A finite element method for vibration analysis of elastic plate-plate structures

Article *in* Discrete and Continuous Dynamical Systems - Series B · March 2008

DOI: 10.3934/dcdsb.2009.11.387

2 authors:

Junjiang Lai
Minjiang University

Jianguo Huang
Shanghai Jiao Tong University

DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **11**, Number **2**, March **2009**

doi:10.3934/dcdsb.2009.11.387

pp. **387–419**

A FINITE ELEMENT METHOD FOR VIBRATION ANALYSIS OF ELASTIC PLATE-PLATE STRUCTURES

Junjiang Lai

Department of Mathematics, Shanghai Jiao Tong University
Shanghai 200240, China

Jianguo Huang [1]

Department of Mathematics, Shanghai Jiao Tong University
Shanghai 200240, China
and
Scientific Computing Key Laboratory of Shanghai Universities
Shanghai Normal University, China
and
Division of Computational Science, E–Institute of Shanghai Universities
Shanghai Normal University, China

(Communicated by Zhimin Zhang)

Abstract. The semi and fully discrete finite element methods are proposed for investigating vibration analysis of elastic plate-plate structures. In the space directions, the longitudinal displacements on plates are discretized by conforming linear elements, and the corresponding transverse displacements are discretized by the Morley element, leading to a semi-discrete finite element method for the problem under consideration. Applying the second order central difference to discretize the time derivative, a fully discrete scheme is obtained, and two approaches for choosing the initial functions are also introduced. The error analysis in the energy norm for the semi and fully discrete methods are established, and some numerical examples are included to validate the theoretical analysis.

1. **Introduction.** Elastic multi-structures are usually composed of a number of elastic substructures with the same or different dimensions (three-dimensional bodies, plates, rods, etc.) coupled by some proper junctions, frequently used in automobile and aeroplane structures and motion- and force-transmitting machines and mechanisms. During the past few decades, many researchers were devoted to mathematical modeling and numerical solution for static elastic multi-structure problems (cf. [5, 6, 7, 11, 14, 15, 16, 17, 18, 19, 20, 22, 28]). We refer to the monograph [11] for practical applications of such structures and mention the following words of P. G. Ciarlet to emphasize the importance of such studies: "A challenging

2000 *Mathematics Subject Classification.* Primary: 65M15, 65M60; Secondary: 70J35, 70J50.
Key words and phrases. Vibration analysis, Plate-plate structures, Error analysis.
The work was partly supported by The National Basic Research Program (2005CB321701), The NCET of China (NCET-06-0391), NNSFC (10771138), and E-Institutes of Shanghai Municipal Education Commission (E03004).
[1]Corresponding author. E-mail address: jghuang@sjtu.edu.cn (Jianguo Huang).

program consists in numerically approximating the mathematical models of elastic multi-structures that comprise many substructures." [11, p. 180].

Since vibration analysis of elastic structures plays important roles in engineering applications [12, 30], in this paper we are concerned with a finite element method for vibration analysis of elastic plate-plate structures. In the space directions, the longitudinal displacements on plates are discretized by conforming linear elements, and the corresponding transverse displacements are discretized by the Morley element, leading to a semi-discrete finite element method for the problem under consideration. Applying the second order central difference to discretize the time derivative, a fully discrete scheme is obtained, and two approaches for choosing the initial functions are also introduced. We remark that vibration analysis of a single Kirchhoff plate was discussed in [21] by the Morley element method, which is the foundation of our method developed here. Moreover, a mesh-free Galerkin method for free vibration analysis of folded plate structures was presented in [24], but the related error analysis has not been developed.

Following the techniques in [3, 13, 19, 21], we establish error estimates in the energy norm for our methods and provide some numerical examples to show the computational performance of the method proposed.

We end up this section with some notation and convention for later uses. As in [18, 19, 20], Latin index i takes values in the set $\{1, 2, 3\}$, while the capital Latin indices I, J, L take their values in the set $\{1, 2\}$. The summation is implied when a Latin index (or a capital Latin index) appears exactly two times.

The standard notation is used for Sobolev norms and semi-norms [2], e.g., for $v \in H^1(\beta)$,

$$\|v\|_{0,\,\beta} := \left(\int_\beta |v|^2 \, \mathrm{d}x \right)^{1/2}, \ \|v\|_{1,\,\beta} := \left(\int_\beta (|v|^2 + |\mathrm{grad} v|^2) \, \mathrm{d}x \right)^{1/2}.$$

If A is a Banach space with norm $\|\cdot\|_A$ and semi-norm $|\cdot|_A$, $v : [0, T] \to A$ is a Lebesgue measurable function, define [23]

$$L^p(0,\,T;\,A) = \left\{ v : \ [0,\,T] \to A; \ \|v\|_{L^p(0,\,T;\,A)} < \infty \right\}, \quad 1 \le p \le \infty,$$

where

$$\|v\|_{L^p(0,\,T;\,A)} := \left(\int_0^T \|v(\cdot, t)\|_A^p \, \mathrm{d}t \right)^{1/p}, \ |v|_{L^p(0,\,T;\,A)} := \left(\int_0^T |v(\cdot, t)|_A^p \, \mathrm{d}t \right)^{1/p},$$

$1 \le p < \infty$, with the norms extended to the case $p = \infty$ in the usual way. In addition, all the notation above may be extended naturally to the case of vector-valued functions.

For ease of exposition, we write

$$\dot{(\ )} = (\ )_t := \frac{\partial (\ )}{\partial t}, \quad \ddot{(\ )} = (\ )_{tt} := \frac{\partial^2 (\ )}{\partial t^2},$$

and use the symbol "$\lesssim \cdots$" to denote "$\le C \cdots$" with a generic constant $C > 0$ independent of the corresponding parameters and the functions under considerations, which may take different values in different appearances.

2. Preliminaries.

2.1. Mathematical model.

Let β_1 and β_2 be two bounded polygon plate members, which are rigidly connected to form an elastic plate-plate structure $\Omega := \{\beta_1, \beta_2\}$ (see Figure 1). Denote all proper boundary lines of plates by Γ^1. An element of Ω and Γ^1 is called respectively an area and line element.

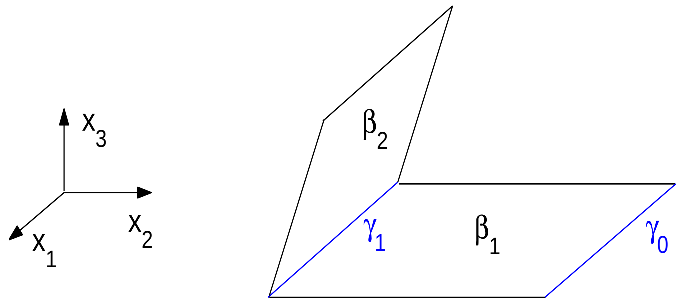

FIGURE 1. An elastic plate-plate structure

Introduce a right-handed orthogonal system (x_1, x_2, x_3) in the space $\mathbb{R}^3$, whose orthonormal basis vectors are denoted by $\{\boldsymbol{e}_i\}_{i=1}^3$. For each plate member $\beta \in \Omega$, we associate a local right-handed coordinate system $(x_1^\beta, x_2^\beta, x_3^\beta)$ as follows ($\{\boldsymbol{e}_i^\beta\}_{i=1}^3$ represent the related orthonormal basis vectors). x_1^β and x_2^β are its longitudinal directions, and x_3^β the transverse direction. Moreover, along the boundary $\partial\beta$ of β, a unit tangent vector $\boldsymbol{t}^\beta$ is selected such that $\{\boldsymbol{n}^\beta, \boldsymbol{t}^\beta, \boldsymbol{e}_3^\beta\}$ forms a right-handed coordinate system, where $\boldsymbol{n}^\beta$ denotes the unit outward normal to $\partial\beta$ in the longitudinal plane, and $\boldsymbol{e}_3^\beta$ the unit transverse vector of β. For a line element $\gamma \in \Gamma^1$, let $\boldsymbol{e}_1^\gamma$ be a unit vector representing the longitudinal direction of γ.

For any two elements $\gamma \in \Gamma^1$ and $\beta \in \Omega$, define

$$\varepsilon(\beta,\gamma) = \begin{cases} 0, & \text{if } \gamma \notin \partial\beta, \\ 1, & \text{if } \gamma \in \partial\beta,\ \boldsymbol{e}_1^\gamma \text{ and } \boldsymbol{t}^\beta \text{ have the same direction on } \gamma, \\ -1, & \text{if } \gamma \in \partial\beta,\ \boldsymbol{e}_1^\gamma \text{ and } \boldsymbol{t}^\beta \text{ have the opposite direction on } \gamma. \end{cases}$$

Since Ω is rigidly connected, it holds the interface conditions:

$$\boldsymbol{u}^{\beta_1} = \boldsymbol{u}^{\beta_2}, \quad \varepsilon(\beta_1, \gamma_1)\partial_{\boldsymbol{n}^{\beta_1}} u_3^{\beta_1} = \varepsilon(\beta_2, \gamma_1)\partial_{\boldsymbol{n}^{\beta_2}} u_3^{\beta_2} \text{ on } \gamma_1. \tag{2.1}$$

Here, for each $\beta \in \Omega$, $\boldsymbol{u}^\beta := u_i^\beta \boldsymbol{e}_i^\beta$ with u_i^β denoting the displacement on β along the direction $\boldsymbol{e}_i^\beta$, and $\boldsymbol{f}^\beta := f_i^\beta \boldsymbol{e}_i^\beta$ with f_i^β denoting the force load on β along the direction $\boldsymbol{e}_i^\beta$.

We impose the clamped conditions on the line element $\gamma_0 \in \partial\beta_1$:

$$\boldsymbol{u}^{\beta_1} = \boldsymbol{0}, \quad \partial_{\boldsymbol{n}^{\beta_1}} u_3^{\beta_1} = 0 \text{ on } \gamma_0,$$

and impose the force and moment free conditions on all other proper boundaries of Ω. It is noted that all derivations in this paper can be extended naturally to problems with other boundary conditions after some straightforward modifications.

Thus, using the d'Alembert's principle in mechanics, under the action of the time-dependent applied generalized load field $\boldsymbol{f} := \{\boldsymbol{f}^{\beta_1}, \boldsymbol{f}^{\beta_2}\}$, the generalized displacement field $\boldsymbol{u} := \{\boldsymbol{u}^{\beta_1}, \boldsymbol{u}^{\beta_2}\}$ of Ω is governed by the following problem: Find $\boldsymbol{u} : t \in [0, T] \longrightarrow \boldsymbol{u}(t) \in \boldsymbol{V}$ such that

$$\begin{cases} B(\ddot{\boldsymbol{u}}, \boldsymbol{v}) + D(\boldsymbol{u}, \boldsymbol{v}) = F(\boldsymbol{v}), \ \forall \boldsymbol{v} \in \boldsymbol{V}, \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0,\ \dot{\boldsymbol{u}}|_{t=0} = \boldsymbol{u}_1, \end{cases} \tag{2.2}$$

where

$$\boldsymbol{V} := \Big\{\boldsymbol{v} = \{\boldsymbol{v}^{\beta_1},\, \boldsymbol{v}^{\beta_2}\};\, \boldsymbol{v}^{\beta_1} \in (H_0^1(\beta_1;\,\gamma_0))^2 \times H_0^2(\beta_1;\,\gamma_0),$$
$$\boldsymbol{v}^{\beta_2} \in (H^1(\beta_2))^2 \times H^2(\beta_2),\, \boldsymbol{v} \text{ satisfies } (2.1)\Big\},$$

$$H_0^1(\beta_1;\,\gamma_0) := \left\{v \in H^1(\beta_1);\, v = 0 \ \text{ on } \gamma_0\right\},$$
$$H_0^2(\beta_1;\,\gamma_0) := \left\{v \in H^2(\beta_1);\, v = \partial_{\boldsymbol{n}^{\beta_1}} v = 0 \text{ on } \gamma_0\right\},$$

$\boldsymbol{u}_0,\, \boldsymbol{u}_1 \in \boldsymbol{V}$ denote the initial generalized displacement and velocity fields, and for $\boldsymbol{w} = \{\boldsymbol{w}^{\beta_1},\, \boldsymbol{w}^{\beta_2}\}$,

$$F(\boldsymbol{w}) := \sum_{\beta\in\Omega} \int_\beta \boldsymbol{f}^\beta \cdot \boldsymbol{w}^\beta \,\mathrm{d}\beta,$$

$$B(\boldsymbol{v},\, \boldsymbol{w}) := \sum_{\beta\in\Omega} \int_\beta \rho_{\beta} t_{\beta} \boldsymbol{v}^\beta \cdot \boldsymbol{w}^\beta \,\mathrm{d}\beta, \tag{2.3}$$

$$D(\boldsymbol{v},\, \boldsymbol{w}) := \sum_{\beta\in\Omega} \int_\beta \Big(\mathcal{Q}_{IJ}^\beta(\boldsymbol{v})\varepsilon_{IJ}^\beta(\boldsymbol{w}) + \mathcal{M}_{IJ}^\beta(\boldsymbol{v})\mathcal{K}_{IJ}^\beta(\boldsymbol{w})\Big) \,\mathrm{d}\beta, \tag{2.4}$$

$$\varepsilon_{IJ}^\beta(\boldsymbol{v}) := (\partial_I \boldsymbol{v}_J^\beta + \partial_J \boldsymbol{v}_I^\beta)/2, \quad \partial_I \boldsymbol{v}_J^\beta := \boldsymbol{v}_{J,\,I}^\beta = \frac{\partial \boldsymbol{v}_J^\beta}{\partial x_I^\beta},$$

$$\mathcal{Q}_{IJ}^\beta := \frac{E_\beta t_{\beta}}{1-\nu_{\beta}^2}\Big((1-\nu_{\beta})\varepsilon_{IJ}^\beta(\boldsymbol{v}) + \nu_{\beta}\big(\varepsilon_{LL}^\beta(\boldsymbol{v})\big)\delta_{IJ}\Big),\ 1 \le I,\, J \le 2,$$

$$\mathcal{K}_{IJ}^\beta(\boldsymbol{v}) := -\partial_{IJ} v_3^\beta = -\frac{\partial^2 v_3^\beta}{\partial x_I^\beta \partial x_J^\beta},$$

$$\mathcal{M}_{IJ}^\beta(\boldsymbol{v}) := \frac{E_\beta t_{\beta}^3}{12(1-\nu_{\beta}^2)}\Big((1-\nu_{\beta})\mathcal{K}_{IJ}^\beta(\boldsymbol{v}) + \nu_{\beta}\big(\mathcal{K}_{LL}^\beta(\boldsymbol{v})\big)\delta_{IJ}\Big).$$

Here $E_\beta > 0$, $\nu_{\beta} \in (0,\, 1/2)$ denote Young's modulus, Poisson's ratio of the elastic plate member β, $\rho_{\beta} > 0$ is the density (mass per unit volume) of the material, t_{β} is the thickness of plate β, and δ_{IJ} stands for the usual Kronecker delta.

Equip the function space $\boldsymbol{V}$ with a norm $\|\cdot\|_{\boldsymbol{V}}$ given by

$$\|\boldsymbol{v}\|_{\boldsymbol{V}} := \left\{\sum_{\beta\in\Omega}\left(\sum_{I=1}^2 \|v_I^\beta\|_{1,\,\beta}^2 + \|v_3^\beta\|_{2,\,\beta}^2\right)\right\}^{1/2},\ \forall\, \boldsymbol{v} \in \boldsymbol{V},$$

and define

$$\|\boldsymbol{v}\|_0 = \left\{\sum_{\beta\in\Omega}\sum_{i=1}^3 \|v_i^\beta\|_{0,\,\beta}^2\right\}^{1/2}, \quad \forall\, \boldsymbol{v} = \{\boldsymbol{v}^{\beta_1},\, \boldsymbol{v}^{\beta_2}\} \in \prod_{\beta\in\Omega}(L^2(\beta))^3.$$

It is easy to check that the bilinear form $D(\cdot,\,\cdot)$ is symmetric, bounded, and $\boldsymbol{V}$-coercive [18], so the problem (2.2) has a unique solution with [29]

$$\boldsymbol{u} \in L^2(0,\, T; \boldsymbol{V}), \quad \dot{\boldsymbol{u}} \in L^2\big(0,\, T;\, \prod_{\beta\in\Omega}(L^2(\beta))^3\big),$$

provided that $\boldsymbol{f} \in L^2\big(0, T; \prod_{\beta\in\Omega} (L^2(\beta))^3\big)$. From now on, we will always use $\boldsymbol{u} = \{\boldsymbol{u}^{\beta_1}, \boldsymbol{u}^{\beta_2}\}$ to denote the unique solution of problem (2.2), and assume that for a.e. $t \in [0, T]$,

$$\boldsymbol{u}^\beta \in (H^2(\beta))^2 \times H^3(\beta), \quad \ddot{\boldsymbol{u}}^\beta,\ \boldsymbol{f}^\beta \in (L^2(\beta))^3, \qquad \forall\, \beta \in \Omega. \tag{2.5}$$

2.2. **Finite element spaces.** For each $\beta \in \Omega$, let $\mathcal{T}_h^\beta := \{K^\beta\}$ be a shape-regular triangulation of β, so that we obtain a total triangulation of Ω, $\mathcal{T}_h^\Omega := \{\mathcal{T}_h^{\beta_1}, \mathcal{T}_h^{\beta_2}\}$. For ease of exposition, assume that the mesh sizes of all triangulations for individual elastic members are of the same size h. Moreover, the triangulation $\mathcal{T}_h^\Omega$ matches across interfaces among different geometric elements.

Let $V_h^1(\beta)$ and $V_h^M(\beta)$ be the space of continuous piecewise linear functions and the usual Morley element space associated with the triangulations $\mathcal{T}_h^\beta$, respectively. That means, for each $K^\beta \in \mathcal{T}_h^\beta$, the local shape function space related to $V_h^M(\beta)$ is $P_2(K^\beta)$ equipped with the nodal variables

$$\Sigma_{K^\beta} := \big\{v(p_i^\beta),\ \partial_{\boldsymbol{n}^{K^\beta}} v(m_i^\beta),\ 1 \le i \le 3\big\}.$$

Here and below, $P_k(G)$ stands for the space of all polynomials with total degree no more than k on G, with G being an open set. The symbols $\boldsymbol{n}^{K^\beta}$ and $\boldsymbol{t}^{K^\beta}$ denote the unit outward normal and tangent direction on ∂K^β respectively, such that $\big\{\boldsymbol{n}^{K^\beta}, \boldsymbol{t}^{K^\beta}, \boldsymbol{e}_3^\beta\big\}$ forms a right-handed coordinate system. The symbol p (resp. m) with or without indices is used to denote a vertex (resp. the midpoint of a side) of some individual element of a triangulation. For an area element $\beta \in \Omega$ (resp. a line element $\gamma \in \Gamma^1$), $p \in \beta$ (resp. $p \in \gamma$) means that $p \in \bar{\beta}$ (resp. $p \in \bar{\gamma}$) is a vertex of some individual element of a triangulation; similarly, $m \in \gamma$ means that $m \in \bar{\gamma}$ is an edge midpoint of some individual element of a triangulation.

