Well-posedness Analysis of Elliptic Mixed Variational-Hemivariational Inequalities

Min Ling

June 20, 2023

Contents

Let V be a Banach space and denote by V^* its dual.

Definition 0.1

Let $\psi : V \to \mathbb{R}$ be a locally Lipschitz functional. The generalized directional derivative of ψ at $u \in V$ in the direction $v \in V$ is defined by

$$\psi^0(u; v) = \limsup_{w \to u,\ \lambda \downarrow 0} \frac{\psi(w + \lambda v) - \psi(w)}{\lambda}.$$

The Clarke subdifferential of ψ at u is defined by

$$\partial \psi(u) = \{\zeta \in V^* : \psi^0(u; v) \geq \langle \zeta, v \rangle \ \forall\, v \in V\}.$$

Remark. If $\psi'(u) \in V^*$, then $\partial\psi(u) = \psi'(u)$ and $\psi^0(u; v) = \langle \psi'(u), v \rangle$;

Moreover, if $\psi \in V^*$, then $\psi'(u) = \psi$ for any $u \in V$.

Contents

Introduce function spaces

$$V = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v}|_{\Gamma_D} = \boldsymbol{0},\ v_n|_{\Gamma_S} = 0\}, \quad Q = L_0^2(\Omega).$$

The mixed formulation of the Stokes HVI is as follows:

Problem 1.1

Find $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ *such that*

$$a(\boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) + \int_{\Gamma_S} \psi^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, ds \geq \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{1.1}$$

$$b(\boldsymbol{u}, q) = 0 \quad \forall\, q \in Q, \tag{1.2}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) = 2\nu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, d\boldsymbol{x} \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V},$$

$$b(\boldsymbol{v}, q) = \int_\Omega q \operatorname{div} \boldsymbol{v}\, d\boldsymbol{x} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},\ q \in Q,$$

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\boldsymbol{x} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}.$$

Introduce a subspace of $\boldsymbol{V}$

$$\boldsymbol{V}_0 = \{\boldsymbol{v} \in \boldsymbol{V} : \operatorname{div} \boldsymbol{v} = 0\}.$$

Problem 1.2

Find $\boldsymbol{u} \in \boldsymbol{V}_0$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) + \int_{\Gamma_S} \psi^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, ds \geq \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_0. \tag{1.3}$$

- HAN W. Minimization principles for elliptic hemivariational inequalities[J]. Nonlinear Analysis: Real World Applications, 2020, 54: 103114.

Contents

Problem 2.1

Find $(u, \lambda) \in K_V \times K_\Lambda$ *such that*

$$a(u, v-u) + b(v-u, \lambda) + \Phi(v) - \Phi(u) + \Psi^0(u; v-u) \geq \langle f, v-u \rangle \quad \forall\, v \in K_V, \tag{2.1}$$

$$b(u, \mu - \lambda) \leq 0 \quad \forall\, \mu \in K_\Lambda. \tag{2.2}$$

$H(K_V)$ V is a real Hilbert space, $K_V \subset V$ is non-empty, closed and convex.

$H(K_\Lambda)$ Λ is a real Hilbert space, $K_\Lambda \subset \Lambda$ is non-empty, closed and convex.

$H(a)$ $a : V \times V \to \mathbb{R}$ is bilinear, bounded and coercive:

$$|a(u, v)| \leq M_a \|u\|_V \|v\|_V \quad \text{and} \quad a(v, v) \geq m_a \|v\|_V^2 \quad \forall\, u, v \in V.$$

$H(b)$ $b : V \times \Lambda \to \mathbb{R}$ is bilinear, bounded and satisfies the inf-sup condition:

$$\sup_{0 \neq v \in K_V} \frac{b(v, \mu)}{\|v\|_V} \geq m_b \|\mu\|_\Lambda \quad \forall\, \mu \in \Lambda.$$

$H(\Phi)$ $\Phi : V \to \mathbb{R}$ is convex and continuous.

