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THE CONTACT SET OF A RIGID BODY PARTIALLY SUPPORTED BY A MEMBRANE

CHARLES M. ELLIOTT
Imperial College, Department of Mathematics, London, U.K.

and

AVNER FRIEDMAN†
Northwestern University, Department of Mathematics, Evanston, Illinois, U.S.A.

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INTRODUCTION

CONSIDER an elastic membrane $\mathcal{M}$ spanned over a domain Ω and clamped at the boundary. Let y be a point in Ω. We take a ball B_ρ of radius ρ and center lying precisely above the point y and drop it down slowly until it sits on the membrane in an equilibrium position. This position is characterized by a variational inequality with obstacle given by the lower hemisphere Σ of ∂B_ρ, say

$$\Sigma : z = \psi(x) \qquad (|x-y| \leq \rho).$$

We are interested in studying the contact set of $\mathcal{M}$ and Σ and, in particular, in proving that

$$\text{the projection of } \mathcal{M} \cap \Sigma \text{ on } \Omega \text{ lies in the interior of the disc } \{|x-y| \leq \rho\}; \tag{0.1}$$

this implies $C^{1,1}$ regularity of the solution.

Suppose next we let the ball move freely over $\mathcal{M}$ until it reaches an equilibrium position. Denoting the projection of its center on Ω by y_ρ, we are interested in determining the location of y_ρ.

Introducing Green's function

$$G(x,y) = \frac{1}{2\pi} \log \frac{1}{|x-y|} - h(x,y)$$

for $-\Delta$ in Ω, we denote by S the (finite) set of points y^* in Ω such that

$$\min_{x\in\Omega} h(x,x) = h(y^*,y^*).$$

We prove that, for some $\gamma > 0$, $C > 0$,

$$\operatorname{dist}(y_\rho, S) \leq C\left(\rho \log \frac{1}{\rho}\right)^{\gamma} \to 0 \qquad \text{if } \rho \to 0. \tag{0.2}$$

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In Section 1 we describe the physical problems in more detail. In Sections 2–4 we study the first question and, in particular, establish (0.1). We shall actually consider here more general obstacles ψ as well as two versions of the physical problem. In Section 5 we prove the assertion (0.2). A result of this type was first established by Caffarelli and Friedman [2] for a plasma problem.

In Section 6 we shall study a related problem for the Hele–Shaw model.

Contact problems for a rigid punch indenting an elastic body have been studied by various authors; see [3, 5, 6, 10, 12, 13] and the references given there. In particular existence results have been established in [3, 5, 6, 10].

1. THE PHYSICAL MODEL

Let $\mathcal{M}$ be an elastic membrane, clamped at its edge $\partial\Omega$, which in its undeformed state occupies the domain Ω in the horizontal plane $\Pi = \{\mathbf{r} = (x_1, x_2, x_3); x_3 = 0\}$. The membrane is deformed by a rigid body (or obstacle) which is at rest above $\mathcal{M}$ and in contact with a portion $\Omega^{\mathcal{M}}_c$ of the membrane's surface $\Omega^{\mathcal{M}}$. In static equilibrium the force on the body due to the tension in the membrane balances an applied external force $\mathbf{F}_a$. It is assumed that the tension σ in the membrane is uniform and that

$$\Omega^{\mathcal{M}} = \{\mathbf{r}; x_3 = u(x_1, x_2), (x_1, x_2) \in \Omega\} \tag{1.1}$$

where u is a single valued continuous function.

Take a portion $S^{\mathcal{M}}$ of $\Omega^{\mathcal{M}}$ with projection S on Π ($S \subset \Omega$) and let $\mathbf{k}$ denote the unit vector pointing out of $S^{\mathcal{M}}$ which is normal to $\partial S^{\mathcal{M}}$ and in the tangent plane to $S^{\mathcal{M}}$. The unit tangent vector to $\partial S^{\mathcal{M}}$ with anti-clockwise orientation is denoted by $\mathbf{t}$. Thus, on $\partial S^{\mathcal{M}}$,

$$\mathbf{k} = \mathbf{t} \times \mathbf{n}, \quad \mathbf{t} = \mathbf{n} \times \mathbf{k} \quad \text{and} \quad \mathbf{n} = \mathbf{k} \times \mathbf{t} \tag{1.2}$$

where $\mathbf{n}$ is the unit normal to $\Omega^{\mathcal{M}}$,

$$\mathbf{n} = (-u_{x_1}, -u_{x_2}, 1)/(1 + |\nabla u|^2)^{1/2}. \tag{1.3}$$

Since the membrane is in a state of uniform tension, the force exerted on $S^{\mathcal{M}}$ by the tension in the membrane outside $S^{\mathcal{M}}$ is

$$\mathbf{F} = \sigma \int_{\partial S^{\mathcal{M}}} \mathbf{k}\, \mathrm{d}s. \tag{1.4}$$

The line integral in (1.4) may be replaced by a surface integral, upon using the divergence theorem,

$$\int_{\partial S^{\mathcal{M}}} \mathbf{k}\, \mathrm{d}s = \int_{S^{\mathcal{M}}} J\mathbf{n}\, \mathrm{d}S \tag{1.5}$$

where J is the mean curvature of the surface, that is,

$$J = \frac{\partial}{\partial x_1} \frac{u_{x_1}}{\sqrt{(1 + |\nabla u|^2)}} + \frac{\partial}{\partial x_2} \frac{u_{x_2}}{\sqrt{(1 + |\nabla u|^2)}} = -\operatorname{div} \mathbf{n} \tag{1.6}$$

where "div" is the divergence in $\mathbb{R}^3$.

