Numerical Analysis Of The Cahn-Hilliard Equation And Approximation For The Hele-Shaw Problem, Part II: Error Analysis And Convergence Of The Interface

NUMERICAL ANALYSIS OF THE CAHN-HILLIARD EQUATION AND APPROXIMATION FOR THE HELE-SHAW PROBLEM, PART II: ERROR ANALYSIS AND CONVERGENCE OF THE INTERFACE *

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Abstract. In this second part of the series, we focus on approximating the Hele-Shaw problem via the Cahn-Hilliard equation $u_t + \Delta(\varepsilon\Delta u - \varepsilon^{-1} f(u)) = 0$ as $\varepsilon \searrow 0$. The primary goal of this paper is to establish the convergence of the solution of the fully discrete mixed finite element scheme proposed in [21] to the solution of the Hele-Shaw (Mullins-Sekerka) problem, provided that the Hele-Shaw (Mullins-Sekerka) problem has a global (in time) classical solution. This is accomplished by establishing some improved a priori solution and error estimates, in particular, an $L^\infty(L^\infty)$-error estimate, and making full use of the convergence result of [2]. Like in [20, 21], the cruxes of the analysis are to establish stability estimates for the discrete solutions, use a spectrum estimate result of Alikakos and Fusco [3] and Chen [12], and establish a discrete counterpart of it for a linearized Cahn-Hilliard operator to handle the nonlinear term.

Key words. Cahn-Hilliard equation, Hele-Shaw (Mullins-Sekerka) problem, phase transition, biharmonic problem, fully discrete mixed finite element method, Ciarlet-Raviart element

AMS subject classifications. 65M60, 65M12, 65M15, 35B25, 35K57, 35Q99, 53A10

1. Introduction. In the first part [21] of this series, we proposed and analyzed a semi-discrete (in time) and a fully discrete mixed finite element method for the Cahn-Hilliard equation:

$$u_t + \Delta(\varepsilon\Delta u - \frac{1}{\varepsilon} f(u)) = 0 \quad \text{in } \Omega_T := \Omega \times (0,T)\,, \tag{1.1}$$

$$\frac{\partial u}{\partial n} = \frac{\partial}{\partial n}(\varepsilon\Delta u - \frac{1}{\varepsilon} f(u)) = 0 \quad \text{in } \partial\Omega_T := \partial\Omega \times (0,T)\,, \tag{1.2}$$

$$u = u_0^\varepsilon \quad \text{in } \Omega \times \{0\}\,. \tag{1.3}$$

Note that the super-index ε on the solution u^ε is suppressed for notation brevity. We established a priori solution estimates and optimal and quasi-optimal error estimates under minimum assumptions on the domain Ω and the initial datum function u_0^ε. Special attention was given to the dependence of the error bounds on ε. It was shown that all the error bounds depend on $\frac{1}{\varepsilon}$ only in some low polynomial order for small ε.

In this second part of the series, we are concerned with the second stage of the evolution of the concentration, that is, the motion of the interface. We focus on

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approximating the Hele-Shaw (Mullins-Sekerka) problem:

$$\Delta w = 0 \qquad \text{in } \Omega \setminus \Gamma_t,\ t \in [0,T], \tag{1.4}$$

$$\frac{\partial w}{\partial n} = 0 \qquad \text{on } \partial\Omega,\ t \in [0,T], \tag{1.5}$$

$$w = \sigma\kappa \qquad \text{on } \Gamma_t,\ t \in [0,T], \tag{1.6}$$

$$V = \frac{1}{2}\Big[\frac{\partial w}{\partial n}\Big]_{\Gamma_t} \qquad \text{on } \Gamma_t,\ t \in [0,T], \tag{1.7}$$

$$\Gamma_0 = \Gamma_{00} \qquad \text{when } t = 0 \tag{1.8}$$

via the Cahn-Hilliard equation as $\varepsilon \searrow 0$. Here

$$\sigma = \int_{-1}^{1} \sqrt{\frac{F(s)}{2}}\,\mathrm{d}s,$$

and κ and V are, respectively, the mean curvature and the normal velocity of the interface Γ_t, n is the unit outward normal to either $\partial\Omega$ or Γ_t, $[\frac{\partial w}{\partial n}]_{\Gamma_t} := \frac{\partial w^+}{\partial n} - \frac{\partial w^-}{\partial n}$, and w^+ and w^- are respectively the restriction of w to Ω_t^+ and Ω_t^-, the exterior and interior of Γ_t in Ω (cf. [2, 21]).

The main objective of this paper is to establish the convergence of the fully discrete mixed finite element method proposed in [21] to the solution of the Hele-Shaw problem (1.4)-(1.8), provided that the Hele-Shaw problem has a global (in time) classical solution. To our knowledge, such a numerical convergence result is not yet known in the literature for the Cahn-Hilliard equation. We also note that the convergence of the Cahn-Hilliard equation to the Hele-Shaw model was established in [2] under the same assumption.

To show convergence, we need to establish stronger error estimates, in particular, a $L^\infty(J;L^\infty)$ estimate. We are able to obtain the desired error estimates by first proving some improved a priori solution estimates, and then an improved discrete spectrum estimate under the assumption that the Hele-Shaw problem admits a global (in time) classical solution. Like in [21], the cruxes of the analysis are to establish stability estimates for a discrete solution, use a spectrum estimate result of Alikakos and Fusco [3] and Chen [12], and establish a discrete counterpart of it for a linearized Cahn-Hilliard operator to handle the nonlinear term.

The analysis of this paper is carried out for a general class of admissible double equal well potentials and initial data $u_0^\varepsilon \in H^4$ which can be bounded in terms of negative powers of ε in $H^j(\Omega)$ norm for $j = 1, 2, 3, 4$; see the general assumptions (GA_1)-(GA_3) in Section 2 and 3, and the fully space-time discretization of (1.1)-(1.3) in Section 3.

We exemplify our results for our fully discrete scheme for the case $f(u) = u^3 - u$. The fully discrete scheme, based on a mixed variational formulation for u and the chemical potential $w := -\varepsilon\Delta u + \frac{1}{\varepsilon} f(u)$, is defined as

$$\big(d_t U^m, \psi_h\big) + \big(\nabla W^m, \nabla\psi_h\big) = 0 \quad \forall \psi_h \in \mathcal{S}_h\,, \tag{1.9}$$

$$\varepsilon\,\big(\nabla U^m, \nabla v_h\big) + \frac{1}{\varepsilon}\big(f(U^m), v_h\big) = (W^m, v_h) \quad \forall v_h \in \mathcal{S}_h\,, \tag{1.10}$$

with some starting value $U^0 \in \mathcal{S}_h$. Here $\mathcal{S}_h \subset H^1(\Omega)$ denotes the finite element space of globally continuous, piecewise affine functions.

We assume there exist positive ε-independent constants m_0 and σ_j for $j = 1, 2, \cdots, 5$ such that

$$m_0 := \frac{1}{|\Omega|} \int_\Omega u_0^\varepsilon(x)\, \mathrm{d}x \, \in (-1, 1)\,, \tag{1.11}$$

$$\mathcal{J}_\varepsilon(u_0^\varepsilon) := \frac{\varepsilon}{2} \| \nabla u_0^\varepsilon \|_{L^2}^2 + \frac{1}{\varepsilon} \| F(u_0^\varepsilon) \|_{L^1} \le C\, \varepsilon^{-2\sigma_1}\,, \tag{1.12}$$

$$\| w_0^\varepsilon \|_{H^\ell} := \| -\varepsilon \Delta u_0^\varepsilon + \frac{1}{\varepsilon} f(u_0^\varepsilon) \|_{H^\ell} \le C\, \varepsilon^{-\sigma_{2+\ell}}, \quad \ell = 0, 1, 2\,. \tag{1.13}$$

$$\lim_{s \to 0^+} \| \nabla u_t(s) \|_{L^2} \le C\, \varepsilon^{-\sigma_5}. \tag{1.14}$$

Our first main result for the case $f(u) = u^3 - u$ is the following one:

Theorem 1.1. *Let $\{(U^m, W^m)\}_{m=0}^M$ solve (1.9)-(1.10) on a quasi-uniform space mesh $\mathcal{T}_h$ of size $O(h)$, allowing for inverse inequalities and H^1-stability of the L^2-finite element projection, and a quasi-uniform time mesh J_k of size $O(k)$. Suppose the Hele-Shaw problem (1.4)-(1.8) has a global (in time) classical solution and u_0^ε be the ones constructed in [2]. Define for some $\nu > 0$*

$$\begin{aligned}
\mu := \mu(N, \nu) &= \min\big\{\, \nu, \tfrac{8-N}{8} \,\big\}\,,\\
\rho_1(\varepsilon, N) &:= \varepsilon \,\big(\varepsilon^{-\max\{2\sigma_1+7, 2\sigma_3+4\}}\\
&\qquad + \varepsilon^{-\frac{2}{6-N}\max\{2\sigma_1+5, 2\sigma_3+2\} - \max\{2\sigma_1+\frac{13}{2}, 2\sigma_3+\frac{7}{2}, 2\sigma_2+4\}} + \varepsilon^{-2\sigma_5}\big),\\
\rho_2(\varepsilon) &:= \varepsilon^{-\max\{\sigma_1+5, \sigma_3+\frac{7}{2}, \sigma_2+\frac{5}{2}, \sigma_4+1\}},\\
\rho_4(\varepsilon) &:= \varepsilon^{-\max\{2\sigma_1+\frac{13}{2}, 2\sigma_3+\frac{7}{2}, 2\sigma_2+4, 2\sigma_4\}}\,,\\
\rho_5(\varepsilon) &:= \varepsilon^{-\max\{2\sigma_1+9, 2\sigma_3+6, 2\sigma_2+4, 2\sigma_4+1\}},\\
\pi_1(k; \varepsilon, N, \sigma_i) &:= \rho_1(\varepsilon, N) + k^{\frac{16+(8-N)}{16-(8-N)}}\, \varepsilon^{-\frac{32(\sigma_1+3)+2(8-N)(2\sigma_1-1)}{16-(8-N)}}\\
&\qquad \times \rho_2(\varepsilon)^{\frac{4N}{16-(8-N)}}\,,\\
\pi_2(k, h; \varepsilon, N, \sigma_i, \nu) &:= \Big[\, h^{\frac{(8-N)}{4} - 2\mu}\, \varepsilon^{-\frac{(2\sigma_1+1)[16+(8-N)]}{16}} + h^{-2\mu} k^2 \rho_5(\varepsilon) \,\Big]\, \rho_2(\varepsilon)^{\frac{N}{8}}\\
&\qquad + \Big\{\, h^{2(1-\mu)} \,\big[\, \rho_2(\varepsilon)^{\frac{2(6-N)}{8-N}} + \rho_4(\varepsilon) \,\big] + h^{2(\nu-\mu)} \varepsilon^{-2(\sigma_2+1)} \Big\}\,,\\
r(h, k; \varepsilon, N, \sigma_i, \nu) &:= k^2\, \pi_1(k; \varepsilon, N, \sigma_i) + h^{2(2+\mu)}\, \pi_2(k, h; \varepsilon, N, \sigma_i, \nu)\,.
\end{aligned}$$

Then, under the following mesh and starting value constraints

1). $k \le \varepsilon^3$,
2). $h^2\, |\ln h| \le \varepsilon^2\, \rho_2(\varepsilon)^{-\frac{4}{8-N}}$,
3). $r(h, k; \varepsilon, N, \sigma_i, \nu) \le \varepsilon^{[2+\frac{48}{(8-N)}]} \rho_2(\varepsilon)^{-\frac{2N}{8-N}}$,
4). $(U^0, 1) = (u_0^\varepsilon, 1)$
5). $\| u_0^\varepsilon - U^0 \|_{H^{-1}} \le C\, h^{2+\nu} \| u_0^\varepsilon \|_{H^2}$,

the solution of (1.9)-(1.10) converges to the solution of (1.1)-(1.3) and satisfies for some positive constant $\tilde{C} = \tilde{C}(u_0^\varepsilon; C_0, T; \Omega)$

$$\text{(i)} \quad \max_{0 \le m \le M} \| u(t_m) - U^m \|_{H^{-1}} + \Big(k \sum_{m=0}^M k \| d_t(u(t_m) - U^m) \|_{H^{-1}}^2 \Big)^{\frac{1}{2}} \le \tilde{C}\, \big[\, r(h, k; \varepsilon, N, \sigma_i, \nu) \,\big]^{\frac{1}{2}}\,,$$

$$\text{(ii)} \quad \Big(k \sum_{m=0}^{M} \| u(t_m) - U^m \|_{L^2}^2\Big)^{\frac{1}{2}} \\ \leq \tilde{C} \Big\{ h^2 \, \varepsilon^{-(\sigma_1+\frac{1}{2})} + \varepsilon^{-2} \, \big[\, r(h,k;\varepsilon,N,\sigma_i,\nu) \, \big]^{\frac{1}{2}} \Big\},$$

$$\text{(iii)} \quad \Big(k \sum_{m=0}^{M} \| \nabla(u(t_m) - U^m) \|_{L^2}^2\Big)^{\frac{1}{2}} \\ \leq \tilde{C} \Big\{ h \, \varepsilon^{-(\sigma_1+\frac{1}{2})} + \varepsilon^{-2} \, \big[\, r(h,k;\varepsilon,N,\sigma_i,\nu) \, \big]^{\frac{1}{2}} \Big\}.$$

Moreover, if U^0 satisfies

$$\| u_0^\varepsilon - U^0 \|_{L^2} \leq C \, h^2 \| u_0^\varepsilon \|_{H^2} \, ,$$

then there also hold

$$\text{(iv)} \quad \max_{0\leq m\leq M} \| u(t_m) - U^m \|_{L^2} + \Big(k \sum_{m=0}^{M} k \| d_t(u(t_m) - U^m) \|_{L^2}^2\Big)^{\frac{1}{2}} \\ \leq \tilde{C} \Big\{ h^2 \varepsilon^{-\max\{\sigma_1+\frac{5}{2},\sigma_3+1\}} + k^{-\frac{1}{4}} \varepsilon^{-1} \big[r(h,k;\varepsilon,N,\sigma_i,\nu) \big]^{\frac{1}{2}} \Big\},$$

$$\text{(v)} \quad \max_{0\leq m\leq M} \| u(t_m) - U^m \|_{L^\infty} \\ \leq \tilde{C} \Big\{ h^2 \, |\ln h| \, \rho_2(\varepsilon)^{\frac{4}{8-N}} + h^{-\frac{N}{2}} k^{-\frac{1}{4}} \varepsilon^{-1} \big[r(h,k;\varepsilon,N,\sigma_i,\nu) \big]^{\frac{1}{2}} \Big\}.$$

Furthermore, if $k = O(h^q)$ for some $\frac{2N}{3} < q < (8-2N) + 4\mu$, then there hold the following additional estimates

$$\text{(vi)} \quad \max_{0\leq m\leq M} \| U^m \|_{L^\infty} \leq \; 3 \, C_0,$$

$$\text{(vii)} \quad \max_{0\leq m\leq M} \| u(t_m) - U^m \|_{L^2} + \Big(k \sum_{m=0}^{M} k \| d_t(u(t_m) - U^m) \|_{L^2}^2\Big)^{\frac{1}{2}} \\ + \Big(\frac{k}{\varepsilon} \sum_{m=0}^{M} \| w(t_m) - W^m) \|_{L^2}^2\Big)^{\frac{1}{2}} \\ \leq \tilde{C} \Big\{ h^2 \, \varepsilon^{-\max\{\sigma_1+\frac{5}{2},\sigma_3+1,\sigma_2\}} + \varepsilon^{\frac{7}{2}} \, \big[r(h,k;\varepsilon,N,\sigma_i,\nu) \big]^{\frac{1}{2}} \Big\},$$

$$\text{(viii)} \quad \max_{0\leq m\leq M} \| u(t_m) - U^m \|_{L^\infty} \\ \leq \tilde{C} \Big\{ h^2 \, |\ln h| \, \rho_2(\varepsilon)^{\frac{4}{8-N}} + h^{\frac{4-N}{2}} \varepsilon^{-\max\{\sigma_1+\frac{5}{2},\sigma_3+1,\sigma_2\}} \\ + h^{-\frac{N}{2}} \varepsilon^{-\frac{7}{2}} \big[r(h,k;\varepsilon,N,\sigma_i,\nu) \big]^{\frac{1}{2}} \Big\}.$$

In addition, for some $\beta > 1$, let W^0 be a value satisfying

$$\| w_0^\varepsilon - W^0 \|_{L^2} \leq C \, h^\beta,$$

Then $w(t_m) - W^m$ also satisfies

$$\text{(ix)} \quad \max_{0\leq m\leq M} \| w(t_m) - W^m \|_{L^2} + \Big(k \sum_{m=0}^{M} k \| d_t(w(t_m) - W^m) \|_{L^2}^2\Big)^{\frac{1}{2}} \\ \leq \tilde{C} \Big\{ h^2 \rho_2(\varepsilon) + k^{-\frac{1}{2}} \, \varepsilon^{-\frac{5}{2}} \, \big[\, r(h,k;\varepsilon,N,\sigma_i,\nu) \, \big]^{\frac{1}{2}} + h^\beta \Big\},$$

(x) $\max_{0\leq m\leq M} \| w(t_m) - W^m \|_{L^\infty}$
$$\leq \tilde{C}\left\{ h^{\frac{4-N}{2}} |\ln h|^{\frac{3-N}{2}} \rho_2(\varepsilon) + h^{-\frac{N}{2}} \big[k^{-\frac12} \varepsilon^{-\frac52} r(h,k;\varepsilon,N,\sigma_i,\nu)^{\frac12} + h^{\beta} \big] \right\}.$$

Remark: (a). Theorem 1.1 is the combination of Theorem 3.3, Theorem 3.4 and Corollary 3.5 in the case of the special potential function $F(u) = \frac14(u^2-1)^2$, or $f(u) = u^3 - u$.