$H(\Psi)$ $\Psi : V \to \mathbb{R}$ is locally Lipschitz continuous (generally non-convex), and for a constant $\alpha_\Psi \geq 0$,

$$\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) \leq \alpha_\Psi \|v_1 - v_2\|_V^2 \quad \forall\, v_1, v_2 \in V.$$

$H(0)$ $f \in V^*$ and $m_a > \alpha_\Psi$.

Contents

Introduce a Lagrangian functional $L : K_V \times K_\Lambda \to \mathbb{R}$ by the formula

$$L(v,\mu) = \frac{1}{2}a(v,v) + \Phi(v) + \Psi(v) - \langle f, v\rangle + b(v,\mu) \quad \forall\, v \in K_V, \mu \in K_\Lambda. \tag{2.3}$$

Then, we consider a saddle-point problem corresponding to Problem 2.1.

Problem 2.2

Find $(u,\lambda) \in K_V \times K_\Lambda$ *such that*

$$L(u,\mu) \le L(u,\lambda) \le L(v,\lambda) \quad \forall\, v \in K_V, \mu \in K_\Lambda. \tag{2.4}$$

For any $\mu \in K_\Lambda$, we denote

$$E_\mu(v) = L(v,\mu) = \frac{1}{2}a(v,v) + \Phi(v) + \Psi(v) - \langle f, v\rangle + b(v,\mu) \quad \forall\, v \in K_V.$$

Theorem 2.3

Assume $H(K_V), H(K_\Lambda), H(a), H(b), H(\Phi), H(\Psi)$ and $m_a > \alpha_\Psi$. Then $E_\mu : K_V \to \mathbb{R}$ satisfies

- *locally Lipschitz continuous*
- *strongly convex*
- *coercive*

Proof: By the summation rule [1]

$$\partial E_\mu(v) \subset Av + \partial\Phi(v) + \partial\Psi(v) - f + B^T\mu \quad \forall\, v \in K_V.$$

For $i = 1, 2$, with $v_i \in V$ and $\zeta_i \in \partial E_\mu(v_i)$, we have

$$\zeta_i = Av_i + \xi_i + \eta_i - f + B^T\mu, \quad \xi_i \in \partial\Phi(v_i), \eta_i \in \partial\Psi(v_i).$$

Then we have

$$\begin{aligned}\langle \zeta_1 - \zeta_2, v_1 - v_2\rangle &= a(v_1 - v_2, v_1 - v_2) + \langle \xi_1 - \xi_2, v_1 - v_2\rangle + \langle \eta_1 - \eta_2, v_1 - v_2\rangle \\ &\geq (m_a - \alpha_\Psi)\|v_1 - v_2\|_V^2.\end{aligned}$$

$\partial E_\mu(\cdot)$ is strongly monotone [2] and thus $E_\mu(\cdot)$ is strongly convex on V.

[1] $\partial(\psi_1 + \psi_2)(u) \subset \partial\psi_1(u) + \partial\psi_2(u)$

[2] $\langle \zeta_1 - \zeta_2, v_1 - v_2\rangle \geq \alpha\|v_1 - v_2\|_V^2 \quad \forall\, v_i \in V,\ \zeta_i \in \partial E_\mu(v_i),\ i = 1, 2.$
$\alpha = 0$ (monotone); $\alpha > 0$ (strongly monotone); $\alpha < 0$ (relaxed monotone).