The following finite element spaces are then introduced to describe displacement fields on individual elastic members.

$$\begin{aligned} \boldsymbol{W}_h(\beta_2) &:= (V_h^1(\beta_2))^2 \times V_h^M(\beta_2), \\ \boldsymbol{W}_h(\beta_1) &:= (V_h^1(\beta_1;\, \gamma_0))^2 \times V_h^M(\beta_1;\, \gamma_0), \end{aligned}$$

where

$$\begin{aligned} V_h^1(\beta_1;\, \gamma_0) &:= \big\{v_h \in V_h^1(\beta_1);\ v_h(p) = 0,\ \forall\, p \in \gamma_0\big\}, \\ V_h^M(\beta_1;\, \gamma_0) &:= \big\{v_h \in V_h^M(\beta_1);\ v_h(p) = 0,\ \partial_{\boldsymbol{n}^{\beta_1}} v_h(m) = 0,\ \forall\, p, m \in \gamma_0\big\}. \end{aligned}$$

The discrete rigid conditions related to (2.1) are given below.

$$\begin{aligned} v_i^{\beta_1}(p)\boldsymbol{e}_i^{\beta_1} &= v_i^{\beta_2}(p)\boldsymbol{e}_i^{\beta_2}, \quad \forall\, p \in \gamma_1, \\ \varepsilon(\beta_1, \gamma_1)\partial_{\boldsymbol{n}^{\beta_1}} v_3^{\beta_1}(m) &= \varepsilon(\beta_2, \gamma_1)\partial_{\boldsymbol{n}^{\beta_2}} v_3^{\beta_2}(m), \quad \forall\, m \in \gamma_1. \end{aligned} \tag{2.6}$$

With these preparations, we get a total finite element space on Ω as follows.

$$\boldsymbol{V}_h := \big\{\boldsymbol{v} \in \boldsymbol{W}_h(\beta_1) \times \boldsymbol{W}_h(\beta_2);\ \boldsymbol{v} \text{ satisfies } (2.6)\big\}.$$

Equip the finite element space $\boldsymbol{V}_h$ with a norm:

$$\|\boldsymbol{v}_h\|_h := \left\{\sum_{\beta\in\Omega} \sum_{K^\beta \in \mathcal{T}_h^\beta} \left(\sum_{I=1}^{2} |v_{h,I}^\beta|_{1,K^\beta}^2 + |v_{h,3}^\beta|_{2,K^\beta}^2\right)\right\}^{1/2},$$

for each $\boldsymbol{v}_h = \{\boldsymbol{v}_h^{\beta_1}, \boldsymbol{v}_h^{\beta_2}\} \in \boldsymbol{V}_h$. The notation can be extended to functions $\boldsymbol{v}$ which are piecewise smooth with respect to the triangulation to $\mathcal{T}_h^\Omega$. In addition, define

$$\|\!| \boldsymbol{v} \|\!|_h := \left\{ \sum_{\beta\in\Omega} \sum_{K^\beta\in\mathcal{T}_h^\beta} \left(\sum_{I=1}^{2} \|v_I^\beta\|_{1,\,K^\beta}^2 + \|v_3^\beta\|_{2,\,K^\beta}^2 \right) \right\}^{1/2}. \tag{2.7}$$

2.3. **Fundamental results.** The following results are contained in [19].

$$\|\boldsymbol{v}_h\|_h^2 \lesssim D_h(\boldsymbol{v}_h,\, \boldsymbol{v}_h), \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.8}$$

where for $\boldsymbol{v}_h = \{\boldsymbol{v}_h^{\beta_1}, \boldsymbol{v}_h^{\beta_2}\}$, $\boldsymbol{w}_h = \{\boldsymbol{w}_h^{\beta_1}, \boldsymbol{w}_h^{\beta_2}\}$,

$$D_h(\boldsymbol{v}_h,\, \boldsymbol{w}_h) := \sum_{\beta\in\Omega} \sum_{K^\beta\in\mathcal{T}_h^\beta} \int_{K^\beta} \left(\mathcal{Q}_{IJ}^\beta(\boldsymbol{v}_h)\varepsilon_{IJ}^\beta(\boldsymbol{w}_h) + \mathcal{M}_{IJ}^\beta(\boldsymbol{v}_h)\mathcal{K}_{IJ}^\beta(\boldsymbol{w}_h) \right) \mathrm{d}K^\beta; \tag{2.9}$$

For $K^\beta \in \mathcal{T}_h^\Omega$,

$$\int_{K^\beta} \mathcal{Q}_{IJ}^\beta(\boldsymbol{u})\varepsilon_{IJ}^\beta(\boldsymbol{v}_h)\,\mathrm{d}K^\beta = -\int_{K^\beta} \mathcal{Q}_{IJ,\,J}^\beta(\boldsymbol{u}) v_{h,\,I}^\beta\,\mathrm{d}K^\beta + \int_{\partial K^\beta} \mathcal{Q}_{IJ}^\beta(\boldsymbol{u}) n_J^{K^\beta} v_{h,\,I}^\beta\,\mathrm{d}s^\beta, \tag{2.10}$$

$$\begin{aligned}\int_{K^\beta} \mathcal{M}_{IJ}^\beta(\boldsymbol{u})\mathcal{K}_{IJ}^\beta(\boldsymbol{v}_h)\,\mathrm{d}K^\beta = &\int_{K^\beta} \mathcal{M}_{IJ,\,J}^\beta(\boldsymbol{u})\partial_I v_{h,\,3}^\beta\,\mathrm{d}K^\beta \\ &- \int_{\partial K^\beta} \Big(\mathcal{M}_{\boldsymbol{nn}}^{K^\beta}(\boldsymbol{u})\partial_{\boldsymbol{n}^{K^\beta}} v_{h,\,3}^\beta \\ &+ \mathcal{M}_{\boldsymbol{nt}}^{K^\beta}(\boldsymbol{u})\partial_{\boldsymbol{t}^{K^\beta}} v_{h,\,3}^\beta \Big)\,\mathrm{d}s^\beta,\end{aligned} \tag{2.11}$$

where

$$\boldsymbol{n}^{K^\beta} := n_I^{K^\beta}\boldsymbol{e}_I^\beta, \quad \boldsymbol{t}^{K^\beta} := t_I^{K^\beta}\boldsymbol{e}_I^\beta,$$

$$\mathcal{M}_{\boldsymbol{nn}}^{K^\beta}(\boldsymbol{u}) := \mathcal{M}_{IJ}^\beta(\boldsymbol{u}) n_I^{K^\beta} n_J^{K^\beta}, \quad \mathcal{M}_{\boldsymbol{nt}}^{K^\beta}(\boldsymbol{u}) := \mathcal{M}_{IJ}^\beta(\boldsymbol{u}) n_I^{K^\beta} t_J^{K^\beta}.$$

For each $\beta \in \Omega$, let $I_{1,\,h}^\beta$ and $I_{M,\,h}^\beta$ be the interpolation operators related to $V_h^1(\beta)$ and $V_h^M(\beta)$, respectively. Then, define the interpolation operator $\boldsymbol{I}_h^\beta$ by

$$\boldsymbol{I}_h^\beta \boldsymbol{v}^\beta := (I_{1,\,h}^\beta v_I^\beta)\boldsymbol{e}_I^\beta + (I_{M,\,h}^\beta v_3^\beta)\boldsymbol{e}_3^\beta, \quad \forall\, \boldsymbol{v}^\beta \in (H^2(\beta))^2 \times H^3(\beta),$$

which induces a global interpolation operator $\boldsymbol{I}_h$ on Ω below.

$$(\boldsymbol{I}_h\boldsymbol{v})^\beta := \boldsymbol{I}_h^\beta\boldsymbol{v}^\beta \text{ on } \beta,\ \forall\, \boldsymbol{v}^\beta \in (H^2(\beta))^2 \times H^3(\beta),\ \forall\, \beta \in \Omega.$$

From error estimates for interpolation operators $I_{1,\,h}^\beta$ and $I_{M,\,h}^\beta$ [9, 10], we have

$$\|\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}\|_0 + h\|\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}\|_h \lesssim h^2 \left\{ \sum_{\beta\in\Omega} \left(\sum_{I=1}^{2} |u_I^\beta|_{2,\,\beta}^2 + |u_3^\beta|_{3,\,\beta}^2 \right) \right\}^{1/2}. \tag{2.12}$$

Next, introduce an elliptic projection operator $\boldsymbol{R}_h : \boldsymbol{V} \longrightarrow \boldsymbol{V}_h$ such that for each $\boldsymbol{v} \in \boldsymbol{V}$, $\boldsymbol{R}_h\boldsymbol{v} \in \boldsymbol{V}_h$ with

$$D_h(\boldsymbol{v} - \boldsymbol{R}_h\boldsymbol{v},\, \boldsymbol{w}_h) = 0, \quad \forall\, \boldsymbol{w}_h \in \boldsymbol{V}_h. \tag{2.13}$$

The well-definiteness of $\boldsymbol{R}_h\boldsymbol{v}$ can be guaranteed by the Lax-Milgram lemma [10] and (2.8)-(2.9). Moreover, we derive from (2.8) and (2.13) that

$$\begin{aligned}\|\boldsymbol{R}_h\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u}\|_h^2 &\lesssim D_h(\boldsymbol{R}_h\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u},\, \boldsymbol{R}_h\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u})\\ &\lesssim \|\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u}\|_h\|\boldsymbol{R}_h\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u}\|_h,\end{aligned}$$

i.e.,

$$\|\boldsymbol{R}_h\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u}\|_h \lesssim \|\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u}\|_h,$$

so

$$\|\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u}\|_h \le \|\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u}\|_h + \|\boldsymbol{R}_h\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u}\|_h \lesssim \|\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u}\|_h,$$

which together with (2.12) implies the following lemma readily.

Lemma 2.1. *Let $\boldsymbol{R}_h$ be an elliptic projection operator defined by (2.13). Then*

$$\|\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u}\|_h \lesssim h\left\{\sum_{\beta\in\Omega}\left(\sum_{I=1}^{2}|u_I^\beta|_{2,\,\beta}^2 + |u_3^\beta|_{3,\,\beta}^2\right)\right\}^{1/2}.$$

Lemma 2.2. *There holds the discrete Poincaré-Friedrichs inequality:*

$$\|\boldsymbol{v}_h\|_0 \lesssim \|\boldsymbol{v}_h\|_h, \qquad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

Proof. Denote by $V_h^{AR}(\beta)$ the Argyris element space associated with the triangulation $\mathcal{T}_h^\beta$; that is, for each $K^\beta \in \mathcal{T}_h^\beta$ with $\{p_i^\beta\}_{i=1}^3$ and $\{m_i^\beta\}_{i=1}^3$ as three vertices and midpoints of sides respectively, the shape function space is $P_5(K^\beta)$ equipped with the nodal variables

$$\begin{aligned}\Sigma_{K^\beta} := \big\{&v(p_i^\beta),\, \partial_1 v(p_i^\beta),\, \partial_2 v(p_i^\beta),\, \partial_{11} v(p_i^\beta),\, \partial_{12} v(p_i^\beta),\\ &\partial_{22} v(p_i^\beta),\, \partial_{\boldsymbol{n}^{K^\beta}} v(m_i^\beta),\, 1\le i\le 3\big\}.\end{aligned}$$

Here and below, the corresponding partial derivatives are associated with the local coordinate system involved, e.g., $\partial_1 v(p_i^\beta) := \partial_{x_1^\beta} v(p_i^\beta)$ in the present situation.

Next, introduce a connection operator E_h^β from $V_h^M(\beta)$ into $V_h^{AR}(\beta)$ as follows [8, 26]. For each $v_3^\beta \in V_h^M(\beta)$, $E_h^\beta v_3^\beta$ is uniquely determined by

$$\begin{cases}(E_h^\beta v_3^\beta)(p) = v_3^\beta(p), & \forall\, p\in\beta,\\ (\partial_I E_h^\beta v_3^\beta)(p) = (\partial_I v_3^\beta)(e_p), & 1\le I\le 2,\, \forall\, p\in\beta,\\ (\partial_{IJ} E_h^\beta v_3^\beta)(p) = 0, & 1\le I,\, J\le 2,\, \forall\, p\in\beta,\\ (\partial_{\boldsymbol{n}^{K^\beta}} E_h^\beta v_3^\beta)(m) = (\partial_{\boldsymbol{n}^{K^\beta}} v_3^\beta)(m), & \forall\, m\in\beta,\end{cases}$$

where e_p is a midpoint of an edge of $\mathcal{T}_h^\beta$ with $p\in\beta$ as one vertex. We remark that there is some freedom for the choice of e_p. For our purpose here, we assume that the related e_p should belong to $\bar\gamma_0$ if $p\in\bar\gamma_0$. For the connection operator E_h^β, it holds that [8, 26], for each $K^\beta\in\mathcal{T}_h^\beta$,

$$\sum_{k=0}^{2} h_{K^\beta}^{2k}|v_3^\beta - E_h^\beta v_3^\beta|_{k,\,K^\beta}^2 \lesssim h_{K^\beta}^4 \sum_{K^\beta\in\mathcal{T}_h^\beta}|v_3^\beta|_{2,\,\tilde K^\beta}^2, \quad \forall\, v_3^\beta\in V_h^M(\beta), \tag{2.14}$$

where $h_{K^\beta} := \mathrm{diam}(K^\beta)$, and $\tilde K^\beta$ is a macro-element formed by all triangles in $\mathcal{T}_h^\beta$ which are adjacent to K^β.

For each $\boldsymbol{v}_h \in \boldsymbol{V}_h$, choose a function $\boldsymbol{w}_h := \{\boldsymbol{w}_h^{\beta_1}, \boldsymbol{w}_h^{\beta_2}\}$ in the form

$$\boldsymbol{w}_h^{\beta} = v_{h,\,I}^{\beta}\boldsymbol{e}_I^{\beta} + (E_h^{\beta} v_{h,\,3}^{\beta})\boldsymbol{e}_3^{\beta}, \quad \forall\, \beta \in \Omega. \tag{2.15}$$

Then, it follows from (2.7) and Lemmas 3.1 and 3.2 in [19] that

$$\begin{aligned}|||\boldsymbol{w}_h|||_h^2 &\lesssim \sum_{\beta\in\Omega}\sum_{I,\,J=1}^{2}\left(\|\varepsilon_{IJ}^{\beta}(\boldsymbol{w}_h)\|_{0,\,\beta}^2 + \|\mathcal{K}_{IJ}^{\beta}(\boldsymbol{w}_h)\|_{0,\,\beta}^2\right)\\ &\lesssim \sum_{\beta\in\Omega}\sum_{I,\,J=1}^{2}\left(\|\varepsilon_{IJ}^{\beta}(\boldsymbol{v}_h)\|_{0,\,\beta}^2 + \|\mathcal{K}_{IJ}^{\beta}(\boldsymbol{v}_h)\|_{0,\,\beta}^2\right).\end{aligned}$$

Combining this with (2.14)-(2.15) yields

$$\begin{aligned}\|\boldsymbol{v}_h\|_0 &\le |||\boldsymbol{v}_h|||_h\\ &\lesssim |||\boldsymbol{v}_h - \boldsymbol{w}_h\,|||_h + |||\,\boldsymbol{w}_h|||_h\\ &\lesssim \left\{\sum_{\beta\in\Omega}\sum_{K^{\beta}\in\mathcal{T}_h^{\beta}}\left(\sum_{I=1}^{2}|v_{h,\,I}^{\beta}|_{1,\,K^{\beta}}^2 + |v_{h,\,3}^{\beta}|_{2,\,K^{\beta}}^2\right)\right\}^{1/2}\\ &\lesssim \|\boldsymbol{v}_h\|_h,\end{aligned}$$

as required. □

Applying the similar arguments for deriving Theorems 3.1 and 4.1, and Lemma 3.1 in [18] , we can get the corresponding equilibrium equations and two important identities from (2.2), described as follows.

Lemma 2.3. *Let $\boldsymbol{u}$ be the solution of problem (2.2) which satisfies the regularity assumption (2.5). Then*

$$\rho_{\beta} t_{\beta}\ddot{\boldsymbol{u}}_i^{\beta}\boldsymbol{e}_i^{\beta} - \mathcal{Q}_{IJ,\,J}^{\beta}(\boldsymbol{u})\boldsymbol{e}_I^{\beta} - \mathcal{M}_{IJ,\,IJ}^{\beta}(\boldsymbol{u})\boldsymbol{e}_3^{\beta} = \boldsymbol{f}_i^{\beta}\boldsymbol{e}_i^{\beta} \;\text{ in } (L^2(\beta))^3,\; \beta = \beta_1,\, \beta_2, \tag{2.16}$$

$$\mathcal{M}_{\boldsymbol{nn}}^{\beta}(\boldsymbol{u}) = 0 \;\text{ in }\; H^{1/2}(\gamma), \quad \forall\, \gamma \in \partial\beta\backslash(\gamma_0\cup\gamma_1), \quad \beta = \beta_1,\, \beta_2, \tag{2.17}$$

$$\varepsilon(\beta_1,\,\gamma_1)\mathcal{M}_{\boldsymbol{nn}}^{\beta_1}(\boldsymbol{u}) + \varepsilon(\beta_2,\,\gamma_1)\mathcal{M}_{\boldsymbol{nn}}^{\beta_2}(\boldsymbol{u}) = 0 \;\text{ in }\; H^{1/2}(\gamma_1), \tag{2.18}$$

where $\boldsymbol{n}^{\beta} := n_I^{\beta}\boldsymbol{e}_I^{\beta}$ and $\mathcal{M}_{\boldsymbol{nn}}^{\beta}(\boldsymbol{u}) := \mathcal{M}_{IJ}^{\beta}(\boldsymbol{u})n_I^{\beta}n_J^{\beta}$.

Lemma 2.4. *Let $\boldsymbol{u}$ be the solution of problem (2.2) which satisfies the regularity assumption (2.5). Then for each $\boldsymbol{v}^{\beta} \in (H^1(\beta))^3$,*

$$\begin{aligned}\int_{\beta}\rho_{\beta} t_{\beta}\ddot{u}_i^{\beta}v_i^{\beta}\,\mathrm{d}\beta - \int_{\beta}\mathcal{Q}_{IJ,\,J}^{\beta}(\boldsymbol{u})v_I^{\beta}\,\mathrm{d}\beta + \int_{\beta}\mathcal{M}_{IJ,\,J}^{\beta}(\boldsymbol{u})\partial_I v_3^{\beta}\,\mathrm{d}\beta - \int_{\beta} f_i^{\beta}v_i^{\beta}\,\mathrm{d}\beta\\ = \langle\mathcal{M}_{IJ,\,J}^{\beta}(\boldsymbol{u})n_I^{\beta},\, v_3^{\beta}\rangle_{H^{-1/2}(\partial\beta)\times H^{1/2}(\partial\beta)}, \quad \beta = \beta_1,\, \beta_2.\end{aligned}$$

Lemma 2.5. *Let $\boldsymbol{u}$ be the solution of problem (2.2). Assume that the regularity assumption (2.5) holds true. Then for each $\boldsymbol{v}^{\partial\Omega} \in \boldsymbol{H}(\partial\Omega)$,*

$$\begin{aligned}\sum_{\beta\in\Omega}\Bigg\{&\langle\mathcal{M}_{IJ,\,J}^{\beta}(\boldsymbol{u})n_I^{\beta},\, v_3^{\partial\beta}\rangle_{H^{-1/2}(\partial\beta)\times H^{1/2}(\partial\beta)}\\ &+ \sum_{\gamma\in\partial\beta}\int_{\gamma}\left(\mathcal{Q}_{IJ}^{\beta}(\boldsymbol{u})n_J^{\beta}v_I^{\partial\beta} - \mathcal{M}_{\boldsymbol{nt}}^{\beta}(\boldsymbol{u})\partial_{\boldsymbol{t}^{\beta}}v_3^{\partial\beta}\right)\mathrm{d}\gamma\Bigg\} = 0,\end{aligned}$$

where $\boldsymbol{t}^\beta := t_I^\beta \boldsymbol{e}_I^\beta$, $\mathcal{M}^\beta_{\boldsymbol{nt}}(\boldsymbol{u}) := \mathcal{M}^\beta_{IJ}(\boldsymbol{u}) n_I^\beta t_J^\beta$, *and*

$$\boldsymbol{H}(\partial\Omega) := \Big\{ \boldsymbol{v}^{\partial\Omega} = \big\{\boldsymbol{v}^{\partial\beta_1},\, \boldsymbol{v}^{\partial\beta_2}\big\} \in \prod_{\beta\in\Omega} (H(\partial\beta))^3;\; v_i^{\partial\beta_1}\boldsymbol{e}_i^{\beta_1} = v_i^{\partial\beta_2}\boldsymbol{e}_i^{\beta_2} \text{ on } \gamma_1 \Big\}$$

with

$$H(\partial\beta_1) := \big\{ v \in H^1(\partial\beta_1);\; v = 0 \text{ on } \gamma_0 \big\}, \quad H(\partial\beta_2) := H^1(\partial\beta_2).$$

3. **The semi-discrete finite element method and error analysis.** The semi-discrete finite element method for problem (2.2) is to find $\boldsymbol{u}_h(t) : [0, T] \longrightarrow \boldsymbol{V}_h$ such that

$$\begin{cases} B(\ddot{\boldsymbol{u}}_h,\, \boldsymbol{v}_h) + D_h(\boldsymbol{u}_h,\, \boldsymbol{v}_h) = (\boldsymbol{f},\, \boldsymbol{v}_h), \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ \boldsymbol{u}_h(0) = \boldsymbol{u}_{0h}, \quad \dot{\boldsymbol{u}}_h(0) = \boldsymbol{u}_{1h}, \end{cases} \tag{3.1}$$

where

$$(\boldsymbol{f},\, \boldsymbol{v}_h) := \sum_{\beta\in\Omega} \int_\beta \boldsymbol{f}^\beta \cdot \boldsymbol{v}_h^\beta \,\mathrm{d}\beta,$$

and $\boldsymbol{u}_{0h}$ and $\boldsymbol{u}_{1h}$ are two approximate functions of $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$, respectively.

