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A mathematical model of coupled plates and its finite element method

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This paper investigates the mathematical modeling of coupled plates and its finite element approximations following the approach proposed by Feng Kang and Shi Zhong-ci.

1. Introduction

The mathematical modeling of elastic composite structures, i.e., elastic structures that comprise substructures of the same or different dimensions (three-dimensional substructures, plates, rods, etc., usually made of different elastic materials), is a problem of practical importance, since such elastic structures are very common in engineering problems. Nevertheless, it is only recently that this problem has been studied from a mathematical viewpoint and numerical analysis.

About 10 years ago, mathematical modeling of such elastic composite structures was proposed by Feng and Shi [1, 2], from a mechanical and mathematical viewpoint. Very recently, mathematical modeling of such elastic composite structures, called elastic multistructures, has been investigated by Ciarlet et al. [3, 4], from a theoretical viewpoint using asymptotic analysis.

In this paper, we consider the mathematical modeling of coupled plates and its finite element approximations following the approach of [1, 2]. The notation is as follows: (cf. Fig. 1)

Ω_1 the mid-surface of the elastic plate
$\Omega_1^{t_1}$ the elastic thin plate, with thickness $2t_1$ $(0 < t_1 \ll 1)$
$f^1 = (f_i^1)$, $f_i^1 = f_i^1(x_1, x_2)$, $i = 1, 2, 3$, the applied surface force in Ω_1
$u^1 = (u_i^1)$, $u_i^1 = u_i^1(x_1, x_2)$, $i = 1, 2, 3$, the displacement vector in Ω_1, with $u_\alpha^1 = 0$, $\alpha = 1, 2$, $u_3^1 = \partial u_3^1/\partial n = 0$ on Γ_0
Ω_2 the mid-surface of the elastic plate
$\Omega_2^{t_2}$ the elastic thin plate, with thickness $2t_2$ $(0 < t_2 \ll 1)$
$f'^2 = (f_i^2)$, $f_i^2 = f_i^2(x_2, x_3)$, $i = 1, 2, 3$, the applied surface force in Ω_2
$u^2 = (u_i^2)$, $u_i^2 = u_i^2(x_2, x_3)$, $i = 1, 2, 3$, the displacement vector in Ω_2

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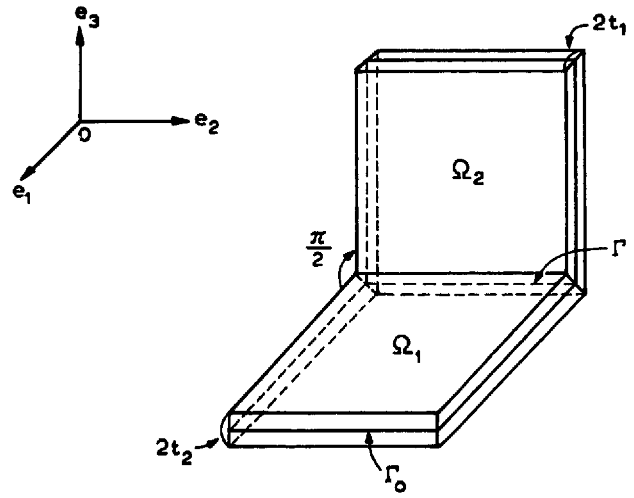

Fig. 1.

Assume that there exists a rigid junction on the part Γ between these two plates Ω_1 and Ω_2, i.e. (cf. [4]),

$$u^1 = u^2 \quad \text{on } \Gamma\,, \tag{1.1}$$

$$-\partial_1 u_3^1 = \partial_3 u_1^2 \quad \text{on } \Gamma\,. \tag{1.2}$$

The equality (1.1) means that the displacements u^1 and u^2 of plates Ω_1 and Ω_2 are continuous on Γ, and the equality (1.2) means that the rotational angles of plates Ω_1 and Ω_2 about Γ are continuous on Γ.

Consider the total energy of the elastic composite structure consisting of these two plates under a virtual displacement $\mathcal{V} = (v^1, v^2)$:

$$\mathcal{J}(\mathcal{V}) = \tfrac{1}{2}D(\mathcal{V}, \mathcal{V}) - F(\mathcal{V})\,, \tag{1.3}$$

where (with $\mathcal{U} = (u^1, u^2)$),

$$\begin{aligned} D(\mathcal{U}, \mathcal{V}) &= \left\{\int_{\Omega_1} Q_{\alpha\beta}(u^1)\varepsilon_{\alpha\beta}(v^1)\,\mathrm{d}x_1\,\mathrm{d}x_2 + \int_{\Omega_1} M_{\alpha\beta}(u^1)K_{\alpha\beta}(v^1)\,\mathrm{d}x_1\,\mathrm{d}x_2\right\} \\ &\quad + \left\{\int_{\Omega_2} Q_{\alpha'\beta'}(u^2)\varepsilon_{\alpha'\beta'}(v^2)\,\mathrm{d}x_2\,\mathrm{d}x_3 + \int_{\Omega_2} M_{\alpha'\beta'}(u^2)K_{\alpha'\beta'}(v^2)\,\mathrm{d}x_2\,\mathrm{d}x_3\right\} \\ &= D_1(u^1, v^1) + D_2(u^2, v^2)\,; \end{aligned} \tag{1.4}$$

$$\begin{cases} \varepsilon_{\alpha\beta}(v^1) = \tfrac{1}{2}(\partial_\alpha v_\beta^1 + \partial_\beta v_\alpha^1)\,, \\ Q_{\alpha\beta}(v^1) = \left(\dfrac{2E_1 t_1}{1 - \nu_1^2}\right)\{(1 - \nu_1)\varepsilon_{\alpha\beta}(v^1) + \nu_1 \varepsilon_{\gamma\gamma}(v^1)\delta_{\alpha\beta}\}\,; \end{cases} \tag{1.5}$$

$$\begin{cases} K_{\alpha\beta}(\boldsymbol{v}^1) = -\partial_{\alpha\beta} v_3^1 , \\ M_{\alpha\beta}(\boldsymbol{v}^1) = \dfrac{2E_1 t_1^3}{3(1-\nu_1^2)} \{(1-\nu_1) K_{\alpha\beta}(\boldsymbol{v}^1) + \nu_1 K_{\gamma\gamma}(\boldsymbol{v}^1)\delta_{\alpha\beta}\} , \quad \alpha, \beta, \gamma = 1, 2 ; \end{cases} \tag{1.6}$$

$$\begin{cases} \varepsilon_{\alpha'\beta'}(\boldsymbol{v}^2) = \tfrac{1}{2}(\partial_{\alpha'} v_{\beta'}^2 + \partial_{\beta'} v_{\alpha'}^2) , \\ Q_{\alpha'\beta'}(\boldsymbol{v}^2) = \dfrac{2E_2 t_2}{1-\nu_2^2} \{(1-\nu_2)\varepsilon_{\alpha'\beta'}(\boldsymbol{v}^2) + \nu_2 \varepsilon_{\gamma'\gamma'}(\boldsymbol{v}^2)\delta_{\alpha'\beta'}\} ; \end{cases} \tag{1.7}$$

$$\begin{cases} K_{\alpha'\beta'}(\boldsymbol{v}^2) = -\partial_{\alpha'\beta'} v_1^2 , \\ M_{\alpha'\beta'}(\boldsymbol{v}^2) = \dfrac{2E_2 t_2^3}{3(1-\nu_2^2)} \{(1-\nu_2) K_{\alpha'\beta'}(\boldsymbol{v}^2) + \nu_2 K_{\gamma'\gamma'}(\boldsymbol{v}^2)\delta_{\alpha'\beta'}\} , \quad \alpha', \beta', \gamma' = 2, 3 ; \end{cases} \tag{1.8}$$

$$F(\mathscr{V}) = \int_{\Omega_1} f^1 \boldsymbol{v}^1 \, \mathrm{d}x_1 \, \mathrm{d}x_2 + \int_{\Omega_2} f^2 \boldsymbol{v}^2 \, \mathrm{d}x_2 \, \mathrm{d}x_3 ; \tag{1.9}$$

$E_1 > 0$, $E_2 > 0$ denote Young's modulus of the plates Ω_1 and Ω_2, respectively, and $0 < \nu_1 < 1$, $0 < \nu_2 < 1$ denote the associated Poisson ratios.

