DOI: 10.1007/s11425-005-0118-x

Finite element analysis for general elastic multi-structures

HUANG Jianguo[1,2], SHI Zhongci[3] & XU Yifeng[4]

1. Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China;
2. Division of Computational Science, E-Institute of Shanghai Universities, Shanghai Normal University, Shanghai 200234, China;
3. Institute of Computational Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China;
4. Department of Mathematics, Chinese University of Hong Kong, Shatin, N. T., Hong Kong, China

Correspondence should be addressed to Huang Jianguo (email: jghuang@sjtu.edu.cn)

Received April 19, 2005; accepted August 14, 2005

Abstract A finite element method is introduced to solve the general elastic multi-structure problem, in which the displacements on bodies, the longitudinal displacements on plates and the longitudinal displacements on beams are discretized using conforming linear elements, the rotational angles on beams are discretized using conforming elements of second order, the transverse displacements on plates and beams are discretized by the Morley elements and the Hermite elements of third order, respectively. The generalized Korn's inequality is established on related nonconforming element spaces, which implies the unique solvability of the finite element method. Finally, the optimal error estimate in the energy norm is derived for the method.

Keywords: elastic multi-structures, finite elements, generalized Korn's inequality, unique solvability, error estimates.

1 Introduction

Elastic multi-structures are widely used in engineering applications. In the past few decades, most existing literatures[1−8] were devoted to mathematical modeling and numerical solutions for simple elastic multi-structures composed of only two elastic members. However, Feng and Shi established mathematical models in refs. [9,10] for general elastic multi-structures by the variational principle after presenting reasonable interface conditions in terms of mechanical interpretation. The corresponding mathematical theory was developed in ref. [11] by Huang et al. As the continuation of the former reference, this paper is intended to provide a systematic finite element analysis for the general elastic multi-structure problem. We would like to mention the following words by Ciarlet (see ref. [1], p. 180), the member of French Academy of Sciences, to emphasize the importance of such studies: "A challenging program consists in numerically approximating the mathematical models of elastic multi-structures that comprise many substructures."

At first, we briefly review the mathematical model[10,11] for a general elastic multi-structure for later uses. Let there be given N_3 body members $\Omega^3 := \{\alpha_1, \cdots, \alpha_{N_3}\}$, N_2 plate members $\Omega^2 := \{\beta_1, \cdots, \beta_{N_2}\}$, and N_1 rod (beam) members $\Omega^1 := \{\gamma_1, \cdots, \gamma_{N_1}\}$, which are rigidly connected to form an elastic multi-structure:

$$\Omega = \{\alpha_1, \cdots, \alpha_{N_3}; \beta_1, \cdots, \beta_{N_2}; \gamma_1, \cdots, \gamma_{N_1}\}$$

satisfying the four conditions in Section 1 of ref. [11]. So each body member α is a bounded polyhedron and each plate member β is a bounded polygon.

We denote all proper boundary area elements of bodies by

$$\Gamma^2 := \{\beta_{N_2+1}, \cdots, \beta_{N_2'}\} = \Gamma_1^2 \cup \Gamma_2^2,$$

where

$$\Gamma_1^2 := \{\beta_{N_2+1}, \cdots, \beta_{N_2+M_2}\}, \quad \Gamma_2^2 := \{\beta_{N_2+M_2+1}, \cdots, \beta_{N_2'}\}.$$

Here Γ_1^2 consists of all external proper boundary area elements while Γ_2^2 consists of all interfaces of bodies. Similarly, we denote all the proper boundary lines of plates by

$$\Gamma^1 := \{\gamma_{N_1+1}, \cdots, \gamma_{N_1'}\} = \Gamma_1^1 \cup \Gamma_2^1,$$

where

$$\Gamma_1^1 := \{\gamma_{N_1+1}, \cdots, \gamma_{N_1+M_1}\}, \quad \Gamma_2^1 := \{\gamma_{N_1+M_1+1}, \cdots, \gamma_{N_1'}\}.$$

Here Γ_1^1 consists of all external boundary lines while Γ_2^1 consists of all interfaces of plates. We denote all boundary points of the rod members by $\Gamma^0 := \{\delta_1, \cdots, \delta_{N_0}\}$, and all corner points of proper boundaries of plate members by $\Gamma_3^0 := \{\delta_{N_0+1}, \cdots, \delta_{N_0'}\}$. Let $\Gamma^0 = \Gamma_1^0 \cup \Gamma_2^0$ with

$$\Gamma_1^0 := \{\delta_1, \cdots, \delta_{M_0}\}, \quad \Gamma_2^0 := \{\delta_{M_0+1}, \cdots, \delta_{N_0}\}.$$

Here Γ_1^0 consists of all external boundary points while Γ_2^0 consists of all common boundary points. An element of Ω^3, $\Omega^2 \cup \Gamma^2$, $\Omega^1 \cup \Gamma^1$ and $\Gamma^0 \cup \Gamma_3^0$ are called respectively a body, area, line and point element.

A right-handed orthogonal system (x_1, x_2, x_3) is fixed in the space $\mathbb{R}^3$, with its orthonormal basis vectors being $\{\vec{e}_i\}_{i=1}^3$. For each element ω, a local right-handed coordinate system $(x_1^\omega, x_2^\omega, x_3^\omega)$ is prescribed as follows ($\{\vec{e}_i^{\,\omega}\}_{i=1}^3$ represent the related orthonormal basis vectors). The above coordinate system is used as the local coordinate system for any body member α or any point element δ. For an area element β, x_1^β and x_2^β are its longitudinal directions, and x_3^β the transverse direction. For a line element γ, x_1^γ is the longitudinal direction, x_2^γ and x_3^γ are the transverse directions, and the origin of the local coordinates is located at an endpoint of γ. Moreover, along the boundary $\partial\beta$ of an area element β we choose a unit tangent vector $\vec{t}^\beta$ such that $\{\vec{n}^\beta, \vec{t}^\beta, \vec{e}_3^{\,\beta}\}$ forms a right-handed coordinate system, where $\vec{n}^\beta$ denotes the unit outward normal to $\partial\beta$ in the longitudinal plane, and $\vec{e}_3^{\,\beta}$ the unit transverse vector of the area element.

For a body member $\alpha \in \Omega^3$, let $\vec{n}^\alpha$ be the unit outward normal to the boundary $\partial\alpha$. For any two elements $\beta \in \Omega^2 \cap \Gamma^2$ and $\alpha \in \Omega^3, \alpha \in \partial^{-1}\beta$ means the β is a boundary element of α. For any two elements $\beta \in \Omega^2$ and $\gamma \in \Omega^1 \cup \Gamma^1$, we define

$$\varepsilon(\beta,\gamma) := \begin{cases} 0 & \text{if } \gamma \notin \partial\beta, \\ 1 & \text{if } \gamma \in \partial\beta,\ \vec{e}_1^{\,\gamma} \text{ and } \vec{t}^{\beta} \text{ have the same direction on } \gamma, \\ -1 & \text{if } \gamma \in \partial\beta,\ \vec{e}_1^{\,\gamma} \text{ and } \vec{t}^{\beta} \text{ have the opposite direction on } \gamma. \end{cases}$$

The symbols $\gamma \in \partial^{-1}\beta$, $\delta \in \partial^{-1}\gamma$ and $\varepsilon(\beta,\delta)$ are understood in the similar manners.

We assume that there exist the clamped conditions on a line element $\gamma_{N_1+1} \in \partial\beta_1$:

$$\vec{u}^{\beta_1} = \vec{0},\ \partial_{\vec{n}^{\beta_1}} u_3^{\beta_1} = 0 \quad \text{on } \gamma_{N_1+1},$$

and impose the force and moment free conditions on all kinds of proper boundaries of Ω except γ_{N_1+1}. Here $\vec{u}^{\beta_1}$ denotes the displacement field on the plate member β_1 (see the following descriptions for details). It is mentioned that all derivations in this paper can be extended to problems with other boundary conditions after some straightforward modifications.

Since Ω is rigidly connected, the admissible space $\vec{V}$ of generalized displacement fields consists of all functions

$$\vec{v} := \{\{\vec{v}^\alpha\}_{\alpha\in\Omega^3},\ \{\vec{v}^\beta\}_{\beta\in\Omega^2},\ \{\vec{v}^\gamma\}_{\gamma\in\Omega^1},\ \{v_4^\gamma\}_{\gamma\in\Omega^1}\}$$

in $\prod_{\alpha\in\Omega^3} \vec{W}(\alpha) \times \prod_{\beta\in\Omega^2} \vec{W}(\beta) \times \prod_{\gamma\in\Omega^1} \vec{W}(\gamma) \times \prod_{\gamma\in\Omega^1} H^1(\gamma)$, which satisfy the following interface conditions:

$$\vec{v}^\alpha = \vec{v}^{\alpha'} \quad \text{on } \beta,\ \forall\, \alpha,\alpha' \in \partial^{-1}\beta,\ \forall\, \beta \in \Gamma_2^2; \tag{1.1}$$

$$\vec{v}^\beta = \vec{v}^{\beta'}, \quad \varepsilon(\beta,\gamma)\partial_{\vec{n}^\beta} v_3^\beta = \varepsilon(\beta',\gamma)\partial_{\vec{n}^{\beta'}} v_3^{\beta'} \quad \text{on } \gamma,\ \forall\, \beta,\beta' \in \partial^{-1}\gamma,\ \forall\, \gamma \in \Gamma_2^1; \tag{1.2}$$

$$v_i^\gamma \vec{e}_i^{\,\gamma} = v_i^{\gamma'} \vec{e}_i^{\,\gamma'}, \quad v_{i+3}^\gamma \vec{e}_i^{\,\gamma} = v_{i+3}^{\gamma'} \vec{e}_i^{\,\gamma'} \quad \text{on } \delta,\ \forall\, \gamma,\gamma' \in \partial^{-1}\delta \cap \Omega^1,\ \forall\, \delta \in \Gamma_2^0, \tag{1.3}$$

where $v_5^\gamma := -dv_3^\gamma/dx_1^\gamma$, $v_6^\gamma := dv_2^\gamma/dx_1^\gamma$;

$$\vec{v}^\alpha = \vec{v}^\beta \quad \text{on } \beta,\ \forall\, \alpha \in \partial^{-1}\beta,\ \forall\, \beta \in \Omega^2; \tag{1.4}$$

$$\vec{v}^\beta = \vec{v}^\gamma, \quad -\varepsilon(\beta,\gamma)\partial_{\vec{n}^\beta} v_3^\beta = v_4^\gamma \quad \text{on } \gamma,\ \forall\, \beta \in \partial^{-1}\gamma,\ \forall\, \gamma \in \Omega^1. \tag{1.5}$$

Here,

$$\vec{v}^\alpha := v_i^\alpha \vec{e}_i^{\,\alpha}, \quad \vec{v}^\beta := v_i^\beta \vec{e}_i^{\,\beta}, \quad \vec{v}^\gamma := v_i^\gamma \vec{e}_i^{\,\gamma},$$

$$\vec{W}(\alpha) := (H^1(\alpha))^3,\ \vec{W}(\beta) := (H_*^1(\beta))^2 \times H_*^2(\beta),\ \vec{W}(\gamma) := H^1(\gamma) \times (H^2(\gamma))^2,$$

$$H_*^1(\beta_1) := H_0^1(\beta_1;\gamma_{N_1+1}),\ H_*^2(\beta_1) := H_0^2(\beta_1;\gamma_{N_1+1}),$$

$$H_*^1(\beta) := H^1(\beta),\ H_*^2(\beta) := H^2(\beta) \text{ for each } \beta \in \Omega^2\backslash\beta_1.$$

It is noted that we adopt the standard notations for Sobolev spaces[12–14], and also use the same index and summation conventions as described in ref. [11]. That means, Latin indices i, j, l take their values in the set $\{1, 2, 3\}$, while the capital Latin indices I, J, L (resp. K) take their values in the set $\{1, 2\}$ (resp. $\{2, 3\}$). The summation is implied when a Latin index (or a capital Latin index) is repeated exactly two times.