Theorem 3.1. *Let $\boldsymbol{u}$ and $\boldsymbol{u}_h$ be the solutions of problems (2.2) and (3.1), respectively. Assume that*

$$\boldsymbol{u} \in L^\infty\big(0, T;\, \prod_{\beta\in\Omega} ((H^2(\beta))^2 \times H^3(\beta))\big),$$

$$\boldsymbol{u}_{tt} \in L^2\big(0, T;\, \prod_{\beta\in\Omega} ((H^2(\beta))^2 \times H^3(\beta))\big) \bigcap L^\infty\big(0, T;\, \prod_{\beta\in\Omega} (L^2(\beta))^3\big),$$

$$\boldsymbol{u}_{ttt},\, \boldsymbol{f},\, \boldsymbol{f}_t \in L^2\big(0, T;\, \prod_{\beta\in\Omega} (L^2(\beta))^3\big),$$

$$f_3^\beta \in L^\infty\big(0, T;\, L^2(\beta)\big), \quad \forall\, \beta \in \Omega.$$

Then

$$\begin{aligned} \max_{0\le t\le T} \|\boldsymbol{u}(t) - \boldsymbol{u}_h(t)\|_h \lesssim{}& \|\boldsymbol{u}_0 - \boldsymbol{u}_{0h}\|_h + \|\boldsymbol{u}_1 - \boldsymbol{u}_{1h}\|_h \\ & + h\, |\boldsymbol{u}|_{L^\infty\left(0, T; \prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)} \\ & + h\, |\boldsymbol{u}_{tt}|_{L^2\left(0, T; \prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)} \\ & + h\, \|\boldsymbol{u}_{tt}\|_{L^\infty\left(0, T; \prod_{\beta\in\Omega}(L^2(\beta))^3\right)} \\ & + h\, \|\boldsymbol{u}_{ttt}\|_{L^2\left(0, T; \prod_{\beta\in\Omega}(L^2(\beta))^3\right)} \\ & + h^2 \sum_{\beta\in\Omega} \Big(\|f_3^\beta\|_{L^\infty(0, T;\, L^2(\beta))} + \|\dot{f}_3^\beta\|_{L^2(0, T;\, L^2(\beta))} \Big). \end{aligned} \tag{3.2}$$

In particular, if we select $\boldsymbol{u}_{0h} = \boldsymbol{R}_h\boldsymbol{u}_0$ *or* $\boldsymbol{I}_h\boldsymbol{u}_0$, $\boldsymbol{u}_{1h} = \boldsymbol{R}_h\boldsymbol{u}_1$ *or* $\boldsymbol{I}_h\boldsymbol{u}_1$, *then*

$$\begin{aligned}\max_{0\le t\le T}\|\boldsymbol{u}(t)-\boldsymbol{u}_h(t)\|_h \lesssim\, & h\,|\boldsymbol{u}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}\\ &+h\,|\boldsymbol{u}_{tt}|_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}\\ &+h\,\|\boldsymbol{u}_{tt}\|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)}\\ &+h\,\|\boldsymbol{u}_{ttt}\|_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)}\\ &+h^2\sum_{\beta\in\Omega}\left(\|f_3^\beta\|_{L^\infty(0,\,T;\,L^2(\beta))}+\|\dot f_3^\beta\|_{L^2(0,\,T;\,L^2(\beta))}\right). \end{aligned}\tag{3.3}$$

Proof. Decompose the error as

$$\boldsymbol{u}-\boldsymbol{u}_h = (\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u}) + (\boldsymbol{R}_h\boldsymbol{u}-\boldsymbol{u}_h) =: \boldsymbol{\rho}+\boldsymbol{\theta}. \tag{3.4}$$

For each $\boldsymbol{v}_h\in\boldsymbol{V}_h$, from (3.4), (3.1), and (2.13), it follows that

$$\begin{aligned}B(\ddot{\boldsymbol{\theta}},\,\boldsymbol{v}_h) &= B\big((\boldsymbol{R}_h\boldsymbol{u}-\boldsymbol{u}_h)_{tt},\,\boldsymbol{v}_h\big)\\ &= -B(\ddot{\boldsymbol{\rho}},\,\boldsymbol{v}_h)+B(\ddot{\boldsymbol{u}},\,\boldsymbol{v}_h)-B(\ddot{\boldsymbol{u}}_h,\,\boldsymbol{v}_h)\\ &= -B(\ddot{\boldsymbol{\rho}},\,\boldsymbol{v}_h)+B(\ddot{\boldsymbol{u}},\,\boldsymbol{v}_h)+D_h(\boldsymbol{u}_h,\,\boldsymbol{v}_h)-(\boldsymbol{f},\,\boldsymbol{v}_h)\\ &= -B(\ddot{\boldsymbol{\rho}},\,\boldsymbol{v}_h)+B(\ddot{\boldsymbol{u}},\,\boldsymbol{v}_h)-D_h(\boldsymbol{\theta},\,\boldsymbol{v}_h)+D_h(\boldsymbol{u},\,\boldsymbol{v}_h)-(\boldsymbol{f},\,\boldsymbol{v}_h),\end{aligned}$$

so rearrangement of the third term on the right yields

$$\begin{aligned}B(\ddot{\boldsymbol{\theta}},\,\boldsymbol{v}_h)+D_h(\boldsymbol{\theta},\,\boldsymbol{v}_h) =& -B(\ddot{\boldsymbol{\rho}},\,\boldsymbol{v}_h)+B(\ddot{\boldsymbol{u}},\,\boldsymbol{v}_h)\\ &+D_h(\boldsymbol{u},\,\boldsymbol{v}_h)-(\boldsymbol{f},\,\boldsymbol{v}_h),\ \forall\,\boldsymbol{v}_h\in\boldsymbol{V}_h.\end{aligned}\tag{3.5}$$

Choosing $\boldsymbol{v}_h=\dot{\boldsymbol{\theta}}$ in (3.5) we find

$$\begin{aligned}B(\ddot{\boldsymbol{\theta}},\,\dot{\boldsymbol{\theta}})+D_h(\boldsymbol{\theta},\,\dot{\boldsymbol{\theta}}) =& -B(\ddot{\boldsymbol{\rho}},\,\dot{\boldsymbol{\theta}})+B(\ddot{\boldsymbol{u}},\,\dot{\boldsymbol{\theta}})+D_h(\boldsymbol{u},\,\dot{\boldsymbol{\theta}})-(\boldsymbol{f},\,\dot{\boldsymbol{\theta}})\\ =& -B(\ddot{\boldsymbol{\rho}},\,\dot{\boldsymbol{\theta}})+\frac{\mathrm{d}}{\mathrm{d}t}\big(B(\ddot{\boldsymbol{u}},\,\boldsymbol{\theta})+D_h(\boldsymbol{u},\,\boldsymbol{\theta})-(\boldsymbol{f},\,\boldsymbol{\theta})\big)\\ &-\big(B(\boldsymbol{u}_{ttt},\,\boldsymbol{\theta})+D_h(\dot{\boldsymbol{u}},\,\boldsymbol{\theta})-(\dot{\boldsymbol{f}},\,\boldsymbol{\theta})\big),\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(B(\dot{\boldsymbol{\theta}},\,\dot{\boldsymbol{\theta}})+D_h(\boldsymbol{\theta},\,\boldsymbol{\theta})\big) =& -B(\ddot{\boldsymbol{\rho}},\,\dot{\boldsymbol{\theta}})+\frac{\mathrm{d}}{\mathrm{d}t}\big(B(\ddot{\boldsymbol{u}},\,\boldsymbol{\theta})+D_h(\boldsymbol{u},\,\boldsymbol{\theta})-(\boldsymbol{f},\,\boldsymbol{\theta})\big)\\ &-\big(B(\boldsymbol{u}_{ttt},\,\boldsymbol{\theta})+D_h(\dot{\boldsymbol{u}},\,\boldsymbol{\theta})-(\dot{\boldsymbol{f}},\,\boldsymbol{\theta})\big).\end{aligned}$$

Integrating the above equation with respect to the time variable from 0 to t, we obtain

$$\frac{1}{2}\big(B(\dot{\boldsymbol{\theta}},\,\dot{\boldsymbol{\theta}})+D_h(\boldsymbol{\theta},\,\boldsymbol{\theta})\big) = \frac{1}{2}\big(B(\dot{\boldsymbol{\theta}}(0),\,\dot{\boldsymbol{\theta}}(0))+D_h(\boldsymbol{\theta}(0),\,\boldsymbol{\theta}(0))\big)+\sum_{n=1}^{4}I_n, \tag{3.6}$$

where

$$\begin{aligned}I_1 &:= -\int_0^t B(\ddot{\boldsymbol{\rho}},\,\dot{\boldsymbol{\theta}})\,\mathrm{d}s,\\ I_2 &:= B(\ddot{\boldsymbol{u}},\,\boldsymbol{\theta})+D_h(\boldsymbol{u},\,\boldsymbol{\theta})-(\boldsymbol{f},\,\boldsymbol{\theta}),\\ I_3 &:= -\big(B(\ddot{\boldsymbol{u}}(0),\,\boldsymbol{\theta}(0))+D_h(\boldsymbol{u}(0),\,\boldsymbol{\theta}(0))-(\boldsymbol{f}(0),\,\boldsymbol{\theta}(0))\big),\\ I_4 &:= -\int_0^t\big(B(\boldsymbol{u}_{ttt},\,\boldsymbol{\theta})+D_h(\dot{\boldsymbol{u}},\,\boldsymbol{\theta})-(\dot{\boldsymbol{f}},\,\boldsymbol{\theta})\big)\,\mathrm{d}s.\end{aligned}$$

By the Cauchy-Schwarz inequality and the geometric-arithmetic mean inequality,

$$|I_1| \le \int_0^t |B(\ddot{\boldsymbol{\rho}}, \dot{\boldsymbol{\theta}})| \,\mathrm{d}s \lesssim \int_0^t \|\ddot{\boldsymbol{\rho}}\|_0 \|\dot{\boldsymbol{\theta}}\|_0 \,\mathrm{d}s \lesssim \int_0^t \|\ddot{\boldsymbol{\rho}}\|_0^2 \mathrm{d}s + \int_0^t \|\dot{\boldsymbol{\theta}}\|_0^2 \,\mathrm{d}s. \tag{3.7}$$

As in [19], we have by (2.9)-(2.11) that

$$\begin{aligned} I_2 &= \sum_{\beta\in\Omega} \sum_{K^\beta \in \mathcal{T}_h^\beta} \int_{K^\beta} \Big(\mathcal{Q}_{IJ}^\beta(\boldsymbol{u}) \varepsilon_{IJ}^\beta(\boldsymbol{\theta}) + \mathcal{M}_{IJ}^\beta(\boldsymbol{u}) \mathcal{K}_{IJ}^\beta(\boldsymbol{\theta}) \Big) \,\mathrm{d}K^\beta \\ &\quad + B(\ddot{\boldsymbol{u}}, \boldsymbol{\theta}) - \sum_{\beta\in\Omega} \int_\beta f_i^\beta \theta_i^\beta \,\mathrm{d}\beta \\ &=: I_{21} + I_{22} + I_{23}, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} I_{21} &:= -\sum_{\beta\in\Omega} \sum_{K^\beta\in\mathcal{T}_h^\beta} \sum_{F^\beta \subset \partial K^\beta \setminus \partial\beta} \int_{F^\beta} \mathcal{M}_{\boldsymbol{nn}}^{K^\beta}(\boldsymbol{u}) \partial_{\boldsymbol{n}^{K^\beta}} \theta_3^\beta \,\mathrm{d}s^\beta, \\ I_{22} &:= -\sum_{\beta\in\Omega} \sum_{F^\beta \subset \partial\beta} \int_{F^\beta} \mathcal{M}_{\boldsymbol{nn}}^{K^\beta}(\boldsymbol{u}) \partial_{\boldsymbol{n}^{K^\beta}} \theta_3^\beta \,\mathrm{d}s^\beta, \\ I_{23} &:= B(\ddot{\boldsymbol{u}}, \boldsymbol{\theta}) + \sum_{\beta\in\Omega} \int_{\partial\beta} \mathcal{Q}_{IJ}^\beta(\boldsymbol{u}) n_J^\beta \theta_I^\beta \,\mathrm{d}\gamma \\ &\quad + \sum_{\beta\in\Omega} \sum_{K^\beta\in\mathcal{T}_h^\beta} \Big(-\int_{K^\beta} \mathcal{Q}_{IJ,\,J}^\beta(\boldsymbol{u}) \theta_I^\beta \,\mathrm{d}K^\beta \\ &\quad + \int_{K^\beta} \mathcal{M}_{IJ,\,J}^\beta(\boldsymbol{u}) \partial_I \theta_3^\beta \,\mathrm{d}K^\beta - \int_{K^\beta} f_i^\beta \theta_i^\beta \,\mathrm{d}K^\beta \\ &\quad - \sum_{F^\beta \subset \partial K^\beta} \int_{F^\beta} \mathcal{M}_{\boldsymbol{nt}}^{K^\beta}(\boldsymbol{u}) \partial_{\boldsymbol{t}^{K^\beta}} \theta_3^\beta \,\mathrm{d}s^\beta \Big) \end{aligned}$$

with F^β representing an edge of a triangle K^β in $\mathcal{T}_h^\beta$. We can estimate the term I_{21} in the standard way to get [10, 25]

$$|I_{21}| \lesssim h \Big(\sum_{\beta\in\Omega} |u_3^\beta|_{3,\,\beta}^2 \Big)^{1/2} \|\boldsymbol{\theta}\|_h. \tag{3.9}$$

Following [19] and using the equilibrium equations (2.17)-(2.18), we see that

$$|I_{22}| \lesssim h \Big(\sum_{\beta\in\Omega} |u_3^\beta|_{3,\,\beta}^2 \Big)^{1/2} \|\boldsymbol{\theta}\|_h. \tag{3.10}$$

We proceed with the bound of the term I_{23}. We first rewrite it as

$$
\begin{aligned}
I_{23} =& \Big\{ -\sum_{\beta\in\Omega}\sum_{K^\beta\in\mathcal{T}_h^\beta}\sum_{F^\beta\subset\partial K^\beta\setminus\partial\beta}\int_{F^\beta}\mathcal{M}^{K^\beta}_{\boldsymbol{nt}}(\boldsymbol{u})\partial_{\boldsymbol{t}_{K^\beta}}\theta_3^\beta\,\mathrm{d}s^\beta\Big\} \\
&+\Big\{B(\ddot{\mathbf{u}},\,\boldsymbol{\theta})+\sum_{\beta\in\Omega}\sum_{K^\beta\in\mathcal{T}_h^\beta}\Big(-\int_{K^\beta}\mathcal{Q}^\beta_{IJ,\,J}(\boldsymbol{u})\theta_I^\beta\,\mathrm{d}K^\beta \\
&+\int_{K^\beta}\mathcal{M}^\beta_{IJ,\,J}(\boldsymbol{u})\partial_I\theta_3^\beta\,\mathrm{d}K^\beta-\int_{K^\beta}f_i^\beta\theta_i^\beta\,\mathrm{d}K^\beta\Big) \\
&-\sum_{\beta\in\Omega}\sum_{F^\beta\subset\partial\beta}\int_{F^\beta}\mathcal{M}^\beta_{\boldsymbol{nt}}(\boldsymbol{u})\partial_{\boldsymbol{t}^\beta}\theta_3^\beta\,\mathrm{d}s^\beta+\sum_{\beta\in\Omega}\int_{\partial\beta}\mathcal{Q}^\beta_{IJ}(\boldsymbol{u})n_J^\beta\theta_I^\beta\,\mathrm{d}\gamma\Big\} \\
=&:\, I_{231}+I_{232}.
\end{aligned}
\tag{3.11}
$$

The term I_{231} can be estimated in the standard way to get [10, 25]

$$
|I_{231}|\lesssim h\Big(\sum_{\beta\in\Omega}|u_3^\beta|^2_{3,\,\beta}\Big)^{1/2}\|\boldsymbol{\theta}\|_h. \tag{3.12}
$$

For the estimate of term I_{232}, introduce an auxiliary function $\boldsymbol{w}_h=\{\boldsymbol{w}_h^{\beta_1},\,\boldsymbol{w}_h^{\beta_2}\}$ from $\boldsymbol{\theta}\in\boldsymbol{V}_h$ in the form

$$
\boldsymbol{w}_h^\beta:=(I^\beta_{1,\,h}\theta_i^\beta)\boldsymbol{e}_i^\beta,\quad \forall\,\beta\in\Omega.
$$

It is easy to check that $\boldsymbol{w}_h^\beta\in(H^1(\beta))^3$ and its restriction to $\partial\beta_1\cup\partial\beta_2$ lies in $\boldsymbol{H}(\partial\Omega)$, so by Lemmas 2.5 and 2.4,

$$
\begin{aligned}
\sum_{\beta\in\Omega}\Big\{&\langle\mathcal{M}^\beta_{IJ,\,J}(\boldsymbol{u})n_I^\beta,\,w^{\partial\beta}_{h,\,3}\rangle_{H^{-1/2}(\partial\beta)\times H^{1/2}(\partial\beta)} \\
&+\sum_{\gamma\in\partial\beta}\int_\gamma\Big(\mathcal{Q}^\beta_{IJ}(\boldsymbol{u})n_J^\beta w^{\partial\beta}_{h,\,I}-\mathcal{M}^\beta_{\boldsymbol{nt}}(\boldsymbol{u})\partial_{\boldsymbol{t}^\beta}w^{\partial\beta}_{h,\,3}\Big)\,\mathrm{d}\gamma\Big\}=0,
\end{aligned}
$$

$$
\begin{aligned}
\int_\beta\rho_\beta t_\beta\ddot{u}_i^\beta w^\beta_{h,\,i}\,\mathrm{d}\beta-\int_\beta\mathcal{Q}^\beta_{IJ,\,J}(\boldsymbol{u})w^\beta_{h,\,I}\,\mathrm{d}\beta+\int_\beta\mathcal{M}^\beta_{IJ,\,J}(\boldsymbol{u})\partial_I w^\beta_{h,\,3}\,\mathrm{d}\beta-\int_\beta f_i^\beta w^\beta_{h,\,i}\,\mathrm{d}\beta \\
=\langle\mathcal{M}^\beta_{IJ,\,J}(\boldsymbol{u})n_I^\beta,\,w^\beta_{h,\,3}\rangle_{H^{-1/2}(\partial\beta)\times H^{1/2}(\partial\beta)}.
\end{aligned}
$$

The combination of the last two identities leads to

$$
\begin{aligned}
\sum_{\beta\in\Omega}\Big\{&\int_\beta\rho_\beta t_\beta\ddot{u}_i^\beta w^\beta_{h,\,i}\,\mathrm{d}\beta-\int_\beta\mathcal{Q}^\beta_{IJ,\,J}(\boldsymbol{u})w^\beta_{h,\,I}\,\mathrm{d}\beta+\int_\beta\mathcal{M}^\beta_{IJ,\,J}(\boldsymbol{u})\partial_I w^\beta_{h,\,3}\,\mathrm{d}\beta \\
&-\int_\beta f_i^\beta w^\beta_{h,\,i}\,\mathrm{d}\beta+\sum_{\gamma\in\partial\beta}\int_\gamma\Big(\mathcal{Q}^\beta_{IJ}(\boldsymbol{u})n_J^\beta w^{\partial\beta}_{h,\,I}-\mathcal{M}^\beta_{\boldsymbol{nt}}(\boldsymbol{u})\partial_{\boldsymbol{t}^\beta}w^{\partial\beta}_{h,\,3}\Big)\,\mathrm{d}\gamma\Big\}=0.
\end{aligned}
\tag{3.13}
$$

On the other hand, it is clear that

$$
w^{\partial\beta}_{h,\,I}=\theta_I^\beta,\ w^{\partial\beta}_{h,\,3}=I^\beta_{1,\,h}\theta_3^\beta\ \text{ on } \partial\beta,\ I=1,\,2,
$$

and hence subtracting the equation (3.13) from I_{232} shows that

$$\begin{aligned} I_{232} &= B(\ddot{\boldsymbol{u}}, \boldsymbol{\theta} - \boldsymbol{w}_h) \\ &\quad + \Big\{ \sum_{\beta\in\Omega} \sum_{K^\beta \in \mathcal{T}_h^\beta} \Big(\int_{K^\beta} \mathcal{M}^\beta_{IJ,\,J}(\boldsymbol{u}) \partial_I (\theta_3^\beta - I^\beta_{1,\,h}\theta_3^\beta)\,\mathrm{d}K^\beta \\ &\quad - \int_{K^\beta} f_3^\beta (\theta_3^\beta - I^\beta_{1,\,h}\theta_3^\beta)\,\mathrm{d}K^\beta \Big) \Big\} \\ &\quad + \Big\{ -\sum_{\beta\in\Omega} \sum_{F^\beta\subset\partial\beta} \int_{F^\beta} \mathcal{M}^\beta_{\boldsymbol{nt}}(\boldsymbol{u}) \partial_{\boldsymbol{t}^\beta} (\theta_3^\beta - I^\beta_{1,\,h}\theta_3^\beta)\,\mathrm{d}s^\beta \Big\} \\ &=: I_{2321} + I_{2322} + I_{2323}. \end{aligned} \tag{3.14}$$

Applying the error estimate for the interpolation operators $I^\beta_{1,\,h}$, we have

$$|I_{2321}| = |B(\ddot{\boldsymbol{u}}, \boldsymbol{\theta} - \boldsymbol{w}_h)| \lesssim h\|\ddot{\boldsymbol{u}}\|_0 \|\boldsymbol{\theta}\|_h, \tag{3.15}$$

$$|I_{2322}| \lesssim h\Big\{ \sum_{\beta\in\Omega} (|u_3^\beta|^2_{3,\,\beta} + h^2\|f_3^\beta\|^2_{0,\,\beta}) \Big\}^{1/2} \|\boldsymbol{\theta}\|_h. \tag{3.16}$$

Following [19], we easily know

$$|I_{2323}| \lesssim h\Big(\sum_{\beta\in\Omega} |u_3^\beta|^2_{3,\,\beta} \Big)^{1/2} \|\boldsymbol{\theta}\|_h. \tag{3.17}$$

From (3.8)-(3.12), (3.14)-(3.17) and using the ε-inequality [27], we see that

$$|I_2| \le C\Big(h^2 \sum_{\beta\in\Omega} |u_3^\beta|^2_{3,\,\beta} + h^2\|\ddot{\boldsymbol{u}}\|_0^2 + h^4 \sum_{\beta\in\Omega} \|f_3^\beta\|^2_{0,\,\beta} \Big) + \varepsilon\|\boldsymbol{\theta}\|_h^2, \tag{3.18}$$

where $\varepsilon > 0$ is some sufficiently small constant as wished.

