SIAM J. NUMER. ANAL.
Vol. 39, No. 2, pp. 363–384

ON THE NECESSITY AND SUFFICIENCY OF THE PATCH TEST FOR CONVERGENCE OF NONCONFORMING FINITE ELEMENTS*

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Abstract. This paper proposes a weak patch test that improves both the patch test and the generalized patch test. Under a weak superapproximation assumption, which can be obtained from certain approximation and weak continuity property, it is shown that the weak patch test is equivalent to the generalized patch test. Thus if a nonconforming finite element passes the patch test and satisfies some approximation and weak continuity property, its convergence is then guaranteed. Furthermore, the consistency term of such a nonconforming element can be proved to be of order $O(h)$. It is shown that if a nonconforming element does not pass the patch test and a condition relative to subdividing a domain by the element is true, then one can find a family of triangulations for which the element is divergent.

Key words. patch test, nonconforming finite element, superapproximation

AMS subject classification. 65N30

PII. S003614299936473X

1. Introduction.

This paper discusses the necessity and sufficiency of the patch test for the convergence of nonconforming finite elements. The patch test was first proposed by Irons [1] in 1965 from a physical perspective. Roughly speaking, in terms of mechanics, an element passes the patch test means that it gives exact solution for constant strain. Numerical experiments have demonstrated that the patch test is effective for many applications. The patch test is very simple, it is a local test, and it can be verified by numerical computation. It is popular among engineers, to the point that some believe that it is a necessary and sufficient condition for the convergence of nonconforming elements (see [5, 6, 9, 10, 24]).

The mathematical results about the patch test, however, did not quite meet engineers' expectations. A mathematical description of the patch test was given by Strang [19] in 1972. In 1980, Stummel [22] gave two examples that pass the patch test but are not convergent, showing that the patch test is not sufficient. Shi [15] demonstrated that an element on some very special mesh can be convergent even though it fails to pass the patch test, establishing that the patch test is not necessary in general.

A generalized patch test was proposed by Stummel [21]. It is both a necessary and sufficient condition provided certain approximability property holds. However, the proposed test is not a local one and it is often impossible to verify practically. Some easier and sufficient conditions were suggested, for instance, the F-E-M test [18] and the IPT test [30].

Besides Irons' patch test, some other versions were also suggested by engineers [2, 10, 24]. All these can lead to Irons' patch test (see [26]).

It is worth pointing out that Stummel's examples were obtained in a nonstandard fashion and they have no weak continuity which usual elements possess. It is a con-

*Received by the editors November 21, 1999; accepted for publication (in revised form) December 12, 2000; published electronically April 3, 2001. This work was supported by the China State Major Key Project for Basic Researches and by the Center for Computational Mathematics and Applications, Pennsylvania State University.

http://www.siam.org/journals/sinum/39-2/36473.html

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jecture that under the approximation, the weak continuity and the patch test lead to the convergence of nonconforming elements (see [4]). In this paper, we will prove the conjecture.

We propose a new patch test—the weak patch test that refines both the patch test and the generalized patch test. It is equivalent to the generalized patch test under the so-called weak superapproximation assumption condition. Thus the patch test together with the weak superapproximation is sufficient for the convergence of nonconforming elements.

The weak superapproximation is a weak form of the superapproximation. The superapproximation of finite elements for second order problems can be found in [8, 11, 12, 25]. Similar properties of finite elements for plate bending problems have also been used in [27, 28]. We will show that the superapproximation is true if some approximation and weak continuity hold. Therefore, the patch test is sufficient for the convergence of nonconforming elements provided some approximation and weak continuity are satisfied.

We also consider the necessity of the patch test. We show that if a nonconforming element fails to pass the patch test and a condition relative to subdividing a domain by the element is true, then there is a family of triangulations for which the element is divergent. That is, the patch test is necessary in the sense of arbitrary family of triangulations.

The discussions in this paper are restricted to the cases that the element shapes are n-simplex or n-cube. Further analysis is needed for the other elements discussed in [14, 15] such as quadrilateral nonconforming elements.

The rest of the paper is organized as follows. Section 2 gives some basic descriptions of nonconforming elements and related assumptions. Section 3 discusses the superapproximation property. Section 4 describes the patch test, the weak patch test, and the generalized patch test and establishes the sufficiency of the patch test. Under the assumptions of the approximation, the weak continuity and the patch test, section 5 shows that the consistency errors of nonconforming elements are of order $O(h)$. Section 6 discusses the necessity of the patch test. Stummel's examples are discussed in section 7. The last section is devoted to the convergence of the Morley element for second order problems.

2. Nonconforming elements. Let Ω be a bounded, connected domain in R^n with Lipschitz continuous boundary $\partial\Omega$. For nonnegative integer s, let $H^s(\Omega)$, $\|\cdot\|_{s,\Omega}$, and $|\cdot|_{s,\Omega}$ denote the usual Sobolev space, norm, and seminorm, respectively, $H_0^s(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ with respect to the norm $\|\cdot\|_{s,\Omega}$, and $(\cdot,\cdot)$ the inner product of $L^2(\Omega)$.

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, set $|\alpha| = \sum_{i=1}^n \alpha_i$ and

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Especially, denote by e_i the multi-index with the ith component 1 and the others 0.

For $f \in L^2(\Omega)$ and m a positive integer, we consider the following boundary value problem of order $2m$:

$$\begin{cases} (-1)^m \displaystyle\sum_{|\alpha|=m} \partial^\alpha(a_\alpha \partial^\alpha u) + a_0 u = f \qquad \text{in } \Omega, \\ u|_{\partial\Omega} = \cdots = \dfrac{\partial^{m-1}}{\partial N^{m-1}} u|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where a_α are positive constants, a_0 is a nonnegative constant, and N is the unit outer normal to $\partial\Omega$.

Define the bilinear functional $a(\cdot,\cdot)$ on $H^m(\Omega)$ by

$$a(v,w) = \int_\Omega \left(\sum_{|\alpha|=m} a_\alpha \partial^\alpha v \, \partial^\alpha w + a_0 vw \right) dx \qquad \forall v,w \in H^m(\Omega) \tag{2.2}$$

and consider the following variational problem: find $u \in H_0^m(\Omega)$ such that

$$a(u,v) = (f,v) \qquad \forall v \in H_0^m(\Omega). \tag{2.3}$$

Problem (2.3) is the weak form of problem (2.1).

Let (T, P_T, Φ_T) be a finite element where T is the geometric shape, P_T the shape function space, and Φ_T the set of degrees of freedom and let Φ_T be P_T-unisolvent (see [3, p. 78]).

Let $\{\mathcal{T}_h\}$ be a family of shape regular triangulations with mesh size $h \to 0$. For each $\mathcal{T}_h$, let V_h and V_{h0} be the corresponding finite element spaces with respect to $H^m(\Omega)$ and $H_0^m(\Omega)$, respectively. This defines two families of finite element spaces $\{V_h\}$ and $\{V_{h0}\}$. In the case of a nonconforming element, $V_h \not\subset H^m(\Omega)$ and $V_{h0} \not\subset H_0^m(\Omega)$.

For $v,w \in H^m(\Omega) + V_h$, we define

$$a_h(v,w) = \sum_{T\in\mathcal{T}_h} \int_T \left(\sum_{|\alpha|=m} a_\alpha \partial^\alpha v \, \partial^\alpha w + a_0 vw \right) dx. \tag{2.4}$$

The nonconforming finite element method for problem (2.3) corresponding to the element (T, P_T, Φ_T) is as follows: find $u_h \in V_{h0}$ such that

$$a_h(u_h, v_h) = (f, v_h) \qquad \forall v_h \in V_{h0}. \tag{2.5}$$

We introduce the following mesh dependent norm $\|\cdot\|_{m,h}$ and seminorm $|\cdot|_{m,h}$:

$$\begin{cases} \|v\|_{m,h} = \left(\sum_{T\in\mathcal{T}_h} \|v\|_{m,T}^2 \right)^{1/2} \\ |v|_{m,h} = \left(\sum_{T\in\mathcal{T}_h} |v|_{m,T}^2 \right)^{1/2} \end{cases} \qquad \forall v \in V_h + H^m(\Omega). \tag{2.6}$$

For convenience, following [29], the symbols $\lesssim$, $\gtrsim$, and $\eqsim$ will be used in this paper. That $X_1 \lesssim Y_1$ and $X_2 \gtrsim Y_2$ mean that $X_1 \le c_1 Y_1$ and $c_2 X_2 \ge Y_2$ for some positive constants c_1 and c_2 that are independent of mesh size h. That $X_3 \eqsim Y_3$ means that $X_3 \lesssim Y_3$ and $X_3 \gtrsim Y_3$.

$a_h(\cdot,\cdot)$ is said to be uniformly V_{h0}-elliptic if

$$\|v_h\|_{m,h}^2 \lesssim a_h(v_h, v_h) \quad \forall v_h \in V_{h0}. \tag{2.7}$$

For a nonconforming element, the following basic error estimates hold (see [19] or [20]).

THEOREM 2.1. *Let $a_h(\cdot,\cdot)$ be uniformly V_{h0}-elliptic and let $u \in H_0^m(\Omega)$ and $u_h \in V_{h0}$ be the solutions of problems* (2.3) *and* (2.5)*, respectively. Then*

$$\|u - u_h\|_{m,h} \eqsim \inf_{v_h \in V_{h0}} \|u - v_h\|_{m,h} + \sup_{0 \neq v_h \in V_{h0}} \frac{|a_h(u, v_h) - (f, v_h)|}{\|v_h\|_{m,h}} \tag{2.8}$$

holds for all h.

One direction of the estimate in (2.8) with " $\lesssim$ " in place of " $\eqsim$ " can be found in [19] or [20, p. 178] while the other direction is easy to prove. The first term on the right of (2.8) is the approximation term and the second one is the consistency term caused by the nonconforming property.

