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A NONCONFORMING FINITE-ELEMENT METHOD FOR THE TWO-DIMENSIONAL CAHN-HILLIARD EQUATION*

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Abstract. The Cahn-Hilliard equation is a nonlinear evolutionary equation that is fourth order in space. In this paper a continuous in-time finite-element Galerkin approximation is considered. We use the nonconforming Morley element and derive optimal order error bounds in L^2.

Key words. Cahn-Hilliard equation, nonconforming finite-element method, Morley element, biharmonic, nonlinear evolution equation

AMS(MOS) subject classifications. 35K22, 35K55, 35Q99, 65M60

1. Introduction.

We consider the Cahn-Hilliard equation

$$u_t+\Delta^2u=\Delta\phi(u),\qquad (x,t)\in\Omega\times(0,T) \tag{1.1a}$$

for $u(x,t)$, subject to the boundary conditions

$$\frac{\partial u}{\partial \nu}=0,\qquad \frac{\partial}{\partial \nu}(\phi(u)-\Delta u)=0\quad \text{on } \partial\Omega \tag{1.1b}$$

and the initial condition

$$u(\cdot,0)=u_0 \tag{1.1c}$$

where $\phi(\cdot)=\psi'(\cdot)$, $\psi(u)=\gamma(u^2-\beta^2)^2/4$, $\gamma>0$, Ω is the interior of a rectangle, and ν is the outward pointing normal to $\partial\Omega$. This initial-boundary value problem arises in the study of phase separation in binary mixtures (see Novick-Cohen and Segel [8] and Elliott and French [2] and the references cited therein). We use the notation

$$H_E^2(\Omega)=\left\{\eta\in H^2(\Omega): \frac{\partial\eta}{\partial\nu}=0\quad \text{on } \partial\Omega\right\} \tag{1.2a}$$

and observe that there exist $C>0$ and $C_\varepsilon>0$ such that for each $w\in H_E^2(\Omega)$

$$|w|^2_{2,2,\Omega}\leqq C(|\Delta w|^2_{0,2,\Omega}+|w|^2_{0,2,\Omega}), \tag{1.2b}$$

$$|w|^2_{1,2,\Omega}\leqq (\varepsilon|\Delta w|^2_{2,2,\Omega}+C_\varepsilon|w|^2_{0,2,\Omega}) \tag{1.2c}$$

where for a set $A\subseteq\mathbb{R}^2$,

$$\|z\|^p_{0,p,A}=\int_A|z|^p,\qquad 1\leqq p<\infty,$$

$$\|z\|_{0,\infty,A}=\operatorname*{ess\,sup}_A|z|,$$

$$|z|^p_{m,p,A}=\sum_{|\alpha|=m}\|D^\alpha z\|^p_{0,p,A},\qquad 1\leqq p<\infty,$$

$$\|z\|^p_{m,p,A}=\sum_{j=0}^m|z|^p_{j,p,A}.$$

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We have the following weak form of (1.1). Find $u(\cdot, t) \in H_E^2(\Omega)$ such that

$$(u_t, v) + a(u, v) = (\Delta\phi(u), v) \quad \forall v \in H_E^2(\Omega), \tag{1.3a}$$

$$u(\cdot, 0) = u_0 \in H_E^2(\Omega) \tag{1.3b}$$

where $a(\cdot, \cdot)$ is a bilinear form commonly used in fourth-order problems:

$$a(w, z) = \int_\Omega \left(\Delta w \, \Delta z + \frac{\partial^2 w}{\partial x_1 \, \partial x_2} \frac{\partial^2 z}{\partial x_1 \, \partial x_2} - \frac{1}{2} \frac{\partial^2 w}{\partial x_1^2} \frac{\partial^2 z}{\partial x_1^2} - \frac{1}{2} \frac{\partial^2 w}{\partial x_2^2} \frac{\partial^2 z}{\partial x_1^2} \right)$$

with Poisson's coefficient set to $\frac{1}{2}$. Note that

$$a(w, w) = \tfrac{1}{2}(\|\Delta w\|_{0,2,\Omega}^2 + |w|_{2,2,\Omega}^2) \quad \forall w \in H^2(\Omega). \tag{1.4}$$

For $w, z \in H_E^2(\Omega)$, we have by Green's formula, since $\partial z/\partial \nu = \partial/\partial s(\partial w/\partial \nu) = 0$,

$$a(w, z) = \int_\Omega \Delta^2 w \cdot z + \int_{\partial\Omega} \left(\Delta w - \frac{1}{2} \frac{\partial^2 w}{\partial s^2} \right) \frac{\partial z}{\partial \nu} - \int_{\partial\Omega} \frac{\partial}{\partial \nu} (\Delta w) \cdot z + \frac{1}{2} \int_{\partial\Omega} \frac{\partial^2 w}{\partial \nu \, \partial s} \frac{\partial z}{\partial s},$$

so that if u satisfies (1.3) and is sufficiently smooth, then it also solves (1.1).

Regularity. In the Appendix we prove that for each $T > 0$ and $u_0 \in H_E^2(\Omega)$ there exists a unique solution such that

$$u \in L^\infty(0, T; H_E^2(\Omega)), \quad u \in L^2(0, T; H^4(\Omega)), \quad u_t \in L^2(0, T; L^2(\Omega)) \tag{1.5}$$

(see also Elliott and Zheng [3]). Furthermore, if $-\Delta^2 u_0 + \Delta\phi(u_0) \in H_E^2(\Omega)$, then

$$u_t \in L^2(0, T; H^4(\Omega)). \tag{1.6}$$

Finite-element approximation. Let $\bar{\Omega} = \bigcup_{\tau \in \mathcal{T}^h} \bar{\tau}$, where $\mathcal{T}^h$ is a family of quasiuniform triangulations of Ω with $0 < h \leqq h_0 < 1$. This means there exist constants β_0 and β_1 *independent of* h such that for any $\tau \in \mathcal{T}^h$, $\beta_0 h \leqq \operatorname{diam}(\tau) \leqq \beta_1 h$ and $\beta_0 h^2 \leqq \operatorname{meas}(\tau)$. Let S^h be the finite-element space consisting of Morley's nonconforming shape functions defined on $\mathcal{T}^h$:

> $S^h = \{\chi \in L^\infty(\Omega): \chi|_\tau$ is a quadratic polynomial, χ is continuous at the vertices of $\mathcal{T}^h$, the normal derivative $\partial\chi|\partial\nu$ is continuous at the midpoints of all edges of triangles $\tau \in \mathcal{T}^h\}$.

To approximate $H_E^2(\Omega)$ we use

$$S_E^h = \{\chi \in S^h : \partial\chi|\partial\nu = 0 \text{ at the midpoints of edges on } \partial\Omega\}.$$

The following approximation property holds for each $p \in [1, \infty]$:

$$|v - I_h v|_{j,p,\tau} \leqq C h^{3-j} |v|_{3,p,\tau} \quad \forall \tau \in \mathcal{T}^h, \quad j = 0, 1, 2 \tag{1.7}$$

where I_h is the usual interpolation operator into S^h and C is a positive constant dependent only on mesh parameters.

It is convenient to use the mesh-dependent seminorms and norms

$$|v|_{j,p;h} = \left(\sum_{\tau \in \mathcal{T}^h} |v|_{j,p,\tau}^p \right)^{1/p},$$

$$\|v\|_{j,p;h} = \left(\sum_{\tau \in \mathcal{T}^h} \|v\|_{j,p,\tau}^p \right)^{1/p},$$

the mesh-dependent inner product

$$(w, v)_h = \sum_{\tau \in \mathcal{T}^h} \int_\tau w(x) v(x) \, dx,$$

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and the notation

$$H^{2,h}(\Omega)=S^h\oplus H^2(\Omega),\qquad H^{1,h}(\Omega)=S^h\oplus H^1(\Omega),$$
$$H_E^{2,h}(\Omega)=S_E^h\oplus H_E^2(\Omega),\qquad H_E^{1,h}(\Omega)=S_E^h\oplus H^1(\Omega)$$

where, for instance,

$$S^h\oplus H^2(\Omega)=\{v+\chi\colon v\in H^2(\Omega) \text{ and } \chi\in S^h\}.$$

Our semidiscrete Galerkin method is as follows. Find $u_h(\cdot,t)\in S_E^h$ such that

$$(u_{h,t},\chi)+a_h(u_h,\chi)+(\nabla\phi(u_h)\cdot\nabla\chi)_h=0\quad \forall\chi\in S_E^h, \tag{1.8a}$$

$$u_h(\cdot,0)=u_0^h\in S_E^h \tag{1.8b}$$

where

$$a_h(w,z)\equiv\sum_{\tau\in\mathcal{T}^h}\int_\tau\left\{\Delta w\,\Delta z+\frac{\partial^2 w}{\partial x_1\,\partial x_2}\frac{\partial^2 z}{\partial x_1\,\partial x_2}-\frac{1}{2}\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 z}{\partial x_2^2}-\frac{1}{2}\frac{\partial^2 w}{\partial x_2^2}\frac{\partial^2 z}{\partial x_1^2}\right\} \tag{1.8c}$$

and

$$a_h(w,w)=\tfrac{1}{2}(\|\Delta w\|_{0,2;h}^2+|w|_{2,2;h}^2)\quad\forall w\in H^{2,h}(\Omega). \tag{1.9}$$

We use $a_h(w,z)$ instead of a form with $(\Delta w,\Delta z)_h$ on $H^{2,h}(\Omega)$ to guarantee the equivalence, independent of h, with respect to $|\cdot|_{2,2;h}$.

Since (1.8) is a nonlinear system of ordinary differential equations there exists a unique solution, at least locally. Taking $\chi=u_h$ in (1.8a) and using the fact that $\phi'(\cdot)\geqq-\gamma\beta^2$, we may obtain an a priori estimate for u_h when h is sufficiently small. The argument is similar to the one in Elliott and Zheng [3]. Our Lemma 2.4 must be used in this argument. Thus, for h sufficiently small (1.8) has a unique, global-in-time solution.