Therefore, under the action of the applied generalized load field

$$\vec{f} := \{\{\vec{f}^\alpha\}_{\alpha\in\Omega^3}, \{\vec{f}^\beta\}_{\beta\in\Omega^2}, \{\vec{f}^\gamma\}_{\gamma\in\Omega^1}, \{f_4^\gamma\}_{\gamma\in\Omega^1}\},$$

the generalized displacement field of the equilibrium configuration

$$\vec{u} := \{\{\vec{u}^\alpha\}_{\alpha\in\Omega^3}, \{\vec{u}^\beta\}_{\beta\in\Omega^2}, \{\vec{u}^\gamma\}_{\gamma\in\Omega^1}, \{u_4^\gamma\}_{\gamma\in\Omega^1}\}$$

of Ω is governed by the following problem[10,11]: Find $\vec{u} \in \vec{V}$ such that

$$D(\vec{u}, \vec{v}) = F(\vec{v}), \quad \forall\, \vec{v} \in \vec{V}, \tag{1.6}$$

where

$$F(\vec{v}) := \sum_{\alpha\in\Omega^3} F^\alpha(\vec{v}) + \sum_{\beta\in\Omega^2} F^\beta(\vec{v}) + \sum_{\gamma\in\Omega^1} F^\gamma(\vec{v}),$$

$$F^\alpha(\vec{v}) := \int_\alpha \vec{f}^\alpha\cdot\vec{v}^\alpha d\alpha, \quad F^\beta(\vec{v}) := \int_\beta \vec{f}^\beta\cdot\vec{v}^\beta d\beta,\ F^\gamma(\vec{v}) := \int_\gamma \vec{f}^\gamma\cdot\vec{v}^\gamma d\gamma + \int_\gamma f_4^\gamma v_4^\gamma d\gamma;$$

moreover, for $\vec{w} = \{\{\vec{w}^\alpha\}_{\alpha\in\Omega^3}, \{\vec{w}^\beta\}_{\beta\in\Omega^2}, \{\vec{w}^\gamma\}_{\gamma\in\Omega^1}, \{w_4^\gamma\}_{\gamma\in\Omega^1}\} \in \vec{V}$,

$$D(\vec{v}, \vec{w}) := \sum_{\alpha\in\Omega^3} D^\alpha(\vec{v}, \vec{w}) + \sum_{\beta\in\Omega^2} D^\beta(\vec{v}, \vec{w}) + \sum_{\gamma\in\Omega^1} D^\gamma(\vec{v}, \vec{w}),$$

where

$$D^\alpha(\vec{v}, \vec{w}) := \int_\alpha \sigma_{ij}^\alpha(\vec{v})\varepsilon_{ij}^\alpha(\vec{w}) d\alpha,$$

$$\varepsilon_{ij}^\alpha(\vec{v}) := (\partial_i v_j^\alpha + \partial_j v_i^\alpha)/2, \quad \partial_i v_j^\alpha := v_{j,i}^\alpha = \partial v_j^\alpha / \partial x_i^\alpha,$$

$$\sigma_{ij}^\alpha(\vec{v}) := \frac{E_\alpha}{1+\nu_\alpha}\varepsilon_{ij}^\alpha(\vec{v}) + \frac{E_\alpha\nu_\alpha}{(1+\nu_\alpha)(1-2\nu_\alpha)}(\varepsilon_{ll}^\alpha(\vec{v}))\delta_{ij},\ 1 \leqslant i, j \leqslant 3,$$

$$D^\beta(\vec{v}, \vec{w}) := \int_\beta [\mathcal{Q}_{IJ}^\beta(\vec{v})\varepsilon_{IJ}^\beta(\vec{w}) d\beta + \mathcal{M}_{IJ}^\beta(\vec{v})\mathcal{K}_{IJ}^\beta(\vec{w})] d\beta,$$

$$\varepsilon_{IJ}^\beta(\vec{v}) := (\partial_I v_J^\beta + \partial_J v_I^\beta)/2,\ \partial_I v_J^\beta := v_{J,I}^\beta = \frac{\partial v_J^\beta}{\partial x_I^\beta},$$

$$\mathcal{Q}_{IJ}^\beta(\vec{v}) := \frac{E_\beta h_\beta}{1-\nu_\beta^2}((1-\nu_\beta)\varepsilon_{IJ}^\beta(\vec{v}) + \nu_\beta(\varepsilon_{LL}^\beta(\vec{v}))\delta_{IJ}),\ 1 \leqslant I, J \leqslant 2,$$

$$\mathcal{K}_{IJ}^\beta(\vec{v}) := -\partial_{IJ} v_3^\beta = -\frac{\partial^2 v_3^\beta}{\partial x_I^\beta \partial x_J^\beta},$$

$$\mathcal{M}_{IJ}^\beta(\vec{v}) := \frac{E_\beta h_\beta^3}{12(1-\nu_\beta^2)}((1-\nu_\beta)\mathcal{K}_{IJ}^\beta(\vec{v}) + \nu_\beta(\mathcal{K}_{LL}^\beta(\vec{v}))\delta_{IJ}),$$

$$D^\gamma(\vec{v}, \vec{w}) := \int_\gamma [\mathcal{Q}_1^\gamma(\vec{v})\varepsilon_{11}^\gamma(\vec{w}) + \mathcal{M}_i^\gamma(\vec{v})\mathcal{K}_i^\gamma(\vec{w})] d\gamma,$$

$$\varepsilon_{11}^\gamma(\vec{v}) := dv_1^\gamma / dx_1^\gamma, \quad \mathcal{Q}_1^\gamma(\vec{v}) := E_\gamma A_\gamma \varepsilon_{11}^\gamma(\vec{v}),$$

$$\mathcal{K}_2^\gamma(\vec{v}) := -d^2 v_3^\gamma / (dx_1^\gamma)^2, \quad \mathcal{K}_3^\gamma := d^2 v_2^\gamma / (dx_1^\gamma)^2,$$

$$\mathcal{M}_2^\gamma(\vec{v}) := E_\gamma I_{22}^\gamma \mathcal{K}_2^\gamma(\vec{v}) + E_\gamma I_{23}^\gamma \mathcal{K}_3^\gamma(\vec{v}),$$
$$\mathcal{M}_3^\gamma(\vec{v}) := E_\gamma I_{32}^\gamma \mathcal{K}_2^\gamma(\vec{v}) + E_\gamma I_{33}^\gamma \mathcal{K}_3^\gamma(\vec{v}),$$
$$I_{23}^\gamma = I_{32}^\gamma,$$
$$\mathcal{K}_1^\gamma(\vec{v}) := dv_4^\gamma/dx_1^\gamma, \quad \mathcal{M}_1^\gamma(\vec{v}) := \frac{E_\gamma}{2(1+\nu_\gamma)} J_\gamma \mathcal{K}_1^\gamma(\vec{v}).$$

Here $E_\omega > 0, \nu_\omega \in (0, 1/2)$ denote Young's modulus, Poisson's ratio of the elastic member $\omega = \alpha$, β, γ, respectively; h_β is the thickness of plate β; A_γ is the area of the cross section, I_{ij}^γ the moment of inertia of the cross section, and J_γ the geometric torsional rigidity of the cross section; δ_{ij} and δ_{IJ} stand for the usual Kronecker delta.

It was proved in ref. [11] that problem (1.6) has a unique solution $\vec{u} \in \vec{V}$. From now on, we will always use $\vec{u} = \{\{\vec{u}^\alpha\}_{\alpha\in\Omega^3}, \{\vec{u}^\beta\}_{\beta\in\Omega^2}, \{\vec{u}^\gamma\}_{\gamma\in\Omega^1}, \{u_4^\gamma\}_{\gamma\in\Omega^1}\}$ to denote the solution of (1.6), and assume that for all $\alpha \in \Omega^3$, $\beta \in \Omega^2$ and $\gamma \in \Omega^1$,

$$\vec{u}^\alpha \in (H^2(\alpha))^3,\ \vec{u}^\beta \in (H^2(\beta))^2 \times H^3(\beta),\ \vec{u}^\gamma \in H^2(\gamma) \times (H^3(\gamma))^2,\ u_4^\gamma \in H^2(\gamma),$$
$$\vec{f}^\alpha \in (L^2(\alpha))^3,\ \vec{f}^\beta \in (L^2(\beta))^3,\ \vec{f}^\gamma \in (L^2(\gamma))^3,\ f_4^\gamma \in L^2(\gamma). \tag{1.7}$$

In this paper, we attempt to propose a finite element method for solving problem (1.6). For each $\alpha \in \Omega^3$, let $\overline{\alpha} = \bigcup_{K^\alpha \in \mathcal{T}_h^\alpha} \overline{K}^\alpha$ denote a quasi-uniform triangulation[15,16] of α, with each element K^α being an open tetrahedron. Similarly, let $\bar{\beta} = \bigcup_{K^\beta \in \mathcal{T}_h^\beta} \overline{K}^\beta$ and $\bar{\gamma} = \bigcup_{K^\gamma \in \mathcal{T}_h^\gamma} \overline{K}^\gamma$ be the quasi-uniform triangulations of the plate member $\beta \in \Omega^2$ and the beam member $\gamma \in \Omega^1$, respectively. So we obtain a total triangulation of Ω,

$$\mathcal{T}_h^\Omega := \{\{\mathcal{T}_h^\alpha\}_{\alpha\in\Omega^3},\ \{\mathcal{T}_h^\beta\}_{\beta\in\Omega^2},\ \{\mathcal{T}_h^\gamma\}_{\gamma\in\Omega^1}\}.$$

For ease of exposition, we assume that all triangulations for individual elastic members are of the same size h. Moreover, the triangulation $\mathcal{T}_h^\Omega$ is matching across interfaces among different geometric elements. For instance, if $\beta \in \Omega^2$ and $\alpha \in \partial^{-1}\beta$, the restriction of the triangulation $\mathcal{T}_h^\alpha$ to β is nothing but the triangulation $\mathcal{T}_h^\beta$; if $\beta \in \Gamma_2^2$, all the triangulations $\mathcal{T}_h^\alpha$ for $\alpha \in \partial^{-1}\beta$ induce the same triangulation on β.

Let $V_h^1(\alpha)$, $V_h^1(\beta)$ and $V_h^1(\gamma)$ be the spaces of continuous piecewise linear functions associated with the triangulations $\mathcal{T}_h^\alpha$, $\mathcal{T}_h^\beta$ and $\mathcal{T}_h^\gamma$, respectively. Likewise, $V_h^2(\gamma)$ denotes the Lagrange element space of second order. Let $V_h^M(\beta)$ be the usual Morley element space while $V_h^H(\gamma)$ the Hermite element space[15,16]; that means, for each $K^\beta \in \mathcal{T}_h^\beta$, the local shape function space related to $V_h^M(\beta)$ is $P_2(K^\beta)$ equipped with the nodal variables

$$\Sigma_{K^\beta} := \{v(p_i^\beta),\ \partial_{\vec{n}_\beta^{\partial K^\beta}} v(m_i^\beta), \quad 1 \leqslant i \leqslant 3\};$$

for each $K^\gamma \in \mathcal{T}_h^\gamma$, the local shape function space related to $V_h^H(\gamma)$ is $P_3(K^\gamma)$ equipped with the nodal variables

$$\Sigma_{K^\gamma} := \{v(p_I^\gamma),\ v'(p_I^\gamma), \quad I = 1, 2\}.$$

Here and in what follows, $P_k(\omega)$ stands for the space of all polynomials with total degree no more than k on ω, with ω being an open set. The symbols

$\vec{n}_\beta^{\partial K^\beta}$ and $\vec{t}_\beta^{\partial K^\beta}$ denote the unit outward normal and tangent direction on ∂K^β respectively, such that $\{\vec{n}_\beta^{\partial K^\beta}, \vec{t}_\beta^{\partial K^\beta}, \vec{e}_3^\beta\}$ forms a right-handed coordinate system. The symbol p (resp. m) with or without indices is used to denote a vertex (resp. the midpoint of a side) of some individual element of a triangulation. For an area element $\beta \in \Omega^2 \cup \Gamma^2$ (resp. a line element $\gamma \in \Omega^1 \cup \Gamma^1$), $p \in \beta$ (resp. $p \in \gamma$) means that $p \in \bar{\beta}$ (resp. $p \in \bar{\gamma}$) is a vertex of some individual element of a triangulation; similarly, $m \in \gamma$ means that $m \in \bar{\gamma}$ is an edge midpoint of some individual element of a triangulation.