Similarly, we can obtain the following two estimates.

$$|I_3| \lesssim h^2 \sum_{\beta\in\Omega} |u_3^\beta(0)|^2_{3,\,\beta} + h^2\|\ddot{\boldsymbol{u}}(0)\|_0^2 + h^4 \sum_{\beta\in\Omega} \|f_3^\beta(0)\|^2_{0,\,\beta} + \|\boldsymbol{\theta}(0)\|_h^2, \tag{3.19}$$

$$|I_4| \lesssim \int_0^t \Big(h^2 \sum_{\beta\in\Omega} |\dot{u}_3^\beta|^2_{3,\,\beta} + h^2\|\boldsymbol{u}_{ttt}\|_0^2 + h^4 \sum_{\beta\in\Omega} \|\dot{f}_3^\beta\|^2_{0,\,\beta} \Big)\,\mathrm{d}s + \int_0^t \|\boldsymbol{\theta}\|_h^2\,\mathrm{d}s. \tag{3.20}$$

Plugging (3.7), (3.18)-(3.20) into (3.6) and using the usual absorbing technique [27], we find

$$\begin{aligned} \|\dot{\boldsymbol{\theta}}\|_0^2 + \|\boldsymbol{\theta}\|_h^2 &\lesssim \|\dot{\boldsymbol{\theta}}(0)\|_0^2 + \|\boldsymbol{\theta}(0)\|_h^2 + \int_0^t \|\ddot{\boldsymbol{\rho}}\|_0^2\,\mathrm{d}s + h^2 \sum_{\beta\in\Omega} \big(|u_3^\beta|^2_{3,\,\beta} + |u_3^\beta(0)|^2_{3,\,\beta} \big) \\ &\quad + h^4 \sum_{\beta\in\Omega} \big(\|f_3^\beta\|^2_{0,\,\beta} + \|f_3^\beta(0)\|^2_{0,\,\beta} \big) + h^2 \big(\|\ddot{\boldsymbol{u}}\|_0^2 + \|\ddot{\boldsymbol{u}}(0)\|_0^2 \big) \\ &\quad + \int_0^t \Big(h^2 \sum_{\beta\in\Omega} |\dot{u}_3^\beta|^2_{3,\,\beta} + h^2\|\boldsymbol{u}_{ttt}\|_0^2 + h^4 \sum_{\beta\in\Omega} \|\dot{f}_3^\beta\|^2_{0,\,\beta} \Big)\,\mathrm{d}s \\ &\quad + \int_0^t (\|\dot{\boldsymbol{\theta}}\|_0^2 + \|\boldsymbol{\theta}\|_h^2)\,\mathrm{d}s, \end{aligned}$$

which along with the well-known Gronwall's Lemma [29] implies

$$
\begin{aligned}
\|\dot{\boldsymbol{\theta}}\|_0^2 + \|\boldsymbol{\theta}\|_h^2 \lesssim\ & \|\dot{\boldsymbol{\theta}}(0)\|_0^2 + \|\boldsymbol{\theta}(0)\|_h^2 + \int_0^t \|\ddot{\boldsymbol{\rho}}\|_0^2 \, \mathrm{d}s + h^2 \sum_{\beta\in\Omega} \big(|u_3^\beta|_{3,\,\beta}^2 + |u_3^\beta(0)|_{3,\,\beta}^2\big) \\
& + h^4 \sum_{\beta\in\Omega} \big(\|f_3^\beta\|_{0,\,\beta}^2 + \|f_3^\beta(0)\|_{0,\,\beta}^2\big) + h^2\big(\|\ddot{\boldsymbol{u}}\|_0^2 + \|\ddot{\boldsymbol{u}}(0)\|_0^2\big) \\
& + \int_0^t \Big(h^2 \sum_{\beta\in\Omega} |\dot{u}_3^\beta|_{3,\,\beta}^2 + h^2\|\boldsymbol{u}_{ttt}\|_0^2 + h^4 \sum_{\beta\in\Omega} \|\dot{f}_3^\beta\|_{0,\,\beta}^2\Big) \, \mathrm{d}s, \qquad (3.21)
\end{aligned}
$$

for a.e. $t \in (0,\, T]$.

Now, let us consider the estimate for $\boldsymbol{\rho}$. It follows from the triangle inequality, result (2.12), and Lemmas 2.2 and 2.1 that

$$
\begin{aligned}
\|\ddot{\boldsymbol{\rho}}\|_0 &\le \|\ddot{\boldsymbol{u}} - \boldsymbol{I}_h\ddot{\boldsymbol{u}}\|_0 + \|\boldsymbol{I}_h\ddot{\boldsymbol{u}} - \boldsymbol{R}_h\ddot{\boldsymbol{u}}\|_0 \\
&\lesssim \|\ddot{\boldsymbol{u}} - \boldsymbol{I}_h\ddot{\boldsymbol{u}}\|_0 + \|\boldsymbol{I}_h\ddot{\boldsymbol{u}} - \boldsymbol{R}_h\ddot{\boldsymbol{u}}\|_h \\
&\lesssim \|\ddot{\boldsymbol{u}} - \boldsymbol{I}_h\ddot{\boldsymbol{u}}\|_0 + \|\boldsymbol{I}_h\ddot{\boldsymbol{u}} - \ddot{\boldsymbol{u}}\|_h + \|\ddot{\boldsymbol{u}} - \boldsymbol{R}_h\ddot{\boldsymbol{u}}\|_h \\
&\lesssim h \left\{ \sum_{\beta\in\Omega} \left(\sum_{I=1}^{2} |\ddot{u}_I^\beta|_{2,\,\beta}^2 + |\ddot{u}_3^\beta|_{3,\,\beta}^2 \right) \right\}^{1/2}. \qquad (3.22)
\end{aligned}
$$

Using the operator interpolation theory for Hilbert spaces [4] combined with (3.21) and (3.22), we find

$$
\begin{aligned}
\max_{0\le t\le T} \|\boldsymbol{\theta}(t)\|_h \lesssim\ & \|\dot{\boldsymbol{\theta}}(0)\|_0 + \|\boldsymbol{\theta}(0)\|_h \\
& + h \sum_{\beta\in\Omega} \Big(\sum_{I=1}^{2} |\ddot{u}_I^\beta|_{L^2(0,\,T;\,H^2(\beta))} + |u_3^\beta|_{L^\infty(0,\,T;\,H^3(\beta))} \\
& + |\dot{u}_3^\beta|_{L^2(0,\,T;\,H^3(\beta))} + |\ddot{u}_3^\beta|_{L^2(0,\,T;\,H^3(\beta))} \Big) \\
& + h\|\ddot{\boldsymbol{u}}\|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)} + h\|\boldsymbol{u}_{ttt}\|_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)} \\
& + h^2 \sum_{\beta\in\Omega} \|f_3^\beta\|_{L^\infty(0,\,T;\,L^2(\beta))} + h^2 \sum_{\beta\in\Omega} \|\dot{f}_3^\beta\|_{L^2(0,\,T;\,L^2(\beta))},
\end{aligned}
$$

and note that

$$
\begin{aligned}
\|\boldsymbol{\theta}(0)\|_h &= \|\boldsymbol{u}_{0h} - \boldsymbol{R}_h\boldsymbol{u}_0\|_h \\
&\le \|\boldsymbol{u}_{0h} - \boldsymbol{u}_0\|_h + \|\boldsymbol{u}_0 - \boldsymbol{R}_h\boldsymbol{u}_0\|_h \\
&\lesssim \|\boldsymbol{u}_{0h} - \boldsymbol{u}_0\|_h + h \left\{ \sum_{\beta\in\Omega} \left(\sum_{I=1}^{2} |u_I^\beta(0)|_{2,\,\beta}^2 + |u_3^\beta(0)|_{3,\,\beta}^2 \right) \right\}^{1/2},
\end{aligned}
$$

$$
\begin{aligned}
\|\dot{\boldsymbol{\theta}}(0)\|_0 &= \|\boldsymbol{u}_{1h} - \boldsymbol{R}_h\boldsymbol{u}_1\|_0 \\
&\lesssim \|\boldsymbol{u}_{1h} - \boldsymbol{u}_1\|_h + \|\boldsymbol{u}_1 - \boldsymbol{R}_h\boldsymbol{u}_1\|_h \\
&\lesssim \|\boldsymbol{u}_{1h} - \boldsymbol{u}_1\|_h + h \left\{ \sum_{\beta\in\Omega} \left(\sum_{I=1}^{2} |\dot{u}_I^\beta(0)|_{2,\,\beta}^2 + |\dot{u}_3^\beta(0)|_{3,\,\beta}^2 \right) \right\}^{1/2},
\end{aligned}
$$

$$\|\boldsymbol{\rho}\|_h = \|\boldsymbol{u} - \boldsymbol{R}_h \boldsymbol{u}\|_h \lesssim h \left\{ \sum_{\beta\in\Omega} \left(\sum_{I=1}^{2} |u_I^\beta|^2_{2,\,\beta} + |u_3^\beta|^2_{3,\,\beta} \right) \right\}^{1/2},$$

we see that

$$\begin{aligned}
\max_{0\le t\le T} \|\boldsymbol{u}(t) - \boldsymbol{u}_h(t)\|_h \le{}& \max_{0\le t\le T} \|\boldsymbol{\rho}(t)\|_h + \max_{0\le t\le T} \|\boldsymbol{\theta}(t)\|_h \\
\lesssim{}& \|\boldsymbol{u}_0 - \boldsymbol{u}_{0h}\|_h + \|\boldsymbol{u}_1 - \boldsymbol{u}_{1h}\|_h \\
&+ h\,|\boldsymbol{u}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)} \\
&+ h\,|\boldsymbol{u}_{tt}|_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)} \\
&+ h\,\|\boldsymbol{u}_{tt}\|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)} \\
&+ h\,\|\boldsymbol{u}_{ttt}\|_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)} \\
&+ h^2 \sum_{\beta\in\Omega} \left(\|f_3^\beta\|_{L^\infty(0,\,T;\,L^2(\beta))} + \|\dot f_3^\beta\|_{L^2(0,\,T;\,L^2(\beta))} \right),
\end{aligned}$$

which yields (3.2). (3.3) is a direct consequence of Lemma 2.1 and (2.12). The proof is completed. □

4. The fully discrete finite element method and error estimates.

4.1. The fully discrete scheme.

We apply the second-order central difference scheme to discretize the time-derivative term $\ddot{\boldsymbol{u}}_h$ in (3.1) to obtain a fully discrete method. Let N be a positive integer, $\tau := T/N$ be the step size of time, $t^n := n\tau$, $0 \le n \le N$. And for a continuous function $\phi \in C^0[0,\,T]$, let

$$\begin{aligned}
\phi^n :={}& \phi(t^n), \qquad D_t\phi^n := \frac{\phi^{n+1} - \phi^n}{\tau}, \\
D_t^2\phi^n :={}& \frac{\phi^{n+1} - 2\phi^n + \phi^{n-1}}{\tau^2} = \frac{D_t\phi^n - D_t\phi^{n-1}}{\tau}, \\
\phi_{n,\,\frac14} :={}& \frac{\phi^{n+1} + 2\phi^n + \phi^{n-1}}{4} = \frac{\phi_{n+\frac12} + \phi_{n-\frac12}}{2}, \\
\phi_{n+\frac12} :={}& \frac{\phi^{n+1} + \phi^n}{2}, \quad \phi_{n-\frac12} := \frac{\phi^n + \phi^{n-1}}{2}.
\end{aligned}$$

The same notation is also used for vector-valued functions. Thus, the fully discrete method based on the semi-discrete scheme (3.1) is to find $\{\boldsymbol{U}^n\}_{n=0}^N \in \boldsymbol{V}_h$ such that

$$\begin{cases} B(D_t^2\boldsymbol{U}^n,\,\boldsymbol{v}_h) + D_h(\boldsymbol{U}_{n,\,\frac14},\,\boldsymbol{v}_h) = (\boldsymbol{f}_{n,\,\frac14},\,\boldsymbol{v}_h),\ \forall\,\boldsymbol{v}_h \in \boldsymbol{V}_h,\ n = 1,\,2,\,\cdots,\,N-1, \\ \boldsymbol{U}^0 = \boldsymbol{u}_{0h}, \quad \boldsymbol{U}^1 \text{ is the approximation of } \boldsymbol{u}(\tau) \text{ in } \boldsymbol{V}_h. \end{cases} \tag{4.1}$$

The methods for choosing the initial functions $\boldsymbol{U}^0$ and $\boldsymbol{U}^1$ will be discussed later.

Theorem 4.1. *Let* $\boldsymbol{u}$ *and* $\{\boldsymbol{U}\}_{n=0}^{N}$ *be the solutions of problems (2.2) and (4.1), respectively. Assume that*

$$\boldsymbol{u} \in L^{\infty}\big(0,\, T;\, \prod_{\beta\in\Omega}((H^2(\beta))^2 \times H^3(\beta))\big),$$
$$\boldsymbol{u}_{tt} \in L^{2}\big(0,\, T;\, \prod_{\beta\in\Omega}((H^2(\beta))^2 \times H^3(\beta))\big) \bigcap L^{\infty}\big(0,\, T;\, \prod_{\beta\in\Omega}((H^1(\beta))^2 \times H^2(\beta))\big),$$
$$\boldsymbol{u}_{tttt},\, \boldsymbol{f},\, \boldsymbol{f}_t \in L^{2}\big(0,\, T;\, \prod_{\beta\in\Omega}(L^2(\beta))^3\big),$$
$$f_3^{\beta} \in L^{\infty}\big(0,\, T;\, L^2(\beta)\big), \quad \forall\, \beta \in \Omega.$$

Then

$$\begin{aligned}
\max_{1\le M\le N} &\|(\boldsymbol{U}^{M-1}+\boldsymbol{U}^{M})/2 - \boldsymbol{u}(t^{M-1/2})\|_h \\
&\lesssim \|(\boldsymbol{u}-\boldsymbol{U})_{\frac12}\|_h + \|D_t(\boldsymbol{u}-\boldsymbol{U})^0\|_0 \\
&\quad + h\,|\boldsymbol{u}|_{L^{\infty}\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)} \\
&\quad + h\,|\boldsymbol{u}_{tt}|_{L^{2}\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)} \\
&\quad + h\,\|\boldsymbol{u}_{tt}\|_{L^{\infty}\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)} \\
&\quad + h\,\|\boldsymbol{u}_{ttt}\|_{L^{2}\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)} \\
&\quad + h^2 \sum_{\beta\in\Omega}\Big(\|f_3^{\beta}\|_{L^{\infty}(0,\,T;\,L^2(\beta))} + \|\dot f_3^{\beta}\|_{L^{2}(0,\,T;\,L^2(\beta))}\Big). \\
&\quad + \tau^2\,|\boldsymbol{u}_{tt}|_{L^{\infty}\left(0,\,T;\,\prod_{\beta\in\Omega}((H^1(\beta))^2\times H^2(\beta))\right)} \\
&\quad + \tau^2\,\|\boldsymbol{u}_{tttt}\|_{L^{2}\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)}.
\end{aligned}$$

Proof. As in (3.4), write

$$\boldsymbol{u}^n - \boldsymbol{U}^n = (\boldsymbol{u}^n - \boldsymbol{R}_h\boldsymbol{u}^n) + (\boldsymbol{R}_h\boldsymbol{u}^n - \boldsymbol{U}^n) =: \boldsymbol{\rho}^n + \boldsymbol{\theta}^n. \tag{4.2}$$

For each $\boldsymbol{v}_h \in \boldsymbol{V}_h$, we have by (4.2), (4.1), and (2.13) that

$$\begin{aligned}
B(D_t^2\boldsymbol{\theta}^n,\, \boldsymbol{v}_h) =& B\big(D_t^2(\boldsymbol{R}_h\boldsymbol{u}^n - \boldsymbol{U}^n),\, \boldsymbol{v}_h\big) \\
=& - B(D_t^2\boldsymbol{\rho}^n,\, \boldsymbol{v}_h) + B(D_t^2\boldsymbol{u}^n,\, \boldsymbol{v}_h) - B(D_t^2\boldsymbol{U}^n,\, \boldsymbol{v}_h) \\
=& - B(D_t^2\boldsymbol{\rho}^n,\, \boldsymbol{v}_h) + B(D_t^2\boldsymbol{u}^n,\, \boldsymbol{v}_h) + D_h(\boldsymbol{U}_{n,\,\frac14},\, \boldsymbol{v}_h) - (\boldsymbol{f}_{n,\,\frac14},\, \boldsymbol{v}_h) \\
=& - B(D_t^2\boldsymbol{\rho}^n,\, \boldsymbol{v}_h) + B(D_t^2\boldsymbol{u}^n,\, \boldsymbol{v}_h) - D_h(\boldsymbol{\theta}_{n,\,\frac14},\, \boldsymbol{v}_h) \\
& + D_h(\boldsymbol{u}_{n,\,\frac14},\, \boldsymbol{v}_h) - (\boldsymbol{f}_{n,\,\frac14},\, \boldsymbol{v}_h),
\end{aligned}$$

which immediately leads to

$$\begin{aligned}
B(D_t^2\boldsymbol{\theta}^n,\, \boldsymbol{v}_h) + D_h(\boldsymbol{\theta}_{n,\,\frac14},\, \boldsymbol{v}_h) =& - B(D_t^2\boldsymbol{\rho}^n,\, \boldsymbol{v}_h) + B(D_t^2\boldsymbol{u}^n - \ddot{\boldsymbol{u}}_{n,\,\frac14},\, \boldsymbol{v}_h) \\
& + D_h(\boldsymbol{u}_{n,\,\frac14},\, \boldsymbol{v}_h) + B(\ddot{\boldsymbol{u}}_{n,\,\frac14},\, \boldsymbol{v}_h) \\
& - (\boldsymbol{f}_{n,\,\frac14},\, \boldsymbol{v}_h), \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h.
\end{aligned} \tag{4.3}$$