Analysis of the biharmonic equation using this nonconforming method can be found in Lascaux and Lesaint [5], Rannacher [9], and Arnold and Brezzi [1]. In this paper we extend their analysis to the nonlinear time-dependent equation (1.1).

Let $\mathcal{R}\equiv\{\eta\in H_E^2(\Omega)\colon\Delta\eta\in H_E^2(\Omega)\}$ and note that by Theorem A.1 of the Appendix $\mathcal{R}\subset H^4(\Omega)$. It is convenient to introduce an elliptic projection $P^h v\in S_E^h$ for $v\in\mathcal{R}$ defined by

$$b_h(P^h v,\psi)=(\Delta^2 v-\nabla(\phi'(u)\nabla v)+\alpha v,\psi)\quad\forall\psi\in S_E^h \tag{1.10a}$$

where $b_h(\cdot,\cdot)$ is the bilinear form

$$b_h(w,\psi)=a_h(w,\psi)+(\phi'(u)\nabla w,\psi)_h+\alpha(w,\psi) \tag{1.10b}$$

and $\alpha\geqq\alpha_0$ for some positive α_0. Since $\phi'(u)\geqq-\gamma\beta^2$ we have that for $z\in H^{2,h}(\Omega)$

$$b_h(z,z)\geqq\tfrac{1}{2}(|\Delta z|_{0,2;h}^2+|z|_{2,2;h}^2)-\gamma\beta^2|z|_{1,2;h}^2+\alpha|z|_{0,2;h}^2$$

and, from (1.2c) and Lemma 2.4 (of § 2), it is clear that α_0 can be chosen independently of h and u such that

$$b_h(z,z)\geqq c\|z\|_{2,2;h}^2\quad\forall z\in H^{2,h}(\Omega) \tag{1.11}$$

where c is a constant independent of h, u, and z. Thus $b_h(\cdot,\cdot)$ is a continuous, coercive bilinear form on $S_E^h\times S_E^h$ and $P^h v$ is well defined. We will need the following approximation property of P^h that is proved in § 5.

PROPOSITION 1.1. *Suppose u solves* (1.1) *and* $P^h u$ *is defined by* (1.10); *then there exists C independent of h such that*

$$\|u - P^h u\|_{1,2;h} + h|u - P^h u|_{2,2;h} \leqq Ch^2, \tag{1.12}$$

$$\|u_t - (P^h u)_t\|_{1,2;h} + h|u_t - (P_h u)_t|_{2,2;h} \leqq Ch^2, \tag{1.13}$$

provided u and $u_t \in \mathcal{R}$. □

The main result of this paper is the following theorem that is proved in § 3.

THEOREM 1.1. *Let u solve* (1.1) *and* u_h *solve* (1.8). *Suppose* $u_0 \in H_E^2(\Omega)$, $-\Delta^2 u_0 + \Delta\phi(u_0) \in H_E^2(\Omega)$, *and*

$$u_h(\cdot, 0) = P^h u_0; \tag{1.14}$$

then there exists C independent of h and h_0 *such that for* $h < h_0$

$$\|u - u_h\|_{1,2;h} + h|u - u_h|_{2,2;h} \leqq Ch^2 \quad \forall t \in [0, T]. \tag{1.15}$$

We consider this result optimal since it is well known that $O(h^2)$ convergence is optimal in H^1 and L^2 and $O(h)$ convergence is optimal in H^2 for finite-element approximations of the biharmonic equation using the Morley elements. These rates of convergence were found in our one-dimensional numerical experiments with C^1 quadratics (see Elliott and French [2]).

In § 4, we briefly derive optimal-order convergence of u_h to u in L^∞.

We will frequently use the following inverse inequalities that hold for quadratic functions. Let $\tau \in \mathcal{T}^h$ and $\chi \in S^h$; then

$$\|\chi\|_{l,p,\tau} \leqq Ch^r \|\chi\|_{m,q,\tau}$$

where $r = m - l - 2((1/q) - (1/p))$ for $0 \leqq m \leqq l \leqq 2$ and $1 \leqq q \leqq p \leqq \infty$.

2. Some auxiliary lemmas.

LEMMA 2.1. *Let* $z \in H_E^{2,h}(\Omega)$ *and* $w \in W^{1,p}(\Omega)$; *then for* $1 \leqq p, q \leqq \infty$, *and* $(1/p) + (1/q) = 1$:

$$\left| \sum_{\tau \in \mathcal{T}^h} \int_{\partial\tau} w \frac{\partial z}{\partial \nu} \right| \leqq Ch|w|_{1,p,\Omega} |z|_{2,q;h}. \tag{2.1}$$

Proof. The following equation holds:

$$\sum_{\tau \in \mathcal{T}^h} \int_{\partial\tau} w \frac{\partial z}{\partial \nu} = \sum_{\Gamma \not\in \partial\Omega} \int_\Gamma w \left[\frac{\partial z}{\partial \nu}\right] + \sum_{\Gamma \in \partial\Omega} w \frac{\partial z}{\partial \nu} \tag{2.2}$$

where Γ is a typical triangle side and the summations are over triangle sides. $[\partial z/\partial \nu]$ denotes the jump in $\partial z/\partial \nu$ across Γ. We note that at x_m the midpoint of Γ,

$$\left[\frac{\partial z}{\partial \nu}\right] = 0, \quad \Gamma \not\in \partial\Omega, \qquad \frac{\partial z}{\partial \nu} = 0, \quad \Gamma \in \partial\Omega.$$

The estimation of the first sum of (2.2) will be given and the second sum can be treated in a similar way. Change to a local coordinate system such that Γ is $(0, H)$ on the x_2-axis. We first show that

$$\int_\Gamma |w - m| \leqq Ch^{2/p} |w|_{1,p,\tau} \tag{2.3}$$

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where τ is a triangle with side Γ and m is the mean value of w on τ. For any $(\tilde{x}_1, \tilde{x}_2) \in \tau$ we have

$$w(0, x_2) = 2(\tilde{x}_1, \tilde{x}_2) + \int_{\tilde{x}_2}^{x_2} \frac{\partial w}{\partial x_2}(\tilde{x}_1, s)\, ds + \int_{\tilde{x}_1}^{0} \frac{\partial w}{\partial x_1}(t, x_2)\, dt$$

and an integration over τ with respect to $(\tilde{x}_1, \tilde{x}_2)$ yields

$$|w(0, x_2) - m| \leqq \frac{Ch}{\text{meas}\,(\tau)} |w|_{1,1,\tau}.$$

Noting that meas $(\tau) \geqq Ch^2$, applying Hölder's inequality to $|w|_{1,1,\tau}$ and integrating to x_2 we obtain (2.3).

Let τ' and τ'' be the elements with common side Γ and denote by z' and z'' the restrictions of z to these elements. Since $z \in H_E^{2,h}(\Omega)$ it follows that $\partial z'/\partial \nu(x_m) = \partial z''/\partial \nu(x_m) = m_1$ and that $[\partial z/\partial \nu]$ is linear on Γ. Hence we obtain, using an inverse inequality,

$$\begin{aligned}
\left\| \left[\frac{\partial z}{\partial \nu} \right] \right\|_{0,\infty,\Gamma} &= \left\| \left[\frac{\partial z}{\partial \nu} - m_1 \right] \right\|_{0,\infty,\Gamma} \\
&\leqq Ch^{-1} \int_\Gamma \left| \left[\frac{\partial z}{\partial \nu} - m_1 \right] \right| \\
&\leqq Ch^{-1} \left\{ \int_\Gamma \left| \frac{\partial z'}{\partial \nu} - m_1 \right| + \int_\Gamma \left| \frac{\partial z''}{\partial \nu} - m_1 \right| \right\}.
\end{aligned}$$

By an argument similar to the derivation of (2.3), the vanishing of $\partial z'/\partial \nu - m_1$ and $\partial z''/\partial \nu - m_1$ at the midpoint of Γ yields

$$\begin{aligned}
\int_\Gamma \left| \frac{\partial z'}{\partial \nu} - m_1 \right| &\leqq C \left| \frac{\partial z'}{\partial \nu} \right|_{1,1,\tau'} \leqq C|z|_{2,1,\tau'}, \\
\int_\Gamma \left| \frac{\partial z''}{\partial \nu} - m_1 \right| &\leqq C \left| \frac{\partial z''}{\partial \nu} \right|_{1,1,\tau''} \leqq C|z|_{2,1,\tau''},
\end{aligned}$$

so that, by Hölder's inequality,

$$\begin{aligned}
\left\| \left[\frac{\partial z}{\partial \nu} \right] \right\|_{0,\infty,\Gamma} &\leqq Ch^{-1}(|z|_{2,1,\tau'} + |z|_{2,1,\tau''}) \\
&\leqq Ch^{2/p-1}(|z|_{2,q,\tau'}^q + |z|_{2,q,\tau''}^q)^{1/q}.
\end{aligned} \tag{2.4}$$

Since $[\partial z/\partial \nu]$ is linear on Γ and vanishes at the midpoint,

$$\int_\Gamma w \left[\frac{\partial z}{\partial \nu} \right] = \int (w - m) \left[\frac{\partial z}{\partial \nu} \right];$$

therefore (2.1) follows from (2.3), (2.4), and Hölder's inequality, implying that

$$\sum_\tau |w|_{1,p,\tau} |z|_{2,q,\tau} \leqq |w|_{1,p;h} |z|_{2,q;h}. \qquad \square$$

LEMMA 2.2. *Let $z \in H_E^{2,h}(\Omega)$ and $w \in W^{1,p}(\Omega)$; then for $1 \leqq p, q \leqq \infty$ and $1/p + 1/q = 1$*

$$\left| \sum_{\tau \in \mathcal{T}_h} \int_{\partial \tau} w \frac{\partial z}{\partial s} \right| \leqq Ch |w|_{1,p,\Omega} |z|_{2,q;h}. \tag{2.5}$$