Problem 2.1: Find $(u, \lambda) \in K_V \times K_\Lambda$ such that

$$a(u, v-u) + b(v-u, \lambda) + \Phi(v) - \Phi(u) + \Psi^0(u; v-u) \geq \langle f, v-u \rangle \quad \forall\, v \in K_V, \tag{2.5}$$

$$b(u, \mu - \lambda) \leq 0 \quad \forall\, \mu \in K_\Lambda. \tag{2.6}$$

Problem 2.2: Find $(u, \lambda) \in K_V \times K_\Lambda$ such that

$$L(u, \mu) \leq L(u, \lambda) \leq L(v, \lambda) \quad \forall\, v \in K_V, \mu \in K_\Lambda, \tag{2.7}$$

where

$$L(v, \mu) = \frac{1}{2} a(v, v) + \Phi(v) + \Psi(v) - \langle f, v \rangle + b(v, \mu) \quad \forall\, v \in K_V, \mu \in K_\Lambda.$$

Remark. It is easy to see that (2.6) and the first inequality in (2.7) are equivalent.

Theorem 2.4

Assume $H(K_V), H(K_\Lambda), H(a), H(b), H(\Phi), H(\Psi)$ and $m_a > \alpha_\Psi$. Then Problem 2.1 and Problem 2.2 are equivalent.

Proof: Let us prove the equivalence of (2.5) and the second inequality in (2.7). Assume (2.5) is valid. Denote $L_1(\cdot, \lambda) = L(\cdot, \lambda) - \Phi(\cdot)$. Then $L_1(\cdot, \lambda)$ is convex. So for any $v \in K_V$ and any $t \in (0, 1)$,

$$L_1(tv + (1-t)u, \lambda) \leq tL_1(v, \lambda) + (1-t)L_1(u, \lambda).$$

Rewrite the inequality as

$$\frac{1}{t}[L_1(u + t(v-u), \lambda) - L_1(u, \lambda)] \leq L_1(v, \lambda) - L_1(u, \lambda).$$

Denote $F_A(v) = \frac{1}{2}a(v, v)$. By the definition of the functional L_1,

$$\begin{aligned} &\frac{1}{t}[F_A(u + t(v-u)) - F_A(u)] + \frac{1}{t}[\Psi(u + t(v-u)) - \Psi(u)] \\ &\quad - \langle f, v-u \rangle + b(v-u, \lambda) \leq L(v, \lambda) - L(u, \lambda) + \Phi(u) - \Phi(v). \end{aligned}$$

Take the upper limit of both sides of the above inequality as $t \to 0+$ to obtain

$$a(u, v-u) + \Phi(v) - \Phi(u) + \Psi^0(u; v-u) - \langle f, v-u \rangle + b(v-u, \lambda) \leq L(v, \lambda) - L(u, \lambda).$$

Conversely, assume

$$L(u, \lambda) \leq L(v, \lambda) \quad \forall v \in K_V.$$

Denote

$$E_\lambda(v) = L(v, \lambda) = \frac{1}{2} a(v, v) + \Phi(v) + \Psi(v) - \langle f, v \rangle + b(v, \lambda) \quad \forall v \in K_V.$$

The functional E_λ has a unique minimizer u on K_V, which satisfies the relation

$$E_\lambda^0(u; v - u) \geq 0 \quad \forall v \in K_V.$$

By the summation rule [3] ,for any $v \in K_V$, we have

$$E_\lambda^0(u; v - u) \leq a(u, v - u) + \Phi(v) - \Phi(u) + \Psi^0(u; v - u) - \langle f, v - u \rangle + b(v - u, \lambda).$$

[3] $(\psi_1 + \psi_2)^0(u; v) \leq \psi_1^0(u; v) + \psi_2^0(u; v)$.

