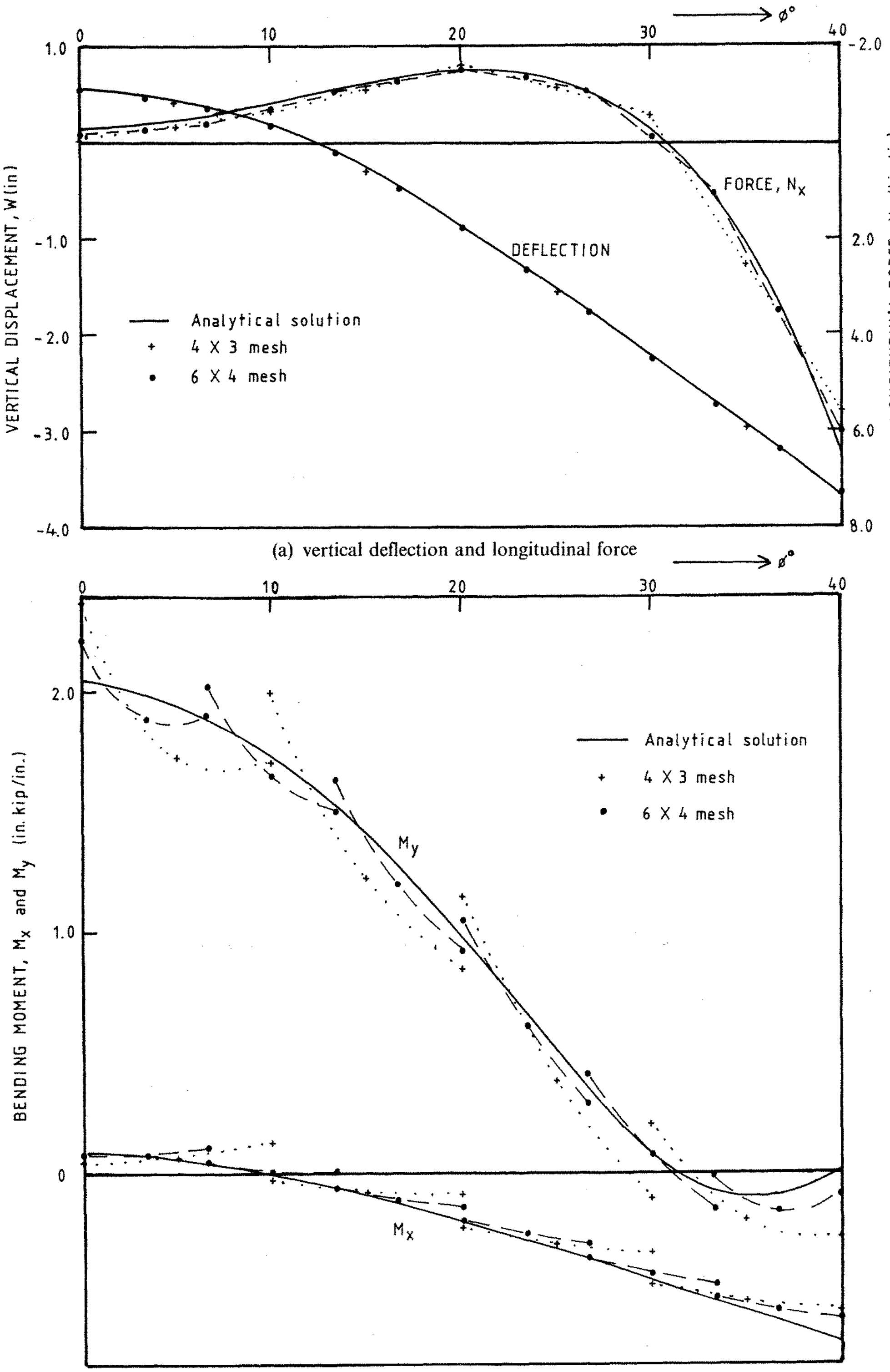


Figure 3. Convergence of deflection versus d.o.f., cylindrical shell problem

It is supported at its two ends by disphragms and the loading considered is due to the self-weight of the shell. Both membrane and bending actions are significant in this example.