Proof. The proof is identical to the one in Lemma 2.1 since

$$\int_\Gamma \left[\frac{\partial z}{\partial s}\right] = 0 \tag{2.6}$$

for any edge $\Gamma \notin \partial\Omega$ since z is continuous at the vertices of τ. Furthermore, $[\partial z/\partial s] = 0$ at the midpoint of Γ. □

LEMMA 2.3. *Let* $w, z \in H_E^{2,h}(\Omega)$; *then*

$$\left|\sum_{\tau\in\mathcal{T}_h}\int_{\partial\tau}\frac{\partial w}{\partial\nu}z\right| \leqq Ch(h\|w\|_{2,2;h}\|z\|_{2,2;h} + \|w\|_{1,2;h}\|z\|_{2,2;h} + \|w\|_{2,2;h}\|z\|_{1,2;h}). \tag{2.7}$$

Proof. The following holds:

$$\sum_{\tau\in\mathcal{T}_h}\int_{\partial\tau}\frac{\partial w}{\partial\nu}z = \sum_{\Gamma\notin\partial\Omega}\int_\Gamma\frac{\partial w}{\partial\nu}[z] + \sum_{\Gamma\in\partial\Omega}\int_\Gamma\frac{\partial w}{\partial\nu}z + \sum_{\Gamma\notin\partial\Omega}\int_\Gamma\left[\frac{\partial w}{\partial\nu}\right]z.$$

The last two terms are estimated by the same argument as in Lemma 2.1 and result in the third term of (2.7). For the first term

$$\begin{aligned}\int_\Gamma\frac{\partial w}{\partial\nu}[z] &\leqq \left(\int_\Gamma\left|\frac{\partial w}{\partial\nu}\right|\right)\|[z]\|_{0,\infty,\Gamma}\\ &\leqq Ch^2\left(\int_\Gamma\left(\left|\frac{\partial w}{\partial\nu} - m\right| + m\right)\right)\left\|\frac{\partial^2 z}{\partial s^2}\right\|_{0,\infty,\Gamma}\end{aligned}$$

where m is the mean value of $\partial w/\partial\nu$ over τ; the argument used to prove (2.3) can be applied here to yield

$$\int_\Gamma\frac{\partial w}{\partial\nu}[z] \leqq Ch(h|w|_{2,2,\tau} + |w|_{1,2,\tau})\|z\|_{2,2,\tau}.$$

The second factor was estimated using an inverse inequality as follows:

$$\left\|\frac{\partial^2 z}{\partial s^2}\right\|_{0,\infty,\Gamma} \leqq |z|_{2,\infty,\tau} \leqq Ch^{-1}\|z\|_{2,2,\tau}.$$ □

LEMMA 2.4. *Let* $\chi\in S_E^h$ *and* $0<\varepsilon\leqq\frac{1}{2}$; *then there exist constants* C_0 *and* C_ε *such that*

$$|\chi|_{1,2;h}^2 \leqq C_0(\varepsilon + h)|\chi|_{2,2;h}^2 + C_\varepsilon\|\chi\|_{0,2;h}^2. \tag{2.8}$$

Proof. By Green's theorem we have

$$|\chi|_{1,2;h}^2 = \sum_{\tau\in\mathcal{T}_h}\left(\int_{\partial\tau}\chi\frac{\partial\chi}{\partial\nu} - \int_\tau\Delta\chi\cdot\chi\right). \tag{2.9}$$

We estimate the boundary term by Lemma 2.3 and obtain

$$\sum_{\tau\in\mathcal{T}_h}\int_{\partial\tau}\chi\frac{\partial\chi}{\partial\nu} \leqq Ch\|\chi\|_{2,2;h}^2. \tag{2.10}$$

For the second term in (2.9) we use the Cauchy-Schwarz inequality:

$$\int_\Gamma\Delta\chi\cdot\chi \leqq \varepsilon\|\Delta\chi\|_{0,2,\tau}^2 + C_\varepsilon\|\chi\|_{0,2,\tau}^2. \tag{2.11}$$

Combining (2.9), (2.10), and (2.11) gives (2.8). □

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The following important boundary term that arises in our analysis:

$$B_h(w, z) = \sum_{\tau \in \mathcal{T}_h} \int_{\partial\tau} \left(\Delta w \frac{\partial z}{\partial \nu} + \frac{1}{2}\left(\frac{\partial^2 w}{\partial \nu\, \partial s}\frac{\partial z}{\partial s} - \frac{\partial^2 w}{\partial s^2}\frac{\partial z}{\partial \nu} \right)\right) \tag{2.12}$$

can be estimated by an application of Lemmas 2.1 and 2.2.

LEMMA 2.5. *Let $z \in H^{2,h}(\Omega)$ and $w \in H^2_E(\Omega) \cap W^{3,p}(\Omega)$; then for $1 \leqq p, q \leqq \infty$, and $1/p + 1/q = 1$*

$$|B_h(w, z)| \leqq Ch|w|_{3,p,\Omega}|z|_{2,q;h}. \tag{2.13}$$

LEMMA 2.6. *Let τ' and τ'' be adjacent elements in $\mathcal{T}^h$ with an intersecting edge Γ whose endpoints are denoted by a and b. Suppose that $w' \in H^3(\tau')$, $w'' \in H^3(\tau'')$, $w'(a) = w''(a)$, $w'(b) = w''(b)$, and*

$$w = \begin{cases} w' & \text{on } \tau', \\ w'' & \text{on } \tau''. \end{cases}$$

It follows that for all $\chi \in S^h$

$$\int_\Gamma [w]\frac{\partial \chi}{\partial \nu} \leqq Ch(h(|w|_{3,2,\tau'} + |w|_{3,2,\tau''}) + |w|_{2,2,\tau'} + |w|_{2,2,\tau''})|\chi|_{1,2,\tau'} \tag{2.14}$$

where $\partial\chi/\partial\nu$ is evaluated on τ'.

Proof. Since $[w] = 0$ at a and b, we have

$$\int_\Gamma [w]\frac{\partial \chi}{\partial \nu} \leqq Ch^2 \left\| \frac{\partial^2}{\partial s^2}[w] \right\|_{0,1,\Gamma} |\chi|_{1,\infty,\tau'}.$$

Let $z' = \partial^2 w'/\partial s^2$ and $m = 1/\text{meas } \tau' \int_{\tau'} z'$; it follows that

$$\begin{aligned} \|z'\|_{0,1,\Gamma} &\leqq \|z' - m\|_{0,1,\Gamma} + Chm \\ &\leqq Ch|z'|_{1,2,\tau'} + C|z'|_{0,2,\tau'} \end{aligned}$$

where (2.3) was applied to the first term. A similar argument applied to $\partial^2 w''/\partial s^2$ and the inverse norm inequality $|\chi|_{1,\infty,\tau'} \leqq Ch^{-1}|\chi|_{1,2,\tau'}$ implies the lemma. □

3. Convergence in $\mathbf{H^1}$ and $\mathbf{H^2}$. In this section we prove Theorem 1.1 concerning the error bounds in $H^{2,h}$ and $H^{1,h}$. Because of the assumptions on the initial data we have the regularity specified by Theorem A.2 of the Appendix which we shall use in this section without comment. Our argument is based on the error decomposition

$$u - u_h = (u - P^h u) + (P^h u - u_h) \equiv \rho + \theta, \tag{3.1}$$

which is often used for parabolic equations (see Wheeler [12] and Thomée [11]). We need only to estimate θ due to the projection error bounds of Proposition 1.1 which imply

$$\|\rho\|_{1,2;h} + h|\rho|_{2,2;h} \leqq Ch^2, \tag{3.2a}$$

$$\|\rho_t\|_{1,2;h} + h|\rho_t|_{2,2;h} \leqq Ch^2, \tag{3.2b}$$

$$\|P^h u\|_{1,\infty;h} + \|(P^h u)_t\|_{1,\infty;h} \leqq C \tag{3.2c}$$

where

$$\|\chi\|_{m,\infty;h} = \max_{\substack{1 \leqq j \leqq m \\ \tau \in \mathcal{T}^h}} |\chi|_{m,\infty,\tau},$$

$$|\chi|_{m,\infty;h} = \max_{\tau \in \mathcal{T}^h} |\chi|_{j,\infty,\tau}.$$

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Inequality (3.2c) is a consequence of (3.2a, b) since

$$\begin{aligned}\left\|\left(\frac{\partial}{\partial t}\right)^j (I_h u - P^h u)\right\|_{1,\infty;h} &\leqq Ch^{-1}\left\|\left(\frac{\partial}{\partial t}\right)^j (I_h u - P^h u)\right\|_{1,2;h}\\ &\leqq Ch^{-1}\left\{\left\|\left(\frac{\partial}{\partial t}\right)^j (I_h u - u)\right\|_{1,2;h} + \left\|\left(\frac{\partial}{\partial t}\right)^j \rho\right\|_{1,2;h}\right\}\\ &\leqq Ch,\end{aligned}$$

$$\left\|\left(\frac{\partial}{\partial t}\right)^j I_h u\right\|_{1,\infty;h} \leqq C\left\|\left(\frac{\partial}{\partial t}\right)^j u\right\|_{1,\infty;h},$$

$$\left\|\left(\frac{\partial}{\partial t}\right)^j P_h u\right\|_{1,\infty;h} \leqq \left\|\left(\frac{\partial}{\partial t}\right)^j (I_h u - P^h u)\right\|_{1,\infty;h} + \left\|\left(\frac{\partial}{\partial t}\right)^j I_h u\right\|_{1,\infty;h}.$$

We shall also assume in the following that

$$\|u^h\|_{1,\infty;h} \leqq C \tag{3.3}$$

where C depends on T. This assumption (3.3) will be justified at the end of the section.