Let V and Λ be two Hilbert spaces. $K_V \subset V$ is non-empty, closed and convex, $K_\Lambda \subset \Lambda$ is non-empty, closed and convex.

$$L(v,\mu) = \frac{1}{2}a(v,v) + \Phi(v) + \Psi(v) - \langle f, v\rangle + b(v,\mu) \quad \forall\, v \in K_V, \mu \in K_\Lambda.$$

Theorem 2.5

Assume $L : K_V \times K_\Lambda \to \mathbb{R}$ has the following properties:

- *for any $\mu \in K_\Lambda$, $v \mapsto L(v,\mu)$ is convex and lower semicontinuous;*
- *for any $v \in K_V$, $\mu \mapsto L(v,\mu)$ is concave and upper semicontinuous;*
- *either K_V is bounded or $\lim\limits_{\|v\|_V \to \infty, v \in K_V} L(v,\mu_*) = \infty$ for some $\mu_* \in K_\Lambda$;*
- *either K_Λ is bounded or $\lim\limits_{\|\mu\|_\Lambda \to \infty, \mu \in K_\Lambda} \inf\limits_{v \in K_V} L(v,\mu) = -\infty$.*

Then, L has at least one saddle point over $K_V \times K_\Lambda$.

- HAN W, MATEI A. Minimax principles for elliptic mixed hemivariational-variational inequalities[J]. Nonlinear Analysis: Real World Applications, 2022, 64: 103448.

Solution Uniqueness

Let (u_1, λ_1) and (u_2, λ_2) be two solutions to Problem 2.1. Then,

$$a(u_1, u_2 - u_1) + b(u_2 - u_1, \lambda_1) + \Phi(u_2) - \Phi(u_1) + \Psi^0(u_1; u_2 - u_1) \geq \langle f, u_2 - u_1 \rangle, \tag{2.8}$$

$$b(u_1, \lambda_2 - \lambda_1) \leq 0, \tag{2.9}$$

and

$$a(u_2, u_1 - u_2) + b(u_1 - u_2, \lambda_2) + \Phi(u_1) - \Phi(u_2) + \Psi^0(u_2; u_1 - u_2) \geq \langle f, u_1 - u_2 \rangle, \tag{2.10}$$

$$b(u_2, \lambda_1 - \lambda_2) \leq 0. \tag{2.11}$$

Add (2.8) and (2.10) to get

$$a(u_1 - u_2, u_1 - u_2) \leq b(u_2 - u_1, \lambda_1 - \lambda_2) + \Psi^0(u_1; u_2 - u_1) + \Psi^0(u_2; u_1 - u_2).$$

Apply the conditions $H(a)$ and $H(\Psi)$,

$$m_a \|u_1 - u_2\|_V^2 \leq \alpha_\Psi \|u_1 - u_2\|_V^2.$$

Since $m_a > \alpha_\Psi$, we deduce from the above inequality that $u_1 = u_2$.

The second component λ of the solution

Mixed Problem:

$$a(u,v) + b(v,\lambda_1) = \langle f, v\rangle \quad \forall\, v \in V,$$
$$a(u,v) + b(v,\lambda_2) = \langle f, v\rangle \quad \forall\, v \in V.$$

Subtracting the second equation from the first one, we obtain

$$b(v,\lambda_1 - \lambda_2) = 0 \quad \forall\, v \in V,$$

which implies that $\lambda_1 = \lambda_2$ as a consequence of the inf-sup condition.

Mixed Hemivariational-variational Inequality:

$$a(u, v-u) + b(v-u,\lambda_1) + \Phi(v) - \Phi(u) + \Psi^0(u; v-u) \geq \langle f, v-u\rangle \quad \forall\, v \in K_V,$$
$$a(u, v-u) + b(v-u,\lambda_2) + \Phi(v) - \Phi(u) + \Psi^0(u; v-u) \geq \langle f, v-u\rangle \quad \forall\, v \in K_V.$$