$\{V_{h0}, H_0^m(\Omega)\}$ is said to have the approximability if for all $v \in H_0^m(\Omega)$

$$\lim_{h \to 0} \inf_{v_h \in V_{h0}} \|v - v_h\|_{m,h} = 0. \tag{2.9}$$

For a given set $B \subset R^n$ and an integer s, we denote by $P_s(B)$ the space of all polynomials on B with degree not greater than s, by $Q_s(B)$ the space of all polynomials on B with the degree of each variable not greater than s. Set $\mathcal{T}_h(B) = \{\, T \in \mathcal{T}_h \,:\, B \cap T \neq \emptyset \,\}$ and $N_h(B)$ the number of the elements in $\mathcal{T}_h(B)$.

Let $\partial\mathcal{T}_h$ be the set of all $n-1$ dimensional faces of $\mathcal{T}_h$ and $\partial\mathcal{T}_h(T) = \{\, F \subset \partial\mathcal{T}_h \,:\, F \subset T\}$. Define

$$\partial\mathcal{T}_h^b = \{\, F \in \partial\mathcal{T}_h \,:\, F \subset \partial\Omega\}, \qquad \partial\mathcal{T}_h^b(T) = \{\, F \in \partial\mathcal{T}_h(T) \,:\, F \subset \partial\Omega\}.$$

Throughout the paper, we assume that T is either an n-simplex or an n-cube. For each element T, let h_T be the diameter of the smallest ball containing T.

For each element $T \in \mathcal{T}_h$, we denote Π_T the interpolation operator of (T, P_T, Φ_T) and define Π_h by $(\Pi_h v)|_T = \Pi_T(v|_T)$, where $T \in \mathcal{T}_h$ and v is piecewise smooth.

Now we list some assumptions for the usual finite elements. Our first assumption is about the finite element (T, P_T, Φ_T). We assume that the degrees of freedom of finite elements are either the values of function or its high order directive derivatives at some points, or their mean values on the element or on the faces of the element. It is known that the affine invariant requirements of affine family do not hold when an element uses the normal derivatives on faces as degree of freedom (see [3, p. 335]). Therefore we assume that the degrees of freedom are continuous with respect to the vertices of the element and certain invariant properties are true in the case of scale and translation transformations (see section 6.2 in [31]). The precise description is the following A1.

A1. There exist nonnegative integers k_1, k_2, $k(i)$, $1 \le i \le L$, and a set $\mathcal{A}$ consisting of L multi-indices such that the following three statements are true:

(1) Let x^0 be the barycenter of T; then $P_T = \text{span}\,\{\, (x_1 - x_1^0)^{\alpha_1} \cdots (x_n - x_n^0)^{\alpha_n} \,:\, \alpha \in \mathcal{A}\}$.

(2) Denote all degrees of freedom in Φ_T by $\varphi_{i,T}$, $1 \le i \le L$, then for all $v \in C^\infty(T)$,

$$\varphi_{i,T}(v) = \begin{cases} \dfrac{\partial^{k(i)} v}{\partial l_{i,T,1} \cdots \partial l_{i,T,k(i)}}(a_{i,T}), & 1 \le i \le k_1, \\[2ex] \dfrac{1}{|F_{i,T}|} \displaystyle\int_{F_{i,T}} \dfrac{\partial^{k(i)} v}{\partial l_{i,T,1} \cdots \partial l_{i,T,k(i)}} ds, & k_1 < i \le k_2, \\[2ex] \dfrac{1}{|T|} \displaystyle\int_T \dfrac{\partial^{k(i)} v}{\partial l_{i,T,1} \cdots \partial l_{i,T,k(i)}} dx, & k_2 < i \le L, \end{cases} \tag{2.10}$$

where $a_{i,T}$ are points in T, $F_{i,T}$ are $n-1$ dimensional faces of T, $|T|$ and $|F_{i,T}|$ are the measures of T and $F_{i,T}$, respectively, and $l_{i,T,j} \in R^n$, $1 \le j \le k(i)$, are unit vectors that vary continuously with respect to the vertices of T.

(3) For two element T and T', if $T' = \{x' : x' = \theta x + b \text{ for all } x \in T\}$ with θ a positive number and $b \in R^n$, then $l_{i,T,j} = l_{i,T',j}$, $1 \le j \le k(i)$, $1 \le i \le L$.

A2. V_h consists of all functions $v_h \in L^2(\Omega)$ such that $v_h|_T \in P_T$ for all $T \in \mathcal{T}_h$, and for all $T \in \mathcal{T}_h$, the following two statements are true:

(1) If $1 \le i \le k_1$ and $a_{i,T} \in \partial T$, then $\dfrac{\partial^{k(i)} v}{\partial l_{i,T,1} \cdots \partial l_{i,T,k(i)}}$ is continuous at point $a_{i,T}$.

(2) If $k_1 < i \le k_2$ and $F_{i,T} = T \cap T'$ with $T' \in \mathcal{T}_h$ and $T' \ne T$, then

$$\int_{F_{i,T}} \frac{\partial^{k(i)} v_h^T}{\partial l_{i,T,1} \cdots \partial l_{i,T,k(i)}} ds = \int_{F_{i,T}} \frac{\partial^{k(i)} v_h^{T'}}{\partial l_{i,T,1} \cdots \partial l_{i,T,k(i)}} ds,$$

where v_h^T and $v_h^{T'}$ are the restrictions of v_h on T and T', respectively.

A2$'$. V_{h0} consists of all functions $v_h \in V_h$ such that for all $T \in \mathcal{T}_h$ and $i \in \{1, \ldots, k_2\}$, if $a_{i,T} \in \partial\Omega$ (or $F_{i,T} \subset \partial\Omega$) and $\varphi_{i,T}(\psi) = 0$ for all $\psi \in H_0^m(\Omega) \cap C^\infty(\bar{\Omega})$, then $\varphi_{i,T}(v_h) = 0$.

A3. *Weak continuity.* For all $F \in \partial\mathcal{T}_h$ there exist pairwise orthogonal unit vectors $\nu_1, \ldots, \nu_n \in R^n$, such that for all $v_h \in V_h$ and α with $|\alpha| < m$, $\frac{\partial^{|\alpha|} v_h}{\partial \nu_1^{\alpha_1} \cdots \partial \nu_n^{\alpha_n}}$ is continuous at least at one point on F.

A3$'$. For all $F \in \partial\mathcal{T}_h^b$ there exist pairwise orthogonal unit vectors $\nu_1, \ldots, \nu_n \in R^n$, such that for all $v_h \in V_{h0}$ and α with $|\alpha| < m$, $\frac{\partial^{|\alpha|} v_h}{\partial \nu_1^{\alpha_1} \cdots \partial \nu_n^{\alpha_n}}$ vanishes at least at one point on F.

A4. *Approximation.* There exists an integer $s > 0$ such that for all h

$$\sum_{j=0}^{m} h_T^j |v - \Pi_T v|_{j,T} \lesssim h_T^{m+s} |v|_{m+s,T} \quad \forall v \in H^{m+s}(T)\ \forall T \in \mathcal{T}_h. \tag{2.11}$$

In the above, A1 is the assumption about the shape function space and the degrees of freedom of finite elements. A2 and A2$'$ are concerned with the finite element spaces. A4 is the requirement of the interpolation operator and leads to the approximability of $\{V_{h0}, H_0^m(\Omega)\}$. They are basic in the finite element methods.

In A3 and A3$'$, $\nu_1, \ldots, \nu_n$ may be the vectors of coordinates or the normal and tangent vectors. For a conforming element, $V_h \subset H^m(\Omega)$ and $V_{h0} \subset H_0^m(\Omega)$, usually, $V_h \subset C^{m-1}(\bar{\Omega})$ and $V_{h0} = \{ v_h \in V_h : \partial^\alpha v_h|_{\partial\Omega} = 0,\ |\alpha| < m\}$. Therefore, A3 and A3$'$ are trivially valid. In the case of usual nonconforming elements, although the C^{m-1} continuity fails, all practically useful nonconforming elements do satisfy A3 and A3$'$.

The Crouzeit–Raviart element and the Wilson element for second order problems, and the Morley element, the Adini element, and the Zienkiewicz element for fourth order problems are well-known nonconforming elements. They all satisfy assumptions A1, A2, A2$'$, A3, A3$'$, and A4.

3. Superapproximation. In this section, we consider the superapproximation property of finite element spaces which will play an important rule in our discussion of the sufficiency of the patch test. We will show the following theorem.

THEOREM 3.1. *Let* A1, A2, A2′, A3, *and* A4 *hold. Assume that* $r \geq \frac{n}{2} + \max\{m + s, k(1), \ldots, k(L)\}$. *Then*

$$\inf_{w_h \in V_{h0}} \sum_{j=0}^{m} \sum_{T \in \mathcal{T}_h} h_T^{2(j-m)} \, |\psi v_h - w_h|_{j,T}^2 \lesssim h^2 \|v_h\|_{m,h}^2 \|\psi\|_{r,\Omega}^2, \quad (v_h, \psi) \in V_h \times H_0^r(\Omega). \tag{3.1}$$

In addition, if A3′ *is true, then*

$$\inf_{w_h \in V_{h0}} \sum_{j=0}^{m} \sum_{T \in \mathcal{T}_h} h_T^{2(j-m)} \, |\psi v_h - w_h|_{j,T}^2 \lesssim h^2 \|v_h\|_{m,h}^2 \|\psi\|_{r,\Omega}^2, \quad (v_h, \psi) \in V_{h0} \times H^r(\Omega). \tag{3.2}$$

We begin with the definition of the superapproximation. $\{V_{h0}\}$ is said to have a superapproximation property if there exists an integer $r \geq 0$ such that

$$\inf_{w_h \in V_{h0}} \|\varphi v_h - w_h\|_{m,h} \lesssim h \|v_h\|_{m,h} \|\varphi\|_{r,\Omega} \quad \forall (v_h, \varphi) \in V_{h0} \times H^r(\Omega). \tag{3.3}$$

$\{V_{h0}\}$ is said to have a weak superapproximation property if

$$\lim_{h \to 0} \sup_{\substack{v_h \in V_{h0} \\ \|v_h\|_{m,h} \leq 1}} \inf_{w_h \in V_{h0}} \|\varphi v_h - w_h\|_{m,h} = 0 \quad \forall \varphi \in C^\infty(\bar{\Omega}). \tag{3.4}$$

Obviously, the superapproximation implies the weak superapproximation.