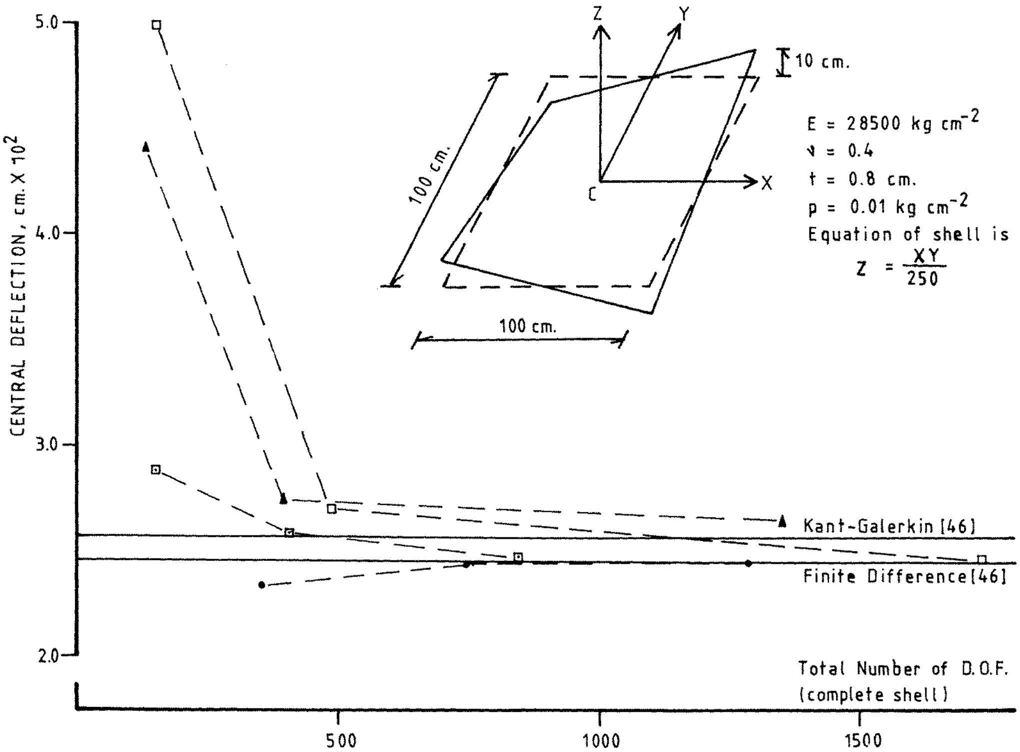
An analytical solution to this problem is given by Scordelis and Lo.[50] This solution is, however, based on shallow shell theory and results in a vertical displacement at the centre of the free edge, W_B of 3·696 in. A deep shell solution originally reported by Forsberg and Hartung[51] gives a result about 3 per cent lower. This corresponds to the results of Parisch,[49] where a convergence



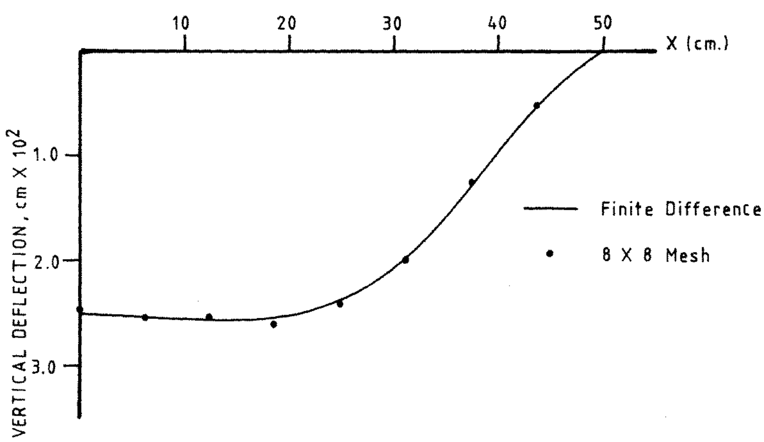
(a) vertical deflection and longitudinal force