We choose $\boldsymbol{v}_h = D_t\boldsymbol{\theta}^n + D_t\boldsymbol{\theta}^{n-1}$ in (4.3) to get

$$\begin{aligned}
&B(D_t^2\boldsymbol{\theta}^n,\, D_t\boldsymbol{\theta}^n + D_t\boldsymbol{\theta}^{n-1}) + D_h(\boldsymbol{\theta}_{n,\frac{1}{4}},\, D_t\boldsymbol{\theta}^n + D_t\boldsymbol{\theta}^{n-1})\\
&\quad = -B(D_t^2\boldsymbol{\rho}^n,\, D_t\boldsymbol{\theta}^n + D_t\boldsymbol{\theta}^{n-1}) + B(D_t^2\boldsymbol{u}^n - \ddot{\boldsymbol{u}}_{n,\frac{1}{4}},\, D_t\boldsymbol{\theta}^n + D_t\boldsymbol{\theta}^{n-1})\\
&\qquad + D_h(\boldsymbol{u}_{n,\frac{1}{4}},\, D_t\boldsymbol{\theta}^n + D_t\boldsymbol{\theta}^{n-1}) + B(\ddot{\boldsymbol{u}}_{n,\frac{1}{4}},\, D_t\boldsymbol{\theta}^n + D_t\boldsymbol{\theta}^{n-1})\\
&\qquad - (\boldsymbol{f}_{n,\frac{1}{4}},\, D_t\boldsymbol{\theta}^n + D_t\boldsymbol{\theta}^{n-1}).
\end{aligned} \tag{4.4}$$

Since $D_t\boldsymbol{\theta}^n + D_t\boldsymbol{\theta}^{n-1} = 2\tau^{-1}(\boldsymbol{\theta}_{n+\frac{1}{2}} - \boldsymbol{\theta}_{n-\frac{1}{2}})$, by reorganizing terms we can put (4.4) in the form

$$\begin{aligned}
&B(D_t\boldsymbol{\theta}^n,\, D_t\boldsymbol{\theta}^n) - B(D_t\boldsymbol{\theta}^{n-1},\, D_t\boldsymbol{\theta}^{n-1}) + D_h(\boldsymbol{\theta}_{n+\frac{1}{2}},\, \boldsymbol{\theta}_{n+\frac{1}{2}}) - D_h(\boldsymbol{\theta}_{n-\frac{1}{2}},\, \boldsymbol{\theta}_{n-\frac{1}{2}})\\
&\quad = -\tau B(D_t^2\boldsymbol{\rho}^n,\, D_t\boldsymbol{\theta}^n + D_t\boldsymbol{\theta}^{n-1}) + \tau B(D_t^2\boldsymbol{u}^n - \ddot{\boldsymbol{u}}_{n,\frac{1}{4}},\, D_t\boldsymbol{\theta}^n + D_t\boldsymbol{\theta}^{n-1})\\
&\qquad + 2D_h(\boldsymbol{u}_{n,\frac{1}{4}},\, \boldsymbol{\theta}_{n+\frac{1}{2}} - \boldsymbol{\theta}_{n-\frac{1}{2}}) + 2B(\ddot{\boldsymbol{u}}_{n,\frac{1}{4}},\, \boldsymbol{\theta}_{n+\frac{1}{2}} - \boldsymbol{\theta}_{n-\frac{1}{2}})\\
&\qquad - 2(\boldsymbol{f}_{n,\frac{1}{4}},\, \boldsymbol{\theta}_{n+\frac{1}{2}} - \boldsymbol{\theta}_{n-\frac{1}{2}}).
\end{aligned}$$

Thus, for a natural number M with $2 \le M \le N$, we take the summation from 1 to $M-1$ to get

$$\begin{aligned}
&B(D_t\boldsymbol{\theta}^{M-1},\, D_t\boldsymbol{\theta}^{M-1}) - B(D_t\boldsymbol{\theta}^0,\, D_t\boldsymbol{\theta}^0) + D_h(\boldsymbol{\theta}_{M-\frac{1}{2}},\, \boldsymbol{\theta}_{M-\frac{1}{2}}) - D_h(\boldsymbol{\theta}_{\frac{1}{2}},\, \boldsymbol{\theta}_{\frac{1}{2}})\\
&\quad = \left\{ -\tau \sum_{n=1}^{M-1} B(D_t^2\boldsymbol{\rho}^n,\, D_t\boldsymbol{\theta}^n + D_t\boldsymbol{\theta}^{n-1}) \right\}\\
&\qquad + \left\{ \tau \sum_{n=1}^{M-1} B(D_t^2\boldsymbol{u}^n - \ddot{\boldsymbol{u}}_{n,\frac{1}{4}},\, D_t\boldsymbol{\theta}^n + D_t\boldsymbol{\theta}^{n-1}) \right\}\\
&\qquad + \left\{ 2\sum_{n=1}^{M-1} D_h(\boldsymbol{u}_{n,\frac{1}{4}},\, \boldsymbol{\theta}_{n+\frac{1}{2}} - \boldsymbol{\theta}_{n-\frac{1}{2}}) + 2\sum_{n=1}^{M-1} B(\ddot{\boldsymbol{u}}_{n,\frac{1}{4}},\, \boldsymbol{\theta}_{n+\frac{1}{2}} - \boldsymbol{\theta}_{n-\frac{1}{2}}) \right.\\
&\qquad \left. - 2\sum_{n=1}^{M-1} (\boldsymbol{f}_{n,\frac{1}{4}},\, \boldsymbol{\theta}_{n+\frac{1}{2}} - \boldsymbol{\theta}_{n-\frac{1}{2}}) \right\}\\
&\quad =: II_1 + II_2 + II_3.
\end{aligned} \tag{4.5}$$

By the Abel identity for summation, it follows that

$$
\begin{aligned}
\sum_{n=1}^{M-1} B(\ddot{\boldsymbol{u}}_{n,\frac14}, \boldsymbol{\theta}_{n+\frac12} - \boldsymbol{\theta}_{n-\frac12}) &= - B(\ddot{\boldsymbol{u}}_{1,\frac14}, \boldsymbol{\theta}_{\frac12}) + B(\ddot{\boldsymbol{u}}_{M-1,\frac14}, \boldsymbol{\theta}_{M-\frac12}) \\
&\quad - \sum_{n=2}^{M-1} B(\ddot{\boldsymbol{u}}_{n,\frac14} - \ddot{\boldsymbol{u}}_{n-1,\frac14}, \boldsymbol{\theta}_{n-\frac12}), \\
\sum_{n=1}^{M-1} D_h(\boldsymbol{u}_{n,\frac14}, \boldsymbol{\theta}_{n+\frac12} - \boldsymbol{\theta}_{n-\frac12}) &= - D_h(\boldsymbol{u}_{1,\frac14}, \boldsymbol{\theta}_{\frac12}) + D_h(\boldsymbol{u}_{M-1,\frac14}, \boldsymbol{\theta}_{M-\frac12}) \\
&\quad - \sum_{n=2}^{M-1} D_h(\boldsymbol{u}_{n,\frac14} - \boldsymbol{u}_{n-1,\frac14}, \boldsymbol{\theta}_{n-\frac12}), \\
\sum_{n=1}^{M-1} (\boldsymbol{f}_{n,\frac14}, \boldsymbol{\theta}_{n+\frac12} - \boldsymbol{\theta}_{n-\frac12}) &= - (\boldsymbol{f}_{1,\frac14}, \boldsymbol{\theta}_{\frac12}) + (\boldsymbol{f}_{M-1,\frac14}, \boldsymbol{\theta}_{M-\frac12}) \\
&\quad - \sum_{n=2}^{M-1} (\boldsymbol{f}_{n,\frac14} - \boldsymbol{f}_{n-1,\frac14}, \boldsymbol{\theta}_{n-\frac12}).
\end{aligned}
$$

Hence, II_3 can be rewritten as

$$
\begin{aligned}
II_3 =& \Big\{ - 2B(\ddot{\boldsymbol{u}}_{1,\frac14}, \boldsymbol{\theta}_{\frac12}) + 2(\boldsymbol{f}_{1,\frac14}, \boldsymbol{\theta}_{\frac12}) - 2D_h(\boldsymbol{u}_{1,\frac14}, \boldsymbol{\theta}_{\frac12}) \Big\} \\
&+ \Big\{ - 2 \sum_{n=2}^{M-1} \Big(B(\ddot{\boldsymbol{u}}_{n,\frac14} - \ddot{\boldsymbol{u}}_{n-1,\frac14}, \boldsymbol{\theta}_{n-\frac12}) - (\boldsymbol{f}_{n,\frac14} - \boldsymbol{f}_{n-1,\frac14}, \boldsymbol{\theta}_{n-\frac12}) \\
&+ D_h(\boldsymbol{u}_{n,\frac14} - \boldsymbol{u}_{n-1,\frac14}, \boldsymbol{\theta}_{n-\frac12}) \Big) \Big\} \\
&+ \Big\{ 2B(\ddot{\boldsymbol{u}}_{M-1,\frac14}, \boldsymbol{\theta}_{M-\frac12}) - 2(\boldsymbol{f}_{M-1,\frac14}, \boldsymbol{\theta}_{M-\frac12}) + 2D_h(\boldsymbol{u}_{M-1,\frac14}, \boldsymbol{\theta}_{M-\frac12}) \Big\} \\
=: & \, II_{31} + II_{32} + II_{33}. \qquad (4.6)
\end{aligned}
$$

We are now ready to estimate the three terms in (4.5). Observing that

$$
D_t^2 \boldsymbol{\rho}^n = \tau^{-2}(\boldsymbol{\rho}^{n+1} - 2\boldsymbol{\rho}^n + \boldsymbol{\rho}^{n-1}) = \tau^{-2} \int_{-\tau}^{\tau} (\tau - |s|) \frac{\partial^2 \boldsymbol{\rho}}{\partial t^2}(t_n + s) \, \mathrm{d}s,
$$

we conclude by the Cauchy-Schwarz inequality that

$$
\|D_t^2 \boldsymbol{\rho}^n\|_0^2 \lesssim \tau^{-1} \int_{t^{n-1}}^{t^{n+1}} \|\ddot{\boldsymbol{\rho}}\|_0^2 \, \mathrm{d}t, \qquad (4.7)
$$

which implies

$$
\begin{aligned}
|II_1| &\lesssim \tau \sum_{n=1}^{M-1} \|D_t^2 \boldsymbol{\rho}^n\|_0 \|D_t \boldsymbol{\theta}^n + D_t \boldsymbol{\theta}^{n-1}\|_0 \\
&\lesssim \tau \left(\sum_{n=1}^{M-1} \|D_t^2 \boldsymbol{\rho}^n\|_0^2 \right)^{1/2} \left(\sum_{n=0}^{M-1} \|D_t \boldsymbol{\theta}^n\|_0^2 \right)^{1/2} \\
&\lesssim \tau^{\frac{1}{2}} \left(\sum_{n=1}^{M-1} \int_{t^{n-1}}^{t^{n+1}} \|\ddot{\boldsymbol{\rho}}\|_0^2 \,\mathrm{d}t \right)^{1/2} \left(\sum_{n=0}^{M-1} \|D_t \boldsymbol{\theta}^n\|_0^2 \right)^{1/2} \\
&\lesssim \tau^{\frac{1}{2}} \|\ddot{\boldsymbol{\rho}}\|_{L^2\left(0,\, T;\, \prod_{\beta \in \Omega} (L^2(\beta))^3\right)} \left(\sum_{n=0}^{M-1} \|D_t \boldsymbol{\theta}^n\|_0^2 \right)^{1/2}.
\end{aligned}
\tag{4.8}
$$

Let $\boldsymbol{r}_n := \ddot{\boldsymbol{u}}_{n,\, \frac{1}{4}} - D_t^2 \boldsymbol{u}^n$, we argue as in the derivation of (4.7) to find [13]

$$
\|\boldsymbol{r}_n\|_0^2 \lesssim \tau^3 \int_{t^{n-1}}^{t^{n+1}} \left\| \frac{\partial^4 \boldsymbol{u}}{\partial t^4} \right\|_0^2 \mathrm{d}t,
$$

from which and the Cauchy-Schwarz inequality we are led to

$$
\begin{aligned}
|II_2| &\lesssim \tau \sum_{n=1}^{M-1} \|\boldsymbol{r}_n\|_0 \|D_t \boldsymbol{\theta}^n + D_t \boldsymbol{\theta}^{n-1}\|_0 \\
&\lesssim \tau \left(\sum_{n=1}^{M-1} \|\boldsymbol{r}_n\|_0^2 \right)^{1/2} \left(\sum_{n=0}^{M-1} \|D_t \boldsymbol{\theta}^n\|_0^2 \right)^{1/2} \\
&\lesssim \tau^{\frac{5}{2}} \|\boldsymbol{u}_{tttt}\|_{L^2\left(0,\, T;\, \prod_{\beta \in \Omega} (L^2(\beta))^3\right)} \left(\sum_{n=0}^{M-1} \|D_t \boldsymbol{\theta}^n\|_0^2 \right)^{1/2}.
\end{aligned}
\tag{4.9}
$$

Applying the same technique for estimating the term I_2 (see (3.8)-(3.18)), we have

$$
|II_{31}| \lesssim \Big(h \sum_{\beta \in \Omega} |(u_3^\beta)_{1,\, \frac{1}{4}}|_{3,\, \beta} + h \|\ddot{\boldsymbol{u}}_{1,\, \frac{1}{4}}\|_0 + h^2 \sum_{\beta \in \Omega} \|(f_3^\beta)_{1,\, \frac{1}{4}}\|_{0,\, \beta} \Big) \|\boldsymbol{\theta}_{\frac{1}{2}}\|_h, \tag{4.10}
$$

$$
|II_{33}| \lesssim \Big(h \sum_{\beta \in \Omega} |(u_3^\beta)_{M-1,\, \frac{1}{4}}|_{3,\, \beta}^2 + h \|\ddot{\boldsymbol{u}}_{M-1,\, \frac{1}{4}}\|_0 + h^2 \sum_{\beta \in \Omega} \|(f_3^\beta)_{M-1,\, \frac{1}{4}}\|_{0,\, \beta} \Big) \|\boldsymbol{\theta}_{M-\frac{1}{2}}\|_h, \tag{4.11}
$$

$$
\begin{aligned}
|II_{32}| \lesssim \sum_{n=2}^{M-1} \Big(& h \sum_{\beta \in \Omega} |(u_3^\beta)_{n,\, \frac{1}{4}} - (u_3^\beta)_{n-1,\, \frac{1}{4}}|_{3,\, \beta} + h \|\ddot{\boldsymbol{u}}_{n,\, \frac{1}{4}} - \ddot{\boldsymbol{u}}_{n-1,\, \frac{1}{4}}\|_0 \\
& + h^2 \sum_{\beta \in \Omega} \|(f_3^\beta)_{n,\, \frac{1}{4}} - (f_3^\beta)_{n-1,\, \frac{1}{4}}\|_{0,\, \beta} \Big) \|\boldsymbol{\theta}_{n-\frac{1}{2}}\|_h.
\end{aligned}
\tag{4.12}
$$

On the other hand, it is easy to show that

$$\begin{aligned}
&\|(f_3^\beta)_{n,\,\frac{1}{4}} - (f_3^\beta)_{n-1,\,\frac{1}{4}}\|_{0,\,\beta}\\
\leq& \frac{1}{4}\|(f_3^\beta)^{n+1} - (f_3^\beta)^{n}\|_{0,\,\beta} + \frac{1}{2}\|(f_3^\beta)^{n} - (f_3^\beta)^{n-1}\|_{0,\,\beta} + \frac{1}{4}\|(f_3^\beta)^{n-1} - (f_3^\beta)^{n-2}\|_{0,\,\beta}\\
\lesssim& \Big\| \int_{t^n}^{t^{n+1}} \dot{f}_3^\beta(t)\,\mathrm{d}t\Big\|_{0,\,\beta} + \Big\| \int_{t^{n-1}}^{t^{n}} \dot{f}_3^\beta(t)\,\mathrm{d}t\Big\|_{0,\,\beta} + \Big\| \int_{t^{n-2}}^{t^{n-1}} \dot{f}_3^\beta(t)\,\mathrm{d}t\Big\|_{0,\,\beta}\\
\lesssim& \int_{t^{n-2}}^{t^{n+1}} \|\dot{f}_3^\beta(t)\|_{0,\,\beta}\,\mathrm{d}t\\
\lesssim& \tau^{\frac{1}{2}} \left(\int_{t^{n-2}}^{t^{n+1}} \|\dot{f}_3^\beta(t)\|_{0,\,\beta}^2\,\mathrm{d}t \right)^{1/2}. \qquad (4.13)
\end{aligned}$$

Similarly,

$$\|\ddot{\boldsymbol{u}}_{n,\,\frac{1}{4}} - \ddot{\boldsymbol{u}}_{n-1,\,\frac{1}{4}}\|_0 \lesssim \tau^{\frac{1}{2}} \left(\int_{t^{n-2}}^{t^{n+1}} \|\boldsymbol{u}_{ttt}(t)\|_0^2\,\mathrm{d}t \right)^{1/2}, \qquad (4.14)$$

$$|(u_3^\beta)_{n,\,\frac{1}{4}} - (u_3^\beta)_{n-1,\,\frac{1}{4}}|_{3,\,\beta} \lesssim \tau^{\frac{1}{2}} \left(\int_{t^{n-2}}^{t^{n+1}} |\dot{u}_3^\beta(t)|_{3,\,\beta}^2\,\mathrm{d}t \right)^{1/2}. \qquad (4.15)$$

Substituting (4.13)-(4.15) into (4.12) and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
|II_{32}| \lesssim \tau^{\frac{1}{2}} \Bigg(& h \sum_{\beta\in\Omega} |\dot{u}_3^\beta|_{L^2\left(0,\,T;\,H^3(\beta)\right)} + h\,\|\boldsymbol{u}_{ttt}\|_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)}\\
& + h^2 \sum_{\beta\in\Omega} |\dot{f}_3^\beta|_{L^2\left(0,\,T;\,L^2(\beta)\right)} \Bigg) \left(\sum_{n=2}^{M-1} \|\boldsymbol{\theta}_{n-\frac{1}{2}}\|_h^2 \right)^{1/2}. \qquad (4.16)
\end{aligned}$$