Latin indices have the values $\{1, 2, 3\}$, Greek indices and primed Greek indices take the values $\{1, 2\}$ and $\{2, 3\}$, respectively; the repeated index convention for summation is systematically used; and c, c_1, c_2 denote generic constants which may have different values in different places.

2. Mathematical model

We now introduce some Sobolev spaces (cf. [5]):

$$\begin{aligned} H = \{\mathscr{V} = (\boldsymbol{v}^1, \boldsymbol{v}^2);\ & \boldsymbol{v}^1 \in H^{112}(\Omega_1) := H^1(\Omega_1) \times H^1(\Omega_1) \times H^2(\Omega_1) , \\ & \boldsymbol{v}^2 \in H^{211}(\Omega_2) := H^2(\Omega_2) \times H^1(\Omega_2) \times H^1(\Omega_2)\} , \end{aligned} \tag{2.1}$$

with norm

$$\|\mathscr{V}\|_H := \Big\{ \sum_{\alpha} \|v_\alpha^1\|_{1,\Omega_1}^2 + \|v_3^1\|_{2,\Omega_1}^2 + \sum_{\alpha'} \|v_{\alpha'}^2\|_{1,\Omega_2}^2 + \|v_1^2\|_{2,\Omega_2}^2 \Big\}^{1/2} , \tag{2.2}$$

$$\begin{aligned} V = \{\mathscr{V} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in H;\ & v_\alpha^1 = 0,\ \alpha = 1, 2, \text{ on } \Gamma_0,\ v_3^1 = \partial v_3^1 / \partial n = 0 \text{ on } \Gamma_0 , \\ & \text{and } \boldsymbol{v}^1 = \boldsymbol{v}^2 \text{ on } \Gamma,\ -\partial_1 v_3^1 = \partial_3 v_1^2 \text{ on } \Gamma\} . \end{aligned} \tag{2.3}$$

Then the mathematical model of the elastic composite structure of two coupled plates is:

Find

$$\mathcal{U} = (u^1, u^2) \in V \,, \quad \text{such that} \quad D(\mathcal{U}, \mathcal{V}) = F(\mathcal{V}) \quad \forall \mathcal{V} \in V \,. \tag{2.4}$$

To prove the existence and uniqueness of the solution of the problem (2.4), we need the following lemmas.

LEMMA 2.1. The subspace V is closed in H with the norm (2.2).

The proof is easy and we omit it.

LEMMA 2.2. The bilinear form $D(\mathcal{U}, \mathcal{V})$ defined in (1.4) is continuous and coercive on $V \times V$.

PROOF. The continuity of $D(\cdot\,,\cdot)$ is obvious. We only need to prove that $D(\cdot\,,\cdot)$ is coercive, i.e., there exists $c = \text{const} > 0$, such that

$$D(\mathcal{V}, \mathcal{V}) \geq c\|\mathcal{V}\|_H^2 \quad \forall \mathcal{V} \in V \,. \tag{2.5}$$

To do this, from (1.4), we firstly consider $D_1(v^1, v^2)$ as follows:

$$\begin{aligned} M_{\alpha\beta}(v^1)K_{\alpha\beta}(v^1) &= \frac{2E_1t_1^3}{3(1-\nu_1^2)}\{(1-\nu_1)(K_{11}^2 + K_{22}^2 + 2K_{12}^2) + \nu_1(K_{11} + K_{22})^2\} \\ &\geq \frac{2E_1t_1^3}{3(1+\nu_1)}(K_{11}^2 + K_{22}^2 + 2K_{12}^2) = \frac{4}{3}\,\mu_1t_1^3(K_{11}^2 + K_{22}^2 + 2K_{12}^2)\,, \end{aligned}$$

with Lamé constants $\lambda > 0$ and $\mu > 0$. Similarly, we have

$$Q_{\alpha\beta}(v^1)\varepsilon_{\alpha\beta}(v^1) \geq 4\mu_1t_1(\varepsilon_{11}^2 + \varepsilon_{22}^2 + 2\varepsilon_{12}^2)\,.$$

Thus we have

$$D_1(v^1, v^1) \geq 4\mu_1t_1\sum_{\alpha,\beta}\|\varepsilon_{\alpha\beta}(v^1)\|_{0,\Omega_1}^2 + \tfrac{4}{3}\mu_1t_1^3|v_3^1|_{2,\Omega_1}^2\,. \tag{2.6}$$

In the same way, we have

$$D_2(v^2, v^2) \geq 4\mu_2t_2\sum_{\alpha',\beta'}\|\varepsilon_{\alpha'\beta'}(v^2)\|_{0,\Omega_2}^2 + \tfrac{4}{3}\mu_2t_2^3|v_1^3|_{2,\Omega_2}^2\,. \tag{2.7}$$

Since $\mathcal{V} \in V$, $v_\alpha^1 = 0$, $\alpha = 1, 2$, on Γ_0. Then from Korn's inequality and Poincare's inequality, we have

$$D_1(v^1, v^1) \geq c_1\Big\{\sum_\alpha\|v_\alpha^1\|_{1,\Omega_1}^2 + \|v_3^1\|_{2,\Omega_1}^2\Big\}\,. \tag{2.8}$$

From (2.7), (2.8) and (1.4), we have

$$D(\mathcal{V},\mathcal{V}) \geq c_1\Big\{\sum_{\alpha}\|v_\alpha^1\|_{1,\Omega_1}^2 + \|v_3^1\|_{2,\Omega_1}^2\Big\} + c_2\Big\{\sum_{\alpha',\beta'}\|\varepsilon_{\alpha'\beta'}(v^2)\|_{0,\Omega_2}^2 + |v_1^2|_{2,\Omega_2}^2\Big\}, \tag{2.9}$$

with c_1, $c_2 = \text{const} > 0$.

If the inequality (2.5) is false, there exists a sequence $(\mathcal{V}_k) \in V$, such that

$$\|\mathcal{V}_k\|_H = 1 \quad \forall k,$$
$$D(\mathcal{V}_k, \mathcal{V}_k) \to 0 \quad \text{as } k \to \infty. \tag{2.10}$$

From the first relation of (2.10) and by the Rellich–Kondrasov compact embedding theorem, there exists a subsequence $(\mathcal{V}_l) \subset (\mathcal{V}_k)$, such that

$$v_{\alpha,l}^1 \to v_\alpha^1, \qquad \alpha = 1,2, \quad \text{in } L^2(\Omega_1), \qquad v_{3,l}^1 \to v_3^1 \quad \text{in } H^1(\Omega_1), \tag{2.11}$$

$$v_{\alpha',l}^2 \to v_\alpha'^2, \qquad \alpha' = 2,3, \quad \text{in } L^2(\Omega_2), \qquad v_{1,l}^2 \to v_1^2 \quad \text{in } H^1(\Omega_2), \tag{2.12}$$

From the second relation of (2.10) and (2.11), (2.12), we have that

$$\nu_\alpha^1 = 0, \quad \alpha = 1,2, \quad \text{in } H^1(\Omega_1), \qquad v_3^1 = 0 \text{ in } H^2(\Omega_1), \tag{2.13}$$

and that $(v_{\alpha',l}^2)$, $\alpha' = 2, 3$, are Cauchy sequences in $H^1(\Omega_2)$, $(v_{1,l}^2)$ is a Cauchy sequence in $H^2(\Omega_2)$. Since V is closed in H, then we can see that

$$\mathcal{V}_l \to \mathcal{V} = (o, v^2) \quad \text{in } H \tag{2.14}$$

and

$$\varepsilon_{\alpha'\beta'}(v^2) = 0, \qquad |v_1^2|_{2,\Omega_2} = 0. \tag{2.15}$$

From the first relation of (2.15),

$$v_2^2(x_2, x_3) = a_2 + bx_3, \qquad v_3^2(x_2, x_3) = a_3 - bx_2, \tag{2.16}$$

and taking into account $v_2^2 = v_2^1 = 0$, $v_3^2 = v_3^1 = 0$ on $\Gamma(x_1 = x_3 = 0)$, we can see that

$$a_2 = a_3 = b = 0,$$

which means that

$$v_2 = v_3^2 = 0 \text{ in } \Omega_2. \tag{2.17}$$

From the second relation of (2.15),

$$v_1^2(x_2, x_3) = a + bx_2 + cx_3,$$

taking into account $v_1^2 = v_1^1 = 0$ and $\partial_3 v_1^2 = -\partial_1 v_3^1 = 0$ on Γ, we can see that

$$v_1^2 = 0 \quad \text{on } \Omega_2 . \tag{2.18}$$

Thus $\mathcal{V} = 0$ in H, but this contradicts the equalities $\|\mathcal{V}_l\|_H = 1$, $\forall l$. The proof is completed. □

By Lemmas 2.1 and 2.2, and Lax–Milgram's Theorem, it can be seen that the problem (2.4) has a unique solution.