(b) Bending moments

Figure 4. Stress and displacement distribution for shell roof at central section



(a) Convergence of central deflection, W_C



(b) Vertical deflection at centre line

Figure 5. Deflection characteristics of clamped hyperbolic shell

study using cubic Lagrange elements yielded for W_B a value of 3·61 in. The convergence behaviour of the proposed element and that of several existing shell elements are plotted in Figure 3. It is observed that monotonic convergence to the exact deep shell solution is attained by the present faceted element and its performance compares well relative to the other elements. We note that the faceted element of Chu and Schnobrich,[17] with 27 d.o.f. per element, yielded a solution corresponding to the shallow shell solution for a 6×9 mesh, which corresponds to a total of 1058 d.o.f.

The stress resultants obtained for some of the meshes employed are plotted in Figure 4. They are shown to be accurate relative to the shallow shell solutions, and without exhibiting significant oscillating behaviour.

Clamped hyperbolic paraboloid

The clamped hypar, loaded with a uniform normal pressure, is analysed with three different meshes (2×2, 4×4 and 8×8) for the whole shell. Two solutions for the central deflection, one obtained using finite differences[46] and the other through the Kantorovich–Galerkin procedure,[46] are given in the literature. The finite element results obtained by different authors as well as by the present element are plotted in Figure 5. Here again, the performance of the present element is excellent relative to the other elements, with convergence to the finite difference solution.

Pinched cylindrical shell

To test the adequacy of the present faceted element in handling deep shell problems, the cylinder loaded with two diametrically opposite point loads as illustrated in Figure 6, is analysed. An analytical solution, using a double Fourier series solution, is given by Lindberg *et al.*[52]

A 4×4 and 6×6 uniform mesh is used here. Good agreement with the analytical solution by the 6×6 mesh pattern, for both the displacements and stress resultants, is observed in Figure 7. More effective use of the element could be made by refining the meshes near to the point load.