(b). $L^2(J;H^1)$- and $L^\infty(J;L^2)$-estimates for U^m and the $L^2(J;L^2)$-estimate for W^m all are optimal with respect to h and k, and $L^\infty(J;H^{-1})$- and $L^\infty(J;L^\infty)$-estimates for U^m are quasi-optimal.

(c). Both, L^2-projection $Q_h u_0^\varepsilon$ and the elliptic projection $P_h u_0^\varepsilon$ of u_0^ε are valid choices for starting value U^0. Also, both $Q_h w_0^\varepsilon$ and $P_h w_0^\varepsilon$ are valid candidates for W^0.

To establish the above error estimates, the following three ingredients play a crucial role in our analysis.

- To establish stability estimates for the solution of the fully discrete scheme.
- To handle the (nonlinear) potential term in the error equation using a spectrum estimate result due to Alikakos and Fusco [3] and Chen [12] for the linearized Cahn-Hilliard operator

$$\mathcal{L}_{CH} := \Delta(\varepsilon\Delta - \frac{1}{\varepsilon} f'(u) I)\,, \tag{1.15}$$

  where I denotes the identity operator and u is a solution of the Cahn-Hilliard equation (1.1), see Proposition 2.5 for details.
- To establish a discrete counterpart of above spectrum estimate, see Proposition 3.2 for details.

The above $L^\infty\big(J;L^\infty(\Omega)\big)$ error estimate combined with the convergence result on the Cahn-Hilliard equation to the Hele-Shaw problem proved in [2] then immediately allows us to establish the convergence of the numerical solution of the fully discrete scheme to the solution of the Hele-Shaw problem.

Our second main result for the case $f(u) = u^3 - u$ is the following convergence theorem.

Theorem 1.2. *Let Ω be a given smooth domain and Γ_{00} be a smooth closed hypersurface in Ω. Suppose that the Hele-Shaw problem (1.4)-(1.8) starting from Γ_{00} has a classical solution $(w, \Gamma := \cup_{0\leq t\leq T}(\Gamma_t \times \{t\}))$ in the time interval $[0,T]$ such that $\Gamma_t \subset \Omega$ for all $t\in[0,T]$. Let $\{u_0^\varepsilon(x)\}_{0<\varepsilon\leq 1}$ be the family of smooth uniformly bounded functions as in Theorem 5.1 of [2]. Let $(U_{\varepsilon,h,k}(x,t), W_{\varepsilon,h,k}(x,t))$ denote the piecewise linear interpolation (in time) of the fully discrete solution $\{(U^m,W^m)\}_{m=0}^M$. Also, let $\mathcal{I}$ and $\mathcal{O}$ stand for the "inside" and "outside" (in Ω_T) of Γ. Then, under the mesh and starting value constraints of Theorem 1.1 with $\nu = 1$ and $k = O(h^q)$ for some $\frac{2N}{3} < q < (8-2N) + 4\mu$ we have*

$$\text{(i)}\quad U_{\varepsilon,h,k}(x,t) \overset{\varepsilon\searrow 0}{\longrightarrow} 1 \quad \textit{uniformly on compact subset of } \mathcal{O}\,,$$
$$\text{(ii)}\quad U_{\varepsilon,h,k}(x,t) \overset{\varepsilon\searrow 0}{\longrightarrow} -1 \quad \textit{uniformly on compact subset of } \mathcal{I}\,.$$

Moreover, when $N = 2$, let $k = O(h^q)$ for some $N < q < (4-N) + 2\mu$ and choose W^0 such that $\| w_0^\varepsilon - W^0 \|_{L^2} \leq C h^\beta$ for some $\beta > \frac{q}{2}$, then we also have

$$\text{(iii)}\quad W_{\varepsilon,h,k}(x,t) \overset{\varepsilon\searrow 0}{\longrightarrow} -w(x,t) \quad \textit{uniformly on } \overline{\Omega}_T\,.$$

Remark: It can be shown that the assertion (iii) also holds when $N = 3$ under a stronger constraint on the starting value U^0 (and some higher regularity assumptions on u), see Section 4 for more discussions.

Our third main result for the case $f(u) = u^3 - u$ is the following convergence theorem for the numerical interface.

THEOREM 1.3. *Let* $\Gamma_t^{\varepsilon,h,k} := \{x \in \Omega\,;\, U_{\varepsilon,h,k}(x,t) = 0\,\}$ *denote the zero level set of* $U_{\varepsilon,h,k}$. *Then under the assumptions (i) and (ii) of Theorem 1.2, we have*

$$\sup_{x\in\Gamma_t^{\varepsilon,h,k}} \bigl(dist\,(x,\Gamma_t)\bigr) \overset{\varepsilon\searrow 0}{\longrightarrow} 0 \quad \textit{uniformly on } [0,T]\,.$$

Our last main result for the case $f(u) = u^3 - u$ is the following convergence rate estimate result for numerical interface.

THEOREM 1.4. *Let* δ^* *be a positive constant such that* $dist\,(\Gamma_t, \partial\Omega) > 2\delta^*$ *for all* $t \in [0,T]$. *Then, under the assumptions for (i) and (ii) of Theorem 1.2, there holds*

$$\sup_{x\in\Gamma_t^{\varepsilon,h,k}} \bigl(dist\,(x,\Gamma_t)\bigr) \le \frac{\delta^*}{2} \quad \textit{uniformly on } [0,T]\,.$$

We remark that using a similar approach a parallel study was carried out by the authors in [20] for the Allen-Cahn equation and the related curvature driven flows. On the other hand, unlike the Allen-Cahn equation which is a gradient flow in L^2, the Cahn-Hilliard equation is a gradient flow only in H^{-1}, which makes the analysis for the Cahn-Hilliard equation in this paper much more delicate and complicated than that for the Allen-Cahn equation given in [20].

The paper is organized as follows: In Section 2, we shall derive some improved a priori estimates for the solution of (1.1)-(1.3) under the condition that the Hele-Shaw problem has a global (in time) classical solution. Special attention is given to dependence of the solution on ε in various norms. In Section 3, we analyze the fully discrete mixed finite element method proposed in [21] for the Cahn-Hilliard equation, which consists of the backward Euler discretization in time and the lowest order Ciarlet-Raviart mixed finite element (for the biharmonic operator) discretization in space. Optimal and quasi-optimal error estimates in stronger norms, including $L^\infty(J; L^\infty)$ norm, are obtained for the fully discrete solution. It is shown that all the error bounds depend on $\frac{1}{\varepsilon}$ only in low polynomial orders for small ε. Like in [20, 21], the spectrum estimate and its discrete counterpart play a crucial role in the proofs. Finally, Section 4 is devoted to establishing the convergence of the fully discrete solution to the solution of the Hele-Shaw problem. Using the $L^\infty(J; L^\infty)$ error estimate and the convergence result of [2], we show that the fully discrete numerical solution converges to the solution (including the free boundary) of the Hele-Shaw problem, provided that the latter admits a global (in time) classical solution.

2. Energy estimates for the differential problem. In this section, we derive some energy estimates in various function spaces up to $L^\infty(J; H^4(\Omega)) \cap H^1(J; H^3(\Omega))$ in terms of negative powers of ε for the solution u of the Cahn-Hilliard problem (1.1)-(1.3) for given $u_0^\varepsilon \in H^4(\Omega)$. The basic estimates are derived under general (minimum) regularities, while the improved estimates are established under the assumption that the Hele-Shaw problem admits a global (in time) solution. The same notation as in Part I of the series [21] are used throughout this paper, in particular, $\Delta^{-1}v$ and $\Delta^{-\frac{1}{2}}v$

are same as there. For the detailed definitions, we refer to Section 2 of [21]. Again, C and $\widetilde{C}$ are used to denote generic positive constants which are independent of ε and the time and space mesh sizes k and h.

In this paper, we are mainly concerned with the second stage of the evolution of the concentration u, that is, the motion of the interface, and focus on approximating the Hele-Shaw problem via the Cahn-Hilliard equation discretized by a fully discrete mixed finite element method. For these purposes, we rewrite (1.1)-(1.3) as

$$u_t = \Delta w \qquad \text{in } \ \Omega_T\,, \tag{2.1}$$

$$w = \frac{1}{\varepsilon} f(u) - \varepsilon\, \Delta u \quad \text{in } \ \Omega_T\,, \tag{2.2}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \ \partial\Omega \times (0,T)\,, \tag{2.3}$$

$$u(x,0) = u_0^\varepsilon(x) \qquad \forall\, x \in \Omega\,, \tag{2.4}$$

where w physically represents the chemical potential. We refer to [18, 7] and references therein for more discussions on well-posedness and regularities of the Cahn-Hilliard and the biharmonic problems.

Like in [20, 21], we consider the following general double equal-well potential function F:

General Assumption 1 (GA_1)

1) $f = F'$, for $F \in C^4(\mathbf{R})$, such that $F(\pm 1) = 0$, and $F > 0$ elsewhere.
2) $f'(u)$ satisfies for some finite $p > 2$ and positive numbers $\tilde{c}_i > 0$, $i = 0, .., 3$,

$$\tilde{c}_1 \,|\, u \,|^{p-2} - \tilde{c}_0 \le f'(u) \le \tilde{c}_2 \,|\, u \,|^{p-2} + \tilde{c}_3\,.$$

3) There exist $0 < \gamma_1 \le 1$, $\gamma_2 > 0$, $\delta > 0$ and $C_0 > 0$ such that

$$\begin{aligned} &\text{(i)} \quad \big(f(a) - f(b), a-b\big) \ge \gamma_1 \big(f'(a)(a-b), a-b\big) \\ &\qquad\qquad -\gamma_2 \,|\, a-b \,|^{2+\delta} \quad \text{for all } |a| \le 2C_0\,, \\ &\text{(ii)} \quad a f''(a) \ge 0 \quad \text{for all } |a| \ge C_0\,. \end{aligned}$$

Remark: We note that the above (GA_1) differs slightly from those of [21] in 2) and 3). It is trivial to check that $(\mathrm{GA}_1)_2$ implies

$$-(f'(u)v, v) \le \tilde{c}_0 \,\|\, v \,\|_{L^2}^2\,, \quad \forall\, v \in L^2(\Omega)\,, \tag{2.5}$$

which will be utilized several times in the paper.

Example: The potential function $F(u) = \frac{1}{4}(u^2-1)^2$, consequently, $f(u) = u^3 - u$, is often used in physical and geometrical applications [4, 9, 5, 2, 13]. For convenience, we verify $(\mathrm{GA}_1)_1$-$(\mathrm{GA}_1)_3$. First, $(\mathrm{GA}_1)_1$ holds trivially. Since $f'(u) = 3u^2 - 1$, $(\mathrm{GA}_1)_2$ holds with $\tilde{c}_1 = \tilde{c}_2 = 3$ and $\tilde{c}_0 = \tilde{c}_3 = 1$. A direct calculation gives

$$f(a) - f(b) = (a-b)\big[f'(a) + (a-b)^2 - 3(a-b)a\big]\,. \tag{2.6}$$

Hence, $(\mathrm{GA}_1)_3$ holds with $\gamma_1 = 1, \gamma_2 = 3$, $\delta = 1$ and any constant $C_0 \ge 0$. Also, (2.5) holds with $\tilde{c}_0 = 1$.

In the rest of this section, we shall establish some basic and improved a priori estimates for the solution of the Cahn-Hilliard equation under the assumption that the Hele-Shaw problem (1.4)-(1.8) has a global (in time) classical solution (cf. [21]).

These improved a priori estimates are necessary for us to obtain error estimates in stronger norms in the next section.

LEMMA 2.1. *Suppose that f satisfies (GA_1), then the solution of (1.1)-(1.3) satisfies the following estimates:*

$$\text{(i)} \qquad \operatorname*{ess\,sup}_{[0,\infty)} \Big\{ \frac{\varepsilon}{2} \, \| \nabla u \|_{L^2}^2 + \frac{1}{\varepsilon} \, \| F(u) \|_{L^1} \Big\} + \left\{ \begin{array}{l} \int_0^\infty \| u_t \|_{H^{-1}}^2 \, \mathrm{d}s \\ \\ \int_0^\infty \| \nabla w \|_{L^2}^2 \, \mathrm{d}s \end{array} \right\} = \mathcal{J}_\varepsilon(u_0^\varepsilon) \, ,$$

$$\text{(ii)} \qquad \operatorname*{ess\,sup}_{[0,\infty)} \| u \|_{L^p}^p \leq C \Big(1 + \mathcal{J}_\varepsilon(u_0^\varepsilon) \Big) \qquad (p \text{ as in } (GA_1)_2) \, ,$$

$$\text{(iii)} \qquad \operatorname*{ess\,sup}_{[0,\infty)} \| \, |u| - 1 \, \|_{L^2}^2 \leq C \, \varepsilon \, \mathcal{J}_\varepsilon(u_0^\varepsilon) \, .$$

Proof. The assertion (i) is the immediate consequence of the basic energy law associated with the Cahn-Hilliard equation

$$\frac{d}{dt} \mathcal{J}_\varepsilon(u(t)) = \left\{ \begin{array}{l} -\| \, u_t(t) \, \|_{H^{-1}}^2 \, , \\ \\ -\| \, \nabla w(t) \, \|_{L^2}^2 , \end{array} \right. \tag{2.7}$$

where

$$\mathcal{J}_\varepsilon(u) := \int_\Omega \Big[\varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} F(u) \Big] \, \mathrm{d}x \qquad \forall \, t \geq 0 \, . \tag{2.8}$$

The conclusions of (ii) and (iii) are in turn the immediate corollaries of (i). ◻

The next proposition is a corollary of Theorem 2.1 and 2.3 of [2]. It shows the boundedness of the solution of the Cahn-Hilliard equation, provided that the Hele-Shaw problem (1.4)-(1.8) has a global (in time) classical solution. This boundedness result is the key for us to be able to establish improved a priori estimates for the solution of the Cahn-Hilliard equation. We remark that the estimates in [21] were obtained without assuming existence of a global (in time) classical solution for the Hele-Shaw problem, hence, we could not be able to show the boundedness of the solution of the Cahn-Hilliard equation there.

PROPOSITION 2.2. *Suppose that f satisfies (GA_1), and the Hele-Shaw problem (1.4)-(1.8) has a global (in time) classical solution. Then there exists a family of smooth initial datum functions $\{u_0^\varepsilon\}_{0<\varepsilon\leq 1}$ and constants $\varepsilon_0 \in (0,1]$ and $C_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ the solution u of the Cahn-Hilliard equation (1.1)-(1.3) with the above initial data u_0^ε satisfies*

$$\| \, u \, \|_{L^\infty(\Omega_T)} \leq \frac{3}{2} \, C_0 \, . \tag{2.9}$$

Proof. A proof of the assertion is buried in the middle of the proof of Theorem 2.3 of [2]. In fact, the assertion of Theorem 2.3 of [2] was proved by establishing (2.9) first. Here we only sketch the main idea of the proof.