3. Boundary value problems and junction conditions

We now formulate boundary value problems and junction conditions for the problem (2.4), which will be useful in the error estimate of finite element approximations for the problem (2.4). Rewrite the problem (2.4) in detail: Find

$$u^1 \in H^{112}(\Omega_1),\ u^2 \in H^{211}(\Omega_2);\ u^1 = 0,\ \partial_1 u_3^1 = 0 \text{ on } \Gamma_0,\ u^1 = u^2,\ -\partial_1 u_3^1 = \partial_3 u_1^2 \text{ on } \Gamma ,$$

such that

$$\begin{aligned} &\int_{\Omega_1} Q_{\alpha\beta}(u^1)\varepsilon_{\alpha\beta}(v^1)\,\mathrm{d}x_1\,\mathrm{d}x_2 + \int_{\Omega_1} M_{\alpha\beta}(u^1)K_{\alpha\beta}(v^1)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &\quad + \int_{\Omega_2} Q_{\alpha'\beta'}(u^2)\varepsilon_{\alpha'\beta'}(v^2)\,\mathrm{d}x_2\,\mathrm{d}x_3 + \int_{\Omega_2} M_{\alpha'\beta'}(u^2)K_{\alpha'\beta'}(v^2)\,\mathrm{d}x_2\,\mathrm{d}x_3 \\ &= \int_{\Omega_1} f^1 v^1\,\mathrm{d}x_1 \mathrm{d}x_2 + \int_{\Omega_2} f^2 v^2\,\mathrm{d}x_2\,\mathrm{d}x_3 \\ &\quad \forall v^1 \in H^{112}(\Omega_1),\ v^2 \in H^{211}(\Omega_2);\ v^1 = 0,\ \partial_1 v_3^1 = 0 \text{ on } \Gamma_0,\ v^1 = v^2,\ -\partial_1 v_3^1 = \partial_3 v_1^2 \text{ on } \Gamma . \end{aligned} \tag{3.1}$$

Let $v^1 = 0$ in (3.1). Then

$$\begin{aligned} &\int_{\Omega_2} Q_{\alpha'\beta'}(u^2)\varepsilon_{\alpha'\beta'}(v^2)\,\mathrm{d}x_2\,\mathrm{d}x_3 + \int_{\Omega_2} M_{\alpha'\beta'}(u^2)K_{\alpha'\beta'}(v^2)\,\mathrm{d}x_2\,\mathrm{d}x_3 \\ &\quad = \int_{\Omega_2} f^2 v^2\,\mathrm{d}x_2\,\mathrm{d}x_3 \quad \forall v^2 \in H^{211}(\Omega_2);\ v^2 = 0,\ \partial_3 v_1^2 = 0 \text{ on } \Gamma . \end{aligned} \tag{3.2}$$

By Green's formula, from (3.2) it can be seen that (cf. Fig. 2.)

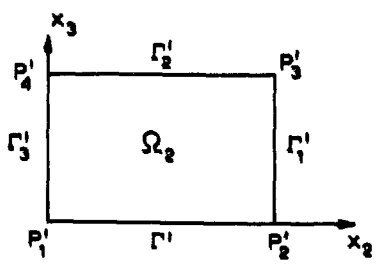

Fig. 2.

$$
\begin{aligned}
&-\int_{\Omega_2} \partial_{\beta'} Q_{\alpha'\beta'}(u^2) v^2_{\alpha'} \,\mathrm{d}x_2\,\mathrm{d}x_3 + \int_{\partial\Omega_2\setminus\Gamma} Q_{\alpha'\beta'}(u^2) n_{\beta'} v^2_{\alpha'} \,\mathrm{d}s \\
&-\int_{\Omega_2} \partial_{\alpha'\beta'} M_{\alpha'\beta'}(u^2) v^2_1 \,\mathrm{d}x_2\,\mathrm{d}x_3 - \int_{\partial\Omega_2\setminus\Gamma} M_{\alpha'\beta'}(u^2) n_{\alpha'} n_{\beta'}\, \partial_{n'} v^2_1 \,\mathrm{d}s \\
&+\int_{\partial\Omega_2\setminus\Gamma} \{\partial_{\alpha'} M_{\alpha'\beta'}(u^2) n_{\beta'} + \partial_{s'} M_{\alpha'\beta'}(u^2) n_{\alpha'} s_{\beta'}\} v^2_1 \,\mathrm{d}s \\
&+ [M_{\alpha'\beta'}(u^2) n_{\alpha'} s_{\beta'}]^{\Gamma'_2}_{\Gamma'_1}(P'_3) v^2_1(P_3') + [M_{\alpha'\beta'}(u^2) n_{\alpha'} s_{\beta'}]^{\Gamma'_3}_{\Gamma'_2}(P'_4) v^2_1(P'_4) \\
&= \int_{\Omega_2} f^2 v^2 \,\mathrm{d}x_2\,\mathrm{d}x_3 \quad \forall v^2 \in H^{211}(\Omega_2);\ v^2 = 0,\ \partial_3 v^2_1 = 0 \text{ on } \Gamma\,,
\end{aligned} \tag{3.3}
$$

where $n' = (n_2, n_3)$ and $s' = (s_2, s_3)$ are the outward unit normal and associated unit tangential vectors on $\partial\Omega_2$, respectively, $\partial_{s'} = s_{\alpha'}\,\partial_{\alpha'}$, $\partial_{n'} = n_{\alpha'}\,\partial_{\alpha'}$ and

$$
[\phi]^{\Gamma'_i}_{\Gamma'_j} = \phi|_{\Gamma'_i} - \phi|_{\Gamma'_j}\,. \tag{3.4}
$$

Then we have

$$
\begin{cases}
-\partial_{\beta'} Q_{\alpha'\beta'}(u^2) = f^2_{\alpha'}\,, & \alpha' = 2,3, \quad \text{in } \Omega_2\,, \\
Q_{\alpha'\beta'}(u^2) n_{\beta'} = 0\,, & \alpha' = 2,3, \quad \text{on } \partial\Omega_2\setminus\Gamma\,;
\end{cases} \tag{3.5}
$$

$$
\begin{cases}
-\partial_{\alpha'\beta'} M_{\alpha'\beta'}(u^2) = f^2_1 & \text{in } \Omega_2\,, \\
M_{\alpha'\beta'}(u^2) n_{\alpha'} n_{\beta'} = 0 & \text{on } \partial\Omega_2\setminus\Gamma\,, \\
\partial_{\alpha'} M_{\alpha'\beta'}(u^2) n_{\beta'} + \partial_{s'} M_{\alpha'\beta'}(u^2) n_{\alpha'} s_{\beta'} = 0 & \text{on } \partial\Omega_2\setminus\Gamma\,,
\end{cases} \tag{3.6a}
$$

and

$$
M_{\alpha'\beta'}(u^2) n_{\alpha'} s_{\beta'}|^{\Gamma'_2}_{\Gamma'_1}(P'_3) = 0\,, \qquad M_{\alpha'\beta'}(u^2) n_{\alpha'} s_{\beta'}|^{\Gamma'_3}_{\Gamma'_2}(P'_4) = 0\,,
$$