Inequality (3.3) holds when inequality (3.2) is true. Therefore the superapproximation holds when A1, A2, A2′, A3, A3′, and A4 are true.

To show Theorem 3.1, we need some lemmas. First, by the affine argument (see [3, pp. 123–124] or [31, pp. 284–285]) we have the following lemma.

LEMMA 3.2. *If* A1 *holds, then*

$$\|p\|_{0,T}^2 \eqsim h_T^n \sum_{i=1}^{L} h_T^{2k(i)} |\varphi_{i,T}(p)|^2 \qquad \forall p \in P_T. \tag{3.5}$$

LEMMA 3.3. *Let* A1 *and* A3 *hold. If* $|\alpha| < m$, *then for* $x \in \bar{\Omega}$ *and* $v_h \in V_h$,

$$|\partial^\alpha v_h^{T'}(x) - \partial^\alpha v_h^{T''}(x)|^2 \lesssim \sum_{T \in \mathcal{T}_h(x)} h_T^{2(m-|\alpha|)-n} |v_h|_{m,T}^2 \quad \forall T', T'' \in \mathcal{T}_h(x). \tag{3.6}$$

Proof. Let $v_h \in V_h$ and $x \in \bar{\Omega}$. If x is in the interior of an element $T \in \mathcal{T}_h$, then $\mathcal{T}_h(x)$ contains only one element and (3.6) is obvious. Now let $x \in F$ with $F \in \partial\mathcal{T}_h$. We need only to consider the case that x is contained in more than one element. If $T', T'' \in \mathcal{T}_h(x)$ and $T' \neq T''$ there exist $T_1, \ldots, T_J \in \mathcal{T}_h(x)$ such that $T_1 = T'$, $T_J = T''$, and $T_j \cap T_{j+1} \in \partial\mathcal{T}_h$, $1 \leq j < J$.

Set $F_j = T_j \cap T_{j+1}$. Let $\nu_1, \ldots, \nu_n$ be the vectors described in A3 for F_j. For multi-index β define

$$\partial_\nu^\beta = \frac{\partial^{|\beta|}}{\partial \nu_1^{\beta_1} \cdots \partial \nu_n^{\beta_n}}.$$

If $0 \le |\alpha| < m$, then there exists $x' \in F_j$ from A3 such that $\partial_\nu^\alpha v_h^{T_j}(x') - \partial_\nu^\alpha v_h^{T_{j+1}}(x') = 0$; this leads to

$$\max_{y\in F_j}\left|\partial_\nu^\alpha v_h^{T_j}(y) - \partial_\nu^\alpha v_h^{T_{j+1}}(y)\right| \le h_{T_j}\max_{y\in F_j}\left|\frac{\partial}{\partial\tau}\partial_\nu^\alpha v_h^{T_j}(y) - \frac{\partial}{\partial\tau}\partial_\nu^\alpha v_h^{T_{j+1}}(y)\right|$$
$$\lesssim h_{T_j}\sum_{|\beta|=|\alpha|+1}\max_{y\in F_j}\left|\partial_\nu^\beta v_h^{T_j}(y) - \partial_\nu^\beta v_h^{T_{j+1}}(y)\right|,$$

where τ is a unit tangent of F_j. Using the same argument, we have

$$\max_{y\in F_j}\left|\partial_\nu^\alpha v_h^{T_j}(y) - \partial_\nu^\alpha v_h^{T_{j+1}}(y)\right| \lesssim h_T^{m-|\alpha|}\sum_{|\beta|=m}\max_{y\in F_j}\left|\partial_\nu^\beta v_h^{T_j}(y) - \partial_\nu^\beta v_h^{T_{j+1}}(y)\right|.$$

Since $\nu_1, \ldots, \nu_n$ are unit vectors and pairwise orthogonal,

$$\max_{y\in F_j}\left|\partial^\alpha v_h^{T_j}(y) - \partial^\alpha v_h^{T_{j+1}}(y)\right| \lesssim h_T^{m-|\alpha|}\sum_{|\beta|=m}\max_{y\in F_j}\left|\partial^\beta v_h^{T_j}(y) - \partial^\beta v_h^{T_{j+1}}(y)\right|.$$

By the inverse inequality, we get

$$\left|\partial^\alpha v_h^{T_j}(x) - \partial^\alpha v_h^{T_{j+1}}(x)\right|^2 \lesssim \left(h_{T_j}^{2(m-|\alpha|)-n} + h_{T_{j+1}}^{2(m-|\alpha|)-n}\right)\left(|v_h|_{m,T_j}^2 + |v_h|_{m,T_{j+1}}^2\right).$$

Since J is bounded and

$$|\partial^\alpha v_h^{T'}(x) - \partial^\alpha v_h^{T''}(x)|^2 \lesssim \sum_{j=1}^{J-1}|\partial^\alpha v_h^{T_j}(x) - \partial^\alpha v_h^{T_{j+1}}(x)|^2,$$

inequality (3.6) follows. ∎

LEMMA 3.4. *Let* A1, A3, *and* A3′ *hold. If* $|\alpha| < m$, *then for* $x \in \partial\Omega$ *and* $v_h \in V_{h0}$,

$$|\partial^\alpha v_h^{T'}(x)|^2 \lesssim \sum_{T\in\mathcal{T}_h(x)} h_T^{2(m-|\alpha|)-n}|v_h|_{m,T}^2 \quad \forall T' \in \mathcal{T}_h(x). \tag{3.7}$$

Proof. Let $v_h \in V_{h0}$, $x \in \partial\Omega$, and $T' \in \mathcal{T}_h(x)$. There must be an $T^b \in \mathcal{T}_h(x)$ and a face $F \in \partial\mathcal{T}_h^b(T^b)$. Let $\nu_1, \ldots, \nu_n$ be the vectors described in A3′ for F. If $0 \le |\alpha| < m$, then there exists $x' \in F$ such that $\partial_\nu^\alpha v_h^{T^b}(x') = 0$. By similar arguments used in the proof of Lemma 3.3, we get

$$|\partial^\alpha v_h^{T^b}(x)|^2 \lesssim h_{T^b}^{2(m-|\alpha|)-n}|v_h|_{m,T^b}^2.$$

Evidently,

$$|\partial^\alpha v_h^{T'}(x)|^2 \lesssim |\partial^\alpha v_h^{T^b}(x)|^2 + |\partial^\alpha v_h^{T'}(x) - \partial^\alpha v_h^{T^b}(x)|^2.$$

The lemma follows from Lemma 3.3. ∎

For $v_h \in V_h$ and $|\alpha| < m$, $\partial^\alpha v_h$ may not be continuous on whole Ω due to the nonconforming property. The discontinuity may appear at the points on the element boundaries. The weak continuity guarantees that the jumps at the discontinuous points are bounded by a certain order of h described in (3.6). It is possible to find a continuous finite element function with a suitable error to $\partial^\alpha v_h$. This can lead us to getting the superapproximation property.

Now let l be a nonnegative integer. For $T \in \mathcal{T}_h$, if T is an n-simplex, then we take $S_{l,T} = P_l(T)$ and $\Lambda_{l,T}$, the interpolating operator corresponding to the element of n-simplex of type (l); otherwise take $S_{l,T} = Q_l(T)$ and $\Lambda_{l,T}$, the interpolating operator corresponding to the element of n-cube of type (l) (see [3, pp. 48, 57]). Set $\Xi_{l,T}$ the set of nodal points of $\Lambda_{l,T}$.

For $|\alpha| < m$, define linear operators $\Lambda_h^\alpha : V_h \to H^1(\Omega)$ and $\Lambda_{h0}^\alpha : V_h \to H_0^1(\Omega)$ as follows. For $v_h \in V_h$ and $T \in \mathcal{T}_h$, $\Lambda_h^\alpha v_h|_T$ and $\Lambda_{h0}^\alpha v_h|_T$ are all in $S_{m-|\alpha|,T}$ and

$$\Lambda_h^\alpha v_h(x) = \frac{1}{N_h(x)} \sum_{T' \in \mathcal{T}_h(x)} \partial^\alpha v_h^{T'}(x), \quad x \in \Xi_{m-|\alpha|,T}, \tag{3.8}$$

$$\Lambda_{h0}^\alpha v_h(x) = \begin{cases} \dfrac{1}{N_h(x)} \displaystyle\sum_{T' \in \mathcal{T}_h(x)} \partial^\alpha v_h^{T'}(x), & x \in \Xi_{m-|\alpha|,T} \cap \Omega, \\ 0, & x \in \Xi_{m-|\alpha|,T} \cap \partial\Omega. \end{cases} \tag{3.9}$$

LEMMA 3.5. *Let* A1 *and* A3 *hold. If* $|\alpha| < m$ *and* $j \in \{0, 1, \ldots, m - |\alpha|\}$, *then*

$$\sum_{T \in \mathcal{T}_h} h_T^{2(|\alpha|+j-m)} |\partial^\alpha v_h - \Lambda_h^\alpha v_h|_{j,T}^2 \lesssim |v_h|_{m,h}^2 \quad \forall v_h \in V_h. \tag{3.10}$$

In addition, if A3$'$ *is also true, then for* $|\alpha| < m$ *and* $j \in \{0, 1, \ldots, m - |\alpha|\}$

$$\sum_{T \in \mathcal{T}_h} h_T^{2(|\alpha|+j-m)} |\partial^\alpha v_h - \Lambda_{h0}^\alpha v_h|_{j,T}^2 \lesssim |v_h|_{m,h}^2 \quad \forall v_h \in V_{h0}. \tag{3.11}$$