The following finite element spaces are then introduced to describe discrete displacement fields on individual elastic members.

$$\vec{W}_h(\alpha) := (V_h^1(\alpha))^3, \ \forall\, \alpha \in \Omega^3;$$
$$\vec{W}_h(\beta) := (V_h^1(\beta))^2 \times V_h^M(\beta), \ \forall\, \beta \in \Omega^2 \backslash \beta_1,$$

and

$$\vec{W}_h(\beta_1) := (V_h^1(\beta_1; \gamma_{N_1+1}))^2 \times V_h^M(\beta_1; \gamma_{N_1+1}),$$

where

$$V_h^1(\beta_1; \gamma_{N_1+1}) := \{v_h \in V_h^1(\beta_1);\ v_h(p) = 0, \quad \forall\, p \in \gamma_{N_1+1}\}$$

and

$$V_h^M(\beta_1; \gamma_{N_1+1}) := \{v_h \in V_h^M(\beta_1);\ v_h(p) = 0, \quad \partial_{\vec{n}^{\beta_1}} v_h(m) = 0, \quad \forall\, p,\ m \in \gamma_{N_1+1}\};$$
$$\vec{W}_h(\gamma) := V_h^1(\gamma) \times (V_h^H(\gamma))^2, \ \forall\, \gamma \in \Omega^1.$$

The discrete rigid conditions related to (1.1)-(1.5) are given below.

$$v_i^\alpha(p)\vec{e}_i^\alpha = v_i^{\alpha'}(p)\vec{e}_i^{\alpha'} \quad \forall\, p \in \beta,\ \forall\, \alpha, \alpha' \in \partial^{-1}\beta,\ \forall\, \beta \in \Gamma_2^2; \tag{1.8}$$
$$v_i^\alpha(p)\vec{e}_i^\alpha = v_i^\beta(p)\vec{e}_i^\beta \quad \forall\, p \in \beta,\ \forall\, \alpha \in \partial^{-1}\beta,\ \forall\, \beta \in \Omega^2; \tag{1.9}$$

for any line element $\gamma \in \Gamma_2^1$ and any two plate members $\beta, \beta' \in \partial^{-1}\gamma$,

$$v_i^\beta(p)\vec{e}_i^\beta = v_i^{\beta'}(p)\vec{e}_i^{\beta'},\ \varepsilon(\beta,\gamma)\partial_{\vec{n}^\beta} v_3^\beta(m) = \varepsilon(\beta',\gamma)\partial_{\vec{n}^{\beta'}} v_3^{\beta'}(m), \forall\, p, m \in \gamma; \tag{1.10}$$

for any rod member γ and any plate member $\beta \in \partial^{-1}\gamma$,

$$v_i^\beta(p)\vec{e}_i^\beta = v_i^\gamma(p)\vec{e}_i^\gamma \quad \forall\, p \in \gamma, \quad -\varepsilon(\beta,\gamma)\partial_{\vec{n}^\beta} v_3^\beta(m) = v_4^\gamma(m), \quad \forall\, m \in \gamma; \tag{1.11}$$
$$v_i^\gamma(\delta)\vec{e}_i^\gamma = v_i^{\gamma'}(\delta)\vec{e}_i^{\gamma'},\ v_{i+3}^\gamma(\delta)\vec{e}_i^\gamma = v_{i+3}^{\gamma'}(\delta)\vec{e}_i^{\gamma'},\ \forall\, \gamma, \gamma' \in \partial^{-1}\delta \cap \Omega^1,\ \forall\, \delta \in \Gamma_2^0. \tag{1.12}$$

With these notations, we obtain a total finite element space on Ω,

$$\vec{V}_h := \left\{ \vec{v} \in \prod_{\alpha\in\Omega^3} \vec{W}_h(\alpha) \times \prod_{\beta\in\Omega^2} \vec{W}_h(\beta) \times \prod_{\gamma\in\Omega^1} \vec{W}_h(\gamma) \times \prod_{\gamma\in\Omega^1} V_h^2(\gamma), \right. \left. \vec{v} \text{ satisfies (1.8)–(1.12)} \right\}. \tag{1.13}$$

Hence the finite element method for problem (1.6) is to find $\vec{u}_h \in \vec{V}_h$ such that

$$D_h(\vec{u}_h, \vec{v}_h) = F(\vec{v}_h), \quad \forall\, \vec{v}_h \in \vec{V}_h, \tag{1.14}$$

where

$$D_h(\vec{u}_h, \vec{v}_h) := \sum_{\alpha\in\Omega^3} D_h^\alpha(\vec{u}_h, \vec{v}_h) + \sum_{\beta\in\Omega^2} D_h^\beta(\vec{u}_h, \vec{v}_h) + \sum_{\gamma\in\Omega^1} D_h^\gamma(\vec{u}_h, \vec{v}_h),$$

$$D_h^\alpha(\vec{u}_h, \vec{v}_h) := \sum_{K^\alpha \in \mathcal{T}_h^\alpha} \int_{K^\alpha} \sigma_{ij}^\alpha(\vec{u}_h)\varepsilon_{ij}^\alpha(\vec{v}_h) dK^\alpha,$$

$$D_h^\beta(\vec{u}_h, \vec{v}_h) := \sum_{K^\beta \in \mathcal{T}_h^\beta} \int_{K^\beta} [\mathcal{Q}_{ij}^\beta(\vec{u}_h)\varepsilon_{ij}^\beta(\vec{v}_h) + \mathcal{M}_{ij}^\beta(\vec{u}_h)\mathcal{K}_{ij}^\beta(\vec{v}_h)] dK^\beta$$

and

$$D_h^\gamma(\vec{u}_h, \vec{v}_h) := \sum_{K^\gamma \in \mathcal{T}_h^\gamma} \int_{K^\gamma} [\mathcal{Q}_1^\gamma(\vec{u}_h)\varepsilon_{11}^\gamma(\vec{v}_h) + \mathcal{M}_i^\gamma(\vec{u}_h)\mathcal{K}_i^\gamma(\vec{v}_h)] dK^\gamma.$$

We equip the finite element space $\vec{V}_h$ with a norm $\|\cdot\|_h$ given by

$$\|\vec{v}_h\|_h := \left\{ \sum_{\alpha\in\Omega^3} |\vec{v}_h^\alpha|_{1,\alpha}^2 + \sum_{\beta\in\Omega^2} |\vec{v}_h^\beta|_{h,\beta}^2 + \sum_{\gamma\in\Omega^1} |\vec{v}_h^\gamma|_{h,\gamma}^2 + \sum_{\gamma\in\Omega^1} |v_{h,4}^\gamma|_{1,\gamma}^2 \right\}^{1/2},$$

for each $\vec{v}_h = \{\{\vec{v}_h^\alpha\}_{\alpha\in\Omega^3}, \{\vec{v}_h^\beta\}_{\beta\in\Omega^2}, \{\vec{v}_h^\gamma\}_{\gamma\in\Omega^1}, \{v_{h,4}^\gamma\}_{\gamma\in\Omega^1}\}$. Here,

$$|\vec{v}_h^\beta|_{h,\beta} := \left\{ \sum_{K^\beta\in\mathcal{T}_h^\beta} \left(\sum_{I=1}^2 |v_{h,I}^\beta|_{1,K^\beta}^2 + |v_{h,3}^\beta|_{2,K^\beta}^2 \right) \right\}^{1/2},$$

$$|\vec{v}_h^\gamma|_{h,\gamma} := \left\{ |v_{h,1}^\gamma|_{1,\gamma}^2 + \sum_{K=2}^3 |v_{h,K}^\gamma|_{2,\gamma}^2 \right\}^{1/2}.$$

It is clear that the norm $\|\cdot\|_h$ still makes sense for piecewise smooth vector fields related to $\mathcal{T}_h^\Omega$. In what follows, for a piecewise smooth function v related to the triangulation $\mathcal{T}_h^\beta$, $\|v\|_{k,\beta}$ and $|v|_{k,\beta}$ are understood in the similar manners. We also use the symbol "$\lesssim \cdots$" to denote "$\leqslant C\cdots$" with a generic positive constant C independent of corresponding parameters and functions under considerations, which may take different values in different appearances.

The main results of this paper read as follows.

Theorem 1.1. For all $\vec{v}_h \in \vec{V}_h$,

$$\begin{aligned}\|\vec{v}_h\|_h^2 \lesssim &\sum_{\alpha\in\Omega^3}\sum_{i,j=1}^3 \|\varepsilon_{ij}^\alpha(\vec{v}_h)\|_{0,\alpha}^2 + \sum_{\beta\in\Omega^2}\sum_{I,J=1}^2 (\|\varepsilon_{IJ}^\beta(\vec{v}_h)\|_{0,\beta}^2 + \|\mathcal{K}_{IJ}^\beta(\vec{v}_h)\|_{0,\beta}^2)\\ &+ \sum_{\gamma\in\Omega^1} (\|\varepsilon_{11}^\gamma(\vec{v}_h)\|_{0,\gamma}^2 + \sum_{i=1}^3 \|\mathcal{K}_i^\gamma(\vec{v}_h)\|_{0,\gamma}^2) \end{aligned} \tag{1.15}$$

and

$$\|\vec{v}_h\|_h^2 \lesssim D_h(\vec{v}_h, \vec{v}_h). \tag{1.16}$$

(1.15) can be viewed as a generalized Korn's inequality on the nonconforming finite element space $\vec{V}_h$.

Theorem 1.2. The discrete problem (1.14) has a unique solution in $\vec{V}_h$.

Theorem 1.3. Let $\vec{u}$ and $\vec{u}_h$ be the solutions of problem (1.6) and the finite element method (1.14), respectively. Then

$$\|\vec{u}-\vec{u}_h\|_h \lesssim h\Bigg\{\sum_{\alpha\in\Omega^3}\sum_{i=1}^{3}|u_i^\alpha|_{2,\alpha}^2+\sum_{\beta\in\Omega^2}\left(\sum_{I=1}^{2}|u_I^\beta|_{2,\beta}^2+|u_3^\beta|_{3,\beta}^2\right)$$
$$+\sum_{\gamma\in\Omega^1}\left(|u_1^\gamma|_{2,\gamma}^2+\sum_{K=2}^{3}|u_K^\gamma|_{3,\gamma}^2+|u_4^\gamma|_{2,\gamma}^2\right)$$
$$+\sum_{\beta\in\Omega^2}h^2\|f_3^\beta\|_{0,\beta}^2+\sum_{\gamma\in\Omega_2^1}\left(\sum_{K=2}^{3}h^2\|f_K^\gamma\|_{0,\gamma}^2+\|f_4^\gamma\|_{0,\gamma}^2\right)\Bigg\}^{1/2},$$

provided that the regularity assumption (1.7) holds true. Here Ω_2^1 consists of all rod members which are adjacent to one plate member at least.

The rest of this paper is organized as follows. Some basic results are given in sec. 2 which will be used for error analysis of the finite element method in later sections. Theorems 1.1–1.3 are proved in secs. 3 and 4, respectively.

2 Some basic results

By integration by parts, we have the following identities. For $K^\alpha\in\mathcal{T}_h^\alpha$,

$$\int_{K^\alpha}\sigma_{ij}^\alpha(\vec{u})\varepsilon_{ij}^\alpha(\vec{v}_h)dK^\alpha=-\int_{K^\alpha}\sigma_{ij,j}^\alpha(\vec{u})v_{h,i}^\alpha dK^\alpha+\int_{\partial K^\alpha}\sigma_{ij}^\alpha(\vec{u})n_j^{\alpha,\partial K^\alpha}v_{h,i}^\alpha ds^\alpha; \tag{2.1}$$

for $K^\beta\in\mathcal{T}_h^\beta$,

$$\int_{K^\beta}\mathcal{Q}_{IJ}^\beta(\vec{u})\varepsilon_{IJ}^\beta(\vec{v}_h)dK^\beta=-\int_{K^\beta}\mathcal{Q}_{IJ,J}^\beta(\vec{u})v_{h,I}^\beta dK^\beta+\int_{\partial K^\beta}\mathcal{Q}_{IJ}^\beta(\vec{u})n_J^{\beta,\partial K^\beta}v_{h,I}^\beta ds^\beta, \tag{2.2}$$

$$\int_{K^\beta}\mathcal{M}_{IJ}^\beta(\vec{u})\mathcal{K}_{IJ}^\beta(\vec{v}_h)dK^\beta=\int_{K^\beta}\mathcal{M}_{IJ,J}^\beta(\vec{u})\partial_I v_{h,3}^\beta dK^\beta-\int_{\partial K^\beta}\{\mathcal{M}_{\vec{n}\vec{n}}^{\beta,\partial K^\beta}(\vec{u})*\partial_{\vec{n}_\beta^{\partial K^\beta}}v_{h,3}^\beta+\mathcal{M}_{\vec{n}\vec{t}}^{\beta,\partial K^\beta}(\vec{u})\partial_{\vec{t}_\beta^{\partial K^\beta}}v_{h,3}^\beta\}ds^\beta, \tag{2.3}$$

where $\vec{n}_\beta^{\partial K^\beta}:=n_I^{\beta,\partial K^\beta}\vec{e}_I^\beta$, $\vec{t}_\beta^{\partial K^\beta}:=t_I^{\beta,\partial K^\beta}\vec{e}_I^\beta$ and

$$\mathcal{M}_{\vec{n}\vec{n}}^{\beta,\partial K^\beta}(\vec{u}):=\mathcal{M}_{IJ}^\beta(\vec{u})n_I^{\beta,\partial K^\beta}n_J^{\beta,\partial K^\beta},\quad \mathcal{M}_{\vec{n}\vec{t}}^{\beta,\partial K^\beta}(\vec{u}):=\mathcal{M}_{IJ}^\beta(\vec{u})n_I^{\beta,\partial K^\beta}t_J^{\beta,\partial K^\beta};$$

for $K^\gamma\in\mathcal{T}_h^\gamma$,

$$\int_{K^\gamma}\mathcal{Q}_1^\gamma(\vec{u})\varepsilon_{11}^\gamma(\vec{v}_h)dK^\gamma=-\int_{K^\gamma}\mathcal{Q}_{1,1}^\gamma(\vec{u})v_{h,1}^\gamma dK^\gamma+\sum_{\delta\in\partial K^\gamma}\varepsilon(K^\gamma,\delta)[\mathcal{Q}_1^\gamma(\vec{u})v_{h,1}^\gamma](\delta), \tag{2.4}$$

$$\int_{K^\gamma}\mathcal{M}_K^\gamma(\vec{u})\mathcal{K}_K^\gamma(\vec{v}_h)dK^\gamma=\int_{K^\gamma}\mathcal{Q}_K^\gamma(\vec{u})(v_{h,K}^\gamma)'dK^\gamma+\sum_{\delta\in\partial K^\gamma}\varepsilon(K^\gamma,\delta)[\mathcal{M}_K^\gamma(\vec{u})v_{h,K+3}^\gamma](\delta) \tag{2.5}$$

and

$$\int_{K^\gamma} \mathcal{M}_1^\gamma(\vec{u})\mathcal{K}_1^\gamma(\vec{v}_h)dK^\gamma = -\int_{K^\gamma} \mathcal{M}_{1,1}^\gamma(\vec{u})v_{h,4}^\gamma dK^\gamma + \sum_{\delta\in\partial K^\gamma} \varepsilon(K^\gamma,\delta)[\mathcal{M}_1^\gamma(\vec{u})v_{h,4}^\gamma](\delta). \tag{2.6}$$