First, using a matched asymptotic expansion technique, a family of smooth approximate solutions $(u_A^\varepsilon, w_A^\varepsilon)$ to the solution (u, w) of (2.1)-(2.4) satisfying the assumption of Theorem 2.1 of [2] was constructed in Section 4 of [2]. One condition is $\| \, u_A^\varepsilon \, \|_{L^\infty(\Omega_T)} \leq C_0$ for some $C_0 > 0$. Second, it was proved in Theorem 2.1 of [2] that

$(u_A^\varepsilon, w_A^\varepsilon)$ is very "close" to (u, w) in $L^p(\Omega_T)$ for some $p > 2$ (see (2.7) on page 169 of [2]). Finally, (2.9) was proved using a regularization argument. The argument goes as follows in three steps: (i) f is modified into $\overline{f}$ such that $\overline{f} = f$ in $(-\frac{3}{2}\,C_0, \frac{3}{2}\,C_0)$ and $\overline{f}$ is linear for $|u| > 2\,C_0$; (ii) It was shown that the solution $\overline{u}$ of the Cahn-Hilliard equation with the new nonlinearity $\overline{f}$ satisfies the estimate (2.9) when $\varepsilon \in (0, \varepsilon_0)$ for some small $\varepsilon_0 \in (0, 1]$; (iii) It follows from the uniqueness of the solution of the Cahn-Hilliard equation that $u \equiv \overline{u}$. $\square$

Remark: As in [2], the result of Proposition 2.2 is proved for a special family of initial data $\{u_0^\varepsilon(x)\}_{0<\varepsilon\leq 1}$. On the other hand, as explained in the introduction of [2], this is not a serious restriction for the purpose of approximating the Hele-Shaw problem since (i) at the end of the first stage of the evolution of the concentration, u has the required profile, and (ii) the solution of the Hele-Shaw problem (1.4)-(1.8) depends only on Γ_{00} and Ω.

The next lemma states a Poincaré-Friedrichs' type inequality for any function w which has the form (2.2), it was first proved in Lemma 3.4 of [13]. We note that u^ε in the lemma does not have to be the solution of the Cahn-Hilliard equation.

Lemma 2.3. *Suppose that u^ε satisfies*

$$\frac{1}{|\Omega|}\int_\Omega u^\varepsilon(t)\,\mathrm{d}x = m_0 \in (-1, 1) \quad \forall\, t \geq 0\,, \tag{2.10}$$

where m_0 is independent of ε. Let $\mathcal{J}_\varepsilon(u^\varepsilon)$ be defined by (2.8) and w^ε be defined by (2.2). Then there exist a (large) positive constant C and a (small) positive constant ε_0 such that for every $\varepsilon \in (0, \varepsilon_0]$

$$\|\, w^\varepsilon(\cdot, t)\,\|_{L^2} \leq C\Big(\mathcal{J}_\varepsilon(u^\varepsilon(\cdot, t)) + \|\,\nabla w^\varepsilon(\cdot, t)\,\|_{L^2}\Big) \qquad \forall\, t \geq 0\,. \tag{2.11}$$

To derive a priori estimates in high norms we need to require u_0^ε satisfies the following conditions:

General Assumption 2 (GA$_2$)

There exist positive ε-independent constants m_0 and σ_j for $j = 1, 2, \cdots, 4$ such that

$$\begin{aligned}
&1) \quad m_0 := \frac{1}{|\Omega|}\int_\Omega u_0^\varepsilon(x)\,\mathrm{d}x \,\in (-1, 1)\,,\\
&2) \quad \mathcal{J}_\varepsilon(u_0^\varepsilon) = \frac{\varepsilon}{2}\,\|\,\nabla u_0^\varepsilon\,\|_{L^2}^2 + \frac{1}{\varepsilon}\,\|\,F(u_0^\varepsilon)\,\|_{L^1} \leq C\,\varepsilon^{-2\sigma_1}\,,\\
&3) \quad \|\, w_0^\varepsilon\,\|_{H^\ell} := \|\,\varepsilon\,\Delta u_0^\varepsilon - \frac{1}{\varepsilon}\,f(u_0^\varepsilon)\,\|_{H^\ell} \leq C\,\varepsilon^{-\sigma_{2+\ell}}, \quad \ell = 0, 1, 2.
\end{aligned}$$

Lemma 2.4. *Suppose f satisfies (GA$_1$), u_0^ε satisfies (GA$_2$), and $\partial\Omega$ is of class $C^{3,1}$. Assume the solution u of (1.1)-(1.3) satisfies (2.9). Then (u, w) satisfies the* *following estimates:*

$$\text{(i)}\qquad \frac{1}{|\Omega|}\int_\Omega u(t)\,\mathrm{d}x = m_0 \in (-1,1) \quad \forall\, t \geq 0\,,$$

$$\text{(ii)}\qquad \int_0^\infty \|\,\Delta u\,\|_{L^2}^2\,\mathrm{d}s \leq C\,\varepsilon^{-(2\sigma_1+3)}\,,$$

$$\text{(iii)}\qquad \int_0^\infty \|\,\nabla\Delta u\,\|_{L^2}^2\,\mathrm{d}s \leq C\,\varepsilon^{-(2\sigma_1+5)}\,,$$

$$\text{(iv)}\qquad \operatorname*{ess\,sup}_{[0,\infty)} \left\{ \begin{array}{c} \|\,u_t\,\|_{H^{-1}}^2 \\ \\ \|\,\nabla w\,\|_{L^2}^2 \end{array} \right\} + \varepsilon \int_0^\infty \|\,\nabla u_t\,\|_{L^2}^2\,\mathrm{d}s \leq C\,\varepsilon^{-\max\{2\sigma_1+3,2\sigma_3\}}\,,$$

$$\text{(v)}\qquad \operatorname*{ess\,sup}_{[0,\infty)} \|\,\Delta u\,\|_{L^2} \leq C\,\varepsilon^{-\max\{\sigma_1+\frac{5}{2},\sigma_3+1\}}\,,$$

$$\text{(vi)}\qquad \operatorname*{ess\,sup}_{[0,\infty)} \|\,\nabla\Delta u\,\|_{L^2} \leq C\,\varepsilon^{-\max\{\sigma_1+\frac{5}{2},\sigma_3+1\}}\,,$$

$$\text{(vii)}\qquad \left\{ \begin{array}{c} \int_0^\infty \|\,u_t\,\|_{L^2}^2\,\mathrm{d}s \\ \\ \int_0^\infty \|\,\Delta w\,\|_{L^2}^2\,\mathrm{d}s \end{array} \right\} + \operatorname*{ess\,sup}_{[0,\infty)} \varepsilon \|\,\Delta u\,\|_{L^2}^2 \leq C\,\varepsilon^{-\max\{2\sigma_1+\frac{7}{2},2\sigma_3+\frac{1}{2},2\sigma_2+1\}}\,,$$

$$\text{(viii)}\qquad \operatorname*{ess\,sup}_{[0,\infty)} \|\,u_t\,\|_{L^2}^2 + \varepsilon \int_0^\infty \|\,\Delta u_t\,\|_{L^2}^2\,\mathrm{d}s \leq C\,\varepsilon^{-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4,2\sigma_4\}}\,.$$

Moreover, in addition to (GA$_2$) suppose that there exists $\sigma_5 > 0$ such that

$$\lim_{s\to 0^+} \|\nabla u_t(s)\|_{L^2} \leq C\varepsilon^{-\sigma_5}\,, \tag{2.12}$$

then the solution of (1.1)-(1.3) also satisfies the following estimates: for $N = 2,3$

$$\begin{aligned} \text{(ix)}\quad & \operatorname*{ess\,sup}_{[0,\infty)} \|\,\nabla u_t\,\|_{L^2}^2 + \varepsilon \int_0^\infty \|\,\nabla\Delta u_t\,\|_{L^2}^2\,\mathrm{d}s \\ & \leq C\,\Big(\varepsilon^{-\max\{2\sigma_1+7,2\sigma_3+4\}} + \varepsilon^{-\frac{2}{6-N}\max\{2\sigma_1+5,2\sigma_3+2\}-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4\}} \\ & \qquad + \varepsilon^{-2\sigma_5}\Big) \equiv C\,\rho_0(\varepsilon,N)\,, \end{aligned}$$

$$\text{(x)}\quad \int_0^\infty \|\,u_{tt}\,\|_{H^{-1}}^2\,\mathrm{d}s \leq C\,\varepsilon\rho_0(\varepsilon,N) \equiv C\rho_1(\varepsilon,N)\,,$$

$$\text{(xi)}\quad \operatorname*{ess\,sup}_{[0,\infty)} \|\,\Delta^2 u\,\|_{L^2} \leq C\,\varepsilon^{-\max\{\sigma_1+5,\sigma_3+\frac{7}{2},\sigma_2+\frac{5}{2},\sigma_4+1\}} \equiv C\,\rho_2(\varepsilon)\,.$$

Proof. Since most of the proofs follow the same line as in the proofs of Lemma 2.1 and 2.2 of [21], we only point out the main steps and differences, in particular, we like to mention that (2.9) holds under the assumption that the Hele-Shaw problem (1.4)-(1.8) has a global (in time) classical solution, it will be used below in the proofs of (iii)-(xi).

The proof of (i) is trivial. The assertion (ii) is obtained from testing (1.1) with u and applying (2.5) to the nonlinear term. To show (iii), we write from (2.2)

$$\Delta u = \frac{1}{\varepsilon^2}\,f(u) - \frac{1}{\varepsilon}\,w\,. \tag{2.13}$$

The assertion follows from applying the operator ∇ to (2.13), assumption $(\mathrm{GA}_1)_2$, triangle inequality, (2.9) and (i) of Lemma 2.1.

To derive the remaining estimates, we differentiate (1.1) with respect to t

$$u_{tt} + \varepsilon\,\Delta^2 u_t - \frac{1}{\varepsilon}\,\Delta\big(f'(u)u_t\big) = 0\,. \tag{2.14}$$

The assertion (iv) results from testing (2.14) with $-\Delta^{-1}u_t$ and using (2.5) as well as interpolating $\|\, u_t \,\|_{L^2}$ by $\|\, u_t \,\|_{H^{-1}}$ and $\|\, u_t \,\|_{H^1}$. Notice that $\|\, \nabla w \,\|_{L^2} = \|\, u_t \,\|_{H^{-1}}$ in view of (2.1) and the fact that $(u_t, 1) = 0$. Assertion (v) follows from (2.13) and using the triangle inequality, (2.9), (2.11), Lemma 2.1, (i) and the assertion (iv). Then (vi) is obtained similarly after first applying the operator ∇ to (2.13). Testing (1.1) with u_t, and using (i) of Lemma 2.1 and the assertion (iv) give (vii). Now (viii) follows from testing (2.14) with u_t, integrating by parts on the nonlinear term and using the assertion (vii).

To show (ix), we multiply (2.14) by $-\Delta u_t$ to get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\, \nabla u_t \,\|_{L^2}^2 + \varepsilon\|\, \nabla\Delta u_t \,\|_{L^2}^2 &= \frac{1}{\varepsilon}(\nabla(f'(u)u_t), \nabla\Delta u_t) \\
&= \frac{1}{\varepsilon}(f''(u)\nabla u u_t + f'(u)\nabla u_t, \nabla\Delta u_t) \\
&\le \frac{\varepsilon}{2}\|\, \nabla\Delta u_t \,\|_{L^2}^2 + \frac{C}{\varepsilon^3}\Big(\|\, \nabla u \,\|_{L^\infty}^2 \|\, u_t \,\|_{L^2}^2 + \|\, \nabla u_t \,\|_{L^2}^2\Big)\,.
\end{aligned}$$

Using the following Gagliardo-Nirenberg inequality [1],

$$\|\, \nabla u \,\|_{L^\infty} \le C\left(\|\, \nabla\Delta u \,\|_{L^2}^{\frac{2}{6-N}} \|\, u \,\|_{L^\infty}^{\frac{4-N}{6-N}} + \|\, u \,\|_{L^\infty}\right) \tag{2.15}$$

we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\, \nabla u_t \,\|_{L^2}^2 + \frac{\varepsilon}{2}\|\, \nabla\Delta u_t \,\|_{L^2}^2 \\
&\qquad \le \frac{C}{\varepsilon^3}\Big[\|\, \nabla u_t \,\|_{L^2}^2 + \Big(\|\, \nabla\Delta u \,\|_{L^2}^{\frac{4}{6-N}} + 1\Big)\|\, u_t \,\|_{L^2}^2\Big].
\end{aligned} \tag{2.16}$$

The assertion follows from (2.16) and (iv), (vi) and (vii).

From (2.14) we have

$$\begin{aligned}
\|\, u_{tt} \,\|_{H^{-1}} &= \sup_{0\not\equiv\phi\in H^1} \frac{< u_{tt}, \phi >}{\|\, \phi \,\|_{H^1}} \\
&\le \varepsilon\|\, \nabla\Delta u_t \,\|_{L^2} + \frac{1}{\varepsilon}\|\, \nabla(f'(u)u_t) \,\|_{L^2}\,.
\end{aligned}$$

Hence

$$\|\, u_{tt} \,\|_{H^{-1}}^2 \le 2\varepsilon^2\|\, \nabla\Delta u_t \,\|_{L^2}^2 + \frac{2}{\varepsilon^2}\Big(\|\, \nabla u \,\|_{L^\infty}^2 \|\, u_t \,\|_{L^2}^2 + \|\, \nabla u_t \,\|_{L^2}^2\Big)\,.$$

Statement (x) then follows from integrating the above inequality over $(0, T)$ and using (2.15) and the assertion (ix).

Finally, we rewrite (1.1) as

$$\begin{aligned}
\Delta^2 u &= \frac{1}{\varepsilon^2}\Delta f(u) - \frac{1}{\varepsilon}u_t \\
&= \frac{1}{\varepsilon^2}\Big[f''(u)\Delta u + f'(u)|\nabla u|^2\Big] - \frac{1}{\varepsilon}u_t\,.
\end{aligned}$$

The assertion (xi) follows from using the triangle inequality, (2.9), and the assertions (v), (vi) and (viii). The proof is complete. □

Remark: From the construction of $(u_A^\varepsilon, w_A^\varepsilon)$ in Section 4 of [2] we know that $\{u_0^\varepsilon\}_{0<\varepsilon\leq 1}$ obtained in Proposition 2.2 satisfy (GA_2). In addition, the corresponding solution u satisfies (2.9), see Proposition 2.2, provided that the Hele-Shaw problem (1.4)-(1.8) has a global (in time) classical solution.

We conclude this section by citing the following result of [3, 12] on low bound estimate of the spectrum of the linearized Cahn-Hilliard operator $\mathcal{L}_{CH}$ in (1.15). The estimate plays an important role in our error analysis.

PROPOSITION 2.5. *Suppose that (GA_1) holds. Then there exists a positive constant C_0 such that the principle eigenvalue of the linearized Cahn-Hilliard operator $\mathcal{L}_{CH}$ in (1.15) satisfies for small $\varepsilon > 0$*

$$\lambda_{CH} \equiv \inf_{0\not\equiv\psi\in H^1(\Omega)} \frac{\varepsilon \,\|\, \nabla\psi \,\|_{L^2}^2 + \frac{1}{\varepsilon}\,(f'(u)\psi, \psi)}{\|\, \Delta^{-\frac{1}{2}}\psi \,\|_{L^2}^2} \geq -C_0\,,$$

or equivalently

$$\lambda_{CH} \equiv \inf_{\substack{0\not\equiv\psi\in H^1(\Omega)\\ \Delta w=\psi}} \frac{\varepsilon \,\|\, \nabla\psi \,\|_{L^2}^2 + \frac{1}{\varepsilon}\,(f'(u)\psi, \psi)}{\|\, \nabla w \,\|_{L^2}^2} \geq -C_0\,.$$

3. Error analysis for a fully discrete mixed finite element approximation. In this section we analyze the fully discrete mixed finite element method proposed in [21] for (2.1)-(2.4) under the condition that the Hele-Shaw problem has a global (in time) classical solution (cf. [2]). Under this assumption, we establish stronger error bounds than those of [21], which were shown under general (minimum) regularity assumptions, for the solution of the fully discrete mixed finite element method. In particular, we obtain $L^\infty(J; L^\infty)$-error estimate, which is necessary for us to establish the convergence to the solution of the fully discrete mixed finite element scheme to the solution of the Hele-Shaw problem in the next section.

We recall that the weak formulation of (2.1)-(2.4) is defined as: Find $(u, w) \in [H^1(\Omega)]^2$ such that for almost every $t \in (0, T)$

$$(u_t, \eta) + (\nabla w, \nabla\eta) = 0 \qquad \forall\, \eta \in H^1(\Omega)\,, \tag{3.1}$$

$$\varepsilon\,(\nabla u, \nabla v) + \frac{1}{\varepsilon}\,(f(u), v) = (w, v) \quad \forall\, v \in H^1(\Omega)\,, \tag{3.2}$$

$$u(x, 0) = u_0^\varepsilon(x) \quad \forall\, x \in \Omega\,. \tag{3.3}$$

Note that $(u_t, 1) = 0$, that is, the mass $\big(u(t), 1\big) = \big(u_0^\varepsilon, 1\big)$ is conserved for all $t \geq 0$.