Proof. Take $|\alpha| < m$ and $j \in \{0, 1, \ldots, m - |\alpha|\}$. Set $l = m - |\alpha|$. First let $v_h \in V_h$ and $T \in \mathcal{T}_h$. Then

$$|\partial^\alpha v_h - \Lambda_h^\alpha v_h|_{j,T} \le |\partial^\alpha v_h - \Lambda_{l,T} \partial^\alpha v_h|_{j,T} + |\Lambda_{l,T} \partial^\alpha v_h - \Lambda_h^\alpha v_h|_{j,T}. \tag{3.12}$$

By the interpolating theory,

$$h_T^{|\alpha|+j-m} |\partial^\alpha v_h - \Lambda_{l,T} \partial_h^\alpha v_h|_{j,T} \lesssim |v_h|_{m,T}. \tag{3.13}$$

Using the affine argument, we can show the following inequality:

$$|p|_{j,T}^2 \lesssim h_T^{n-2j} \sum_{x \in \Xi_{l,T}} |p(x)|^2 \quad \forall p \in S_{l,T}. \tag{3.14}$$

Since $\Lambda_{l,T} \partial^\alpha v_h - \Lambda_h^\alpha v_h \in S_{l,T}$,

$$|\Lambda_{l,T} \partial^\alpha v_h - \Lambda_h^\alpha v_h|_{j,T}^2 \lesssim h_T^{n-2j} \sum_{x \in \Xi_{l,T}} |\Lambda_{l,T} \partial^\alpha v_h^T(x) - \Lambda_h^\alpha v_h(x)|^2. \tag{3.15}$$

For $x \in \Xi_{l,T}$, by (3.8) and Lemma 3.3 we have

$$\begin{aligned} |\Lambda_{l,T} \partial^\alpha v_h^T(x) - \Lambda_h^\alpha v_h(x)|^2 &= \left| \frac{1}{N_h(x)} \sum_{T' \in \mathcal{T}_h(x)} \left(\partial^\alpha v_h^T(x) - \partial^\alpha v_h^{T'}(x) \right) \right|^2 \\ &\lesssim \frac{1}{N_h(x)} \sum_{T' \in \mathcal{T}_h(x)} \left| \partial^\alpha v_h^T(x) - \partial^\alpha v_h^{T'}(x) \right|^2 \\ &\lesssim \sum_{T' \in \mathcal{T}_h(x)} h_{T'}^{2(m-|\alpha|)-n} |v_h|_{m,T'}^2. \end{aligned}$$

From (3.15) we obtain

$$h_T^{2(|\alpha|+j-m)}|\Lambda_{l,T}\partial^\alpha v_h - \Lambda_h^\alpha v_h|_{j,T}^2 \lesssim \sum_{T'\in\mathcal{T}_h(T)} |v_h|_{m,T'}^2. \tag{3.16}$$

Combining (3.12), (3.13), and (3.16), we get

$$\sum_{T\in\mathcal{T}_h} h_T^{2(|\alpha|+j-m)}|\partial^\alpha v_h - \Lambda_h^\alpha v_h|_{j,T}^2 \lesssim \sum_{T\in\mathcal{T}_h}\sum_{T'\in\mathcal{T}_h(T)} |v_h|_{m,T'}^2. \tag{3.17}$$

Then (3.10) follows from (3.17).

Now let $v_h \in V_{h0}$. For $x \in \Xi_{l,T}$, if $x \in \partial\Omega$, then from (3.9) and Lemma 3.4

$$|\partial^\alpha v_h^T(x) - \Lambda_{h0}^\alpha v_h(x)|^2 = |\partial^\alpha v_h^T(x)|^2 \lesssim \sum_{T'\in\mathcal{T}_h(x)} h_{T'}^{2(m-|\alpha|)-n}|v_h|_{m,T'}^2;$$

otherwise, by the same argument used above we can get

$$|\partial^\alpha v_h^T(x) - \Lambda_{h0}^\alpha v_h(x)|^2 \lesssim \sum_{T'\in\mathcal{T}_h(x)} h_{T'}^{2(m-|\alpha|)-n}|v_h|_{m,T'}^2.$$

These lead to

$$h_T^{2(|\alpha|+j-m)}|\Lambda_{l,T}\partial^\alpha v_h - \Lambda_{h0}^\alpha v_h|_{j,T}^2 \lesssim \sum_{T'\in\mathcal{T}_h(T)} |v_h|_{m,T'}^2. \tag{3.18}$$

Inequality (3.11) follows. $\square$

LEMMA 3.6. *Let* A4 *hold and* $r \geq m+s+\frac{n}{2}$. *Then for all* $T\in\mathcal{T}_h$,

$$\sum_{j=0}^{m} h_T^{j-m}|\psi p - \Pi_T(\psi p)|_{j,T} \lesssim h_T\|p\|_{m,T}\|\psi\|_{r,T} \ \forall p\in P_T \ \forall \psi\in H^r(T). \tag{3.19}$$

Proof. Take $p \in P_T$ and $\psi \in H^r(T)$. Let P_T^0 be the orthogonal projection operator from $L^2(T)$ to $P_0(T)$. Since $\Pi_T p = p$,

$$\psi p - \Pi_T(\psi p) = (\psi - P_T^0\psi)p - \Pi_T((\psi - P_T^0\psi)p).$$

From A4 and the interpolating theory,

$$\begin{aligned}\sum_{j=0}^{m} h_T^{j-m}|\psi p - \Pi_T(\psi p)|_{j,T} &= \sum_{j=0}^{m} h_T^{j-m}|(\psi - P_T^0\psi)p - \Pi_T((\psi - P_T^0\psi)p)|_{j,T} \\ &\lesssim h_T^s|(\psi - P_T^0\psi)p|_{m+s,T} \\ &\lesssim h_T^{s+1}|\psi|_{1,\infty,T}|p|_{m+s.T} + h_T^s\|\psi\|_{m+s,\infty,T}\|p\|_{m+s-1.T},\end{aligned}$$

where for $l \geq 0$

$$|\psi|_{l,\infty,T} = \max_{|\alpha|=l}\operatorname*{esssup}_{x\in T}|\partial^\alpha\psi(x)|, \quad \|\psi\|_{l,\infty,T} = \max_{|\alpha|\leq l}\operatorname*{esssup}_{x\in T}|\partial^\alpha\psi(x)|.$$

By the inverse inequality

$$\sum_{j=0}^{m} h_T^{j-m}|\psi p - \Pi_T(\psi p)|_{j,T} \lesssim h_T\|\psi\|_{m+s,\infty,T}\|p\|_{m,T}.$$

From the Sobolev embedding theorem, we obtain (3.19). ∎

Proof of Theorem 3.1. For $T \in \mathcal{T}_h$, corresponding to $\varphi_{i,T}$ we can read

$$\frac{\partial^{k(i)} v}{\partial l_{i,T,1} \cdots \partial l_{i,T,k(i)}} = \sum_{|\alpha|=k(i)} \eta_{i,\alpha} \partial^\alpha v \quad \forall v \in H^{k(i)}(T), \tag{3.20}$$

where $\eta_{i,\alpha}$ are constants dependent on $l_{i,T,j}$, $1 \le j \le k(i)$, and $|\eta_{i,\alpha}| \lesssim 1$.

Given two multi-indices β and γ, $\gamma \le \beta$ means that $\gamma_j \le \beta_j$, $1 \le j \le n$, and $\gamma < \beta$ means that $\gamma \le \beta$ and there exists $j \in \{1, \dots, n\}$ with $\gamma_j < \beta_j$. It can be proved that for multi-index α,

$$\partial^\alpha (vw) = \sum_{\beta \le \alpha} b_{\alpha\beta} \partial^\beta v \partial^{\alpha-\beta} w \quad \forall v, w \in H^{|\alpha|}(T), \tag{3.21}$$

where $b_{\alpha\beta}$ are fixed nonnegative integers. Obviously, $b_{\alpha\alpha} = b_{\alpha 0} = 1$.

Let $v_h \in V_h$, $\psi \in H_0^r(\Omega)$. We define function $\psi_h \in V_{h0}$ such that for all $T \in \mathcal{T}_h$, $\psi_h|_T \in P_T$ and

$$\varphi_{i,T}(\psi_h) = \begin{cases} \psi(a_{i,T})\varphi_{i,T}(v_h) \\ \quad + \left(\sum_{|\alpha|=k(i)} \eta_{i,\alpha} \sum_{\substack{|\beta|<m \\ \beta<\alpha}} b_{\alpha\beta} \Lambda_h^\beta v_h \partial^{\alpha-\beta}\psi \right)(a_{i,T}), \\ \hfill 1 \le i \le k_1, \\ P^0_{F_{i,T}} \psi \varphi_{i.T}(v_h) \\ \quad + \sum_{|\alpha|=k(i)} \eta_{i,\alpha} P^0_{F_{i,T}} \left((\psi - P^0_{F_{i,T}}\psi) \Lambda_h^\alpha v_h \right) \\ \quad + \sum_{|\alpha|=k(i)} \eta_{i,\alpha} \sum_{\beta<\alpha} b_{\alpha\beta} P^0_{F_{i,T}} (\Lambda_h^\beta v_h \partial^{\alpha-\beta}\psi), \\ \hfill k_1 < i \le k_2,\ k(i) < m, \\ P^0_{F_{i,T}} \psi \varphi_{i.T}(v_h) \\ \quad + \sum_{|\alpha|=k(i)} \eta_{i,\alpha} \sum_{\substack{|\beta|<m \\ \beta<\alpha}} b_{\alpha\beta} P^0_{F_{i,T}} (\Lambda_h^\beta v_h \partial^{\alpha-\beta}\psi), \\ \hfill k_1 < i \le k_2,\ k(i) \ge m, \\ \varphi_{i,T}(\psi v_h), \hfill k_2 < i \le L, \end{cases} \tag{3.22}$$

where $P^0_{F_{i,T}}$ is the orthogonal projection operator from $L^2(F_{i,T})$ to $P_0(F_{i,T})$.

