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A SIMPLE AND EFFICIENT TRIANGULAR FINITE ELEMENT FOR PLATE BENDING

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ABSTRACT: A simple and efficient triangular finite element is introduced for plate bending application. The element is a three-node triangular one with three basic degrees of freedom per node and two internal rotation degrees of freedom, using selective reduced integration. Numerical examples indicate that, despite its simplicity, the element is not only competitively accurate, but also useful as a thick/thin triangular plate bending element. It is also pointed out that this element using selective reduced integration is in fact a mixed element.

KEY WORDS: reduced integration, selective reduced integration, element using reduced integration, element using selective reduced integration, mixed element.

I. INTRODUCTION

Whether or not an element can be extensively applied in engineering depends to a large extent on cost-effective and accurate computer programs. At present, there are elements whose displacement models are derived from the theory of plates with transverse shear deformations, and which use reduced integration or selectice reduced integration. Sometimes these elements are simple and efficient[1-3]. However, reduced integration has only been applied in the quadrilateral elements. Reference 4 applied selective reduced integration to a three-node triangular element with the three basic degrees of freedom per node (with one numerical integration point on the shear energy term) and it was found that the result was not satisfactory.

In this paper, the author changed the shape functions of the triangular element, put forward a three-node element with eleven parameters containing two internal rotations, using reduced integration on the shear energy term, and found that this element is rather simple and numerical examples indicate that not only it is competitively accurate, but also it is thick/thin triangular plate bending element, and that the element using selective reduced integration is in fact a mixed element.

II. TRIANGULAR PLATE BENDING ELEMENT

The strain energy for an linear elastic plate, including shear deformation, is

$$\pi_p = \frac{1}{2}\int \bar{k}^T D_b \bar{k} dxdy + \frac{1}{2}\int \bar{r}^T D_s \bar{r} dxdy$$

$$= \frac{1}{2} U^T [\bar{B}_b^T D_b \bar{B}_b dxdy + \int \bar{B}_s^T D_s \bar{B}_s dxdy] \cdot U \qquad (1)$$

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where

$$\bar{k}=\begin{bmatrix} -\dfrac{\partial\psi_x}{\partial x} \\[2ex] -\dfrac{\partial\psi_y}{\partial y} \\[2ex] -\left(\dfrac{\partial\psi_x}{\partial y}+\dfrac{\partial\psi_y}{\partial x}\right) \end{bmatrix}=\bar{B}_b\cdot U \tag{2}$$

$$\bar{r}=\begin{bmatrix} \dfrac{\partial w}{\partial x}-\psi_x \\[2ex] \dfrac{\partial w}{\partial y}-\psi_y \end{bmatrix}=\bar{B}_s\cdot U \tag{3}$$

$$D_b=\frac{Eh^3}{12(1-\nu^2)}\begin{bmatrix} & \nu & 0 \\ & 1 & 0 \\ (\text{sym}) & & \dfrac{1-\nu}{2} \end{bmatrix} \tag{4}$$

$$D_s=\frac{Ehk}{2(1+\nu)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5}$$

The variable w is displacement, the variables ψ_x and ψ_y are rotations, positive directions of w, ψ_x and ψ_y are shown in Figure 1. The variables h, E and ν are respectively the thickness of the plate, Young's modulus and Poisson's ratio and k is the shear correction factor usually taken as 5/6.

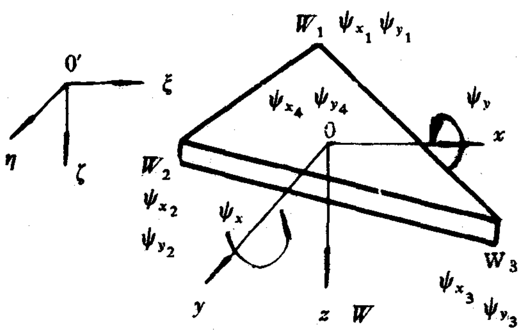

Fig.1

$\bar{B}_b$ and $\bar{B}_s$ are respectively the strain-displacement transformation matrix and U is displacement matrix.