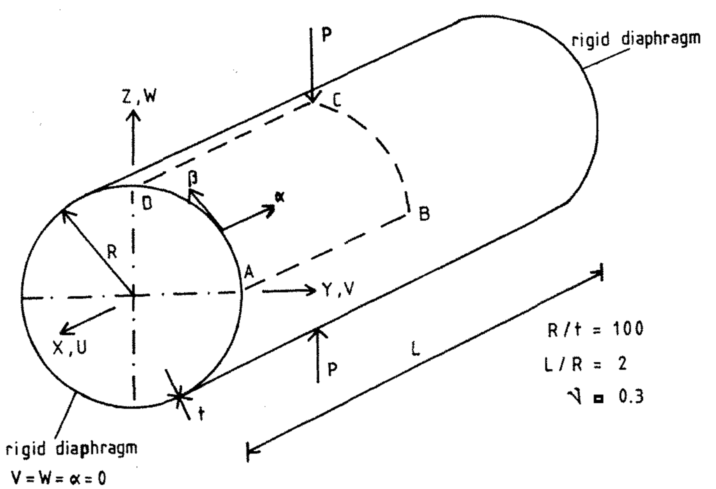
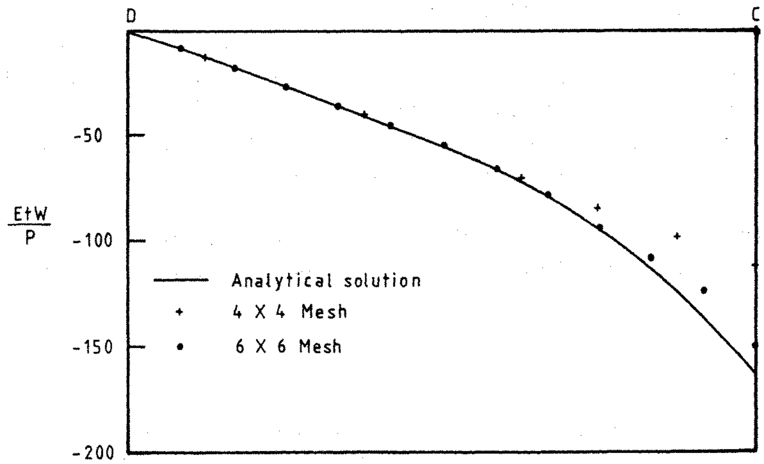
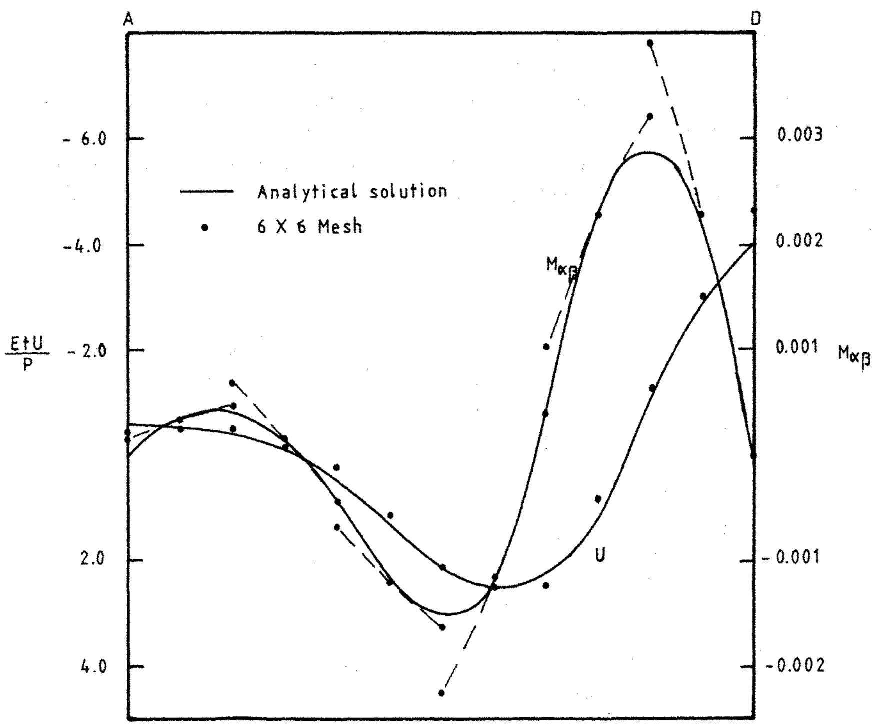