We also recall that the fully discrete mixed finite element discretization of (3.1)-(3.3) is defined as: Find $\big\{(U^m, W^m)\big\}_{m=1}^M \in \big[\mathcal{S}_h\big]^2$ such that

$$(d_t U^{m+1}, \eta_h) + (\nabla W^{m+1}, \nabla\eta_h) = 0 \qquad \forall\, \eta_h \in \mathcal{S}_h\,, \tag{3.4}$$

$$\varepsilon\,(\nabla U^{m+1}, \nabla v_h) + \frac{1}{\varepsilon}\,(f(U^{m+1}), v_h) = (W^{m+1}, v_h) \quad \forall\, v_h \in \mathcal{S}_h\,, \tag{3.5}$$

with some suitable starting value U^0. Here $J_k := \{t_m\}_{m=0}^M$ is a quasi-uniform partition of $[0, T]$ of mesh size $k := \frac{T}{M}$ and $\mathcal{T}_h$ be a quasi-uniform “triangulation” of Ω. Also,

$d_tU^{m+1} := (U^{m+1} - U^m)/k$ and $\mathcal{S}_h$ denotes the P_1 conforming finite element space defined by

$$\mathcal{S}_h := \Big\{ v_h \in C(\overline{\Omega})\,; v_h|_K \in P_1(K),\ \forall\, K \in \mathcal{T}_h \Big\}\,.$$

The mixed finite element space $\mathcal{S}_h \times \mathcal{S}_h$ is the lowest order element among a family of stable mixed finite spaces known as the Ciarlet-Raviart mixed finite elements for the biharmonic problem (cf. [14, 24]). In fact, it is not hard to check that the following *inf-sup* condition holds

$$\inf_{0\not\equiv\eta_h\in\mathcal{S}_h} \sup_{0\not\equiv\psi_h\in\mathcal{S}_h} \frac{(\nabla\psi_h, \nabla\eta_h)}{\|\,\psi_h\,\|_{H^1}\,\|\,\eta_h\,\|_{H^1}} \geq c_0 \tag{3.6}$$

for some $c_0 > 0$.

Also, we note that $(d_tU^{m+1}, 1) = 0$, which implies that $(U^{m+1}, 1) = (U^0, 1)$ for $m = 0, 1, \cdot, M - 1$. Hence, the mass is also conserved by the fully discrete solution at each time step.

We define the $L^2(\Omega)$-projection $Q_h : L^2(\Omega) \to \mathcal{S}_h$ by

$$(Q_hv - v, \eta_h) = 0 \quad \forall\, \eta \in \mathcal{S}_h\,, \tag{3.7}$$

and the elliptic projection $P_h : H^1(\Omega) \to \mathcal{S}_h$ by

$$(\nabla[P_hv - v], \nabla\eta_h) = 0 \quad \forall\, \eta_h \in \mathcal{S}_h\,, \tag{3.8}$$

$$(P_hv - v, 1) = 0\,. \tag{3.9}$$

We refer to Section 4 of [20] for a list of approximation properties of Q_h and P_h. In the sequel, we confine to the meshes $\mathcal{T}_h$ that result in H^1-stability of the Q_h, see [10] and reference therein for the details.

We also introduce space notation

$$\begin{aligned} \mathring{\mathcal{S}}_h &:= \Big\{ v_h \in \mathcal{S}_h\,; (v_h, 1) = 0 \Big\}\,, \\ L^2_0(\Omega) &:= \Big\{ v \in L^2(\Omega)\,; (v, 1) = 0 \Big\}\,, \end{aligned}$$

and define the discrete inverse Laplace operator: $-\Delta_h^{-1} : L^2_0(\Omega) \to \mathring{\mathcal{S}}_h$ such that

$$\big(\nabla(-\Delta_h^{-1}v), \nabla\eta_h\big) = (v, \eta_h) \quad \forall\, \eta_h \in \mathcal{S}_h\,. \tag{3.10}$$

To establish stability estimates for the solution of the fully discrete scheme (3.4)-(3.5) for general potential function $F(u)$, we make the last structural assumption on $f(u)$.

General Assumption 3 (GA_3)

Suppose that there exists $\alpha_0 \geq 0$, $0 < \gamma_3 < 1$, and $\tilde{c}_4 > 0$ such that f satisfies for any $0 < k \leq \varepsilon^{\alpha_0}$ and any set of discrete (in time) functions $\{\phi^m\}_{m=0}^M \in H^1(\Omega)$

$$\begin{aligned} \gamma_3 k \sum_{m=1}^{\ell} \Big(\|\, d_t\phi^m \,\|_{H^{-1}}^2 + k\varepsilon \|\, \nabla d_t\phi^m \,\|_{L^2}^2 \Big) \\ + \frac{k}{\varepsilon} \sum_{m=1}^{\ell} \big(f(\phi^m), d_t\phi^m \big) + \tilde{c}_4 \mathcal{J}_\varepsilon(\phi^0) \geq \frac{\tilde{c}_4}{\varepsilon} \|\, F(\phi^\ell) \,\|_{L^1} \quad \forall\, \ell \leq M. \end{aligned} \tag{3.11}$$

We remark that the validity of (GA_3) was proved in [21] for the case of the quartic potential $F(u) = \frac{1}{4}(u^2-1)^4$ with $\alpha_0 = 3, \gamma_3 = \frac{1}{4}$ and $\tilde{c}_4 = 2$. With help of (GA_3) we are able to show that the solution of (3.4)-(3.5) satisfies the following stability estimates.

Lemma 3.1. *The solution* $\big\{(U^m, W^m)\big\}_{m=0}^M$ *of (3.4)-(3.5) satisfies*

(i) $$\frac{1}{|\Omega|}\int_\Omega U^m \, \mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega U^0 \, \mathrm{d}x\,, \quad m = 1,2,\cdots,M,$$

(ii) $$\| \, d_t U^m \, \|_{H^{-1}} \leq C \, \| \, \nabla W^m \, \|_{L^2}, \quad m = 1,2,\cdots,M,$$

(iii) $$\max_{0\le m\le M}\Big\{\varepsilon \, \| \, \nabla U^m \, \|_{L^2}^2 + \frac{1}{\varepsilon} \, \| \, F(U^m) \, \|_{L^1}\Big\} + k\sum_{m=1}^M \Big(\| \, \nabla W^m \, \|_{L^2}^2 + \varepsilon k \, \| \, \nabla d_t U^m \, \|_{L^2}^2 \Big) \leq C\mathcal{J}_\varepsilon(U^0)\,,$$

(iv) $$k\sum_{m=1}^M \| \, d_t U^m \, \|_{H^{-1}}^2 \leq C\mathcal{J}_\varepsilon(U^0)\,,$$

(v) $$\max_{0\le m\le M} \| \, U^m \, \|_{L^p} \leq C\,\Big(1 + \mathcal{J}_\varepsilon(U^0)\Big) \quad (p \textit{ as in } (GA_1)_2),$$

(vi) $$\operatorname*{ess\,sup}_{0\le m\le M} \| \, \nabla W^m \, \|_{L^2}^2 + k\sum_{m=1}^M \Big(k \| \, d_t \nabla W^m \, \|_{L^2}^2 + \varepsilon \| \, \nabla d_t U^m \, \|_{L^2}^2 \Big) \leq C\,\varepsilon^{-3}\mathcal{J}_\varepsilon(U^0)\,,$$

(vii) $$\operatorname*{ess\,sup}_{0\le m\le M} \| \, d_t U^m \, \|_{L^2}^2 + k\sum_{m=1}^M k \| \, d_t^2 U^m \, \|_{L^2}^2 \leq C\,(k\varepsilon)^{-\frac{1}{2}}\varepsilon^{-3}\mathcal{J}_\varepsilon(U^0)\,.$$

Proof. The assertion (i) is an immediate consequence of setting $\eta_h = 1$ in (3.4).

For any $\phi \in H^1(\Omega)$, from (3.4), (3.7), and the stability of Q_h in $H^1(\Omega)$ (cf. [10] and references therein) we have

$$\begin{aligned} (d_t U^m, \phi) &= (d_t U^m, Q_h\phi) + (d_t U^m, \phi - Q_h\phi) \\ &= (\nabla W^m, \nabla Q_h\phi) \leq C \, \| \, \nabla W^m \, \|_{L^2} \| \, \nabla\phi \, \|_{L^2}\,. \end{aligned} \tag{3.12}$$

Assertion (ii) then follows from

$$\| \, d_t U^m \, \|_{H^{-1}} = \sup_{0\not\equiv\phi\in H^1} \frac{(d_t U^m, \phi)}{\| \, \phi \, \|_{H^1}} \leq C \, \| \, \nabla W^m \, \|_{L^2}\,.$$

To show assertion (iii), setting $\eta_h = W^{m+1}$ in (3.4) and $v_h = d_t U^{m+1}$ in (3.5) and adding the resulting equations give

$$\| \, \nabla W^{m+1} \, \|_{L^2}^2 + \frac{\varepsilon}{2}\frac{d}{dt} \| \, \nabla U^{m+1} \, \|_{L^2}^2 + \frac{\varepsilon k}{2} \| \, d_t \nabla U^{m+1} \, \|_{L^2}^2 + \frac{1}{\varepsilon}(f(U^{m+1}), d_t U^{m+1}) = 0\,. \tag{3.13}$$

The statement then follows from (GA_3) and (ii) after multiplying (3.13) by k and taking sum over m from 0 to $\ell \, (\leq M-1)$.

The assertions (iv) and (v) are immediate consequences of (ii) and (iii). To show (vi), choose $\eta_h = d_t W^{m+1}$ in (3.4) and $v_h = d_t U^{m+1}$ in (3.5) after applying the difference operator d_t to (3.5), and add the resulting equations we get

$$d_t\| \nabla W^{m+1} \|_{L^2}^2 + \frac{k}{2}\| d_t\nabla W^{m+1} \|_{L^2}^2 + \varepsilon\| \nabla d_t U^{m+1} \|_{L^2}^2 + \frac{1}{\varepsilon}(d_t f(U^{m+1}), d_t U^{m+1}) = 0. \tag{3.14}$$

By the Mean Value Theorem and (2.5) we bound the last term on the left hand side of (3.14) as follows,

$$\begin{aligned} \frac{1}{\varepsilon}(d_t f(U^{m+1}), d_t U^{m+1}) &= \frac{1}{\varepsilon}\big(f'(\xi), |d_t U^{m+1}|^2\big) \\ &\geq -\frac{\widetilde{c}_0}{\varepsilon}\| d_t U^{m+1} \|_{L^2}^2 \\ &\geq -\frac{\varepsilon}{2}\| \nabla d_t U^{m+1} \|_{L^2}^2 - \frac{C}{\varepsilon^3}\| d_t U^{m+1} \|_{H^{-1}}^2 . \end{aligned} \tag{3.15}$$

The assertion then follows from (3.14)-(3.15) and the assertion (iv) after multiplying (3.14) by k and taking sum over m from 0 to $\ell\,(\leq M-1)$.

Finally, to show statement (vii), we first apply the difference operator d_t to both sides of (3.4), then take $\eta_h = d_t U^{m+1}$ in the resulting equation to have

$$\begin{aligned} \frac{1}{2}d_t\| d_t U^{m+1} \|_{L^2}^2 + \frac{k}{2}\| d_t^2 U^{m+1} \|_{L^2}^2 &= -\big(\nabla d_t W^{m+1}, \nabla d_t U^{m+1}\big) \\ &\leq \frac{1}{\sqrt{k\varepsilon}}\Big(k\| \nabla d_t W^{m+1} \|_{L^2}^2 + \varepsilon\| \nabla d_t U^{m+1} \|_{L^2}^2\Big) . \end{aligned} \tag{3.16}$$

Multiplying (3.16) by k and taking sum over m from 0 to $\ell\,(\leq M-1)$, the assertion then follows from (vi). □

Remark: In view of (i) of Lemma 2.4 and (i) of Lemma 3.1, in order for the scheme (3.4)-(3.5) to conserve the mass of the underlying physical problem, it is necessary to require $(U^0 - u_0^\varepsilon, 1) = 0$ for the starting value U^0. This condition will be assumed in the rest of this section.

As is shown in [21], in order to establish error bounds that depend on low order polynomials of $\frac{1}{\varepsilon}$, the crucial idea is to utilize the spectrum estimate result of Proposition 2.5 for the linearized Cahn-Hilliard operator. In the following we show that the spectrum estimate still holds if the function u, which is the solution of (1.1)-(1.3), is replaced by its elliptic projection $P_h u$ and the nonlinear term is scaled by a factor $1-\varepsilon$, provided that the mesh size h is small enough. As expected, this result plays a critical role in our error analysis for the fully discrete finite element discretization.

For u the solution of (1.1)-(1.3), let C_0 be same as in (2.9) and define

$$C_1 = \max_{|v|\leq 2C_0} |f''(v)|\,, \tag{3.17}$$

and C_2 be the smallest positive ε-independent constant such that (cf. Chapter 7 of [8])

$$\begin{aligned} \| u - P_h u \|_{L^\infty(J;L^\infty)} &\leq C_2\, h^2 |\ln h|\, \| u \|_{L^\infty(J;W^{2,\infty})} \\ &\leq C_2\, h^2 |\ln h|\, \rho_3(\varepsilon, N) \end{aligned} \tag{3.18}$$

for some (low order) polynomial function $\rho_3(\varepsilon, N)$ in $\frac{1}{\varepsilon}$. We remark that the existence of C_2 and $\rho_3(\varepsilon, N)$ follows easily from (xi) of Lemma 2.4 and the following Gagliardo-Nirenberg inequality [1]

$$\| u \|_{W^{2,\infty}} \leq C\Big(\| D^4 u \|_{L^2}^{\frac{4}{8-N}} \| u \|_{L^\infty}^{\frac{4-N}{8-N}} + \| u \|_{L^\infty} \Big), \quad N = 2, 3.$$

In fact, the above inequality, (2.9) and (xi) of Lemma 2.4 imply

$$\rho_3(\varepsilon, N) \leq \rho_2(\varepsilon)^{\frac{4}{8-N}}, \tag{3.19}$$

where $\rho_2(\varepsilon)$ is defined in (xi) of Lemma 2.4.

Proposition 3.2. *Let the assumptions of Proposition 2.5 hold and C_0 be same as there. Let u be the solution of (1.1)-(1.2) satisfying (2.9) and $P_h u$ be its elliptic projection. Then there holds for small $\varepsilon > 0$*

$$\lambda_{CH}^h \equiv \inf_{\substack{0 \not\equiv \psi \in L_0^2(\Omega) \\ \Delta w = \psi,\, \frac{\partial w}{\partial n} = 0}} \frac{\varepsilon \| \nabla \psi \|_{L^2}^2 + \frac{1-\varepsilon}{\varepsilon} \left(f'(P_h u)\psi, \psi\right)}{\| \nabla w \|_{L^2}^2} \geq -(1-\varepsilon)(C_0 + 1), \tag{3.20}$$

provided that h satisfies the constraint

$$h^2 \, |\ln h| \leq (C_1 \, C_2 \, \rho_3(\varepsilon, N))^{-1} \, \varepsilon^2 \, . \tag{3.21}$$

Proof. From the definitions of C_1 and C_2, we immediately have

$$\| P_h u \|_{L^\infty(J;L^\infty)} \leq \| u \|_{L^\infty(J;L^\infty)} + \| P_h u - u \|_{L^\infty(J;L^\infty)} \leq \frac{4}{3} \| u \|_{L^\infty(J;L^\infty)} \leq 2\, C_0$$

if h satisfies (3.21).

It then follows from the Mean Value Theorem that

$$\begin{aligned} \| f'(P_h u) - f'(u) \|_{L^\infty(J;L^\infty)} &\leq \max_{|\xi| \leq 2C_0} |f''(\xi)| \| P_h u - u \|_{L^\infty(J;L^\infty)} \\ &\leq C_1 \, C_2 \, h^2 \, |\ln h| \, \rho(\varepsilon, N) \\ &\leq \varepsilon^2 \, . \end{aligned} \tag{3.22}$$

Using the inequality $a \geq b - |a - b|$ and (3.22) we get

$$\begin{aligned} f'(P_h u) &\geq f'(u) - \| f'(P_h u) - f'(u) \|_{L^\infty(J;L^\infty)} \\ &\geq f'(u) - \varepsilon^2. \end{aligned} \tag{3.23}$$

In addition, we have

$$\| \psi \|_{L^2}^2 = \big(\nabla \psi, \nabla w\big) \leq \Big(\frac{\varepsilon}{1-\varepsilon} \| \nabla \psi \|_{L^2}^2 + \frac{1-\varepsilon}{\varepsilon} \| \nabla w \|_{L^2}^2 \Big) \, . \tag{3.24}$$

Substituting (3.23)-(3.24) into the definition of λ_{CH}^h we get

$$\lambda_{CH}^h \geq \inf_{\substack{0 \not\equiv \psi \in L_0^2(\Omega) \\ \Delta w = \psi,\, \frac{\partial w}{\partial n} = 0}} \frac{(1-\varepsilon)\big[\varepsilon \| \nabla \psi \|_{L^2}^2 + \frac{1}{\varepsilon} \left(f'(u)\psi, \psi\right)\big]}{\| \nabla w \|_{L^2}^2} - (1-\varepsilon)^2 \, .$$

The proof is completed by applying Proposition 2.5. □

The first main result of this section is stated in the following theorem.