It follows from (1.8) and (1.10) that for any $\chi \in S_E^h$

$$\begin{aligned}(\theta_t, \chi) + a_h(\theta, \chi) &= [((P^h u)_t, \chi) + a_h(P^h u, \chi)] - [(u_{h,t}, \chi) + a_h(u_h, \chi)]\\ &= -(\rho_t, \chi) + (u_t + \Delta^2 u - \Delta\phi(u) + \alpha u, \chi)\\ &\quad - [(\phi'(u)\nabla P_h u, \nabla\chi)_h + \alpha(P^h u, \chi)] + (\nabla\phi(u_h), \nabla\chi)_h.\end{aligned}$$

Hence we obtain the fundamental equation for the error

$$(\theta_t, \chi) + a_h(\theta, \chi) = (-\rho_t + \alpha\rho, \chi) - (\phi'(u)\nabla P^h u - \nabla\phi(u^h), \nabla\chi)_h \quad \forall \chi \in S_E^h. \tag{3.4}$$

Taking $\chi = \theta$ we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|_{0,2,\Omega}^2 + a_h(\theta, \theta) &\leqq \|\rho_t - \alpha\rho\|_{0,2,\Omega}\|\theta\|_{0,2,\Omega}\\ &\quad + \|\phi'(u)\nabla P^h u - \nabla\phi(u^h)\|_{0,2,\Omega}|\theta|_{1,2;h}\end{aligned}$$

and noting that, by (3.2c) and (3.3),

$$\begin{aligned}\|\phi'(u)\nabla P^h u - \nabla\phi(u^h)\|_{0,2,\Omega} &\leqq \|(\phi'(u) - \phi'(P^h u))\nabla P^h u\|_{0,2,\Omega}\\ &\quad + \|(\phi'(P^h u) - \phi'(u^h))\nabla P^h u\|_{0,2,\Omega}\\ &\quad + \|\phi'(u^h)(\nabla P^h u - \nabla u^h)\|_{0,2,\Omega}\\ &\leqq C(\|\rho\|_{0,2,\Omega} + \|\theta\|_{1,2;h})\end{aligned}$$

we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|_{0,2,\Omega}^2 + a_h(\theta, \theta) &\leqq (\|\rho_t\|_{0,2,\Omega} + \alpha\|\rho\|_{0,2,\Omega})\|\theta\|_{0,2,\Omega}\\ &\quad + C(\|\rho\|_{0,2,\Omega}|\theta|_{1,2;h} + |\theta|_{1,2;h}\|\theta\|_{1,2;h}).\end{aligned}$$

Applying the Cauchy-Schwarz inequality, Lemma 2.4, and (1.9), we obtain

$$\frac{d}{dt}\|\theta\|_{0,2,\Omega}^2 + |\theta|_{2,2;h}^2 \leqq C(\|\rho\|_{0,2,\Omega}^2 + \|\rho\|_{0,2,\Omega}^2 + \|\theta\|_{0,2,\Omega}^2). \tag{3.5}$$

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It now follows from Gronwall's inequality and (3.2a, b) that

$$\|\theta(\cdot,t)\|_{0,2,\Omega}^2+\int_0^T |\theta|_{2,2;h}^2\,dt' \leqq Ch^4 \tag{3.6}$$

where the fact that $\theta(\cdot,0)=0$ has been used.

Taking $\chi=\theta_t$ in (3.4) we obtain

$$\|\theta_t\|_{0,2,\Omega}^2+\frac{1}{2}\frac{d}{dt}a_h(\theta,\theta)\leqq(\|\rho_t\|_{0,2,\Omega}+\alpha\|\rho\|_{0,2,\Omega})\|\theta_t\|_{0,2,\Omega}$$
$$-(\phi'(u)\nabla P^h u-\nabla\phi(u_h),\nabla\theta_t)_h$$

and after an integration with respect to t, we have

$$\int_0^t \|\theta_t\|_{0,2,\Omega}^2\,dt'+a_h(\theta,\theta)\leqq I_1+I_2 \tag{3.7}$$

where

$$I_1\leqq C_\varepsilon\int_0^t\{\|\rho_t\|_{0,2,\Omega}^2+\alpha^2\|\rho\|_{0,2,\Omega}^2\}\,dt'+\varepsilon\int_0^t\|\theta_t\|_{0,2,\Omega}^2\,dt',$$
$$I_2=-\int_0^t(\phi'(u)\nabla P^h u-\nabla\phi(u^h),\nabla\theta_t)_h\,dt'$$

where $\varepsilon>0$.

We will need the following inequalities that may be verified by a straightforward calculation. Recall that $|\lambda|_{3,2;h}\equiv 0$ for all $\lambda\in S^h$. Using the differentiability of ϕ, we have

$$\|\phi(\mu)-\phi(\lambda)\|_{3,2;h}\leqq C\|\mu-\lambda\|_{2,2;h}\quad\forall\mu,\lambda\in S^h \tag{3.8}$$

where $C=C(\|\mu\|_{2,\infty;h},\|\lambda\|_{1,\infty;h})$;

$$\left|\frac{\partial}{\partial t}(\phi(u)-\phi(P^h u))\right|_{2,2;h}\leqq C(\|\rho\|_{2,2;h}+\|\rho_t\|_{2,2;h}) \tag{3.9a}$$

and

$$\left\|\frac{\partial}{\partial t}(\phi(u)-\phi(P^h u))\right\|_{3,2;h}\leqq C\sum_{j=0}^{1}\left(\left\|\left(\frac{\partial}{\partial t}\right)^j\rho\right\|_{2,2;h}+\left\|\left(\frac{\partial}{\partial t}\right)^j u\right\|_{3,2;h}\right) \tag{3.9b}$$

where

$$C=C\left(\left\|\left(\frac{\partial}{\partial t}\right)^j u\right\|_{2,\infty;h},\left\|\left(\frac{\partial}{\partial t}\right)^j P^h u\right\|_{1,\infty;h};j=0,1\right).$$

Since (3.2c) holds, (3.8) and (3.9) can be used appropriately so that the constants C are independent of h.

We first estimate the more difficult term I_2:

$$\begin{aligned}I_2&=\int_0^t((\phi'(u)\nabla P^h u-\nabla\phi(u_h))_t,\nabla\theta)_h\,dt'+(\phi'(u)\nabla P^h u-\nabla\phi(u_h),\nabla\theta)_h\\&=\int_0^t((\phi'(u)\nabla(P^h u-u))_t,\nabla\theta)_h\,dt'-\int_0^t((\phi(u)-\phi(u_h))_t,\Delta\theta)_h\,dt'\\&\quad+\int_0^t\sum_{\tau\in\mathcal{T}_h}\int_{\partial\tau}(\phi(u)-\phi(u_h))_t\frac{\partial\theta}{\partial\nu}\,dt'+(\phi'(u)\nabla P^h u-\nabla\phi(u_h),\nabla\theta)_h\\&=J_1+J_2+J_3+J_4\end{aligned} \tag{3.10}$$

where first we integrated by parts with respect to t. Again we handle the most difficult

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term first:

$$J_3=\int_0^t\left(\sum_{\Gamma\not\in\partial\Omega}\left(\int_\Gamma\left[(\phi(u)-\phi(u_h))_t\right]\frac{\partial\theta}{\partial\nu}+\int_\Gamma(\phi(u)-\phi(u_h))_t\left[\frac{\partial\theta}{\partial\nu}\right]\right)\right.$$
$$\left.+\sum_{\Gamma\in\partial\Omega}\int_\Gamma(\phi(u)-\phi(u_h))_t\frac{\partial\theta}{\partial\nu}\right)dt'$$
$$=A_1+A_2+A_3.$$

We estimate A_1:

$$A_1=\int_0^t\left(\sum_{\Gamma\not\in\partial\Omega}\int_\Gamma[(\phi(u)-\phi(P_hu))_t]\frac{\partial\theta}{\partial\nu}+\int_\Gamma[(\phi(P^hu)-\phi(u_h))]_t\right)\frac{\partial\theta}{\partial\nu}\,dt'. \tag{3.11}$$

Taking $w=\partial/\partial t(\phi(u)-\phi(P^hu))$ and $\chi=\theta$ in (2.14), we have from (3.9) that the first term in (3.11) is bounded as follows:

$$\begin{aligned}\sum_{\Gamma\not\in\partial\Omega}\int_\Gamma[w]\frac{\partial\theta}{\partial\nu}&\leqq Ch\sum_{\tau\in\mathcal{T}^h}\{h|w|_{3,2,\tau}+|w|_{2,2,\tau}\}|\theta|_{1,2,\tau}\\&\leqq Ch^2\sum_{j=0}^{1}\left\{\left\|\left(\frac{\partial}{\partial t}\right)^j\rho\right\|^2_{2,2;h}+h^2\left\|\left(\frac{\partial}{\partial t}\right)^j u\right\|^2_{3,2,\Omega}\right\}+\|\theta\|^2_{1,2;h}.\end{aligned}\tag{3.12}$$

To bound the second term in (3.11), let $v=(\phi(P^hu)-\phi(u_h))$ and integrate by parts in time to obtain

$$\int_0^t\sum_{\Gamma\not\in\partial\Omega}\int_\Gamma[v_t]\frac{\partial\theta}{\partial\nu}\,dt'=\sum_{\Gamma\not\in\partial\Omega}\left\{\int_0^t\int_\Gamma[v]\frac{\partial\theta_t}{\partial\nu}\,dt'-\int_\Gamma[v]\frac{\partial\theta}{\partial\nu}\right\}$$

where we recall that $\theta(\cdot,0)=0$. Applying Lemma 2.6 and (3.8) we obtain

$$\begin{aligned}\int_0^t\sum_{\Gamma\not\in\partial\Omega}\int_\Gamma[v_t]\frac{\partial\theta}{\partial\nu}\,dt'&\leqq Ch\left\{\int_0^t\|\theta\|_{2,2;h}\|\theta_t\|_{1,2;h}\,dt'+\|\theta\|_{2,2;h}\|\theta\|_{1,2;h}\right\}\\&\leqq C\left\{\int_0^t\|\theta\|_{2,2;h}\|\theta_t\|_{0,2,\Omega}\,dt'+\|\theta\|_{2,2;h}\|\theta\|_{0,2,\Omega}\right\}\end{aligned}\tag{3.13}$$

where an inverse norm inequality was used on the last step.