Obviously, ψ_h is well defined. The continuous properties of v_h and ψ lead to $\psi_h \in V_h$. We conclude $\psi_h \in V_{h0}$ from the fact that $\psi \in H_0^r(\Omega)$.

By (3.5) and the inverse inequality, we conclude that for $0 \le j \le m$

$$h_T^{2(j-m)} |\Pi_T(\psi v_h) - \psi_h|^2_{j,T} \lesssim h_T^{n-2m} \sum_{i=1}^{L} h_T^{2k(i)} |\varphi_{i,T}(\psi v_h - \psi_h)|^2. \tag{3.23}$$

When $1 \le i \le k_1$, from (2.10), (3.20), (3.21), and (3.22) we have

$$\varphi_{i,T}(\psi v_h - \psi_h)$$

$$= \sum_{|\alpha|=k(i)} \eta_{i,\alpha} \Bigg(\sum_{\substack{m \le |\beta| \\ \beta < \alpha}} b_{\alpha\beta} \partial^\beta v_h \partial^{\alpha-\beta}\psi + \sum_{\substack{|\beta|<m \\ \beta<\alpha}} b_{\alpha\beta} (\partial^\beta v_h - \Lambda_h^\beta v_h) \partial^{\alpha-\beta}\psi \Bigg)(a_{i,T}).$$

Furthermore, by the inverse inequality

$$\begin{aligned} &| \varphi_{i,T}(\psi v_h - \psi_h)|^2 \\ &\quad \lesssim h_T^{-n} \|\psi\|^2_{k(i),\infty,T} \Bigg(\sum_{l=m}^{k(i)-1} |v_h|^2_{l,T} + \sum_{\substack{|\beta|<m \\ |\beta|<k(i)}} |\partial^\beta v_h - \Lambda_h^\beta v_h|^2_{0,T} \Bigg). \end{aligned} \tag{3.24}$$

When $k_1 < i \le k_2$ and $k(i) < m$,

$$\begin{aligned} \varphi_{i,T}(\psi v_h - \psi_h) = &\sum_{|\alpha|=k(i)} \eta_{i,\alpha} P^0_{F_{i,T}} \big((\psi - P^0_{F_{i,T}}\psi)(\partial^\alpha v_h - \Lambda_h^\alpha v_h) \big) \\ &+ \sum_{|\alpha|=k(i)} \eta_{i,\alpha} \sum_{\beta<\alpha} b_{\alpha\beta} P^0_{F_{i,T}} \big((\partial^\beta v_h - \Lambda_h^\beta v_h) \partial^{\alpha-\beta}\psi \big). \end{aligned}$$

From then interpolating theory and the inverse inequality we obtain

$$\begin{aligned} |\varphi_{i,T}(\psi v_h - \psi_h)|^2 \lesssim\ &h_T^{2-n} \|\psi\|^2_{1,\infty,T} \sum_{|\alpha|=k(i)} |\partial^\alpha v_h - \Lambda_h^\alpha v_h|^2_{0,T} \\ &+ h_T^{-n} \|\psi\|^2_{k(i),\infty,T} \sum_{|\beta|<k(i)} |\partial^\beta v_h - \Lambda_h^\beta v_h|^2_{0,T}. \end{aligned} \tag{3.25}$$

When $k_1 < i \le k_2$ and $k(i) \ge m$,

$$\begin{aligned} \varphi_{i,T}(\psi v_h - \psi_h) = &\sum_{|\alpha|=k(i)} \eta_{i,\alpha} P^0_{F_{i,T}} \big((\psi - P^0_{F_{i,T}}\psi) \partial^\alpha v_h \big) \\ &+ \sum_{|\alpha|=k(i)} \eta_{i,\alpha} \sum_{\substack{m \le |\beta| \\ \beta<\alpha}} b_{\alpha\beta} P^0_{F_{i,T}} (\partial^\beta v_h \partial^{\alpha-\beta}\psi) \\ &+ \sum_{|\alpha|=k(i)} \eta_{i,\alpha} \sum_{\substack{|\beta|<m \\ \beta<\alpha}} b_{\alpha\beta} P^0_{F_{i,T}} \big((\partial^\beta v_h - \Lambda_h^\beta v_h) \partial^{\alpha-\beta}\psi \big). \end{aligned}$$

From the interpolating theory and the inverse inequality we obtain

$$\begin{aligned} |\varphi_{i,T}(\ \psi v_h - \psi_h)|^2 \lesssim\ &h_T^{2-n} \|\psi\|^2_{k(i),\infty,T} \sum_{|\alpha|=k(i)} |\partial^\alpha v_h|^2_{0,T} \\ &+ h_T^{-n} \|\psi\|^2_{k(i),\infty,T} \Bigg(\sum_{l=m}^{k(i)-1} |v_h|^2_{l,T} + \sum_{|\beta|<m} |\partial^\beta v_h - \Lambda_h^\beta v_h|^2_{0,T} \Bigg). \end{aligned} \tag{3.26}$$

Combining (3.23), (3.24), (3.25), and (3.26) and the fact that $\varphi_{i,T}(\psi v_h - \psi_h) = 0$ when $k_2 < i \le L$, we get that for $j \in \{0, 1, \dots, m\}$

$$h_T^{2(j-m)}|\Pi_T(\psi v_h)-\psi_h|_{j,T}^2$$

$$\lesssim \sum_{\substack{k_1<i\le k_2\\ k(i)<m}} \|\psi\|_{1,\infty,T}^2 h_T^{2k(i)+2-2m} \sum_{|\alpha|=k(i)} |\partial^\alpha v_h-\Lambda_h^\alpha v_h|_{0,T}^2$$

$$+\sum_{i=1}^{k_2}\|\psi\|_{k(i),\infty,T}^2 h_T^{2k(i)-2m}\left(\sum_{l=m}^{k(i)-1}|v_h|_{l,T}^2+\sum_{\substack{|\beta|<m\\ |\beta|<k(i)}}|\partial^\beta v_h-\Lambda_h^\beta v_h|_{0,T}^2\right).$$

By A1 and the inverse inequality,

$$h_T^{2(j-m)}\;|\Pi_T(\psi v_h)-\psi_h|_{j,T}^2 \lesssim\; h_T^2\|\psi\|_{r,\Omega}^2\left(\|v_h\|_{m,T}^2+\sum_{|\beta|<m}h_T^{2(|\beta|-m)}|\partial^\beta v_h-\Lambda_h^\beta v_h|_{0,T}^2\right). \tag{3.27}$$

From (3.19),

$$\sum_{T\in\mathcal{T}_h}h_T^{2(j-m)}|\psi v_h-\psi_h|_{j,T}^2$$

$$\lesssim h^2\|\psi\|_{r,\Omega}^2\left(\|v_h\|_{m,h}^2+\sum_{|\beta|<m}\sum_{T\in\mathcal{T}_h}h_T^{2(|\beta|-m)}|\partial^\beta v_h-\Lambda_h^\beta v_h|_{0,T}^2\right).$$

Lemma 3.5 leads to

$$\sum_{j=0}^{m}\sum_{T\in\mathcal{T}_h}h_T^{2(j-m)}|\psi v_h-\psi_h|_{j,T}^2\lesssim h^2\|\psi\|_{r,\Omega}^2\|v_h\|_{m,h}^2. \tag{3.28}$$

Inequality (3.1) follows from (3.28) and the fact that $\psi_h\in V_{h0}$.

Now let $v_h\in V_{h0}$ and $\psi\in H^r(\Omega)$. Repeating the previous arguments and replacing Λ_h^β by Λ_{h0}^β, we can obtain inequality (3.2). ∎

4. Sufficiency of the local patch test. Let $\Omega_P\subset\Omega$ be a bounded and connected domain consisting of a union of elements of $\mathcal{T}_P\subset\mathcal{T}_h$. Denote by V_P and V_{P0} the corresponding finite element spaces with respect to $H^m(\Omega_P)$ and $H_0^m(\Omega_P)$, respectively.

An element (T,P_T,Φ_T) is said to pass the patch test on the triangulation $\mathcal{T}_P$ if

$$\sum_{T\in\mathcal{T}_P}\sum_{|\alpha|=m}\int_T a_\alpha\partial^\alpha p\,\partial^\alpha v\,dx=0\quad\forall p\in P_m(\Omega_P)\;\forall v\in V_{P0} \tag{4.1}$$

holds (see [19] or [20, p. 176]). This is the patch test originally introduced by Irons.

Since $a_\alpha>0$, we can take $p=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $|\alpha|=m$, in (4.1) and obtain

$$\sum_{T\in\mathcal{T}_P}\int_T\partial^\alpha v dx=0\qquad\forall v\in V_{P0},\;|\alpha|=m. \tag{4.2}$$

It is also easy to see that (4.2) actually implies (4.1). Hence we get the following lemma.

LEMMA 4.1. *Element (T, P_T, Φ_T) passes the patch test on $\mathcal{T}_P$ if and only if* (4.2) *is true.*

For $v_h \in V_h$ and $\varphi \in C^\infty(\bar{\Omega})$, we define

$$T_{\alpha,i}(\varphi, v_h) = \sum_{T\in\mathcal{T}_h} \int_T \left(\varphi \frac{\partial}{\partial x_i}\partial^\alpha v_h + \frac{\partial \varphi}{\partial x_i}\partial^\alpha v_h\right) dx, \quad |\alpha| < m,\ 1 \le i \le n. \tag{4.3}$$

We say that $\{V_{h0}, H_0^m(\Omega)\}$ passes the generalized patch test if

$$\lim_{h\to 0} \sup_{\substack{w_h\in V_{h0}\\ \|w_h\|_{m,h}\le 1}} |T_{\alpha,i}(\varphi, w_h)| = 0, \quad |\alpha| < m,\ 1 \le i \le n\ \forall \varphi \in C^\infty(\bar{\Omega}). \tag{4.4}$$

Let $f_\alpha \in L^2(\Omega)$, $|\alpha| \le m$. Define

$$\begin{cases} F(v) = \sum_{|\alpha|\le m} (f_\alpha, \partial^\alpha v) & \forall v \in H_0^m(\Omega), \\ F_h(v_h) = \sum_{T\in\mathcal{T}_h} \sum_{|\alpha|\le m} \int_T f_\alpha \partial^\alpha v_h dx & \forall v_h \in V_h. \end{cases} \tag{4.5}$$

Stummel showed the following result [21].

