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CONSTRAINED ANISOTROPIC ELASTIC MATERIALS IN UNILATERAL CONTACT WITH OR WITHOUT FRICTION

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INTRODUCTION

We consider time independent problems of linear homogeneous anisotropic elasticity for materials with internal constraints that are in frictional, or frictionless, contact with a rigid support.

Such problems, but without contact or friction, were recently considered by Arnold and Falk [1] where the well posedness of the classical boundary value problems has been proved under the assumption that the internal constraints satisfy the Arnold–Falk nonsingularity criterion.

We show the well posedness of three problems with contact, two with friction as well, under the same nonsingularity assumption.

The special problem of an incompressible isotropic material with frictionless contact was considered by Kikuchi and Oden [11].

Let $\Omega \subset R^3$ be a bounded domain, representing the elastic body, with a smooth boundary $\partial\Omega$. Let $\mathbf{u} = (u_1, u_2, u_3) : \Omega \to R^3$ be the displacement vector and $\underset{\sim}{\sigma} = (\sigma_{kl}) : \Omega \to R^9$ be the symmetric 3×3 stress tensor, then the equations of equilibrium of linear elasticity are

$$A\underset{\sim}{\sigma} = \underset{\sim}{\varepsilon}(\mathbf{u}) \qquad \text{in } \Omega, \tag{1.1}$$

$$-\operatorname{div} \underset{\sim}{\sigma} = \mathbf{f} \qquad \text{in } \Omega \tag{1.2}$$

where $\mathbf{f} = (f_1, f_2, f_3) : \Omega \to R^3$ is the prescribed force density in Ω, the tensor $\underset{\sim}{\varepsilon}(\mathbf{u})$ is the symmetric part of the gradient tensor $\nabla\mathbf{u}$, i.e. $\varepsilon_{ij} = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ and

$$(\operatorname{div} \underset{\sim}{\sigma})_i = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}.$$

The compliance tensor A, which characterizes the elastic properties of the material, is a fourth order tensor and acts as a self-adjoint linear operator on the six dimensional space S of all 3×3 symmetric tensors. It is determined by specifying 21 coefficients or elastic moduli.

Let $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$, where Γ_i, $i = 1, 2, 3$, are relatively open parts of the boundary, then we assume that

$$\mathbf{u} = \mathbf{g} \qquad \text{on } \Gamma_1, \tag{1.3}$$

$$-\underset{\sim}{\sigma}\mathbf{n} = \mathbf{t}_r \qquad \text{on } \Gamma_2, \tag{1.4}$$

where $\mathbf{t}_r$ is a given traction, $(\underset{\sim}{\sigma}\mathbf{n})_i = \sigma_{ij}n_j$, $\mathbf{n} = (n_1, n_2, n_3)$ is the unit outward normal to Γ_i and $\mathbf{g}$ is given on Γ_1.

The three problems that we consider differ in the conditions on Γ_3, the part of the boundary that is the candidate for contact with the rigid support.

First consider Signorini's condition for frictionless contact, see e.g. Fichera [5] or Duvaut and Lions [3]. At a point $x \in \Gamma_3$ either there is no contact and then $\sigma_n = \sigma_{ij}n_in_j = 0$ and $u_n = u_in_i \leq s(x)$, where $s(x)$ is the initial gap between Γ_3 and the rigid support, or there is contact $u_n = s$, and $\sigma_n \leq 0$. Since the contact is frictionless $\underset{\sim}{\sigma}_t = \underset{\sim}{\sigma}\mathbf{n} - \sigma_n\mathbf{n} = 0$. Thus (on Γ_3)

$$\underset{\sim}{\sigma}_t = 0, \qquad \sigma_n \leq 0, \qquad u_n \leq s \qquad \text{and} \qquad \sigma_n(u_n - s) = 0. \tag{1.5}$$

Secondly we consider the problem with prescribed friction, Duvaut and Lions [3, pp. 134–140]. It is assumed that the normal force t_* is given and then

$$\sigma_n = t_*, \qquad |\underset{\sim}{\sigma}_t| \leq g_*, \tag{1.6}$$

if $|\underset{\sim}{\sigma}_t| < g_*$ then $\mathbf{u}_t = 0$, if $|\underset{\sim}{\sigma}_t| = g_*$ then $\exists\, \lambda \geq 0$ such that $\mathbf{u}_t = -\lambda\underset{\sim}{\sigma}_t$. The region where $|\underset{\sim}{\sigma}_t| < g_*$ is called the *stick region* and where $|\underset{\sim}{\sigma}_t| = g_*$ the *slip region*. $g_*\colon \Gamma_3 \to R$ is given, usually taken as $\mu(x)|t_*(x)|$ where $\mu(x)$ is the *friction coefficient*.

We remark that in Coulomb's law of friction g_* is not prescribed but $g_* = -\mu\sigma_n$ where μ is the coefficient of friction. Nevertheless for small external loads the restricted version (1.6) seems to be a good approximation, (see e.g. Kalker [7]).

Finally we consider the problem of frictional contact with normal compliance

$$\underset{\sim}{\sigma}_n = c_N(u_n - s)^m_+, \qquad |\sigma_t| \leq c_T(u_n - s)^m_+, \tag{1.7}$$

if $|\underset{\sim}{\sigma}_t| < c_T(u_n - s)^m_+$ then $\mathbf{u}_t = 0$, if $|\underset{\sim}{\sigma}_t| = c_T(u_n - s)^m_+$ then $\exists\, \lambda \geq 0$ such that $\mathbf{u}_t = -\lambda\underset{\sim}{\sigma}_t$. The problem is obtained from (1.6) by setting $t_* = c_N(u_n - s)^m_+$ and $g_* = c_T(u_n - s)^m_+$, where c_N and c_T are given bounded and positive functions defined on Γ_3 and $m \geq 1$. Also $(x)_+ = \max(0, x)$. Conditions of normal compliance were introduced recently by Oden and Martins [14] instead of the usual Signorini's conditions as more realistically representing the actual contact with friction since it takes into account the material properties of the surface of the elastic body. They were considered in Martins and Oden [14] and in [8, 9, and 10], see also [12].

Below we refer to Signorini's problem, (1.1)–(1.5), as (P_S), to the friction problem, (1.1)–(1.4) and (1.6), as (P_F) and to the problem with normal compliance, (1.1)–(1.4) and (1.7), as (P_N).

It is usually assumed that the compliance tensor A is positive definite (and therefore the equations are uniformly elliptic) and so invertible and thus $\underset{\sim}{\sigma}$ can be eliminated and the problem written in terms of $\mathbf{u}$ only. In such cases the three problems, (P_S), (P_F) and (P_N) are well-posed (see the references above). For many important materials, however, the compliance tensor is positive semidefinite and therefore singular or almost singular. It is clear that the question of the continuous dependence on the data is of importance in justifying the use of constrained problems to describe almost constrained problems.

In this paper we assume that the compliance tensor A is positive semidefinite. If $\underset{\sim}{\sigma}_0$ is a nonzero tensor in A's null space, i.e. $A\underset{\sim}{\sigma}_0 = 0$, then it follows from (1.1) that

$$\underset{\sim}{\varepsilon}(\mathbf{u}) : \underset{\sim}{\sigma}_0 = 0$$

This relation is called the *material constraint*, the material is said to be (internally) constrained, and $\underset{\sim}{\sigma}_0$ is called a *constraint tensor*. An incompressible material has the identity matrix as a constraint tensor and the material constraint is $\operatorname{div} \mathbf{u} = 0$. A material which is inextensible in the direction $\mathbf{l}$ has the constraint tensor $\mathbf{l}\mathbf{l}^T$ and satisfies the constraint $\mathbf{l} \cdot \nabla(\mathbf{l} \cdot \mathbf{u}) = 0$. Even without contact or friction the boundary value problems for constrained materials may or may not be well posed. It has been shown in [1] that this depends on whether the constraint tensor is of full rank, i.e. *nonsingular*, or is of deficient rank, that is *singular*.

Under the assumption that the material admits only nonsingular constraints Arnold and Falk [1] proved the well posedness of the fundamental boundary value problems of linear elasticity without contact or friction. We prove our results under the same hypothesis.

To state the various necessary estimates we use the Arnold–Falk [1] nonsingularity criterion $\chi(A)$ which we proceed to describe. Let $\mathcal{C}$ be the set of positive semidefinite self-adjoint linear operators on $\mathbf{S}$. If $A \in \mathcal{C}$ let $0 \le \lambda_1(A) \le \lambda_2(A) \le \cdots \lambda_6(A)$ denote the eigenvalues of A and let $\underset{\sim}{\sigma}_1(A), \ldots, \underset{\sim}{\sigma}_6(A) \in \mathbf{S}$ be the corresponding orthonormal basis of eigenvectors. Then the quantity that measures how close is a material to having a singular constraint is defined by

$$\chi(A) = \max[\lambda_1(A), \lambda_2(A)|\underset{\sim}{\sigma}_1(A)^{-1}|^{-2}]. \tag{1.8}$$

It is proved in [1] that this definition is independent of the choice of the eigenbasis, that $\chi : \mathcal{C} \to [0, +\infty)$ is continuous and it vanishes if and only if A admits a singular constraint. For further discussion see [1].

The principal results of this paper are the following theorems.

THEOREM 1.1. Assume that the compliance tensor A is positive semidefinite and admits no singular constraints. Then for any data

$$(\mathbf{f}, \mathbf{g}, \mathbf{t}_r, s) \in L^2(\Omega; R^3) \times \mathbf{H}_{00}^{1/2}(\Gamma_1) \times \mathbf{L}^2(\Gamma_2) \times H_{00}^{1/2}(\Gamma_3)$$

there exists a unique solution $(\underset{\sim}{\sigma}, \mathbf{u}) \in \mathbf{L}(\Omega) \times \mathbf{H}^1(\Omega)$ to the boundary value problem (P_S), (1.1)–(1.5).

Moreover if $(\underset{\sim}{\sigma}^i, \mathbf{u}^i)$ are the solutions corresponding to the data $(\mathbf{f}^i, \mathbf{g}^i, \mathbf{t}_r^i, s^i)$, $i = 1, 2$ then the *a priori* estimate

$$\begin{aligned} &\|\underset{\sim}{\sigma}^1 - \underset{\sim}{\sigma}^2\|_{L^2(\Omega)} + \|\mathbf{u}^1 - \mathbf{u}^2\|_{H^1(\Omega)} \\ &\quad \le c(\|\mathbf{f}^1 - \mathbf{f}^2\|_{V_D'} + \|\mathbf{g}^1 - \mathbf{g}^2\|_{H_{00}^{1/2}(\Gamma_1)} + \|\mathbf{t}_r^1 - \mathbf{t}_r^2\|_{H^{-1/2}(\Gamma_2)} + \|s^1 - s^2\|_{H_{00}^{1/2}(\Gamma_3)}), \end{aligned} \tag{1.9}$$

holds, where $c > 0$ is a constant depending only on Ω, the upper bounds of the compliances and the lower bound of $\chi(A)$. The solution $(\underset{\sim}{\sigma}, \mathbf{u})$ depends continuously on A.

For the notation we refer the reader to Section 2. A similar result holds for (P_F), where we take $\mathbf{g} = 0$ and $s = 0$.

THEOREM 1.2. Assume that the compliance tensor A is positive semidefinite and admits no singular constraints. Then for any data $(\mathbf{f}, \mathbf{t}_r, t_*, g_*) \in \mathbf{L}(\Omega) \times \mathbf{L}^2(\Gamma_2) \times L^\infty(\Gamma_3) \times L^\infty(\Gamma_3)$, $g_* \geq 0$ a.e. on Γ_3, there exists a unique solution $(\underset{\sim}{\sigma}, \mathbf{u}) \in \mathbf{L}(\Omega) \times \mathbf{H}^1(\Omega)$ to (P_F), (1.1)–(1.4) and (1.6). Moreover if $(\underset{\sim}{\sigma}^i, \mathbf{u}^i)$ are the solutions corresponding to the data $(\mathbf{f}^i, \mathbf{t}_r^i, t_*^i, g_*^i)$, $i = 1, 2$, then there holds

$$\begin{aligned} &\|\underset{\sim}{\sigma}^1 - \underset{\sim}{\sigma}^2\|_{L^2(\Omega)} + \|\mathbf{u}^1 - \mathbf{u}^2\|_{H^1(\Omega)} \\ &\quad \leq c(\|\mathbf{f}^1 - \mathbf{f}^2\|_{V_D'} + \|\mathbf{t}_r^1 - \mathbf{t}_r^2\|_{H^{-1/2}(\Gamma_2)} + \|t_*^1 - t_*^2\|_{L^\infty(\Gamma_3)} + \|g_*^1 - g_*^2\|_{L^\infty(\Gamma_3)}), \end{aligned} \tag{1.10}$$

where $c > 0$ is a constant depending only on Ω, the upper bounds of the compliances and the lower bound of $\chi(A)$. The solution $(\underset{\sim}{\sigma}, \mathbf{u})$ depends continuously on A.

A similar result holds for (P_N) as well.

THEOREM 1.3. Assume A is as above. Then for any data

$$(\mathbf{f}, \mathbf{t}_r, \mathbf{g}, s, c_N, c_T) \in \mathbf{L}(\Omega) \times \mathbf{L}^2(\Gamma_2) \times (\mathbf{H}^{1/2}(\Gamma_1))^3 \times (L^\infty(\Gamma_3))^3,$$

$c_T, c_N > 0$ a.e. on Γ_3, there exists a locally unique solution $(\underset{\sim}{\sigma}, \mathbf{u}) \in L(\Omega) \times \mathbf{H}^1(\Omega)$ to problem (P_N), (1.1)–(1.4) and (1.7) provided c_N and c_T are sufficiently small.

By locally unique solution we mean a unique solution in a ball B_R of radius R and center 0 in $\mathbf{H}^1(\Omega)$.

The proofs of these theorems are based on the construction of appropriate Lagrangians with saddle points. This in turn is based on the dual formulations in the sense of Ekeland and Temam [4] and proofs, using convexity, of the existence of solutions to the dual problems. In addition for (P_N) we use an iterative argument. Our method differs essentially from that of Arnold and Falk [1] who use Brezzi's [2] saddle point theorem.

The outline of the paper is as follows. Notation and various results from Ekeland and Temam [4] are given in Section 2. In Section 3, theorem 1.1 is proved. First the mixed formulation is given, then the existence for its dual problem is proved. The connection between the two is via a Lagrangian whose saddle point corresponds to the solution to the mixed problem once it is known to solve the dual problem.

In the case of Dirichlet conditions as well as Signorini's but without any tractions, i.e. $\Gamma_2 = \emptyset$, Section 4, appropriate compatibility conditions are needed to guarantee the existence of a unique solution.

Following [1] we consider, in Section 5, a displacement-pressure formulation of the problem (P_S), the main advantage of which is the need to solve for $\mathbf{u}$ and the scalar pressure p, i.e. 4 variables instead of the 9 variables $\mathbf{u}$ and $\underset{\sim}{\sigma}$.

In Section 6 we consider the problem with friction, (P_F), and using arguments similar to those of Section 3 we prove theorem 1.2. Finally theorem 1.3 is proved in Section 7. We linearize the problem with normal compliance and use the results of Section 6 to prove that an appropriately operator T is a contraction on a ball B_R of radius R centered at the origin of $\mathbf{H}^1$.

2. NOTATION AND PRELIMINARIES

Let $\mathbf{S}$ denote the linear normed space of 3×3 symmetric tensors with inner product $\underset{\sim}{\sigma} : \underset{\sim}{\tau} = \sigma_{ij}\tau_{ij}$ and the induced norm $|\underset{\sim}{\sigma}|_F$. Here and below summation is implied for repeated indices and always $i, j = 1, 2, 3$. If $\mathbf{l} \in R^3$ then the action of $\underset{\sim}{\sigma} \in \mathbf{S}$ on $\mathbf{l}$ is denoted by $\underset{\sim}{\sigma}\mathbf{l}$ where $(\underset{\sim}{\sigma}\mathbf{l})_i = \sigma_{ij} l_j$.