Figure 1 shows a typical triangular element. The centre of the triangular element is taken as the origin of its local coordinate XOY and directions of the local coordinate agree with those of the global coordinate $\xi 0'\eta$. The element has a total of eleven degrees of freedom (dof), that is: 3 dof at the corner nodes and 2 dof for internal rotations.

The shape function of the displacement w is taken from reference 8, it satisfies the requirement for continuity. It is:

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \frac{1}{2}[\beta_2 x^2 + (\beta_3 + \gamma_2)xy + \gamma_3 y^2] \tag{6}$$

where

$$\beta_2 = \frac{1}{2A}(a_1\psi_{x_1} + a_2\psi_{x_2} + a_3\psi_{x_3}) \qquad \beta_3 = \frac{1}{2A}(b_1\psi_{x_1} + b_2\psi_{x_2} + b_3\psi_{x_3})$$

$$\gamma_2 = \frac{1}{2A}(a_1\psi_{y_1} + a_2\psi_{y_2} + a_3\psi_{y_3}) \qquad \gamma_3 = \frac{1}{2A}(b_1\psi_{y_1} + b_2\psi_{y_2} + b_3\psi_{y_3})$$

$$\alpha_1 = \frac{1}{2A}(c_1 w'_1 + c_2 w'_2 + c_3 w'_3) \qquad \alpha_2 = \frac{1}{2A}(a_1 w'_1 + a_2 w'_2 + a_3 w'_3)$$

$$\alpha_3 = \frac{1}{2A}(b_1 w'_1 + b_2 w'_2 + b_3 w'_3)$$

$$w'_i = w_i - d_i \qquad i = 1,2,3$$

$$d_i = \frac{1}{2}[\beta_2 x_i^2 + (\beta_3 + \gamma_2)x_i y_i + \gamma_3 y_i^2] \qquad i = 1,2,3$$

$$a_1 = y_2 - y_3 \qquad a_2 = y_3 - y_1 \qquad a_3 = y_1 - y_3$$

$$b_1 = x_3 - x_2 \qquad b_2 = x_1 - x_3 \qquad b_3 = x_2 - x_1$$

$$c_1 = x_2 y_3 - x_3 y_2 \qquad c_2 = x_3 y_1 - x_1 y_3 \qquad c_3 = x_1 y_2 - x_2 y_1$$

Because the origin of the local coordinate is at the center, so c_1, c_2 and c_3 are equal to $\frac{2}{3}A$. A is the area of the element, x_i and y_i are the coordinates at the corner nodes and w_i, ψ_{x_i} and ψ_{y_i} are the displacement and rotations at the corner nodes. $(i = 1,2,3)$

The shape functions of the rotations are:

$$\psi_x = \xi_1\psi_{x_1} + \xi_2\psi_{x_2} + \xi_3\psi_{x_3} + \xi_1\xi_2\xi_3\psi_{x_4} \tag{7}$$

$$\psi_y = \xi_1\psi_{y_1} + \xi_2\psi_{y_2} + \xi_3\psi_{y_3} + \xi_1\xi_2\xi_3\psi_{y_4} \tag{8}$$

The variables ξ_1, ξ_2 and ξ_3 are the triangular area coordinates.

Using reduced integration on the shear energy term of equation 1, the strain energy of the element is:

$$\pi'_p = \frac{1}{2}U^T k'_{p_b} U + \frac{1}{2}U^T k'_{p_s} U \tag{9}$$

where

$$k'_{p_b} = \int \bar{B}_b^T D_b \bar{B}_b dxdy \tag{10}$$

$$k'_{p_s} = \int B_s^T D_s B_s dxdy \tag{11}$$

where B_s consists of the constant terms in $\bar{B}_s$, that is,

$$B_s^T = \frac{1}{2A}\begin{bmatrix} a_1 & b_1 \\ -\frac{1}{4A}[a_1(a_1x_1^2 + b_1x_1y_1) + a_2(a_1x_2^2 + b_1x_2y_2) + a_3(a_1x_3^2 + b_1x_3y_3)] - c_1 & -\frac{1}{4A}[b_1(a_1x_1^2 + b_1x_1y_1) + b_2(a_1x_2^2 + b_1x_2y_2) + b_3(a_1x_3^2 + b_1x_3y_3)] \\ -\frac{1}{4A}[a_1(a_1x_1y_1 + b_1y_1^2) + a_2(a_1x_2y_2 + b_1y_2^2) + a_3(a_1x_3y_3 + b_1y_3^2)] & -\frac{1}{4A}[b_1(a_1x_1y_1 + b_1y_1^2) + b_2(a_1x_2y_2 + b_1y_2^2) + b_3(a_1x_3y_3 + b_1y_3^2)] - c_1 \\ a_2 & b_2 \\ -\frac{1}{4A}[a_1(a_2x_1^2 + b_2x_1y_1) + a_2(a_2x_2^2 + b_2x_2y_2) + a_3(a_2x_3^2 + b_2x_3y_3)] - c_2 & -\frac{1}{4A}[b_1(a_2x_1^2 + b_2x_1y_1) + b_2(a_2x_2^2 + b_2x_2y_2) + b_3(a_2x_3^2 + b_2x_3y_3)] \\ -\frac{1}{4A}[a_1(a_2x_1y_1 + b_2y_1^2) + a_2(a_2x_2y_2 + b_2y_2^2) + a_3(a_2x_3y_3 + b_2y_3^2)] & -\frac{1}{4A}[b_1(a_2x_1y_1 + b_2y_1^2) + b_2(a_2x_2y_2 + b_2y_2^2) + b_3(a_2x_3y_3 + b_2y_3^2)] - c_2 \\ a_3 & b_3 \\ -\frac{1}{4A}[a_1(a_3x_1^2 + b_3x_1y_1) + a_2(a_3x_2^2 + b_3x_2y_2) + a_3(a_3x_3^2 + b_3x_3y_3)] - c_3 & -\frac{1}{4A}[b_1(a_3x_1^2 + b_3x_1y_1) + b_2(a_3x_2^2 + b_3x_2y_2) + b_3(a_3x_3^2 + b_3x_3y_3)] \\ -\frac{1}{4A}[a_1(a_3x_1y_1 + b_3y_1^2) + a_2(a_3x_2y_2 + b_3y_2^2) + a_3(a_3x_3y_3 + b_3y_3^2)] & -\frac{1}{4A}[b_1(a_3x_1y_1 + b_3y_1^2) + b_2(a_3x_2y_2 + b_3y_2^2) + b_3(a_3x_3y_3 + b_3y_3^2)] - c_3 \\ -\frac{2A}{27} & 0 \\ 0 & -\frac{2A}{27} \end{bmatrix} \tag{12}$$

and

$$U^T = [w_1 \psi_{x_1} \psi_{y_1} w_2 \psi_{x_2} \psi_{y_2} w_3 \psi_{x_3} \psi_{y_3} \psi_{x_4} \psi_{y_4}] \tag{13}$$

III. EQUIVALENCE BETWEEN THE ELEMENT USING SELECTIVE REDUCED INTEGRATION AND THE MIXED ELEMENT

For plate bending problems, reference 5 proposes a mixed Hellinger-Reissner principle. The strain energy is

$$\pi_R = \int\left(k^T D_b \bar{k} - \frac{1}{2}k^T D_6 k\right)dxdy + \int\left(r^T D_s \bar{r} - \frac{1}{2}r^T D_s r\right)dxdy \tag{14}$$

k and r are respectively independent strain components that are not related to the displacements w, ψ_x and ψ_y.

For the triangular element shown in Figure 1, the assumed shape function of the displacement w is given by equations (6).

