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Analytic Singular Perturbations of Elliptic Systems

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We study a singular perturbation problem for a system defined under a variational form. We show the analytic dependence of the solution of the equation with respect to a small, nonnull parameter ε, and make explicit the terms of the power series. This result improves a theorem of Chap. I of J. L. Lions ("Perturbations singulières dans les problèmes aux limites et en contrôle optimal," Springer–Verlag, Berlin 1973) in which the variational forms are supposed to be symmetric and no analycity result is given. We give an application to the study of a stationary thermical system with a small convection coefficient.

I. SETTING OF THE PROBLEM

Let V be a complex Hilbert space and V_1, V_2 two closed subspaces of V such that $V = V_1 \oplus V_2$. Let us denote by $\|\cdot\|$ the norm of V. Let $a_i(\cdot, \cdot)$, $i = 1, 2$, be two sesquilinear continuous forms on V such that, for two numbers $\gamma_1 > 0$, $\gamma_2 > 0$,

$$\begin{aligned} &\forall v \in V, \quad v = v_1 + v_2, \quad v_i \in V_i, \quad i = 1, 2, \qquad \text{we have} \\ &\operatorname{Re} a_i(v, v) \geqslant \gamma_i \|v_i\|^2, \quad i = 1, 2. \end{aligned} \tag{1}$$

$$a_1(u, v_2) = a_1(v_2, u) = 0, \qquad \forall u \in V,\ V_2 \in V_2. \tag{2}$$

Let L be an antilinear continuous form on V. The state equation is

$$\text{find } u \in V \text{ such that } a_1(u, v) + \varepsilon a_2(u, v) = L(v),\ \forall v \in V. \tag{3}$$

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The problem is to analyse the behaviour of the solution of system (3) when $\varepsilon \to 0$. We notice that, in general, the system has no solution if $\varepsilon = 0$.

II. Existence and Unicity of the Solution for a Small, Nonnull ε

For any $\alpha > 0$, we put

$$D_\alpha = \{\varepsilon \in \mathbb{C},\ \varepsilon \neq 0,\ |\varepsilon| < \alpha\}.$$

We show that system (3) has a unique solution if ε is in some D_α:

THEOREM 1. *Under hypotheses* (1) *and* (2) *there exists* $\alpha_0 > 0$ *such that* (3) *has a unique solution for any* ε *in* D_{α_0}.

Proof. Let us write any v of V as $v_1 + v_2$, with $v_i \in V_i$, $i = 1, 2$. Using (2), we see that (3) is equivalent to:

$$\begin{aligned} &\text{(i)} \quad u_\varepsilon = u_1 + u_2;\ u_i \in V_i,\ i = 1, 2, \\ &\text{(ii)} \quad a_1(u_1, v_1) + \varepsilon a_2(u_1 + u_2, v_1) = L(v_1),\ \forall v_1 \in V_1, \\ &\text{(iii)} \quad \varepsilon a_2(u_1 + u_2, v_2) = L(v_2),\ \forall v_2 \in V_2. \end{aligned} \tag{4}$$

Let us write (4)(iii), for $\varepsilon \neq 0$, under the form

$$a_2(u_2, v_2) = \frac{1}{\varepsilon} L(v_2) - a_2(u_1, v_2),\ \forall v_2 \in V_2.$$

Hypothesis (1) and Lax–Milgram's lemma allow to write u_2 as

$$u_2 = \frac{1}{\varepsilon} v_0 + T u_1, \tag{5}$$

where $v_0 \in V_2$ and $T \in \mathscr{L}(V)$ do not depend on ε. From (5) and (4)(ii) it follows that u_1 is a solution to

$$a_1(u_1, v_1) + \varepsilon a_2(u_1 + T u_1, v_1) = L(v_1) - a_2(v_0, v_1),\ \forall v_1 \in V_1. \tag{6}$$

Thanks to (1), Lax–Milgram's lemma can be used if $|\varepsilon|$ is small. This implies the existence and unicity of the solution of (6), from which the theorem follows. ▮

III. Expansion of u_ε in Laurent Series

We establish the analycity of the mapping $\varepsilon \to u_\varepsilon$, for ε in D_{α_0}, and caracterize the terms of the Laurent series.

THEOREM 2. *For $\alpha_0 > 0$ small enough, the mapping $\varepsilon \to u_\varepsilon$ is analytical from D_{α_0} into V. The point $\varepsilon = 0$ is at most a simple pole of u_ε, i.e.,*

$$u_\varepsilon = \sum_{k=-1}^{\infty} \varepsilon^k u^k, \tag{7}$$

and $\{u^k\}_{k=-1}^{\infty}$ is the solution of

$$\begin{aligned} &u^{-1} \in V_2, \\ &a_2(u^{-1}, v_2) = L(v_2), \qquad \forall v_2 \in V_2; \end{aligned} \tag{8}$$

$$\begin{aligned} &a_1(u^0, v) + a_2(u^{-1}, v) = L(v), \qquad \forall v \in V, \\ &a_2(u^0, v_2) = 0, \qquad \forall v_2 \in V_2; \end{aligned} \tag{9}$$

$$\begin{aligned} &a_1(u^k, v) + a_2(u^{k-1}, v) = 0, \qquad \forall v \in V, \\ &a_2(u^k, v_2) = 0, \qquad \forall v_2 \in V_2;\ k = 1 \text{ to } \infty. \end{aligned} \tag{10}$$