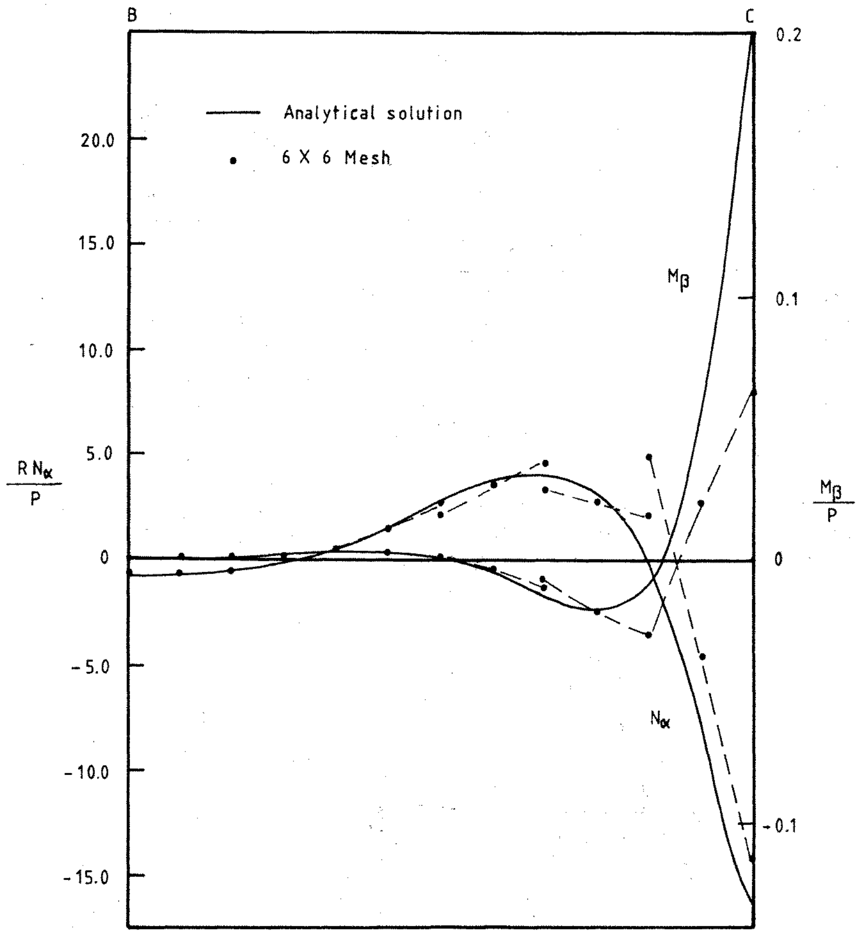
Figure 6. Pinched cylindrical shell



(a) Distribution of vertical displacement along DC.



(b) Distribution of longitudinal displacement and twist moment along AD.



(c) Stress distribution along BC

Figure 7. Displacement and stress distribution in pinched cylindrical shell

Northlight folded plate

Experimental results as well as an elasticity solution can be found in a paper by Scordelis *et al.*[53] for the folded plate depicted in Figure 8. This example was chosen to illustrate the ability of the present element to handle fold-lines. Since the regions between the folds are not curved, there is no geometric approximation involved in this structure. The 36-element mesh used in analysing the folded plate is also shown in Figure 8.

Results of the present analysis, the elasticity solution of Scordelis and the finite element analysis of Carr[25] are given in Table II. Excellent agreement with the elasticity solution is observed. The 36-element mesh using the present element provided a more accurate analysis than the 192-element mesh of the 27 d.o.f. element of Carr.

Table II. Midspan results folded plate

Joint	a	b	c	d	e	f
Method	Longitudinal stresses, σ(lb/in.^{2})					
Experimental	950	−530	80	30	480	−900
Elasticity	1214	−807	103	−103	807	−1214
Carr[25]	1220	−810	95·5	−96·5	807	−1220
Present element	1220	−801	95	−95	800	−1220
	Transverse moment (in/lb.in.)					
Experimental	0	0·11	−0·21	−0·11	0·05	0
Elasticity	0	0·048	−0·208	−0·208	0·048	0
Carr[25]	0·0066	0·0263	−0·1893	−0·1855	0·0293	0·0022
Present element	0·006	0·045	−0·209	−0·209	0·049	0·006
	Vertical deflection (in.)					
Experimental	—	0·0058	0·0013	0·0012	0·0063	—
Elasticity	0·0136	0·0080	0·0008	0·0008	0·0080	0·0136
Carr[25]	0·0136	0·0080	0·0008	0·0008	0·0080	0·0137
Present element	0·0134	0·0080	0·0008	0·0008	0·0080	0·0137