which is equivalent to

$$
M_{23}(u^2)(P'_3) = 0\,, \qquad M_{23}(u^2)(P'_4) = 0\,. \tag{3.6b}
$$

Substituting (3.5), (3.6) into (3.1), it can be seen that

$$
\begin{aligned}
&\int_{\Omega_1} Q_{\alpha\beta}(u^1) \varepsilon_{\alpha\beta}(v^1) \,\mathrm{d}x_1\,\mathrm{d}x_2 + \int_{\Omega_1} M_{\alpha\beta}(u^1) K_{\alpha\beta}(v^1) \,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&\quad + \int_{\Gamma} Q_{\alpha'\beta'}(u^2) n_{\beta'} v^2_{\alpha'} \,\mathrm{d}s + \int_{\Gamma} M_{\alpha'\beta'}(u^2) n_{\alpha'} n_{\beta'}\, \partial_3 v^2_1 \,\mathrm{d}s \\
&\quad + \int_{\Gamma} \{\partial_{\alpha'} M_{\alpha'\beta'}(u^2) n_{\beta'} + \partial_2 M_{\alpha'\beta'}(u^2) n_{\alpha'} s_{\beta'}\} v^2_1 \,\mathrm{d}s \\
&\quad + [M_{\alpha'\beta'}(u^2) n_{\alpha'} s_{\beta'}]^{\Gamma'_1}_{\Gamma'}(P'_2) v^2_1(P'_2) + [M_{\alpha'\beta'}(u^2) n_{\alpha'} s_{\beta'}]^{\Gamma'}_{\Gamma'_3}(P'_1) v^2_1(P'_1) \\
&= \int_{\Omega_1} f^1 v^1 \,\mathrm{d}x_1\,\mathrm{d}x_2 \quad \forall v^1 \in H^{112}(\Omega_1);\ v^1 = 0,\ \partial_1 v^1_3 = 0 \text{ on } \Gamma_0\,,
\end{aligned}
$$

or equivalently, in our case (cf. Fig. 2), taking into account $v^2 = v^1$, $\partial_3 v_1^2 = -\partial_1 v_3^1$ on Γ, we can see that

$$\begin{aligned}
&\int_{\Omega_1} Q_{\alpha\beta}(u^1)\varepsilon_{\alpha\beta}(v^1)\,dx_1\,dx_2 + \int_{\Omega_1} M_{\alpha\beta}(u^1)K_{\alpha\beta}(v^1)\,dx_1\,dx_2 \\
&\quad - \int_\Gamma Q_{\alpha'3}(u^2)v^1_{\alpha'}\,dx_2 - \int_\Gamma M_{33}(u^2)\,\partial_1 v_3^1\,dx_2 \\
&\quad - \int_\Gamma \{\partial_{\alpha'}M_{\alpha'3}(u^2) + \partial_2 M_{32}(u^2)\}v_1^1\,dx_2 \\
&\quad + 2M_{23}(u^2)(P_2')v_1^1(P_2') - 2M_{23}(u^2)(P_1')v_1^1(P_1') \\
&= \int_{\Omega_1} f^1 v^1\,dx_1\,dx_2 \quad \forall v^1 \in H^{112}(\Omega_1);\ v^1 = 0,\ \partial_1 v_3^1 = 0 \text{ on } \Gamma_0\,.
\end{aligned} \tag{3.7}$$

By Green's formula, from (3.7), it can be seen that (cf. Fig. 3)

$$\begin{aligned}
&-\int_{\Omega_1} \partial_\beta Q_{\alpha\beta}(u^1)v^1_\alpha\,dx_1\,dx_2 + \int_{\partial\Omega_1} Q_{\alpha\beta}(u^1)n_\beta v^1_\alpha\,ds \\
&-\int_{\Omega_1} \partial_{\alpha\beta}M_{\alpha\beta}(u^1)v_3^1\,dx_1\,dx_2 - \int_{\partial\Omega_1} M_{\alpha\beta}(u^1)n_\alpha n_\beta\,\partial_n v_3^1\,ds \\
&+\int_{\partial\Omega_1} \{\partial_\alpha M_{\alpha\beta}(u^1)n_\beta + \partial_s M_{\alpha\beta}(u^1)n_\alpha s_\beta\}v_3^1\,ds \\
&+2M_{12}(u^1)(P_2)v_3^1(P_2) - 2M_{12}(u^1)(P_1)v_3^1(P_1) \\
&-\int_\Gamma Q_{\alpha'3}(u^2)v^1_{\alpha'}\,dx_2 - \int_\Gamma M_{33}(u^2)\,\partial_1 v_3^1\,dx_2 - \int_\Gamma \{\partial_{\alpha'}M_{\alpha'3}(u^2) + \partial_2 M_{32}(u^2)\}v_1^2\,dx_2 \\
&+2M_{23}(u^2)(P_2')v_1^1(P_2') - 2M_{23}(u^2)(P_1')v_1^1(P_1') \\
&= \int_{\Omega_1} f^1 v^1\,dx_1\,dx_2 \quad \forall v^1 \in H^{112}(\Omega_1);\ v^1 = 0,\ \partial_1 v_3^1 = 0 \text{ on } \Gamma_0\,,
\end{aligned} \tag{3.8}$$

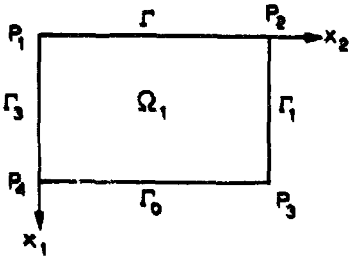

Fig. 3.

from which and taking into account that $\Gamma=\Gamma'$, $P_1=P_1'$, $P_2=P_2'$, we have

$$\begin{cases} -\partial_\beta Q_{\alpha\beta}(u^1)=f_\alpha^1, & \alpha=1,2, \quad \text{in } \Omega_1, \\ Q_{\alpha\beta}(u^1)n_\beta=0, & \alpha=1,2, \quad \text{on } \partial\Omega_1\backslash(\Gamma\cup\Gamma_0), \\ u_\alpha^1=0, & \alpha=1,2, \quad \text{on } \Gamma_0; \end{cases} \tag{3.9}$$

$$\begin{cases} -\partial_{\alpha\beta}M_{\alpha\beta}(u^1)=f_3^1, & \text{in } \Gamma_1, \\ M_{\alpha\beta}(u^1)n_\alpha n_\beta=0, & \text{on } \partial\Omega_1\backslash(\Gamma\cup\Gamma_0), \\ \partial_\alpha M_{\alpha\beta}(u^1)n_\beta+\partial_s M_{\alpha\beta}(u^1)n_\alpha s_\beta=0, & \text{on } \partial\Omega_1\backslash(\Gamma\cup\Gamma_0), \\ u_3^1=\partial_1 u_3^1=0, & \text{on } \Gamma_0. \end{cases} \tag{3.10}$$

$$M_{12}(u^1)(P_\alpha)=0, \qquad M_{23}(u^2)(P_\alpha)=0, \quad \alpha=1,2. \tag{3.11}$$

We now formulate the coupled conditions on Γ. From (3.8)–(3.11), it can be seen that (cf. Figs. 2 and 3)

$$\begin{aligned} &-\int_\Gamma Q_{\alpha 1}(u^1)v_\alpha^1\,dx_2+\int_\Gamma M_{11}(u^1)\,\partial_1 v_3^1\,dx_2-\int_\Gamma Q_{\alpha' 3}(u^2)v_{\alpha'}^1\,dx_2 \\ &-\int_\Gamma M_{33}(u^2)\,\partial_1 v_3^1\,dx_2-\int_\Gamma\{\partial_\alpha M_{\alpha 1}(u^1)+\partial_2 M_{12}(u^1)\}v_3^1\,dx_2 \\ &-\int_\Gamma\{\partial_{\alpha'}M_{\alpha' 3}(u^2)+\partial_2 M_{32}(u^2)\}v_1^1\,dx_2=0 \quad \forall v^1\in H^{112}(\Omega_1);\ v^1=0,\ \partial_1 v_3^1=0 \text{ on } \Gamma_0, \end{aligned} \tag{3.12}$$

from which we have

$$\begin{cases} -Q_{11}(u^1)-\{\partial_{\alpha'}M_{\alpha' 3}(u^2)+\partial_2 M_{32}(u^2)\}=0, \\ -Q_{21}(u^1)-Q_{23}(u^2)=0, \\ -\{\partial_\alpha M_{\alpha 1}(u^1)+\partial_2 M_{12}(u^1)\}-Q_{33}(u^2)=0, \\ M_{11}(u^1)-M_{33}(u^2)=0, \quad \text{on } \Gamma. \end{cases} \tag{3.13}$$

Notice that conditions (3.13) and

$$u^1=u^2, \qquad -\partial_1 u_3^1=\partial_3 u_1^3 \quad \text{on } \Gamma, \tag{3.14}$$

are the junction conditions arising from the rigid junction between two coupled plates, while the boundary value problems (3.5), (3.6), (3.9), (3.10) and (3.11) are normal formulas as in the usual single elastic plate.