In considering vector valued functions $\mathbf{u}: \Omega \to R^3$ we set

$$H^1(\Omega; R^3) = \mathbf{H}^1(\Omega) = \{\mathbf{u} = (u_1, u_2, u_3);\ u_i \in H^1(\Omega),\ i = 1, 2, 3\}$$

and

$$\|\mathbf{u}\|_1^2 = \sum_{i=1}^{3} \|u_i\|_{H^1(\Omega)}^2.$$

Similarly we set

$$\mathbf{L}(\Omega) = L^2(\Omega; R^6) = \{\underset{\sim}{\sigma};\ \sigma_{ij} \in L^2(\Omega),\ \sigma_{ij} = \sigma_{ji},\ 1 \le i, j \le 3\}, \tag{2.1}$$

and

$$\|\underset{\sim}{\sigma}\|_0^2 = \sum_{\substack{i,j=1\\ i \le j}}^{3} \|\sigma_{ij}\|_{L^2(\Omega)}^2.$$

Let $\Gamma = \partial\Omega$ and Γ' be an open subset of Γ with a smooth boundary, then we set $\mathbf{H}^{1/2}(\Gamma) = \{\mathbf{u};\ u_i \in H^{1/2}(\Gamma),\ i = 1, 2, 3\}$ and $\mathbf{H}^{1/2}(\Gamma')$ is defined similarly, where $H^{1/2}(\Gamma)$ and $H^{1/2}(\Gamma')$ are the standard Sobolev spaces on the boundary (see e.g. Lions and Magenes [13, Chapter 1]). The subspace of $\mathbf{H}^{1/2}(\Gamma')$ consisting of functions on Γ' whose extension by zero to Γ belongs to $\mathbf{H}^{1/2}(\Gamma)$ is denoted by $\mathbf{H}_{00}^{1/2}(\Gamma')$ and $\mathbf{H}^{-1/2}(\Gamma') = (\mathbf{H}_{00}^{1/2}(\Gamma'))'$ is its dual. On occasions we denote their norms by $|\cdot|_{1/2,\Gamma'}$ and $|\cdot|_{-1/2,\Gamma'}$ as well. Let

$$\begin{aligned} V_0 &= H_0^1(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega);\ \mathbf{u} = 0 \text{ on } \Gamma\},\\ V_D &= \{\mathbf{u} \in \mathbf{H}^1(\Omega);\ \mathbf{u} = 0 \text{ on } \Gamma_1\} \end{aligned} \tag{2.2}$$

where here and below $\Gamma = \partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$, $\Omega \subset R^3$ is a bounded domain, $\partial\Omega$ is in $C^{1,1}$ and Γ_i are relatively open in Γ and unless indicated to the contrary, $\Gamma_i \neq \emptyset$. The duals are $V_0' = (H_0^1(\Omega; R^3))'$ and V_D' with norms $\|\cdot\|_{-1,0}$ and $\|\cdot\|_{-1,D}$ respectively.

Next we shall need the following Green's formula. Let

$$\mathbf{L}_{\text{div}}^2(\Omega) = \{\underset{\sim}{\tau} \in \mathbf{L}(\Omega);\ \operatorname{div} \underset{\sim}{\tau} \in L^2(\Omega; R^3)\} \tag{2.3}$$

then it is well known (e.g. Kikuchi and Oden [12]) that the mappings $\pi_n: \mathbf{L}_{\text{div}}^2(\Omega) \to H^{-1/2}$ and $\pi_t: \mathbf{L}_{\text{div}}^2(\Omega) \to \mathbf{H}^{1/2}(\Gamma)$ given, for $\underset{\sim}{\tau} \in (C(\bar{\Omega}))^6$, by

$$\pi_n(\underset{\sim}{\tau}) = \underset{\sim}{\tau}\mathbf{n} \cdot \mathbf{n}|_\Gamma, \tag{2.4}$$

$$\pi_t(\underset{\sim}{\tau}) = (\underset{\sim}{\tau}\mathbf{n} - (\underset{\sim}{\tau}\mathbf{n} \cdot \mathbf{n})\mathbf{n})|_\Gamma, \tag{2.5}$$

are well defined. Here and everywhere below $\mathbf{n}$ is the unit normal to Γ pointing out of Ω. Below we also use the notation

$$\underset{\sim}{\tau}_t = \pi_t(\underset{\sim}{\tau}),\ \tau_n = \pi_n(\underset{\sim}{\tau}), \qquad v_n = \mathbf{v} \cdot \mathbf{n} \qquad \text{and} \qquad \mathbf{v}_t = \mathbf{v} - v_n\mathbf{n}.$$

Then Green's formula holds

$$\int_\Omega \underset{\sim}{\tau} : \underset{\sim}{\varepsilon}(\mathbf{v})\,dx + \int_\Omega \operatorname{div} \underset{\sim}{\tau} \cdot \mathbf{v}\,dx = \langle \pi_n(\underset{\sim}{\tau}), v_n \rangle_\Gamma + \langle \pi_t(\underset{\sim}{\tau}), \mathbf{v}_t \rangle_\Gamma, \tag{2.6}$$

for all $\underset{\sim}{\tau} \in \mathbf{L}_{\text{div}}^2(\Omega)$ and $\mathbf{v} \in H^1(\Omega; R^3)$, where $\langle \cdot, \cdot \rangle_\Gamma$ is the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

Since we use the duality theory of Ekeland and Temam extensively we recall some definitions and results from [4].

Let V be a reflexive Banach space and V' its dual. Let $F \in \Gamma_0(V) = \{$set of all proper, convex and l.s.c. functions $V' \to \bar{R}\}$, $(\bar{R} = R \cup \{+\infty\})$, and consider the minimization problem:

$$\text{(P)} \qquad u \in V: F(u) \leq F(v), \qquad \forall\, v \in V. \tag{2.7}$$

Let Y be another reflexive Banach space with dual Y'. Let ϕ: $V \times Y \to \bar{R}$ be such that

$$\phi(v, 0) = F(v). \tag{2.8}$$

Then for $p \in Y$ we may consider the problem:

$$(\mathrm{P}_p) \qquad u \in V: \phi(u, p) \leq \phi(v, p), \qquad \forall\, v \in V. \tag{2.9}$$

Define

$$h(p) = \inf_{v \in V} \phi(v, p), \tag{2.10}$$

then we have the following definition.

Definition 2.1 ([4, p. 50]). (i) The problem (P) is said to be *stable* if h is finite and subdifferentiable at zero.

(ii) The *dual problem* for (P) is given by

$$(\mathrm{P}^*) \qquad \sup_{p^* \in Y'} \{-\phi^*(0, p^*)\}, \tag{2.11}$$

where ϕ^* is the Fenchel conjugate function of ϕ.

(iii) The *Lagrangian* for (P_p) is the functional L: $V \times Y' \to R$ defined by

$$-L(u, p^*) = \sup_{p \in Y} [(p^*, p) - \phi(u, p)], \qquad \forall\, u \in V, \qquad \forall\, p^* \in Y'. \tag{2.12}$$

We shall need the following.

THEOREM 2.2 ([4, p. 57]). Let $\phi \in \Gamma_0(V \times Y)$ and assume that (P) is stable. Then $\bar{u} \in V$ is a solution to (P) iff there exists $\bar{p}^* \in Y'$ such that $(\bar{u}, \bar{p}^*)$ is a saddle point for L. Then $\bar{p}^*$ solves (P*).

3. THE PROBLEM WITH SIGNORINI'S CONDITIONS

In this section we prove theorem 1.1 using duality. First we prove the existence of the unique solution to the dual problem and then an appropriate Lagrangian L is constructed. The unique saddle point of L furnishes the solution to the original problem via theorem 2.2.

It is convenient to deal with zero Dirichlet data on Γ_1 (recall $\Gamma_1 \neq \emptyset$), so we introduce $\xi \in \mathbf{H}^1(\Omega)$ being the unique solution of

$$\begin{aligned} \operatorname{div} \underset{\sim}{\varepsilon}(\xi) &= 0, && \text{in } \Omega, \\ \xi &= \mathbf{g}, && \text{on } \Gamma_1, \\ \underset{\sim}{\varepsilon}(\xi)\mathbf{n} &= 0, && \text{on } \Gamma_2, \\ \xi_n &= s, && \pi_t(\underset{\sim}{\varepsilon}(\xi)) = 0 \qquad \text{on } \Gamma_3. \end{aligned} \tag{3.1}$$

From the standard theory (e.g. [3]) there holds

$$\|\xi\|_1 \le c(\Omega)[|\mathbf{g}|_{1/2,\Gamma_1} + |s|_{1/2,\Gamma_3}]. \tag{3.2}$$

Now problem (P_S) can be written as:

Find a pair $\{\underset{\sim}{\sigma}, \mathbf{w}\}$ such that

$$\begin{aligned} A\underset{\sim}{\sigma} - \underset{\sim}{\varepsilon}(\mathbf{w}) &= \underset{\sim}{\varepsilon}(\xi), && \text{in } \Omega,\\ -\operatorname{div}\underset{\sim}{\sigma} &= \mathbf{f}, && \text{in } \Omega,\\ \mathbf{w} &= 0, && \text{on } \Gamma_1,\\ \underset{\sim}{\sigma}\mathbf{n} &= \mathbf{t}_r, && \text{on } \Gamma_2.\\ \pi_t(\underset{\sim}{\sigma}) &= 0, &&\\ \pi_n(\underset{\sim}{\sigma}) &\le 0, && w_n \le 0,\\ \text{and}\quad \pi_n(\underset{\sim}{\sigma})w_n &= 0, && \text{on } \Gamma_3. \end{aligned} \tag{3.3}$$

It is clear that if $\{\underset{\sim}{\sigma}, \mathbf{w}\}$ is a solution to (3.3) and if we set $\mathbf{u} = \mathbf{w} + \xi$, then $\{\underset{\sim}{\sigma}, \mathbf{u}\}$ is a solution to (P_S).

Problem (3.3) can be set in the following variational form, the so called *mixed formulation*:

$$(\mathrm{P}_1)\qquad \begin{cases} \text{Find } (\underset{\sim}{\sigma}, \mathbf{w}) \in \mathbf{L}(\Omega) \times K \text{ such that} \\ a(\underset{\sim}{\sigma}, \underset{\sim}{\tau}) - b(\underset{\sim}{\tau}, \mathbf{w}) = b(\underset{\sim}{\tau}, \xi), \qquad \forall\, \underset{\sim}{\tau} \in \mathbf{L}(\Omega). & (3.4)\\ b(\underset{\sim}{\sigma}, \mathbf{v} - \mathbf{w}) \ge l(\mathbf{v} - \mathbf{w}), \qquad \forall\, \mathbf{v} \in K & (3.5) \end{cases}$$

where

$$a(\underset{\sim}{\sigma}, \underset{\sim}{\tau}) = \int_\Omega A\underset{\sim}{\sigma} : \underset{\sim}{\tau}\, \mathrm{d}x, \tag{3.6}$$

$$b(\underset{\sim}{\tau}, \mathbf{v}) = \int_\Omega \underset{\sim}{\tau} : \underset{\sim}{\varepsilon}(\mathbf{v})\, \mathrm{d}x, \tag{3.7}$$

$$l(\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\, \mathrm{d}x + \langle \mathbf{t}_r, \mathbf{v}\rangle_{\Gamma_2}, \tag{3.8}$$

and K is the (nonempty) closed convex set

$$K = \{\mathbf{v} \in V_D;\ \mathbf{v}\cdot\mathbf{n} = v_n \le 0 \text{ on } \Gamma_3 \text{ and } v_n \in H_{00}^{1/2}(\Gamma_3)\}. \tag{3.9}$$

Any solution $\{\underset{\sim}{\sigma}, \mathbf{w}\}$ to (P_1) is a weak solution to (3.3). In order to study the well-posedness of (P_1), and therefore of (3.3), we consider the *dual problem*:

Find $\underset{\sim}{\sigma} \in K^*$ such that

$$(\mathrm{P}_2)\qquad F^*(\underset{\sim}{\sigma}) \le F^*(\underset{\sim}{\tau}), \qquad \forall\, \underset{\sim}{\tau} \in K^*, \tag{3.10}$$

where

$$F^*(\underset{\sim}{\tau}) = \tfrac{1}{2}a(\underset{\sim}{\tau}, \underset{\sim}{\tau}) - b(\underset{\sim}{\tau}, \xi), \tag{3.11}$$

and K^* is the (nonempty) closed convex set

$$K^* = \{\underset{\sim}{\tau} \in \mathbf{L}(\Omega);\ b(\underset{\sim}{\tau}, \mathbf{v}) \ge l(\mathbf{v}),\ \forall\, \mathbf{v} \in K\}. \tag{3.12}$$

Remark 3.1. Since $V_0 = H_0^1(\Omega) \subset K$ it follows that if $\tau \in K^*$ then

$$b(\underset{\sim}{\tau}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx, \qquad \forall \, \mathbf{v} \in V_0.$$

In order to simplify the problem we use a translation of K^*. We shall use it below. Let $\psi \in V_D$ be the unique solution of

$$\int_\Omega \underset{\sim}{\varepsilon}(\psi) : \underset{\sim}{\varepsilon}(\mathbf{v}) \, dx = l(\mathbf{v}), \qquad \forall \, \mathbf{v} \in V_D. \tag{3.13}$$

That is $-\operatorname{div} \underset{\sim}{\varepsilon}(\psi) = \mathbf{f}$ in Ω, $\psi = 0$ on Γ_1, $\underset{\sim}{\varepsilon}(\psi)\mathbf{n} = \mathbf{t}_r$ on Γ_2 and $\underset{\sim}{\varepsilon}(\psi)\mathbf{n} = 0$ or Γ_3. Then we have the decomposition

$$K^* = \underset{\sim}{\varepsilon}(\psi) + \hat{K}^* \tag{3.14}$$

i.e. $\underset{\sim}{\tau} \in K^* \Leftrightarrow \underset{\sim}{\tau} = \underset{\sim}{\varepsilon}(\psi) + \underset{\sim}{\nu}$ where $\underset{\sim}{\nu} \in \hat{K}^*$ and

$$\hat{K}^* = \{\underset{\sim}{\nu} \in \mathbf{L}(\Omega); b(\underset{\sim}{\nu}, \mathbf{v}) \geq 0, \forall \, \mathbf{v} \in K\}. \tag{3.15}$$

Since $V_0 \subset K$ there holds

$$b(\underset{\sim}{\nu}, \mathbf{v}) = 0, \qquad \forall \, \mathbf{v} \in V_0,$$

which implies $\operatorname{div} \underset{\sim}{\nu} = 0$ and so $\hat{K}^*$ is a closed convex subset of $\mathbf{L}^2_{\text{div}}(\Omega)$. It follows by using Green's formula, (2.6), and (3.15), that

$$\hat{K}^* = \{\underset{\sim}{\nu} \in \mathbf{L}(\Omega); \operatorname{div} \underset{\sim}{\nu} = 0, \langle \pi_n(\underset{\sim}{\nu}), v_n \rangle_\Gamma + \langle \pi_t(\underset{\sim}{\nu}), \mathbf{v}_t \rangle_\Gamma \geq 0, \forall \, \mathbf{v} \in K\}.$$

Hence for smooth $\underset{\sim}{\nu} \in \hat{K}^*$

$$\int_{\Gamma_2 \cup \Gamma_3} \pi_n(\underset{\sim}{\nu}) v_n \, ds + \int_{\Gamma_2 \cup \Gamma_3} \pi_t(\underset{\sim}{\nu}) \cdot \mathbf{v}_t \, ds \geq 0, \qquad \forall \, \mathbf{v} \in K,$$

and choosing $\mathbf{v} = 0$ on Γ_3 yields $\pi_n(\underset{\sim}{\nu}) = 0$ and $\pi_t(\underset{\sim}{\nu}) = 0$ on Γ_2 and therefore

$$\underset{\sim}{\nu} \in \hat{K}^* \Rightarrow \underset{\sim}{\nu}\mathbf{n} = 0 \qquad \text{on } \Gamma_2.$$

So finally

$$\hat{K}^* = \{\underset{\sim}{\nu} \in \mathbf{L}(\Omega); \operatorname{div} \underset{\sim}{\nu} = 0 \text{ in } \Omega, \underset{\sim}{\nu}\mathbf{n} = 0 \text{ on } \Gamma_2 \text{ and } \pi_t(\underset{\sim}{\nu}) = 0, \langle \pi_n(\underset{\sim}{\nu}), v_n \rangle_{\Gamma_3} \geq 0, \forall \, \mathbf{v} \in K\}. \tag{3.16}$$

The dual problem (P_2) is equivalent to the variational inequality

$$\underset{\sim}{\sigma} \in K^* : a(\underset{\sim}{\sigma}, \underset{\sim}{\tau} - \underset{\sim}{\sigma}) \geq b(\underset{\sim}{\tau} - \underset{\sim}{\sigma}, \xi), \qquad \forall \, \underset{\sim}{\tau} \in K^*. \tag{3.17}$$

In turn, bearing in mind remark 3.1, this is equivalent to

$$\begin{cases} \text{find } \hat{\underset{\sim}{\sigma}} \in \hat{K}^* \quad \text{such that } \forall \, \underset{\sim}{\nu} \in \hat{K}^*, \\ a(\hat{\underset{\sim}{\sigma}}, \underset{\sim}{\nu} - \hat{\underset{\sim}{\sigma}}) \geq b(\underset{\sim}{\nu} - \hat{\underset{\sim}{\sigma}}, \xi) - a(\underset{\sim}{\varepsilon}(\psi), \underset{\sim}{\nu} - \hat{\underset{\sim}{\sigma}}). \end{cases} \tag{3.18}$$

Clearly if $\hat{\underset{\sim}{\sigma}}$ solves (3.18) then $\underset{\sim}{\sigma} = \hat{\underset{\sim}{\sigma}} + \underset{\sim}{\varepsilon}(\psi)$ solves (3.17) and vice versa.