Combining (3.12) and (3.13), applying Lemma 2.4 to $|\theta|_{1,2;h}$ terms, and noting (3.2), we obtain

$$|A_1|\leqq Ch^4+\varepsilon|\theta|^2_{2,2;h}+C_\varepsilon|\theta|^2_{0,2,\Omega}+\int_0^t\{\varepsilon|\theta_t|^2_{0,2,\Omega}+C_\varepsilon|\theta|^2_{2,2;h}\}\,dt'\tag{3.14}$$

where $\varepsilon>0$ is to be chosen later in this section. (We assume $\varepsilon>h$.)

For both A_2 and A_3 a result identical to that of Lemma 2.1 can be proved, since the argument holds on each $\tau\in\mathcal{T}^h$; hence

$$|A_2+A_3|\leqq Ch\int_0^t\left\{\|\theta\|_{2,2;h}\sum_{j=0}^{1}\left\|\left(\frac{\partial}{\partial t}\right)^j(u-u_h)\right\|_{1,2;h}\right\}dt'$$

where C is independent of h by (3.3). It follows that

$$\begin{aligned}|A_2+A_3|&\leqq Ch\int_0^t(|\theta|_{2,2;h}+\|\theta\|_{0,2,\Omega})\sum_{j=0}^{1}\left(\left\|\left(\frac{\partial}{\partial t}\right)^j\theta\right\|_{1,2;h}+\left\|\left(\frac{\partial}{\partial t}\right)^j\rho\right\|_{1,2;h}\right)dt'\\&\leqq C_\varepsilon h^4+\varepsilon\int_0^t\|\theta_t\|^2_{0,2,\Omega}\,dt'+C_\varepsilon\int_0^t(\|\theta\|^2_{0,2,\Omega}+|\theta|^2_{2,2;h})\,dt'.\end{aligned}\tag{3.15}$$

Combining our results for A_1, A_2, and A_3, we have

$$|J_3| \leqq \varepsilon|\theta|_{2,2;h}^2 + C_\varepsilon \|\theta\|_{0,2,\Omega}^2 + \int_0^t (\varepsilon\|\theta_t\|_{0,2,\Omega}^2 + C_\varepsilon|\theta|_{2,2;h}^2)\, dt' + Ch^4.$$

The remaining terms in (3.10) are less complicated to estimate. J_1 and J_2 can be treated using arguments similar to those above:

$$\begin{aligned}|J_1| &\leqq C\int_0^t |\rho_t|_{1,2;h}|\theta|_{1,2;h}\, dt' \\ &\leqq Ch^4 + \int_0^t (\varepsilon|\theta|_{2,2;h}^2 + C_\varepsilon\|\theta\|_{0,2,\Omega}^2)\, dt', \\ |J|_2 &\leqq \int_0^t \|(\phi(u)-\phi(u_h))_t\|_{0,2,\Omega}\|\Delta\theta\|_{0,2,\Omega}\, dt' \\ &\leqq C\int_0^t (h^2 + |\theta|_{0,2,\Omega} + |\theta_t|_{0,2,\Omega})|\theta|_{2,2;h}\, dt' \\ &\leqq Ch^4 + \int_0^t (C_\varepsilon|\theta|_{2,2;h}^2 + \varepsilon\|\theta_t\|_{0,2,\Omega}^2 + \varepsilon\|\theta\|_{0,2,\Omega}^2)\, dt'.\end{aligned}$$

For J_4 we have by Lemma 2.4 that

$$|J_4| \leqq Ch^4 + \varepsilon|\theta|_{2,2;h}^2 + C_\varepsilon\|\theta\|_{0,2,\Omega}^2.$$

Using these results in (3.7) we have

$$\begin{aligned}\int_0^t \|\theta_t\|_{0,2,\Omega}^2\, dt' + \frac{1}{2}|\theta|_{2,2;h}^2 &\leqq Ch^4 + \varepsilon|\theta|_{2,2;h}^2 + \varepsilon\int_0^t \|\theta_t\|_{0,2,\Omega}^2\, dt' \\ &\quad + C_\varepsilon\|\theta\|_{0,2,\Omega}^2 + C_\varepsilon\int_0^t |\theta|_{2,2;h}^2\, dt',\end{aligned}$$

and hence, choosing $\varepsilon = \frac{1}{4}$, we have

$$|\theta|_{2,2;h}^2 + \int_0^t \|\theta_t\|_{0,2,\Omega}^2\, dt' \leqq C\left(h^4 + \|\theta\|_{0,2,\Omega}^2 + \int_0^t |\theta|_{2,2,2;h}^2\, dt'\right). \tag{3.16}$$

Combining (3.2a, b), (3.6), and (3.16) gives the desired error bounds of Theorem 1.1. □

We turn to justifying (3.3). The preceding argument has shown that if

$$\|u_h(\cdot, t)\|_{1,\infty;h} \leqq M \quad \text{for } 0 \leqq t \leqq \tau \leqq T, \tag{3.17}$$

then

$$\|u(\cdot, t) = u_h(\cdot, t)\|_{1,s;h} \leqq Ch^2, \qquad 0 \leqq t \leqq \tau \tag{3.18a}$$

and by an inverse inequality

$$\|u(\cdot, t) - u_h(\cdot, t)\|_{1,\infty;h} \leqq C(M, T)h. \tag{3.18b}$$

Suppose that

$$K = \sup_{0 \leqq t \leqq T} \|u(\cdot, t)\|_{1,\infty} \tag{3.19}$$

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and that $M > K$. Choose $h_0 > 0$ so that $h_0 < (M-K)/C(M, T)$. Suppose that there exists $t_h < T$ such that

$$\|u_h(\cdot, t)\|_{1,\infty;h} < M, \qquad 0 \leqq t < t_h, \tag{3.20a}$$

$$\|u_h(\cdot, t)\|_{1,\infty;h} = M. \tag{3.20b}$$

It follows from (3.18b) and (3.19) that

$$\|u_h(\cdot, t_h)\|_{1,\infty;h} \leqq K + C(M, T)h, \qquad 0 \leqq t \leqq t_h$$

and if $h < h_0$ then $\|u_h(\cdot, t_h)\|_{1,\infty;h} < M$. Hence for each $T > 0$ we can choose h_0 so that (3.3) holds for all $h < h_0$.

4. L^∞ convergence. In this section, we show that an "almost" optimal-order error bound holds for our method. Our result is not optimal due to a $(\ln 1/h)^{1/2}$ factor. The key to the proof is a subspace Sobolev inequality for the Morley elements.

LEMMA 4.1. *If $\chi \in S^h$, then*

$$\|\chi\|_{0,\infty,\Omega} \leqq C\left(\ln\frac{1}{h}\right)^{1/2} \|\chi\|_{1,2;h}. \tag{4.1}$$

Proof. If $\tau \in \mathcal{T}^h$ then

$$\begin{aligned}\|\chi\|_{0,\infty,\tau} &\leqq \|\chi^L\|_{0,\infty,\tau} + \|\chi - \chi^L\|_{0,\infty,\tau}\\ &\leqq C\left(\ln\frac{1}{h}\right)^{1/2} \|\chi^L\|_{1,2;h} + Ch^2|\chi|_{2,\infty,\tau}\end{aligned}$$

where χ^L is the piecewise linear interpolant of χ and the subspace Sobolev inequality for piecewise linear functions is applied to the first term (see Schatz and Wahlbin [10]). Applying inverse inequalities, we finish the verification of (4.1):

$$\begin{aligned}\|\chi\|_{0,\infty,\tau} &\leqq C\left(\ln\frac{1}{h}\right)^{1/2} (\|\chi\|_{1,2;h} + \|\chi - \chi^L\|_{1,2;h}) + C\|\chi\|_{1,2,\tau}\\ &\leqq C\left(\ln\frac{1}{h}\right)^{1/2} (\|\chi\|_{1,2;h} + Ch\|\chi\|_{2,2;h})\\ &\leqq C\left(\ln\frac{1}{h}\right)^{1/2} \|\chi\|_{1,2;h}. \qquad \square\end{aligned}$$

We now present the short statement and proof of the L^∞ convergence.

THEOREM 4.1. *Suppose u is sufficiently smooth and* (1.14) *holds; then*

$$\|(u-u_h)(\cdot, t)\|_{0,\infty,\Omega} \leqq C\left(\ln\frac{1}{h}\right)^{1/2} h^2 \tag{4.2}$$

where $t \in [0, T]$ and C depends on u, T, and the mesh parameters.