4. Finite element approximation

In this section, the finite element approximation to problem (2.4) is considered.

Let $\mathscr{T}_h^\alpha$, $\alpha = 1, 2$, be regular rectangular subdivisions with the inverse hypotheses of Ω_1 and Ω_2, respectively, which have the same nodes on Γ, and let $V_h^1(\Omega_\alpha)$ be the bilinear finite element spaces associated with the subdivisions $\mathscr{T}_h^\alpha$ on Ω_α, $\alpha = 1, 2$, $V_h^2(\Omega_\alpha)$ be Adini's finite element spaces associated with the subdivisions $\mathscr{T}_h^\alpha$ on Ω_α, $\alpha = 1, 2$,

$$\begin{aligned} V_h = \{\mathscr{V} = (v_h^1, v_h^2);\ & v_{\alpha h}^1 \in V_h^1(\Omega_1),\ \alpha = 1,2,\ v_{3h}^1 \in V_h^2(\Omega_1)\,, \\ & v_{\alpha' h}^2 \in V_h^1(\Omega_2),\ \alpha' = 2,3,\ v_{1h}^2 \in V_h^2(\Omega_2)\,, \\ & v_{\alpha h}^1(Q) = 0,\ v_{3h}^1(Q) = 0,\ \partial_1 v_{3h}^1(Q) = 0\ \forall \text{ nodes } Q \in \Gamma_0\,, \\ & \text{and } v_{ih}^1(P) = v_{ih}^2(P),\ -\partial_1 v_{3h}^1(P) = \partial_3 v_{1h}^2(P)\ \forall \text{ nodes } P \in \Gamma\}\,. \end{aligned} \tag{4.1}$$

Then the finite element approximation of the problem (2.4) is the following: Find $\mathscr{U}_h \in V_h$, such that

$$D_h(\mathscr{U}_h, \mathscr{V}_h) = F(\mathscr{V}_h) \quad \forall \mathscr{V}_h \in V_h\,, \tag{4.2}$$

where

$$\begin{aligned} D_h(\mathscr{U}_h, \mathscr{V}_h) = & \left\{\int_{\Omega_1} Q_{\alpha\beta}(u_h^1)\varepsilon_{\alpha\beta}(v_h^1)\,\mathrm{d}x_1\,\mathrm{d}x_2 + \sum_{\tau_1}\int_{\tau_1} M_{\alpha\beta}(u_{3h}^1)K_{\alpha\beta}(v_{3h}^1)\,\mathrm{d}x_1\,\mathrm{d}x_2\right\} \\ & + \left\{\int_{\Omega_2} Q_{\alpha'\beta'}(u_h^2)\varepsilon_{\alpha'\beta'}(v_h^2)\,\mathrm{d}x_2\,\mathrm{d}x_3 \right. \\ & \left. + \sum_{\tau_2}\int_{\tau_2} M_{\alpha'\beta'}(u_{1h}^2)K_{\alpha'\beta'}(v_{1h}^2)\,\mathrm{d}x_2\,\mathrm{d}x_3\right\}\,, \end{aligned} \tag{4.3}$$

$$F(\mathscr{V}_h) = \int_{\Omega_1} f_i^1 v_{ih}^1\,\mathrm{d}x_1\,\mathrm{d}x_2 + \int_{\Omega_2} f_i^2 v_{ih}^2\,\mathrm{d}x_2\,\mathrm{d}x_3\,. \tag{4.4}$$

Let

$$\begin{aligned} & \mathscr{V} = (v^1, v^2) \in V_h \\ & \to \|\mathscr{V}\|_h = \left\{\left(\sum_\alpha \|v_\alpha^1\|_{1,\Omega_1}^2 + \sum_{\tau_1} |v_3^1|_{2,\tau_1}^2\right) + \left(\sum_{\alpha'} \|v_{\alpha'}^2\|_{1,\Omega_2}^2 + \sum_{\tau_2} |v_1^1|_{2,\tau_2}^2\right)\right\}\,, \end{aligned} \tag{4.5}$$

which is a norm on V_h.

We have the following error estimate.

THEOREM 4.1. Assume that $\mathscr{U} = (u^1, u^2)$ is the solution of the problem (2.4), with $u_\alpha^1 \in H^2(\Omega_1)$, $\alpha = 1, 2$, $u_3^1 \in H^3(\Omega_1)$ and $u_{\alpha'}^2 \in H^2(\Omega_2)$, $\alpha' = 2, 3$, $u_1^2 \in H^3(\Omega_2)$. Let $\mathscr{U}_h = (u_h^1, u_h^2)$ be the solution of the problem (4.3). Then the following error estimate holds:

$$\|\mathscr{U} - \mathscr{U}_h\|_h \leq ch\left(\sum_\alpha \|u_\alpha^1\|_{2,\Omega_1} + \|u_3^1\|_{3,\Omega_1} + \sum_{\alpha'} \|u_{\alpha'}^2\|_{2,\Omega_2} + \|u_1^2\|_{3,\Omega_2}\right)\,. \tag{4.6}$$

PROOF. By the abstract error estimate for the nonconforming finite element approximation (cf. [6]), we have

$$\|\mathcal{U}-\mathcal{U}_h\|_h \leqslant c\left\{\inf_{\mathcal{V}_h\in V_h}\|\mathcal{U}-\mathcal{V}_h\|_h + \sup_{\mathcal{W}_h\in V_h}\frac{D_h(\mathcal{U},\mathcal{W}_h)-F(\mathcal{W}_h)}{\|\mathcal{W}_h\|_h}\right\}. \tag{4.7}$$

By using the usual interpolation error estimates, the first term on the right-hand side of (4.7) can be estimated as follows:

$$\inf_{\mathcal{V}_h\in V_h}\|\mathcal{U}-\mathcal{V}_h\|_h \leqslant ch\left\{\sum_{\alpha}|u_\alpha^1|_{2,\Omega_1}+|u_3^1|_{3,\Omega_1}+\sum_{\alpha'}|u_{\alpha'}^2|_{2,\Omega_2}+|u_1^2|_{3,\Omega_2}\right\}. \tag{4.8}$$

It remains to estimate the second term on the right-hand side of (4.8). Using Green's formula repeatedly, and taking into account the boundary value problems and the junction conditions in Section 3, after some straightforward calculations, it can be seen that