Contents

Problem 2.6

Find $(u,\lambda) \in K_V \times K_\Lambda$ *such that*

$$a(u, v-u) + b(v-u, \lambda) + \Phi(v) - \Phi(u) + \Psi^0(u; v-u) \geq \langle f, v-u \rangle \quad \forall\, v \in K_V, \tag{2.12}$$

$$b(u, \mu - \lambda) \leq 0 \quad \forall\, \mu \in K_\Lambda. \tag{2.13}$$

For any $\theta > 0$, Problem 2.6 is equivalent to

$$\begin{aligned} &(u, v-u)_V + \theta \left[b(v-u, \lambda) + \Phi(v) - \Phi(u) + \Psi^0(u; v-u) \right] \\ &\qquad \geq (u, v-u)_V - \theta a(u, v-u) + \theta \langle f, v-u \rangle \quad \forall\, v \in K_V, \end{aligned} \tag{2.14}$$

$$\theta b(u, \mu - \lambda) \leq 0 \quad \forall\, \mu \in K_\Lambda. \tag{2.15}$$

Let $\theta \in (0, \alpha_\Psi^{-1})$. Then for any $w \in K_V$, there exists $(u,\lambda) \in K_V \times K_\Lambda$, u being unique, such that

$$\begin{aligned} &(u, v-u)_V + \theta \left[b(v-u, \lambda) + \Phi(v) - \Phi(u) + \Psi^0(u; v-u) \right] \\ &\qquad \geq (w, v-u)_V - \theta a(w, v-u) + \theta \langle f, v-u \rangle \quad \forall\, v \in K_V, \end{aligned} \tag{2.16}$$

$$\theta b(u, \mu - \lambda) \leq 0 \quad \forall\, \mu \in K_\Lambda. \tag{2.17}$$

Theorem 2.7

Assume $H(K_V), H(K_\Lambda), H(a), H(b), H(\Phi), H(\Psi)$ and $m_a > \alpha_\Psi$. Then Problem 2.6 has a solution $(u, \lambda) \in K_V \times K_\Lambda$, and the first component u of the solution is unique.

Proof. Define a mapping $P_\theta : K_V \to K_V$ by the formula

$$u = P_\theta(w), \quad w \in K_V.$$

Let us prove that the mapping $P_\theta : K_V \to K_V$ is a contraction. For any $w_1, w_2 \in K_V$, denote $u_1 = P_\theta(w_1)$ and $u_2 = P_\theta(w_2)$. Then there exist $\lambda_1, \lambda_2 \in K_\Lambda$ such that

$$\begin{aligned}
&(u_1, v - u_1)_V + \theta\left[b(v - u_1, \lambda_1) + \Phi(v) - \Phi(u_1) + \Psi^0(u_1; v - u_1)\right] \\
&\qquad \geq (w_1, v - u_1)_V - \theta a(w_1, v - u_1) + \theta\langle f, v - u_1\rangle \quad \forall\, v \in K_V, && (2.18)\\
&\theta b(u_1, \mu - \lambda_1) \leq 0 \quad \forall\, \mu \in K_\Lambda. && (2.19)
\end{aligned}$$

and

$$\begin{aligned}
&(u_2, v - u_2)_V + \theta\left[b(v - u_2, \lambda_2) + \Phi(v) - \Phi(u_2) + \Psi^0(u_2; v - u_2)\right] \\
&\qquad \geq (w_2, v - u_2)_V - \theta a(w_2, v - u_2) + \theta\langle f, v - u_2\rangle \quad \forall\, v \in K_V, && (2.20)\\
&\theta b(u_2, \mu - \lambda_2) \leq 0 \quad \forall\, \mu \in K_\Lambda. && (2.21)
\end{aligned}$$

Take $v = u_2$ in (2.18), $v = u_1$ in (2.20), we obtain

$$\|u_1 - u_2\|_V^2 = \|P_\theta(w_1) - P_\theta(w_2)\|_V^2 \leq \frac{1 - 2\theta m_a + \theta^2 M_a^2}{(1 - \theta\alpha_\Psi)^2}\|w_1 - w_2\|_V^2.$$

Note that

$$\frac{1 - 2\theta m_a + \theta^2 M_a^2}{(1 - \theta\alpha_\Psi)^2} < 1 \tag{2.22}$$

if and only if

$$\theta(M_a^2 - \alpha_\Psi^2) < 2(m_a - \alpha_\Psi). \tag{2.23}$$

For $\theta > 0$ sufficiently small, (2.22) holds and the mapping $P_\theta : K_V \to K_V$ is a contraction. By the Banach fixed-point theorem, P_θ has a unique fixed-point $u \in K_V$: $P_\theta(u) = u$. Then for some $\lambda \in K_\Lambda$, the pair (u, λ) is a solution of Problem 2.6.