Since $a(\cdot, \cdot)$ is a continuous symmetric bilinear form on $\mathbf{L}(\Omega) \times \mathbf{L}(\Omega)$, $\hat{K}^*$ is a closed convex subset of $\mathbf{L}(\Omega)$ and

$$l^*(\cdot) \equiv b(\cdot, \xi) - a(\underset{\sim}{\varepsilon}(\psi), \cdot) \tag{3.19}$$

is a continuous linear functional on $\mathbf{L}(\Omega)$, it follows from the standard theory of variational inequalities (see e.g. [4, Chapter 3]) that (3.18) has a unique solution provided a is coercive on $\hat{K}^*$ i.e. there exists a constant $c_0 > 0$ such that

$$a(\underset{\sim}{v}, \underset{\sim}{v}) \geq c_0 \|\underset{\sim}{v}\|_0^2, \qquad \forall\, \underset{\sim}{v} \in \hat{K}^*, \tag{3.20}$$

(since $\operatorname{div} \underset{\sim}{v} = 0$, $\forall\, \underset{\sim}{v} \in \hat{K}^*$), i.e. that $a(\cdot, \cdot)$ is coercive on $\hat{K}^*$.

The proof of (3.20) is essentially contained in the proof of lemma 3.2 in [1, pp. 148–150]. Let us give the details. Decompose an arbitrary element $\underset{\sim}{v} \in \hat{K}^*$ as

$$\underset{\sim}{v} = \underset{\sim}{v}_1 + \underset{\sim}{v}_D,$$

where $\underset{\sim}{v}_1 = (\underset{\sim}{v} : \underset{\sim}{\sigma}_1)\underset{\sim}{\sigma}_1$, $\underset{\sim}{\sigma}_1$ is as in (1.8), so that

$$\|\underset{\sim}{v}\|_0^2 = \|\underset{\sim}{v} : \underset{\sim}{\sigma}_1\|_0^2 + \|\underset{\sim}{v}_D\|_0^2.$$

Recalling that $A\underset{\sim}{\sigma}_1 = \lambda_1 \underset{\sim}{\sigma}_1$, $\lambda_1 \geq 0$ and $\underset{\sim}{\sigma}_1^{-1}$ exists (nonsingular constraint) we have

$$a(\underset{\sim}{v}, \underset{\sim}{v}) \geq \max\{\lambda_1 \|\underset{\sim}{v}\|_0^2, \lambda_2 \|\underset{\sim}{v}_D\|_0^2\}. \tag{3.21}$$

When $\lambda_1 > 0$ we have strict coercivity, but this is the standard case. Otherwise let $\mathbf{p} \in V_0$ be the solution of

$$\operatorname{div} \mathbf{p} = \underset{\sim}{v} : \underset{\sim}{\sigma}_1 \qquad \text{in } \Omega,$$

and so $\|\mathbf{p}\|_1 \leq c_1(\Omega)\|\underset{\sim}{v} : \underset{\sim}{\sigma}_1\|_0$. Setting $\mathbf{q} = \underset{\sim}{\sigma}_1^{-1}\mathbf{p}$ gives

$$\underset{\sim}{\sigma}_1 : \underset{\sim}{\varepsilon}(\mathbf{q}) = \underset{\sim}{v} : \underset{\sim}{\sigma}_1 \qquad \text{in } \Omega, \qquad \mathbf{q} = 0 \qquad \text{on } \Gamma,$$

and

$$\|\mathbf{q}\|_1 \leq c_1(\Omega)|\underset{\sim}{\sigma}_1^{-1}|_F \|\underset{\sim}{v} : \underset{\sim}{\sigma}_1\|_0.$$

Consequently

$$\|\underset{\sim}{v} : \underset{\sim}{\sigma}\|_0^2 = \int_\Omega (\underset{\sim}{\sigma}_1 : \underset{\sim}{\varepsilon}(\mathbf{q}))\underset{\sim}{v} : \underset{\sim}{\sigma}_1)\, dx = \int_\Omega \underset{\sim}{v}_1 : \underset{\sim}{\varepsilon}(\mathbf{q})\, dx$$

$$= \int_\Omega (\underset{\sim}{v} - \underset{\sim}{v}_D) : \underset{\sim}{\varepsilon}(\mathbf{q})\, dx$$

and using Green's formula, (2.6), noting that $\mathbf{q} \in V_0$ and $\underset{\sim}{v} \in K^*$ then

$$= -\int_\Omega \underset{\sim}{v}_D : \underset{\sim}{\varepsilon}(\mathbf{q})\, dx + \int_\Gamma \underset{\sim}{v}\mathbf{n} \cdot \mathbf{q}\, ds - \int_\Omega \mathbf{q} \cdot \operatorname{div} \underset{\sim}{v}\, dx$$

$$= -\int_\Omega \underset{\sim}{v}_D : \underset{\sim}{\varepsilon}(\mathbf{q})\, dx.$$

Hence

$$\|\underset{\sim}{v} : \underset{\sim}{\sigma}_1\|_0^2 \leq \|\underset{\sim}{v}_D\|_0 \|\mathbf{q}\|_1 \leq c_1 |\underset{\sim}{\sigma}_1^{-1}|_F \|\underset{\sim}{v} : \underset{\sim}{\sigma}_1\|_0 \|\underset{\sim}{v}_D\|_0,$$

and thus

$$\|\underset{\sim}{v}\|_0^2 \leq (c_1^2 |\underset{\sim}{\sigma}_1^{-2}|_F^2 + 1)\|\underset{\sim}{v}_D\|_0^2, \tag{3.22}$$

which yields for all $\underset{\sim}{v} \in \hat{K}^*$

$$a(\underset{\sim}{v}, \underset{\sim}{v}) \geq \max\left[\lambda_1, \frac{\lambda_2}{1 + c_1^2 |\underset{\sim}{\sigma}_1^{-1}|_F^2}\right] \|\underset{\sim}{v}\|_0^2. \tag{3.23}$$

Thus we may take $c_0 = \max[\lambda_1, \lambda_2(1 + c_2^2|\underset{\sim}{\sigma}_1^{-1}|_F^2)^{-1}]$ and so (3.20) holds true. Notice that (3.22) implies that

$$\hat{K}^* \cap \{\mu\underset{\sim}{\sigma}_1; \mu \in R\} = \{0\}$$

since for each $\underset{\sim}{v} \in \hat{K}^*$ if $\underset{\sim}{v} \neq 0$ then $\underset{\sim}{v}_D \neq 0$. Therefore the following theorem applies.

THEOREM 3.2. *Problem* (P_2) *has a unique solution. Moreover if* $\underset{\sim}{\sigma}^i$, $i = 1, 2$ *are two solutions corresponding to data* $\{\xi^i, \mathbf{t}_r^i, \mathbf{f}^i\}$, $i = 1, 2$, *then*

$$\|\underset{\sim}{\sigma}^1 - \underset{\sim}{\sigma}^2\|_0 \leq c[\|\xi^1 - \xi^2\|_1 + |\mathbf{t}_r^1 - \mathbf{t}_r^2|_{-1/2,\Gamma_2} + \|\mathbf{f}^1 - \mathbf{f}^2\|_{V_D'}]. \tag{3.24}$$

Proof. It remains to prove (3.24) since the existence of the unique solution follows from the discussion above.

Let $\psi_i (i = 1, 2$ here and below) be the corresponding solutions to (3.13) then $K_i^* = \underset{\sim}{\varepsilon}(\psi_i) + \hat{K}^*$ and so let $\underset{\sim}{\sigma}^i = \underset{\sim}{\varepsilon}(\psi_i) + \hat{\underset{\sim}{\sigma}}^i$ where $\hat{\underset{\sim}{\sigma}}^i \in \hat{K}^*$. Then we take $\underset{\sim}{v} = \hat{\underset{\sim}{\sigma}}^2$ in (3.18) for $\hat{\underset{\sim}{\sigma}}^1$ and take $\underset{\sim}{v} = \hat{\underset{\sim}{\sigma}}^1$ in (3.18) for $\hat{\underset{\sim}{\sigma}}^2$, subtracting gives

$$a(\hat{\underset{\sim}{\sigma}}^1 - \hat{\underset{\sim}{\sigma}}^2, \hat{\underset{\sim}{\sigma}}^1 - \hat{\underset{\sim}{\sigma}}^2) \leq b(\hat{\underset{\sim}{\sigma}}^1 - \hat{\underset{\sim}{\sigma}}^2, \xi^1 - \xi^2) + a(\underset{\sim}{\varepsilon}(\psi_1) - \underset{\sim}{\varepsilon}(\psi_2), \hat{\underset{\sim}{\sigma}}^1 - \hat{\underset{\sim}{\sigma}}^2).$$

By the coercivity property (3.20) the left hand side is not less than $c_0\|\hat{\underset{\sim}{\sigma}}^1 - \hat{\underset{\sim}{\sigma}}^2\|_0^2$ and since $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are bounded it follows that

$$\|\hat{\underset{\sim}{\sigma}}^1 - \underset{\sim}{\sigma}^2\|_0 \leq c(\|\xi^1 - \xi^2\|_1 + \|\underset{\sim}{\varepsilon}(\psi_1) - \underset{\sim}{\varepsilon}(\psi_2)\|_0).$$

Then (3.24) follows from the fact that $\underset{\sim}{\sigma}^i = \underset{\sim}{\varepsilon}(\psi_i) + \hat{\underset{\sim}{\sigma}}^i$ and the estimate

$$\|\underset{\sim}{\varepsilon}(\psi_1) - \underset{\sim}{\varepsilon}(\psi_2)\|_0 \leq c\{|\mathbf{t}_r^1 - \mathbf{t}_r^2|_{-1/2,\Gamma_2} + \|\mathbf{f}^1 - \mathbf{f}^2\|_{V_D'}\}, \tag{3.25}$$

for the solutions of (3.13).

Thus the dual problem (P_2) is well posed.

It remains to relate the dual problem to the mixed formulation (P_1). In order to do this we introduce a Lagrangian and consider the appropriate saddle-point problem.

Note that (3.18) may be rewritten in the equivalent form:

$$\begin{cases} \text{find } \underset{\sim}{\sigma} \in \mathbf{L}(\Omega), \quad \text{such that } \forall\, \underset{\sim}{\tau} \in \mathbf{L}(\Omega), \\ \qquad \Phi(\underset{\sim}{\sigma}) \leq \Phi(\underset{\sim}{\tau}), \end{cases} \tag{3.26}$$

where

$$\begin{aligned} \Phi(\underset{\sim}{\tau}) &= F(\underset{\sim}{\tau}) + \chi_1(\Lambda_1\underset{\sim}{\tau}) + \chi_2(\Lambda_2\underset{\sim}{\tau}) \\ &= \tilde{\Phi}(\underset{\sim}{\tau}, \Lambda_1\underset{\sim}{\tau}, \Lambda_2\underset{\sim}{\tau}), \end{aligned} \tag{3.27}$$

and

$$F(\underset{\sim}{\tau}) = \tfrac{1}{2}a(\underset{\sim}{\tau}, \underset{\sim}{\tau}) - l^*(\underset{\sim}{\tau}), \tag{3.28}$$

where $l^*(\cdot)$ is given in (3.19), and $\Lambda_1: \mathbf{L}(\Omega) \to V_D'$, is given by

$$\langle \Lambda_1\underset{\sim}{\tau}, \mathbf{v}\rangle = \int_\Omega \underset{\sim}{\tau} : \underset{\sim}{\varepsilon}(\mathbf{v})\, dx - \langle \pi_n(\underset{\sim}{\tau}), v_n\rangle_{\Gamma_3}, \qquad \forall\, \mathbf{v} \in V_D, \tag{3.29}$$

and $\Lambda_2: \mathbf{L}(\Omega) \to H^{-1/2}(\Gamma_3)$, is given by

$$\Lambda_2\underset{\sim}{\tau} = \pi_n(\underset{\sim}{\tau}) = \tau_n \qquad \text{on } \Gamma_3, \tag{3.30}$$

also

$$\chi_1(q_1) = \begin{cases} 0 & q_1 = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$\chi_2(q_2) = \begin{cases} 0 & q_2 \le 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Note that $\underset{\sim}{\tau} \in \mathbf{L}(\Omega)$ is an element of $\hat{K}^*$ iff $\Lambda_1 \underset{\sim}{\tau} = 0$ and $\Lambda_2 \underset{\sim}{\tau} \le 0$ that is

$$\begin{aligned} \operatorname{div} \underset{\sim}{\tau} &= 0 && \text{in } \mathbf{H}^1(\Omega) \\ \pi_t(\underset{\sim}{\tau}) &= 0 && \text{in } \mathbf{H}^{-1/2}(\Gamma_2 \cup \Gamma_3), \\ \pi_n(\underset{\sim}{\tau}) &= 0 && \text{in } H^{-1/2}(\Gamma_2), \\ \pi_n(\underset{\sim}{\tau}) &\le 0 && \text{in } H^{-1/2}(\Gamma_3). \end{aligned} \tag{3.31}$$

Let $Y' = V_D' \times H^{-1/2}(\Gamma_3)$, setting

$$\phi(\underset{\sim}{\tau}, p) = \hat{\Phi}(\underset{\sim}{\tau}, \Lambda_1 \underset{\sim}{\tau} - p_1, \Lambda_2 \underset{\sim}{\tau} - p_2)$$

for $\underset{\sim}{\tau} \in \mathbf{L}(\Omega)$ and $p = (p_1, p_2) \in Y'$, consider the functional

$$h(p) = \inf_{\underset{\sim}{\tau} \in L(\Omega)} \phi(\underset{\sim}{\tau}, p).$$

Obviously $h(0) = F(\underset{\sim}{\sigma})$, $\underset{\sim}{\sigma}$ the solution of (3.26) i.e. of (P_2), and it is finite. Taking $p \neq 0$ corresponds to perturbing the convex sets K^* and $\hat{K}^*$, (3.9) and (3.15) respectively.

A direct consequence of (3.24) is that $h(\cdot)$ is continuous at zero. Since h is a convex functional it follows that it is subdifferentiable at zero [4, p. 22]. Therefore $\phi \in \Gamma_0(\mathbf{L}(\Omega) \times Y')$ and hence problem (3.26) is stable in the sense of definition 2.1.