THEOREM 4.2. *Let $a_h(\cdot,\cdot)$ be uniformly V_{h0}-elliptic and let $u \in H_0^m(\Omega)$ and $u_h \in V_{h0}$ be the solutions of the following problems, respectively:*

$$a(u, v) = F(v) \qquad \forall v \in H_0^m(\Omega), \tag{4.6}$$

$$a_h(u_h, v_h) = F_h(v_h) \qquad \forall v_h \in V_{h0}. \tag{4.7}$$

Then $\lim_{h\to 0} \|u - u_h\|_{m,h} = 0$ for all $f_\alpha \in L^2(\Omega)$ if and only if $\{V_{h0}, H_0^m(\Omega)\}$ has the approximability and passes the generalized patch test.

Now we give the definition of the weak patch test. Define

$$T^w_{h,\alpha}(v) = \sum_{T\in\mathcal{T}_h} \int_T \partial^\alpha v dx, \quad 0 < |\alpha| \le m. \tag{4.8}$$

$\{V_{h0}\}$ is said to pass the weak patch test if

$$\lim_{h\to 0} \sup_{\substack{w_h\in V_{h0}\\ \|w_h\|_{m,h}\le 1}} |T^w_{h,\alpha}(w_h)| = 0, \quad 0 < |\alpha| \le m. \tag{4.9}$$

If (T, P_T, Φ_T) passes the patch test on $\mathcal{T}_h$, then (4.2) leads to

$$T^w_{h,\alpha}(v_h) = 0, \quad |\alpha| = m\ \forall v_h \in V_{h0}.$$

That is, the following lemma holds.

LEMMA 4.3. *Let $m = 1$. If (T, P_T, Φ_T) passes the patch test on each of $\{\mathcal{T}_h\}$, then $\{V_{h0}\}$ passes the weak patch test.*

For the nonconforming elements of higher order problems ($m > 1$), we have the following lemma.

LEMMA 4.4. *Let $m > 1$ and let* A1, A3, *and* A3′ *be true. If (T, P_T, Φ_T) passes the patch test on each of $\{\mathcal{T}_h\}$, then $\{V_{h0}\}$ passes the weak patch test.*

Proof. Let $v_h \in V_{h0}$, $1 \le i \le n$, and $|\alpha| < m-1$; then

$$T^w_{h,\alpha+e_i}(v_h) = \sum_{T\in\mathcal{T}_h} \int_T \frac{\partial}{\partial x_i}(\partial^\alpha v_h - \Lambda^\alpha_{h0} v_h)dx + \int_\Omega \frac{\partial}{\partial x_i}\Lambda^\alpha_{h0} v_h dx$$
$$= \sum_{T\in\mathcal{T}_h} \int_T \frac{\partial}{\partial x_i}(\partial^\alpha v_h - \Lambda^\alpha_{h0} v_h)dx.$$

By Schwarz inequality we have

$$|T^w_{h,\alpha+e_i}(v_h)| \le \sum_{T\in\mathcal{T}_h} |T|^{1/2} |\partial^\alpha v_h - \Lambda^\alpha_{h0} v_h|_{1,T}$$
$$\le \left(\sum_{T\in\mathcal{T}_h} |T|\right)^{1/2} \left(\sum_{T\in\mathcal{T}_h} |\partial^\alpha v_h - \Lambda^\alpha_{h0} v_h|^2_{1,T}\right)^{1/2} \lesssim |\partial^\alpha v_h - \Lambda^\alpha_{h0} v_h|_{1,h}.$$

Finally, from Lemma 3.5 we get

$$|T^w_{h,\alpha+e_i}(v_h)| \lesssim h|v_h|_{m,h} \quad \forall v_h \in V_{h0},\ 1 \le i \le n,\ |\alpha| < m-1. \tag{4.10}$$

Combining (4.10) and Lemma 4.1, we obtain the lemma. □

LEMMA 4.5. *If $\{V_{h0}, H^m_0(\Omega)\}$ passes the generalized patch test, then $\{V_{h0}\}$ passes the weak patch test.*

Proof. Take $\psi \equiv 1$ on Ω. Given a function $v_h \in V_{h0}$, it is obvious that

$$T^w_{h,e_i+\alpha}(v_h) = T_{\alpha,i}(1, v_h) = T_{\alpha,i}(\psi, v_h), \quad 1 \le i \le n,\ |\alpha| < m.$$

Then the lemma follows from the definitions of the generalized patch test and the weak patch test. □

THEOREM 4.6. *Let $\{V_{h0}\}$ have the weak superapproximation. Then the weak patch test is equivalent to the generalized patch test.*

Proof. From Lemma 4.5, we need only to show that the weak patch test implies the generalized patch test.

Let $|\alpha| < m$; we need to show

$$\lim_{h\to 0} \sup_{\substack{v_h\in V_{h0} \\ \|v_h\|_{m,h}\le 1}} |T_{\alpha,i}(\varphi, v_h)| = 0, \quad 1 \le i \le n\ \forall \varphi \in C^\infty(\bar{\Omega}). \tag{4.11}$$

First, we prove

$$\lim_{h\to 0} \sup_{\substack{w_h\in V_{h0} \\ \|w_h\|_{m,h}\le 1}} |T^w_{h,\alpha}(w_h\phi)| = 0, \quad 0 < |\alpha| \le m\ \forall \phi \in C^\infty(\bar{\Omega}). \tag{4.12}$$

Let $0 < |\alpha| \le m$ and $\phi \in C^\infty(\bar{\Omega})$, and let $v_h \in V_{h0}$ such that $\|v_h\|_{m,h} \le 1$ and

$$|T^w_{h,\alpha}(v_h\phi)| = \sup_{\substack{w_h\in V_{h0} \\ \|w_h\|_{m,h}\le 1}} |T^w_{h,\alpha}(w_h\phi)|.$$

Now let $\phi_h \in V_{h0}$ be the function satisfying

$$\|\phi v_h - \phi_h\|_{m,h} = \inf_{w_h\in V_{h0}} \|\phi v_h - w_h\|_{m,h}.$$

By the weak superapproximation we have

$$\lim_{h \to 0} \|\phi v_h - \phi_h\|_{m,h} = 0. \tag{4.13}$$

It is obvious that

$$T^w_{h,\alpha}(v_h\phi) = T^w_{h,\alpha}(v_h\phi - \phi_h) + T^w_{h,\alpha}(\phi_h). \tag{4.14}$$

Since $\|v_h\|_{m,h} \leq 1$, $\|\phi_h\|_{m,h}$ is bounded. The weak patch test leads to

$$\lim_{h \to 0} T^w_{h,\alpha}(\phi_h) = 0. \tag{4.15}$$

Therefore, we obtain (4.12) by (4.13), (4.14), and (4.15).

To establish (4.11) we use induction. By (4.3) and (4.8),

$$T_{0,i}(\varphi, v_h) = T^w_{h,e_i}(v_h\varphi), \quad 1 \leq i \leq n, \ \varphi \in C^\infty(\bar{\Omega}).$$

Hence, (4.11) is true for $\alpha = 0$. Assume that (4.11) is true for $|\alpha| \leq j$ with $0 \leq j < m - 1$; we prove that it is also true for $|\alpha| = j + 1$. Take $\varphi \in C^\infty(\bar{\Omega})$ and $1 \leq i \leq n$. Obviously,

$$T_{\alpha,i}(\varphi, v_h) = \sum_{T \in \mathcal{T}_h} \int_T \frac{\partial}{\partial x_i}(\varphi \partial^\alpha v_h) dx. \tag{4.16}$$

Using (3.21) we have

$$T_{\alpha,i}(\varphi, v_h) = T^w_{h,\alpha+e_i}(v_h\varphi) - \sum_{0<\beta\leq\alpha} b_{\alpha\beta} T_{i,\alpha-\beta}(\partial^\beta\varphi, v_h). \tag{4.17}$$

From (4.12) and the inductive assumption, it follows that (4.11) is true for $|\alpha| = j+1$.

Finally, we conclude that $\{V_{h0}, H^m_0(\Omega)\}$ passes the generalized patch test. ∎

Now let u and u_h be the solutions of (4.6) and (4.7), respectively. The following theorem about the sufficiency of the patch test follows from Lemma 4.3, Lemma 4.4, Theorem 4.2, Theorem 4.6, and Theorem 3.1.

THEOREM 4.7. *Let* A1, A2, A2′, A3, A3′, *and* A4 *hold. Assume that* $a_h(\cdot, \cdot)$ *is uniformly* V_{h0}*-elliptic. If* (T, P_T, Φ_T) *passes the patch test on each of* $\{\mathcal{T}_h\}$*, then* $\lim_{h\to 0} \|u - u_h\|_{m,h} = 0$ *for all* $f_\alpha \in L^2(\Omega)$.

Theorem 4.7 leads to the conjecture that the patch test is sufficient for the convergence of nonconforming elements under the approximation (A4) and the weak continuity (A3 and A3′) when $a_h(\cdot, \cdot)$ is uniformly V_{h0}-elliptic and A1, A2, and A2′ hold.

It is well known that the Crouzeit–Raviart element and the Wilson element for second order problems and the Morley element and the Adini element for fourth order problems are convergent. They all pass the patch test; the mathematical proofs can be found in [7, 21, 23]. Therefore, we can get their convergence from Theorem 4.7.