Solution Uniqueness

Let (u_1, λ_1) and (u_2, λ_2) be two solutions to Problem 2.6. Then,

$$a(u_1, u_2 - u_1) + b(u_2 - u_1, \lambda_1) + \Phi(u_2) - \Phi(u_1) + \Psi^0(u_1; u_2 - u_1) \geq \langle f, u_2 - u_1 \rangle, \tag{2.24}$$

$$b(u_1, \lambda_2 - \lambda_1) \leq 0, \tag{2.25}$$

and

$$a(u_2, u_1 - u_2) + b(u_1 - u_2, \lambda_2) + \Phi(u_1) - \Phi(u_2) + \Psi^0(u_2; u_1 - u_2) \geq \langle f, u_1 - u_2 \rangle, \tag{2.26}$$

$$b(u_2, \lambda_1 - \lambda_2) \leq 0. \tag{2.27}$$

Add (2.24) and (2.26) to get

$$a(u_1 - u_2, u_1 - u_2) \leq b(u_2 - u_1, \lambda_1 - \lambda_2) + \Psi^0(u_1; u_2 - u_1) + \Psi^0(u_2; u_1 - u_2).$$

Apply the conditions $H(a)$ and $H(\Psi)$,

$$m_a \|u_1 - u_2\|_V^2 \leq \alpha_\Psi \|u_1 - u_2\|_V^2.$$

Since $m_a > \alpha_\Psi$, we deduce from the above inequality that $u_1 = u_2$.

Contents

Φ has two independent variables

Problem 2.8

Find $(u, \lambda) \in K_V \times K_\Lambda$ *such that*

$$a(u, v-u) + b(v-u, \lambda) + \Phi(u, v) - \Phi(u, u) + \Psi^0(u; v-u) \geq \langle f, v-u \rangle \quad \forall\, v \in K_V, \tag{2.28}$$

$$b(u, \mu - \lambda) \leq 0 \quad \forall\, \mu \in K_\Lambda. \tag{2.29}$$

In the study of Problem 2.8, we modify $H(\Phi)$ to $H(\Phi)_2$;

$H(\Phi)_2$ $\Phi : V \times V \to \mathbb{R}$; for any $u \in V$, $\Phi(u, \cdot) : V \to \mathbb{R}$ is convex and continuous; and for a constant $\alpha_\Phi \geq 0$,

$$\Phi(u_1, v_2) - \Phi(u_1, v_1) + \Phi(u_2, v_1) - \Phi(u_2, v_2) \leq \alpha_\Phi \|u_1 - u_2\|_V \|v_1 - v_2\|_V \quad \forall\, u_1, u_2, v_1, v_2 \in V.$$

Theorem 2.9

Assume $H(K_V), H(K_\Lambda), H(a), H(b), H(\Phi)_2, H(\Psi)$ and $\alpha_\Phi + \alpha_\Psi < m_a$. Then Problem 2.8 has a solution $(u, \lambda) \in K_V \times K_\Lambda$, and the first component u of the solution is unique.