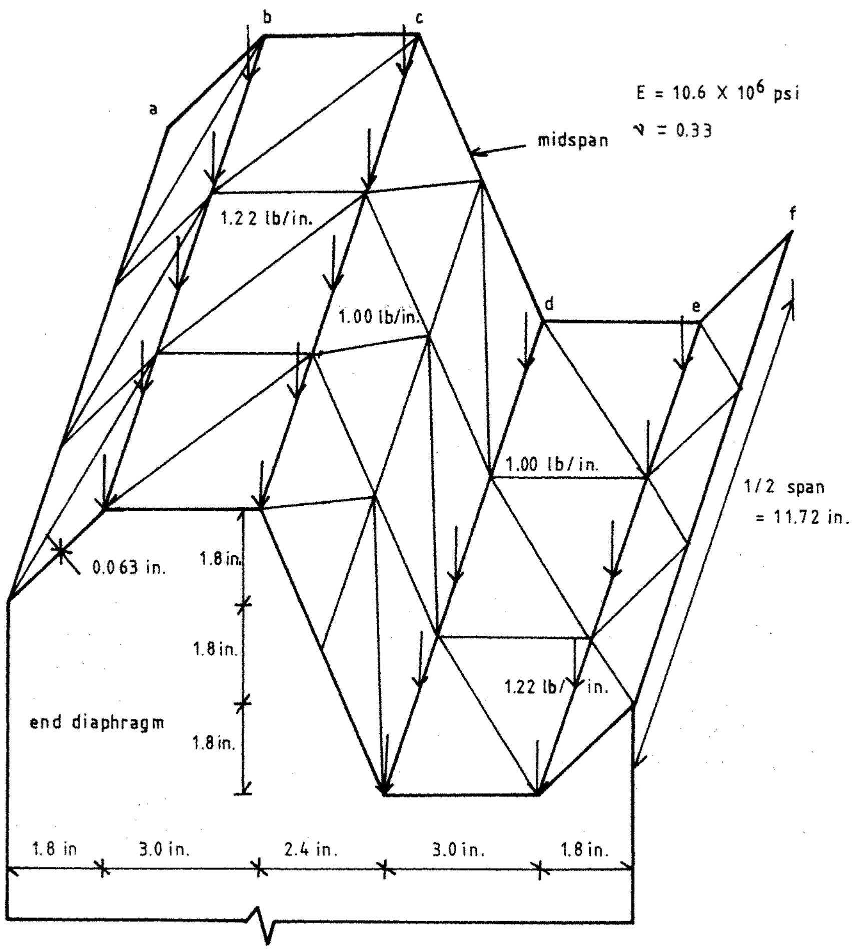


Figure 8. Northlight folded plate

CONCLUSIONS

A flat shell element with engineering d.o.f. only as nodal parameters and which is (a) displacement compatible, (b) possesses at least a linear stress variation, and (c) is capable of modelling folds, has been presented. The term 'displacement compatibility', as used in this paper, implies continuity only in the translational displacements. The rotations are not compatible along adjacent element boundaries, but this is not required as the formulation employed here need only be C^0 continuous in the displacements. The numerical examples indicate that the performance of this element relative to other elements in the literature, both faceted and curved, is good. In particular, the stress distributions computed for the element exhibits a reasonably smooth distribution close to the exact solution without significant oscillations. This effect has been highlighted in Figures 4, 7(b) and (7c) by plotting raw rather than smoothed element stresses.

A practical difficulty associated with the present element, and as is the case with all elements with Loof nodes, is the need to define a positive sense along the interface of the elements for the rotational d.o.f. As described in the paper, this can be achieved quite easily. However, this manner of defining the rotational axes is often only allowed in very few general finite element programs. In addition, the beam element to be used in conjunction with the present faceted element must have equivalent connectors, and such a beam element has been proposed by Irons.[41] Consequently, due to the rather unconventional choice of the rotational connectors, the introduction of the proposed element into most existing general finite element programs, will necessitate some modifications.

Knowles *et al.*[54] noted that, for practical problems, not one class of elements for shell analysis has yet been found to be pre-eminent. The proposed faceted shell element, which has proved to be both accurate and efficient in the examples studied, is relatively easy and cheap to formulate, is simple to use, does not exhibit membrane or shear locking and does not have any spurious mechanisms; it should thus be a useful addition to any finite element shell library.

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