For each $\alpha \in \Omega^3$, let $I_{1,h}^\alpha$ be the usual interpolation operator related to $V_h^1(\alpha)$; for each $\beta \in \Omega^2$, let $I_{1,h}^\beta$ and $I_{M,h}^\beta$ be the interpolation operators related to $V_h^1(\beta)$ and $V_h^M(\beta)$, respectively; for each $\gamma \in \Omega^1$, let $I_{1,h}^\gamma$, $I_{2,h}^\gamma$ and $I_{H,h}^\gamma$ be the interpolation operators related to $V_h^1(\gamma)$, $V_h^2(\gamma)$ and $V_h^H(\gamma)$, respectively. Then we define the interpolation operators $\vec{I}_h^\alpha$, $\vec{I}_h^\beta$ and $\vec{I}_h^\gamma$ by

$$\vec{I}_h^\alpha\vec{v}^\alpha := (I_{1,h}^\alpha v_i^\alpha)\vec{e}_i^\alpha, \quad \forall\, \vec{v}^\alpha \in (H^2(\alpha))^3,$$
$$\vec{I}_h^\beta\vec{v}^\beta := (I_{1,h}^\beta v_I^\beta)\vec{e}_I^\beta + (I_{M,h}^\beta v_3^\beta)\vec{e}_3^\beta, \quad \forall\, \vec{v}^\beta \in (H^2(\beta))^2 \times H^3(\beta),$$
$$\vec{I}_h^\gamma\vec{v}^\gamma := (I_{1,h}^\gamma v_1^\gamma)\vec{e}_1^\gamma + (I_{H,h}^\gamma v_K^\gamma)\vec{e}_K^\gamma, \quad \forall\, \vec{v}^\gamma \in H^2(\gamma) \times (H^3(\gamma))^2,$$

which result in a global interpolation operator $\vec{I}_h$ on Ω as follows:

$$(\vec{I}_h\vec{v})^\alpha := \vec{I}_h^\alpha\vec{v}^\alpha \quad \text{on } \alpha, \quad \forall\, \vec{v}^\alpha \in (H^2(\alpha))^3, \forall\, \alpha \in \Omega^3,$$
$$(\vec{I}_h\vec{v})^\beta := \vec{I}_h^\beta\vec{v}^\beta \quad \text{on } \beta, \quad \forall\, \vec{v}^\beta \in (H^2(\beta))^2 \times H^3(\beta), \forall\, \beta \in \Omega^2,$$
$$(\vec{I}_h\vec{v})^\gamma := \{\vec{I}_h^\gamma\vec{v}^\gamma, I_{2,h}^\gamma v_4^\gamma\} \text{ on } \gamma, \forall\, \{\vec{v}^\gamma,\ v_4^\gamma\} \in H^2(\gamma)\times(H^3(\gamma))^2\times H^2(\gamma), \forall\, \gamma \in \Omega^1.$$

Applying the well-known error estimates[15,16] for interpolation operators $I_{1,h}^\alpha$, $I_{1,h}^\beta$, $I_{M,h}^\beta$, $I_{1,h}^\gamma$, $I_{2,h}^\gamma$ and $I_{H,h}^\gamma$, we obtain

Lemma 2.1. For the above-mentioned interpolation operator $\vec{I}_h$,

$$\|\vec{u}-\vec{I}_h\vec{u}\|_h \lesssim h\left\{\sum_{\alpha\in\Omega^3}\sum_{i=1}^3 |u_i^\alpha|_{2,\alpha}^2 + \sum_{\beta\in\Omega^2}\left(\sum_{I=1}^2 |u_I^\beta|_{2,\beta}^2 + |u_3^\beta|_{3,\beta}^2\right) + \sum_{\gamma\in\Omega^1}\left(|u_1^\gamma|_{2,\gamma}^2 + \sum_{K=2}^3 |u_K^\gamma|_{3,\gamma}^2 + |u_4^\gamma|_{2,\gamma}^2\right)\right\}^{1/2}.$$

A few results derived in ref. [11] will be used to bound the consistency error for the finite element method (1.14) in Section 4.

Lemma 2.2[11]**.** Let $\vec{u} \in \vec{V}$ be the solution of problem (1.6). Assume that the regularity assumption (1.7) holds true. Then

$$-\sigma_{ij,j}^\alpha(\vec{u}) = f_i^\alpha \quad \text{in } L^2(\alpha), \quad \forall\, \alpha \in \Omega^3, \tag{2.7}$$
$$\sum_{\alpha\in\partial^{-1}\beta} \sigma_{ij}^\alpha(\vec{u})n_j^\alpha\vec{e}_i^\alpha = \vec{0} \quad \text{in } (H^{1/2}(\beta))^3, \quad \forall\, \beta \in \Gamma^2, \tag{2.8}$$
$$-\mathcal{M}_{1,1}^\gamma(\vec{u}) + \sum_{\beta\in\partial^{-1}\gamma} \varepsilon(\beta,\gamma)\mathcal{M}_{\vec{n}\vec{n}}^\beta(\vec{u}) = f_4^\gamma \quad \text{in } L^2(\gamma), \quad \forall\, \gamma \in \Omega^1, \tag{2.9}$$
$$\sum_{\beta\in\partial^{-1}\gamma} \varepsilon(\beta,\gamma)\mathcal{M}_{\vec{n}\vec{n}}^\beta(\vec{u}) = 0 \quad \text{in } H^{1/2}(\gamma), \quad \forall\, \gamma \in \Gamma^1\backslash\gamma_{N_1+1}, \tag{2.10}$$
$$\sum_{\gamma\in\partial^{-1}\delta} \varepsilon(\gamma,\delta)\mathcal{M}_i^\gamma(\vec{u})(\delta)\vec{e}_i^\gamma = \vec{0}, \quad \forall\, \delta \in \Gamma^0. \tag{2.11}$$

We remark that in the above equations and henceforth, we use the convention implied in ref. [11], i.e. for a sum $\Sigma_{t\in\Lambda} a_t$, if a_t is not defined for some $t_0 \in \Lambda$, a_{t_0} is taken to be zero automatically.

Lemma 2.3[11]**.** Let $\beta \in \Omega^2$ be a plate member in Ω. Then for each $\vec{v}^\beta \in (H^1(\beta))^3$,

$$-\int_\beta \mathcal{Q}^\beta_{IJ,J}(\vec{u}) v^\beta_I d\beta + \int_\beta \mathcal{M}^\beta_{IJ,J}(\vec{u}) \partial_I v^\beta_3 d\beta + \int_\beta \sum_{\alpha\in\partial^{-1}\beta} \sigma^\alpha_{ij}(\vec{u}) n^\alpha_j \vec{e}^\alpha_i \cdot v^\beta_l \vec{e}^\beta_l d\beta$$
$$- \int_\beta f^\beta_i v^\beta_i d\beta = \langle \mathcal{M}^\beta_{IJ,J}(\vec{u}) n^\beta_I, v^\beta_3 \rangle_{H^{-1/2}(\partial\beta)\times H^{1/2}(\partial\beta)}.$$

We define some auxiliary spaces by

$$H(\partial\beta_1) := \{v \in H^1(\partial\beta_1);\ v = 0 \text{ on } \gamma_{N_1+1}\},\ H(\partial\beta) := H^1(\partial\beta),\ \forall\, \beta \in \Omega^2 \backslash \beta_1.$$

$$\vec{H}(\Omega^1) := \Bigg\{ \vec{v}^{\Omega^1} = \{\vec{v}^\gamma\}_{\gamma\in\Omega^1} \in \prod_{\gamma\in\Omega^1} ((H^1(\gamma))^3;\ v^\gamma_i \vec{e}^\gamma_i = v^{\gamma'}_i \vec{e}^{\gamma'}_i \text{ on } \delta,$$
$$\forall\, \delta \in \Gamma^0_2,\ \gamma, \gamma' \in \partial^{-1}\delta \cap \Omega^1 \Bigg\}$$

and

$$\vec{H}(\partial\Omega^2) := \Bigg\{ \vec{v}^{\partial\Omega^2} = \{\vec{v}^{\partial\beta}\}_{\beta\in\Omega^2} \in \prod_{\beta\in\Omega^2} (H(\partial\beta))^3;\ v^{\partial\beta}_i \vec{e}^\beta_i = v^{\partial\beta'}_i \vec{e}^{\beta'}_i \text{ on } \gamma,$$
$$\forall\, \gamma \in \Gamma^1_2,\ \beta, \beta' \in \partial^{-1}\gamma \Bigg\}.$$

Let $\Omega^1 = \Omega^1_1 \cup \Omega^1_2$, where Ω^1_2 is the set of all rod members which are adjacent to one plate member at least (cf. Theorem 1.3), while Ω^1_1 is the set of the remaining rod members. Then we further define a trace space on $\partial\Omega^2 \cup \Omega^1$ by

$$\vec{H}(\partial\Omega^2 \cup \Omega^1) := \{\vec{v} = (\vec{v}^{\partial\Omega^2}, \vec{v}^{\Omega^1}) \in \vec{H}(\partial\Omega^2) \times \vec{H}(\Omega^1);\ \vec{v}^{\partial\beta} = \vec{v}^\gamma \text{ on } \gamma,$$
$$\forall\, \gamma \in \Omega^1_2,\ \beta \in \partial^{-1}\gamma\}.$$

The following identity will be used to alleviate the regularity requirement for the solution $\vec{u}$ of problem (1.6) in our finite element analysis.

Lemma 2.4[11]**.** Let $\vec{u} \in \vec{V}$ be the solution of problem (1.6). Assume that the regularity assumption (1.7) holds true. Then for each $(\vec{v}^{\partial\Omega^2}, \vec{v}^{\Omega^1}) \in \vec{H}(\partial\Omega^2 \cup \Omega^1)$,

$$\sum_{\beta\in\Omega^2} \Bigg\{ < \mathcal{M}^\beta_{IJ,J}(\vec{u}) n^\beta_I, v^{\partial\beta}_3 >_{H^{-1/2}(\partial\beta)\times H^{1/2}(\partial\beta)} + \sum_{\gamma\in\partial\beta} \int_\gamma (\mathcal{Q}^\beta_{IJ}(\vec{u}) n^\beta_J v^{\partial\beta}_I$$
$$- \mathcal{M}^\beta_{\vec{n}\vec{t}}(\vec{u}) \partial_{\vec{t}^\beta} v^{\partial\beta}_3) d\gamma \Bigg\} + \sum_{\gamma\in\Omega^1} \int_\gamma \mathcal{Q}^\gamma_i(\vec{u}) (v^\gamma_i)' d\gamma = \sum_{\gamma\in\Omega^1} \int_\gamma f^\gamma_i v^\gamma_i d\gamma.$$

3 Unique solvability of the finite element method

We introduce a broken Sobolev space by

$$\vec{W} := \Big\{ \vec{v} \in \prod_{\alpha\in\Omega^3} \vec{W}(\alpha) \times \prod_{\beta\in\Omega^2} \vec{W}(\beta) \times \prod_{\gamma\in\Omega^1} \vec{W}(\gamma) \times \prod_{\gamma\in\Omega^1} H^1(\gamma);$$
$$\vec{v} \text{ satisfies } (1.12) \Big\},$$

which is equipped with the norm

$$\|\vec{v}\|_{\vec{W}} := \Bigg\{ \sum_{\alpha\in\Omega^3}\sum_{i=1}^{3} \|v_i^\alpha\|_{1,\alpha}^2 + \sum_{\beta\in\Omega^2}\left(\sum_{I=1}^{2} \|v_I^\beta\|_{1,\beta}^2 + \|v_3^\beta\|_{2,\beta}^2\right)$$
$$+ \sum_{\gamma\in\Omega^1}\left(\|v_1^\gamma\|_{1,\gamma}^2 + \sum_{K=2}^{3}\|v_K^\gamma\|_{2,\gamma}^2 + \|v_4^\gamma\|_{1,\gamma}^2 \right) \Bigg\}^{1/2}$$

and the seminorm

$$|\vec{v}|_{\vec{W}} := \Bigg\{ \sum_{\alpha\in\Omega^3}\sum_{i=1}^{3} |v_i^\alpha|_{1,\alpha}^2 + \sum_{\beta\in\Omega^2}\left(\sum_{I=1}^{2} |v_I^\beta|_{1,\beta}^2 + |v_3^\beta|_{2,\beta}^2\right)$$
$$+ \sum_{\gamma\in\Omega^1}\left(|v_1^\gamma|_{1,\gamma}^2 + \sum_{K=2}^{3}|v_K^\gamma|_{2,\gamma}^2 + |v_4^\gamma|_{1,\gamma}^2 \right) \Bigg\}^{1/2}.$$