Proof. For any $w \in K_V$, we consider the auxiliary problem of finding $(u, \lambda) \in K_V \times K_\Lambda$ such that

$$a(u, v - u) + b(v - u, \lambda) + \Phi(w, v) - \Phi(w, u) + \Psi^0(u; v - u) \geq \langle f, v - u \rangle \quad \forall\, v \in K_V, \tag{2.30}$$
$$b(u, \mu - \lambda) \leq 0 \quad \forall\, \mu \in K_\Lambda. \tag{2.31}$$

Under the stated assumptions, there is a pair $(u, \lambda) \in K_V \times K_\Lambda$ satisfying (2.30)–(2.31) and u is unique. Define an operator $P : K_V \to K_V$ by

$$P(w) = u.$$

Let us prove that the mapping $P : K_V \to K_V$ is a contraction. For any $w_1, w_2 \in K_V$, denote $u_1 = P(w_1)$ and $u_2 = P(w_2)$. Then there exist $\lambda_1, \lambda_2 \in K_\Lambda$ such that for any $(v, \mu) \in K_V \times K_\Lambda$

$$a(u_1, v - u_1) + b(v - u_1, \lambda_1) + \Phi(w_1, v) - \Phi(w_1, u_1) + \Psi^0(u_1; v - u_1) \geq \langle f, v - u_1 \rangle, \tag{2.32}$$
$$b(u_1, \mu - \lambda_1) \leq 0, \tag{2.33}$$

and

$$a(u_2, v - u_2) + b(v - u_2, \lambda_2) + \Phi(w_2, v) - \Phi(w_2, u_2) + \Psi^0(u_2; v - u_2) \geq \langle f, v - u_2 \rangle, \tag{2.34}$$
$$b(u_2, \mu - \lambda_2) \leq 0. \tag{2.35}$$

Take $v = u_2$ in (2.32), $v = u_1$ in (2.34), we obtain

$$a(u_1 - u_2, u_1 - u_2) \leq b(u_2 - u_1, \lambda_1 - \lambda_2) + \Psi^0(u_1; u_2 - u_1) + \Psi^0(u_2; u_1 - u_2) \\ + \Phi(w_1, u_2) - \Phi(w_1, u_1) + \Phi(w_2, u_1) - \Phi(w_2, u_2).$$

Apply the conditions $H(a)$, $H(\Phi)_2$ and $H(\Psi)$,

$$m_a \|u_1 - u_2\|_V^2 \leq \alpha_\Psi \|u_1 - u_2\|_V^2 + \alpha_\Phi \|w_1 - w_2\|_V \|u_1 - u_2\|_V.$$

Then

$$\|u_1 - u_2\|_V = \|P(w_1) - P(w_2)\|_V \leq \frac{\alpha_\Phi}{m_a - \alpha_\Psi} \|w_1 - w_2\|_V.$$

The mapping $P : K_V \to K_V$ is a contraction. By the Banach fixed-point theorem, P has a unique fixed-point $u \in K_V$: $Pu = u$. Then for some $\lambda \in K_\Lambda$, the pair (u, λ) is a solution of Problem 2.8.

- HAN W, MATEI A. Well-posedness of a general class of elliptic mixed hemivariational-variational inequalities[J]. Nonlinear Analysis: Real World Applications, 2022, 66: 103553.

Contents

Find a displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ such that

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{3.1}$$

$$\mathrm{div}\boldsymbol{\sigma} + \boldsymbol{f}_1 = \mathbf{0} \quad \text{in } \Omega, \tag{3.2}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma_1, \tag{3.3}$$

$$\boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2, \tag{3.4}$$

$$u_n = 0, \quad -\boldsymbol{\sigma}_\tau \in \partial j_\tau(\boldsymbol{u}_\tau) \quad \text{on } \Gamma_3, \tag{3.5}$$

$$\boldsymbol{\sigma}_\tau = \mathbf{0}, \quad \sigma_n \le 0, \quad u_n \le 0, \quad \sigma_n u_n = 0 \quad \text{on } \Gamma_4. \tag{3.6}$$

Introduce a function space

$$V = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1,\ v_n = 0 \text{ on } \Gamma_3\}.$$