Recalling (2.12), if $\underset{\sim}{\tau} \in \mathbf{L}(\Omega)$, $\mathbf{v} \in V_D$ and $z \in H_{00}^{1/2}(\Gamma_3)$, the Lagrangian L for ϕ is

$$\begin{aligned} L(\underset{\sim}{\tau}, \mathbf{v}, z) &= - \sup_{(\mathbf{v}^*, z^*) \in Y'} \{\langle \mathbf{v}^*, \mathbf{v}\rangle + \langle z^*, z\rangle - \hat{\Phi}(\underset{\sim}{\tau}, \Lambda_1 \underset{\sim}{\tau} - \mathbf{v}^*, \Lambda_2 \underset{\sim}{\tau} - z)\} \\ &= F(\underset{\sim}{\tau}) - \sup_{\mathbf{v}^* \in V_D'} \{\langle \mathbf{v}^*, \mathbf{v}\rangle - \chi_1(\Lambda_1 \underset{\sim}{\tau} - \mathbf{v}^*)\} \\ &\quad - \sup_{z^* \in H^{-1/2}(\Gamma_3)} \{\langle z^*, z\rangle - \chi_2(\Lambda_2 \underset{\sim}{\tau} - z^*)\}. \end{aligned}$$

Let $\mathbf{v}^* = \Lambda_1 \underset{\sim}{\tau} - \mathbf{q}_1$ and $z^* = \Lambda_2 \underset{\sim}{\tau} - q_2$ then

$$\begin{aligned} &= F(\underset{\sim}{\tau}) - \langle \Lambda_1 \underset{\sim}{\tau}, \mathbf{v}\rangle - \langle \Lambda_2 \underset{\sim}{\tau}, z\rangle_{\Gamma_3} \\ &\quad - \sup_{\mathbf{q}_1 \in V_D'} \{-\langle \mathbf{q}_1, \mathbf{v}\rangle - \chi_1(\mathbf{q}_1)\} - \sup_{q_2 \in H^{-1/2}(\Gamma_3)} \{-\langle q_2, z\rangle_{\Gamma_3} - \chi_2(q_2)\}. \end{aligned}$$

But

$$\sup_{\mathbf{q}_1 \in V_D'} \{-\langle \mathbf{q}_1, \mathbf{v}\rangle - \chi_1(\mathbf{q}_1)\} = 0,$$

and

$$\begin{aligned} - \sup_{q_2 \in H^{-1/2}(\Gamma_3)} \{-\langle q_2, z\rangle - \chi_2(q_2)\} &= \inf_{q_2}\{\langle q_1, z\rangle_{\Gamma_3} + \chi_2(q_2)\} \\ &= \begin{cases} -\infty & \text{if } z > 0 \\ 0 & \text{if } z < 0 \end{cases} = -\chi_2(z), \end{aligned}$$

therefore the Lagrangian is

$$
\begin{aligned}
L(\underset{\sim}{\tau}, \mathbf{v}, z) &= F(\underset{\sim}{\tau}) - \langle \Lambda_1 \underset{\sim}{\tau}, \mathbf{v} \rangle - \langle \Lambda_2 \underset{\sim}{\tau}, z \rangle - \chi_2(z) \\
&= \tfrac{1}{2} a(\underset{\sim}{\tau}, \underset{\sim}{\tau}) + l^*(\underset{\sim}{\tau}) - (\underset{\sim}{\tau}, \underset{\sim}{\varepsilon}(\mathbf{v}))_{L^2} + \langle \pi_n(\underset{\sim}{\tau}), v_n - z \rangle_{\Gamma_3} - \chi_2(z).
\end{aligned} \tag{3.32}
$$

Applying theorem 2.2 we have that $(\underset{\sim}{\sigma}, \mathbf{u}, y)$ is a saddle point of L, that is it solves

$$
\inf_{\underset{\sim}{\tau} \in \mathbf{L}(\Omega)} \sup_{(\mathbf{v}, z) \in Y} L(\underset{\sim}{\tau}, \mathbf{v}, z), \tag{3.33}
$$

if and only if $\underset{\sim}{\sigma}$ solves (3.18). Here $Y = V_D \times H_{00}^{1/2}(\Gamma_3)$. By considering

$$
\sup_{(\mathbf{v}, z) \in Y} L(\underset{\sim}{\tau}, \mathbf{v}, z),
$$

for any $\underset{\sim}{\tau} \in \mathbf{L}(\Omega)$ it is clear that (3.33) has a solution if $\chi_2(z) = 0$ hence $z \leq 0$ and since $v_n \leq 0$ then

$$
\langle \pi_n(\underset{\sim}{\tau}), v_n - z \rangle_{\Gamma_3} = 0, \qquad \forall\, \underset{\sim}{\tau} \in \mathbf{L}(\Omega),
$$

hence $z = v_n$ and (3.33) becomes

$$
\inf_{\underset{\sim}{\tau} \in \mathbf{L}(\Omega)} \sup_{(\mathbf{v}, v_n) \in Y} L(\underset{\sim}{\tau}, \mathbf{v}, v_n). \tag{3.34}
$$

We use below the notation

$$
L(\underset{\sim}{\tau}, \mathbf{v}) = L(\underset{\sim}{\tau}, \mathbf{v}, v_n), \tag{3.35}
$$

and $\mathbf{W} = \{\mathbf{v} \in V_D;\, \mathbf{v} \cdot \mathbf{n} \in H_{00}^{1/2}(\Gamma_3)\}$, therefore

$$
L(\underset{\sim}{\tau}, \mathbf{v}) = \tfrac{1}{2} a(\underset{\sim}{\tau}, \underset{\sim}{\tau}) - l^*(\underset{\sim}{\tau}) - b(\underset{\sim}{\tau}, \mathbf{v}) - \chi_2(v_n).
$$

The *saddle point formulation* is

$$
(\mathrm{P}_3) \qquad \begin{cases} \text{find } (\hat{\underset{\sim}{\sigma}}, \mathbf{w}) \in \mathbf{L}(\Omega) \times \mathbf{W} \quad \text{such that } \forall\, (\underset{\sim}{\tau}, \mathbf{v}) \in \mathbf{L}(\Omega) \times \mathbf{W} \\ L(\hat{\underset{\sim}{\sigma}}, \mathbf{v}) \leq L(\hat{\underset{\sim}{\sigma}}, \mathbf{w}) \leq L(\underset{\sim}{\tau}, \mathbf{w}). \end{cases} \tag{3.36}
$$

The considerations above together with theorem 2.2 yield the following theorem.

THEOREM 3.3. There exists a unique solution $(\hat{\underset{\sim}{\sigma}}, \mathbf{w})$ to (P_3). Furthermore $\hat{\underset{\sim}{\sigma}}$ solves (3.18), $\underset{\sim}{\sigma} = \hat{\underset{\sim}{\sigma}} + \underset{\sim}{\varepsilon}(\psi)$ is the unique solution of (P_2) and $(\underset{\sim}{\sigma}, \mathbf{w})$ is the unique solution of (P_1).

Proof. (P_3) is equivalent to (3.34) and theorem 2.2 implies the existence of $(\hat{\underset{\sim}{\sigma}}, \mathbf{w})$ and the uniqueness of $\hat{\underset{\sim}{\sigma}}$ where $\underset{\sim}{\sigma} = \hat{\underset{\sim}{\sigma}} + \underset{\sim}{\varepsilon}(\psi)$ is the unique solution of (P_2). The uniqueness in the second variable follows from (3.36) since

$$
a(\hat{\underset{\sim}{\sigma}}, \underset{\sim}{\tau}) + l^*(\underset{\sim}{\tau}) - b(\underset{\sim}{\tau}, \mathbf{w}) = 0, \qquad \forall\, \underset{\sim}{\tau} \in \mathbf{L}(\Omega), \tag{3.37}
$$

and therefore if $\mathbf{w}_1$ and $\mathbf{w}_2$ are two solutions then $b(\underset{\sim}{\tau}, \mathbf{w}_1 - \mathbf{w}_2) = 0$, $\forall\, \underset{\sim}{\tau} \in \mathbf{L}(\Omega)$ so that $\underset{\sim}{\varepsilon}(\mathbf{w}_1 - \mathbf{w}_2) = 0 \Rightarrow \mathbf{w}_1 = \mathbf{w}_2$. It remains to be shown the relationship between (P_3) and the mixed formulation (P_1). We rewrite (3.37) as

$$
a(\underset{\sim}{\sigma} - \underset{\sim}{\varepsilon}(\psi), \underset{\sim}{\tau}) - b(\underset{\sim}{\tau}, \xi) + a(\underset{\sim}{\varepsilon}(\psi), \underset{\sim}{\tau}) - b(\underset{\sim}{\tau}, \mathbf{w}) = 0, \qquad \forall\, \underset{\sim}{\tau} \in \mathbf{L}(\Omega),
$$

or

$$
a(\underset{\sim}{\sigma}, \underset{\sim}{\tau}) - b(\underset{\sim}{\tau}, \xi) = b(\underset{\sim}{\tau}, \mathbf{w}), \qquad \forall\, \tilde{\underset{\sim}{\tau}} \in \mathbf{L}(\Omega),
$$

and we obtain (3.4). It follows from $L(\hat{\sigma}, \mathbf{v}) \leq L(\hat{\sigma}, \mathbf{w})$, $\forall\, \mathbf{v} \in \mathbf{W}$ that

$$b(\hat{\sigma}, \mathbf{v} - \mathbf{w}) + \chi_2(v_n) - \chi_2(w_n) \geq 0, \qquad \forall\, \mathbf{v} \in \mathbf{W},$$

hence $w_n \leq 0$ on Γ_3 and so $\mathbf{w} \in K$ and then $b(\hat{\sigma}, \mathbf{v} - \mathbf{w}) \geq 0$, $\forall\, \mathbf{v} \in K$. Equation (3.8) implies that

$$b(\sigma - \hat{\sigma}, \mathbf{v}) = l(\mathbf{v}), \qquad \forall\, \mathbf{v} \in V_D,$$

and for $\mathbf{v} \in K$, we obtain, using Green's formula (2.6),

$$b(\sigma, \mathbf{v} - \mathbf{w}) - l(\mathbf{v} - \mathbf{w}) = b(\hat{\sigma}, \mathbf{v} - \mathbf{w}) \geq 0$$

and so (3.5) holds. Similarly one can show that any solution to (P_1) solves (P_3).

Thus the unique solvability of (3.3) is proved. In order to prove theorem 1.1 it remains to prove estimate (1.9).

LEMMA 3.4. The estimate (1.9) holds for the solution $(\sigma, \mathbf{w})$.

Proof. Let $(\sigma^i, \mathbf{w}^i)$ be two solutions corresponding to the data $(\xi^i, \mathbf{t}_r^i, \mathbf{f}^i)$ for $i = 1, 2$. Then the part of the estimate related to σ^i is given in (3.24). So it remains to estimate $\mathbf{w}^1 - \mathbf{w}^2$, but this estimate follows easily from

$$a(\sigma^i, \tau) - b(\tau, \xi^i) = b(\tau, \mathbf{w}^i), \; i = 1, 2,$$

combined with (3.24), the boundedness of $a(\cdot, \cdot)$ and of $b(\cdot, \cdot)$, and the fact that $\mathbf{w}^i \in \mathbf{W}$ there follows that $\mathbf{w}^1 - \mathbf{w}^2 = 0$ on Γ_1.

Therefore the assumption that A admits only nonsingular constraints, i.e. $\chi(A) > 0$ and σ_1^{-1} exists, leads to theorem 1.1.

4. THE CASE $\Gamma_2 = \emptyset$

We consider the case where $\Gamma_2 = \emptyset$, that is, we have only displacement and Signorini's conditions on the boundary. It is complicated, due to the necessity of a compatibility condition that depends on the compliance tensor, more precisely they apply only to the internal data σ_1, the external data $\mathbf{f}$, $\mathbf{g}$ and s is unaffected. The case of pure Dirichlet condition, i.e. $\Gamma_3 = \emptyset$ and $\Gamma = \Gamma_1$ was considered in [1] where the necessary compatibility condition is

$$\int_\Gamma \mathbf{g} \cdot \sigma_1 \mathbf{n} \, ds = 0$$

and the solution σ is unique if we impose the condition $\int_\Omega \sigma : \sigma_1 \, dx = 0$. Since in our case $\Gamma_3 \neq \emptyset$ our compatibility conditions are different. Indeed we require that either $\pi_n(\sigma_1^{-1}) \leq 0$ in $H^{-1/2}(\Gamma_3)$ or $\pi_n(\sigma_1^{-1}) \geq 0$ in $H^{-1/2}(\Gamma_3)$ and these are restrictions both on the constraint tensor σ_1 and on the contact surface Γ_3 but not on the data. For incompressible materials $\sigma_1 = I$ (the identity tensor) and hence $\pi_n(I) = 1 > 0$ on Γ.

We proceed as follows. First we prove the existence of the unique solution of the dual problem (P_2) and its stability and then construct a Lagrangian, the saddle point of which is the solution to the original problem (P_1).

Let

$$K^*_- = \left\{ \underset{\sim}{\tau} \in K^*; \int_\Omega \underset{\sim}{\tau} : \underset{\sim}{\sigma}_1 \, dx \le 0 \right\}, \tag{4.1}$$

and

$$K^*_+ = \left\{ \underset{\sim}{\tau} \in K^*; \int_\Omega \underset{\sim}{\tau} : \underset{\sim}{\sigma}_1 \, dx \ge 0 \right\}, \tag{4.2}$$

where K^* is given in (3.12). Then we have the following theorem.

THEOREM 4.1. Let the assumptions of theorem 1.1 hold with $\Gamma_2 = \emptyset$ and assume in addition that

$$\pi_n(\underset{\sim}{\sigma}_1^{-1}) \le 0 \qquad (\pi_n(\underset{\sim}{\sigma}_1^{-1}) \ge 0), \text{ in } H^{-1/2}(\Gamma_3). \tag{4.3}$$

Then there exists a unique solution $\underset{\sim}{\sigma} \in K^*_-(\underset{\sim}{\sigma} \in K^*_+$ resp.) for the problem

$$\begin{aligned} &(\mathrm{P}_-) \qquad \inf\{F^*(\underset{\sim}{\tau}); \underset{\sim}{\tau} \in K^*_-\}, \\ &(\mathrm{P}_+) \qquad (\text{respectively } \inf\{F^*(\underset{\sim}{\tau}); \underset{\sim}{\tau} \in K^*_+\}). \end{aligned} \tag{4.4}$$

Proof. First we translate K^* to obtain homogeneous boundary conditions. Since $\underset{\sim}{\sigma}_1$ is a constant tensor there exists $\mathbf{u}_1 \in \mathbf{H}^1(\Omega)$ such that

$$\mathbf{u}_1 = \underset{\sim}{\sigma}_1 \mathbf{x}. \tag{4.5}$$

Next define $\mathbf{G}$ by

$$\langle \mathbf{G}, \mathbf{r} \rangle_{\Gamma_1} = -\int_\Omega \mathbf{f} \cdot \mathbf{r} \, dx, \qquad \forall \, \mathbf{r} \in \operatorname{span}\{\mathbf{u}_1\}. \tag{4.6}$$

This vector can be extended to $\mathbf{G} \in \mathbf{H}^{-1/2}(\Gamma_1)$. Let $\mathbf{z} \in \mathbf{H}^1_\perp(\Omega)$ (the orthogonal complement of $\operatorname{span}\{\mathbf{u}_1\}$ in $\mathbf{H}^1(\Omega)$) be such that

$$b(\underset{\sim}{\varepsilon}(\mathbf{z}), \mathbf{v}) = l(\mathbf{v}) + \langle \mathbf{G}, \mathbf{v} \rangle_{\Gamma_1}, \qquad \forall \, \mathbf{v} \in \mathbf{H}^1_\perp(\Omega), \tag{4.7}$$

where $b(\cdot, \cdot)$ is given in (3.7) and $l(\cdot)$ in (3.8) with $\Gamma_2 = \emptyset$. So $\mathbf{z}$ is a weak solution of $-\operatorname{div} \underset{\sim}{\varepsilon}(\mathbf{z}) = \mathbf{f}$ in Ω, $\underset{\sim}{\varepsilon}(\mathbf{z})\mathbf{n} = \mathbf{G}$ on Γ_1 and $\underset{\sim}{\varepsilon}(\mathbf{z})\mathbf{n} = 0$ on Γ_3. Then we set

$$K^* = \underset{\sim}{\varepsilon}(\mathbf{z}) + \tilde{K}^*, \qquad \underset{\sim}{\tau} = \underset{\sim}{\varepsilon}(\mathbf{z}) + \tilde{\underset{\sim}{\tau}} \tag{4.8}$$

where $\underset{\sim}{\tau} \in K^*$ and $\tilde{\underset{\sim}{\tau}} \in \tilde{K}^*$ and

$$\tilde{K}^* = \{\underset{\sim}{\tau} \in \mathbf{L}(\Omega); \operatorname{div} \underset{\sim}{\tau} = 0, \pi_t(\underset{\sim}{\tau}) = 0 \text{ in } \mathbf{H}^{-1/2}(\Gamma_3) \text{ and } \pi_n(\underset{\sim}{\tau}) \le 0 \text{ in } H^{-1/2}(\Gamma_3)\}. \tag{4.9}$$