Proof. Let us define the operators A_i in $\mathscr{L}(V, V')$, $i = 1, 2$, V' being the antidual of V, by

$$\langle A_i u, v \rangle = a_i(u, v), \qquad \forall u, v \in V,\ i = 1, 2.$$

Let α_0 be such that Theorem 1 holds and let ε belongs to D_{α_0}. Put $T = A_1 + \varepsilon A_2$. Theorem 1 implies that T is an isomorphism between V and V'. For any $\varepsilon' \neq \varepsilon$ in D_{α_0}, we have

$$u_{\varepsilon'} - u_\varepsilon = [(I + (\varepsilon' - \varepsilon)\, T^{-1} A_2)^{-1} - I]\, T^{-1} L$$

and, with the resolvant identity

$$u_{\varepsilon'} - u_\varepsilon = -(\varepsilon' - \varepsilon)[I + (\varepsilon' - \varepsilon)\, T^{-1} A_2]^{-1}\, T^{-1} A_2 T^{-1} L,$$

hence,

$$\frac{du_\varepsilon}{d\varepsilon} = -T^{-1} A_2 T^{-1} L,$$

which implies the analyticity of u_ε in D_{α_0} (see [2]). Consequently there exists a unique expansion in Laurent series of u_ε around 0. Let us prove that $\varepsilon = 0$ is at most a simple pole of u_ε. System (8) has a unique solution u^{-1}; put $v^\varepsilon = u_\varepsilon - (1/\varepsilon)\, u^{-1}$. Then, using (3), (8), we get

$$a_1(v^\varepsilon, v) + \varepsilon a_2(v^\varepsilon, v) = L(v) - a_2(u^{-1}, v), \qquad \forall v \in V. \tag{11}$$

Put $v^\varepsilon = v_1^\varepsilon + v_2^\varepsilon$, $v_i^\varepsilon \in V_i$, $i = 1, 2$. Take $v = v_2^\varepsilon$ in (11). From (8) we deduce that $a_2(v^\varepsilon, v_2^\varepsilon) = 0$, which implies with (1), the existence of $C_1 > 0$ such that $\|v_2^\varepsilon\| \leqslant C_1 \|v_1^\varepsilon\|$. Then, taking $v = v_1^\varepsilon$ in (11), we deduce from (1) the existence of C_2, $C_3 > 0$ such that

$$\gamma_1 \|v_1^\varepsilon\|^2 \leqslant |\varepsilon| \; C_2(1 + C_1)\|v_1^\varepsilon\|^2 + C_3\|v_1^\varepsilon\|.$$

This proves that v^ε is bounded uniformly near zero. Hence $\varepsilon = 0$ is at most a simple pole of u_ε (see [1, 2]). This proves (7), (8). Replacing u_ε by its expansion in (3) we deduce (9), (10). ▮

Remark. The sequence $\{u^k\}$ can be computed in a recurrent way from (8), (9), (10).

IV. An Application

The functional spaces considered here are complex. Let Ω be an open bounded subset of $\mathbb{R}^n$, with smooth boundary Γ. Consider the system

$$\begin{aligned} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) &= f \qquad \text{in} \quad \Omega, \\ \varepsilon u + \partial_{n_A} u &= g \qquad \text{on} \quad \Gamma, \end{aligned} \tag{12}$$

where a_{ij}, $i, j = 1$ to n, are in $C(\bar{\Omega})$, ∂_{n_A} being defined by

$$\partial_{n_A} u = \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} n_j,$$

and f, g being given in $L^2(\Omega) \times L^2(\Gamma)$. The variational formulation corresponding to (12) is

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i}\frac{\partial \bar{v}}{\partial x_j} + \varepsilon \int_\Gamma u\bar{v} = \int_\Omega f\bar{v} + \int_\Gamma g\bar{v}; \quad \forall v \in H^1(\Omega).$$

Under the hypothesis of the existence of some $\beta > 0$ such that

$$\operatorname{Re}\left(\sum_{i,j=1}^{n} a_{ij}(x)\, \zeta_i \bar{\zeta}_j\right) \geqslant \beta \sum_{i=1}^{n} |\zeta_i|^2, \quad \forall x \in \Omega, \forall \zeta \in \mathbb{C}^n,$$

we can apply the general result with $V = H^1(\Omega)$ and

$$V_1 = \left\{u \in H^1(\Omega); \int_\Gamma u = 0\right\}; \; V_2 \equiv \mathbb{C},$$

and

$$a_1(u, v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \bar{v}}{\partial x_j}\, dx; \quad a_2(u, v) = \int_\Gamma u\bar{v}\, d\gamma,$$

$$L(v) = \int_\Omega f\bar{v}\, dx + \int_\Gamma g\bar{v}\, d\gamma.$$

If $\varepsilon \neq 0$ is small enough, (12) has a unique solution u satisfying (7), with

$$u^{-1} = \frac{1}{m(\Gamma)} \left[\int_\Omega f\, dx + \int_\Gamma g\, d\gamma \right].$$

Then u^0 is the solution of

$$-\sum_{i,j} \frac{\partial}{\partial x_j}\left(a_{ij}(x) \frac{\partial u^0}{\partial x_i} \right) = f \qquad \text{in} \quad \Omega,$$

$$\frac{\partial u^0}{\partial n_A} = -u^{-1} \qquad \text{on} \quad \Gamma; \quad \int_\Gamma u^0 = 0.$$

Finally the equation of u^k, $k \geqslant 1$, is

$$-\sum_{i,j} \frac{\partial}{\partial x_j}\left(a_{ij}(x) \frac{\partial u^k}{\partial x_i} \right) = 0 \qquad \text{in} \quad \Omega,$$

$$\frac{\partial u^k}{\partial n_A} = -u^{k-1} \qquad \text{on} \quad \Gamma; \quad \int_\Gamma u^k = 0.$$

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