It is easy to check that $(\vec{W}, \|\cdot\|_{\vec{W}})$ is a Hilbert space. Moreover, using the similar argument for proving estimate (2.4) in ref. [11], we have

Lemma 3.1. For all $\vec{v} \in \vec{W}$,

$$|\vec{v}|_{\vec{W}}^2 \leqslant \|\vec{v}\|_{\vec{W}}^2 \lesssim \sum_{\alpha\in\Omega^3}\sum_{i,j=1}^{3} \|\varepsilon_{ij}^\alpha(\vec{v})\|_{0,\alpha}^2 + \sum_{\beta\in\Omega^2}\sum_{I,J=1}^{2} (\|\varepsilon_{IJ}^\beta(\vec{v})\|_{0,\beta}^2 + \|\mathcal{K}_{IJ}^\beta(\vec{v})\|_{0,\beta}^2)$$
$$+ \sum_{\gamma\in\Omega^1}\left(\|\varepsilon_{11}^\gamma(\vec{v})\|_{0,\gamma}^2 + \sum_{i=1}^{3}\|\mathcal{K}_i^\gamma(\vec{v})\|_{0,\gamma}^2 \right) + J_\Omega(\vec{v}), \tag{3.1}$$

where

$$J_\Omega(\vec{v}) := J_1(\vec{v}) + J_2(\vec{v}) + J_3(\vec{v}) + J_4(\vec{v}),$$

$$J_1(\vec{v}) := \sum_{\beta\in\Gamma_2^2}\sum_{\alpha,\alpha'\in\partial^{-1}\beta} \|\vec{v}^\alpha - \vec{v}^{\alpha'}\|_{0,\beta}^2,\ J_2(\vec{v}) := \sum_{\beta\in\Omega_2^2}\sum_{\alpha\in\partial^{-1}\beta} \|\vec{v}^\alpha - \vec{v}^\beta\|_{0,\beta}^2,$$

$$J_3(\vec{v}) := \sum_{\gamma\in\Gamma_2^1}\sum_{\beta,\beta'\in\partial^{-1}\gamma} \|\varepsilon(\beta,\gamma)\partial_{\vec{n}^\beta} v_3^\beta - \varepsilon(\beta',\gamma)\partial_{\vec{n}^{\beta'}} v_3^{\beta'}\|_{0,\gamma}^2$$
$$+ \sum_{\gamma\in\Gamma_2^1}\sum_{\beta,\beta'\in\partial^{-1}\gamma} \|\vec{v}^\beta - \vec{v}^{\beta'}\|_{0,\gamma}^2 = J_{31}(\vec{v}) + J_{32}(\vec{v}),$$

$$J_4(\vec{v}) := \sum_{\gamma\in\Omega_2^1}\sum_{\beta\in\partial^{-1}\gamma} \|\vec{v}^\beta - \vec{v}^\gamma\|_{0,\gamma}^2 + \sum_{\gamma\in\Omega_2^1}\sum_{\beta,\beta'\in\partial^{-1}\gamma} \|\varepsilon(\beta,\gamma)\partial_{\vec{n}^\beta} v_3^\beta + v_4^\gamma\|_{0,\gamma}^2$$
$$=: J_{41}(\vec{v}) + J_{42}(\vec{v}),$$

and Ω_2^2 consists of all plate members connected with body members.

We denote by $V_h^{AR}(\beta)$ the Argyris element space[15,16] associated with the triangulation $\mathcal{T}_h^\beta$; that is, for each $K^\beta \in \mathcal{T}_h^\beta$ with $\{p_i^\beta\}_{i=1}^3$ and $\{m_i^\beta\}$ as three vertices and midpoints of sides respectively, the shape function space is $P_5(K^\beta)$ equipped with the nodal variables

$$\begin{aligned}\Sigma_{K^\beta} := \{&v(p_i^\beta),\ \partial_1 v(p_i^\beta),\ \partial_2 v(p_i^\beta),\ \partial_{11} v(p_i^\beta),\ \partial_{12} v(p_i^\beta),\ \partial_{22} v(p_i^\beta),\\ &\partial_{\vec{n}_\beta^{\partial K^\beta}} v(m_i^\beta), \quad 1 \leqslant i \leqslant 3\}.\end{aligned}$$

Here and in what follows, the corresponding partial derivatives are associated with the local coordinate system involved, e.g., $\partial_1 v(p_i^\beta) := \partial_{x_1^\beta} v(p_i^\beta)$ in the present situation.

For each $\beta \in \Omega^2 \backslash \beta_1$, we define

$$\vec{W}_h^A(\beta) := (V_h^1(\beta))^2 \times V_h^{AR}(\beta)$$

and

$$\vec{W}_h^A(\beta_1) := (V_h^1(\beta_1; \gamma_{N_1+1}))^2 \times V_h^{AR}(\beta_1; \gamma_{N_1+1}),$$

where

$$V_h^{AR}(\beta_1; \gamma_{N_1+1}) := \{v_h \in V_h^{AR}(\beta_1);\ v_h = \partial_{\vec{n}^{\beta_1}} v_h = 0 \quad \text{on } \gamma_{N_1+1}\}.$$

Then we have a finite-dimensional subspace of $\vec{W}$ given by

$$\vec{W}_h := \left\{ \vec{v}_h \in \prod_{\alpha\in\Omega^3} \vec{W}_h(\alpha) \times \prod_{\beta\in\Omega^2} \vec{W}_h^A(\beta) \times \prod_{\gamma\in\Omega^1} \vec{W}_h(\gamma) \times \prod_{\gamma\in\Omega^1} V_h^2(\gamma);\right.$$
$$\left. \vec{v}_h \text{ satisfies (1.8)–(1.12)} \right\}.$$

Lemma 3.2. For all $\vec{v}_h \in \vec{W}_h$,

$$\begin{aligned}|\vec{v}_h|_{\vec{W}}^2 \lesssim &\sum_{\alpha\in\Omega^3}\sum_{i,j=1}^3 \|\varepsilon_{ij}^\alpha(\vec{v}_h)\|_{0,\alpha}^2 + \sum_{\beta\in\Omega^2}\sum_{I,J=1}^2 (\|\varepsilon_{IJ}^\beta(\vec{v}_h)\|_{0,\beta}^2 + \|\mathcal{K}_{IJ}^\beta(\vec{v}_h)\|_{0,\beta}^2)\\ &+ \sum_{\gamma\in\Omega^1}\left(\|\varepsilon_{11}^\gamma(\vec{v}_h)\|_{0,\gamma}^2 + \sum_{i=1}^3 \|\mathcal{K}_i^\gamma(\vec{v}_h)\|_{0,\gamma}^2\right). \qquad (3.2)\end{aligned}$$

Proof. By virtue of Lemma 3.1, it suffices to get desired estimate for the additional term $J_\Omega(\vec{v})$ which implies result (3.2). To this end, we introduce a set of interpolation operators $\vec{\Pi}_h^\beta$ and $\vec{\Pi}_h^\gamma$ by

$$\vec{\Pi}_h^\beta \vec{v}_h^\beta := (I_{1,h}^\beta v_{h,i}^\beta)\vec{e}_i^\beta, \quad \forall\, \vec{v}_h^\beta \in \vec{W}_h^A(\beta) \qquad (3.3)$$

and

$$\vec{\Pi}_h^\gamma \vec{v}_h^\gamma := (I_{1,h}^\gamma v_{h,i}^\gamma)\vec{e}_i^\gamma, \quad \forall\, \vec{v}_h^\gamma \in \vec{W}_h(\gamma).$$

It follows from condition (1.8) that

$$J_1(\vec{v}_h) = 0. \qquad (3.4)$$

In terms of (1.9), (3.3) and usual error estimates for the interpolation operators

$I_{1,h}^{\beta}$ and $I_{1,h}^{\gamma}$ [15], we know

$$\begin{aligned} J_2(\vec{v}_h) &\leqslant 2 \sum_{\beta\in\Omega_2^2} \|\vec{v}_h^{\beta} - \vec{\Pi}_h^{\beta}\vec{v}_h^{\beta}\|_{0,\beta}^2 \\ &= 2 \sum_{\beta\in\Omega_2^2} \|v_{h,3}^{\beta} - I_{1,h}^{\beta} v_{h,3}^{\beta}\|_{0,\beta}^2 \\ &\lesssim \sum_{\beta\in\Omega_2^2} h^4 |v_{h,3}^{\beta}|_{2,\beta}^2 \lesssim h^4 \sum_{\beta\in\Omega^2} \sum_{I,J=1}^{2} \|\mathcal{K}_{IJ}^{\beta}(\vec{v}_h)\|_{0,\beta}^2. \end{aligned} \tag{3.5}$$

We have by (1.10) and (3.3) that

$$\vec{\Pi}_h^{\beta}\vec{v}_h^{\beta} = \vec{\Pi}_h^{\beta'}\vec{v}_h^{\beta'}, \ \forall\, \beta,\beta' \in \partial^{-1}\gamma, \ \forall\, \gamma \in \Gamma_2^1,$$

from which and the error estimate for the interpolation operator $\vec{\Pi}_h^{\beta}$ we obtain

$$\begin{aligned} J_{32}(\vec{v}_h) &\lesssim \sum_{\gamma\in\Gamma_2^1} \sum_{\beta\in\partial^{-1}\gamma} \|\vec{v}_h^{\beta} - \vec{\Pi}_h^{\beta}\vec{v}_h^{\beta}\|_{0,\gamma}^2 \\ &\lesssim \sum_{\gamma\in\Gamma_2^1} \sum_{\beta\in\partial^{-1}\gamma} \|v_{h,3}^{\beta} - I_{1,h}^{\beta} v_{h,3}^{\beta}\|_{0,\infty,\beta}^2 \\ &\lesssim h^2 \sum_{\gamma\in\Gamma_2^1} \sum_{\beta\in\partial^{-1}\gamma} |v_{h,3}^{\beta}|_{2,\beta}^2 \\ &\lesssim h^2 \sum_{\beta\in\Omega^2} \sum_{I,J=1}^{2} \|\mathcal{K}_{IJ}^{\beta}(\vec{v}_h)\|_{0,\beta}^2. \end{aligned} \tag{3.6}$$

Similarly, by the mean value theorem and the inverse inequality for finite element functions, it follows from (1.10) and (3.3) that

$$\begin{aligned} J_{31}(\vec{v}_h) &\lesssim \sum_{\gamma\in\Gamma_2^1} \sum_{\beta\in\partial^{-1}\gamma} \sum_{F^{\beta}\subset\gamma} \|\varepsilon(\beta,\gamma)\partial_{\vec{n}^{\beta}} v_{h,3}^{\beta} - (\varepsilon(\beta,\gamma)\partial_{\vec{n}^{\beta}} v_{h,3}^{\beta})(m^{F^{\beta}})\|_{0,F^{\beta}}^2 \\ &\lesssim h^2 \sum_{\gamma\in\Gamma_2^1} \sum_{\beta\in\partial^{-1}\gamma} |v_{h,3}^{\beta}|_{2,\infty,\beta}^2 \lesssim \sum_{\gamma\in\Gamma_2^1} \sum_{\beta\in\partial^{-1}\gamma} |v_{h,3}^{\beta}|_{2,\beta}^2 \\ &\lesssim \sum_{\beta\in\Omega^2} \sum_{I,J=1}^{2} \|\mathcal{K}_{IJ}^{\beta}(\vec{v}_h)\|_{0,\beta}^2. \end{aligned} \tag{3.7}$$

Here and in what follows, for a point set $\mathcal{E}$, $F^{\beta} \subset \mathcal{E}$ indicates that F^{β} is some edge of a triangle K^{β} in $\mathcal{T}_h^{\beta}$, and some subset of $\mathcal{E}$ as well.