Next we need

$$\tilde{K}^*_- = \left\{ \underset{\sim}{\tau} \in \tilde{K}^*; \int_\Omega \underset{\sim}{\tau} : \underset{\sim}{\sigma}_1 \, dx \le 0 \right\}. \tag{4.10}$$

If $\underset{\sim}{\tau} \in K^*_-$ then $\underset{\sim}{\tau} = \hat{\underset{\sim}{\tau}} + \underset{\sim}{\varepsilon}(\mathbf{z})$ for some $\hat{\underset{\sim}{\tau}}$ and

$$\begin{aligned} 0 \ge \int_\Omega \underset{\sim}{\tau} : \underset{\sim}{\sigma}_1 \, dx &= \int_\Omega \hat{\underset{\sim}{\tau}} : \underset{\sim}{\sigma}_1 \, dx + \int_\Omega \underset{\sim}{\varepsilon}(\mathbf{z}) : \underset{\sim}{\sigma}_1 \, dx \\ &= \int_\Omega \hat{\underset{\sim}{\tau}} : \sigma_1 \, dx + \int_\Omega \mathbf{f} \cdot \mathbf{u}_1 \, dx + \langle \mathbf{G}, \mathbf{u}_1 \rangle_{\Gamma_1} = \int_\Omega \hat{\underset{\sim}{\tau}} : \sigma_1 \, dx, \end{aligned}$$

where we used (4.7) and then (4.6), and therefore $\hat{\underset{\sim}{\tau}} \in \hat{K}^*_\sim$ i.e.

$$K^*_{\sim} = \underset{\sim}{\varepsilon}(\mathbf{z}) + \tilde{K}^*_{\sim} \tag{4.11}$$

In order to prove the theorem it is enough to show coercivity as lower semicontinuity is obvious. But then it is enough to show that

$$a(\underset{\sim}{\tau}, \underset{\sim}{\tau}) \geq c_0 \|\underset{\sim}{\tau}\|_0^2, \qquad \forall\, \underset{\sim}{\tau} \in \tilde{K}^*_{\sim}. \tag{4.12}$$

The proof of (4.12) is very similar to the proof of (3.20) so we only outline it. Let $\underset{\sim}{\tau} = \underset{\sim}{\tau}_1 + \underset{\sim}{\tau}_D$ where $\underset{\sim}{\tau}_1 = (\underset{\sim}{\tau} : \underset{\sim}{\sigma}_1)\sigma_1$ and then

$$a(\underset{\sim}{\tau}, \underset{\sim}{\tau}) \geq \max\{\lambda_1 \|\underset{\sim}{\tau}\|_0^2, \lambda_2 \|\underset{\sim}{\tau}_D\|_0^2\}.$$

So we need the estimate

$$\|\underset{\sim}{\tau}_D\|_0^2 \geq c \|\underset{\sim}{\tau}\|_0^2 \tag{4.13}$$

and obviously (4.13) implies (4.12).

Let $\mathbf{p} \in V_D$ be a solution of

$$\operatorname{div} \mathbf{p} = \underset{\sim}{\tau} : \underset{\sim}{\sigma}_1 \quad \text{in } \Omega, \qquad \pi_t(\mathbf{p}) = 0 \qquad \text{and} \qquad \pi_n(\mathbf{p}) = m \qquad \text{on } \Gamma_3, \tag{4.14}$$

where $m \in C^1(\bar{\Gamma}_3)$, $m = 0$ on $\bar{\Gamma}_3 \cap \bar{\Gamma}_1$, $m \leq 0$ in Γ_3 and $\int_{\Gamma_3} m \, ds = \int_{\Gamma_3} \underset{\sim}{\tau} : \underset{\sim}{\sigma}_1 \, ds$. The construction of m is given below. It follows from the standard theory that $\mathbf{p}$ exists and is unique (see e.g. Duvaut and Lions [3]) and satisfies the estimate

$$\|\mathbf{p}\|_1 \leq c \|\tau : \sigma_1\|_0, \tag{4.15}$$

where the constant c depend only on Ω. Let $\mathbf{q} = \underset{\sim}{\sigma}_1^{-1}\mathbf{p}$ then

$$\|\mathbf{q}\|_1 \leq c |\underset{\sim}{\sigma}_1^{-1}|_F \|\underset{\sim}{\tau} : \underset{\sim}{\sigma}_1\|_0 \tag{4.16}$$

and $\underset{\sim}{\sigma}_1 : \underset{\sim}{\varepsilon}(\mathbf{q}) = \operatorname{div} \mathbf{p} = \underset{\sim}{\tau} : \underset{\sim}{\sigma}_1$ and proceeding as above (the argument that follows (3.21)) but noticing that now $\mathbf{p} \in V_D$ only, we obtain

$$\|\underset{\sim}{\tau} : \underset{\sim}{\sigma}_1\|_0^2 = -\int_\Omega \underset{\sim}{\tau}_D : \underset{\sim}{\varepsilon}(\mathbf{q}) \, dx + \int_\Omega \underset{\sim}{\tau} : \underset{\sim}{\varepsilon}(\mathbf{q}) \, dx.$$

First we show that the last term on the right hand side is nonpositive, indeed

$$\int_\Omega \underset{\sim}{\varepsilon}(\mathbf{q}) : \underset{\sim}{\tau} \, dx = -\int_\Omega \mathbf{q} \cdot \operatorname{div} \underset{\sim}{\tau} \, dx + \int_\Gamma \underset{\sim}{\tau}\mathbf{n} \cdot \mathbf{q} \, ds$$

using Green's formula (2.6) and since $\mathbf{q} = 0$ on Γ_1 then

$$= \int_{\Gamma_3} \underset{\sim}{\tau}\mathbf{n} \cdot \mathbf{q} \, ds = \int_{\Gamma_3} \underset{\sim}{\tau}\mathbf{n} \cdot (\underset{\sim}{\sigma}_1^{-1}\mathbf{p}) \, ds$$

but $\pi_t(\mathbf{p}) = 0$ on Γ_3 by (4.14) hence

$$= \int_{\Gamma_3} \pi_n(\underset{\sim}{\tau})\pi_n(\underset{\sim}{\sigma}_1^{-1})\mathbf{p} \cdot \mathbf{n} \, ds$$

where we used the equality

$$\mathbf{q} \cdot \mathbf{n} = (\underset{\sim}{\sigma}_1^{-1}\mathbf{p}) \cdot \mathbf{n} = \underset{\sim}{\sigma}_1^{-1}\mathbf{n} \cdot \mathbf{n}(\mathbf{p} \cdot \mathbf{n}) + \text{tangential part} = \pi_n(\underset{\sim}{\sigma}_1^{-1})(\mathbf{p} \cdot \mathbf{n})$$

which holds on Γ_3 by virtue of (4.14), and so

$$= \int_{\Gamma_3} \pi_n(\underset{\sim}{\tau})\pi_n(\underset{\sim}{\sigma}_1^{-1}) m \, ds \le 0 \tag{4.17}$$

since $\pi_n(\underset{\sim}{\sigma}_1^{-1}) \le 0$ by assumption, $m \le 0$ by construction and $\pi_n(\underset{\sim}{\tau}) \le 0$ in $H^{-1/2}(\Gamma_3)$ since $\underset{\sim}{\tau} \in \tilde{K}_-^*$. Therefore

$$\|\underset{\sim}{\tau} : \underset{\sim}{\sigma}_1\|_0^2 \le -\int_\Omega \underset{\sim}{\varepsilon}(\mathbf{q}) : \underset{\sim}{\tau}_D \, dx \le \|\mathbf{q}\|_1 \|\underset{\sim}{\tau}_D\|_0 \le c|\underset{\sim}{\sigma}_1^{-1}|_F \|\underset{\sim}{\tau} : \underset{\sim}{\sigma}_1\|_0 \|\underset{\sim}{\tau}_D\|_0$$

and consequently

$$\|\underset{\sim}{\tau}_D\|_0^2 \ge \frac{c}{(1 + |\underset{\sim}{\sigma}_1^{-1}|^2)} \|\underset{\sim}{\tau}\|_0^2$$

which proves (4.13) and thus the coercivity and therefore the existence of a unique solution to (P_-). The case (P_+) is similar, the only changes are $\pi_n(\underset{\sim}{\sigma}_1^{-1}) \ge 0$ on Γ_3, $\pi_n(\underset{\sim}{\tau}) \ge 0$ in $H^{-1/2}(\Gamma_3)$ and in (4.14) we choose $m \le 0$ such that $\int_{\Gamma_3} m \, ds = -\int_\Omega \underset{\sim}{\tau} : \underset{\sim}{\sigma}_1 \, dx \le 0$ since $\underset{\sim}{\tau} \in \tilde{K}_+^*$. Thus it remains to construct m.

Let ρ be a continuous nonnegative function on $\bar{\Omega}$, C^1 in a neighbourhood of Γ in $\bar{\Omega}$, $\rho > 0$ in Ω, $\rho = 0$ on $\bar{\Gamma}_1 \cap \bar{\Gamma}_3 \equiv \Gamma_*$ and of the order of $d(x, \Gamma_*)$, where $d(x, D)$ is the distance of x from the set D, i.e.

$$\lim_{x \to x_0} \frac{\rho(x)}{d(x, \Gamma_*)} = d \ne 0 \qquad \text{if } x_0 \in \Gamma_*$$

(see Lions and Magenes [13, p. 57]). Then we define

$$m(x) = \rho(x)\left(\int_\Omega \underset{\sim}{\tau} : \underset{\sim}{\sigma}_1 \, dx\right) \Big/ \int_{\Gamma_3} \rho(s) \, ds$$

and then $m \in H_0^s(\Gamma_3)$ for $s > \frac{1}{2}$ and therefore can be extended as zero to Γ and clearly $|m|_{1/2,\Gamma} \le c|\int_\Omega \underset{\sim}{\tau} : \underset{\sim}{\sigma}_1 \, dx|$.

Notice that in the case of pure Dirichlet condition on Γ, i.e. $\Gamma_3 = \emptyset$, one has to impose the condition $\int_\Omega \underset{\sim}{\tau} : \underset{\sim}{\sigma}_1 \, dx = 0$ for any admissible $\underset{\sim}{\tau}$, as follows from (4.17).

In order to construct a Lagrangian for the problem we rewrite it as follows. Let

$$\Phi(\underset{\sim}{\tau}) = F^*(\underset{\sim}{\tau}) + \chi_1(\Lambda_1 \underset{\sim}{\tau} - \mathbf{G}) + \chi_2(\Lambda_2 \underset{\sim}{\tau}) + \chi_2(\Lambda_3 \underset{\sim}{\tau}) \equiv \hat{\Phi}(\underset{\sim}{\tau}, \Lambda_1 \underset{\sim}{\tau}, \Lambda_2 \underset{\sim}{\tau}, \Lambda_3 \underset{\sim}{\tau}), \tag{4.18}$$

where F^* is given in (3.11), Λ_1 is defined in (3.29), Λ_2 in (3.30) and $\Lambda_3 : \mathbf{L}(\Omega) \to R$ is defined as

$$\Lambda_3 \underset{\sim}{\tau} = \int_\Omega \underset{\sim}{\tau} : \underset{\sim}{\sigma}_1 \, dx, \qquad \underset{\sim}{\tau} \in \mathbf{L}(\Omega). \tag{4.19}$$

Note that $\tau \in K_-^*$ iff $\langle \Lambda_1 \underset{\sim}{\tau} - \mathbf{G}, \mathbf{v} \rangle = 0 \ \forall \, \mathbf{v} \in V_D$, $\Lambda_2 \underset{\sim}{\tau} \le 0$ and $\Lambda_3 \underset{\sim}{\tau} \le 0$. Then the problem is:

$$(P_4) \qquad \begin{cases} \text{find } \underset{\sim}{\sigma} \in \mathbf{L}(\Omega) \text{ such that } \forall \, \underset{\sim}{\tau} \in \mathbf{L}(\Omega) \\ \Phi(\underset{\sim}{\sigma}) \le \Phi(\underset{\sim}{\tau}). \end{cases} \tag{4.20}$$

Let

$$Y = \Lambda_D \times H_{00}^{1/2}(\Gamma_3) \times R, \qquad (Y' = \Lambda_D' \times H^{-1/2}(\Gamma_3) \times R), \tag{4.21}$$

(notice the slight change of notation from Section 3) and set

$$\phi(\underset{\sim}{\tau}, \mathbf{p}^*) = \hat{\Phi}(\underset{\sim}{\tau}, \Lambda_1 \underset{\sim}{\tau} - p_1^*, \Lambda_2 \underset{\sim}{\tau} - p_2^*, \Lambda_3 \underset{\sim}{\tau} - p_3^*) \tag{4.22}$$

for $\underset{\sim}{\tau} \in \mathbf{L}(\Omega)$ and $\mathbf{p}^* = (p_1^*, p_2^*, p_3^*) \in Y'$. As in Section 3, it is easy to show that problem (4.20) is stable (in the sense of definition (2.1)). So we turn to calculate the Lagrangian,

$$\tilde{L}(\underset{\sim}{\tau}, \mathbf{v}, z, c) = -\sup_{p^* \in Y'} \{\langle \mathbf{v}^*, \mathbf{v}\rangle + \langle z^*, z\rangle + c^*c - \phi(\underset{\sim}{\tau}, p^*)\}, \tag{4.23}$$

where $p^* = (\mathbf{v}^*, z^*, c^*)$. We need only calculate the new terms with c^* as the rest was calculated above. So

$$\begin{aligned} -\sup_{c^* \in R} \{c^*c - \chi_2(\Lambda_3\underset{\sim}{\tau} - c^*)\} &= -c\Lambda_3\underset{\sim}{\tau} + \inf_{q_3 \in R} \{cq_3 + \lambda_2(q_3)\} \\ &= -c\Lambda_3\underset{\sim}{\tau} - \lambda_2(c). \end{aligned}$$

Therefore the Lagrangian for (4.20) is

$$\tilde{L}(\underset{\sim}{\tau}, \mathbf{v}, z, c) = F^*(\underset{\sim}{\tau}) - \langle \Lambda_1\underset{\sim}{\tau} - \mathbf{G}, \mathbf{v}\rangle - \langle \Lambda_2\underset{\sim}{\tau}, z\rangle_{\Gamma_3} - \chi_2(z) - c\Lambda_3\underset{\sim}{\tau} - \chi_2(c). \tag{4.24}$$

Notice that

$$\sup_{c \in R} \{-c\Lambda_3\underset{\sim}{\tau} - \chi_2(c)\} = \chi_2(\Lambda_3\underset{\sim}{\tau}),$$

and therefore

$$\inf_{\tau}\ \sup_{(x,z,c)}\ \tilde{L}(\underset{\sim}{\tau}, \mathbf{v}, z, c) = \inf_{\tau}\sup_{\mathbf{v}, v_n} \{L(\underset{\sim}{\tau}, \mathbf{v}) + \chi_2(\Lambda_3\underset{\sim}{\tau})\},$$

where L is the Lagrangian (3.32). The saddle point problem is

$$(\mathrm{P}_1) \qquad \inf_{\tau}\sup_{\mathbf{v}} \{L(\underset{\sim}{\tau}, \mathbf{v}) + \chi_2(\Lambda_3\underset{\sim}{\tau})\}, \tag{4.25}$$

where $\tau \in \mathbf{L}(\Omega)$ and $\mathbf{v} \in \mathbf{W} = \{\mathbf{v} \in V_D;\ v_n \in H_{00}^{1/2}(\Gamma_3)\}$, and then the following theorem applies.