Now by the ε-inequality and the usual absorbing technique, it follows from (2.8), (4.5)-(4.6), (4.8)-(4.11), and (4.16) that

$$\begin{aligned}
&\|D_t\boldsymbol{\theta}^{M-1}\|_0^2 + \|\boldsymbol{\theta}_{M-\frac{1}{2}}\|_h^2\\
&\quad\lesssim \|D_t\boldsymbol{\theta}^0\|_0^2 + \|\boldsymbol{\theta}_{\frac{1}{2}}\|_h^2 + \tau \sum_{n=1}^{M-1} \left(\|D_t\boldsymbol{\theta}^{n-1}\|_0^2 + \|\boldsymbol{\theta}_{n-\frac{1}{2}}\|_h^2 \right)\\
&\qquad + \|\ddot{\boldsymbol{\rho}}\|^2_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)} + \tau^4 \|\boldsymbol{u}_{tttt}\|^2_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)}\\
&\qquad + h^2 \sum_{\beta\in\Omega} |(u_3^\beta)_{1,\,\frac{1}{4}}|_{3,\,\beta}^2 + h^2 \|\ddot{\boldsymbol{u}}_{1,\,\frac{1}{4}}\|_0^2 + h^4 \sum_{\beta\in\Omega} \|(f_3^\beta)_{1,\,\frac{1}{4}}\|_{0,\,\beta}^2\\
&\qquad + h^2 \sum_{\beta\in\Omega} |(u_3^\beta)_{M-1,\,\frac{1}{4}}|_{3,\,\beta}^2 + h^2 \|\ddot{\boldsymbol{u}}_{M-1,\,\frac{1}{4}}\|_0^2 + h^4 \sum_{\beta\in\Omega} \|(f_3^\beta)_{M-1,\,\frac{1}{4}}\|_{0,\,\beta}^2\\
&\qquad + h^2 \sum_{\beta\in\Omega} |\dot{u}_3^\beta|^2_{L^2\left(0,\,T;\,H^3(\beta)\right)} + h^2\,\|\boldsymbol{u}_{ttt}\|^2_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)}\\
&\qquad + h^4 \sum_{\beta\in\Omega} |\dot{f}_3^\beta|^2_{L^2\left(0,\,T;\,L^2(\beta)\right)},
\end{aligned}$$

which with the discrete Gronwall's Lemma [27, p. 157] implies

$$\begin{aligned}
\|D_t\boldsymbol{\theta}^{M-1}\|_0^2+&\|\boldsymbol{\theta}_{M-\frac{1}{2}}\|_h^2 \\
&\lesssim \|D_t\boldsymbol{\theta}^0\|_0^2+\|\boldsymbol{\theta}_{\frac{1}{2}}\|_h^2 \\
&\quad+\|\ddot{\boldsymbol{\rho}}\|^2_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)}+\tau^4\|\boldsymbol{u}_{tttt}\|^2_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)} \\
&\quad+h^2\sum_{\beta\in\Omega}|(u_3^\beta)_{1,\,\frac{1}{4}}|^2_{3,\,\beta}+h^2\|\ddot{\boldsymbol{u}}_{1,\,\frac{1}{4}}\|_0^2+h^4\sum_{\beta\in\Omega}\|(f_3^\beta)_{1,\,\frac{1}{4}}\|^2_{0,\,\beta} \\
&\quad+h^2\sum_{\beta\in\Omega}|(u_3^\beta)_{M-1,\,\frac{1}{4}}|^2_{3,\,\beta}+h^2\|\ddot{\boldsymbol{u}}_{M-1,\,\frac{1}{4}}\|_0^2+h^4\sum_{\beta\in\Omega}\|(f_3^\beta)_{M-1,\,\frac{1}{4}}\|^2_{0,\,\beta} \\
&\quad+h^2\sum_{\beta\in\Omega}|\dot{u}_3^\beta|^2_{L^2\left(0,\,T;\,H^3(\beta)\right)}+h^2\,\|\boldsymbol{u}_{ttt}\|^2_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)} \\
&\quad+h^4\sum_{\beta\in\Omega}|\dot{f}_3^\beta|^2_{L^2\left(0,\,T;\,L^2(\beta)\right)}.
\end{aligned}$$

By virtue of (3.22) and the operator interpolation theory in Hilbert spaces, the above estimate can be recast as

$$\begin{aligned}
\max_{1\le M\le N}&\|\boldsymbol{\theta}_{M-\frac{1}{2}}\|_h^2 \\
&\lesssim \|D_t\boldsymbol{\theta}^0\|_0^2+\|\boldsymbol{\theta}_{\frac{1}{2}}\|_h^2 \\
&\quad+h^2|\ddot{\boldsymbol{u}}|^2_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}+\tau^4\|\boldsymbol{u}_{tttt}\|^2_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)} \\
&\quad+h^2\sum_{\beta\in\Omega}|u_3^\beta|^2_{L^\infty\left(0,\,T;\,H^3(\beta)\right)}+h^2\|\ddot{\boldsymbol{u}}\|^2_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)} \\
&\quad+h^4\sum_{\beta\in\Omega}\|f_3^\beta\|^2_{L^\infty\left(0,\,T;\,L^2(\beta)\right)}+h^2\sum_{\beta\in\Omega}|\dot{u}_3^\beta|^2_{L^2\left(0,\,T;\,H^3(\beta)\right)} \\
&\quad+h^2\,\|\boldsymbol{u}_{ttt}\|^2_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)}+h^4\sum_{\beta\in\Omega}|\dot{f}_3^\beta|^2_{L^2\left(0,\,T;\,L^2(\beta)\right)}.
\end{aligned}\tag{4.17}$$

It is clear that

$$\begin{aligned}
\|\boldsymbol{\theta}_{\frac{1}{2}}\|_h &\le \|(\boldsymbol{u}-\boldsymbol{U})_{\frac{1}{2}}\|_h+\|\boldsymbol{u}_0-\boldsymbol{R}_h\boldsymbol{u}_0\|_h+\|\boldsymbol{u}^1-\boldsymbol{R}_h\boldsymbol{u}^1\|_h \\
&\lesssim \|(\boldsymbol{u}-\boldsymbol{U})_{\frac{1}{2}}\|_h+h|\boldsymbol{u}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)},
\end{aligned}\tag{4.18}$$

and

$$\begin{aligned}
\|D_t\boldsymbol{\theta}^0\|_0 \le&\|D_t(\boldsymbol{u}-\boldsymbol{U})^0\|_0+\|D_t(\boldsymbol{u}-\boldsymbol{R}_h\boldsymbol{u})^0\|_0 \\
\lesssim&\|D_t(\boldsymbol{u}-\boldsymbol{U})^0\|_0+\|\dot{\boldsymbol{u}}-\boldsymbol{R}_h\dot{\boldsymbol{u}}\|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)} \\
\lesssim&\|D_t(\boldsymbol{u}-\boldsymbol{U})^0\|_0+\|\dot{\boldsymbol{u}}-\boldsymbol{I}_h\dot{\boldsymbol{u}}\|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)} \\
&+\max_{0\le t\le T}\|\boldsymbol{I}_h\dot{\boldsymbol{u}}-\boldsymbol{R}_h\dot{\boldsymbol{u}}\|_h \\
\lesssim&\|D_t(\boldsymbol{u}-\boldsymbol{U})^0\|_0+h^2|\dot{\boldsymbol{u}}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)} \\
&+\max_{0\le t\le T}\|\dot{\boldsymbol{u}}-\boldsymbol{I}_h\dot{\boldsymbol{u}}\|_h+\max_{0\le t\le T}\|\dot{\boldsymbol{u}}-\boldsymbol{R}_h\dot{\boldsymbol{u}}\|_h \\
\lesssim&\|D_t(\boldsymbol{u}-\boldsymbol{U})^0\|_0+h|\dot{\boldsymbol{u}}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}.
\end{aligned}\tag{4.19}$$

Similarly,

$$\begin{aligned}\|\boldsymbol{\rho}_{M-\frac{1}{2}}\|_h &\lesssim \|\boldsymbol{u}^{M-1}-\boldsymbol{R}_h\boldsymbol{u}^{M-1}\|_h+\|\boldsymbol{u}^M-\boldsymbol{R}_h\boldsymbol{u}^M\|_h\\ &\lesssim h|\boldsymbol{u}|_{L^\infty\left(0,\,T;\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)},\end{aligned}\tag{4.20}$$

which with (4.2), (4.17)-(4.20) yields

$$\begin{aligned}\max_{1\le M\le N}\|(\boldsymbol{u}-\boldsymbol{U})_{M-\frac{1}{2}}\|_h &\le \max_{1\le M\le N}\|\boldsymbol{\rho}_{M-\frac{1}{2}}\|_h+\max_{1\le M\le N}\|\boldsymbol{\theta}_{M-\frac{1}{2}}\|_h\\ &\lesssim \|(\boldsymbol{u}-\boldsymbol{U})_{\frac{1}{2}}\|_h+\|D_t(\boldsymbol{u}-\boldsymbol{U})^0\|_0\\ &\quad + h\,|\boldsymbol{u}|_{L^\infty\left(0,\,T;\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}\\ &\quad + h\,|\boldsymbol{u}_{tt}|_{L^2\left(0,\,T;\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}\\ &\quad + h\,\|\boldsymbol{u}_{tt}\|_{L^\infty\left(0,\,T;\prod_{\beta\in\Omega}(L^2(\beta))^3\right)}\\ &\quad + h\,\|\boldsymbol{u}_{ttt}\|_{L^2\left(0,\,T;\prod_{\beta\in\Omega}(L^2(\beta))^3\right)}\\ &\quad + h^2\sum_{\beta\in\Omega}\left(\|f_3^\beta\|_{L^\infty(0,\,T;\,L^2(\beta))}+\|\dot{f}_3^\beta\|_{L^2(0,\,T;\,L^2(\beta))}\right)\\ &\quad + \tau^2\,\|\boldsymbol{u}_{tttt}\|_{L^2\left(0,\,T;\prod_{\beta\in\Omega}(L^2(\beta))^3\right)}.\end{aligned}$$

Therefore, it follows from Taylor's formula in Banach spaces [1, p. 86] that

$$\|(\boldsymbol{u}^{M-1}+\boldsymbol{u}^M)/2-\boldsymbol{u}(t^{M-\frac{1}{2}})\|_h \lesssim \tau^2\,|\boldsymbol{u}_{tt}|_{L^\infty\left(0,\,T;\prod_{\beta\in\Omega}((H^1(\beta))^2\times H^2(\beta))\right)}.$$

The proof is completed. □

4.2. **The methods for selecting the initial functions.** Let us introduce two kinds of methods to select the initial functions $\boldsymbol{U}^0$ and $\boldsymbol{U}^1$ in $\boldsymbol{V}_h$. If the initial functions $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ are sufficiently smooth, we may take

$$\boldsymbol{U}^0=\boldsymbol{I}_h\boldsymbol{u}_0,\quad \boldsymbol{U}^1=\boldsymbol{I}_h(\boldsymbol{u}_0+\tau\boldsymbol{u}_1+\frac{\tau^2}{2}\ddot{\mathbf{u}}(0))\tag{4.21}$$

with $\ddot{\mathbf{u}}(0)$ being given by the equilibrium equation (2.16), i.e., for $\beta=\beta_1,\,\beta_2$,

$$\begin{cases}\ddot{u}_1^\beta(0)=\frac{1}{\rho_\beta t_\beta}\left(f_1^\beta(0)+\mathcal{Q}^\beta_{1J,\,J}(\boldsymbol{u}_0)\right),\\ \ddot{u}_2^\beta(0)=\frac{1}{\rho_\beta t_\beta}\left(f_2^\beta(0)+\mathcal{Q}^\beta_{2J,\,J}(\boldsymbol{u}_0)\right),\\ \ddot{u}_3^\beta(0)=\frac{1}{\rho_\beta t_\beta}\left(f_3^\beta(0)+\mathcal{M}^\beta_{IJ,\,IJ}(\boldsymbol{u}_0)\right).\end{cases}\tag{4.22}$$

If the initial functions are selected by (4.21)-(4.22), applying Taylor's formula for Hilbert spaces we can easily find

$$\begin{aligned}\|D_t(\boldsymbol{u}-\boldsymbol{U})^0\|_0 \lesssim\, & h^2|\boldsymbol{u}_t|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)} \\ & + \tau h^2|\boldsymbol{u}_{tt}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)} \\ & + \tau^2\|\boldsymbol{u}_{ttt}\|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)}, \\ \|(\boldsymbol{u}-\boldsymbol{U})_{\frac{1}{2}}\|_h \lesssim\, & h|\boldsymbol{u}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)} \\ & + h|\boldsymbol{u}_{tt}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)} \\ & + \tau^3|\boldsymbol{u}_{ttt}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^1(\beta))^2\times H^2(\beta))\right)},\end{aligned}$$

which together with Theorem 4.1 implies the following theorem.

Theorem 4.2. *Let $\boldsymbol{u}$ and $\{\boldsymbol{U}^n\}_{n=0}^N$ be the solutions of problems (2.2) and (4.1), respectively. Assume that*

$$\boldsymbol{u}\in L^\infty\big(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\big),\quad \boldsymbol{u}_{tt}\in L^\infty\big(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\big),$$

$$\boldsymbol{u}_{ttt}\in L^\infty\big(0,\,T;\,\prod_{\beta\in\Omega}((H^1(\beta))^2\times H^2(\beta))\big),\quad \boldsymbol{u}_{tttt},\,\boldsymbol{f},\,\boldsymbol{f}_t\in L^2\big(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\big),$$

$$f_3^\beta\in L^\infty\big(0,\,T;\,L^2(\beta)\big),\quad \forall\,\beta\in\Omega.$$

If the initial functions $\boldsymbol{U}^0$ and $\boldsymbol{U}^1$ are given by (4.21)-(4.22), then

$$\begin{aligned}\max_{1\le M\le N}\|\boldsymbol{U}_{M-\frac{1}{2}}-\boldsymbol{u}(t^{M-\frac{1}{2}})\|_h & \\ \lesssim\, & h|\boldsymbol{u}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)} \\ & + h|\boldsymbol{u}_{tt}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)} \\ & + h\|\boldsymbol{u}_{tt}\|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)} \\ & + h\|\boldsymbol{u}_{ttt}\|_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)} \\ & + h^2\sum_{\beta\in\Omega}\Big(\|f_3^\beta\|_{L^\infty\left(0,\,T;\,L^2(\beta)\right)}+\|\dot f_3^\beta\|_{L^2\left(0,\,T;\,L^2(\beta)\right)}\Big) \\ & + \tau^2|\boldsymbol{u}_{tt}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^1(\beta))^2\times H^2(\beta))\right)} \\ & + \tau^2\|\boldsymbol{u}_{ttt}\|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^1(\beta))^2\times H^2(\beta))\right)} \\ & + \tau^2\|\boldsymbol{u}_{tttt}\|_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)}.\end{aligned}$$

For the choice (4.21)-(4.22) it is inconvenient to compute the high order partial derivatives of the function $\boldsymbol{u}_0$ in practical applications, so we give the next method to overcome this difficulty. We also choose $\boldsymbol{U}^0=\boldsymbol{I}_h\boldsymbol{u}_0$, but by introducing an auxiliary function $\boldsymbol{U}^{-1}\in\boldsymbol{V}_h$ at the artificial time step t^{-1}, we get the function $\boldsymbol{U}^1\in\boldsymbol{V}_h$ by

$$\begin{cases}\frac{\boldsymbol{U}^1-\boldsymbol{U}^{-1}}{2\tau}=\boldsymbol{I}_h\boldsymbol{u}_1=\boldsymbol{I}_h\dot{\boldsymbol{u}}(0),\\ B\left(\frac{\boldsymbol{U}^1-2\boldsymbol{U}^0+\boldsymbol{U}^{-1}}{\tau^2},\,\boldsymbol{v}\right)+D_h\left(\frac{\boldsymbol{U}^1+2\boldsymbol{U}^0+\boldsymbol{U}^{-1}}{4},\,\boldsymbol{v}\right)=(\boldsymbol{f}(0),\,\boldsymbol{v}),\,\forall\,\boldsymbol{v}\in\boldsymbol{V}_h.\end{cases} \tag{4.23}$$

Let

$$\boldsymbol{\xi}_1(\tau) := \frac{1}{\tau}\left(\boldsymbol{u}(\tau) - \boldsymbol{u}(0) - \tau\dot{\boldsymbol{u}}(0) - \frac{\tau^2}{2}\ddot{\boldsymbol{u}}(0)\right),$$
$$\boldsymbol{\xi}_2(\tau) := \boldsymbol{u}(\tau) - \boldsymbol{u}(0) - \tau\dot{\boldsymbol{u}}(0).$$

We then have the following result by Taylor's expansion in Banach spaces.

Lemma 4.3. *Assume that the solution $\boldsymbol{u}$ of problem (2.2) satisfies that $\boldsymbol{u}_{tt} \in L^\infty\big(0, T; \prod_{\beta\in\Omega}((H^1(\beta))^2 \times H^2(\beta))\big)$, and $\boldsymbol{u}_{ttt} \in L^\infty\big(0, T; \prod_{\beta\in\Omega}(L^2(\beta))^3\big)$. Then*

$$\|\boldsymbol{\xi}_1(\tau)\|_0 \lesssim \tau^2\|\boldsymbol{u}_{ttt}\|_{L^\infty\left(0,\, T;\, \prod_{\beta\in\Omega}(L^2(\beta))^3\right)},$$
$$\|\boldsymbol{\xi}_2(\tau)\|_h \lesssim \tau^2|\boldsymbol{u}_{tt}|_{L^\infty\left(0,\, T;\, \prod_{\beta\in\Omega}((H^1(\beta))^2\times H^2(\beta))\right)}.$$

Lemma 4.4. *Let $\boldsymbol{u}$ be the solution of problem (2.2). Assume that*

$$\boldsymbol{u} \in L^\infty\big(0, T; \prod_{\beta\in\Omega}((H^2(\beta))^2 \times H^3(\beta))\big),$$
$$\boldsymbol{u}_{tt} \in L^2\big(0, T; \prod_{\beta\in\Omega}((H^2(\beta))^2 \times H^3(\beta))\big) \bigcap L^\infty\big(0, T; \prod_{\beta\in\Omega}((H^1(\beta))^2 \times H^2(\beta))\big),$$
$$\boldsymbol{u}_{ttt} \in L^\infty\big(0, T; \prod_{\beta\in\Omega}(L^2(\beta))^3\big), \quad \boldsymbol{f} \in L^2\big(0, T; \prod_{\beta\in\Omega}(L^2(\beta))^3\big),$$
$$f_3^\beta \in L^\infty\big(0, T; L^2(\beta)\big), \quad \forall\, \beta \in \Omega.$$

If the initial functions $\boldsymbol{U}^0$ and $\boldsymbol{U}^1$ are given by (4.23), then

$$\begin{aligned}\|D_t(\boldsymbol{u} - \boldsymbol{U})^0\|_0 \lesssim{}& h|\boldsymbol{u}|_{L^\infty\left(0, T; \prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}\\
&+ h|\boldsymbol{u}_t|_{L^\infty\left(0, T; \prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}\\
&+ h\|\boldsymbol{u}_{tt}\|_{L^\infty\left(0, T; \prod_{\beta\in\Omega}(L^2(\beta))^3\right)}\\
&+ h^2\sum_{\beta\in\Omega}\|f_3^\beta\|_{L^\infty\left(0, T; L^2(\beta)\right)}\\
&+ \tau^2|\boldsymbol{u}_{tt}|_{L^\infty\left(0, T; \prod_{\beta\in\Omega}((H^1(\beta))^2\times H^2(\beta))\right)}\\
&+ \tau^2\|\boldsymbol{u}_{ttt}\|_{L^\infty\left(0, T; \prod_{\beta\in\Omega}(L^2(\beta))^3\right)},\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}\|(\boldsymbol{u} - \boldsymbol{U})_{\frac{1}{2}}\|_h \lesssim{}& (\tau^{-1}h^2 + h)|\boldsymbol{u}|_{L^\infty\left(0, T; \prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}\\
&+ h|\boldsymbol{u}_t|_{L^\infty\left(0, T; \prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}\\
&+ h\|\boldsymbol{u}_{tt}\|_{L^\infty\left(0, T; \prod_{\beta\in\Omega}(L^2(\beta))^3\right)}\\
&+ h^2\sum_{\beta\in\Omega}\|f_3^\beta\|_{L^\infty\left(0, T; L^2(\beta)\right)}\\
&+ \tau^2|\boldsymbol{u}_{tt}|_{L^\infty\left(0, T; \prod_{\beta\in\Omega}((H^1(\beta))^2\times H^2(\beta))\right)}\\
&+ \tau^2\|\boldsymbol{u}_{ttt}\|_{L^\infty\left(0, T; \prod_{\beta\in\Omega}(L^2(\beta))^3\right)}.\end{aligned} \tag{4.25}$$

Proof. By the first equation of (4.23) we get

$$\boldsymbol{U}^{-1} = \boldsymbol{U}^1 - 2\tau \boldsymbol{I}_h \dot{\boldsymbol{u}}(0),$$

and insert it into the second equation of (4.23) we have

$$\begin{aligned} B\Big(\frac{2\boldsymbol{U}^1 - 2\boldsymbol{U}^0 - 2\tau \boldsymbol{I}_h \dot{\boldsymbol{u}}(0)}{\tau^2}, \boldsymbol{v}\Big) &+ D_h\Big(\frac{\boldsymbol{U}^1 + \boldsymbol{U}^0 - \tau \boldsymbol{I}_h \dot{\boldsymbol{u}}(0)}{2}, \boldsymbol{v}\Big) \\ &= (\boldsymbol{f}(0), \boldsymbol{v}), \forall \boldsymbol{v} \in \boldsymbol{V}_h. \end{aligned} \tag{4.26}$$