$$\begin{aligned}
E_h(\mathcal{U},\mathcal{W}_h) &= D_h(\mathcal{U},\mathcal{W}_h)-F(\mathcal{W}_h)\\
&=\left\{\int_{\Omega_1} Q_{\alpha\beta}(u^1)\varepsilon_{\alpha\beta}(w_h^1)\,\mathrm{d}x_1\,\mathrm{d}x_2+\sum_{\tau_1\in\mathcal{T}_h^1}\int_{\tau_1} M_{\alpha\beta}(u^1)K_{\alpha\beta}(w_h^1)\,\mathrm{d}x_1\,\mathrm{d}x_2\right.\\
&\quad\left.-\int_{\Omega_1} f^1 w_h^1\,\mathrm{d}x_1\,\mathrm{d}x_2\right\}+\left\{\int_{\Omega_2} Q_{\alpha'\beta'}(u^2)\varepsilon_{\alpha'\beta'}(w_h^2)\,\mathrm{d}x_2\,\mathrm{d}x_3\right.\\
&\quad\left.+\sum_{\tau_2\in\mathcal{T}_h^2}\int_{\tau_2} M_{\alpha'\beta'}(u^2)K_{\alpha'\beta'}(w_h^2)\,\mathrm{d}x_2\,\mathrm{d}x_3-\int_{\Omega_2} f^2 w_h^2\,\mathrm{d}x_2\,\mathrm{d}x_3\right\}\\
&=-\int_\Gamma Q_{11}(u^1)w_{1h}^1\,\mathrm{d}s-\int_\Gamma Q_{21}(u^1)w_{2h}^1\,\mathrm{d}s-\int_\Gamma Q_{23}(u^2)w_{2h}^2\,\mathrm{d}s\\
&\quad-\int_\Gamma Q_{33}(u^2)w_{3h}^2\,\mathrm{d}s-\int_\Gamma\{\partial_\alpha M_{\alpha 1}(u^1)+\partial_2 M_{12}(u^1)\}w_{3h}^1\,\mathrm{d}s\\
&\quad-\int_{\Gamma}\{\partial_{\alpha'}M_{\alpha'3}(u^2)+\partial_2 M_{32}(u^2)\}w_{1h}^2\,\mathrm{d}s\\
&\quad-\sum_{\tau_1}\int_{\partial\tau_1} M_{\alpha\beta}(u)n_\alpha n_\beta\,\partial_n w_{3h}^1\,\mathrm{d}s-\sum_{\tau_2}\int_{\partial\tau_2} M_{\alpha'\beta'}(u^2)n'_{\alpha'}n'_{\beta'}\,\partial_{n'}w_{1h}^2\,\mathrm{d}s\\
&=\int_\Gamma Q_{11}(u^1)(w_{1h}^2-w_{1h}^1)\,\mathrm{d}x_2+\int_\Gamma Q_{33}(u^2)(w_{3h}^1-w_{3h}^2)\,\mathrm{d}x_2\\
&\quad-\sum_{F_1\subset\Gamma}\int_{F_1} M_{\alpha\beta}(u^1)n_\alpha n_\beta\,\partial_n w_{3h}^1\,\mathrm{d}s-\sum_{F_2\subset\Gamma'}\int_{F_2} M_{\alpha'\beta'}(u^2)n'_{\alpha'}n'_{\beta'}\,\partial_{n'}w_{1h}^2\,\mathrm{d}s\\
&\quad-\sum_{\tau_1}\sum_{\substack{F_1\in\partial\tau_1\\ F_1\not\subset\partial\Omega_1}}\int_{F_1} M_{\alpha\beta}(u^1)n_\alpha n_\beta\,\partial_n w_{3h}^1\,\mathrm{d}s\\
&\quad-\sum_{\tau_2}\sum_{\substack{F_2\in\partial\tau_2\\ F_2\not\subset\partial\Omega_2}}\int_{F_2} M_{\alpha'\beta'}(u^2)n'_{\alpha'}n'_{\beta'}\,\partial_{n'}w_{1h}^2\,\mathrm{d}s\,,
\end{aligned} \tag{4.9}$$

from the junction conditions (3.13), in particular in the last equality. Let v^I denote the piecewise linear interpolation of v and $R_1(v) = v - v^I$. Then

$$\begin{aligned}
&-\sum_{F_1\subset\Gamma}\int_{F_1} M_{\alpha\beta}(u^1)n_\alpha n_\beta\, \partial_n w_{3h}^1\,\mathrm{d}s - \sum_{F_2\subset\Gamma'}\int_{F_2} M_{\alpha'\beta'}(u^2)n'_{\alpha'}n'_{\beta'}\,\partial_{n'}w_{1h}^2\,\mathrm{d}s \\
&= \sum_{F\subset\Gamma}\int_F M_{11}(u^1)\,\partial_1 w_{3h}^1\,\mathrm{d}x_2 + \sum_{F\subset\Gamma}\int_F M_{33}(u^2)\,\partial_3 w_{1h}^2\,\mathrm{d}x_2 \\
&= \Big\{\sum_{F\subset\Gamma}\int_F M_{11}(u^1)(\partial_1 w_{3h}^1)^I\,\mathrm{d}x_2 + \sum_{F\subset\Gamma}\int_F M_{33}(u^2)(\partial_3 w_{1h}^2)^I\,\mathrm{d}x_2\Big\} \\
&\quad + \Big\{\sum_{F\subset\Gamma}\int_F M_{11}(u^1)(\partial_1 w_{3h}^1 - (\partial_1 w_{3h}^1)^I)\,\mathrm{d}x_2\Big\} \\
&\quad + \Big\{\sum_{F\subset\Gamma}\int_F M_{33}(u^2)(\partial_3 w_{1h}^2 - (\partial_3 w_{1h}^2)^I)\,\mathrm{d}x_2\Big\} \\
&= \delta_h + \sum_{F\subset\Gamma}\int_F M_{11}(u^1)R_1(\partial_1 w_{3h}^1)\,\mathrm{d}x_2 + \sum_{F'\subset\Gamma'}\int_{F'} M_{33}(u^2)R_1(\partial_3 w_{1h}^2)\,\mathrm{d}x_2\,.
\end{aligned}$$

Since $\partial_3 w_{1h}^2(P) = -\partial_1 w_{3h}^1(P)$ for all nodes $P\in\Gamma$, which in turn imply that $(\partial_3 w_{1h}^2)^I = -(\partial_1 w_{3h}^1)$ on Γ, and taking into account the last equality of (3.13), then $\delta_h = 0$. Thus we have (cf. [7]),

$$\begin{aligned}
E_h(\mathscr{U}, \mathscr{W}_h) &= \int_{\Gamma'} Q_{11}(u^1)(w_{1h}^2 - w_{1h}^I)\,\mathrm{d}x_2 + \int_\Gamma Q_{33}(u^2)(w_{3h}^1 - w_{3h}^2)\,\mathrm{d}x_2 \\
&\quad - \sum_{\tau_1}\sum_{F_1\in\partial\tau_1}\int_{F_1} M_{\alpha\beta}(u^1)n_\alpha n_\beta R_1(\partial_n w_{3h}^1)\,\mathrm{d}s \\
&\quad - \sum_{\tau_2}\sum_{F_2\in\partial\tau_2}\int_{F_2} M_{\alpha'\beta'}(u^2)n_{\alpha'}n_{\beta'}R_1(\partial_{n'} w_{1h}^2)\,\mathrm{d}s\,.
\end{aligned} \tag{4.10}$$

By the interpolation error estimates (cf. [6, 7]), and taking into account that w_{1h}^I and w_{3h}^2 are the piecewise linear interpolations of w_{1h}^2 and w_{3h}^1 on Γ, respectively, we have

$$\begin{aligned}
\Big|\int_{\Gamma} Q_{11}(u^1)(w_{1h}^2 - w_{1h}^I)\,\mathrm{d}x_2\Big| &\le ch\Big(\sum_\alpha \|u_\alpha^1\|_{2,\Omega_1}\Big)\Big(\sum_{\tau_2}|w_{1h}^2|_{2,\tau_2}^2\Big)^{1/2}, \\
\Big|\int_{\Gamma} Q_{33}(u^2)(w_{3h}^1 - w_{3h}^2)\,\mathrm{d}x_2\Big| &\le ch\Big(\sum_{\alpha'} \|u_{\alpha'}^2\|_{2,\Omega_2}\Big)\Big(\sum_{\tau_1}|w_{3h}^1|_{2,\tau_1}^2\Big)^{1/2}.
\end{aligned} \tag{4.11}$$

By the error estimate for Adini's element [7], we have

$$\left\{\begin{aligned}
\Big|\sum_{\tau_1}\sum_{F_1\in\partial\tau_1}\int_{F_1} M_{\alpha\beta}(u^1)n_\alpha n_\beta R_1(\partial_n w_{3h}^1)\,\mathrm{d}s\Big| &\le ch\|u_3^1\|_{3,\Omega_1}\Big(\sum_{\tau_1}|w_{3h}^1|_{2,\tau_1}^2\Big)^{1/2}, \\
\Big|\sum_{\tau_2}\sum_{F_2\in\partial\tau_2}\int_{F_2} M_{\alpha'\beta'}(u^2)n_{\alpha'}n_{\beta'}R_1(\partial_{n'} w_{1h}^2)\,\mathrm{d}s\Big| &\le ch\|u_1^2\|_{3,\Omega_2}\Big(\sum_{\tau_2}|w_{1h}^2|_{2,\tau_2}^2\Big)^{1/2}.
\end{aligned}\right. \tag{4.12}$$