THEOREM 4.2. There exists a unique saddle point $(\sigma, \mathbf{w}) \in \mathbf{L}(\Omega) \times \mathbf{W}$ of

$$L(\underset{\sim}{\tau}, \mathbf{v}) + \chi_2(\Lambda_3\underset{\sim}{\tau}).$$

Proof. It follows from theorems 2.2 and 4.1 that $(\underset{\sim}{\sigma}, \mathbf{w}) \in \mathbf{L}(\Omega) \times \mathbf{W}$ exists and $\underset{\sim}{\sigma}$ is unique. So we have to show the uniqueness of $\mathbf{w}$. We may rewrite (4.25) as

$$L(\underset{\sim}{\sigma}, \mathbf{v}) + \chi_2(\Lambda_3\underset{\sim}{\sigma}) \le L(\underset{\sim}{\sigma}, \mathbf{w}) + \chi_2(\Lambda_3\underset{\sim}{\sigma}) \le L(\underset{\sim}{\tau}, \mathbf{w}) + \chi_2(\Lambda_3\underset{\sim}{\tau}), \tag{4.26}$$

$\forall\ (\underset{\sim}{\tau}, \mathbf{v}) \in \mathbf{L}(\Omega) \times \mathbf{W}$. Then

$$a(\underset{\sim}{\sigma}, \underset{\sim}{\tau} - \underset{\sim}{\sigma}) - b(\mathbf{w} + \xi, \underset{\sim}{\tau} - \underset{\sim}{\sigma}) \ge 0. \tag{4.27}$$

Let $K_* = \{\underset{\sim}{\tau} \in \mathbf{L}(\Omega);\ \Lambda_3\underset{\sim}{\tau} \le 0\}$, then if $\underset{\sim}{\tau} \in K_*$ it can be decomposed as

$$\underset{\sim}{\tau} = \theta\underset{\sim}{\sigma}_1 + \hat{\underset{\sim}{\tau}}, \qquad \theta = \Lambda_3\underset{\sim}{\tau} \le 0 \qquad \text{and} \qquad \Lambda_3\hat{\underset{\sim}{\tau}} = 0,$$

and similarly for $\underset{\sim}{\sigma}$ we write $\underset{\sim}{\sigma} = \theta_0\underset{\sim}{\sigma}_1 + \hat{\underset{\sim}{\sigma}}$ where $\theta_0 = \Lambda_3\underset{\sim}{\sigma} \le 0$ and $\Lambda_3\hat{\underset{\sim}{\sigma}} = 0$. Then (4.27) can be written as

$$a(\underset{\sim}{\sigma}, \underset{\sim}{\tau}) - b(\mathbf{w} + \xi, \hat{\underset{\sim}{\tau}}) = 0,$$

$$a(\underset{\sim}{\sigma}, \underset{\sim}{\sigma}_1) - b(\mathbf{w} + \xi, \underset{\sim}{\sigma}_1) \le 0,$$

and

$$\theta_0[a(\underset{\sim}{\sigma}, \underset{\sim}{\sigma}_1) - b(\mathbf{w} + \xi, \underset{\sim}{\sigma}_1)] = 0,$$

for all $\underset{\sim}{\tau} \in \mathbf{L}(\Omega)$. There arise two cases $\theta_0 < 0$ and $\theta_0 = 0$. If $\theta_0 < 0$ then the bracket in the last equality vanishes and therefore

$$a(\underset{\sim}{\sigma}, \underset{\sim}{\tau}) - b(\mathbf{w} + \xi, \underset{\sim}{\tau}) = 0, \qquad \forall\, \underset{\sim}{\tau} \in \mathbf{L}(\Omega),$$

and the uniqueness of $\mathbf{w}$ follows. If $\theta_0 = 0$ we conclude that if $\mathbf{w}_1$ and $\mathbf{w}_2$ are two solutions then $\underset{\sim}{\varepsilon}(\mathbf{w}) = c\underset{\sim}{\sigma}_1$ for $\mathbf{w} = \mathbf{w}_1 - \mathbf{w}_2$ but $\mathbf{w} = 0$ or Γ_1 hence $c = 0$ and so $\mathbf{w} \equiv 0$.

The solution to the saddle point problem solves the mixed problem as well.

Theorem 4.3. Problem (4.20) is equivalent to the problem to find $(\underset{\sim}{\sigma}, \mathbf{w}) \in K_* \times K$ such that

$$\begin{aligned} &a(\underset{\sim}{\sigma}, \underset{\sim}{\tau} - \underset{\sim}{\sigma}) - b(\mathbf{w} + \xi, \underset{\sim}{\tau} - \underset{\sim}{\sigma}) \ge 0, \qquad \forall\, \underset{\sim}{\tau} \in K_* \\ &b(\mathbf{v} - \mathbf{w}, \underset{\sim}{\sigma}) \ge \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{w})\, \mathrm{d}x, \qquad \forall\, \mathbf{v} \in K. \end{aligned} \tag{4.28}$$

Proof. Follows directly from the results of [4, Chapter 6].

In order to establish theorem 1.1 it remains to be shown the estimate (1.9) appropriately modified ($\Gamma_2 = \varnothing$).

Theorem 4.4. Let the assumptions of theorem 1.1 hold as well as (4.3). Let $(\underset{\sim}{\sigma}^i, \mathbf{w}^i)$ be solutions to the problem (4.28) for data $(\xi^i, \mathbf{f}^i)$, $i = 1, 2$, respectively. Then there holds

$$\|\underset{\sim}{\sigma}^1 - \underset{\sim}{\sigma}^2\|_0 + \|\underset{\sim}{\varepsilon}(\mathbf{w}^1) - \underset{\sim}{\varepsilon}(\mathbf{w}^2)\|_* \le c\{\|\underset{\sim}{\varepsilon}(\xi^1 - \xi^2)\|_0 + \|\mathbf{f}^1 - \mathbf{f}^2\|_{V_D'}\}, \tag{4.29}$$

where $\|\cdot\|_*$ is the norm of $\mathbf{L}(\Omega)/\{\sigma_1\}$ and c depends only on Ω, the upper bound for $|A|$ and the lower bound for $\chi(A)$.

Proof. It follows from theorem 4.3 that the $\underset{\sim}{\sigma}^i$ are solutions to problem (P_-), (4.4), hence by the variational inequality characterization of such problems

$$a(\underset{\sim}{\sigma}^i, \underset{\sim}{\tau} - \underset{\sim}{\sigma}^i) - b(\underset{\sim}{\tau} - \underset{\sim}{\sigma}^i, \xi^i) \ge 0, \qquad \forall\, \underset{\sim}{\tau} \in \varepsilon(\mathbf{z}^i) + \tilde{K}^*_-. \tag{4.30}$$

Then let $\underset{\sim}{\sigma}^i = \varepsilon(\mathbf{z}^i) + \hat{\sigma}^i$, $i = 1, 2$, so that $\hat{\sigma}^i \in \tilde{K}^*_-$ and let $\tau = \varepsilon(\mathbf{z}^i) + \hat{\sigma}^2$ in (4.30) with $i = 1$ and let $\tau = \varepsilon(\mathbf{z}^2) + \hat{\sigma}^1$ in (4.30) with $i = 2$. Adding the expressions and using the coerciveness of $a(\cdot, \cdot)$ over $\tilde{K}^*_-$ we obtain

$$\|\underset{\sim}{\hat{\sigma}}^1 - \underset{\sim}{\hat{\sigma}}^2\|_0 \le C\{\|\varepsilon(\xi^1 - \xi^2)\|_0 + \|\mathbf{f}^1 - \mathbf{f}^2\|_{V_D'}\}. \tag{4.31}$$

Obviously $\|\underset{\sim}{\sigma}^1 - \underset{\sim}{\sigma}^2\|_0$ satisfies a similar estimate.

We turn to estimate $\mathbf{w}^1 - \mathbf{w}^2$. It follows from (4.28) that

$$b(\mathbf{w}^i, \underset{\sim}{\tau}) \le a(\underset{\sim}{\sigma}^i, \underset{\sim}{\tau} - \underset{\sim}{\sigma}^i) + b(\xi^i, \underset{\sim}{\sigma}^i - \underset{\sim}{\tau}) + b(\mathbf{w}^i, \underset{\sim}{\sigma}^i)$$

and hence

$$b(\mathbf{w}^i, \underset{\sim}{\tau}) \le a(\underset{\sim}{\sigma}^i, \underset{\sim}{\tau}) - b(\xi^i, \underset{\sim}{\tau}), \tag{4.32}$$

$$b(\mathbf{w}^i, \underset{\sim}{\sigma}^i) = a(\underset{\sim}{\sigma}^i, \underset{\sim}{\sigma}^i) - b(\xi^i, \underset{\sim}{\sigma}^i). \tag{4.33}$$

The last equality follows from (4.28) by taking $\underset{\sim}{\tau} = 0$ and $\underset{\sim}{\tau} = 2\underset{\sim}{\sigma}$ in the first inequality. Let $\underset{\sim}{\tau} \in K_* = \{\underset{\sim}{\tau} \in \mathbf{L}(\Omega); \Lambda_3\underset{\sim}{\tau} \le 0\}$, then it can be written as

$$\underset{\sim}{\tau} = \theta\underset{\sim}{\sigma}_1 + \hat{\underset{\sim}{\tau}} \qquad \text{where } \theta = \Lambda_3\underset{\sim}{\tau} \le 0 \qquad \text{and} \qquad \Lambda_3\hat{\underset{\sim}{\tau}} = 0.$$

Similarly $\underset{\sim}{\sigma}^i = \theta_0^i\underset{\sim}{\sigma}_1 + \hat{\underset{\sim}{\sigma}}^i$, $\theta_0^i = \Lambda_3\underset{\sim}{\sigma}^i \le 0$ and $\Lambda_3\hat{\underset{\sim}{\sigma}}^i = 0 (i = 1, 2)$. Notice that $\underset{\sim}{\sigma}^1$ is a solution while $\underset{\sim}{\sigma}_1$ is the eigentensor. Then (4.32) implies

$$\theta[b(\mathbf{w}^i, \underset{\sim}{\sigma}_1) - a(\underset{\sim}{\sigma}^i, \underset{\sim}{\sigma}_1) + b(\underset{\sim}{\xi}^i, \underset{\sim}{\sigma}_1)] \le a(\underset{\sim}{\sigma}^i, \hat{\underset{\sim}{\tau}}) - b(\mathbf{w}^i, \hat{\underset{\sim}{\tau}}) - b(\underset{\sim}{\xi}^i, \hat{\underset{\sim}{\tau}}), \tag{4.34}$$

and since we may choose $\theta < 0$ arbitrarily small while $\hat{\underset{\sim}{\tau}}$ is held fixed

$$b(\mathbf{w}^i, \hat{\underset{\sim}{\tau}}) \le a(\underset{\sim}{\sigma}^i, \hat{\underset{\sim}{\tau}}) - b(\underset{\sim}{\xi}^i, \hat{\underset{\sim}{\tau}}), \qquad \forall\, \hat{\underset{\sim}{\tau}} \in K_*, \tag{4.35}$$

but $\hat{\underset{\sim}{\tau}}$ is arbitrary, hence

$$\theta[b(\mathbf{w}^i + \underset{\sim}{\xi}^i, \underset{\sim}{\sigma}_1) - a(\underset{\sim}{\sigma}^i, \underset{\sim}{\sigma}_1)] \le 0, \tag{4.36}$$

for all $\theta \le 0$, and also there holds

$$b(\mathbf{w}^i + \underset{\sim}{\xi}^i, \underset{\sim}{\sigma}_1) - a(\underset{\sim}{\sigma}^i, \underset{\sim}{\sigma}_1) \ge 0, \tag{4.37}$$

otherwise the left hand side of (4.34) can be made as large as we wish while the right hand side is fixed.

Next we set

$$\underset{\sim}{\varepsilon}(\mathbf{w}^i + \underset{\sim}{\xi}^i) = \theta^i\underset{\sim}{\sigma}_1 + \underset{\sim}{\rho}^i, \qquad \theta^i = \Lambda_3\underset{\sim}{\varepsilon}(\mathbf{w}^i + \underset{\sim}{\xi}^i) \le 0, \qquad \Lambda_3\underset{\sim}{\rho}^i = 0, \tag{4.38}$$

where $i = 1, 2$. Clearly $\underset{\sim}{\rho}^i \in K_*$ and so we may take $\hat{\underset{\sim}{\tau}} = \underset{\sim}{\rho}^1 - \underset{\sim}{\rho}^2$ in (4.35) with $i = 1$ and $\hat{\underset{\sim}{\tau}} = \underset{\sim}{\rho}^2 - \underset{\sim}{\rho}^1$ in (4.35) with $i = 2$. Then adding the expressions we obtain

$$b(w^1 + \underset{\sim}{\xi}^1 - w^2 - \underset{\sim}{\xi}^2, \underset{\sim}{\rho}^1 - \underset{\sim}{\rho}^2) \le a(\underset{\sim}{\sigma}^1 - \underset{\sim}{\sigma}^2, \underset{\sim}{\rho}^1 - \underset{\sim}{\rho}^2)$$

and using the fact that $\Lambda_3(\underset{\sim}{\rho}^1 - \underset{\sim}{\rho}^2) = 0$, the definition of $b(\cdot, \cdot)$, (3.7), and the boundedness of $a(\cdot, \cdot)$ we obtain

$$\|\underset{\sim}{\rho}^1 - \underset{\sim}{\rho}^2\|_0 \le C\|\underset{\sim}{\sigma}^1 - \underset{\sim}{\sigma}^2\|_0, \tag{4.39}$$

but then it follows from (4.38) and (4.31) that

$$\|\underset{\sim}{\varepsilon}(\mathbf{w}^1 - \mathbf{w}^2)\|_* \le C\{\|\varepsilon(\underset{\sim}{\xi}^1 - \underset{\sim}{\xi}^2)\|_0 + \|\mathbf{f}^1 - \mathbf{f}^2\|_{V_D'}\},$$

which together with (4.31) leads to (4.29).

Combining theorems, 4.1, 4.2, and 4.3 gives theorem 1.1 for the case $\Gamma_2 = \varnothing$. Finally we remark that when $\Gamma_3 = \varnothing$ the condition (4.36) is changed to an equality as was noted at the beginning of this section.

5. THE DISPLACEMENT-PRESSURE FORMULATION

The system (1.1) and (1.2) involves nine independent scalar unknown functions. From the point of view of numerical solutions to the problem this number of unknowns is too large and a way of reducing this number is of importance. We are able to reduce the number of variables to four, three displacements and one stress function, usually called pressure. We follow Arnold and Falk [1] (see references there for earlier works). When A is invertible one can reduce the problem to that for the (three) displacements only, but when A is not invertible, as in our case, or when A is almost singular (ill conditioned) then the approach below should be useful.