As a nonconforming element for a fourth order problem, the Zienkiewicz element passes the patch test only under the condition of parallel lines (see [6, 7]) which requires the triangulations generated by three sets of parallel lines. Apart from the proofs in [7, 16], Theorem 4.7 also leads to the convergence result.

Remark 4.1. The result of Theorem 4.2 is also true for more general problems, for example, the variable coefficient problems. From Theorems 4.2 and 4.6

we can conclude that under V_{h0}-elliptic property and the weak superapproximation, $\lim_{h\to 0}\|u-u_h\|_{m,h}=0$ for all $f_\alpha\in L^2(\Omega)$ if and only if $\{V_{h0},H_0^m(\Omega)\}$ has the approximability and $\{V_{h0}\}$ passes the weak patch test. The weak superapproximation, the approximation, and the weak patch test are only dependent on the order and the dimension of the problem to solve. On the other hand, passing the patch test for a special problem implies passing the weak patch test. Therefore we can get the convergence result for more problems from a simpler one.

5. Error estimate of the consistency term. In this section, let u and u_h be the solutions of problems (2.3) and (2.5), respectively. We consider the error estimates of the consistency term.

THEOREM 5.1. *Assume that* A1, A2, A2′, A3, A3′, *and* A4 *hold. If* (T,P_T,Φ_T) *passes the patch test on each of* $\{\mathcal{T}_h\}$, *then*

$$\sup_{0\neq v_h\in V_{h0}}\frac{|a_h(u,v_h)-(f,v_h)|}{\|v_h\|_{m,h}}\lesssim h\|u\|_{r+m,\Omega} \tag{5.1}$$

provided $u\in H^{r+m}(\Omega)$ *with* $r\geq\frac{n}{2}+\max\{m+s,k(1),\dots,k(L)\}$.

Proof. Let $v_h\in V_{h0}$. Define

$$E_h(v_h)=a_h(u,v_h)-(f,v_h),$$

$$E_{h,\alpha}(v_h)=\sum_{T\in\mathcal{T}_h}\int_T\left(\partial^\alpha u\partial^\alpha v_h+(-1)^{m+1}\partial^{2\alpha}uv_h\right)dx,\quad |\alpha|=m.$$

Then

$$E_h(v_h)=\sum_{|\alpha|=m}a_\alpha E_{h,\alpha}(v_h). \tag{5.2}$$

For α with $|\alpha|=m$, there exist $i_1,\dots,i_m\in\{1,\dots,n\}$ such that $\alpha=e_{i_1}+\cdots+e_{i_m}$. Set $w=\partial^\alpha u$, $\beta(0)=\alpha-e_{i_1}$ and $\beta(j)=e_{i_1}+\cdots+e_{i_j}$, $1\leq j\leq m$. It can be derived that

$$w\partial^\alpha v_h+(-1)^{m+1}\partial^\alpha wv_h=\partial^{e_{i_1}}\left(w\partial^{\beta(0)}v_h\right)-\sum_{j=2}^m(-1)^j\partial^{e_{i_j}}\left(\partial^{\beta(j-1)}w\partial^{\alpha-\beta(j)}v_h\right).$$

By (3.21), we have

$$\partial^{e_{i_1}}\left(w\partial^{\beta(0)}v_h\right)=\partial^\alpha(wv_h)-\sum_{0<\gamma\leq\beta(0)}b_{\beta(0),\gamma}\partial^{e_{i_1}}\left(\partial^\gamma w\partial^{\beta(0)-\gamma}v_h\right).$$

Furthermore,

$$\begin{aligned}E_{h,\alpha}(v_h)=&\sum_{T\in\mathcal{T}_h}\int_T\partial^\alpha(\partial^\alpha uv_h)dx\\&-\sum_{0<\gamma\leq\beta(0)}b_{\beta(0),\gamma}\sum_{T\in\mathcal{T}_h}\int_T\partial^{e_{i_1}}\left(\partial^{\gamma+\alpha}u\partial^{\beta(0)-\gamma}v_h\right)dx\\&-\sum_{j=2}^m(-1)^j\sum_{T\in\mathcal{T}_h}\int_T\partial^{e_{i_j}}\left(\partial^{\beta(j-1)+\alpha}u\partial^{\alpha-\beta(j)}v_h\right)dx.\end{aligned} \tag{5.3}$$

Taking $\mathcal{T}_P = \mathcal{T}_h$ in (4.2), we have

$$\sum_{T\in\mathcal{T}_h}\int_T \partial^\alpha(v_h\partial^\alpha u)dx = \sum_{T\in\mathcal{T}_h}\int_T \partial^\alpha(v_h\partial^\alpha u - w_h)dx \quad \forall w_h \in V_{h0}.$$

Inequality (3.2) leads to

$$\left|\sum_{T\in\mathcal{T}_h}\int_T \partial^\alpha(v_h\partial^\alpha u)dx\right| \lesssim h\|u\|_{r+m,\Omega}\|v_h\|_{m,h}. \tag{5.4}$$

On the other hand,

$$\sum_{0<\gamma\le\beta(0)} b_{\beta(0),\gamma}\sum_{T\in\mathcal{T}_h}\int_T \partial^{e_{i_1}}\big(\partial^{\gamma+\alpha}u\partial^{\beta(0)-\gamma}v_h\big)dx$$
$$= \sum_{0<\gamma\le\beta(0)} b_{\beta(0),\gamma}\sum_{T\in\mathcal{T}_h}\int_T \partial^{e_{i_1}}\big(\partial^{\gamma+\alpha}u(\partial^{\beta(0)-\gamma} - \Lambda_{h0}^{\beta(0)-\gamma})v_h\big)dx.$$

Then from Schwarz inequality and (3.11) we get

$$\left|\sum_{0<\gamma\le\beta(0)} b_{\beta(0),\gamma}\sum_{T\in\mathcal{T}_h}\int_T \partial^{e_{i_1}}\big(\partial^{\gamma+\alpha}u\partial^{\beta(0)-\gamma}v_h\big)dx\right| \lesssim h\|u\|_{2m,\Omega}\|v_h\|_{m,h}. \tag{5.5}$$

Similarly, we have

$$\left|\sum_{j=2}^{m}(-1)^j\sum_{T\in\mathcal{T}_h}\int_T \partial^{e_{i_j}}\big(\partial^{\beta(j-1)+\alpha}u\partial^{\alpha-\beta(j)}v_h\big)dx\right| \lesssim h\|u\|_{2m,\Omega}\|v_h\|_{m,h}. \tag{5.6}$$

From (5.3), (5.4), (5.5), and (5.6) we obtain

$$|E_{h,\alpha}(v_h)| \lesssim h\big(\|u\|_{r+m,\Omega} + \|u\|_{2m,\Omega}\big)\|v_h\|_{m,h}.$$

Consequently, (5.1) is true. □

From the above discussion, we can see that the consistency term is of order $O(h)$ at least when the element passes the patch test and that A1, A2, A2′, A3, A3′, and A4 hold.

6. Necessity of patch test. Now we turn to the necessity of the patch test. Let $\mathcal{T}_P \subset \mathcal{T}_{h_0}$ for some mesh size h_0.

For a positive number ϑ and a mesh size h, let $\mathcal{T}_h(\mathcal{T}_P, \vartheta)$ be the set of all patches T_M with the following properties:

(1) There exists a mapping $F_{T_M} : x \in R^n \to \theta_{T_M}x + b_{T_M} \in R^n$, with θ_{T_M} a positive number and $b_{T_M} \in R^n$, such that $T_M = \{\,T \mid T = F_{T_M}T' \text{ for all } T' \in \mathcal{T}_P\}$.

(2) $T_M \subset \mathcal{T}_h$.

(3) $\vartheta h \le \theta_{T_M} \le h$.

A subset of $\mathcal{T}_h(\mathcal{T}_P, \vartheta)$ is said to be nonoverlapping if the intersection of its arbitrary two different patches is empty. Obviously, the number of T_M contained in each nonoverlapping subset is finite. Let $N_h(\mathcal{T}_P, \vartheta)$ be the largest among all nonoverlapping subsets.

THEOREM 6.1. *Let* A1 *be true. Assume that the following three statements are true:*

(1) *All degrees of freedom of $v \in V_{P0}$ vanish on $\partial\Omega_P$.*
(2) (T, P_T, Φ_T) *does not pass the patch test on $\mathcal{T}_P$.*
(3) *There exists a positive constant ϑ such that*

$$N_h(\mathcal{T}_P, \vartheta) \eqsim h^{-n}. \tag{6.1}$$

Then $\{V_{h0}, H_0^m(\Omega)\}$ does not pass the generalized patch test.

Proof. Since the element does not pass the patch test on $\mathcal{T}_P$, there is a function $v_p \in V_{P0}$ such that

$$\xi \equiv \sum_{T \in \mathcal{T}_P} \int_T \partial^\alpha v_p dx \neq 0 \tag{6.2}$$

for some α with $|\alpha| = m$. Define

$$\xi_i = \left(\sum_{T \in \mathcal{T}_P} |v_p|_{i,T}^2 \right)^{1/2}, \quad 0 \le i \le m. \tag{6.3}$$

Then $\xi_m > 0$ because of (6.2).

Let $\psi(x) \equiv 1$ on Ω. Let S_h be the subset of $\mathcal{T}_h(\mathcal{T}_P, \vartheta)$ with its patch number equal to $N_h(\mathcal{T}_P, \vartheta)$. Define v_h on $\bar{\Omega}$ by

$$v_h|_T = \begin{cases} v_p \cdot F_{T_M}^{-1}, & T \in T_M \text{ and } T_M \in S_h, \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

From A1 it follows that $v_h \in V_{h0}$.