THEOREM 3.3. *Let $\{(U^m, W^m)\}_{m=0}^{M}$ solve (3.4)-(3.5) on a quasi-uniform space mesh $\mathcal{T}_h$ of size $O(h)$, allowing for inverse inequalities and H^1-stability of Q_h, and a quasi-uniform time mesh J_k of size $O(k)$. Suppose the assumptions of Lemma 2.4 and 3.1, Proposition 2.5 and 3.2 hold, in particular, α_0, σ_i, $\rho_i(\varepsilon, N)$ are the same as there. Let $0 < \delta < \frac{16}{8-N}$ for $N = 2, 3$, and define for some $\nu > 0$*

$$(3.25)\quad \mu := \mu(N,\delta,\nu) = \min\left\{\, \delta, \nu, \tfrac{8-N}{8} \,\right\},$$

$$\rho_4(\varepsilon) := \varepsilon^{-\max\{2\sigma_1+\frac{13}{2}, 2\sigma_3+\frac{7}{2}, 2\sigma_2+4, 2\sigma_4\}},$$

$$\rho_5(\varepsilon) := \varepsilon^{-\max\{2\sigma_1+9, 2\sigma_3+6, 2\sigma_2+4, 2\sigma_4+1\}},$$

$$(3.26)\quad \pi_1(k;\varepsilon,N,\delta,\sigma_i) := \rho_1(\varepsilon,N) + k^{\frac{16+(8-N)\delta}{16-(8-N)\delta}}\, \varepsilon^{-\frac{32(\sigma_1+3)+2\delta(8-N)(2\sigma_1-1)}{16-(8-N)\delta}} \times \rho_2(\varepsilon)^{\frac{4N\delta}{16-(8-N)\delta}},$$

$$(3.27)\quad \pi_2(k,h;\varepsilon,N,\delta,\sigma_i,\nu) := \left[h^{\frac{(8-N)\delta}{4}-2\mu}\, \varepsilon^{-\frac{(2\sigma_1+1)[16+(8-N)\delta]}{16}} + h^{-2\mu}k^2\rho_5(\varepsilon)\right] \times \rho_2(\varepsilon)^{\frac{\delta N}{8}} + \left[h^{2(\delta-\mu)}\left(\rho_2(\varepsilon)^{\frac{4(2+\delta)-2N}{8-N}} + \rho_4(\varepsilon)\right) + h^{2(\nu-\mu)}\varepsilon^{-2(\sigma_2+1)}\right],$$

$$(3.28)\quad r(h,k;\varepsilon,N,\delta,\sigma_i,\nu) := k^2\, \pi_1(k;\varepsilon,N,\delta,\sigma_i) + h^{2(2+\mu)}\, \pi_2(k,h;\varepsilon,N,\delta,\sigma_i,\nu).$$

Then, under the following mesh and starting value constraints

1). $k \le \varepsilon^{\alpha_0}$,
2). $h^2\, |\ln h| \le \varepsilon^2\, \rho_2(\varepsilon)^{-\frac{4}{8-N}}$,
3). $r(h,k;\varepsilon,N,\delta,\sigma_i,\nu) \le \varepsilon^{[2+\frac{48}{(8-N)\delta}]}\rho_2(\varepsilon)^{-\frac{2N}{8-N}}$,
4). $(U^0, 1) = (u_0^\varepsilon, 1)$,
5). $\|\, u_0^\varepsilon - U^0\,\|_{H^{-1}} \le C\, h^{2+\nu}\|\, u_0^\varepsilon\,\|_{H^2}$,

the solution of (3.4)-(3.5) converges to the solution of (2.1)-(2.4) and satisfies

$$\text{(i)}\quad \max_{0\le m\le M} \|\, u(t_m) - U^m\,\|_{H^{-1}} + \Big(k \sum_{m=0}^{M} k\|\, d_t(u(t_m) - U^m)\,\|_{H^{-1}}^2\Big)^{\frac{1}{2}} \le \tilde{C}\, \big[\, r(h,k;\varepsilon,N,\delta,\sigma_i,\nu)\,\big]^{\frac{1}{2}},$$

$$\text{(ii)}\quad \Big(k \sum_{m=0}^{M} \|\, u(t_m) - U^m\,\|_{L^2}^2\Big)^{\frac{1}{2}} \le \tilde{C}\left\{ h^2\, \varepsilon^{-(\sigma_1+\frac{1}{2})} + \varepsilon^{-2}\, \big[\, r(h,k;\varepsilon,N,\delta,\sigma_i,\nu)\,\big]^{\frac{1}{2}} \right\},$$

$$\text{(iii)}\quad \Big(k \sum_{m=0}^{M} \|\, \nabla(u(t_m) - U^m)\,\|_{L^2}^2\Big)^{\frac{1}{2}} \le \tilde{C}\left\{ h\, \varepsilon^{-(\sigma_1+\frac{1}{2})} + \varepsilon^{-2}\, \big[\, r(h,k;\varepsilon,N,\delta,\sigma_i,\nu)\,\big]^{\frac{1}{2}} \right\}$$

for some positive constant $\tilde{C} = \tilde{C}(u_0^\varepsilon; \gamma_1, \gamma_2, C_0, T; \Omega)$.

Proof. The proof is divided into four steps. Step one deals with consistency error due to the time discretization. Steps two and three use Proposition 3.2 and stability estimates of Lemma 2.4 and 3.1 to avoid exponential blow-up in $\frac{1}{\varepsilon}$ of the error constants. In the final step, an inductive argument is used to handle the difficulty caused by the super-quadratic term in $(\mathrm{GA}_1)_3$.

Step 1: Let $E^m := u(t_m) - U^m$ and $G^m := w(t_m) - W^m$. Subtracting (3.4)-(3.5) from (3.1)-(3.2), respectively, we get the error equations

$$(d_t E^m, \eta_h) + (\nabla G^m, \nabla \eta_h) = (\mathcal{R}(u_{tt}; m), \eta_h)\,, \tag{3.29}$$

$$\varepsilon(\nabla E^m, \nabla v_h) + \frac{1}{\varepsilon}\big(f(u(t_m)) - f(U^m), v_h\big) = (G^m, v_h)\,, \tag{3.30}$$

where

$$\mathcal{R}(u_{tt}; m) = -\frac{1}{k}\int_{t_{m-1}}^{t_m} (s - t_{m-1})\, u_{tt}(s)\mathrm{d}s\,. \tag{3.31}$$

It is easy to check that

$$\begin{aligned} k\sum_{m=0}^{M} \| \mathcal{R}(u_{tt}; m) \|_{H^{-1}}^2 & \\ &\leq \frac{1}{k}\sum_{m=0}^{M}\Big[\int_{t_{m-1}}^{t_m} (s - t_{m-1})^2\,\mathrm{d}s\Big]\Big[\int_{t_{m-1}}^{t_m} \| u_{tt} \|_{H^{-1}}^2\,\mathrm{d}s\Big] \\ &\leq C\,k^2\,\rho_1(\varepsilon, N)\,, \end{aligned} \tag{3.32}$$

where $\rho_1(\varepsilon, N)$ is defined in (x) of Lemma 2.4.

Step 2: Introduce the decompositions: $E^m := \Theta^m + \Phi^m$ and $G^m := \Lambda^m + \Psi^m$, where

$$\begin{aligned} \Theta^m &:= u(t_m) - P_h u(t_m)\,, \qquad & \Phi^m &:= P_h u(t_m) - U^m\,, \\ \Lambda^m &:= w(t_m) - P_h w(t_m)\,, \qquad & \Psi^m &:= P_h w(t_m) - W^m\,. \end{aligned}$$

Then from the definition of P_h in (3.8)-(3.9) we can rewrite (3.29)-(3.30) as follows

$$(d_t \Phi^m, \eta_h) + (\nabla \Psi^m, \nabla \eta_h) = (\mathcal{R}(u_{tt}; m), \eta_h) - (d_t \Theta^m, \eta_h)\,, \tag{3.33}$$

$$\begin{aligned} \varepsilon(\nabla \Phi^m, \nabla v_h) + \frac{1}{\varepsilon}\big(f(P_h u(t_m)) - f(U^m), v_h\big) &= (\Psi^m, v_h) + (\Lambda^m, v_h) \\ &\quad -\frac{1}{\varepsilon}\big(f(u(t_m)) - f(P_h u(t_m)), v_h\big)\,. \end{aligned} \tag{3.34}$$

Since $E^m, \Phi^m \in L_0^2(\Omega)$ for $0 \leq m \leq M$, setting $\eta_h = -\Delta_h^{-1}\Phi^m$ in (3.33) and $v_h = \Phi^m$ in (3.34) and taking summation over m from 1 to ℓ $(\leq M)$, after adding the equations we conclude

$$
\begin{aligned}
(3.35)\quad & \| \nabla \Delta_h^{-1} \Phi^\ell \|_{L^2}^2 + k \sum_{m=1}^{\ell} \frac{k}{2} \| \nabla \Delta_h^{-1} d_t \Phi^m \|_{L^2}^2 \\
& \qquad + k \sum_{m=1}^{\ell} \Big[\varepsilon \| \nabla \Phi^m \|_{L^2}^2 + \frac{1}{\varepsilon} \big(f(P_h u(t_m)) - f(U^m), \Phi^m \big) \Big] \\
& = k \sum_{m=1}^{\ell} \Big[(\mathcal{R}(u_{tt}; m), -\Delta_h^{-1} \Phi^m) - (d_t \Theta^m, -\Delta_h^{-1} \Phi^m) + (\Lambda^m, \Phi^m) \Big] \\
& \qquad + \frac{k}{\varepsilon} \sum_{m=1}^{\ell} \big(f(u(t_m)) - f(P_h u(t_m)), \Phi^m \big) + \| \nabla \Delta_h^{-1} \Phi^0 \|_{L^2}^2 .
\end{aligned}
$$

The first sum on the right hand side can be bounded as follows

$$
\begin{aligned}
(3.36)\quad & k \sum_{m=1}^{\ell} \Big[(\mathcal{R}(u_{tt}; m), -\Delta_h^{-1} \Phi^m) - (d_t \Theta^m, -\Delta_h^{-1} \Phi^m) + (\Lambda^m, \Phi^m) \Big] \\
& \leq C\, k \sum_{m=1}^{\ell} \Big[\| \mathcal{R}(u_{tt}; m) \|_{H^{-1}}^2 + \| d_t \Theta^m \|_{H^{-1}}^2 + \varepsilon^{-1} \| \Lambda^m \|_{H^{-1}}^2 \Big] \\
& \qquad + k \sum_{m=1}^{\ell} \Big\{ \| \nabla \Delta_h^{-1} \Phi^m \|_{L^2}^2 + \frac{\varepsilon(1-\varepsilon-\gamma_1)}{2(1-\varepsilon)} \| \nabla \Phi^m \|_{L^2}^2 \Big\} \\
& \leq k \sum_{m=1}^{\ell} \Big\{ \| \nabla \Delta_h^{-1} \Phi^m \|_{L^2}^2 + \frac{\varepsilon(1-\varepsilon-\gamma_1)}{2(1-\varepsilon)} \| \nabla \Phi^m \|_{L^2}^2 \Big\} \\
& \qquad + C \Big[k^2 \rho_1(\varepsilon, N) + h^6 \rho_4(\varepsilon) \Big] ,
\end{aligned}
$$

where $\rho_1(\varepsilon, N)$ is defined in (x) of Lemma 2.4 and

$$
\rho_4(\varepsilon) := \varepsilon^{-\max\{2\sigma_1 + \frac{13}{2}, 2\sigma_3 + \frac{7}{2}, 2\sigma_2 + 4, 2\sigma_4\}} .
$$

Here we have used the following approximation properties in H^{-1} of the elliptic projection P_h (cf. [16])

$$
\begin{aligned}
\| u - P_h u \|_{H^{-1}} &\leq C\, h^3 \| u \|_{H^2} , \\
\| (u - P_h u)_t \|_{H^{-1}} &\leq C\, h^3 \| u_t \|_{H^2} , \\
\| w - P_h w \|_{H^{-1}} &\leq C\, h^3 \| w \|_{H^2} .
\end{aligned}
$$

In view of (2.9) and the inequality at the beginning of the proof of Proposition 3.2, the second sum on the right hand side of (3.35) can be bounded by

$$
\begin{aligned}
(3.37)\quad & \frac{k}{\varepsilon} \sum_{m=1}^{\ell} \big(f(u(t_m)) - f(P_h u(t_m)), \Phi^m \big) = \frac{k}{\varepsilon} \sum_{m=1}^{\ell} \big(f'(\xi) \Theta^m, \Phi^m \big) \\
& \leq k \sum_{m=1}^{\ell} \Big[\frac{\varepsilon(1-\varepsilon-\gamma_1)}{(1-\varepsilon)} \| \nabla \Phi^m \|_{L^2}^2 + C\, \varepsilon^{-3} \| \Theta^m \|_{H^{-1}}^2 \Big] \\
& \leq k \sum_{m=1}^{\ell} \frac{\varepsilon(1-\varepsilon-\gamma_1)}{2(1-\varepsilon)} \| \nabla \Phi^m \|_{L^2}^2 + C\, h^6\, \varepsilon^{-\max\{2\sigma_1 + 8, 2\sigma_3 + 3\}} .
\end{aligned}
$$

By $(\mathrm{GA}_1)_3$, the last term on the left hand side of (3.35) is bounded from below by

$$
k\sum_{m=1}^{\ell}\frac{1}{\varepsilon}\big(f(P_h u(t_m))-f(U^m),\Phi^m\big) \geq \frac{k}{\varepsilon}\sum_{m=1}^{\ell}\Big[\gamma_1\big(f'(P_h u(t_m))\Phi^m,\Phi^m\big)-\gamma_2\|\,\Phi^m\,\|_{L^{2+\delta}}^{2+\delta}\Big]. \tag{3.38}
$$

Substituting (3.36)-(3.38) into (3.35) we arrive at

$$
\begin{aligned}
&\|\,\nabla\Delta_h^{-1}\Phi^\ell\,\|_{L^2}^2+k\sum_{m=1}^{\ell}\frac{k}{2}\|\,\nabla\Delta_h^{-1}d_t\Phi^m\,\|_{L^2}^2 \\
&\qquad+\frac{\gamma_1 k}{1-\varepsilon}\sum_{m=1}^{\ell}\Big[\varepsilon\|\,\nabla\Phi^m\,\|_{L^2}^2+\frac{1-\varepsilon}{\varepsilon}\big(f'(P_h u(t_m))\Phi^m,\Phi^m\big)\Big] \\
&\leq C\Big[k^2\,\rho_1(\varepsilon,N)+h^6\,\rho_4(\varepsilon)\Big]+2k\sum_{m=0}^{\ell}\|\,\nabla\Delta_h^{-1}\Phi^m\,\|_{L^2}^2 \\
&\qquad+\frac{\gamma_2 k}{\varepsilon}\sum_{m=1}^{\ell}\|\,\Phi^m\,\|_{L^{2+\delta}}^{2+\delta}.
\end{aligned} \tag{3.39}
$$

We could bound the last term on the left hand side from below using Proposition 3.2, however, this will consume all the contribution of $\varepsilon\|\,\nabla\Phi^m\,\|_{L^2}^2$ on the left hand side. On the other hand, in order to bound the super-quadratic term on the right hand side in Step 3 below, we do need some (small) help from this $\varepsilon\|\,\nabla\Phi^m\,\|_{L^2}^2$. For that reason, we only apply Proposition 3.2 with a scaling factor $1-\varepsilon^3$ (we remark that any scaling of form $1-\varepsilon^\gamma$ ($\gamma\geq 3$) should work, but $\gamma=3$ seems to give the "best" constant), that is,

$$
\begin{aligned}
(1-\varepsilon^3)\Big[\,\varepsilon\|\,\nabla\Phi^m\,\|_{L^2}^2+\frac{1-\varepsilon}{\varepsilon}\big(f'(P_h u(t_m))\Phi^m,\Phi^m\big)\,\Big]
&\geq -(1-\varepsilon^3)(1-\varepsilon)(C_0+1)\|\,\nabla\Delta^{-1}\Phi^m\,\|_{L^2}^2\,, \\
&\geq -(C_0+1)\|\,\nabla\Delta^{-1}\Phi^m\,\|_{L^2}^2\,.
\end{aligned} \tag{3.40}
$$

We keep the leftover term $\frac{\gamma_1\varepsilon^4}{1-\varepsilon}\|\,\nabla\Phi^m\,\|_{L^2}^2$ on the left hand side, and move the leftover term $\gamma_1\varepsilon^2\big(f'(P_h u(t_m))\Phi^m,\Phi^m\big)$ to the right side to bound it from above by

$$
\gamma_1\varepsilon^2\big(f'(P_h u(t_m))\Phi^m,\Phi^m\big) \leq \frac{\gamma_1\varepsilon^4}{2(1-\varepsilon)}\|\,\nabla\Phi^m\,\|_{L^2}^2+2\gamma_1(1-\varepsilon)\|\,\nabla\Delta_h^{-1}\Phi^m\,\|_{L^2}^2\,. \tag{3.41}
$$

Combining (3.39)-(3.41) we finally get

$$
\begin{aligned}
&\|\,\nabla\Delta_h^{-1}\Phi^\ell\,\|_{L^2}^2+k\sum_{m=1}^{\ell}\Big[\frac{k}{2}\|\,\nabla\Delta_h^{-1}d_t\Phi^m\,\|_{L^2}^2+\frac{\gamma_1\varepsilon^4}{2}\|\,\nabla\Phi^m\,\|_{L^2}^2\Big] \\
&\leq C\Big[k^2\,\rho_1(\varepsilon,N)+h^6\,\rho_4(\varepsilon)\Big]+\big(C_0+2\gamma_1+3\big)k\sum_{m=0}^{\ell}\|\,\nabla\Delta_h^{-1}\Phi^m\,\|_{L^2}^2 \\
&\qquad+\frac{\gamma_2 k}{\varepsilon}\sum_{m=1}^{\ell}\|\,\Phi^m\,\|_{L^{2+\delta}}^{2+\delta}\,,
\end{aligned} \tag{3.42}
$$

where we have used the fact that $0 < \varepsilon < 1$ and $\| \nabla\Delta^{-1} v_h \|_{L^2} = \| \nabla\Delta_h^{-1} v_h \|_{L^2}$ for any $v_h \in \overset{\circ}{\mathcal{S}}_h$.