Finally it turns out that

$$|E_h(\mathcal{U}, \mathcal{W}_h)| \leq ch\Big(\sum_{\alpha} \|u^1_\alpha\|_{2,\Omega_1} + \|u^1_3\|_{3,\Omega_1} + \sum_{\alpha'} \|u^2_{\alpha'}\|_{2,\Omega_2} + \|u^2_1\|_{3,\Omega_2}\Big)\|\mathcal{W}_h\|_h\,, \tag{4.13}$$

and the proof is completed. □

REMARK 1. In fact, we can apply other nonconforming finite element spaces for $V^2_h(\Omega_\alpha)$. For the case of the space $V^2_h(\Omega_\alpha) \subset H^1(\Omega_\alpha)$, such as Zienkiewicz's element space, the conclusion of Theorem 4.1 holds with the same proof as that of Theorem 4.1. For the case of the space $V^2_h(\Omega_\alpha) \not\subset H^1(\Omega_\alpha)$, such as Morely's and De Veubeke's element space, the conclusion of Theorem 4.1 holds essentially. The proof is given in Remark 2.

REMARK 2. As an example, we consider Morley's element space $V^2_h(\Omega_\alpha)$. Let $\mathcal{T}^\alpha_h$, $\alpha = 1, 2$, be regular triangulations with the inverse hypotheses of Ω_1 andΩ_2, respectively, which have the same nodes on Γ. Let $V^1_h(\Omega_\alpha)$ be the linear finite element spaces associated with the triangulations $\mathcal{T}^\alpha_h$ on Ω_α, $\alpha = 1, 2$, and let $V^2_h(\Omega_\alpha)$ be Morley's finite element spaces associated with the triangulations $\mathcal{T}^\alpha_h$, on Ω_α, $\alpha = 1, 2$, and

$$\begin{aligned} V_h = \Big\{ \mathcal{V}_h = (v^1_h, v^2_h);\ & v^1_{\alpha h} \in V^1_h(\Omega_1),\ \alpha = 1, 2,\ v^1_{3h} \in V^2_h(\Omega_1)\,, \\ & v^2_{\alpha' h} \in V^1_h(\Omega_2),\ \alpha' = 2, 3,\ v^2_{1h} \in V^2_h(\Omega_2)\,, \\ & v^1_{ih}(Q) = 0,\ \forall \text{nodes } Q \in \Gamma_0,\ \int_F \partial_1 v^1_{3h}\, \mathrm{d}s = 0,\ \forall \text{edges } F \subset \Gamma_0\,, \\ & \text{and } v^1_{ih}(P) = v^2_{ih}(P),\ \forall \text{nodes } P \in \Gamma, \\ & -\int_F \partial_1 v^1_{3h}\, \mathrm{d}s = \int_F \partial_3 v^2_{1h}\, \mathrm{d}s\,,\ \forall \text{edges } F \subset \Gamma \Big\}\,. \end{aligned} \tag{4.15}$$

The finite element approximation of (2.4) for Morley's element is the same as above. Then the following error estimate holds.

THEOREM 4.2. Under the same assumptions as in Theorem 4.1, the following error estimate holds:

$$\begin{aligned} \|\mathcal{U} - \mathcal{U}_h\|_h \leq ch\Big\{ & \sum_{\alpha} \|u^1_\alpha\|_{2,\Omega_1} + \|u^1_3\|_{3,\Omega_1} + \sum_{\alpha'} \|u^2_{\alpha'}\|_{2,\Omega_2} + \|u^2_1\|_{3,\Omega_2} \\ & + h(\|f^1_3\|_{0,\Omega_1} + \|f^2_1\|_{0,\Omega_2})\Big\}\,. \end{aligned} \tag{4.16}$$

PROOF. Similar to the proof of Theorem 4.1, it is sufficient to note that, for Morely's element $w_h \in V^2_h \not\subset H^1(\Omega)$, by Green's formula, we have

$$\begin{aligned}\Delta &= \sum_T \int_T M_{\alpha\beta}(u) K_{\alpha\beta}(w_h)\,dx - \int_\Omega f w_h\,dx \\ &= -\sum_T \int_T M_{\alpha\beta}(u)\,\partial_{\alpha\beta} w_h\,dx - \int_\Omega f w_h\,dx \\ &= \sum_T \int_T \partial_\alpha M_{\alpha\beta}(u)\,\partial_\beta w_h\,dx - \sum_T \int_{\partial T} M_{\alpha\beta}(u) n_\alpha\,\partial_\beta w_h\,ds - \int_\Omega f w_h\,dx\,. \end{aligned} \tag{4.17}$$

Let v^I denote the piecewise interpolation of v, since $v^I \in H^1(\Omega)$. Then

$$\begin{aligned}\Delta = &\sum_T \int_T \partial_\alpha M_{\alpha\beta}(u)\,\partial_\beta w_h^I\,dx - \int_\Omega f w_h^I\,dx - \sum_T \int_{\partial T} M_{\alpha\beta}(u) n_\alpha\,\partial_\beta w_h\,ds \\ &+ \sum_T \int_T \partial_\alpha M_{\alpha\beta}(u)\,\partial_\beta (w_h - w_h^I)\,dx + \int_\Omega f(w_h^I - w_h)\,dx\,. \end{aligned} \tag{4.18}$$

Let

$$e_{1h} = \sum_T \int_T \partial_\alpha M_{\alpha\beta}(u)\,\partial_\beta (w_h - w_h^I)\,dx\,, \qquad e_{2h} = \int_\Omega f(w_h^I - w_h)\,dx\,,$$

by the interpolate error estimate,

$$|e_{1h}| \le ch|u|_{3,\Omega}\Big(\sum_T |w_h|_{2,T}^2\Big)^{1/2}, \qquad |e_{2h}| \le ch^2\|f\|_{0,\Omega}\Big(\sum_T |w_h|_{2,T}^2\Big)^{1/2}. \tag{4.19}$$

Since $w_h^I \in H^1(\Omega)$, and by Green's formula, then

$$\begin{aligned}\Delta_1 &= \sum_T \int_T \partial_\alpha M_{\alpha\beta}(u)\,\partial_\beta w_h^I\,dx - \int_\Omega f w_h^I\,dx - \sum_T \int_{\partial T} M_{\alpha\beta}(u) n_\alpha\,\partial_\beta w_h\,ds \\ &= \int_\Omega \partial_\alpha M_{\alpha\beta}(u)\,\partial_\beta w_h^I\,dx - \int_\Omega f w_h^I\,dx - \sum_T \int_{\partial T} M_{\alpha\beta}(u) n_\alpha\,\partial_\beta w_h\,ds \\ &= -\Big\{\int_\Omega \partial_{\alpha\beta} M_{\alpha\beta}(u) w_h^I\,dx + \int_\Omega f w_h^I\,dx\Big\} + \int_{\partial\Omega} \partial_\alpha M_{\alpha\beta}(u) n_\beta w_h^I\,ds \\ &\quad - \sum_T \int_{\partial T} M_{\alpha\beta}(u) n_\alpha (n_\beta\,\partial_n w_h + s_\beta\,\partial_s w_h)\,ds \\ &= -\int_\Omega (\partial_{\alpha\beta} M_{\alpha\beta}(u) + f) w_h^{I}\,dx + \int_{\partial\Omega} \partial_\alpha M_{\alpha\beta}(u) n_\beta w_h^I\,ds \\ &\quad - \sum_T \int_{\partial T} M_{\alpha\beta}(u) n_\alpha n_\beta\,\partial_n w_h\,ds - \sum_T \int_{\partial T} M_{\alpha\beta}(u) n_\alpha s_\beta\,\partial_s w_h^I\,ds \\ &\quad + \sum_T \int_{\partial T} M_{\alpha\beta}(u) n_\alpha s_\beta\,\partial_s (w_h^I - w_h)\,ds \\ &= -\int_\Omega (\partial_{\alpha\beta} M_{\alpha\beta}(u) + f) w_h^I\,dx + \int_{\partial\Omega} \partial_\alpha M_{\alpha\beta}(u) n_\beta w_h^I\,ds \\ &\quad - \sum_T \int_{\partial T} M_{\alpha\beta}(u) n_\alpha n_\beta\,\partial_n w_h\,ds + \int_{\partial\Omega} \partial_s M_{\alpha\beta}(u) n_\alpha s_\beta w_h^I\,ds \\ &\quad + \sum_T \int_{\partial T} M_{\alpha\beta}(u) n_\alpha s_\beta\,\partial_s (w_h^I - w_h)\,ds\,. \end{aligned} \tag{4.20}$$