Arguing as in the above deduction, we also find

$$\begin{aligned} J_{41}(\vec{v}_h) &\lesssim \sum_{\gamma\in\Omega_2^1} \sum_{\beta\in\partial^{-1}\gamma} \|\vec{v}_h^{\beta} - \vec{\Pi}_h^{\beta}\vec{v}_h^{\beta}\|_{0,\gamma}^2 + \sum_{\gamma\in\Omega_2^1} \|\vec{v}_h^{\gamma} - \vec{\Pi}_h^{\gamma}\vec{v}_h^{\gamma}\|_{0,\gamma}^2 \\ &\lesssim \sum_{\gamma\in\Omega_2^1} \sum_{\beta\in\partial^{-1}\gamma} \|v_{h,3}^{\beta} - I_{1,h}^{\beta} v_{h,3}^{\beta}\|_{0,\infty,\beta}^2 + \sum_{\gamma\in\Omega_2^1} \sum_{K=2}^{3} \|v_{h,K}^{\gamma} - I_{1,h}^{\gamma} v_{h,K}^{\gamma}\|_{0,\gamma}^2 \\ &\lesssim h^2 \sum_{\beta\in\Omega^2} \sum_{I,J=1}^{2} \|\mathcal{K}_{IJ}^{\beta}(\vec{v}_h)\|_{0,\beta}^2 + h^4 \sum_{\gamma\in\Omega_2^1} \sum_{K=2}^{3} \|\mathcal{K}_{K}^{\gamma}(\vec{v}_h)\|_{0,\gamma}^2, \end{aligned}$$

$$
\begin{aligned}
J_{42}(\vec{v}_h) \lesssim & \sum_{\gamma\in\Omega_2^1}\sum_{\beta\in\partial^{-1}\gamma}\sum_{F^\beta\subset\gamma} \|\varepsilon(\beta,\gamma)\partial_{\vec{n}^\beta} v_{h,3}^\beta - (\varepsilon(\beta,\gamma)\partial_{\vec{n}^\beta} v_{h,3}^\beta)(m^{F^\beta})\|_{0,F^\beta}^2 \\
& + \sum_{\gamma\in\Omega_2^1}\sum_{K^\gamma\subset\gamma} \|v_{h,4}^\gamma - v_{h,4}^\gamma(m^{K^\gamma})\|_{0,K^\gamma}^2 \\
& \lesssim \sum_{\beta\in\Omega^2}\sum_{I,J=1}^{2} \|\mathcal{K}_{IJ}^\beta(\vec{v}_h)\|_{0,\beta}^2 + h^2 \sum_{\gamma\in\Omega_2^1} \|\mathcal{K}_1^\gamma(\vec{v}_h)\|_{0,\gamma}^2.
\end{aligned}
$$

These with (3.1) and (3.4)–(3.7) yield (3.2). □

We next introduce a transfer operator E_h^β from $V_h^M(\beta)$ into $V_h^{AR}(\beta)$ as follows[17,18]. For each $v_3^\beta \in V_h^M(\beta)$, $E_h^\beta v_3^\beta$ is uniquely determined by

$$
\begin{cases}
(E_h^\beta v_3^\beta)(p) = v_3^\beta(p), \ \forall\, p \in \beta, & \\
(\partial_I E_h^\beta v_3^\beta)(p) = (\partial_I v_3^\beta)(e_p), & 1 \leqslant I \leqslant 2, \ \forall\, p \in \beta, \\
(\partial_{IJ} E_h^\beta v_3^\beta)(p) = 0, & 1 \leqslant I, J \leqslant 2, \forall\, p \in \beta, \\
(\partial_{\vec{n}_\beta^{\partial K^\beta}} E_h^\beta v_3^\beta)(m) = (\partial_{\vec{n}_\beta^{\partial K^\beta}} v_3^\beta)(m), & \forall\, m \in F^\beta \subset K^\beta \in \mathcal{T}_h^\beta,
\end{cases}
$$

where e_p is a midpoint of an edge of $\mathcal{T}_h^\beta$ with $p \in \beta$ as one vertex. We remark that there is some freedom for the choice of e_p. For our purpose here, we assume that the related e_p should belong to $\bar{\gamma}_{N_1+1}$ if $p \in \bar{\gamma}_{N_1+1}$.

Lemma 3.3[17,18]**.** For the above-mentioned transfer operator E_h^β, it holds that

$$
\sum_{K^\beta\in\mathcal{T}_h^\beta} |E_h^\beta v_3^\beta|_{2,K^\beta}^2 \lesssim \sum_{K^\beta\in\mathcal{T}_h^\beta} |v_3^\beta|_{2,K^\beta}^2, \quad \forall\, v_3^\beta \in V_h^M(\beta).
$$

Proof of Theorem 1.1. After obtaining Lemma 3.2, we can prove the result via the function transformation method[8,19]. For each $\vec{v}_h \in \vec{V}_h$, we choose a function $\vec{w}_h$ in $\vec{W}_h$ in the form

$$
\begin{aligned}
&\vec{w}_h^\alpha = \vec{v}_h^\alpha, \ \forall\, \alpha \in \Omega^3; \ \vec{w}_h^\gamma = \vec{v}_h^\gamma, \ w_{h,4}^\gamma = v_{h,4}^\gamma, \ \forall\, \gamma \in \Omega^1; \\
&\vec{w}_h^\beta = v_{h,I}^\beta \vec{e}_I^\beta + (E_h^\beta v_{h,3}^\beta)\vec{e}_3^\beta, \ \forall\, \beta \in \Omega^2.
\end{aligned}
$$

Hence, using Lemma 3.2, Lemma 3.3 and the triangle inequality, we find

$$
\|\vec{v}_h\|_h^2 \lesssim \|\vec{v}_h - \vec{w}_h\|_h^2 + |\vec{w}_h|_{\vec{W}}^2,
$$

$$
\|\vec{v}_h - \vec{w}_h\|_h^2 \lesssim \sum_{\beta\in\Omega^2}\sum_{K^\beta\in\mathcal{T}_h^\beta} |v_{h,3}^\beta|_{2,K^\beta}^2 \lesssim \sum_{\beta\in\Omega^2} \|\mathcal{K}_{IJ}^\beta(\vec{v}_h)\|_{0,\beta}^2
$$

and

$$
\begin{aligned}
|\vec{w}_h|_{\vec{W}}^2 \lesssim & \sum_{\alpha\in\Omega^3}\sum_{i,j=1}^{3} \|\varepsilon_{ij}^\alpha(\vec{v}_h)\|_{0,\alpha}^2 + \sum_{\beta\in\Omega^2}\sum_{I,J=1}^{2} (\|\varepsilon_{IJ}^\beta(\vec{v}_h)\|_{0,\beta}^2 + \|\mathcal{K}_{IJ}^\beta(\vec{w}_h)\|_{0,\beta}^2) \\
& + \sum_{\gamma\in\Omega^1} (\|\varepsilon_{11}^\gamma(\vec{v}_h)\|_{0,\gamma}^2 + \sum_{i=1}^{3} \|\mathcal{K}_i^\gamma(\vec{v}_h)\|_{0,\gamma}^2)
\end{aligned}
$$

$$\lesssim \sum_{\alpha\in\Omega^3}\sum_{i,j=1}^{3}\|\varepsilon_{ij}^{\alpha}(\vec{v}_h)\|_{0,\alpha}^2+\sum_{\beta\in\Omega^2}\sum_{I,J=1}^{2}(\|\varepsilon_{IJ}^{\beta}(\vec{v}_h)\|_{0,\beta}^2+\|\mathcal{K}_{IJ}^{\beta}(\vec{v}_h)\|_{0,\beta}^2)$$
$$+\sum_{\gamma\in\Omega^1}(\|\varepsilon_{11}^{\gamma}(\vec{v}_h)\|_{0,\gamma}^2+\sum_{i=1}^{3}\|\mathcal{K}_{i}^{\gamma}(\vec{v}_h)\|_{0,\gamma}^2).$$

Combining these estimates gives (1.15) immediately.

Result (1.16) can be obtained by virtue of (1.15) and the same technique for proving (2.5) in ref. [11]. The proof is completed. □

Proof of Theorem 1.2. By the Cauchy-Schwarz inequality, it is easy to show that the bilinear form $D_h(\cdot,\cdot)$ is continuous and uniformly coercive in $\vec{V}_h$. So the unique solvability of problem (1.14) follows directly from the Lax-Milgram lemma. □

4 Error analysis

The next result is only the second Strang lemma for the finite element method (1.14).

Lemma 4.1. Let $\vec{u}$ and $\vec{u}_h$ be the solutions of problem (1.6) and the discrete problem (1.14), respectively. Then

$$\|\vec{u}-\vec{u}_h\|_h \lesssim E_a(\vec{u})+E_c(\vec{u}), \tag{4.1}$$

where

$$E_a(\vec{u}):=\inf_{\vec{v}_h\in\vec{V}_h}\|\vec{u}-\vec{v}_h\|_h,\quad E_c(\vec{u}):=\sup_{\vec{0}\neq\vec{v}_h\in\vec{V}_h}\frac{|D_h(\vec{u},\vec{v}_h)-F(\vec{v}_h)|}{\|\vec{v}_h\|_h}.$$

Here $E_a(\vec{u})$ is the approximation error while $E_c(\vec{u})$ the consistency error of the finite element method.

Proof of Theorem 1.3. It is easy to check that $\vec{I}_h\vec{u}\in\vec{V}_h$, so Lemma 2.1 implies

$$E_a(\vec{u})\leqslant\|\vec{u}-\vec{I}_h\vec{u}\|_h\lesssim h\Bigg\{\sum_{\alpha\in\Omega^3}\sum_{i=1}^{3}|u_i^{\alpha}|_{2,\alpha}^2+\sum_{\beta\in\Omega^2}\left(\sum_{I=1}^{2}|u_I^{\beta}|_{2,\beta}^2+|u_3^{\beta}|_{3,\beta}^2\right)$$
$$+\sum_{\gamma\in\Omega^1}\left(|u_1^{\gamma}|_{2,\gamma}^2+\sum_{K=2}^{3}|u_K^{\gamma}|_{3,\gamma}^2+|u_4^{\gamma}|_{2,\gamma}^2\right)\Bigg\}^{1/2}. \tag{4.2}$$

Now it remains to bound the consistency error $E_c(\vec{u})$. For all $\vec{v}_h\in\vec{V}_h$, it follows from (2.1)–(2.6) that

$$D_h(\vec{u},\vec{v}_h)-F(\vec{v}_h)=\sum_{\alpha\in\Omega^3}\sum_{K^{\alpha}\in\mathcal{T}_h^{\alpha}}\int_{K^{\alpha}}\sigma_{ij}^{\alpha}(\vec{u})\varepsilon_{ij}^{\alpha}(\vec{v}_h)dK^{\alpha}$$
$$+\sum_{\beta\in\Omega^2}\sum_{K^{\beta}\in\mathcal{T}_h^{\beta}}\int_{K^{\beta}}\{\mathcal{Q}_{IJ}^{\beta}(\vec{u})\varepsilon_{IJ}^{\beta}(\vec{v}_h)+\mathcal{M}_{IJ}^{\beta}(\vec{u})\mathcal{K}_{IJ}^{\beta}(\vec{v}_h)\}dK^{\beta}$$

$$
\begin{aligned}
&+\sum_{\gamma\in\Omega^1}\sum_{K^\gamma\in\mathcal{T}_h^\gamma}\int_{K^\beta}\{\mathcal{Q}_1^\gamma(\vec{u})\varepsilon_{11}^\gamma(\vec{v}_h)+\mathcal{M}_i^\gamma(\vec{u})\mathcal{K}_i^\gamma(\vec{v}_h)\}dK^\gamma\\
&-\sum_{\alpha\in\Omega^3}\int_\alpha f_i^\alpha v_{h,i}^\alpha d\alpha-\sum_{\beta\in\Omega^2}\int_\beta f_i^\beta v_{h,i}^\beta d\beta\\
&-\sum_{\gamma\in\Omega^1}\int_\gamma f_i^\gamma v_{h,i}^\gamma d\gamma-\sum_{\gamma\in\Omega^1}\int_\gamma f_4^\gamma v_{h,4}^\gamma d\gamma\\
=:&I_1+I_2+I_3+I_4,
\end{aligned}\tag{4.3}
$$

where

$$
\begin{aligned}
I_1:=&\sum_{\alpha\in\Omega^3}\int_\alpha(-\sigma_{ij,j}^\alpha(\vec{u})-f_i^\alpha)v_{h,i}^\alpha d\alpha+\sum_{\beta\in\Gamma^2}\sum_{\alpha\in\partial^{-1}\beta}\int_\beta\sigma_{ij}^\alpha(\vec{u})n_j^\alpha v_{h,i}^\alpha d\beta,\\
I_2:=&-\sum_{\beta\in\Omega^2}\sum_{K^\beta\in\mathcal{T}_h^\beta}\sum_{F^\beta\subset\partial K^\beta}\int_{F^\beta}\mathcal{M}_{\vec{n}\vec{n}}^{\beta,\partial K^\beta}(\vec{u})\partial_{\vec{n}_\beta^{\partial K^\beta}}v_{h,3}^\beta ds^\beta\\
&+\sum_{\gamma\in\Omega^1}\sum_{K^\gamma\in\mathcal{T}_h^\gamma}\int_{K^\gamma}(-\mathcal{M}_{1,1}^\gamma(\vec{u})-f_4^\gamma)v_{h,4}^\gamma dK^\gamma,\\
I_3:=&\sum_{\beta\in\Omega^2}\sum_{K^\beta\in\mathcal{T}_h^\beta}\left(-\int_{K^\beta}\mathcal{Q}_{IJ,J}^\beta(\vec{u})v_{h,I}^\beta dK^\beta+\int_{K^\beta}\mathcal{M}_{IJ,J}^\beta(\vec{u})\partial_I v_{h,3}^\beta dK^\beta\right.\\
&\left.-\int_{K^\beta}f_i^\beta v_{h,i}^\beta dK^\beta-\sum_{F^\beta\subset\partial K^\beta}\int_{F^\beta}\mathcal{M}_{\vec{n}\vec{t}}^{\beta,\partial K^\beta}(\vec{u})\partial_{\vec{t}_\beta^{\partial K^\beta}}v_{h,3}^\beta ds^\beta\right)\\
&+\sum_{\beta\in\Omega^2}\sum_{\alpha\in\partial^{-1}\beta}\int_\beta\sigma_{ij}^\alpha(\vec{u})n_j^\alpha v_{h,i}^\alpha d\beta+\sum_{\beta\in\Omega^2}\int_{\partial\beta}\mathcal{Q}_{IJ}^\beta(\vec{u})n_J^\beta v_{h,I}^\beta d\gamma\\
&+\sum_{\gamma\in\Omega^1}\sum_{K^\gamma\in\mathcal{T}_h^\gamma}\left(\int_{K^\gamma}\mathcal{Q}_i^\gamma(\vec{u})(v_{h,i}^\gamma)'dK^\gamma-\int_{K^\gamma}f_i^\gamma v_{h,i}^\gamma dK^\gamma\right)
\end{aligned}
$$

and

$$
I_4:=\sum_{\gamma\in\Omega^1}\sum_{\delta\in\partial\gamma}\varepsilon(\gamma,\delta)[\mathcal{M}_i^\gamma(\vec{u})v_{h,i+3}^\gamma](\delta).
$$