Proof. We use $I_h u$, the interpolant in S^h. Then for fixed $t \in [0, T]$

$$\|u - u^h\|_{0,\infty,\Omega} \leqq \|u - I_h u\|_{0,\infty,\Omega} + \|I_h u - u_h\|_{0,\infty,\Omega}.$$

From (4.1) and (1.7), we have

$$\|u - u_h\|_{0,\infty} \leqq Ch^3|u|_{3,\infty,\Omega} + C\left(\ln\frac{1}{h}\right)^{1/2} \|I_h u - u_h\|_{1,2;h}$$

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or

$$\|u-u_h\|_{0,\infty,\Omega} \leqq Ch^3|u|_{3,\infty,\Omega} + C\left(\ln\frac{1}{h}\right)^{1/2}(\|u-I_hu\|_{1,2;h}+\|u-u_h\|_{1,2;h})$$

$$\leqq C(u)\left(\ln\frac{1}{h}\right)^{1/2}h^2$$

where the result of Theorem 1.1 was used on the last step. □

5. Error bounds for the elliptic projection. In this section we prove Proposition 1.1. For each $t \in [0, T]$ recall that

$$b_h(w,\psi) = a_h(w,\psi) + (\phi'(u)\nabla w, \nabla\psi)_h + \alpha(w,\psi), \qquad w, \psi \in H^{2,h}(\Omega), \tag{5.1}$$

and using the definition of $P^h u$, we find

$$b_h(P^h u, \chi) = (\Delta^2 u - \Delta\phi(u) + \alpha u, \chi) \quad \forall \chi \in S_E^h, \tag{5.2a}$$

$$b_h(u, \eta) = (\Delta^2 u - \Delta\phi(u) + \alpha u, \eta) \quad \forall \eta \in H_E^2, \tag{5.2b}$$

$$b_h(P^h u_t, \chi) = (\Delta^2 u_t - \Delta\phi_t(u) + \alpha u_t, \chi) + (\phi''(u)u_t\nabla P^h u, \nabla\chi)_h \quad \forall \chi \in S_E^h, \tag{5.2c}$$

$$b_h(u_t, \eta) = (\Delta^2 u_t - \Delta\phi_t(u) + \alpha u_t, \eta) + (\phi''(u)u_t\nabla u, \nabla\eta) \quad \forall \eta \in H_E^2(\Omega). \tag{5.2d}$$

As noted in § 1, choosing α sufficiently large makes $b_h(\cdot,\cdot)$ coercive on S_E^h and $H_E^2(\Omega)$. To obtain bounds for ρ and ρ_t we study the following problems:

$$v \in H_E^2(\Omega)\colon b_h(v,\eta) = F_h(\eta) \quad \forall \eta \in H_E^2(\Omega), \tag{5.3a}$$

$$v_h \in S_E^h\colon b_h(v_h,\chi) = \tilde{F}_h(\chi) \quad \forall \chi \in S_E^h \tag{5.3b}$$

where F_h and $\tilde{F}_h$ are continuous linear functionals on $H^{1,h}(\Omega)$. Note that $(H^{1,h}(\Omega))' \subset (H^1(\Omega))'$. We use the notation

$$\|l\|_{-1} = \sup_{\substack{\psi\in H^1(\Omega)\\ \psi\neq 0}} \frac{l(\psi)}{\|\psi\|_{1,2,\Omega}},$$

$$\|l\|_{-1,h} = \sup_{\substack{\psi\in H^{1,h}(\Omega)\\ \psi\neq 0}} \frac{l(\psi)}{\|\psi\|_{1.2;h}}.$$

The regularity result, Theorem A.2, implies that

$$\|v\|_{3,2,\Omega} \leqq C\|F_h\|_{-1} \leqq C\|F_h\|_{-1,h}. \tag{5.4}$$

For $w \in H^3(\Omega)$ integrating by parts gives us

$$a_h(w,\psi) = (-\nabla\Delta w, \nabla\psi)_h + B_h(w,\psi) \quad \forall \psi \in H^{2,h}(\Omega) \tag{5.5}$$

where $B_h(\cdot,\cdot)$ is defined in (2.12).

Also, by noting that $H_E^2(\Omega)$ is dense in $H^1(\Omega)$ we have

$$(-\nabla\Delta v, \nabla\eta) + (\phi'(u)\nabla v, \nabla\eta) + \alpha(v,\eta) = F_h(\eta) \quad \forall \eta \in H^1(\Omega). \tag{5.6}$$

On $\mathcal{T}^h$ we may define a piecewise linear interpolant $\psi^L \in C(\bar{\Omega})$ of $\psi \in H^{2,h}(\Omega)$ with the approximation property

$$|\psi - \psi^L|_{k,p,\tau} \leqq Ch^{2-k}|\psi|_{2,p,\tau} k = 0, 1 \quad \forall \tau \in \mathcal{T}^h. \tag{5.7}$$

LEMMA 5.1. *For v and v_h defined by* (5.3) *we have that*

$$\|v-v_h\|_{1,2;h} + h|v-v_h|_{2,2;h} \leqq C(h^2\|F_h\|_{-1,h} + \|F_h - \tilde{F}_h\|_{-1,h}). \tag{5.8}$$

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Proof. Let $\psi \in H_E^{2,h}(\Omega)$ so that $\psi^L \in H^1(\Omega)$ and

$$\begin{aligned} b_h(v, \psi) - \tilde{F}_h(\psi) &= (-\nabla\Delta v, \nabla\psi)_h + B_h(v, \psi) + (\phi'(u)\nabla v, \nabla\psi)_h + (\alpha v, \psi) - \tilde{F}_h(\psi) \\ &= (-\nabla\Delta v, \nabla(\psi - \psi^L))_h + (\phi'(u)\nabla v, \nabla(\psi - \psi^L))_h \\ &\quad + (\alpha v, \psi - \psi^L) + B_h(v, \psi) - F_h(\psi - \psi^L) + F_h(\psi) - \tilde{F}_h(\psi) \end{aligned}$$

where we have used (5.5) and (5.6). Hence, using (5.4), (5.7), and Lemma 2.5 we obtain

$$|b_h(v, \psi) - \tilde{F}_h(\psi)| \leqq C[h\|F_h\|_{-1,h}|\psi|_{2,2;h} + \|F_h - \tilde{F}_h\|_{-1,h}\|\psi\|_{1,2;h}]. \tag{5.9}$$

Using (5.3b) and (5.9) we obtain

$$\begin{aligned} b_h(v - v_h, v - v_h) &= b_h(v - v_h, v - I_h v) + b_h(v, I_h v - v_h) - \tilde{F}_h(I_h v - v_h) \\ &\leqq b_h(v - v_h, v - I_h v) + Ch\|F_h\|_{-1,h}|I_h v - v_h|_{2,2;h} \\ &\quad + C\|F_h - \tilde{F}_h\|_{-1,h}\|I_h v - v_h\|_{1,2;h} \end{aligned}$$

and the coercivity of $b_h(\cdot, \cdot)$ (i.e., (1.11)), together with the approximation property (1.7), yields

$$\|v - v_h\|_{2,2;h} \leqq C[h\|F_h\|_{-1,h} + \|F_h - \tilde{F}_h\|_{-1,h}]. \tag{5.10}$$

Let $L_h(\psi)$ be defined by

$$L_h(\psi) = (\nabla I_h e, \nabla\psi)_h + (I_h e, \psi) \quad \forall \psi \in H^{1,h}(\Omega) \tag{5.11}$$

where $e = v - v_h$. It follows that

$$\|L_h\|_{-1,h} = \|I_h e\|_{1,2;h}. \tag{5.12}$$

Hence, defining z and z_h by

$$b_h(z, \eta) = L_h(\eta) \quad \forall \eta \in H_E^2(\Omega), \tag{5.13a}$$

$$b_h(z_h, \chi) = L_h(\chi) \quad \forall \chi \in S_E^h, \tag{5.13b}$$

we have by the previous discussion that

$$h\|z\|_{3,2,\Omega} + \|z - z_h\|_{2,2;h} \leqq Ch\|I_h e\|_{1,2;h}. \tag{5.14}$$

Using the definitions (5.2) and (5.13), we have

$$\begin{aligned} \|I_h e\|_{1,2;h}^2 &= L_h(I_h e) = b_h(z_h, I_h e) \\ &= b_h(z - z_h, v - I_h v) + b_h(z_h - z, v) \\ &\quad + b_h(z, v) - b_h(z_h, v_h) - b_h(z, v - I_h v) \\ &= b_h(z - z_h, v - I_h v) + [b_h(v, z_h - z) - \tilde{F}_h(z_h - z)] \\ &\quad + [F_h(z) - \tilde{F}_h(z)] + (\nabla\Delta z, \nabla(v - I_h v))_h \\ &\quad - (\phi'(u)\nabla z, \nabla(v - I_h v)) - (\alpha z, v - I_h v) - B_h(z, v - I_h v). \end{aligned}$$

It follows from (5.14), (1.7), (5.9), and Lemma 2.5 that

$$\|I_h e\|_{1,2;h}^2 \leqq C[h^2\|I_h e\|_{1,2;h}\|F_h\|_{1,h} + h\|F_h - \tilde{F}_h\|_{-1,h}\|I_h e\|_{1,2;h} + \|F_h - \tilde{F}_h\|_{-1,h}\|I_h e\|_{1,2;h}],$$

and since

$$\begin{aligned} \|e\|_{1,2;h} &\leqq \|e - I_h e\|_{1,2;h} + \|I_h e\|_{1,2;h} \\ &\leqq Ch|e|_{2,2;h} + C[h^2\|F_h\|_{-1,h} + \|F_h - \tilde{F}_h\|_{-1,h}] \end{aligned}$$

we obtain the desired bound (5.8) of the lemma. □

Choosing

$$\begin{aligned} &v = u, \qquad v_h = P^h u, \\ &F_h(\psi) = \tilde{F}_h(\psi) = (\Delta^2 u - \Delta\phi(u) + \alpha u, \psi), \\ &\|F_h\|_{-1,h} \leqq \|\Delta^2 u - \Delta\phi(u) + \alpha u\|_{0,2,\Omega} \leqq C(u), \end{aligned}$$

we obtain from (5.8) the bound (1.12) for ρ. The bound for ρ_t (1.13) follows by choosing $v = u_t$, $v_h = (P_h u)_t$,

$$F_h(\psi) = (\Delta^2 u_t - \Delta\phi_t(u) + \alpha u_t, \psi) - (\phi''(u) u_t \nabla u, \nabla\chi)_h,$$

$$\tilde{F}_h(\psi) = (\Delta^2 u_t - \Delta\phi_t(u) + \alpha u_t, \psi) - (\phi''(u) u_t \nabla P^h u, \nabla\psi)_h,$$

$$\|F_h - \tilde{F}_h\|_{-1,h} \leqq C(\|u\|_{0,\infty,\Omega} \|u_t\|_{0,\infty,\Omega}) |u - P^h u|_{1,2;h} \leqq C(u, u_t) h^2.$$

This completes the proof of Proposition 1.1. □

Appendix. In the following, $\Omega = (0, a) \times (0, b)$.