We now estimate the last term on the right-hand side of (4.20):

$$e_{3h} = \sum_{\tau} \int_{\partial\tau} M_{\alpha\beta}(u) n_\alpha s_\beta \, \partial_s (w_h^I - w_h) \, \mathrm{d}s \, .$$

Let

$$P_0^F w = \frac{1}{|F|} \int_F w \, \mathrm{d}s \, , \qquad |F| = \int_F 1 \, \mathrm{d}s \, .$$

Since $w_h^I = w_h$ at all nodes of $\mathcal{T}_h$, then we have

$$e_{3h} = \sum_{\tau} \sum_{F \in \partial\tau} \int_F (M_{\alpha\beta}(u) - P_0^F[M_{\alpha\beta}(u)]) n_\alpha s_\beta \, \partial_s (w_h^I - w_h) \, \mathrm{d}s \, ,$$

from which by the interpolate error estimates (cf. [7]),

$$|e_{3h}| \leq ch|u|_{3,\Omega} \Big(\sum_{\tau} |w_h|_{2,\tau}^2 \Big)^{1/2} \, . \tag{4.21}$$

Thus by the second equation of (3.13),

$$\begin{aligned}
E_h(\mathcal{U}, \mathcal{W}_h) = & - \int_\Gamma Q_{11}(u^1) w_{1h}^1 \, \mathrm{d}x_2 - \int_\Gamma Q_{21}(u^1) w_{2h}^1 \, \mathrm{d}x_2 \\
& - \int_\Gamma Q_{23}(u^2) w_{2h}^2 \, \mathrm{d}x_2 - \int_\Gamma Q_{33}(u^2) w_{3h}^2 \, \mathrm{d}x_2 \\
& + \int_\Gamma \{ \partial_\alpha M_{\alpha\beta}(u^1) n_\beta + \partial_s M_{\alpha\beta}(u^1) n_\alpha s_\beta \} (w_{3h}^1)' \, \mathrm{d}x_2 \\
& + \int_\Gamma \{ \partial_{\alpha'} M_{\alpha'\beta'}(u^2) n_{\beta'} + \partial_{s'} M_{\alpha'\beta'}(u^2) n_{\alpha'} s_{\beta'} \} (w_{1h}^2)' \, \mathrm{d}x_2 \\
& - \sum_{F_1 \subset \Gamma} \int_{F_1} M_{\alpha\beta}(u^1) n_\alpha n_\beta \, \partial_n w_{3h}^1 \, \mathrm{d}s \\
& - \sum_{F_2 \subset \Gamma} \int_{F_2} M_{\alpha'\beta'}(u^2) n_{\alpha'} n_{\beta'} \, \partial_{n'} w_{1h}^2 \, \mathrm{d}s - I_h^1 - I_h^2 + \sum_{i=1}^{3} (e_{ih}^1 + e_{ih}^2) \\
= & - \int_\Gamma Q_{11}(u^1)(w_{1h}^1 - (w_{1h}^2)' \, \mathrm{d}x_2) - \int_\Gamma Q_{33}(u^2)(w_{3h}^2 - (w_{3h}^1)') \, \mathrm{d}x_2 \\
& - \int_\Gamma Q_{11}(u^1)(w_{1h}^2)' \, \mathrm{d}x_2 - \int_\Gamma \{ \partial_{\alpha'} M_{\alpha'3}(u^2) + \partial_2 M_{32}(u^2) \} (w_{1h}^2)' \, \mathrm{d}x_2 \\
& - \int_\Gamma Q_{33}(u^2)(w_{3h}^1)' \, \mathrm{d}x_2 - \int_\Gamma \{ \partial_\alpha M_{\alpha 1}(u^1) + \partial_2 M_{12}(u^1) \} (w_{3h}^1)' \, \mathrm{d}x_2 \\
& + \sum_{F \subset \Gamma} \int_F \{ M_{11}(u^1) \, \partial_1 w_{3h}^1 + M_{33}(u^2) \, \partial_3 w_{1h}^2 \} \, \mathrm{d}x_2 \\
& - I_h^1 - I_h^2 + \sum_{i=1}^{3} (e_{ih}^1 + e_{ih}^2) \, ,
\end{aligned} \tag{4.22}$$

where

$$I_h^1 = \sum_{\tau_1} \sum_{\substack{F_1 \in \partial\tau_1 \\ F_1 \not\subset \partial\Omega_1}} \int_{F_1} M_{\alpha\beta}(u^1) n_\alpha n_\beta \, \partial_n w_{3h}^1 \, ds ,$$
$$I_h^2 = \sum_{\tau_2} \sum_{\substack{F_2 \in \partial\tau_2 \\ F_2 \not\subset \partial\Omega_2}} \int_{F_2} M_{\alpha'\beta'}(u^2) n_{\alpha'} n_{\beta'} \, \partial_{n'} w_{1h}^2 \, ds . \tag{4.23}$$

By the first and third equations of (3.13), and taking into account that $w_{1h}^1 = (w_{1h}^2)'$, $w_{3h}^2 = (w_{3h}^1)'$ on Γ, we have

$$E_h(\mathcal{U}, \mathcal{W}_h) = \sum_{F \subset \Gamma} \int_F \{M_{11}(u^1)\, \partial_1 w_{3h}^1 + M_{33}(u^2)\, \partial_3 w_{1h}^2\} \, dx_2$$
$$- I_h^1 - I_h^2 + \sum_{i=1}^{3} (e_{ih}^1 + e_{ih}^2) . \tag{4.24}$$

Let $R_0^F w = w - P_0^F w$. By the last equation of (3.13) we have

$$\Big| \sum_{F \subset \Gamma} \int_F \{M_{11}(u^1)\, \partial_1 w_{3h}^1 + M_{33}(u^2)\, \partial_3 w_{1h}^2\} \, dx_2 \Big|$$
$$= \Big| \sum_{F \subset \Gamma} \int_F R_0^F[M_{11}(u^1)] R_0^F[\partial_1 w_{3h}^1] \, dx_2 + \sum_{F \subset \Gamma} \int_F R_0^F[M_{33}(u^2)] R_0^F[\partial_3 w_{1h}^2] \, dx_2 \Big|$$
$$\leq ch(\|u_3^1\|_{3,\Omega_1} + \|u_1^2\|_{3,\Omega_2}) \Big(\sum_{\tau_1} |w_{3h}^1|_{2,\tau_1}^2 + \sum_{\tau_2} |w_{1h}^2|_{2,\tau_2}^2 \Big) , \tag{4.25}$$

where we have used the interpolate error estimates (cf. [7]) in the last inequality. By the error estimate for Morley's element (cf. [7]) it can be seen that

$$I_h^1 \leq ch \|u_3^1\|_{3,\Omega_1} \Big(\sum_{\tau_1} |w_{3h}^1|_{2,\tau_1}^2 \Big)^{1/2} ,$$
$$I_h^2 \leq ch \|u_1^2\|_{3,\Omega_2} \Big(\sum_{\tau_2} |w_{1h}^2|_{2,\tau_2}^2 \Big)^{1/2} , \tag{4.26}$$

Finally we have

$$|E_h(\mathcal{U}, \mathcal{W}_h)| \leq ch\{\|u_3^1\|_{3,\Omega_1} + \|u_1^2\|_{3,\Omega_2} + h(\|f_3^1\|_{0,\Omega_1}) + \|f_1^2\|_{0,\Omega_2})\} \|\mathcal{W}_h\|_h , \tag{4.27}$$

and the proof is completed $\square$.

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