The problem (P_1), (3.4) and (3.5) can be written as

$$A\underset{\sim}{\sigma} = \underset{\sim}{\varepsilon}(\mathbf{w} + \xi), \qquad \underset{\sim}{\sigma} \in \mathbf{L}(\Omega), \tag{5.1}$$

$$b(\underset{\sim}{\sigma}, \mathbf{v} - \mathbf{w}) \geq l(\mathbf{v} - \mathbf{w}), \qquad \forall\, \mathbf{v} \in K. \tag{5.2}$$

The construction is based on the decomposition of R^6 into a one dimensional subspace span $\{\underset{\sim}{\sigma}_1\}$, $\underset{\sim}{\sigma}_1 = \underset{\sim}{\sigma}_1(A)$, and its orthogonal complement $\mathbf{Y} = \{\underset{\sim}{\tau} \in R^6; \underset{\sim}{\tau} : \underset{\sim}{\sigma}_1 = 0\}$. It is clear that $A : \mathbf{Y} \to \mathbf{Y}$ and since $\lambda_2 > 0$, A restricted to $\mathbf{Y}$, $(A|_{\mathbf{Y}})$, is positive definite. Define $A^+ : R^6 \to R^6$ by

$$A^+\underset{\sim}{\tau} = (A|_{\mathbf{Y}})^{-1}\underset{\sim}{\tau}, \qquad \underset{\sim}{\tau} \in \mathbf{Y}$$

$$A^+\underset{\sim}{\sigma}_1 = 0.$$

If $\underset{\sim}{\sigma} \in R^6$ we may write

$$\underset{\sim}{\sigma} = p\underset{\sim}{\sigma}_1 + \underset{\sim}{\sigma}_D, \tag{5.3}$$

where $p = \underset{\sim}{\sigma} : \underset{\sim}{\sigma}_1$ and $\underset{\sim}{\sigma}_D \in \mathbf{Y}$ and then it follows from (5.1) that

$$\underset{\sim}{\varepsilon}(\mathbf{w} + \xi) = \lambda_1 p\underset{\sim}{\sigma}_1 + A\underset{\sim}{\sigma}_D. \tag{5.4}$$

We apply A^+ to this equation and obtain

$$A^+\underset{\sim}{\varepsilon}(\mathbf{w} + \xi) = \underset{\sim}{\sigma}_D$$

and so (5.3) may be written as

$$\underset{\sim}{\sigma} = A^+\underset{\sim}{\varepsilon}(\mathbf{w} + \xi) + p\underset{\sim}{\sigma}_1. \tag{5.5}$$

We multiply (5.4) by $\underset{\sim}{\sigma}_1$ and obtain

$$\lambda_1 p = \operatorname{div}[\underset{\sim}{\sigma}_1(\mathbf{w} + \xi)] \tag{5.6}$$

since $\underset{\sim}{\sigma}_1 : \underset{\sim}{\varepsilon}(\mathbf{v}) = \operatorname{div}[\underset{\sim}{\sigma}_1\mathbf{v}]$. Inserting (5.5) and (5.6) in (5.1) and (5.2) leads to the displacement-pressure formulation

$$\begin{cases} \text{find } (\mathbf{w}, p),\ \mathbf{w} \in K \qquad \text{such that } \forall\, \mathbf{v} \in K. \\ \qquad \operatorname{div}[\underset{\sim}{\sigma}_1(\mathbf{w} + \xi)] = \lambda_1 p, & (5.7) \\ \displaystyle\int_\Omega [A^+\underset{\sim}{\varepsilon}(\mathbf{w} + \xi) + p\underset{\sim}{\sigma}_1] : \underset{\sim}{\varepsilon}(\mathbf{v} - \mathbf{w})\,\mathrm{d}x \geq l(\mathbf{v} - \mathbf{w}). & (5.8) \end{cases}$$

When $\lambda_1 > 0$ this is the regular case with A positive definite.

In the case $\lambda_1 = 0$ when the constraint is nonsingular this can be further reduced to a variational inequality for the Stokes equation. Indeed

$$\begin{aligned} \int_\Omega p\underset{\sim}{\sigma}_1 : \underset{\sim}{\varepsilon}(\mathbf{v} - \mathbf{w})\,\mathrm{d}x &= \int_\Omega \operatorname{div}[p\underset{\sim}{\sigma}_1(\mathbf{v} - \mathbf{w})]\,\mathrm{d}x - \int_\Omega (\mathbf{v} - \mathbf{w})\underset{\sim}{\sigma}_1\nabla p\,\mathrm{d}x \\ &= \int_\Omega p \operatorname{div}[\underset{\sim}{\sigma}_1(\mathbf{v} - \mathbf{w})]\,\mathrm{d}x \end{aligned} \tag{5.9}$$

and if we define

$$K_p = \{\mathbf{v} \in V_D; \mathbf{v}\cdot\mathbf{n} \leq 0 \quad \text{in } H_{00}^{1/2}(\Gamma_3) \quad \text{and} \quad \operatorname{div}[\underset{\sim}{\sigma}_1(\mathbf{v} + \xi)] = 0\}, \tag{5.10}$$

then the right hand side of (5.9) vanishes for all $\mathbf{v} \in K_p$ by (5.6) and therefore (5.7) and (5.8) are reduced to the single variational inequality

$$\mathbf{w} \in K_p, \qquad \int_\Omega A^+ \underset{\sim}{\varepsilon}(\mathbf{w} + \xi) : \underset{\sim}{\varepsilon}(\mathbf{v} - \mathbf{w})\, dx \geq l(\mathbf{v} - \mathbf{w}), \qquad \forall\, \mathbf{v} \in K_p. \tag{5.11}$$

The well posedness of (5.7), (5.8) and of (5.11) follows from theorem 1.1.

6. THE CONTACT PROBLEM WITH FRICTION

In this section we consider the problem with friction, (P_F), and prove its well posedness, i.e. theorem 1.2. The method is the same as in Section 3. First we prove the well posedness of the dual problem (P_F^*) (in the sense of Ekeland and Temam [4]), then we construct a Lagrangian L_F such that the solution of the dual problem leads to a saddle point of L_F and this in turn gives the solution to the problem (P_F).

First we derive the mixed formulation for (1.1)–(1.4) and (1.6). Following Duvaut and Lions [3] let

$$j(\mathbf{v}) = \int_{\Gamma_3} g_*(s)|\mathbf{v}_t(s)|\, ds \tag{6.1}$$

be the *friction functional* [3, p. 138], then the *mixed formulation* is:

$$(\mathrm{P}_F) \qquad \begin{cases} \text{find } (\underset{\sim}{\sigma}, \mathbf{u}) \in \mathbf{L}(\Omega) \times V_D \text{ such that} & \\ \quad a(\underset{\sim}{\sigma}, \underset{\sim}{\tau}) - b(\underset{\sim}{\tau}, \mathbf{u}) = 0, \qquad \forall\, \underset{\sim}{\tau} \in \mathbf{L}(\Omega), & (6.2)\\ b(\underset{\sim}{\sigma}, \mathbf{v} - \mathbf{u}) + j(\mathbf{v}) - j(\mathbf{u}) \geq l_*(\mathbf{v} - \mathbf{u}), \qquad \forall\, \mathbf{v} \in V_D & (6.3) \end{cases}$$

where $a(\cdot, \cdot)$ is given in (3.6), $b(\cdot, \cdot)$ in (3.7) and

$$l_*(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx + \int_{\Gamma_2} \mathbf{t}_r \cdot \mathbf{v}\, ds + \int_{\Gamma_3} t_* v_n\, ds. \tag{6.4}$$

It is shown in [3, pp. 138–150] that (P_F), (6.2) and (6.3) is a weak formulation of (1.1)–(1.4) and (1.6), so we use the notation (P_F) for both problems.

We turn to the dual problem. Let

$$K_F = \{\underset{\sim}{\tau} \in \mathbf{L}(\Omega);\ \operatorname{div} \underset{\sim}{\tau} + \mathbf{f} = 0 \text{ in } \Omega,\ \underset{\sim}{\tau}\mathbf{n} = \mathbf{t}_r \text{ on } \Gamma_r \text{ and } \tau_n = t_*,\ |\tau_t| \leq g_* \text{ in } L^\infty(\Gamma_3)\} \tag{6.5}$$

be a closed convex set and as in ([3, p. 145]) the condition $|\tau_t| \leq g_*$ is understood in the sense that $\tau_t \in L^\infty(\Gamma_3)$ and the inequality holds in $L^\infty(\Gamma_3)$. As in [3] we have to show that K_F is not empty. Then the *dual problem* is

$$(\mathrm{P}_F^*) \qquad \underset{\sim}{\sigma} \in K_F : F^*(\underset{\sim}{\sigma}) \leq F^*(\underset{\sim}{\tau}), \qquad \forall\, \underset{\sim}{\tau} \in K_F \tag{6.6}$$

where in this section

$$F^*(\underset{\sim}{\tau}) = a(\underset{\sim}{\tau}, \underset{\sim}{\tau}). \tag{6.7}$$

THEOREM 6.1. Under the assumptions of theorem 1.2 there exists a unique solution $\underset{\sim}{\sigma} \in K_F$ of problem (P_F^*).

Proof. Since A is semidefinite F^* is bounded from below and $A\underset{\sim}{\tau}:\underset{\sim}{\tau}$ is convex and therefore F^* is l.s.c. It remains to be shown that K_F is not empty and F^* is coercive on K_F. Choose the auxiliary function $\eta \in V_D$ such that

$$b(\underset{\sim}{\varepsilon}(\eta), \mathbf{v}) = l_*(\mathbf{v}), \qquad \forall\, \mathbf{v} \in V_D, \tag{6.8}$$

that is $\underset{\sim}{\varepsilon}(\eta)$ is the solution of

$$\operatorname{div} \underset{\sim}{\varepsilon}(\eta) + \mathbf{f} = 0 \qquad \text{in } \Omega,$$

and $\eta = 0$ on Γ_1, $\underset{\sim}{\varepsilon}(\eta)\mathbf{n} = \mathbf{t}_r$ on Γ_2, $\underset{\sim}{\varepsilon}(\eta)\mathbf{n}\cdot\mathbf{n} = t_*$ and $\underset{\sim}{\varepsilon}(\eta)\mathbf{n}\cdot\mathbf{t} = 0$ on Γ_3. If $\underset{\sim}{\tau} \in K_F$ then it may be decomposed as

$$\underset{\sim}{\tau} = \underset{\sim}{\varepsilon}(\eta) + \hat{\underset{\sim}{\tau}}$$

that is $K_F = \underset{\sim}{\varepsilon}(\eta) + \hat{K}_F$ and

$$\hat{K}_F = \left\{\underset{\sim}{\tau} \in \mathbf{L}(\Omega);\ \int_\Omega \underset{\sim}{\tau} : \underset{\sim}{\varepsilon}(\mathbf{z})\, dx = \int_{\Gamma_3} \underset{\sim}{\tau}_t \cdot \mathbf{z}_t\, ds,\ \forall\, \mathbf{z} \in V_D, \text{ and } |\underset{\sim}{\tau}_t| \le g_*\right\}. \tag{6.9}$$

Clearly it is enough to prove the coerciveness of F^* on $\hat{K}_F$ but the proof is exactly the same as the proof of (3.20). It remains to be shown that $K_F \neq \emptyset$. It is enough to show that the problem

$$\operatorname{div} \underset{\sim}{\tau} + \mathbf{f} = 0 \qquad \text{in } \Omega,$$

$$\underset{\sim}{\tau}\mathbf{n} = \mathbf{t}_r \qquad \text{on } \Gamma_2,$$

$$\tau_n = t_r,\ \tau_{t1} = g_*,\ \tau_{t2} = \tau_{t3} = 0 \qquad \text{on } \Gamma_3,$$

has a solution. But the existence of a solution follows from theorem 1.1 of [1, p. 146].

In order to show uniqueness we notice that $\forall\, \hat{\underset{\sim}{\tau}} \in \hat{K}_F$ there holds

$$a(\hat{\underset{\sim}{\tau}}, \hat{\underset{\sim}{\tau}}) \ge \max\left\{\lambda_1, \frac{\lambda_2 c}{|\underset{\sim}{\sigma}_1^{-1}|^2}\right\} \|\hat{\underset{\sim}{\tau}}\|_0^2.$$

Next we consider the stability of the dual problem.

PROPOSITION 6.2. Let $\underset{\sim}{\sigma}^1$ and $\underset{\sim}{\sigma}^2$ be two solutions of (P_F^*) corresponding to the data $\mathbf{t}_r^i t_*^i$, g_*^i and $\mathbf{f}^i$, $i = 1, 2$, respectively. Then

$$\begin{aligned} &|F^*(\underset{\sim}{\sigma}^1) - F^*(\underset{\sim}{\sigma}^2)| \\ &\quad \le c\{|\mathbf{t}_r^1 - \mathbf{t}_r^2|_{1/2,\Gamma_2} + \|\mathbf{f}^1 - \mathbf{f}^2\|_{V_d'} + |t_*^1 - t_*^2|_{1/2,\Gamma_3} + \|g_*^1 - g_*^2\|_{-1/2,\Gamma_3}\}, \end{aligned} \tag{6.10}$$

where c depends on Ω, A and $\chi(A)$ only.

Proof. We have $K_F^i = \underset{\sim}{\varepsilon}(\eta_i) + \hat{K}_F$ and then (7.10) follows from the boundedness and coercivity of F^* over $\hat{K}_F$.

As a corollary to theorem 6.1 and proposition 6.2 we have the following theorem.

THEOREM 6.3. Under the conditions of theorem 1.2 the dual problem (P_F^*) is well posed.

In order to compute a Lagrangian for the problem we rewrite (P_F^*). Let $\Lambda_4 : \mathbf{L}(\Omega) \to R$ be given by

$$\langle \Lambda_4(\underset{\sim}{\tau}), \mathbf{z} \rangle_{\Gamma_3} = \int_{\Gamma_3} [\underset{\sim}{\tau}_t \mathbf{z}_t + g_* |\mathbf{z}_t|] \, ds, \tag{6.11}$$

$\forall \, \mathbf{z} \in \mathbf{H}_{00}^{1/2}(\Gamma_3)$ ($\mathbf{z}_t$ is the tangential component of $\mathbf{z}$ and likewise $\underset{\sim}{\tau}_t$) and define

$$\chi_3(q) = \begin{cases} 0 & \text{if } q \geq 0, \\ +\infty & \text{if } q < 0, \end{cases} \tag{6.12}$$

then let $\mathbf{G} \in V_D'$ be such that

$$\langle \mathbf{G}, \mathbf{v} \rangle = l_*(\mathbf{v}), \qquad \forall \, \mathbf{v} \in V_D, \tag{6.13}$$

finally we define, for $\underset{\sim}{\tau} \in \mathbf{L}(\Omega)$

$$\begin{aligned} \Phi_F(\underset{\sim}{\tau}) &= F^*(\underset{\sim}{\tau}) + \chi_1(\Lambda_1 \underset{\sim}{\tau} - \mathbf{G}) + \chi_3(\Lambda_4 \underset{\sim}{\tau}) \\ &= \hat{\Phi}_F(\underset{\sim}{\tau}, \Lambda_1 \underset{\sim}{\tau}, \Lambda_4 \underset{\sim}{\tau}), \end{aligned} \tag{6.14}$$

where Λ_1 is given in (3.29), then we may write (P_F^*) in the form

$$\underset{\sim}{\sigma} \in \mathbf{L}(\Omega) : \Phi_F(\underset{\sim}{\sigma}) \leq \Phi_F(\underset{\sim}{\tau}), \qquad \forall \, \underset{\sim}{\tau} \in \mathbf{L}(\Omega). \tag{6.15}$$

Note that $\underset{\sim}{\tau} \in K_F$ iff $\langle \Lambda_1 \underset{\sim}{\tau} - \mathbf{G}, \mathbf{v} \rangle_\Omega = 0$, $\forall \, \mathbf{v} \in V_D$ and $\langle \Lambda_4 \underset{\sim}{\tau}, \mathbf{z} \rangle_{\Gamma_3} \geq 0$, $\forall \, \mathbf{z} \in \mathbf{H}_{00}^{1/2}(\Gamma_3)$.

Define $Y = V_D \times \mathbf{H}_{00}^{1/2}(\Gamma_3)$ and then its dual is $Y' = V_D' \times \mathbf{H}^{-1/2}(\Gamma_3)$ (see before (3.26) above) and set

$$\begin{gathered} \phi_F(\underset{\sim}{\tau}, \mathbf{p}) = \hat{\Phi}_F(\underset{\sim}{\tau}, \Lambda_1 \underset{\sim}{\tau} - \mathbf{p}_1, \Lambda_4 \underset{\sim}{\tau} - \mathbf{p}_2), \\ \forall \, \underset{\sim}{\tau} \in \mathbf{L}(\Omega) \qquad \text{and} \qquad \forall \, \mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2) \in Y'. \end{gathered} \tag{6.16}$$

It is clear that $\Phi_F(\underset{\sim}{\tau}) = \phi_F(\underset{\sim}{\tau}, 0)$ and in order to show that (P_F^*) is stable consider

$$h(\mathbf{p}) = \inf_{\underset{\sim}{\tau} \in \mathbf{LL}(\Omega)} \phi_F(\underset{\sim}{\tau}, \mathbf{p}). \tag{6.17}$$

Obviously $h(0)$ is finite (note that $\chi_3(\Lambda_4 \underset{\sim}{\tau} - \mathbf{p}_2) = 0$ implies $\int_{\Gamma_3} [(\underset{\sim}{\tau}\mathbf{n} - \mathbf{p}_2)_t \mathbf{z}_t + g_* |\mathbf{z}_t|] \, ds \geq 0$, $\forall \, \mathbf{z} \in \mathbf{H}_{00}^{1/2}(\Gamma_3)$) and for $\mathbf{p} \neq 0$ it is a perturbation of the convex set K_F. It follows from proposition 6.2 that h is continuous at zero and since it is a convex function of $\mathbf{p}$ it is subdifferentiable at zero. Finally $\phi_F \in \Gamma_0(\mathbf{L}(\Omega) \times Y')$, i.e. is a proper convex and l.s.c. and therefore the problem (P_F^*) is stable in the sense of definition 2.1 above.