Multiplying by $\frac{\tau}{2}$ and reorganizing terms, we can recast (4.26) as

$$\begin{aligned} B\Big(\frac{\boldsymbol{U}^1 - \boldsymbol{U}^0}{\tau}, \boldsymbol{v}\Big) &+ \frac{\tau^2}{4} D_h\Big(\frac{\boldsymbol{U}^1 - \boldsymbol{U}^0}{\tau}, \boldsymbol{v}\Big) \\ &= \frac{\tau}{2}\Big((\boldsymbol{f}(0), \boldsymbol{v}) - D_h(\boldsymbol{U}^0, \boldsymbol{v})\Big) + B(\boldsymbol{I}_h \dot{\boldsymbol{u}}(0), \boldsymbol{v}) + \frac{\tau^2}{4} D_h(\boldsymbol{I}_h \dot{\boldsymbol{u}}(0), \boldsymbol{v}) \\ &= B(\boldsymbol{I}_h \dot{\boldsymbol{u}}(0) - \dot{\boldsymbol{u}}(0), \boldsymbol{v}) - \frac{\tau}{2} D_h(\boldsymbol{U}^0 - \boldsymbol{u}(0), \boldsymbol{v}) \\ &\quad + \frac{\tau^2}{4} D_h(\boldsymbol{I}_h \dot{\boldsymbol{u}}(0) - \dot{\boldsymbol{u}}(0), \boldsymbol{v}) \\ &\quad + \frac{\tau}{2}\big(- B(\ddot{\boldsymbol{u}}(0), \boldsymbol{v}) - D_h(\boldsymbol{u}(0), \boldsymbol{v}) + (\boldsymbol{f}(0), \boldsymbol{v})\big) \\ &\quad + B\big(\dot{\boldsymbol{u}}(0) + \frac{\tau}{2}\ddot{\boldsymbol{u}}(0), \boldsymbol{v}\big) + \frac{\tau^2}{4} D_h(\dot{\boldsymbol{u}}(0), \boldsymbol{v}), \end{aligned}$$

i.e.,

$$\begin{aligned} B\big(D_t(\boldsymbol{u} - \boldsymbol{U})^0, \boldsymbol{v}\big) &+ \frac{\tau^2}{4} D_h\big(D_t(\boldsymbol{u} - \boldsymbol{U})^0, \boldsymbol{v}\big) \\ &= B(\dot{\boldsymbol{u}}(0) - \boldsymbol{I}_h \dot{\boldsymbol{u}}(0), \boldsymbol{v}) - \frac{\tau}{2} D_h(\boldsymbol{u}(0) - \boldsymbol{U}^0, \boldsymbol{v}) \\ &\quad + \frac{\tau^2}{4} D_h(\dot{\boldsymbol{u}}(0) - \boldsymbol{I}_h \dot{\boldsymbol{u}}(0), \boldsymbol{v}) \\ &\quad + \frac{\tau}{2}\big(B(\ddot{\boldsymbol{u}}(0), \boldsymbol{v}) + D_h(\boldsymbol{u}(0), \boldsymbol{v}) - (\boldsymbol{f}(0), \boldsymbol{v})\big) \\ &\quad + B\big(\boldsymbol{\xi}_1(\tau), \boldsymbol{v}\big) + \frac{\tau}{4} D_h(\boldsymbol{\xi}_2(\tau), \boldsymbol{v}). \end{aligned} \tag{4.27}$$

Let

$$D_t(\boldsymbol{u} - \boldsymbol{U})^0 = D_t(\boldsymbol{u} - \boldsymbol{I}_h \boldsymbol{u})^0 + D_t(\boldsymbol{I}_h \boldsymbol{u} - \boldsymbol{U})^0 =: \boldsymbol{w}_1 + \boldsymbol{w}_2.$$

Then the equation (4.27) becomes

$$\begin{aligned} B(\boldsymbol{w}_2, \boldsymbol{v}) &+ \frac{\tau^2}{4} D_h(\boldsymbol{w}_2, \boldsymbol{v}) \\ &= \Big\{ -B(\boldsymbol{w}_1, \boldsymbol{v}) - \frac{\tau}{2} D_h(\boldsymbol{u}(0) - \boldsymbol{U}^0, \boldsymbol{v}) - \frac{\tau^2}{4} D_h(\boldsymbol{w}_1, \boldsymbol{v}) \Big\} \\ &\quad + \Big\{ B(\dot{\boldsymbol{u}}(0) - \boldsymbol{I}_h \dot{\boldsymbol{u}}(0), \boldsymbol{v}) + \frac{\tau^2}{4} D_h(\dot{\boldsymbol{u}}(0) - \boldsymbol{I}_h \dot{\boldsymbol{u}}(0), \boldsymbol{v}) \Big\} \\ &\quad + \Big\{ \frac{\tau}{2}\big(B(\ddot{\boldsymbol{u}}(0), \boldsymbol{v}) + D_h(\boldsymbol{u}(0), \boldsymbol{v}) - (\boldsymbol{f}(0), \boldsymbol{v})\big) \Big\} \\ &\quad + \Big\{ B\big(\boldsymbol{\xi}_1(\tau), \boldsymbol{v}\big) + \frac{\tau}{4} D_h(\boldsymbol{\xi}_2(\tau), \boldsymbol{v}) \Big\} \\ &=: III_1 + III_2 + III_3 + III_4. \end{aligned} \tag{4.28}$$

From (2.12) we know that

$$\|D_t(\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u})^0\|_0 + h\|D_t(\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u})^0\|_h \lesssim h^2|\boldsymbol{u}_t|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}. \tag{4.29}$$

It follows from (4.29), Lemma 4.3, and the Cauchy-Schwarz inequality that

$$\begin{aligned}|III_1| &\lesssim \|\boldsymbol{w}_1\|_0\|\boldsymbol{v}\|_0 + \frac{\tau}{2}\|\boldsymbol{u}(0)-\boldsymbol{U}^0\|_h\|\boldsymbol{v}\|_h + \frac{\tau^2}{4}\|\boldsymbol{w}_1\|_h\|\boldsymbol{v}\|_h\\ &\lesssim h^2|\boldsymbol{u}_t|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}\|\boldsymbol{v}\|_0\\ &\quad + \tau h|\boldsymbol{u}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}\|\boldsymbol{v}\|_h\\ &\quad + \tau^2 h|\boldsymbol{u}_t|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}\|\boldsymbol{v}\|_h,\end{aligned} \tag{4.30}$$

$$\begin{aligned}|III_2| &\lesssim \|\dot{\boldsymbol{u}}(0)-\boldsymbol{I}_h\dot{\boldsymbol{u}}(0)\|_0\|\boldsymbol{v}\|_0 + \tau^2\|\dot{\boldsymbol{u}}(0)-\boldsymbol{I}_h\dot{\boldsymbol{u}}(0)\|_h\|\boldsymbol{v}\|_h\\ &\lesssim h^2|\boldsymbol{u}_t|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}\|\boldsymbol{v}\|_0\\ &\quad + \tau^2 h|\boldsymbol{u}_t|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}\|\boldsymbol{v}\|_h,\end{aligned} \tag{4.31}$$

$$\begin{aligned}|III_4| &\lesssim \|\boldsymbol{\xi}_1(\tau)\|_0\|\boldsymbol{v}\|_0 + \tau\|\boldsymbol{\xi}_2(\tau)\|_h\|\boldsymbol{v}\|_h\\ &\lesssim \tau^2\|\boldsymbol{u}_{ttt}\|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)}\|\boldsymbol{v}\|_0\\ &\quad + \tau^3|\boldsymbol{u}_{tt}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^1(\beta))^2\times H^2(\beta))\right)}\|\boldsymbol{v}\|_h.\end{aligned} \tag{4.32}$$

Moreover, using the same argument for estimating the term I_2, we deduce that

$$|III_3| \lesssim \tau\Big(h\sum_{\beta\in\Omega}|u_3^\beta(0)|_{3,\,\beta} + h\|\ddot{\boldsymbol{u}}(0)\|_0 + h^2\sum_{\beta\in\Omega}\|f_3^\beta(0)\|_{0,\,\beta}\Big)\|\boldsymbol{v}\|_h. \tag{4.33}$$

Choosing $\boldsymbol{v}=\boldsymbol{w}_2$ in (4.28), from (4.30)-(4.33) and the ε-inequality we have

$$\begin{aligned}\|\boldsymbol{w}_2\|_0 \lesssim{}& h|\boldsymbol{u}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)} + h|\boldsymbol{u}_t|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}\\ &+ h\|\boldsymbol{u}_{tt}\|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)} + h^2\sum_{\beta\in\Omega}\|f_3^\beta\|_{L^\infty\left(0,\,T;\,L^2(\beta)\right)}\\ &+ \tau^2|\boldsymbol{u}_{tt}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^1(\beta))^2\times H^2(\beta))\right)} + \tau^2\|\boldsymbol{u}_{ttt}\|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)},\end{aligned} \tag{4.34}$$

which with (4.29) yields (4.24).

In order to prove (4.25), we first apply the similar technique for deriving (4.27) to rewrite (4.26) as

$$\begin{aligned}4B\big((\boldsymbol{u}-\boldsymbol{U})_{\frac12},\,\boldsymbol{v}\big)&+\tau^2 D_h\big((\boldsymbol{u}-\boldsymbol{U})_{\frac12},\,\boldsymbol{v}\big)\\ &= 4B(\boldsymbol{u}^0-\boldsymbol{U}^0,\,\boldsymbol{v}) + 2\tau B(\dot{\boldsymbol{u}}(0)-\boldsymbol{I}_h\dot{\boldsymbol{u}}(0),\boldsymbol{v})\\ &\quad + \frac{\tau^3}{2}D_h(\dot{\boldsymbol{u}}(0)-\boldsymbol{I}_h\dot{\boldsymbol{u}}(0),\boldsymbol{v})\\ &\quad + \tau^2\big(B(\ddot{\boldsymbol{u}}(0),\,\boldsymbol{v}) + D_h(\boldsymbol{u}(0),\,\boldsymbol{v}) - (\boldsymbol{f}(0),\,\boldsymbol{v})\big)\\ &\quad + 2\tau B(\boldsymbol{\xi}_1(\tau),\,\boldsymbol{v}) + \frac{\tau^2}{2}D_h(\boldsymbol{\xi}_2(\tau),\,\boldsymbol{v}).\end{aligned} \tag{4.35}$$

Write

$$(\boldsymbol{u}-\boldsymbol{U})_{\frac{1}{2}} = (\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u})_{\frac{1}{2}} + (\boldsymbol{I}_h\boldsymbol{u}-\boldsymbol{U})_{\frac{1}{2}} =: \boldsymbol{w}_3+\boldsymbol{w}_4. \tag{4.36}$$

Then (4.35) can be rewritten as

$$\begin{aligned}
4B(\boldsymbol{w}_4,\,\boldsymbol{v}) &+ \tau^2 D_h(\boldsymbol{w}_4,\,\boldsymbol{v})\\
&= -4B(\boldsymbol{w}_3,\,\boldsymbol{v}) - \tau^2 D_h(\boldsymbol{w}_3,\,\boldsymbol{v})\\
&\quad + 4B(\boldsymbol{u}^0-\boldsymbol{U}^0,\,\boldsymbol{v}) + 2\tau B(\dot{\boldsymbol{u}}(0)-\boldsymbol{I}_h\dot{\boldsymbol{u}}(0),\boldsymbol{v})\\
&\quad + \frac{\tau^3}{2} D_h(\dot{\boldsymbol{u}}(0)-\boldsymbol{I}_h\dot{\boldsymbol{u}}(0),\boldsymbol{v})\\
&\quad + \tau^2\big(B(\ddot{\boldsymbol{u}}(0),\,\boldsymbol{v}) + D_h(\boldsymbol{u}(0),\,\boldsymbol{v}) - (\boldsymbol{f}(0),\,\boldsymbol{v})\big)\\
&\quad + 2\tau B(\boldsymbol{\xi}_1(\tau),\,\boldsymbol{v}) + \frac{\tau^2}{2} D_h(\boldsymbol{\xi}_2(\tau),\,\boldsymbol{v}),
\end{aligned}$$

we have by (2.12) that

$$\|\boldsymbol{w}_3\|_0 + h\|\boldsymbol{w}_3\|_h \lesssim h^2|\boldsymbol{u}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}, \tag{4.37}$$

and argue as in the derivation of (4.34) to know

$$\begin{aligned}
\tau\|\boldsymbol{w}_4\|_h &\lesssim \|\boldsymbol{w}_3\|_0 + \tau\|\boldsymbol{w}_3\|_h + \|\boldsymbol{u}^0-\boldsymbol{U}^0\|_0\\
&\quad + \tau\|\dot{\boldsymbol{u}}(0)-\boldsymbol{I}_h\dot{\boldsymbol{u}}(0)\|_0 + \tau^2\|\dot{\boldsymbol{u}}(0)-\boldsymbol{I}_h\dot{\boldsymbol{u}}(0)\|_h\\
&\quad + \tau\Big(h\sum_{\beta\in\Omega}|u_3^\beta(0)|_{3,\,\beta} + h\|\ddot{\boldsymbol{u}}(0)\|_0 + h^2\sum_{\beta\in\Omega}\|f_3^\beta(0)\|_{0,\,\beta}\Big)\\
&\quad + \tau\|\boldsymbol{\xi}_1(\tau)\|_0 + \tau\|\boldsymbol{\xi}_2(\tau)\|_h\\
&\lesssim (h^2+\tau h)|\boldsymbol{u}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}\\
&\quad + \tau h|\boldsymbol{u}_t|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}\\
&\quad + \tau h\|\boldsymbol{u}_{tt}\|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)}\\
&\quad + \tau h^2\sum_{\beta\in\Omega}\|f_3^\beta\|_{L^\infty\left(0,\,T;\,L^2(\beta)\right)}\\
&\quad + \tau^3|\boldsymbol{u}_{tt}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^1(\beta))^2\times H^2(\beta))\right)}\\
&\quad + \tau^3\|\boldsymbol{u}_{ttt}\|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)},
\end{aligned}$$

i.e.,

$$\begin{aligned}
\|\boldsymbol{w}_4\|_h &\lesssim (\tau^{-1}h^2+h)|\boldsymbol{u}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}\\
&\quad + h|\boldsymbol{u}_t|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)}\\
&\quad + h\|\boldsymbol{u}_{tt}\|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)}\\
&\quad + h^2\sum_{\beta\in\Omega}\|f_3^\beta\|_{L^\infty\left(0,\,T;\,L^2(\beta)\right)}\\
&\quad + \tau^2|\boldsymbol{u}_{tt}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^1(\beta))^2\times H^2(\beta))\right)}\\
&\quad + \tau^2\|\boldsymbol{u}_{ttt}\|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)},
\end{aligned}$$

which with (4.36) and (4.37) gives (4.25). □

The following result is a direct consequence of Theorem 4.1 and Lemma 4.4.

Theorem 4.5. *Let $\boldsymbol{u}$ and $\{\boldsymbol{U}^n\}_{n=0}^N$ be the solutions of problems (2.2) and (4.1), respectively. Assume that*

$$\boldsymbol{u} \in L^\infty\big(0,\, T;\, \prod_{\beta\in\Omega}((H^2(\beta))^2 \times H^3(\beta))\big),$$

$$\boldsymbol{u}_{tt} \in L^2\big(0,\, T;\, \prod_{\beta\in\Omega}((H^2(\beta))^2 \times H^3(\beta))\big) \bigcap L^\infty\big(0,\, T;\, \prod_{\beta\in\Omega}((H^1(\beta))^2 \times H^2(\beta))\big),$$

$$\boldsymbol{u}_{tttt},\, \boldsymbol{f},\, \boldsymbol{f}_t \in L^2\big(0,\, T;\, \prod_{\beta\in\Omega}(L^2(\beta))^3\big),$$

$$f_3^\beta \in L^\infty\big(0,\, T;\, L^2(\beta)\big), \quad \forall\, \beta \in \Omega.$$

If the initial functions $\boldsymbol{U}^0$ and $\boldsymbol{U}^1$ are given by (4.23), then

$$\begin{aligned}
\max_{1\le M\le N} \|\boldsymbol{U}_{M-\frac12} - &\boldsymbol{u}(t^{M-\frac12})\|_h \\
&\lesssim (\tau^{-1}h^2 + h)|\boldsymbol{u}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)} \\
&\quad + h|\boldsymbol{u}_{tt}|_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}((H^2(\beta))^2\times H^3(\beta))\right)} \\
&\quad + h\|\boldsymbol{u}_{tt}\|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)} \\
&\quad + h\|\boldsymbol{u}_{ttt}\|_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)} \\
&\quad + h^2 \sum_{\beta\in\Omega}\Big(\|f_3^\beta\|_{L^\infty\left(0,\,T;\,L^2(\beta)\right)} + \|\dot f_3^\beta\|_{L^2\left(0,\,T;\,L^2(\beta)\right)}\Big) \\
&\quad + \tau^2|\boldsymbol{u}_{tt}|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}((H^1(\beta))^2\times H^2(\beta))\right)} \\
&\quad + \tau^2\|\boldsymbol{u}_{ttt}\|_{L^\infty\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)} \\
&\quad + \tau^2\|\boldsymbol{u}_{tttt}\|_{L^2\left(0,\,T;\,\prod_{\beta\in\Omega}(L^2(\beta))^3\right)}.
\end{aligned}$$

5. **Numerical examples.**

We provide some numerical examples to show the computational performance of the fully discrete finite element method (4.1).

We adopt the same notation described as before. As in Figure 2, fix a global coordinate system $(x_1,\, x_2,\, x_3)$ whose orthonormal basis vectors are denoted by $\{\boldsymbol{e}_1,\, \boldsymbol{e}_2,\, \boldsymbol{e}_3\}$. The two plate members are taken as $\beta_1 := (-1,\, 1)^2 \times \{0\}$ and $\beta_2 := \{0\} \times (-1,\, 1) \times (0,\, 1)$. Let $\gamma_1 := \bar\beta_1 \cap \bar\beta_2$ be the interface line, and choose the local coordinate system as

$$(x_1^{\beta_1},\, x_2^{\beta_1},\, x_3^{\beta_1}) = (x_1,\, x_2,\, x_3), \quad (x_1^{\beta_2},\, x_2^{\beta_2},\, x_3^{\beta_2}) = (x_2,\, x_3,\, x_1),$$

which implies that

$$\{\boldsymbol{e}_1^{\beta_1},\, \boldsymbol{e}_2^{\beta_1},\, \boldsymbol{e}_3^{\beta_1}\} = \{\boldsymbol{e}_1,\, \boldsymbol{e}_2,\, \boldsymbol{e}_3\}, \quad \{\boldsymbol{e}_1^{\beta_2},\, \boldsymbol{e}_2^{\beta_2},\, \boldsymbol{e}_3^{\beta_2}\} = \{\boldsymbol{e}_2,\, \boldsymbol{e}_3,\, \boldsymbol{e}_1\}.$$

Moreover, select $\boldsymbol{e}_1^{\gamma_1} = \boldsymbol{e}_1^{\beta_2}$.