Step 3: It remains to bound the super-quadratic term at the end of (3.42). Since $\Phi^m = E^m - \Theta^m$, the triangle inequality implies

$$\| \Phi^m \|_{L^{2+\delta}}^{2+\delta} \le C \left(\| E^m \|_{L^{2+\delta}}^{2+\delta} + \| \Theta^m \|_{L^{2+\delta}}^{2+\delta} \right). \tag{3.43}$$

To bound $\| E^m \|_{L^{2+\delta}}^{2+\delta}$ in (3.43), we first make a shift in the super-index to get

$$\| E^m \|_{L^{2+\delta}}^{2+\delta} \le \sum_{K \in \mathcal{T}_h} \left[\| E^{m-1} \|_{L^{2+\delta}(K)}^{2+\delta} + k^{2+\delta} \| d_t E^m \|_{L^{2+\delta}(K)}^{2+\delta} \right]. \tag{3.44}$$

For each term in the first sum on the right hand side of (3.44), we use the Gagliardo-Nirenberg inequality [1] which interpolates $L^{2+\delta}(K)$ between $L^2(K)$ and $H^4(K)$.

$$\begin{aligned} \| E^{m-1} \|_{L^{2+\delta}(K)}^{2+\delta} &\le C \left[\| D^4 E^{m-1} \|_{L^2(K)}^{\frac{\delta N}{8}} \| E^{m-1} \|_{L^2(K)}^{\frac{16+(8-N)\delta}{8}} + \| E^{m-1} \|_{L^2(K)}^{2+\delta} \right], \\ &\le C \| E^{m-1} \|_{L^2(K)}^{2+\frac{(8-N)\delta}{8}} \left[\| D^4 u(t_{m-1}) \|_{L^2(K)}^{\frac{\delta N}{8}} + \| E^{m-1} \|_{L^2(K)}^{\frac{\delta N}{8}} \right] \\ &\le C \| E^{m-1} \|_{L^2(K)}^{2+\frac{(8-N)\delta}{8}} \rho_2(\varepsilon)^{\frac{\delta N}{8}} . \end{aligned} \tag{3.45}$$

Here we have used the estimates (xi) of Lemma 2.4, (ii) of Lemma 2.1 and (v) of Lemma 3.1 to obtain the last inequality.

Summing (3.45) over all $K \in \mathcal{T}_h$ and using the convexity of the function $g(s) = s^r$ for $r > 1$ and $s \ge 0$ we have

$$\| E^{m-1} \|_{L^{2+\delta}}^{2+\delta} \le C \rho_2(\varepsilon)^{\frac{\delta N}{8}} \| E^{m-1} \|_{L^2}^{2+\frac{(8-N)\delta}{8}} . \tag{3.46}$$

Similarly, we can bound the second sum on the right hand side of (3.44).

$$\begin{aligned} & k^{2+\delta} \| d_t E^m \|_{L^{2+\delta}(K)}^{2+\delta} \\ &\quad \le C k^{2+\delta} \left[\| D^4 d_t E^m \|_{L^2(K)}^{\frac{\delta N}{8}} \| d_t E^m \|_{L^2(K)}^{\frac{16+(8-N)\delta}{8}} + \| d_t E^m \|_{L^2(K)}^{2+\delta} \right] \\ &\quad \le C k^{2+\delta} \| d_t E^m \|_{L^2(K)}^{2+\frac{(8-N)\delta}{8}} \left[\| D^4 d_t u(t_m) \|_{L^2(K)}^{\frac{\delta N}{8}} + \| d_t E^m \|_{L^2(K)}^{\frac{\delta N}{8}} \right] \\ &\quad \le C k^{2+\delta} \| d_t E^m \|_{L^2(K)}^{2+\frac{(8-N)\delta}{8}} \left[k^{-\frac{N\delta}{8}} \| D^4 u(t_m) \|_{L^2(K)}^{\frac{\delta N}{8}} + \| d_t E^m \|_{L^2(K)}^{\frac{\delta N}{8}} \right] \\ &\quad \le C k^{2+\frac{(8-N)\delta}{8}} \| d_t E^m \|_{L^2(K)}^{2+\frac{(8-N)\delta}{8}} \rho_2(\varepsilon)^{\frac{\delta N}{8}} . \end{aligned} \tag{3.47}$$

Here we have used the estimates (viii) and (xi) of Lemma 2.4, and (vii) of Lemma 3.1 to obtain the last inequality.

Taking sum on both sides of (3.47) over all $K \in \mathcal{T}_h$ we get

$$k^{2+\delta} \| d_t E^m \|_{L^{2+\delta}}^{2+\delta} \le C k^{2+\frac{(8-N)\delta}{8}} \rho_2(\varepsilon)^{\frac{\delta N}{8}} \| d_t E^m \|_{L^2}^{2+\frac{(8-N)\delta}{8}} . \tag{3.48}$$

It now follows from the triangle inequality $\| E^m \|_{L^2} \le \| \Theta^m \|_{L^2} + \| \Phi^m \|_{L^2}$, the inequalities (3.43), (3.44), (3.46) and (3.48) that

$$\begin{aligned} & \| \Phi^m \|_{L^{2+\delta}}^{2+\delta} \\ &\quad \le C \rho_2(\varepsilon)^{\frac{\delta N}{8}} \Big(\| \Phi^{m-1} \|_{L^2}^{2+\frac{(8-N)\delta}{8}} + k^{2+\frac{(8-N)\delta}{8}} \| d_t \Phi^m \|_{L^2}^{2+\frac{(8-N)\delta}{8}} \\ &\qquad + \| \Theta^{m-1} \|_{L^2}^{2+\frac{(8-N)\delta}{8}} + k^{2+\frac{(8-N)\delta}{8}} \| d_t \Theta^m \|_{L^2}^{2+\frac{(8-N)\delta}{8}} \Big) + C \| \Theta^m \|_{L^{2+\delta}}^{2+\delta} . \end{aligned} \tag{3.49}$$

From [8, 14, 25] we know that

$$\| u - P_h u \|_{L^2} \le C\, h^2 \| u \|_{H^2}\,, \tag{3.50}$$

$$\| (u - P_h u)_t \|_{L^2} \le C\, h^2 \| u_t \|_{H^2}\,, \tag{3.51}$$

$$\| u - P_h u \|_{L^{2+\delta}} \le C\, h^2 \| u \|_{W^{2,2+\delta}} \le C\, h^2 \, \| \Delta^2 u \|_{L^2}^{\frac{4(2+\delta)-2N}{(8-N)(2+\delta)}}\,. \tag{3.52}$$

To control two terms which involve Φ^{m-1} on the right hand side of (3.49), we only consider the case $\delta < \frac{16}{8-N}$ because (i) it covers most useful ranges of δ, (ii) the analysis for the case $\delta > \frac{16}{8-N}$ is easier to carry out since $\frac{16+(8-N)\delta}{8} > 4$ in this case.

From the definition of $-\Delta_h^{-1}$ in (3.10) and Young's inequality we have

$$\begin{aligned} \| \Phi^{m-1} \|_{L^2}^{\frac{16+(8-N)\delta}{8}} &\le \| \nabla \Phi^{m-1} \|_{L^2}^{\frac{16+(8-N)\delta}{16}} \| \nabla \Delta_h^{-1} \Phi^{m-1} \|_{L^2}^{\frac{16+(8-N)\delta}{16}} \\ &\le C \left[\varepsilon^5 \rho_2(\varepsilon)^{-\frac{\delta N}{8}} \right]^{-\frac{16+(8-N)\delta}{16-(8-N)\delta}} \| \nabla \Delta_h^{-1} \Phi^{m-1} \|_{L^2}^{\frac{2[16+(8-N)\delta]}{16-(8-N)\delta}} \\ &\quad + \frac{\gamma_1 \varepsilon^5 \rho_2(\varepsilon)^{-\frac{\delta N}{8}}}{4\gamma_2} \| \nabla \Phi^{m-1} \|_{L^2}^2 \,. \end{aligned} \tag{3.53}$$

Similarly,

$$\begin{aligned} \| d_t \Phi^m \|_{L^2}^{\frac{16+(8-N)\delta}{8}} &\le \| \nabla d_t \Phi^m \|_{L^2}^{\frac{16+(8-N)\delta}{16}} \| \nabla \Delta_h^{-1} d_t \Phi^m \|_{L^2}^{\frac{16+(8-N)\delta}{16}} \\ &\le C \left[\varepsilon k^{-\frac{8+(8-N)\delta}{8}} \rho_2(\varepsilon)^{-\frac{\delta N}{8}} \right]^{-\frac{16+(8-N)\delta}{16-(8-N)\delta}} \| \nabla d_t \Phi^m \|_{L^2}^{\frac{2[16+(8-N)\delta]}{16-(8-N)\delta}} \\ &\quad + \frac{\varepsilon k^{-\frac{8+(8-N)\delta}{8}} \rho_2(\varepsilon)^{-\frac{\delta N}{8}}}{4\gamma_2} \| \nabla d_t \Delta_h^{-1} \Phi^m \|_{L^2}^2 \,. \end{aligned}$$

Using

$$\begin{aligned} \| \nabla d_t \Phi^m \|_{L^2}^{\frac{2[16+(8-N)\delta]}{16-(8-N)\delta}} &= \| \nabla d_t \Phi^m \|_{L^2}^2 \| \nabla d_t \Phi^m \|_{L^2}^{\frac{4(8-N)\delta}{16-(8-N)\delta}} \\ &\le k^{-\frac{4(8-N)\delta}{16-(8-N)\delta}} \, \varepsilon^{-(\sigma_1+\frac{1}{2})\frac{4(8-N)\delta}{16-(8-N)\delta}} \, \| \nabla d_t \Phi^m \|_{L^2}^2 \,, \end{aligned}$$

we get

$$\begin{aligned} \| d_t \Phi^m \|_{L^2}^{\frac{16+(8-N)\delta}{8}} &\le \frac{\varepsilon k^{-\frac{8+(8-N)\delta}{8}} \rho_2(\varepsilon)^{-\frac{\delta N}{8}}}{4\gamma_2} \| \nabla d_t \Delta_h^{-1} \Phi^m \|_{L^2}^2 \\ &\quad + C\, \varepsilon^{-[1+\frac{4\delta(8-N)(\sigma_1+1)}{16-(8-N)\delta}]} \, \rho_2(\varepsilon)^{-\frac{\delta N[16+(8-N)\delta]}{8[16-(8-N)\delta]}} \\ &\quad \times k^{1+\frac{(8-N)^2\delta^2}{8[16-(8-N)\delta]}} \, \| \nabla d_t \Phi^m \|_{L^2}^2 \,. \end{aligned} \tag{3.54}$$

Now, substituting (3.50)-(3.54) into (3.49) summing over m from 1 to $\ell\,(\le M)$ after multiplying (3.49) by $\frac{\gamma_2 k}{\varepsilon}$ and using (iv) of Lemma 2.4 and (vi) of Lemma 3.1

lead to the following estimate

$$
(3.55)\quad \frac{\gamma_2 k}{\varepsilon}\sum_{m=1}^{\ell}\|\,\Phi^m\,\|_{L^{2+\delta}}^{2+\delta} \le k\sum_{m=1}^{\ell}\Big[\,\frac{\varepsilon^4}{4}\|\,\nabla\Phi^m\,\|_{L^2}^2+\frac{k}{4}\|\,\nabla\Delta_h^{-1}d_t\Phi^m\,\|_{L^2}^2\,\Big]
$$

$$
+C\,\varepsilon^{-\frac{4[24+(8-N)\delta]}{16-(8-N)\delta}}\,\rho_2(\varepsilon)^{\frac{4N\delta}{16-(8-N)\delta}}\,k\sum_{m=1}^{\ell}\|\,\nabla\Delta_h^{-1}\Phi^{m-1}\,\|_{L^2}^{2+\frac{4(8-N)\delta}{16-(8-N)\delta}}
$$

$$
+C\,k^{3+\frac{2(8-N)\delta}{16-(8-N)\delta}}\,\varepsilon^{-[2+\frac{4\delta(8-N)(\sigma_1+1)}{16-(8-N)\delta}]}\,\rho_2(\varepsilon)^{\frac{4N\delta}{16-(8-N)\delta}}\,\varepsilon^{-2(\sigma_1+2)}
$$

$$
+C\,\rho_2(\varepsilon)^{\frac{\delta N}{8}}\,\Big[\,\varepsilon^{-\frac{(2\sigma_1+1)[16+(8-N)\delta]}{16}}\,h^{4+\frac{(8-N)\delta}{4}}
$$

$$
+k^2h^4\varepsilon^{-\max\{2\sigma_1+9,2\sigma_3+6,2\sigma_2+4,2\sigma_2+1\}}\,\Big]+C\,h^{2(2+\delta)}\,\rho_2(\varepsilon)^{\frac{4(2+\delta)-2N}{8-N}}
$$

Finally, substituting (3.55) into (3.42) we get

$$
(3.56)\quad \|\,\nabla\Delta_h^{-1}\Phi^\ell\,\|_{L^2}^2+k\sum_{m=1}^{\ell}\Big[\frac{k}{4}\|\,\nabla\Delta_h^{-1}d_t\Phi^m\,\|_{L^2}^2+\frac{\gamma_1\varepsilon^4}{4}\|\,\nabla\Phi^m\,\|_{L^2}^2\Big]
$$

$$
\le C\,r(h,k;\varepsilon,N,\delta,\sigma_i,\nu)+(C_0+2\gamma_1+3)\,k\sum_{m=0}^{\ell}\|\,\nabla\Delta_h^{-1}\Phi^m\,\|_{L^2}^2
$$

$$
+C\,s(\varepsilon,N,\delta,\sigma_i)\,k\sum_{m=0}^{\ell-1}\|\,\nabla\Delta_h^{-1}\Phi^m\,\|_{L^2}^{2+\frac{4(8-N)\delta}{16-(8-N)\delta}}\,,
$$

where $r(h,k;\varepsilon,N,\delta,\sigma_i,\nu)$ is defined in (3.28) and

$$
(3.57)\qquad s(\varepsilon,N,\delta,\sigma_i)=\varepsilon^{-\frac{4[24+(8-N)\delta]}{16-(8-N)\delta}}\,\rho_2(\varepsilon)^{\frac{4N\delta}{16-(8-N)\delta}}\,.
$$

It is easy to check from (3.28) that for a fixed ε

$$
(3.58)\qquad r(h,k;\varepsilon,N,\delta,\sigma_i,\nu)\le C\,\big(\,k^2+h^{2(2+\mu)}\,\big)\,,
$$

provided that $k\le h$.