Set $\beta + e_i = \alpha$ with $i \in \{1, \dots, n\}$ and $|\beta| = m - 1$. Then the definition of v_h gives

$$T_{\beta,i}(\psi, v_h) = \sum_{T_M \in S_h} \sum_{T \in T_M} \int_T \partial^\alpha v_h dx. \tag{6.5}$$

Taking coordinate transformation $x \to F_{T_M}^{-1} x$ on each $T \in T_M$, we have

$$T_{\beta,i}(\psi, v_h) = \sum_{T_M \in S_h} \theta_{T_M}^{n-m} \sum_{T \in \mathcal{T}_P} \int_T \partial^\alpha v_p dx = \xi \sum_{T_M \in S_h} \theta_{T_M}^{n-m}.$$

Consequently,

$$|T_{\beta,i}(\psi, v_h)| = |\xi| \sum_{T_M \in S_h} \theta_{T_M}^{n-m}.$$

From (6.1) and the fact that $\vartheta h \le \theta_{T_M} \le h$, we conclude

$$|T_{\beta,i}(\psi, v_h)| \gtrsim |\xi| \vartheta^n h^{-m}. \tag{6.6}$$

Similarly, we can obtain

$$|v_h|_{i,h}^2 = \xi_i^2 \sum_{T_M \in S_h} \theta_{T_M}^{n-2i} \lesssim \xi_i^2 \vartheta^{-2i} h^{-2i}, \quad 0 \le i \le m.$$

Therefore

$$\|v_h\|_{m,h} \lesssim h^{-m}(1+O(h)). \tag{6.7}$$

Combining (6.6) and (6.7), we get

$$\frac{1}{\|v_h\|_{m,h}}|T_{\beta,i}(\psi, v_h)| \gtrsim (1+O(h)). \tag{6.8}$$

Inequality (6.8) shows that for the bounded sequence

$$w_h = \frac{v_h}{\|v_h\|_{m,h}} \in V_{h0}$$

and ψ given above, $T_{\beta,i}(\psi, w_h)$ does not tend to zero as $h \to 0$. $\{V_{h0}, H_0^m(\Omega)\}$ fails to pass the generalized patch test. □

By Theorem 4.2, a nonconforming element diverges for $\{\mathcal{T}_h\}$ satisfying the assumption (3) in Theorem 6.1 if it does not pass the patch test on $\mathcal{T}_P$ and all the degrees of freedom of $v \in V_{P0}$ vanish on $\partial\Omega_P$.

Let M_h be the number of elements in $\mathcal{T}_h$. Then $M_h \eqsim h^{-n}$ when $\rho_T \gtrsim h$ for all $T \in \mathcal{T}_h$. Assumption (3) in Theorem 6.1 means $N_h(\mathcal{T}_P, \vartheta) \eqsim M_h$.

For usual nonconforming elements, one can subdivide a domain with assumption (3) of Theorem 6.1 true. Then there are triangulations making an element divergent when it does not pass the patch test. In other words, if an element converges for all families $\{\mathcal{T}_h\}$, then it passes the patch test on arbitrary triangulation. The patch test is necessary in this sense.

For some elements, the degrees of freedom of $v \in V_{P0}$ may not vanish on $\partial\Omega_P$. In these case, we can obtain the following theorem by the similar way.

THEOREM 6.2. *Let* A1 *be true. Assume that statement* (3) *of Theorem* 6.1 *is true. If there exist* $p \in P_m(\Omega_P)$ *and* $v_p \in V_{P0}$ *such that all the degrees of freedom of* v_p *on* $\partial\Omega_P$ *vanish and*

$$\sum_{T\in\mathcal{T}_P}\sum_{|\alpha|=m}\int_T a_\alpha \partial^\alpha p\, \partial^\alpha v_p\, dx \neq 0, \tag{6.9}$$

then $\{V_{h0}, H_0^m(\Omega)\}$ *does not pass the generalized patch test.*

Since the Zienkiewicz element does not pass the patch test when the condition of parallel lines fails, it may be divergent for many meshes. Theorem 6.1 gives a simple method to check the divergence.

7. Stummel's examples. In this section, we consider Stummel's examples. It was shown that Stummel's examples pass the patch test and are not convergent [13, 22]. By Theorem 4.2, Lemma 4.3, and Theorem 4.6 Stummel's examples fail the weak superapproximation. Here we show directly that Stummel's first example does not possess the superapproximation.

Set $m = n = 1$, $\Omega = (0,1)$. Ω is divided into equally spaced subintervals with mesh size

$$h = h_\tau = \frac{1}{\tau}, \quad \tau = 3, 4, \ldots$$

and set $x_j = jh$, $0 \le j \le \tau$, and

$$T_j = [(j-1)h_\tau, jh_\tau], \quad 1 \le j \le \tau.$$

Then the triangulation $\mathcal{T}_h$ consists of all T_j. Let S_{h0} be the linear conforming element space corresponding to $\mathcal{T}_h$.

Let $w_j(x)$, $j = 1, 2, \ldots, \tau$, be the step functions

$$w_j(x) = \begin{cases} 1, & x_{j-1} < x < x_j, \\ 0 & \text{elsewhere.} \end{cases}$$

The finite element space V_{h0} of Stummel's first example [22] is given by

$$V_{h0} = S_{h0} + \operatorname{span}\{w_2, \ldots, w_{\tau-1}\}. \tag{7.1}$$

Now let $\varphi = x$ and $v_h \in V_{h0}$ be given by

$$v_h(x) = \sum_{j=2}^{\tau-1} w_j.$$

Let $\phi_h \in V_{h0}$ satisfy

$$\|\varphi v_h - \phi_h\|_{1,h} = \inf_{w_h \in V_{h0}} \|\varphi v_h - w_h\|_{1,h}; \tag{7.2}$$

then $\phi_h = \tilde{\phi}_h + \bar{\phi}_h$ with $\tilde{\phi}_h \in S_{h0}$ and $\bar{\phi}_h \in \operatorname{span}\{w_2, \ldots, w_{\tau-1}\}$.

It is obvious that

$$\|\varphi v_h - \phi_h\|_{1,h}^2 \le \|\varphi v_h\|_{1,h}^2 < \frac{4}{3}. \tag{7.3}$$

Set $y_j = \tilde{\phi}_h(x_j)$, $1 \le j \le \tau - 1$. Then

$$|\varphi v_h - \phi_h|_{1,h}^2 = h \sum_{j=2}^{\tau-1} \Big(1 - \frac{y_j - y_{j-1}}{h}\Big)^2 + h^{-1}(y_1^2 + y_{\tau-1}^2). \tag{7.4}$$

From (7.3) and (7.4) we have

$$y_1^2 + y_{\tau-1}^2 < \frac{4}{3} h. \tag{7.5}$$

On the other hand,

$$\begin{aligned} h \sum_{j=2}^{\tau-1} \Big(1 - \frac{y_j - y_{j-1}}{h}\Big)^2 &= h \sum_{j=2}^{\tau-1} \Big(1 - 2\frac{y_j - y_{j-1}}{h} + h^{-2}(y_j - y_{j-1})^2\Big) \\ &= 1 - 2h + 2y_1 - 2y_{\tau-1} + h^{-1} \sum_{j=2}^{\tau-1} (y_j - y_{j-1})^2 \\ &\ge 1 - 2h + 2y_1 - 2y_{\tau-1}. \end{aligned}$$

Hence

$$\|\varphi v_h - \phi_h\|_{1,h}^2 \ge 1 - 2h + 2y_1 - 2y_{\tau-1}. \tag{7.6}$$

Because of (7.5) and (7.6), there exists $h_0 \in (0, 1)$ such that

$$\|\varphi v_h - \phi_h\|_{1,h}^2 > \frac{1}{2}, \quad h < h_0. \tag{7.7}$$

That is, Stummel's first example fails the weak superapproximation. $\square$

8. The Morley element.

The Morley element is a well-known convergent nonconforming element for fourth order problems. It is even nonconforming for second order problems. In this section, we discuss its convergent property for second order problems.

Let $m = 1$, $n = 2$, and Ω_P be a polygon domain in R^2. $\mathcal{T}_P$ denotes a triangulation of Ω_P by triangles. Set V_{P0} the finite element space for second order problems corresponding to $\mathcal{T}_P$ and the Morley element.

LEMMA 8.1. *Let T_1 and T_2 be two different triangles with a common side F and let $\mathcal{T}_P$ consist of T_1 and T_2. Assume that $v \in V_{P0}$ and the normal derivative of v is nonzero at the midpoint of F and vanishes at the midpoints of other sides. Then there exists $i \in \{1, 2\}$ such that*

$$\int_{T_1} \frac{\partial v}{\partial x_i} dx + \int_{T_2} \frac{\partial v}{\partial x_i} dx \neq 0.$$

Proof. Let N and s be the normal and tangent of F. Set (ξ, η) be the coordinate in $N - s$-axis. Denote by h_1 and h_2 the heights, orthogonal to F, of T_1 and T_2, respectively. Then v can be written as

$$v(\xi, \eta) = \begin{cases} \dfrac{b}{h_1} \xi (h_1 - \xi), & (\xi, \eta) \in T_1, \\ \dfrac{b}{h_2} \xi (\xi + h_2), & (\xi, \eta) \in T_2 \end{cases}$$

with b a nonzero constant. We can compute that

$$\int_{T_1} \frac{\partial v}{\partial \xi} + \int_{T_2} \frac{\partial v}{\partial \xi} = \frac{b}{3}(|T_2| + |T_1|), \quad \int_{T_1} \frac{\partial v}{\partial \eta} + \int_{T_2} \frac{\partial v}{\partial \eta} = 0.$$

Then the lemma follows. $\square$

From Lemma 8.1, if $\mathcal{T}_P$ consists of two different elements with a common side, then the Morley element does not pass the patch test and there is $v \in V_{P0}$ such that (4.2) does not hold and all degrees of freedom of v on the $\partial\Omega_P$ are zero. Then the Morley element may not be convergent in the general case by Theorem 6.2.

Acknowledgments. The author would like to thank Professor Xu Jinchao and Professor Zhou Aihui for their useful discussions.

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