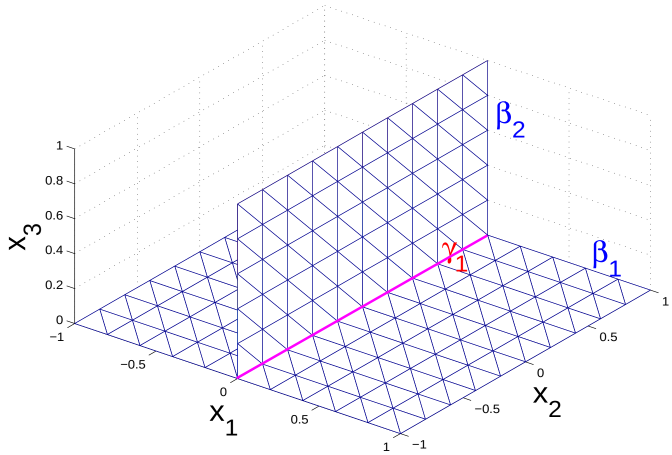

FIGURE 2. The triangulation for the plate-plate structure

Let the displacement fields $\boldsymbol{u}^{\beta_1} := u_i^{\beta_1}\boldsymbol{e}_i^{\beta_1}$ and $\boldsymbol{u}^{\beta_2} := u_i^{\beta_2}\boldsymbol{e}_i^{\beta_2}$ be given respectively by

$$\begin{cases} u_1^{\beta_1} := \frac{1}{100}e^t(1-x_1^2)^2(1-x_2^2)^2, \\ u_2^{\beta_1} := \frac{1}{100}e^t(1-x_1^2)^2(1-x_2^2)^2, \\ u_3^{\beta_1} := \frac{1}{100}e^t(1-x_1^2)^2(1-x_2^2)^2, \end{cases} \quad \text{and} \quad \begin{cases} u_1^{\beta_2} := \frac{1}{100}e^t(1-x_2^2)^2(1-x_3^2)^2, \\ u_2^{\beta_2} := \frac{1}{100}e^t(1-x_2^2)^2(1-x_3^2)^2, \\ u_3^{\beta_2} := \frac{1}{100}e^t(1-x_2^2)^2(1-x_3^2)^2. \end{cases}$$

It is easy to check that $\boldsymbol{u}^{\beta_1}$ and $\boldsymbol{u}^{\beta_2}$ satisfy the following interface conditions:

$$\boldsymbol{u}^{\beta_1} = \boldsymbol{u}^{\beta_2}, \quad \partial_{x_3} u_3^{\beta_2} = -\partial_{x_1} u_3^{\beta_1} \text{ on } \gamma_1. \tag{5.1}$$

Furthermore, using integration by parts we can show that $\boldsymbol{u} := \{\boldsymbol{u}^{\beta_1}, \boldsymbol{u}^{\beta_2}\}$ is just the unique solution of the following problem: Find $\boldsymbol{u} : t \in [0, 1] \longrightarrow \boldsymbol{u}(t) \in \boldsymbol{V}$ such that

$$\begin{cases} B(\ddot{\boldsymbol{u}}, \boldsymbol{v}) + D(\boldsymbol{u}, \boldsymbol{v}) = F(\boldsymbol{v}), \forall\, \boldsymbol{v} \in \boldsymbol{V}, \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0,\, \dot{\boldsymbol{u}}|_{t=0} = \boldsymbol{u}_1, \end{cases} \tag{5.2}$$

where $B(\cdot,\cdot)$ and $D(\cdot,\cdot)$ are given as in (2.3) and (2.4),

$$\boldsymbol{V} := \Big\{ \boldsymbol{v} = \{\boldsymbol{v}^{\beta_1}, \boldsymbol{v}^{\beta_2}\};\, \boldsymbol{v}^{\beta_1} \in (H_0^1(\beta_1))^2 \times H_0^2(\beta_1), \\ \boldsymbol{v}^{\beta_2} \in (H_*^1(\beta_2))^2 \times H_*^2(\beta_2),\, \boldsymbol{v} \text{ satisfies (5.1)} \Big\}$$

with

$$H_*^1(\beta_2) := \left\{ v \in H^1(\beta_2);\, v = 0 \text{ on } \partial\beta_2 \backslash \gamma_1 \right\},$$
$$H_*^2(\beta_2) := \left\{ v \in H^2(\beta_2);\, v = \partial_{\boldsymbol{n}^{\beta_2}} v = 0 \text{ on } \partial\beta_2 \backslash \gamma_1 \right\},$$

and

$$F(\boldsymbol{v}) := \sum_{k=1}^{2} \int_{\beta_k} \boldsymbol{f}^{\beta_k} \cdot \boldsymbol{v}^{\beta_k}\, \mathrm{d}\beta_k + \sum_{i=1}^{3} \int_{\gamma_1} f_i^{\gamma_1} v_i^{\beta_2}\, \mathrm{d}\gamma_1 - \int_{\gamma_1} f_4^{\gamma_1} \partial_{\boldsymbol{n}^{\beta_2}} v_3^{\beta_2}\, \mathrm{d}\gamma_1$$

with

$$\begin{aligned}
&\boldsymbol{f}^{\beta_k} := \rho_{\beta_k} t_{\beta_k} \ddot{\boldsymbol{u}}_i^{\beta_k} \boldsymbol{e}_i^{\beta_k} - \mathcal{Q}_{IJ,\,J}^{\beta_k}(\boldsymbol{u}) \boldsymbol{e}_I^{\beta_k} - \mathcal{M}_{IJ,\,IJ}^{\beta_k}(\boldsymbol{u}) \boldsymbol{e}_3^{\beta_k} \text{ in } \beta_k, \quad k = 1,\, 2,\\
&f_1^{\gamma_1} := \mathcal{Q}_{1J}^{\beta_2}(\boldsymbol{u}) n_J^{\beta_2} = -\mathcal{Q}_{12}^{\beta_2}(\boldsymbol{u}) \text{ on } \gamma_1,\\
&f_2^{\gamma_1} := \mathcal{Q}_{2J}^{\beta_2}(\boldsymbol{u}) n_J^{\beta_2} = -\mathcal{Q}_{22}^{\beta_2}(\boldsymbol{u}) \text{ on } \gamma_1,\\
&f_3^{\gamma_1} := \partial_{\boldsymbol{t}^{\beta_2}} \mathcal{M}_{\boldsymbol{nt}}^{\beta_2}(\boldsymbol{u}) + \mathcal{M}_{IJ,\,J}^{\beta_2}(\boldsymbol{u}) n_I^{\beta_2} \text{ on } \gamma_1,\\
&f_4^{\gamma_1} := \mathcal{M}_{\boldsymbol{nn}}^{\beta_2}(\boldsymbol{u}) \text{ on } \gamma_1.
\end{aligned}$$

Since $\boldsymbol{u}$ is given in advance, $\boldsymbol{f}^{\beta_k}$ $(k = 1,\, 2)$, $f_n^{\gamma_1}$ $(n = 1,\, 2,\, 3,\, 4)$, and $\boldsymbol{u}_0$, $\boldsymbol{u}_1$ can be computed explicitly. We assume that the structure is made of steel material, so

$$\rho_{\beta_k} = 7850,\ t_{\beta_k} = 0.01,\ \nu_{\beta_k} = 0.3,\ E_{\beta_k} = 2 \times 10^{11},\ k = 1,\, 2.$$

As shown in Figure 2, we introduce a family of triangulations $\{\mathcal{T}_h^{\beta_1},\, \mathcal{T}_h^{\beta_2}\}$, whose mesh size is denoted as h. Concretely, we partition β_1 into $(2S)^2$ equal squares with the length $h = 1/S$, and then divide each square into two triangles in the same direction, so that we get the triangulation $\mathcal{T}_h^{\beta_1}$. The subdivision $\mathcal{T}_h^{\beta_2}$ is obtained similarly.

Next, we construct a nonconforming element space $\boldsymbol{V}_h$ related to $\boldsymbol{V}$ by using the procedure given in Section 2, which reads

$$\begin{aligned}
\boldsymbol{V}_h := \Big\{ \boldsymbol{v}_h = \{\boldsymbol{v}_h^{\beta_1},\, \boldsymbol{v}_h^{\beta_2}\};\ &\boldsymbol{v}_h^{\beta_1} \in (V_{h,\,*}^1(\beta_1))^2 \times V_{h,\,*}^M(\beta_1),\\
&\boldsymbol{v}_h^{\beta_2} \in (V_{h,\,*}^1(\beta_2))^2 \times V_{h,\,*}^M(\beta_2),\\
v_{h,\,i}^{\beta_1}(p) \boldsymbol{e}_i^{\beta_1} = v_{h,\,i}^{\beta_2}(p) \boldsymbol{e}_i^{\beta_2},\ \forall\, p \in \gamma_1,\ &\partial_{x_3} v_{h,\,3}^{\beta_2}(m) = -\partial_{x_1} v_{h,\,3}^{\beta_1}(m),\ \forall\, m \in \gamma_1 \Big\},
\end{aligned}$$

where

$$\begin{aligned}
&V_{h,\,*}^1(\beta_1) := \left\{ v_h \in V_h^1(\beta_1);\ v_h(p) = 0,\ \forall\, p \in \partial\beta_1 \right\},\\
&V_{h,\,*}^M(\beta_1) := \left\{ v_h \in V_h^M(\beta_1);\ v_h(p) = 0,\ \partial_{\boldsymbol{n}^{\beta_1}} v_h(m) = 0, \forall\, p,\, m \in \partial\beta_1 \right\},\\
&V_{h,\,*}^1(\beta_2) := \left\{ v_h \in V_h^1(\beta_2);\ v_h(p) = 0,\ \forall\, p \in \partial\beta_2 \backslash \gamma_1 \right\},\\
&V_{h,\,*}^M(\beta_2) := \left\{ v_h \in V_h^M(\beta_2);\ v_h(p) = 0,\ \partial_{\boldsymbol{n}^{\beta_2}} v_h(m) = 0, \forall\, p,\, m \in \partial\beta_2 \backslash \gamma_1 \right\}.
\end{aligned}$$

Hence, the finite element method (4.1) for problem (5.2) is to find $\{\boldsymbol{U}^n\}_{n=0}^N \in \boldsymbol{V}_h$ such that

$$\begin{cases}
B(D_t^2 \boldsymbol{U}^n,\, \boldsymbol{v}_h) + D_h(\boldsymbol{U}_{n,\,\frac{1}{4}},\, \boldsymbol{v}_h) = \sum\limits_{k=1}^{2} \int_{\beta_k} \boldsymbol{f}_{n,\,\frac{1}{4}}^{\beta_k} \cdot \boldsymbol{v}_h^{\beta_k} \,\mathrm{d}\beta_k + \sum\limits_{i=1}^{3} \int_{\gamma_1} (f_i^{\gamma_1})_{n,\,\frac{1}{4}} v_{h,\,i}^{\beta_2} \,\mathrm{d}\gamma_1\\
\qquad\qquad - \int_{\gamma_1} (f_4^{\gamma_1})_{n,\,\frac{1}{4}} \partial_{\boldsymbol{n}^{\beta_2}} v_{h,\,3}^{\beta_2} \,\mathrm{d}\gamma_1,\\
\qquad\qquad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h,\ n = 1,\, 2,\, \cdots,\, N-1,\\
\boldsymbol{U}^0 \text{ is the approximation of } \boldsymbol{u}_0 \text{ in } \boldsymbol{V}_h,\\
\boldsymbol{U}^1 \text{ is the approximation of } \boldsymbol{u}(\tau) \text{ in } \boldsymbol{V}_h.
\end{cases}$$

We adopt two methods to compute approximate initial functions. The Method 1 corresponds to (4.21), and for the Method 2 $\boldsymbol{U}^0 = \boldsymbol{I}_h \boldsymbol{u}_0$ and $\boldsymbol{U}^1$ is taken by (4.23). Let $\beta_{c,\,1}$ and $\beta_{c,\,2}$ be the centers of the plate members β_1 and β_2 , respectively. For showing the convergence behavior of the two methods, define

$$E = \max_{1\le M\le N} \|(\boldsymbol{U}^M + \boldsymbol{U}^{M-1})/2 - \boldsymbol{u}(t^{M-\frac{1}{2}})\|_h,$$

$$Ep1 = \max_{1\le M\le N} \max_{1\le i\le 3} \frac{\left|(U_i^{\beta_1})^M(\beta_{c,1}) - (u_i^{\beta_1})^M(\beta_{c,1})\right|}{\left|(u_i^{\beta_1})^M(\beta_{c,1})\right|},$$

$$Ep2 = \max_{1\le M\le N} \max_{1\le i\le 3} \frac{\left|(U_i^{\beta_2})^M(\beta_{c,2}) - (u_i^{\beta_2})^M(\beta_{c,2})\right|}{\left|(u_i^{\beta_2})^M(\beta_{c,2})\right|},$$

where $Ep1$ and $Ep2$ represent the relative errors of the finite element solution at $\beta_{c,1}$ and $\beta_{c,2}$, respectively. Furthermore, for Method 1 we denote $F_1 := E/(h+\tau^2)$, and for Method 2, $F_2 := E/(\tau^{-1}h^2 + h + \tau^2)$. The computational results and the total number of degrees of freedom (Ndof) are shown in Table 1 and Table 2.

Table 1. The computational results for Method 1

h	τ	Ndof	E	F_1	$Ep1$	$Ep2$
1/6	1/8	1134	1.1732e–1	6.4358e–1	1.7452e–1	3.3643e–1
1/12	1/16	4854	6.2222e–2	7.1323e–1	8.7585e–2	8.7841e–2
1/24	1/24	20070	3.1921e–2	7.3546e–1	3.2126e–2	1.8856e–2
1/36	1/32	45654	2.1417e–2	7.4481e–1	1.8267e–2	6.8873e–3
1/48	1/40	81606	1.6143e–2	7.5231e–1	1.2110e–2	3.7364e–3

Table 2. The computational results for Method 2

h	τ	Ndof	E	F_2	$Ep1$	$Ep2$
1/6	1/8	1134	8.9357e–2	2.2090e–1	5.2683e–2	9.1694e–2
1/12	1/16	4854	4.7543e–2	2.3969e–1	1.8881e–2	2.4152e–2
1/24	1/24	20070	2.4241e–2	2.8496e–1	9.3805e–3	7.9877e–3
1/36	1/32	45654	1.6240e–2	3.0386e–1	6.3220e–3	3.9982e–3
1/48	1/40	81606	1.2220e–2	3.1478e–1	4.7422e–3	2.3441e–3

Moreover, we take $\tau^2 = h$ to examine the convergence rate of these two methods. The numerical results are given in Figure 3 in the log scale. It is seen that both of these methods are first-order accuracy with energy norm in space. Due to the relation $h = \tau^2$, it is natural that these methods are two-order accuracy in time for the present case.

From the above numerical examples, we may also show the validity of Theorem 4.2 and Theorem 4.5, i.e.,

$$\max_{1\le M\le N} \|(\boldsymbol{U}^M + \boldsymbol{U}^{M-1})/2 - \boldsymbol{u}(t^{M-\frac{1}{2}})\|_h = \begin{cases} O(h+\tau^2), & \text{for Method 1,} \\ O(\tau^{-1}h^2 + h + \tau^2), & \text{for Method 2.} \end{cases}$$

Moreover, the two methods have desired computational accuracy.

Acknowledgements. The authors would like to express their sincere thanks to the referees whose valuable comments and suggestions greatly improved the presentation of the manuscript.

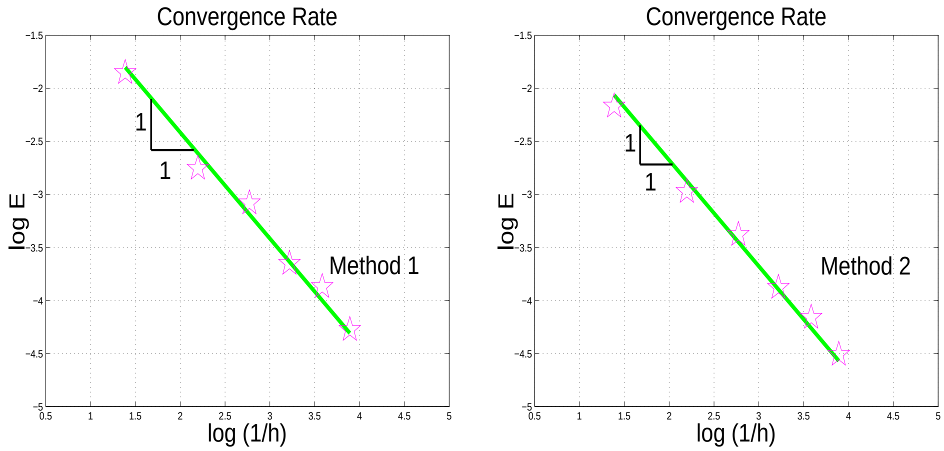

FIGURE 3. Rate of convergence in space in the log scale

REFERENCES

[1] R. Abraham, J. E. Marsden and T. Ratiu, "Manifolds, Tensor Analysis, and Applications," Addison-Wesley, Reading, MA, 1983.

[2] R. A. Adams and J. Fournier, "Sobolev Spaces," 2^{nd} edition, Academic Press, Amsterdam, 2003.

[3] G. A. Baker, *Error estimates for finite element methods for second order hyperbolic equations*, SIAM J. Numer. Anal., **13** (1976), 564–576.

[4] J. Bergh and J. Löfström, "Interpolation Spaces," Springer-Verlag, Berlin, 1976.

[5] M. Bernadou, S. Fayolle and F. Léné, *Numerical analysis of junctions between plates*, Comput. Methods Appl. Mech. Engrg., **74** (1989), 307–326.

[6] M. Bernadou and A. Cubier, *Numerical analysis of junctions between thin shells. Part 1: Continuous problems*, Comput. Methods Appl. Mech. Engrg., **161** (1998), 349–363.

[7] M. Bernadou and A. Cubier, *Numerical analysis of junctions between thin shells. Part 2: Approximation by finite element methods*, Comput. Methods Appl. Mech. Engrg., **161** (1998), 365–387.

[8] S. C. Brenner, *A two-level additive Schwarz preconditioner for nonconforming plate elements*, Numer. Math., **72** (1996), 419–447.

[9] S. C. Brenner and L. R. Scott, "The Mathematical Theory of Finite Element Methods," 3^{rd} edition, Springer-Verlag, New York, 2008.

[10] P. G. Ciarlet, "The Finite Element Method for Elliptic Problems," North-Holland, Amsterdam, 1978.

[11] P. G. Ciarlet, "Mathematical Elasticity II: Theory of Plates," Elsevier, Amsterdam, 1997.

[12] R. W. Clough and T. Penzien, "Dynamics of Structures," 2^{nd} edition, McGraw-Hill, Inc., New York, 1993.

[13] T. Dupont, *L^2-estimates for Galerkin methods for second order hyperbolic equations*, SIAM J. Numer. Anal., **10** (1973), 880–889.

[14] K. Feng, *Elliptic equations on composite manifold and composite elastic structures*, Math. Numer. Sinica, in Chinese, **1** (1979), 199–208.

[15] K. Feng and Z. Shi, "Mathematical Theory of Elastic Structures," Springer-Verlag, Berlin, 1996.

[16] F. d'Hennezel, *Domain decomposition method and elastic multi-structures: the stiffened problem*, Numer. Math., **66** (1993), 181–193.

[17] J. Huang, *Numerical solution of the elastic body-plate problem by nonoverlapping domain decomposition type techniques*, Math. Comp., **73** (2004), 19–34.

[18] J. Huang, Z. Shi and Y. Xu, *Some studies on mathematical models for general elastic multi-structures*, Sci. China Ser. A, **48** (2005), 986–1007.

[19] J. Huang, Z. Shi and Y. Xu, *Finite element analysis for general elastic multi-structures*, Sci. China Ser. A, **49** (2006), 109–129.

[20] J. Huang, L. Guo and Z. Shi, *A finite element method for investigating general elastic multi-structures*, Comput. Math. Appl., **53** (2007), 1867–1895.

[21] J. Huang, L. Guo and Z. Shi, *Vibration analysis of Kirchhoff plates by the Morley element method*, J. Comput. Appl. Math., **213** (2008), 14–34.

[22] V. Kozlov, V. Mazya and A. Movchan, "Asymptotic Analysis of Fields in Multi-Structures," Clarendon Press, Oxford, 1999.

[23] J. L. Lions and E. M. Magenes, "Non-homogeneous Boundary Value Problems and Applications I, II," Springer-Verlag, Berlin, 1972.

[24] L. X. Peng, S. Kitipornchai and K. M. Liew, *Free vibration analysis of folded plate structures by the FSDT mesh-free method*, Comput. Mech., **39** (2007), 799–814.

[25] Z. Shi, *Error estimates for the Morley element*, Numer. Math. Sinica, in Chinese, **12** (1990), 113–118.

[26] Z. Shi and Z. Xie, *Substructure preconditioners for nonconforming plate elements*, J. Comput. Math., **16** (1998), 385–395.

[27] V. Thomée, "Galerkin Finite Element Methods for Parabolic Problems," Springer-Verlag, Berlin, 1997.

[28] I. Titeux and E. Sanchez-Palencia, *Junction of thin plates*, Eur. J. Mech. A Solids, **19** (2000), 377–400.

[29] J. Wloka, "Partial Differential Equations," Cambridge University Press, London, 1987.

[30] O. C. Zienkiewicz and R. L. Taylor, "The Finite Element Method," Vol **2**, McGraw-Hill Book Company, London, 1989.

Received August 2007; revised October 2008.

E-mail address: laijunjiang@sjtu.edu.cn
E-mail address: jghuang@sjtu.edu.cn