Step 4: We now conclude the proof by the following induction argument. Suppose there exist two positive constants

$$
c_1=c_1(t_\ell,\Omega,u_0^\varepsilon,\sigma_i)\,,\qquad c_2=c_2(t_\ell,\Omega,u_0^\varepsilon,\sigma_i;C_0)\,,
$$

independent of k and ε, such that there holds

$$
(3.59)\quad \max_{0\le m\le\ell}\|\,\nabla\Delta_h^{-1}\Phi^m\,\|_{L^2}^2+k\sum_{m=1}^{\ell}\Big[\frac{k}{4}\|\,\nabla\Delta_h^{-1}d_t\Phi^m\,\|_{L^2}^2+\frac{\gamma_1\varepsilon^4}{4}\|\,\nabla\Phi^m\,\|_{L^2}^2\Big]
$$

$$
\le c_1\,r(h,k;\varepsilon,N,\delta,\sigma_i,\nu)\exp(c_2t_\ell)\,.
$$

In view of (xi) of Lemma 2.4 and (3.56), we can choose

$$
c_1=2\,,\qquad c_2=2(C_0+2\gamma_1+3)\,.
$$

Since the exponent in the last term of (3.56) is bigger than 2, we can recover (3.59) at the $(\ell+1)$th time step by using the discrete Gronwall's inequality, provided that h,k satisfy

$$
s(\varepsilon,N,\delta,\sigma_i)\,\cdot r(h,k;\varepsilon,N,\delta,\sigma_i,\nu)^{1+\frac{2(8-N)\delta}{16-(8-N)\delta}}\le\frac{c_1}{2}\,r(h,k;\varepsilon,N,\delta,\sigma_i,\nu)\,\exp(c_2t_{\ell+1})\,.
$$

That is,

$$s(\varepsilon, N, \delta, \sigma_i) \cdot r(h, k; \varepsilon, N, \delta, \sigma_i, \nu)^{\frac{2(8-N)\delta}{16-(8-N)\delta}} \leq C\,,$$

which gives the mesh condition 3) in the theorem. Hence, we have shown that

$$\max_{0\le m\le M} \| \nabla\Delta_h^{-1}\Phi^m \|_{L^2}^2 + k\sum_{m=1}^{\ell}\Big[k\| \nabla\Delta_h^{-1} d_t\Phi^m \|_{L^2}^2 + \varepsilon^4 \| \nabla\Phi^m \|_{L^2}^2 \Big] \leq C\, r(h, k; \varepsilon, N, \delta, \sigma_i, \nu)\,. \tag{3.60}$$

Finally, the assertion (i) follows from (3.60) and applying the triangle inequality on $E^m = \Theta^m + \Phi^m$. The assertions (ii) and (iii) follow in the same way. Note that we need to use Poincaré inequality on Φ^m to show (ii), and since $\Phi^m \in \mathring{\mathcal{S}}_h$, we can bound $\| \Phi^m \|_{L^2}$ by $\| \nabla\Phi^m \|_{L^2}$. The proof is complete. ☐

Remark: (a). The $L^2(J; H^1)$-estimate is optimal with respect to h and k, and $L^\infty(J; H^{-1})$-estimate is quasi-optimal.

(b). The proof clearly shows how the three mesh conditions arise. The condition 1) is for the stability of the time discretization (see (GA_3)), the condition 2) is to ensure the discrete spectrum estimate (see Proposition 3.2), finally, the condition 3) is caused by the super-quadratic nonlinearity of f (see $(\mathrm{GA}_1)_3$). Also notice that only "smallness" of k and h with respect to ε is required but no restriction is imposed on the ratio between k and h in the $L^\infty(J; H^{-1})$- and $L^2(J; H^1)$-norm estimates.

(c). It is well-known [25] that the finite element solutions of all linear and some nonlinear parabolic problems exhibit superconvergence property (in h) when compared with the elliptic projections of the solutions of underlying problems, it is worth pointing out that this superconvergence property also hold for the Cahn-Hilliard equation as showed by the inequality (3.60).

(d). Regarding the choices of the starting value U^0, clearly, both $U^0 = Q_h u_0^\varepsilon$ and $U^0 = P_h u_0^\varepsilon$ satisfy the conditions 4) and 5) with $\nu = 1$ in view of (3.7) and (3.9). In fact, they also satisfy a stronger inequality, see (3.61). The L^2-projection $Q_h u_0^\varepsilon$ has the advantage of being cheaper to be computed compared to the elliptic projection $P_h u_0^\varepsilon$. Note that the condition 4) is necessary in order for the scheme (3.4)-(3.5) to conserve the mass.

In the next theorem we derive error estimates in stronger norms under a slightly stronger requirement on the starting value U^0, which nevertheless is satisfied by both, L^2-projection $U^0 = Q_h u_0^\varepsilon$ and the elliptic projection $U^0 = P_h u_0^\varepsilon$. In addition, a mild constraint on admissible choices of (k, h) is required to assure their validity.

Theorem 3.4. *In addition to the assumptions of Theorem 3.3, if*

$$\| u_0^\varepsilon - U^0 \|_{L^2} \leq C\, h^2 \| u_0^\varepsilon \|_{H^2}\,, \tag{3.61}$$

then the solution of (3.4)-(3.5) also satisfies the following error estimates

(i) $$\max_{0\le m\le M} \| u(t_m) - U^m \|_{L^2} + \Big(k\sum_{m=0}^{M} k\| d_t(u(t_m) - U^m) \|_{L^2}^2 \Big)^{\frac12} \leq \tilde{C}\Big\{ h^2 \varepsilon^{-\max\{\sigma_1+\frac52, \sigma_3+\frac12\}} + k^{-\frac14}\varepsilon^{-1}\big[r(h, k; \varepsilon, N, \delta, \sigma_i, \nu)\big]^{\frac12} \Big\}\,,$$

(ii) $$\max_{0\le m\le M} \| u(t_m) - U^m \|_{L^\infty} \leq \tilde{C}\Big\{ h^2\, |\ln h|\, \rho_2(\varepsilon)^{\frac{4}{8-N}} + h^{-\frac{N}{2}} k^{-\frac14} \varepsilon^{-1}\big[r(h, k; \varepsilon, N, \delta, \sigma_i, \nu)\big]^{\frac12} \Big\}\,.$$

Moreover, if $k = O(h^q)$ for some $\frac{2N}{3} < q < (8-2N) + 4\mu$, then there also hold

$$\text{(iii)} \quad \max_{0\le m\le M} \| U^m \|_{L^\infty} \le 3\, C_0 \,,$$

$$\begin{aligned}\text{(iv)} \quad &\max_{0\le m\le M} \| u(t_m) - U^m \|_{L^2} + \Big(k \sum_{m=0}^{M} k \| d_t(u(t_m) - U^m) \|_{L^2}^2 \Big)^{\frac12} \\ &\qquad + \Big(\frac{k}{\varepsilon} \sum_{m=0}^{M} \| w(t_m) - W^m \|_{L^2}^2 \Big)^{\frac12} \\ &\le \tilde C \Big\{ h^2\, \varepsilon^{-\max\{\sigma_1 + \frac72, \sigma_3 + \frac12\}} + \varepsilon^{-\frac72} \big[r(h,k;\varepsilon,N,\delta,\sigma_i,\nu) \big]^{\frac12} \Big\},\end{aligned}$$

$$\begin{aligned}\text{(v)} \quad &\max_{0\le m\le M} \| u(t_m) - U^m \|_{L^\infty} \\ &\le \tilde C \Big\{ h^2 \,|\ln h|\, \rho_2(\varepsilon)^{\frac{4}{8-N}} + h^{\frac{4-N}{2}} \varepsilon^{-\max\{\sigma_1+\frac72, \sigma_3+\frac12\}} \\ &\qquad + h^{-\frac N2} \varepsilon^{-\frac72} \big[r(h,k;\varepsilon,N,\delta,\sigma_i,\nu) \big]^{\frac12} \Big\}.\end{aligned}$$

for some positive constant $\tilde C = \tilde C(u_0^\varepsilon; \gamma_1, \gamma_2, C_0, T; \Omega)$.

Proof. Since

$$E^m = \Theta^m + \Phi^m \,, \qquad G^m = \Lambda^m + \Psi^m \,,$$

it suffices to show the assertions (i), (ii), (iv), (v) hold for Φ^m and Ψ^m without the first term on the right hand side of each inequality. Notice that Φ^m and Ψ^m satisfy (3.33)-(3.34).

Using the identity

$$(d_t\Phi^m, \Phi^m) = \frac12 d_t \| \Phi^m \|_{L^2}^2 + \frac{k}{2} \| d_t \Phi^m \|_{L^2}^2 \,,$$

the definition of $-\Delta_h^{-1}$ in (3.10) and the estimate (3.60) we have

$$\begin{aligned}(3.62) \quad &\frac12 \| \Phi^\ell \|_{L^2}^2 + k \sum_{m=1}^{\ell} \frac{k}{2} \| d_t \Phi^m \|_{L^2}^2 \\ &= k \sum_{m=1}^{\ell} \big(\nabla(-\Delta_h^{-1} d_t \Phi^m), \nabla \Phi^m \big) + \frac12 \| \Phi^0 \|_{L^2}^2 \\ &\le k^{-\frac12} \varepsilon^{-2} \sum_{m=1}^{\ell} \Big[k^2 \| \nabla \Delta_h^{-1} d_t \Phi^m \|_{L^2}^2 + k\, \varepsilon^4 \| \nabla \Phi^m \|_{L^2}^2 \Big] + \frac12 \| \Phi^0 \|_{L^2}^2 \\ &\le k^{-\frac12} \varepsilon^{-2} \, r(h,k;\varepsilon,N,\delta,\sigma,\nu) + \frac12 \| \Phi^0 \|_{L^2}^2 \,.\end{aligned}$$

The assertion (i) then follows from (3.62) and (3.61).

The assertion (ii) is an immediate consequence of (i), the inverse inequality bounding L^∞ by L^2-norm, and the L^∞-estimate of Θ^m (see Chapter 7 of [8]).

To show (iii), notice that under the mesh conditions of Theorem 3.3 and the assumption that $k = O(h^q)$ for some $\frac{2N}{3} < q < (8-2N) + 4\mu$, there holds for sufficiently small ε

$$(3.63) \qquad \max_{0\le m\le M} \| u(t_m) - U^m \|_{L^\infty} \le \frac32 C_0 \,,$$

which together with (2.9) then implies

$$\max_{0\le m\le M}\|\,U^m\,\|_{L^\infty}\le \max_{0\le m\le M}\big[\,\|\,u(t_m)\,\|_{L^\infty}+\|\,u(t_m)-U^m\,\|_{L^\infty}\,\big]\le 3\,C_0\,. \tag{3.64}$$

Hence the assertion (iii) holds.

Now, taking $\eta_h=\Phi^m$ in (3.33) and $v_h=-\frac{1}{\varepsilon}\Psi^m$ in (3.34) and adding the resulting equations we get

$$\begin{aligned}\frac{1}{2}d_t\|\,\Phi^m\,\|_{L^2}^2+\frac{k}{2}\|\,d_t\Phi^m\,\|_{L^2}^2+\frac{1}{\varepsilon}\|\,\Psi^m\,\|_{L^2}^2&\\ =(\mathcal{R}(u_{tt};m),\Phi^m)-(d_t\Theta^m,\Phi^m)-\frac{1}{\varepsilon}(\Lambda^m,\Psi^m)&\\ +\frac{1}{\varepsilon^2}\big(f(u(t_m))-f(U^m),\Psi^m\big)\,.&\end{aligned} \tag{3.65}$$

The first three terms on the right hand side can be bounded similarly as in (3.36), and the last term can be bounded as follows. By the Mean Value Theorem and Schwarz inequality we obtain

$$\begin{aligned}\frac{1}{\varepsilon^2}\big(f(u(t_m))-f(U^m),\Psi^m\big)&=\frac{1}{\varepsilon^2}\big(f'(\xi)E^m,\Psi^m\big)\\ &\le\frac{1}{2\varepsilon}\|\,\Psi^m\,\|_{L^2}^2+\frac{C}{\varepsilon^3}\|\,E^m\,\|_{L^2}^2\,.\end{aligned} \tag{3.66}$$

The assertion (iv) follows from multiplying (3.65) by k, summing it over m from 1 to $\ell\,(\le M)$ and using (3.66) and the assertion (ii) of Theorem 3.3.

Finally, the assertion (v) is a refinement of (ii), it is based on the estimate (iv), instead of (i). The proof is complete. $\square$

Remark: (a). The estimate in (i) is optimal in h and suboptimal in k due to the factor $k^{-\frac{1}{4}}$ in the second term on the right hand side of the inequality. However, this estimate is important for establishing the $L^\infty(J;L^\infty)$-estimate in (ii), which then leads to the proof of the boundedness of U^m in (3.64), and the improved estimates (iv) and (v).

(b). Optimal estimates in stronger norms also can be obtained for both E^m and G^m under stronger regularity assumptions on the solution u (e.g. $u_{tt}\in L^2(J;L^2)$) of the Cahn-Hilliard equation and on the starting value U^0. These estimates include statements for E^m in $L^\infty(J;H^1)$ and $H^1(J;L^2)$, and in $L^\infty(J;L^2)$ and $L^2(J;H^1)$ for G^m. For more expositions in this direction, we refer to [17] (also see [15]), where a (continuous in time) semi-discrete splitting finite element method was analyzed for a *fixed* $\varepsilon>0$ under the assumption that the semi-discrete finite element approximate solution for u is bounded in L^∞. Note that here we have indeed showed in (iii) that our fully discrete solution U^m is bounded in L^∞.

Corollary 3.5. *Let the assumptions of Theorem 3.4 be valid, and W^0 be a value satisfying for some $\beta>1$*

$$\|\,P_hw(0)-W^0\,\|_{L^2}\le C\,h^\beta. \tag{3.67}$$

Then there hold the following estimates

$$\text{(i)} \quad \max_{0\le m\le M} \| w(t_m) - W^m \|_{L^2} + \Big(k \sum_{m=0}^{M} k \| d_t (w(t_m) - W^m) \|_{L^2}^2 \Big)^{\frac12}$$
$$\le \tilde{C} \Big\{ h^2 \rho_2(\varepsilon) + k^{-\frac12} \left\{ h^2 \varepsilon^{-\max\{\sigma_1+3,\sigma_3\}} + \varepsilon^{-3} \left[r(h,k;\varepsilon,N,\delta,\sigma_i,\nu) \right]^{\frac12} \right\} + h^{\beta} \Big\},$$

$$\text{(ii)} \quad \max_{0\le m\le M} \| w(t_m) - W^m \|_{L^\infty}$$
$$\le \tilde{C} \Big\{ h^{\frac{4-N}{2}} |\ln h|^{\frac{3-N}{2}} \rho_2(\varepsilon)$$
$$+ h^{-\frac{N}{2}} \Big[k^{-\frac12} \left\{ h^2 \varepsilon^{-\max\{\sigma_1+3,\sigma_3\}} + \varepsilon^{-3} \left[r(h,k;\varepsilon,N,\delta,\sigma_i,\nu) \right]^{\frac12} \right\} + h^{\beta} \Big] \Big\}.$$

Proof. First, from [8, 25] we know that

$$\max_{0\le m\le M} \| \Lambda^m \|_{L^2} + \Big(k \sum_{m=0}^{M} k \| d_t \Lambda^m \|_{L^2}^2 \Big)^{\frac12} \le \tilde{C}\, h^2 \rho_2(\varepsilon) . \tag{3.68}$$

Next, using the identity just above (3.62) we get

$$\begin{aligned} \frac12 \| \Psi^\ell \|_{L^2}^2 + k \sum_{m=1}^{\ell} \frac{k}{2} \| d_t \Psi^m \|_{L^2}^2 &= k \sum_{m=1}^{\ell} (d_t \Psi^m, \Psi^m) + \frac12 \| \Psi^0 \|_{L^2}^2 \\ &\le k \sum_{m=1}^{\ell} \Big[\frac{k}{4} \| d_t \Psi^m \|_{L^2}^2 + 4k^{-1} \| \Psi^m \|_{L^2}^2 \Big] + \frac12 \| \Psi^0 \|_{L^2}^2 . \end{aligned} \tag{3.69}$$

The first term on the right hand side can be absorbed by the second term on the left, a desired bound for the second term on the right has been obtained in the proof of (iv) of Theorem 3.4. Hence, assertion (i) follows from combining (3.68) and (3.69).

The assertion (ii) comes from applying the triangle inequality to $G^m = \Lambda^m + \Psi^m$, the following estimate (cf. Section 4 of [20])

$$\| \Lambda^m \|_{L^\infty} \le C\, h^{\frac{4-N}{2}} \, |\ln h|^{\frac{3-N}{2}} \, \| w \|_{H^2} \le C\, h^{\frac{4-N}{2}} \, |\ln h|^{\frac{3-N}{2}} \, \rho_2(\varepsilon) ,$$

and the inverse inequality bounding $\| \Psi^m \|_{L^\infty}$ in terms of $\| \Psi^m \|_{L^2}$. The proof is complete. □

Remark: (a). Clearly, both $Q_h w_0^\varepsilon$ and $P_h w_0^\varepsilon$ are valid candidates for W^0.