The assumed shape functions of rotations are

$$\psi_x = \xi_1\psi_{x_1} + \xi_2\psi_{x_2} + \xi_3\psi_{x_3} + \frac{20}{9}\xi_1\xi_2\xi_3\psi_{x_4} \tag{15}$$

$$\psi_y = \xi_1\psi_{y_1} + \xi_2\psi_{y_2} + \xi_3\psi_{y_3} + \frac{20}{9}\xi_1\xi_2\xi_3\psi_{y_4} \tag{16}$$

Let

$$\gamma = B_s U \tag{17}$$

where B_s is given by equation (12).

Let

$$k = B_b U \tag{18}$$

where:

$$B_b = -\frac{1}{2A}\begin{bmatrix} 0 & a_1 & 0 & 0 & a_2 & 0 & 0 & a_3 & 0 & \zeta(a_1\xi_2\xi_3 + a_2\xi_1\xi_3 + a_3\xi_1\xi_2) & 0 \\ 0 & 0 & b_1 & 0 & 0 & b_2 & 0 & 0 & b_3 & 0 & \zeta(a_1\xi_2\xi_3 + b_2\xi_1\xi_3 + b_3\xi_1\xi_2) \\ 0 & b_1 & a_1 & 0 & b_2 & a_2 & 0 & b_3 & a_3 & \zeta(b_1\xi_2\xi_3 + b_2\xi_1\xi_3 + b_3\xi_1\xi_2) & \zeta(a_1\xi_2\xi_3 + a_2\xi_1\xi_3 + a_3\xi_1\xi_2) \end{bmatrix} \tag{19}$$

and:

$$\zeta = \frac{20 + \sqrt{319}}{9} \quad \left(\text{or} \quad \frac{20 - \sqrt{319}}{9}\right)$$

Using equations (17) and (18) in equation (14), we have

$$\pi_R = \frac{1}{2}U^T k_{R_b} U + \frac{1}{2}U^T k_{R_s} U \tag{20}$$

where:

$$k_{R_b} = \int B_b^T D_b (2\bar{B}_b - B_b) dxdy \tag{21}$$

$$k_{R_s} = \int B_s^T D_s (2\bar{B}_s - B_s) dxdy \tag{22}$$

Because the origin of the local coordinate is at the centre, so

$$\int x dxdy = \int y dxdy = 0$$

and note

$$\int (a_1\xi_2\xi_3 + a_2\xi_1\xi_3 + a_3\xi_1\xi_2) dxdy = \int (b_1\xi_2\xi_3 + b_2\xi_1\xi_3 + b_3\xi_1\xi_2) dxdy = 0$$

through computations, we have:

$$k_{R_b} = k'_{P_b} \tag{23}$$

$$k_{R_s} = k'_{P_s} \tag{24}$$

Equations (23) and (24) mean that the stiffness matrix of the previous element using selective reduced integration is equal to that of this mixed element, that is, the element using selective reduced integration is in fact a mixed element. However, due to the use of selective reduced integration, not only the variational principle of the finite element has been changed, but also some modifications of the original assumed shape function are made.

Reference 5 recommends the elements using reduced integration that have the same stiffness matrix as that of mixed elements, referred to as R24 and MR24. These elements do not change the original assumed shape functions, but the element described here does.

IV. NUMERICAL EXAMPLES

In this section we take several numerical examples which have become more or less standard ones for evaluating plate elements. All computations were performed on a TQ–16 computer.

The equivalent node load may be given by the minimum potential principle. The bending moment at centre of the plate is given by average of the bending moments of the surrounding elements. The bending moments of elements are given by $\psi_x = \xi_1\psi_{x_1} + \xi_2\psi_{x_2} + \xi_2\psi_{x_3}$ and $\psi_y = \xi_1\psi_{y_1} + \xi_2\psi_{y_2} + \xi_2\psi_{y_3}$, and ψ_{x_4} and ψ_{y_4} are omitted.