THEOREM A.1. *If $f \in H^k(\Omega)$ and $\int_\Omega f(x) = 0$, then the boundary value problem*

$$\Delta^2 u = f \quad \text{in } \Omega, \qquad \int_\Omega u = 0, \tag{A.1a}$$

$$\frac{\partial u}{\partial \nu} = 0, \qquad \frac{\partial}{\partial \nu} \Delta u = 0 \quad \text{on } \partial\Omega \tag{A.1b}$$

has a unique solution $u \in H^{k+4}(\Omega)$ and

$$\|u\|_{k+4,2,\Omega} \leqq C \|f\|_{k,2,\Omega} \tag{A.1c}$$

for $k = -1, 0$.

Proof. Consider the boundary value problem

$$\Delta w = g \quad \text{in } \Omega, \qquad \int_\Omega w = 0, \tag{A.2a}$$

$$\frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \qquad \int_\Omega g = 0. \tag{A.2b}$$

There exists a unique solution and by Grisvard [4, p. 149] $w \in H^2(\Omega)$ if $g \in L^2(\Omega)$; furthermore, it also holds that

$$\|w\|_{2,2,\Omega} \leqq C \|g\|_{0,2,\Omega}. \tag{A.2c}$$

See Grisvard [4, p. 199]. Consider the following problem. Find $z = z(x, y)$ such that

$$\Delta z = \frac{\partial g}{\partial y} \quad \text{in } \Omega, \qquad z(\cdot, 0) = z(\cdot, b) = 0,$$

$$\frac{\partial z}{\partial x}(0, \cdot) = \frac{\partial z}{\partial x}(a, \cdot) = 0.$$

By a reflection argument we can show that if $\partial g/\partial y \in L^2(\Omega)$, then $z = \partial w/\partial y$ is the unique solution and $z \in H^2(\Omega)$. Also, from Grisvard [4, p. 199] we have the estimate

$$\|z\|_{2,2,\Omega} \leqq C \left\|\frac{\partial g}{\partial y}\right\|_{0,2,\Omega}.$$

Letting $z = \partial w/\partial x$, we can obtain results similar to the above. Thus we conclude that if $g \in H^1(\Omega)$, then

$$\|w\|_{3,2,\Omega} \leqq C \|g\|_{1,2,\Omega}. \tag{A.3}$$

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Continuing in this manner, we observe that $z=\partial^2 w/\partial x\,\partial y$ solves

$$\Delta z=\frac{\partial^2 g}{\partial x\,\partial y}\quad \text{in } \Omega \quad \text{and} \quad z=0 \quad \text{on } \partial\Omega.$$

If $\partial^2 g/\partial x\,\partial y\in L^2(\Omega)$, then $z\in H^2(\Omega)$ and

$$\|z\|_{2,2,\Omega}\leqq C\left\|\frac{\partial^2 g}{\partial x\,\partial y}\right\|_{0,2,\Omega}.$$

If we add the hypothesis that $\partial g/\partial\nu=0$ on $\partial\Omega$, then $z=\partial^2 w/\partial y^2$ is the unique solution of

$$\Delta z=\frac{\partial^2 g}{\partial y^2}\quad \text{in } \Omega, \qquad \int_\Omega z=0,$$

$$\frac{\partial z}{\partial\nu}=0\quad \text{on } \partial\Omega, \qquad \int_\Omega \frac{\partial^2 g}{\partial y^2}=0$$

and $z=\partial^2 w/\partial x^2$ is the unique solution of

$$\Delta z=\frac{\partial^2 g}{\partial x^2}\quad \text{in } \Omega, \qquad \int_\Omega z=0,$$

$$\frac{\partial z}{\partial\nu}=0\quad \text{on } \partial\Omega, \qquad \int_\Omega \frac{\partial^2 g}{\partial x^2}=0.$$

Furthermore, again from results in Grisvard [4], if $\partial^2 g/\partial y^2\in L^2(\Omega)$, then $\partial^2 w/\partial y^2\in H^2(\Omega)$ and

$$\left\|\frac{\partial^2 w}{\partial y^2}\right\|_{2,2,\Omega}\leqq C\left\|\frac{\partial^2 g}{\partial y^2}\right\|_{0,2,\Omega}.$$

If $\partial^2 g/\partial x^2\in L^2(\Omega)$, then $\partial^2 w/\partial x^2\in H^2(\Omega)$ and

$$\left\|\frac{\partial^2 g}{\partial x^2}\right\|_{2,2,\Omega}\leqq C\left\|\frac{\partial^2 w}{\partial x^2}\right\|_{0,2,\Omega}.$$

We conclude that if $g\in H^2(\Omega)$, then $w\in H^4(\Omega)$ and

$$\|w\|_{4,2,\Omega}\leqq C\|g\|_{2,2,\Omega}. \tag{A.4}$$

Now we consider problem (A.1a, b). By energy arguments there exists a unique solution $u\in H_E^2(\Omega)$. Set $g=\Delta u$. From (A.3) with $w=u$ it follows that if $f\in H^{-1}(\Omega)$, then $g\in H^1(\Omega)$ is the unique weak solution of

$$\Delta z=f\quad \text{in } \Omega, \qquad \int_\Omega z=0,$$

$$\frac{\partial z}{\partial\nu}=0\quad \text{on } \partial\Omega,$$

and $u\in H^3(\Omega)$. Furthermore, it holds that

$$\|u\|_{3,2,\Omega}\leqq C\|g\|_{1,2,\Omega}\leqq C\|f\|_{-1,2,\Omega}. \tag{A.5}$$

If $f\in L^2(\Omega)$, then from (A.4), $g\in H^2(\Omega)$, $u\in H^4(\Omega)$, and

$$\|u\|_{4,2,\Omega}\leqq C\|g\|_{2,2,\Omega}\leqq C\|f\|_{0,2,\Omega}. \tag{A.6}$$

THEOREM A.2. *Let* $u_0 \in H_E^2(\Omega)$ *and* $T>0$. *There exists a unique* $u \in L^2(0, T; \mathcal{R}) \cap C(0, T; H_E^2(\Omega))$ *with* $\partial u/\partial t \in L^2(0, T; L^2(\Omega))$ *such that* $u(\cdot, 0) = u_0$ *and*

$$\frac{\partial u}{\partial t} = \Delta\phi(u) - \gamma\Delta^2 u \quad a.e.\ t \in (0, T). \tag{A.7}$$

Furthermore, $\|u(\cdot, t)\|_{2,2,\Omega} \leqq C$ *for each* $t \in [0, T]$, *where* C *is independent of* T. *If* $-\Delta^2 u_0 + \Delta\phi(u_0) \in H_E^2(\Omega)$, *then it holds that* $\partial u/\partial t \in L^2(0, T; \mathcal{R}) \cap C[0, T; H_E^2(\Omega)]$.

Proof. The proof is based on the methods of compactness and Galerkin approximation (Lions [6]). Let $\{z_j\}$ be the orthogonal basis for $H^1(\Omega)$ defined by

$$z_j \in H_E^2(\Omega), \qquad -\Delta z_j + z_j = \lambda_j z_j \tag{A.8a}$$

and normalized so that

$$(z_i, z_i) = 1. \tag{A.8b}$$

(Note that $\Delta^m z \in H_E^2(\Omega)$ for $m = 1, 2, \cdot$.) Let P^m be the projection defined by

$$P^m v = \sum_{j=1}^{m} (v, z_j) z_j \tag{A.9}$$

so that

$$(P^m v - v, \eta^m) = (\nabla P^m v - \nabla v, \nabla \eta^m) = 0 \quad \forall \eta^m \in V^m, \tag{A.10a}$$

$$\|P^m\|_{\mathcal{L}(H^1(\Omega), V^m)} = \|P^m\|_{\mathcal{L}(L^2(\Omega), V^m)} = 1, \tag{A.10b}$$

$$\lim_{m\to\infty} |P^m v - v|_0 = 0 \quad \forall v \in L^2(\Omega), \qquad \lim_{m\to\infty} \|P^m v - v\|_1 = 0 \quad \forall v \in H^1(\Omega) \tag{A.10c}$$

where V^m is the finite-dimensional subspace of $H^1(\Omega)$ spanned by $\{z_j\}_{j=1}^m$.

Consider the following initial value problem. Find $\{u^m, w^m\}$ such that

$$u^m(\cdot, t) = \sum_{j=1}^{m} c_j(t) z_j, \qquad w^m(\cdot, t) = \sum_{j=1}^{m} d_j(t) z_j, \tag{A.11a}$$

$$\left(\frac{du^m}{dt}, \eta^m\right) + (\nabla w^m, \nabla \eta^m) = 0 \quad \forall \eta^m \in V^m, \tag{A.11b}$$

$$(w^m - \phi(u^m), \eta^m) = \gamma(\nabla u^m, \nabla \eta^m) \quad \forall \eta^m \in V^m, \tag{A.11c}$$

$$u^m(\cdot, 0) = P^m u_0. \tag{A.11d}$$

Clearly, (A.11a–d) can be rewritten as an initial value problem for a finite-dimensional system of ordinary differential equations for the variables $\underset{\sim}{c}(t) = (c_1(t), c_2(t) \cdots c_m(t))$ and $\underset{\sim}{d}(t) = (d_1(t), \cdots, d_m(t))$. Since $\phi(\cdot)$ is continuously differentiable, there exists a positive t_m such that (A.11) has a unique solution with $\underset{\sim}{c}(t)$ and $\underset{\sim}{d}(t)$ being absolutely continuous on $[0, t_m]$ and differentiable on $(0, t_m)$.