By the equations (2.7) and (2.8), it is easy to show that $I_1=0$. We divide the edge set of the finite element triangulation $\mathcal{T}_h^\beta$ into two subsets; one consists of all edges in β, the other consists of all edges on the boundary $\partial\beta$. Then I_2 can be rewritten as

$$
\begin{aligned}
I_2=&\left\{-\sum_{\beta\in\Omega^2}\sum_{K^\beta\in\mathcal{T}_h^\beta}\sum_{F^\beta\subset\partial K^\beta\backslash\partial\beta}\int_{F^\beta}\mathcal{M}_{\vec{n}\vec{n}}^{\beta,\partial K^\beta}(\vec{u})\partial_{\vec{n}_\beta^{\partial K^\beta}}v_{h,3}^\beta ds^\beta\right\}\\
&+\left\{-\sum_{\beta\in\Omega^2}\sum_{F^\beta\subset\Gamma^1}\int_{F^\beta}\mathcal{M}_{\vec{n}\vec{n}}^{\beta,\partial K^\beta}(\vec{u})\partial_{\vec{n}_\beta^{\partial K^\beta}}v_{h,3}^\beta ds^\beta\right\}\\
&+\left\{-\sum_{\beta\in\Omega^2}\sum_{F^\beta\subset\Omega_2^1}\int_{F^\beta}\mathcal{M}_{\vec{n}\vec{n}}^{\beta,\partial K^\beta}(\vec{u})\partial_{\vec{n}_\beta^{\partial K^\beta}}v_{h,3}^\beta ds^\beta\right.
\end{aligned}
$$

$$+\sum_{\gamma\in\Omega_2^1}\sum_{K^\gamma\in\mathcal{T}_h^\gamma}\int_{K^\gamma}(-\mathcal{M}_{1,1}^\gamma(\vec{u})-f_4^\gamma)v_{h,4}^\gamma dK^\gamma\Bigg\}$$

$$+\Bigg\{\sum_{\gamma\in\Omega_1^1}\int_{K^\gamma}(-\mathcal{M}_{1,1}^\gamma(\vec{u})-f_4^\gamma)v_{h,4}^\gamma d\gamma\Bigg\}$$

$$=: I_{21}+I_{22}+I_{23}+I_{24}. \tag{4.4}$$

The term I_{21} can be estimated in the standard way[15,20,21],

$$|I_{21}|\lesssim h\bigg(\sum_{\beta\in\Omega^2}|u_3^\beta|_{3,\beta}^2\bigg)^{1/2}\|\vec{v}_h\|_h. \tag{4.5}$$

For an edge F^β of a triangle K^β in $\mathcal{T}_h^\beta$, let $P_0^{F^\beta}$ be the orthogonal projection operator from $L^2(F^\beta)$ onto the space of constants on F^β, i.e.,

$$P_0^{F^\beta}g:=\frac{1}{|F^\beta|}\int_{F^\beta}gds^\beta,\ |F^\beta|:=\text{meas}(F^\beta),\ \forall\, g\in L^2(F^\beta).$$

Then by the scaling argument and the trace theorem for Sobolev spaces, we have[22,23]

$$\|R_0^{F^\beta}g\|_{0,F^\beta}\lesssim h^{1/2}|g|_{1,K^\beta},\ \forall\, g\in H^1(K^\beta), \tag{4.6}$$

where $R_0^{F^\beta}:=I-P_0^{F^\beta}$.

According to the definition of $\vec{V}_h$ (cf. (1.13)), we know that for each $m\in\gamma\in\Gamma_2^1$, $\beta,\beta'\in\partial^{-1}\gamma$,

$$\varepsilon(\beta,\gamma)\partial_{\vec{n}^\beta}v_{h,3}^\beta(m)=\varepsilon(\beta',\gamma)\partial_{\vec{n}^{\beta'}}v_{h,3}^{\beta'}(m) \tag{4.7}$$

and for each $m\in\gamma_{N_1+1}$,

$$\partial_{\vec{n}^{\beta_1}}v_{h,3}^{\beta_1}(m)=0. \tag{4.8}$$

Since $\partial_{\vec{n}^\beta}v_{h,3}^\beta$ is a first order polynomial on a line element of β, we know

$$P_0^{F^\beta}(\varepsilon(\beta,\gamma)\partial_{\vec{n}^\beta}v_{h,3}^\beta)=\varepsilon(\beta,\gamma)\partial_{\vec{n}^\beta}v_{h,3}^\beta(m^{F^\beta}). \tag{4.9}$$

Therefore, we have from (4.7)–(4.9) and the equilibrium equation (2.10) that

$$I_{22}=-\sum_{\beta\in\Omega^2}\sum_{F^\beta\subset\Gamma^1}\int_{F^\beta}R_0^{F^\beta}(\varepsilon(\beta,\gamma)\mathcal{M}_{\vec{n}\vec{n}}^\beta(\vec{u}))R_0^{F^\beta}(\varepsilon(\beta,\gamma)\partial_{\vec{n}^\beta}v_{h,3}^\beta)ds^\beta,$$

which with the Cauchy-Schwarz inequality and result (4.6) gives

$$|I_{22}|\lesssim h\bigg(\sum_{\beta\in\Omega^2}|u_3^\beta|_{3,\beta}^2\bigg)^{1/2}\|\vec{v}_h\|_h. \tag{4.10}$$

Moreover, due to the interface condition (1.11) and relation (4.9), we see

$$v_{h,4}^\gamma(m^{K^\gamma})=-P_0^{F^\beta}(\varepsilon(\beta,\gamma)\partial_{\vec{n}^\beta}v_{h,3}^\beta),\ \forall\, K^\gamma=F^\beta\subset\partial\beta\cap\Omega_2^1.$$

This along with the equilibrium equation (2.9) leads to

$$I_{23}=-\sum_{\beta\in\Omega^2}\sum_{F^\beta\subset\gamma\in\Omega_2^1}\int_{F^\beta}R_0^{F^\beta}(\varepsilon(\beta,\gamma)\mathcal{M}_{\vec{n}\vec{n}}^\beta(\vec{u}))R_0^{F^\beta}(\varepsilon(\beta,\gamma)\partial_{\vec{n}^\beta}v_{h,3}^\beta)ds^\beta$$

$$+\sum_{\gamma\in\Omega_2^1}\sum_{K^\gamma\in\mathcal{T}_h^\gamma}\int_{K^\gamma}(-\mathcal{M}_{1,1}^\gamma(\vec{u})-f_4^\gamma)\{v_{h,4}^\gamma-v_{h,4}^\gamma(m^{K^\gamma})\}dK^\gamma, \tag{4.11}$$

so that

$$|I_{23}| \lesssim h\Big\{\sum_{\beta\in\Omega^2}|u_3^\beta|_{3,\beta}^2 + \sum_{\gamma\in\Omega_2^1}(|u_4^\gamma|_{2,\gamma}^2 + \|f_4^\gamma\|_{0,\gamma}^2)\Big\}^{1/2}\|\vec{v}_h\|_h, \tag{4.12}$$

by means of the Cauchy-Schwarz inequality and estimate (4.6).

Moreover, it follows from the equilibrium equation (2.9) that

$$-\mathcal{M}_{1,1}^\gamma(\vec{u}) = f_4^\gamma \quad \text{in } L^2(\gamma),\ \forall\,\gamma\in\Omega_1^1,$$

hence $I_{24}=0$. This with (4.4)–(4.12) yields

$$|I_2| \lesssim h\Big\{\sum_{\beta\in\Omega^2}|u_3^\beta|_{3,\beta}^2 + \sum_{\gamma\in\Omega_2^1}(|u_4^\gamma|_{2,\gamma}^2 + \|f_4^\gamma\|_{0,\gamma}^2)\Big\}^{1/2}\|\vec{v}_h\|_h. \tag{4.13}$$

The estimate of the term I_3 is rather technical. As in the derivation of (4.13), we also rewrite I_3 as

$$\begin{aligned}
I_3 :=&\Big\{\sum_{\beta\in\Omega^2}\sum_{K^\beta\in\mathcal{T}_h^\beta}\sum_{F^\beta\subset\partial K^\beta\setminus\partial\beta}\int_{K^\beta}\mathcal{M}_{\vec{n}\vec{t}}^{\beta,\partial K^\beta}(\vec{u})\partial_{\vec{t}_\beta^{\partial K^\beta}}v_{h,3}^\beta ds^\beta\Big\}\\
&+\Big\{\sum_{\beta\in\Omega^2}\sum_{K^\beta\in\mathcal{T}_h^\beta}\Big(-\int_{K^\beta}\mathcal{Q}_{IJ,J}^\beta(\vec{u})v_{h,I}^\beta dK^\beta + \int_{K^\beta}\mathcal{M}_{IJ,J}^\beta(\vec{u})\partial_I v_{h,3}^\beta dK^\beta\\
&-\int_{K^\beta}f_i^\beta v_{h,i}^\beta dK^\beta\Big) + \sum_{\beta\in\Omega^2}\sum_{\alpha\in\partial^{-1}\beta}\int_\beta\sigma_{ij}^\alpha(\vec{u})n_j^\alpha v_{h,i}^\alpha d\beta\\
&-\sum_{\beta\in\Omega^2}\sum_{F^\beta\subset\partial\beta}\int_{F^\beta}\mathcal{M}_{\vec{n}\vec{t}}^\beta(\vec{u})\partial_{\vec{t}^\beta}v_{h,3}^\beta ds^\beta + \sum_{\beta\in\Omega^2}\int_{\partial\beta}\mathcal{Q}_{IJ}^\beta(\vec{u})n_J^\beta v_{h,I}^\beta d\gamma\\
&+\sum_{\gamma\in\Omega^1}\sum_{K^\gamma\in\mathcal{T}_h^\gamma}\Big(\int_{K^\gamma}\mathcal{Q}_i^\gamma(\vec{u})(v_{h,i}^\gamma)'dK^\gamma - \int_{K^\gamma}f_i^\gamma v_{h,i}^\gamma dK^\gamma\Big)\Big\}\\
=:&I_{31}+I_{32}.
\end{aligned}\tag{4.14}$$

I_{31} can be estimated in a standard way[15,20,21],

$$|I_{31}| \lesssim h\Big(\sum_{\beta\in\Omega^2}|u_3^\beta|_{3,\beta}^2\Big)^{1/2}\|\vec{v}_h\|_h. \tag{4.15}$$

We next introduce an auxiliary function $\vec{w}_h$ from $\vec{v}_h\in\vec{V}_h$ as follows.

$$\begin{aligned}
&\vec{w}_h^\alpha := (I_{1,h}^\alpha v_{h,i}^\alpha)\vec{e}_i^\alpha,\ \forall\,\alpha\in\Omega^3;\\
&\vec{w}_h^\beta := (I_{1,h}^\beta v_{h,i}^\beta)\vec{e}_i^\beta,\ \forall\,\beta\in\Omega^2;\\
&\vec{w}_h^\gamma := (I_{1,h}^\gamma v_{h,i}^\gamma)\vec{e}_i^\gamma,\ \forall\,\gamma\in\Omega_2^1;\\
&\vec{w}_h^\gamma := \vec{v}_h^\gamma,\ \forall\,\gamma\in\Omega_1^1;\\
&w_{h,4}^\gamma := v_{h,4}^\gamma,\ \forall\,\gamma\in\Omega^1.
\end{aligned}\tag{4.16}$$

It is easy to check that $\vec{w}_h\in\vec{H}(\partial\Omega^2\cup\Omega^1)$, when it is restricted on $\partial\Omega^2\cup\Omega^1$.