Square plate:

The data for this example consists of the following (see Fig.2)

$$E = 10.92 \times 10^5, \quad \nu = 0.3, \quad t = 0.1, \quad L = 10:$$

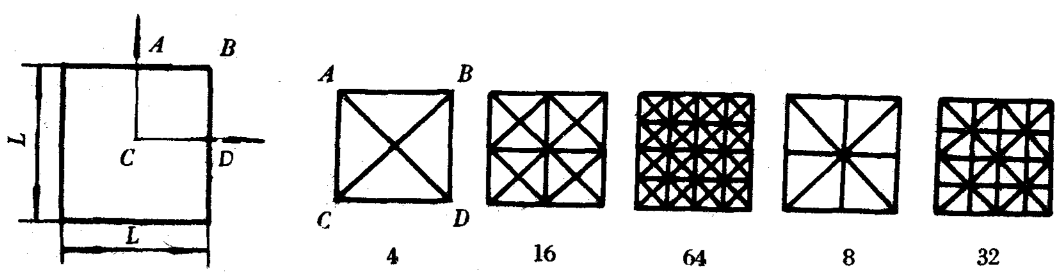

Fig. 2

Simple supported and clamped boundary conditions were considered as well as concentrated and uniformly distributed loadings. Results are presented in Tables I and II. Centre displacements of plates with various thickness under uniformly distributed loadings are presented in Table III. The theoretical value of centre displacement is taken from reference 6 for thin plate and from reference 7 for thick plate.

Table 1

Normalized centre displacement and bending moment for a simple supported square plate.

Number of elements	Displacement concentrated load	Displacement uniform load	Moment-uniform load
4	1.2137	1.3434	0.8218
16	1.0318	1.0461	0.9584
64	1.0095	1.0086	0.9858

Table 2

Normalized centre displacement and bending moment for a clamped square plate.

Number of elements	Displacement concentrated load	Displacement uniform load	Moment-uniform load
4	1.1393	1.2660	0.1158
16	1.0349	1.0475	0.8433
64	1.0137	1.0147	0.9561

Table 3

Maximum nondimensional normal deflection under uniform pressure q. $w_{max}= qL^4/100D(v=0.3)$

B	N.O.E \ t/L α	0.35	0.3	0.25	0.2	0.15	0.1	0.01	0.001	0.0001	0.00001
S.S	8	0.69134	0.6216	0.56197	0.51247	0.47324	0.44457	0.42119	0.42095	0.42094	0.42208
	32	0.67305	0.60442	0.54613	0.49814	0.46036	0.4328	0.40999	0.40976	0.40976	0.40673
	theoretic value	0.6649	0.5967	0.5382	0.4907	0.4537	0.4273	0.4062	0.4062	0.4062	0.4062
C	8	0.3967	0.3279	0.26919	0.22045	0.18165	0.15296	0.12908	0.12884	0.12883	0.12892
	32	0.39478	0.3259	0.2673	0.21887	0.18054	0.15233	0.12855	0.12829	0.12829	0.12802
	theoretic value	0.3951	0.3238	0.2675	0.2167	0.1798	0.1499	0.126	0.126	0.126	0.126

N.O.E—number of elements, B—boundry, S.S——simple supported. C——clamped.

Clamped circular plate:

The data for this example is the same as that given for the previous problem except $R=5$, $t=0.1$; (see figure 3)

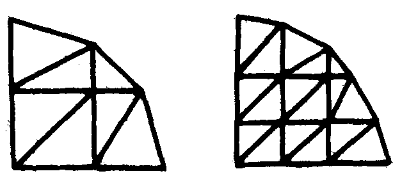

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Fig 3

Results are presented in Table 4 for concentrated and uniform loadings.

Table 4

Normalized centre displacement and bending moment for a clamped circular plate

Number of elements	Displacement concentrated load	Displacement uniform load	Moment-uniform load
7	1.0266	1.0752	0.5986
17	1.0186	1.0361	0.8695

V. CONCLUSIONS

1. The element using selective reduced integration of this paper is in fact a mixed element. However, due to the use of selective reduced integration, not only the variational principle of finite elements has to be changed, but also some modifications of the original assumed shape functions should be made.
2. We have presented an element for the bending of both thin and thick plates, which involves minimal programming and is highly efficient and competitively accurate.

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