Let us define an energy functional $\mathcal{F}(\cdot)$ by

$$\mathcal{F}(\eta) = \int_\Omega \left[\frac{1}{2}|\nabla \eta|^2 + \psi(\eta)\right]. \tag{A.12}$$

Differentiating (A.12) with respect to t we obtain

$$\frac{d}{dt}\mathcal{F}(u^m) = \left(\nabla u^m, \nabla \frac{du^m}{dt}\right) + \left(\phi(u^m), \frac{du^m}{dt}\right)$$

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and taking $\eta^m = u^m$ in (A.11c) and $\eta^m = w^m$ in (A.11b) we obtain

$$\frac{d}{dt}\mathcal{F}(u^m) = \left(w^m, \frac{du^m}{dt}\right)$$
$$= -|w^m|_{1,2,\Omega}^2.$$

Integrating with respect to t yields

$$\mathcal{F}(u^m(t)) + \int_0^t |w^m(\tau)|_{1,2,\Omega}^2 \, d\tau = \mathcal{F}(P^m u_0). \tag{A.13}$$

Since $\psi(\cdot)$ is a quartic polynomial, it follows from (A.10c) and the compact embedding of $H^1(\Omega)$ into $L^4(\Omega)$ that

$$\lim_{m\to\infty} \mathcal{F}(P^m u_0) = \mathcal{F}(u_0)$$

and the right-hand side of (A.13) is bounded independently of m.

Taking $\eta^m = |\Omega|^{1/2} z_1 = 1$ in (A.11b, c) yields

$$(u^m(\cdot, t), 1) = (P^m u_0, 1) = (u_0, 1), \tag{A.14a}$$

$$(w^m(\cdot, t), 1) = (\phi(u^m(\cdot, t)), 1). \tag{A.14b}$$

Since

$$\mathcal{F}(\eta) \geqq \tfrac{1}{2}|\eta|_{1,2,\Omega}^2,$$

it follows from the Poincaré inequality that

$$\|\eta\|_{1,2,\Omega} \leqq C(\Omega)[|\eta|_{1,2,\Omega} + |(\eta, 1)|] \quad \forall \eta \in H^1(\Omega) \tag{A.15}$$

and from (A.12) and (A.14a, b) that for each $t \geqq 0$

$$\|u^m(\cdot, t)\|_{1,2,\Omega} \leqq C, \tag{A.16a}$$

$$\int_0^t \|w^m(\cdot, \tau)\|_{1,2,\Omega}^2 \, d\tau \leqq C(1+t) \tag{A.16b}$$

where C is independent of t and m. The estimate (A.16a) implies an a priori bound for $\|\underline{c}(t)\|$, and hence by the classical theory of ordinary differential equations there exists a solution to (A.11a–d) for all t. Thus we have global existence of $\{u^m, w^m\}$.

Since $V^m \subset H_E^2(\Omega)$ and $\Delta\eta^m \in V^m$, it follows that

$$\frac{du^m}{dt} = \Delta w^m, \tag{A.17a}$$

$$(w^m - \phi(u^m) - \gamma\Delta u^m, \eta^m) = 0 \quad \forall \eta^m \in V^m, \tag{A.17b}$$

$$(\Delta w^m - \Delta\phi(u^m) - \gamma\Delta^2 u^m, \eta^m) = 0 \quad \forall \eta^m \in V^m. \tag{A.17c}$$

Taking $\eta^m = \Delta^2 u^m$ in (A.17c) and using (A.17a) yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Delta u^m\|_{0,2,\Omega}^2 + \|\Delta^2 u^m\|_{0,2,\Omega}^2 &= (\Delta\phi(u^m), \Delta^2 u^m) \\ &\leqq \frac{1}{2}(\|\Delta\phi(u^m)\|_{0,2,\Omega}^2 + |\Delta^2 u^m|_{0,2,\Omega}^2). \end{aligned} \tag{A.18}$$

Since

$$\Delta\phi(u^m) = \phi''(u^m)|\nabla u^m|^2 + \phi'(u^m)\Delta u^m,$$

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we have that

$$|\Delta\phi(u^m)|_0^2 \leqq 2[\|\phi''(u^m)\|_{0,6,\Omega}^2\|\nabla u^m\|_{0,6,\Omega}^4 + \|\phi'(u^m)\|_{0,3,\Omega}^2\|\Delta u^m\|_{0,6,\Omega}^2];$$

recalling that $\phi''(\cdot)$ is linear and $\phi'(\cdot)$ is quadratic and noting that (A.16b) holds, we have from the embedding

$$\|\eta\|_{0,6,\Omega} \leqq C\|\eta\|_{1,2,\Omega} \quad \forall \eta \in H^2(\Omega)$$

that

$$\|\Delta\phi(u^m)\|_{0,2,\Omega}^2 \leqq C[\|\nabla u^m\|_{0,6,\Omega}^4 + \|\Delta u^m\|_{0,6,\Omega}^2] \tag{A.19}$$

where C is independent of m and t.

It follows from the Gagliardo–Nirenberg interpolation inequalities that

$$\|\nabla u^m\|_{0,6,\Omega} \leqq C\|\nabla u^m\|_{3,2,\Omega}^{1/3}\|\nabla u^m\|_{0,2,\Omega}^{2/3}, \tag{A.20a}$$

$$\|\Delta u^m\|_{0,6,\Omega} \leqq C\|\nabla u^m\|_{3,2,\Omega}^{2/3}\|\nabla u^m\|_{0,2,\Omega}^{1/3} \tag{A.20b}$$

and by the elliptic regularity theorem, Theorem A.1, that

$$\|u^m\|_{4,2,\Omega} \leqq C\|\Delta^2 u^m\|_{0,2,\Omega}. \tag{A.20c}$$

Applying (A.20a–c) to (A.19) we obtain

$$\|\Delta\phi(u^m)\|_{0,2,\Omega}^2 \leqq C\|\Delta^2 u^m\|_{0,2,\Omega}^{4/3}$$

and by Young's inequality,

$$\|\Delta\phi(u^m)\|_{0,2,\Omega}^2 \leqq \varepsilon\|\Delta^2 u^m\|_{0,2,\Omega}^2 + C_\varepsilon,$$

which implies upon substituting in (A.18) that

$$\frac{d}{dt}\|\Delta u^m\|_{0,2,\Omega}^2 + \|\Delta^2 u^m\|_{0,2,\Omega}^2 \leqq C. \tag{A.21}$$

Since there exists c_0 such that

$$c_0\|\eta\|_{0,2,\Omega} \leqq \|\Delta\eta\|_{0,2,\Omega} \quad \text{for } (\eta, 1) = 0, \quad \eta \in H_E^2(\Omega)$$

it follows that

$$\frac{d}{dt}\|\Delta u^m\|_{0,2,\Omega}^2 + c_0\|\Delta u^m\|_{0,2,\Omega}^2 \leqq C, \tag{A.22}$$

which implies

$$\|\Delta u^m(\cdot, t)\|_{0,2,\Omega} \leqq C \quad \forall t \in [0, T). \tag{A.23}$$

It follows immediately from (A.23) and (A.21) that u^m is uniformly bounded in $L^\infty(0, T; H_E^2(\Omega)) \cap L^2(0,T; \mathcal{R})$. Recalling the estimate for $\|\Delta\phi(u^m)\|_{0,2,\Omega}$ it follows from (A.14a,b) that w^m and du^m/dt are uniformly bounded, respectively, in $L^2(0, T; H^2(\Omega))$ and $L^2(0, T; L^2(\Omega))$. By interpolation (Lions and Magenes [7]), u^m is uniformly bounded in $C[0, T; H_E^2(\Omega)]$. Therefore there exists $u \in L^2(0, T; \mathcal{R}) \cap C(0, T; H_E^2(\Omega))$ with $du/dt \in L^2(0, T; L^2(\Omega))$ and a subsequence $\{u^{m'}\}$ such that

$$u^{m'} \to u \quad \text{weakly in } L^2(0, T; \mathcal{R}), \qquad \frac{du^{m'}}{dt} \to \frac{du}{dt},$$

$$\Delta\phi(u^{m'}) \to \Delta\phi(u) \quad \text{and} \quad \Delta^2 u^{m'} \to \Delta^2 u \quad \text{weakly in } L^2(0, T; L^2(\Omega));$$

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and passing to the limit in (A.17a, c) we obtain

$$\frac{du}{dt} - \Delta\phi(u) - \Delta^2 u = 0.$$

This completes the proof of the first statement of the theorem. It remains to prove further regularity in the case where $v_0 \equiv -\Delta^2 u_0 + \Delta\phi(u_0) \in H_E^2(\Omega)$. Consider the linear evolution equation: find $v(\cdot, t) \in H_E^2(\Omega)$ such that

$$\left(\frac{dv}{dt}, \eta\right) + (\Delta v, \Delta\eta) = (\Delta(\phi'(u)v), \eta) \quad \forall v \in H_E^2(\Omega), \quad v(\cdot, 0) = v_0 \tag{A.24}$$

and its Galerkin approximation, $v^m(\cdot, t) \in V^m$ such that

$$\begin{gathered}\left(\frac{dv^m}{dt}, \eta^m\right) + (\Delta v^m, \Delta\eta^m) = (\Delta(\phi'(u)v^m), \eta^m) \quad \forall \eta^m \in V^m, \\ v^m(\cdot, 0) = P^m v_0.\end{gathered} \tag{A.25}$$

When we take $\eta^m = v^m$ and $\eta^m = dv^m/dt$ in (A.25), it is straightforward to obtain a priori bounds and thus to pass to the limit and obtain the existence of

$$v \in L^2(0, T; H_E^2(\Omega)), \qquad \Delta(\phi'(u)v) \in L^2(0, T; L^2(\Omega))$$

and $dv/dt \in L^2(0, T; L^2(\Omega))$, where v solves (A.24). The elliptic regularity result Theorem A.1 yields that $v \in L^2(0, T; \mathscr{R})$ (hence $v \in C[0, T; H_E^2(\Omega)]$) and clearly $v = u_t$. $\square$

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