(b). Both estimates are not optimal due to the factor $k^{-\frac12}$ in the second term on the right hand side of each inequality. It can be shown that the estimates will be improved to optimal order (first order in k and second order in h) under some stronger regularity assumptions and starting value constraint. See (b) of the remark after the proof of Theorem 3.4.

We conclude this section by giving a very short proof of Theorem 1.1.

Proof of Theorem 1.1. The assertions follows immediately from setting $\delta = 1, \alpha_0 = 3, \gamma_1 = 1$ and $\gamma_2 = 3$ in Theorem 3.3 and Theorem 3.4. □

4. Approximation for the Hele-Shaw problem. The goal of this section is to establish the convergence of the solution $\{(U^m, W^m)\}_{m=0}^M$ of the fully discrete mixed finite element scheme (3.4)-(3.5) to the solution of the Hele-Shaw problem (1.4)-(1.8), provided that the Hele-Shaw problem has a global (in time) classical solution. It is shown that the fully discrete solution W^m, as $h, k \searrow 0$, converges to the solution w of the Hele-Shaw problem uniformly in $\overline{\Omega}_T$. In addition, the fully discrete solution U^m converges to ± 1 uniformly on every compact subset of the "outside" and "inside" of the free boundary Γ of the Hele-Shaw problem, respectively. Hence, the zero level set of U^m converges to the free boundary Γ. Our main ideas are to make fully use of the convergence result that the Hele-Shaw problem is the distinguished limit, as $\varepsilon \searrow 0$, of the Cahn-Hilliard equation proved by Alikakos, Bates and Chen in [2], and to exploit the "closeness" between the solution u of the Cahn-Hilliard equation and its fully discrete approximation U^m, which is demonstrated by the error estimates in the previous section. We remark that as in [2], our numerical convergence result is also established under the assumption that the Hele-Shaw problem has a global (in time) classical solution. We refer to [2, 13] and references therein for further expositions on this assumption and related theoretical works on the Hele-Shaw problem.

Although the results of this section hold for general potential $F(u)$ satisfying (GA_1), for the sake of clarity of the presentation, we only consider the quartic potential $F(u) = \frac{1}{4}(u^2-1)^2$ in this section. Let u^ε denote the solution of the Cahn-Hilliard problem (1.1)-(1.3). Note that we put back the super index ε on the solution in this section. Let $(U_{\varepsilon,h,k}(x,t), W_{\varepsilon,h,k}(x,t))$ denote the piecewise linear interpolation (in time) of the fully discrete solution (U^m, W^m), that is,

$$U_{\varepsilon,h,k}(\cdot,t) := \frac{t-t_m}{k} U^{m+1}(\cdot) + \frac{t_{m+1}-t}{k} U^m(\cdot)\,, \tag{4.1}$$

$$W_{\varepsilon,h,k}(\cdot,t) := \frac{t-t_m}{k} W^{m+1}(\cdot) + \frac{t_{m+1}-t}{k} W^m(\cdot)\,, \tag{4.2}$$

for $t_m \le t \le t_{m+1}$ and $0 \le m \le M-1$. Note that W^0 is defined in Corollary 3.5, and $U_{\varepsilon,h,k}(x,t)$ and $W_{\varepsilon,h,k}(x,t)$ are a continuous piecewise linear functions in space and time.

Let $\Gamma_{00} \subset \Omega$ be a smooth closed hypersurface and let $(w, \Gamma := \cup_{0\le t\le T}(\Gamma_t \times \{t\}))$ be a smooth solution of the Hele-Shaw problem (1.4)-(1.8) starting from Γ_{00} such that $\Gamma \subset \Omega \times [0,T]$. Let $d(x,t)$ denote the *signed distance function* to Γ_t such that $d(x,t) < 0$ in $\mathcal{I}_t$, the *inside* of Γ_t, and $d(x,t) > 0$ in $\mathcal{O}_t := \Omega \setminus (\Gamma_t \cup \mathcal{I}_t)$, the *outside* of Γ_t. We also define the *inside* $\mathcal{I}$ and the *outside* $\mathcal{O}$ of Γ as follows

$$\mathcal{I} := \{(x,t) \in \Omega \times [0,T]\,;\, d(x,t) < 0\}, \qquad \mathcal{O} := \{(x,t) \in \Omega \times [0,T]\,;\, d(x,t) > 0\}\,.$$

For the numerical solution $U_{\varepsilon,h,k}(x,t)$, we denote its zero level set at time t by $\Gamma_t^{\varepsilon,h,k}$, that is,

$$\Gamma_t^{\varepsilon,h,k} := \{x \in \Omega\,;\, U_{\varepsilon,h,k}(x,t) = 0\,\}\,. \tag{4.3}$$

Before we give a proof of Theorem 1.2, we need to recall the following convergence result of [2], see Theorem 5.1 of [2], which proved that the Hele-Shaw problem is the distinguished limit, as $\varepsilon \searrow 0$, of the Cahn-Hilliard equation.

THEOREM 4.1. *Let Ω be a given smooth domain and Γ_{00} be a smooth closed hypersurface in Ω. Suppose that the Hele-Shaw problem (1.4)-(1.8) starting from Γ_{00} has a smooth solution $(w, \Gamma := \cup_{0\le t\le T}(\Gamma_t \times \{t\}))$ in the time interval $[0,T]$ such that*

$\Gamma_t \subset \Omega$ for all $t \in [0,T]$. Then there exists a family of smooth functions $\{u_0^\varepsilon(x)\}_{0<\varepsilon\leq 1}$ which are uniformly bounded in $\varepsilon \in (0,1]$ and $(x,t) \in \overline{\Omega}_T$, such that if u^ε solves the Cahn-Hilliard equation (1.1)-(1.3), then

$$\text{(4.4)(i)} \quad \lim_{\varepsilon\to 0} u^\varepsilon(x,t) = \begin{cases} 1 & \text{if } (x,t) \in \mathcal{O} \\ -1 & \text{if } (x,t) \in \mathcal{I} \end{cases} \quad \text{uniformly on compact subsets,}$$

$$\text{(4.5)(ii)} \quad \lim_{\varepsilon\to 0} \Big(\frac{1}{\varepsilon} f(u^\varepsilon) - \varepsilon\Delta u^\varepsilon\Big)(x,t) = -w(x,t) \quad \text{uniformly on } \overline{\Omega}_T\,.$$

We now are ready to show Theorem 1.2.

Proof of Theorem 1.2. Let A be any compact subset of $\mathcal{O}$, for any $(x,t) \in A$, using the triangle inequality we have

$$\begin{aligned} |\, U_{\varepsilon,h,k}(x,t) - 1\,| &\leq |\, U_{\varepsilon,h,k}(x,t) - u^\varepsilon(x,t)\,| + |\, u^\varepsilon(x,t) - 1\,| \\ &\leq \|\, U_{\varepsilon,h,k} - u^\varepsilon\,\|_{L^\infty(\Omega_T)} + |\, u^\varepsilon(x,t) - 1\,|\,. \end{aligned} \tag{4.6}$$

Under the assumptions of Theorem 1.2, from (viii) of Theorem 1.1 we know that there exists a constant $0 < \alpha < \frac{4-N}{2}$ such that

$$\|\, U_{\varepsilon,h,k} - u^\varepsilon\,\|_{L^\infty(\Omega_T)} \leq C\, h^\alpha\,. \tag{4.7}$$

Here we have used the assumption $k = O(h^q)$ for some $\frac{2N}{3} < q < \frac{24-5N}{2}$.

Clearly, the first term on the right hand side of (4.6) converges to zero uniformly on A (and on Ω) as $h \searrow 0$. From (i) of Theorem 4.1 we know that the second term on the right hand side of (4.6) also converges to zero uniformly on A. Note that $h \searrow 0$ as $\varepsilon \searrow 0$. Therefore,

$$U_{\varepsilon,h,k} \overset{\varepsilon\searrow 0}{\longrightarrow} 1 \quad \text{uniformly on } A.$$

This then completes the proof of the assertion (i).

The proof of the assertion (ii) is almost same. The only change is to replace $\mathcal{O}$ by $\mathcal{I}$ and 1 by -1 in the above proof. So we omit it.

To show the assertion (iii), first, we notice that

$$w^\varepsilon = \frac{1}{\varepsilon} f(u^\varepsilon) - \varepsilon\Delta u^\varepsilon$$

if the solution u^ε of the Cahn-Hilliard equation (1.1)-(1.3) belongs to $W^{1,\infty}(J;L^2) \cap L^\infty(J;H^4)$. Next, from (x) of Theorem 1.1 we know that under the additional assumptions of Theorem 1.2 there exists a positive constant $0 < \zeta < \frac{q-2}{2}$ such that

$$\|\, W_{\varepsilon,h,k} - w^\varepsilon\,\|_{L^\infty(\Omega_T)} \leq C\, h^\zeta\,. \tag{4.8}$$

Here we have used the assumption $k = O(h^q)$ for some $2 < q < \frac{7}{2}$.

By the triangle inequality we have for any $(x,t) \in \overline{\Omega}_T$

$$\begin{aligned} |\, W_{\varepsilon,h,k}(x,t) - (-w)\,| &\leq |\, W_{\varepsilon,h,k}(x,t) - w^\varepsilon(x,t)\,| + |\, w^\varepsilon(x,t) - (-w)\,| \\ &\leq \|\, W_{\varepsilon,h,k} - w^\varepsilon\,\|_{L^\infty(\Omega_T)} + |\, w^\varepsilon(x,t) - (-w)\,|\,. \end{aligned} \tag{4.9}$$

The first term on the right hand side of (4.9) clearly converges to zero uniformly as $h \searrow 0$, and so does the second term due to (ii) of Theorem 4.1. Hence,

$$W_{\varepsilon,h,k}(x,t) \overset{\varepsilon\searrow 0}{\longrightarrow} -w(x,t) \quad \text{uniformly on } \overline{\Omega}_T\,.$$

The proof is complete. □

Remark: (a). The reason for us to only show assertion (iii) for $N = 2$ is that the current $L^\infty(J;L^\infty)$-estimate for $w^\varepsilon - W_{\varepsilon,h,k}$ in (x) of Theorem 1.1 is not strong enough to give a positive power of h (note $k = O(h^q)$) in the error bound when $N = 3$. To circumvent the difficulty, we need a better $L^\infty(J;L^\infty)$ estimate for $w^\varepsilon - W_{\varepsilon,h,k}$ which is similar to the one for $u^\varepsilon - U_{\varepsilon,h,k}$ in (viii) of Theorem 1.1. This can be done under the assumption $u_{tt} \in L^2(J;L^2)$ (need to derive a priori estimate in $\frac{1}{\varepsilon}$) and that the starting value U^0 satisfies the following stronger constraint

$$\| \nabla(P_h u_0^\varepsilon - U^0) \|_{L^2} \leq C\, h^2 . \tag{4.10}$$

See (b) of the remark after Theorem 3.4.

An immediate corollary of Theorem 1.1 is the following convergence result of the zero level set $\Gamma_t^{\varepsilon,h,k}$ of $U_{\varepsilon,h,k}$ to the free boundary Γ_t as stated in Theorem 1.3.

Proof of Theorem 1.3. For any $\eta \in (0,1)$, define the (open) tabular neighborhood $\mathcal{N}_\eta$ of width 2η of Γ as

$$\mathcal{N}_\eta := \{ (x,t) \in \Omega_T \,;\, d(x,t) < \eta \} . \tag{4.11}$$

Let A and B denote the complements of $\mathcal{N}_\eta$ in $\mathcal{O}$ and $\mathcal{I}$, respectively, that is

$$A = \mathcal{O} \setminus \mathcal{N}_\eta , \qquad B = \mathcal{I} \setminus \mathcal{N}_\eta .$$

Note that A is a compact subset of $\mathcal{O}$ and B is a compact subset of $\mathcal{I}$. Hence, from (i) and (ii) of Theorem 1.2 we know that there exists $\widehat{\varepsilon}_0 > 0$, which only depends on η, such that for all $\varepsilon \in (0, \widehat{\varepsilon}_0)$

$$| U_{\varepsilon,h,k}(x,t) - 1 | \leq \eta \quad \forall\, (x,t) \in A , \tag{4.12}$$

$$| U_{\varepsilon,h,k}(x,t) + 1 | \leq \eta \quad \forall\, (x,t) \in B . \tag{4.13}$$

Now for any $t \in [0,T]$ and $x \in \Gamma_t^{\varepsilon,h,k}$, since $U_{\varepsilon,h,k}(x,t) = 0$, we have

$$| U_{\varepsilon,h,k}(x,t) - 1 | = 1 , \tag{4.14}$$

$$| U_{\varepsilon,h,k}(x,t) + 1 | = 1 . \tag{4.15}$$

Evidently, (4.12) and (4.14) imply that $(x,t) \notin A$, and (4.13) and (4.15) says that $(x,t) \notin B$. Hence (x,t) must reside in the tubular neighborhood $\mathcal{N}_\eta$. Since t is an arbitrary number in $[0,T]$ and x is an arbitrary point on $\Gamma_t^{\varepsilon,h,k}$, therefore, for all $\varepsilon \in (0, \widehat{\varepsilon}_0)$

$$\sup_{x \in \Gamma_t^{\varepsilon,h,k}} \big(\operatorname{dist}(x, \Gamma_t) \big) \leq \eta \quad \text{uniformly on } [0,T] . \tag{4.16}$$

The proof is complete.□

We conclude this section and the paper with some discussions about the rate of convergence of $\Gamma_t^{\varepsilon,h,k}$ to Γ_t.

It is well-known (see [4, 11, 19]) that the solution for the Allen-Cahn equation $u_t^\varepsilon = \Delta u^\varepsilon - \frac{1}{\varepsilon^2} f(u^\varepsilon)$ approaches to ± 1 away from the (curvature driven) interface exponentially fast. This property allows to estimate the rate of convergence for the zero level sets of the solution of the Allen-Cahn equation and its numerical approximations to the true interface (see [6, 20, 22, 23] and references therein).

Unlike the situation for the Allen-Cahn equation, the solution u^ε of the Cahn-Hilliard equation (1.1)-(1.3) *does not* approach to ± 1 away from the (free boundary) interface exponentially fast, and the transition region from 1 to -1 could be "large" (see [2]). In fact, it was shown in Theorem 4.12 of [2] that this transition region is contained in a tubular neighborhood of width δ^* of Γ, where δ^* is a constant such that $\mathrm{dist}\,(\Gamma_t, \partial\Omega) > 2\delta^*$ for all $t \in [0,T]$. The combination of this result with $L^\infty(J; L^\infty)$ estimate for $u^\varepsilon - U_{\varepsilon,h,k}$ immediately leads to the conclusion of Theorem 1.4.

Proof of Theorem 1.4. From Theorem 4.12 and Theorem 5.1 of [2] we know that there exists an $\widehat{\varepsilon}_1 > 0$ and a constant $C^* > 0$ such that for all $\varepsilon \in (0, \widehat{\varepsilon}_1)$

$$\| u^\varepsilon - 1 \|_{C^0(\mathcal{O}\setminus\mathcal{N}_{\frac{\delta^*}{2}})} \leq C^*\,\varepsilon, \tag{4.17}$$

$$\| u^\varepsilon + 1 \|_{C^0(\mathcal{I}\setminus\mathcal{N}_{\frac{\delta^*}{2}})} \leq C^*\,\varepsilon. \tag{4.18}$$

Now for any $x \in \Gamma_t^{\varepsilon,h,k}$, since $U_{\varepsilon,h,k}(x,t) = 0$, from (i) of Theorem 4.1 we know that there exists an $\widehat{\varepsilon}_2 > 0$, which is independent of (x,t), such that

$$|\, u^\varepsilon \pm 1 \,| \geq 1 - |\, u^\varepsilon - U_{\varepsilon,h,k} \,| \geq 2\, C^*\,\varepsilon \tag{4.19}$$

for all $\varepsilon \in (0, \widehat{\varepsilon}_2)$. Then (4.17)-(4.19) implies that (x,t) must be in the tubular neighborhood $\mathcal{N}_{\frac{\delta^*}{2}}$ of Γ for all $\varepsilon \in (0, \min\{\widehat{\varepsilon}_1, \widehat{\varepsilon}_2\})$. The proof is complete. $\square$

Remark: In the case that the interface Γ_t is "close" to the boundary $\partial\Omega$ for all $t \in [0,T]$, say $\delta^* = O(\varepsilon^\gamma)$ for some $\gamma > 0$, then the numerical interface $\Gamma_t^{\varepsilon,h,k}$ approaches to the true interface Γ_t at a rate of no worse than $O(\varepsilon^\gamma)$ order.

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