We multiply (3.2) by a smooth function $\boldsymbol{v} \in \boldsymbol{V}$, integrate over Ω, perform an integration by parts, and apply (3.4), (3.6) and boundary conditions of $\boldsymbol{v} \in \boldsymbol{V}$ to obtain

$$\int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx - \int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau\, d\Gamma - \int_{\Gamma_4} \sigma_n v_n\, d\Gamma = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \tag{3.7}$$

where

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{f}_1 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}\, d\Gamma, \quad \boldsymbol{v} \in \boldsymbol{V}. \tag{3.8}$$

By applying the boundary condition (3.5),

$$-\int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau\, d\Gamma \le \int_{\Gamma_3} j_\tau^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau)\, d\Gamma. \tag{3.9}$$

Let $\boldsymbol{M}$ be the dual space of the trace space $\boldsymbol{W} = \boldsymbol{H}^{1/2}(\Gamma_4)$. We introduce the following subset of $\boldsymbol{M}$

$$\boldsymbol{\Lambda} = \{\boldsymbol{\mu} \in \boldsymbol{M} : \langle \boldsymbol{\mu}, \boldsymbol{v} \rangle_{\Gamma_4} \ge 0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{W} \text{ with } v_n \ge 0\}, \tag{3.10}$$

Next, we define a Lagrange multiplier $\boldsymbol{\lambda} \in \boldsymbol{M}$ as follows:

$$\langle \boldsymbol{\lambda}, \boldsymbol{v} \rangle_{\Gamma_4} = -\int_{\Gamma_4} \boldsymbol{\sigma n} \cdot \boldsymbol{v}\, d\Gamma = -\int_{\Gamma_4} \sigma_n v_n\, d\Gamma, \quad \boldsymbol{v} \in \boldsymbol{W}. \tag{3.11}$$

Furthermore, we define the bilinear form $b : \boldsymbol{V} \times \boldsymbol{M} \to \mathbb{R}$ by

$$b(\boldsymbol{v}, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{v} \rangle_{\Gamma_4}, \quad \boldsymbol{v} \in \boldsymbol{V},\ \boldsymbol{\mu} \in \boldsymbol{M}. \tag{3.12}$$

Moreover, by (3.6) we obtain

$$b(\boldsymbol{u}, \boldsymbol{\lambda}) = -\int_{\Gamma_4} \sigma_n u_n \, d\Gamma = 0, \tag{3.13}$$

and by (3.10)

$$b(\boldsymbol{u}, \boldsymbol{\mu}) = \langle \mu_n, u_n \rangle_{\Gamma_4} \leq 0 \quad \forall\, \boldsymbol{\mu} \in \boldsymbol{\Lambda}. \tag{3.14}$$

In summary, the mixed weak formulation of contact problem is as follows.

Problem 3.1

Find $\boldsymbol{u} \in \boldsymbol{V}$ and $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, \boldsymbol{\lambda}) + \int_{\Gamma_3} j_\tau^0(\boldsymbol{u}_\tau; \boldsymbol{v}_\tau) \, d\Gamma \geq \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{3.15}$$

$$b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) \leq 0 \quad \forall\, \boldsymbol{\mu} \in \boldsymbol{\Lambda}. \tag{3.16}$$

The bilinear form $b(\cdot, \cdot)$ satisfies the inf-sup condition:

$$\sup_{\boldsymbol{v} \in \widetilde{\boldsymbol{V}}} \frac{b(\boldsymbol{v}, \boldsymbol{\mu})}{\|\boldsymbol{v}\|_V} \geq \beta \, \|\boldsymbol{\mu}\|_{-\frac{1}{2}, \Gamma_4} \quad \forall\, \boldsymbol{\mu} \in \boldsymbol{\Lambda}, \tag{3.17}$$

where $\widetilde{\boldsymbol{V}} = \{\boldsymbol{v} \in \boldsymbol{V} : \boldsymbol{v}_\tau = \boldsymbol{0} \text{ on } \Gamma_3\}$.