We claim that L_F is a Lagrangian for problem (P_F^*), where

$$L_F(\underset{\sim}{\tau}, \mathbf{v}) = a(\underset{\sim}{\tau}, \underset{\sim}{\tau}) - b(\underset{\sim}{\tau}, \mathbf{v}) + l_*(\mathbf{v}) - j(\mathbf{v}) \tag{6.18}$$

and $j(\mathbf{v})$ is the friction functional (6.1). Indeed let (following [4, Chapter 6])

$$\begin{aligned} \Phi(\underset{\sim}{\tau}, \mathbf{p}) &= \sup_{\mathbf{v} \in V_D} \{\langle \mathbf{p}, \mathbf{v} \rangle + L_F(\underset{\sim}{\tau}, \mathbf{v})\} \\ &= \sup_{\mathbf{v}} \{(\mathbf{p}, \mathbf{v}) + a(\underset{\sim}{\tau}, \underset{\sim}{\tau}) - b(\underset{\sim}{\tau}, v) + l_*(\mathbf{v}) - \int_{\Gamma_3} g_* |\mathbf{v}_t| \, ds\} \\ &= a(\underset{\sim}{\tau}, \underset{\sim}{\tau}) + \sup_{\mathbf{v}} \{(\mathbf{p}, \mathbf{v}) + \langle \Lambda_1 \underset{\sim}{\tau} - \mathbf{G}, \mathbf{v} \rangle - \langle \Lambda_4 \underset{\sim}{\tau}, \mathbf{v} \rangle_{\Gamma_3}\} \\ &= \begin{cases} a(\underset{\sim}{\tau}, \underset{\sim}{\tau}) & \text{if } \Lambda_1 \underset{\sim}{\tau} - \mathbf{p}_1 - \mathbf{G} = 0 \quad \text{and} \quad \displaystyle\int_{\Gamma_3} [(\underset{\sim}{\tau}\mathbf{n} - \mathbf{p}_2)\mathbf{z}_t + g_* |\mathbf{z}_t|] \, ds \geq 0, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore $\Phi(\underset{\sim}{\tau}, \mathbf{p}) = \tilde{\Phi}_F(\underset{\sim}{\tau}, \mathbf{p})$ for $\underset{\sim}{\tau} \in \mathbf{L}^2_{\text{div}}(\Omega)$ and $\mathbf{p} \in Y'$, and therefore L_F is a Lagrangian for (P_F^*).

We summarize our results as the following theorem.

THEOREM 6.4. The problem

$$\inf_{\underset{\sim}{\tau}} \sup_{\mathbf{v}} \{L_F(\underset{\sim}{\tau}, \mathbf{v}); \underset{\sim}{\tau} \in \mathbf{L}(\Omega), \mathbf{v} \in V_D, v_n \in H^{1/2}_{00}(\Gamma_3)\}, \tag{6.19}$$

has a unique saddle point $(\underset{\sim}{\sigma}, \mathbf{u})$ where $\underset{\sim}{\sigma}$ is the solution of (P_F^*), (6.6).

Proof. It follows from theorems 6.1 and 2.2 that the saddle point $(\underset{\sim}{\sigma}, \mathbf{u})$ exists and that $\underset{\sim}{\sigma}$ is unique. Therefore it remains to be shown that $\mathbf{u}$ is unique as well. We write (6.19) as

$$L_F(\underset{\sim}{\sigma}, \mathbf{v}) \le L_F(\underset{\sim}{\sigma}, \mathbf{u}) \le L_F(\underset{\sim}{\tau}, \mathbf{u}) \tag{6.20}$$

$\underset{\sim}{\tau} \in \mathbf{L}(\Omega)$ and $\mathbf{v} \in V_D$, $v_n \in H^{1/2}_{00}(\Gamma_3)$. But then

$$a(\underset{\sim}{\sigma}, \underset{\sim}{\tau}) - b(\underset{\sim}{\tau}, \mathbf{u}) = 0, \qquad \forall\, \underset{\sim}{\tau} \in \mathbf{L}(\Omega).$$

Therefore if $\mathbf{u}^1$ and $\mathbf{u}^2$ are two solutions then

$$b(\underset{\sim}{\tau}, \mathbf{u}^1 - \mathbf{u}^2) = \int_\Omega \underset{\sim}{\varepsilon}(\mathbf{u}^1 - \mathbf{u}^2) : \underset{\sim}{\tau}\, dx = 0$$

for all $\underset{\sim}{\tau}$ and therefore $\underset{\sim}{\varepsilon}(\mathbf{u}^1 - \mathbf{u}^2) = 0$ a.e. in Ω but $\mathbf{u}^1, \mathbf{u}^2 \in V_D$ hence $\mathbf{u}^1 = \mathbf{u}^2$.

It remains to establish the connection between the solution $(\underset{\sim}{\sigma}, \mathbf{u})$ of (6.19) and of the solution of (P_F), (6.2) and (6.3). But the following proposition applies.

PROPOSITION 6.5. The problem (6.19) is equivalent to the mixed problem (P_F), that is if $(\underset{\sim}{\sigma}, \mathbf{u})$ solves (6.19) then it solves (P_F) and vice versa.

Proof. Ekeland and Temam [4, Chapter 6].

As a corollary of theorem 6.4 and proposition 6.5 the following theorem holds.

THEOREM 6.6. Problem (P_F), (6.2) and (6.3), has a unique solution.

Next we have a stability result.

THEOREM 6.7. Let $(\underset{\sim}{\sigma}^i, \mathbf{u}^i)$ be the solutions to (P_F) corresponding to the data $(\mathbf{f}^i, \mathbf{t}_r^i, t_*^i, g_*^i)$, $i = 1, 2$. Then

$$\begin{aligned}\|\underset{\sim}{\sigma}^1 - \underset{\sim}{\sigma}^2\|_0 + \|\mathbf{u}^1 - \mathbf{u}^2\|_1 \le c\{&\|\mathbf{f}^1 - \mathbf{f}^2\|_{V_D'} + |\mathbf{t}_r^1 - \mathbf{t}_r^2|_{-1/2,\Gamma_2} \\ &+ |t_*^1 - t_*^2|_{-1/2,\Gamma_3} + |g_*^1 - g_*^2|_{-1/2,\Gamma_3}\},\end{aligned} \tag{6.21}$$

where c depends only on Ω, $|A|$ and $\chi(A)$.

Proof. The estimate for $\underset{\sim}{\sigma}^1 - \underset{\sim}{\sigma}^2$ follows from the stability result (6.10). The appropriate estimates for $\mathbf{u}^1 - \mathbf{u}^2$ are easily derived from (6.2) by an appropriate choice of $\underset{\sim}{\tau}$ and from (6.3).

Combining all the results above leads to the existence of a unique solution to (P_F) together with the estimate (6.20) and this concludes the proof of theorem 1.2.

7. THE CONTACT PROBLEM WITH NORMAL COMPLIANCE

We consider the friction problem with normal compliance (P_N). Using the results of the previous section and a fixed point argument (similar to the one in [9]) we prove theorem 1.3. The mixed formulation for (P_N) is as follows.

Define the *normal compliance functional*

$$j_N(\mathbf{v}, \mathbf{z}) = \int_{\Gamma_3} c_N(v_n - s)_+^m \, ds', \tag{7.1}$$

and the *friction functional*

$$j_T(\mathbf{v}, \mathbf{z}) = \int_{\Gamma_3} c_T(v_n - s)_+^m |\mathbf{z}_t| \, ds'. \tag{7.2}$$

We assume that $0 \le c_N \in L^\infty(\Gamma_3)$, $0 \le c_T \in L^\infty(\Gamma_3)$ and that

$$1 \le m < \infty \qquad \text{if } n = 2; \qquad 1 \le m < 3 \qquad \text{if } n = 3. \tag{7.3}$$

These restrictions ensure that j_N and j_T are well defined on $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$. Indeed from the imbedding $\mathbf{H}^1(\Omega) \to \mathbf{L}^q(\Gamma_3)$, $q \le 2(n-1)/(n-2)$ it follows that if $\mathbf{w} \in \mathbf{H}^1(\Omega)$ then $(w_n)_+^m z_n$ belongs to $L^q(\Gamma_3)$, $\forall\, q < \infty$ if $n = 2$ and any $m < \infty$ but when $n = 3$ it belongs to $L^1(\Gamma_3)$ only if $m \le 3$.

We remark that we can generalize j_N and j_T, as in [8], by choosing $m = m_N$ in (7.1) and $m = m_T$ in (7.2) and assume that both m_N and m_T satisfy (7.3). Then everything below holds true for such a choice. For simplicity of notation we choose $m_N = m_T = m$.

Next we use the solution ξ to problem (3.1) so we have zero Dirichlet data for $\mathbf{w} = \mathbf{u} - \xi$, as in Section 3. This simplifies the handling of j_N and j_T.

Proceeding as in Section 7 (see also the derivation in [8] we have that the *mixed formulation* of the problem (P_N) is:

$$(P_N) \quad \begin{cases} \text{find } (\underset{\sim}{\sigma}, \mathbf{w}) \in \mathbf{L}(\Omega) \times V_D \text{ such that} & \\ a(\underset{\sim}{\sigma}, \underset{\sim}{\tau}) - b(\underset{\sim}{\tau}, \mathbf{w}) = b(\underset{\sim}{\tau}, \xi), \qquad \forall\, \underset{\sim}{\tau} \in \mathbf{L}(\Omega), & (7.4) \\ b(\underset{\sim}{\sigma}, \mathbf{v} - \mathbf{w}) + j_T(\mathbf{w}, \mathbf{v}) - j_T(\mathbf{w}, \mathbf{w}) + j_N(\mathbf{w}, \mathbf{v} - \mathbf{w}) \ge l(\mathbf{v} - \mathbf{w}), \qquad \forall\, \mathbf{v} \in V_D, & (7.5) \end{cases}$$

where $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ and $l(\cdot)$ are given in (3.6), (3.7) and (3.8) respectively. By assumption c_N and c_T are nonnegative and let

$$0 \le c_N, \qquad c_T \le \mu. \tag{7.6}$$

Then we prove theorem 1.3 for μ sufficiently small. This follows from a fixed point argument. Let $B_R \subset V_D$ be the ball of radius R, i.e. $B_R = \{\mathbf{v} \in V_D;\ \|v\|_1 \le R\}$ and define the operator $T: B_R \to V_D$ by

$$\mathbf{y} = T\mathbf{z}, \qquad \mathbf{z} \in B_R, \tag{7.7}$$

where $\mathbf{y} \in V_D$ is such that $(\underset{\sim}{\sigma}_y, \mathbf{y})$ is the solution of the linearized problem

$$(P_{NL}) \quad \begin{cases} \text{find } (\underset{\sim}{\sigma}, \mathbf{y}) \in \mathbf{L}(\Omega) \times V_D \text{ such that} & \\ a(\underset{\sim}{\sigma}, \underset{\sim}{\tau}) - b(\underset{\sim}{\tau}, \mathbf{y}) = b(\underset{\sim}{\tau}, \xi), \qquad \forall\, \underset{\sim}{\tau} \in \mathbf{L}(\Omega) & (7.8) \\ b(\underset{\sim}{\sigma}, \mathbf{v} - \mathbf{y}) + j_T(\mathbf{z}, \mathbf{v}) - j_T(\mathbf{z}, \mathbf{y}) + j_N(\mathbf{z}, \mathbf{v} - \mathbf{y}) \ge l(\mathbf{v} - \mathbf{y}), \qquad \forall\, \mathbf{v} \in V_D, & (7.9) \end{cases}$$

The nonlinearities in j_T and j_N are "frozen" and (P_{NL}) is reduced to the form (P_F) of section 6. Then we are able to use the results of the previous section to show the following proposition.

PROPOSITION 7.1. Assume that μ is sufficiently small. Then the operator T, defined in (7.7), is a contraction mapping on B_R for R sufficiently large.

Proof. We use theorem 6.7 and the estimate (6.21), the ideas and method are very similar to those in [9, Section 2].

First we show that $T: B_R \to B_R$. By theorem 6.7 the problem (P_{NL}) has a unique solution therefore for zero data the solution is (0, 0). Then it follows from (6.21) that

$$\|\mathbf{y}\|_1 \leq c\{\|\mathbf{f}\|_{V_D'} + |\mathbf{t}_r|_{-1/2,\Gamma_2} + \mu|(z_n)_+^m|_{-1/2,\Gamma_3}\}. \tag{7.10}$$

But $\mathbf{z} \in V_D$ hence by (7.3) and Sobolev's imbedding theorem $(z_n)_+^m \in L^{4/3}(\Gamma_3)$ and so

$$|(z_n)_+^m|_{-1/2,\Gamma_3} \leq c\|\mathbf{z}\|_{L^{4/3}(\Gamma_3)} \leq c\|\mathbf{z}\|_1.$$

Therefore

$$\|\mathbf{y}\|_1 \leq a + \mu c\|\mathbf{z}\|_1 \tag{7.11}$$

and this implies that if $\mathbf{z} \in B_R$, R large, and μ sufficiently small then $\|\mathbf{y}\|_1 \leq R$, whence $T: B_R \to B_R$. We fix R at such a value. Next we show that T is a contraction. Let $\mathbf{z}_1, \mathbf{z}_2 \in B_R$ and let $\mathbf{y}_1$ and $\mathbf{y}_2$ be the corresponding solutions to (P_{NL}). Then it follows from (6.21) that

$$\begin{aligned}\|\mathbf{y}_1 - \mathbf{y}_2\|_1 &\leq c\mu|(z_{1n})_+^m - (z_{2n})_+^m|_{-1/2,\Gamma_3} \\ &\leq c\mu\|(z_{1n})_1^m - (z_{2n})_+^m\|_{L^{4/3}(\Gamma_3)}.\end{aligned} \tag{7.12}$$

Using the inequality $|a^m - b^m| \leq m|a - b|(a^{m-1} + b^{m-1})$, $a, b, \geq 0$, $m \geq 1$ (see e.g. Hardy *et al.* [6, p. 393]) we obtain

$$\|(z_{1n})_+^m - (z_{2n})_+^m\|_{L^{4/3}(\Gamma_3)} \leq m\|(z_{1n})_+ - (z_{2n})_+\|_{L^4(\Gamma_3)}\|(z_{1n})_+^{m-1} + (z_{2n})_+^{m-1}\|_{L^2(\Gamma_3)}$$

where Hölder's inequality was used together with (7.3). Since $\mathbf{z}_1$ and $\mathbf{z}_2 \in B_R$ the last factor in the product is bounded by cR and therefore using imbedding again

$$\|(z_{1n})_+^m - (z_{2n})_+^m\|_{L^{4/3}(\Gamma_3)} \leq c\|\mathbf{z}_1 - \mathbf{z}_2\|_1. \tag{7.13}$$

Combining (7.12) and (7.13) leads to

$$\|\mathbf{y}_1 - \mathbf{y}_2\|_1 = \|T\mathbf{z}_1 - T\mathbf{z}_2\|_1 \leq C\mu\|\mathbf{z}_1 - \mathbf{z}_2\|_1. \tag{7.14}$$

When μ is sufficiently small this is contraction on $B_R \subset V_D$. Thus we have the following.

COROLLARY 7.2. If μ is sufficiently small then problem (P_N) has a unique solution $(\underset{\sim}{\sigma}, \mathbf{w}) \in \mathbf{L}(\Omega) \times B_R$.

Proof. Since $T: B_R \to B_R$ is a contraction it has a unique fixed point $\mathbf{w} \in B_R$ such that $\mathbf{w} = T\mathbf{w}$. But it then follows from theorem 6.6 that the corresponding $\underset{\sim}{\sigma} \in \mathbf{L}(\Omega)$ is unique.

Thus theorem 1.3 has been proved.

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