So Lemma 2.4 implies

$$\sum_{\beta\in\Omega^2}\Bigg\{\langle \mathcal{M}^\beta_{IJ,J}(\vec{u})n^\beta_I, w^{\partial\beta}_{h,3}\rangle_{H^{-1/2}(\partial\beta)\times H^{1/2}(\partial\beta)}$$
$$+\sum_{\gamma\in\partial\beta}\int_\gamma (\mathcal{Q}^\beta_{IJ}(\vec{u})n^\beta_J w^{\partial\beta}_{h,I} - \mathcal{M}^\beta_{\vec{n}\vec{t}}(\vec{u})\partial_{\vec{t}^\beta} w^{\partial\beta}_{h,3})d\gamma\Bigg\}$$
$$+\sum_{\gamma\in\Omega^1}\int_\gamma \mathcal{Q}^\gamma_i(\vec{u})(w^\gamma_{h,i})' d\gamma - \sum_{\gamma\in\Omega^1}\int_\gamma f^\gamma_i w^\gamma_{h,i} d\gamma = 0.$$

Moreover, since $\vec{w}^\beta_h \in (H^1(\beta))^3$, it follows from Lemma 2.3 that

$$-\int_\beta \mathcal{Q}^\beta_{IJ,J}(\vec{u})w^\beta_{h,I}d\beta + \int_\beta \mathcal{M}^\beta_{IJ,J}(\vec{u})\partial_I w^\beta_{h,3}d\beta + \int_\beta \sum_{\alpha\in\partial^{-1}\beta}\sigma^\alpha_{ij}(\vec{u})n^\alpha_j \vec{e}^\alpha_i\cdot w^\beta_{h,l}\vec{e}^\beta_l d\beta$$
$$-\int_\beta f^\beta_i w^\beta_{h,i} d\beta = \langle \mathcal{M}^\beta_{IJ,J}(\vec{u})n^\beta_I, w^{\partial\beta}_{h,3}\rangle_{H^{-1/2}(\partial\beta)\times H^{1/2}(\partial\beta)}.$$

Combining the last two identities we get

$$\sum_{\beta\in\Omega^2}\Bigg\{\int_\beta \mathcal{M}^\beta_{IJ,J}(\vec{u})\partial_I w^\beta_{h,3}d\beta - \int_\beta \mathcal{Q}^\beta_{IJ,J}(\vec{u})w^\beta_{h,I}d\beta$$
$$+\int_\beta\sum_{\alpha\in\partial^{-1}\beta}\sigma^\alpha_{ij}(\vec{u})n^\alpha_j\vec{e}^\alpha_i\cdot w^\beta_{h,l}\vec{e}^\beta_l d\beta - \int_\beta f^\beta_i w^\beta_{h,i}d\beta$$
$$+\sum_{\gamma\in\partial\beta}\left(\int_\gamma \mathcal{Q}^\beta_{IJ}(\vec{u})n^\beta_J w^{\partial\beta}_{h,I}d\gamma - \int_\gamma \mathcal{M}^\beta_{\vec{n}\vec{t}}(\vec{u})\partial_{\vec{t}^\beta}w^{\partial\beta}_{h,3}d\gamma\right)\Bigg\}$$
$$+\sum_{\gamma\in\Omega^1}\int_\gamma \mathcal{Q}^\gamma_i(\vec{u})(w^\gamma_{h,i})'d\gamma - \sum_{\gamma\in\Omega^1}\int_\gamma f^\gamma_i w^\gamma_{h,i}d\gamma = 0. \tag{4.17}$$

Observe that (cf. (4.16))

$$w^{\partial\beta}_{h,I} = v^\beta_{h,I},\ w^{\partial\beta}_{h,3} = I^\beta_{1,h}v^\beta_{h,3} \quad \text{on } \partial\beta, \quad I=1,2,$$
$$w^\gamma_1 = v^\gamma_{h,1},\ w^\gamma_K = I^\gamma_{1,h}v^\gamma_{h,K} \quad \text{on } \gamma\in\Omega^1_2, \quad K=2,3,$$

hence subtracting the equation (4.17) from I_{32} implies

$$I_{32} = \Bigg\{\sum_{\beta\in\Omega^2}\sum_{K^\beta\in\mathcal{T}^\beta_h}\Bigg[\int_{K^\beta}\mathcal{M}^\beta_{IJ,J}(\vec{u})\partial_I(v^\beta_{h,3} - I^\beta_{1,h}v^\beta_{h,3})dK^\beta$$
$$-\int_{K^\beta} f^\beta_3(v^\beta_{h,3} - I^\beta_{1,h}v^\beta_{h,3})dK^\beta\Bigg]\Bigg\}$$
$$+\Bigg\{-\sum_{\beta\in\Omega^2}\sum_{F^\beta\subset\partial\beta}\int_{F^\beta}\mathcal{M}^\beta_{\vec{n}\vec{t}}(\vec{u})\partial_{\vec{t}^\beta}(v^\beta_{h,3} - I^\beta_{1,h}v^\beta_{h,3})ds^\beta\Bigg\}$$
$$+\Bigg\{\sum_{\gamma\in\Omega^1_2}\sum_{K^\gamma\in\mathcal{T}^\gamma_h}\Bigg[\int_{K^\gamma}\mathcal{Q}^\gamma_i(\vec{u})(v^\gamma_{h,i} - I^\gamma_{1,h}v^\gamma_{h,i})'dK^\gamma$$
$$-\int_{K^\gamma} f^\gamma_i(v^\gamma_{h,i} - I^\gamma_{1,h}v^\gamma_{h,i})dK^\gamma\Bigg]\Bigg\}$$
$$=: I_{321} + I_{322} + I_{323}. \tag{4.18}$$

By virtue of error estimates for the interpolation operators $I_{1,h}^{\beta}$ and $I_{1,h}^{\gamma}$, we can easily obtain

$$|I_{321}| \lesssim h\left\{ \sum_{\beta\in\Omega^2} (|u_3^{\beta}|_{3,\beta}^2 + h^2\|f_3^{\beta}\|_{0,\beta}^2)\right\}^{1/2} \|\vec{v}_h\|_h \tag{4.19}$$

and

$$|I_{323}| \lesssim h\left\{ \sum_{\gamma\in\Omega_2^1} \sum_{K=2}^{3} (|u_K^{\gamma}|_{3,\gamma}^2 + h^2\|f_K^{\gamma}\|_{0,\gamma}^2)\right\}^{1/2} \|\vec{v}_h\|_h. \tag{4.20}$$

Since $\int_{F^\beta} \partial_{\vec{t}^\beta}(v_{h,3}^{\beta} - I_{1,h}^{\beta} v_{h,3}^{\beta})ds^{\beta} = 0$, we have by the Cauchy-Schwarz inequality and estimate (4.6) that

$$\begin{aligned}
|I_{322}| &= \left| \sum_{\beta\in\Omega^2} \sum_{F^\beta\subset\partial\beta} \int_{F^\beta} \mathcal{M}_{\vec{n}\vec{t}}^{\beta}(\vec{u})\partial_{\vec{t}^\beta}(v_{h,3}^{\beta} - I_{1,h}^{\beta} v_{h,3}^{\beta})ds^{\beta}\right| \\
&= \left| \sum_{\beta\in\Omega^2} \sum_{F^\beta\subset\partial\beta} \int_{F^\beta} R_0^{F^\beta}(\mathcal{M}_{\vec{n}\vec{t}}^{\beta}(\vec{u}))\partial_{\vec{t}^\beta}(v_{h,3}^{\beta} - I_{1,h}^{\beta} v_{h,3}^{\beta})ds^{\beta}\right| \\
&\leqslant \sum_{\beta\in\Omega^2} \sum_{F^\beta\subset\partial\beta} \|R_0^{F^\beta}(\mathcal{M}_{\vec{n}\vec{t}}^{\beta}(\vec{u}))\|_{0,F^\beta} \|\partial_{\vec{t}^\beta}(v_{h,3}^{\beta} - I_{1,h}^{\beta} v_{h,3}^{\beta})\|_{0,F^\beta} \\
&\lesssim h\left(\sum_{\beta\in\Omega^2} |u_3^{\beta}|_{3,\beta}^2 \right)^{1/2} \|\vec{v}_h\|_h.
\end{aligned} \tag{4.21}$$

On the other hand, it follows from the interface condition (1.12) and the equilibrium equation (2.11) that

$$\begin{aligned}
I_4 &= \sum_{\gamma\in\Omega^1} \sum_{\delta\in\partial\gamma} \varepsilon(\gamma,\delta)[\mathcal{M}_i^{\gamma}(\vec{u}) v_{h,i+3}^{\gamma}](\delta) \\
&= \sum_{\delta\in\Gamma^0} \left(\sum_{\gamma\in\partial^{-1}\delta} \varepsilon(\gamma,\delta)\mathcal{M}_i^{\gamma}(\vec{u})(\delta)\vec{e}_i^{\gamma} \right)(v_{h,l+3}^{\gamma_\delta}\vec{e}_l^{\,\gamma_\delta}) = 0,
\end{aligned}$$

where γ_δ is any prescribed beam member with δ as one end point. Combining this equation with (4.3), (4.13)–(4.15), (4.18)–(4.16), and the equation $I_1 = 0$ showed before, we obtain

$$\begin{aligned}
|E_c(\vec{u})| \lesssim h\Bigg\{ &\sum_{\beta\in\Omega^2} (|u_3^{\beta}|_{3,\beta}^2 + h^2\|f_3^{\beta}\|_{0,\beta}^2) \\
&+ \sum_{\gamma\in\Omega_2^1}\left[\sum_{K=2}^{3} (|u_K^{\gamma}|_{3,\gamma}^2 + h^2\|f_K^{\gamma}\|_{0,\gamma}^2) + |u_4^{\gamma}|_{2,\gamma}^2 + \|f_4^{\gamma}\|_{0,\gamma}^2 \right]\Bigg\}^{1/2}.
\end{aligned} \tag{4.22}$$

Theorem 1.3 now follows from (4.1), (4.2) and (4.22) directly. The proof is com-
pleted. □

Acknowledgements This work was partly supported by the National Natural Science Foundation (Grant No. 10371076), E-Institutes of Shanghai Municipal Education Commission (Grant No. E03004) and The Science Foundation of Shanghai (Grant No. 04JC14062).

References

1. Ciarlet, P. G., Mathematical Elasticity II: Theory of Plates, Amsterdam: Elsevier, 1997.
2. Kozlov, V., Mazya, V., Movchan, A., Asymptotic Analysis of Fields in Multi-Structures, Oxford: Clarendon Press, 1999.
3. Bernadou, M., Fayolle, S., Lene, F., Numerical analysis of junctions between plates, Comput. Methods Appl. Mech. Engrg., 1989, 74: 307–326.
4. Wang, L., An analysis of the TRUNC elements for coupled plates with rigid junction, Comput. Methods Appl. Mech. Engrg., 1995, 121: 1–14.
5. Wang, L., Mathematical model of coupled plates meeting at an angle $0 < \theta < \pi$ and its finite element method, Numer. Methods for Partial Differential Equations, 1993, 9: 13–22.
6. Wang, L., A mathematical model of coupled plates and its finite element method, Comput. Methods Appl. Mech. Engrg., 1992, 99: 43–59.
7. d'Hennezel, F., Domain decomposition method and elastic multi-structures: the stiffened problem, Numer. Math., 1993, 66: 181–193.
8. Huang, J., Numerical solution of the elastic body-plate problem by nonoverlapping domain decomposition type techniques, Math. Comp., 2004, 73: 19–34.
9. Feng K., Elliptic equations on composite manifold and composite elastic structures, Math. Numer. Sinica (in Chinese), 1979, 1: 199–208.
10. Feng, K., Shi, Z., Mathematical Theory of Elastic Structures (in Chinese), Beijing: Science Press, 1981. (English translation, Berlin-New York: Springer-Verlag & Science Press, 1995.)
11. Huang, J., Shi, Z., Xu, Y., Some studies on mathematical models for general elastic multi-structures, Science in China Ser. A, 2005, 48: 986–1007.
12. Adams, R., Sobolev Spaces, New York: Academic Press, 1975.
13. Grisvard, P., Elliptic Problems in Nonsmooth Domains, London: Pitman, 1985.
14. Lions, J., Magenes, E., Nonhomogeneous Boundary Value Problems and Applications (I), Berlin-New York: Springer-Verlag, 1972.
15. Ciarlet, P. G., The Finite Element Method for Elliptic Problems, Amsterdam: North-Holland, 1978.
16. Brenner, S. C., Scott, L. R., The Mathematical Theory of Finite Element Methods, New York: Springer-Verlag, 1994.
17. Brenner, S. C., A two-level additive Schwarz preconditioner for nonconforming plate elements, Numer. Math., 1996, 72: 419–447.
18. Shi, Z., Xie, Z., Substructure preconditioners for nonconforming plate elements, Journal of Computational Mathematics, 1998, 16: 385–395.
19. Brenner, S. C., Poincaré-Friedrichs inequalities for piecewise H^1 functions, SIAM J. Numer. Anal., 2003, 41: 306–324.
20. Shi, Z., On the error estimate for Morley element, Numer. Math. Sinica (in Chinese), 1990, 12: 113–118.
21. Huang, J., Li, L., Chen, J., On mortar-type Morley element method for plate bending problem, Applied Numerical Mathematics, 2001, 37: 519–533.
22. Shi, Z., Convergence of the TRUNC plate element, Comput. Methods Appl. Mech. Engrg., 1987, 62: 71–88.
23. Stummel, F., The generalized patch test, SIAM J. Numer. Anal., 1979, 16: 449–471.