In order to justify (1.5), let $\mathbf{r} = (x_1, x_2, x_3)$ be a point in $S^{\mathcal{U}}$ and (w, v) local parameters. Define

$$\mathbf{r}_w = \frac{\partial \mathbf{r}}{\partial w}, \qquad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v},$$

$$E = \mathbf{r}_w \cdot \mathbf{r}_u, \qquad F = \mathbf{r}_w \cdot \mathbf{r}_v, \qquad G = \mathbf{r}_v \cdot \mathbf{r}_v,$$

$$H = |\mathbf{r}_w \times \mathbf{r}_v| = \sqrt{(EG - F^2)}.$$

For any vector $\underset{\sim}{W}$ in $\mathbb{R}^3$ define

$$\mathrm{DIV}(\underset{\sim}{W}) = H^{-2}\left\{\left(G\frac{\partial \underset{\sim}{W}}{\partial w} - F\frac{\partial \underset{\sim}{W}}{\partial v}\right)\cdot \mathbf{r}_w + \left(E\frac{\partial \underset{\sim}{W}}{\partial v} - F\frac{\partial \underset{\sim}{W}}{\partial v} - F\frac{\partial \underset{\sim}{W}}{\partial w}\right)\cdot \mathbf{r}_w\right\}.$$

A calculation reveals that for $\underset{\sim}{W} = P\mathbf{r}_w + Q\mathbf{r}_v$,

$$\mathrm{DIV}(\underset{\sim}{W}) = H^{-1}\left\{\frac{\partial}{\partial w}(HP) + \frac{\partial}{\partial v}(HQ)\right\} \equiv \operatorname{div} \underset{\sim}{W}$$

and

$$\mathrm{DIV}(\mathbf{n}) = -J = \operatorname{div} \mathbf{n}.$$

Since $\mathbf{n}$ is orthogonal to $\mathbf{r}_w$ and $\mathbf{r}_v$, also

$$\mathrm{DIV}(R\mathbf{n}) = R\,\mathrm{DIV}(\mathbf{n}) = -JR.$$

If $\underset{\sim}{W} = \underset{\sim}{V} + R\mathbf{n}$ where $\underset{\sim}{V} = P\mathbf{r}_w + Q\mathbf{r}_v$, the divergence theorem gives

$$\int_{S^{\mathcal{U}}} \operatorname{div} \underset{\sim}{V}\, \mathrm{d}S = \int_{\partial S^{\mathcal{U}}} \underset{\sim}{V} \cdot k\, \mathrm{d}s = \int_{\partial S^{\mathcal{U}}} \underset{\sim}{W} \cdot k\, \mathrm{d}s$$

(since $\mathbf{n} \cdot \mathbf{k} = 0$). If $\underset{\sim}{W}$ is a constant vector then clearly $\mathrm{DIV}(\underset{\sim}{W}) = 0$ so that $\operatorname{div} \underset{\sim}{V} = RJ$. Hence, since $R = \underset{\sim}{W} \cdot \mathbf{n}$,

$$\int_{S^{\mathcal{U}}} \underset{\sim}{W} \cdot \mathbf{n} J\, \mathrm{d}S = \int_{\partial S^{\mathcal{U}}} \underset{\sim}{W} \cdot \mathbf{k}\, \mathrm{d}s.$$

Choosing $\underset{\sim}{W} = (1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ we deduce that (1.5) holds.

At all points of $\Omega^{\mathcal{U}}$ not in contact with the body the external force is zero, and from (1.4), (1.5) we then deduce that

$$J = 0 \quad \text{in } \Omega^{\mathcal{U}} \backslash \Omega_c^{\mathcal{U}}. \tag{1.7}$$

Noting next that the vertical component of the force on the membrane due to the contact with the body is nonpositive, we also have

$$J \geqslant 0 \quad \text{in } \Omega_c^{\mathcal{U}}. \tag{1.8}$$

We can rewrite (1.7), (1.8) as a differential equation and inequality in Ω,

$$J = 0 \quad \text{in } \Omega\backslash\Omega_c, \qquad J \geqslant 0 \quad \text{in } \Omega_c. \tag{1.9}$$

We now proceed to calculate the total force on the rigid body due to the tension in the membrane. Equation (1.4) holds for any curve in $\Omega^{\mathcal{U}}$ and in particular it holds for $\partial\Omega$. Denoting by $\mathbf{N}$ the unit outward normal to $\partial\Omega$ (in the plane Π) we have that on $\partial\Omega$

$$\mathbf{t} = (t_1, t_2, 0), \quad \nabla u \cdot \mathbf{t} = 0 \quad \text{and} \quad \mathbf{N} = (t_2, -t_1, 0) \tag{1.10$_a$}$$

which implies

$$\mathbf{k} = \mathbf{t} \times \mathbf{n} = (t_2, -t_1, \nabla u \cdot \mathbf{N})/(1 + |\nabla u|^2)^{1/2}. \tag{1.10$_b$}$$

Thus the total force extended on the rigid body by the membrane is

$$\mathbf{F}_{\mathcal{M}} = \sigma \int_{\partial\Omega} \mathbf{k}\, ds = \sigma \int_{\partial\Omega} (t_2, -t_1, \nabla u \cdot \mathbf{N})(1 + |\nabla u|^2)^{-1/2}\, ds, \tag{1.10}$$

and for static equilibrium

$$\mathbf{F}_a + \mathbf{F}_{\mathcal{M}} = \mathbf{0}. \tag{1.11}$$

Conditions may also be derived which hold on the contact curve $\partial\Omega_c^{\mathcal{M}}$. Consider a "thin" surface element of $\Omega^{\mathcal{M}}$ containing an arc of $\partial\Omega_c^{\mathcal{M}}$ of length δl. The force on this element due to the tension in the membrane is

$$\sigma[\mathbf{k}]\delta l$$

where $[\cdot]$ denotes the jump across $\partial\Omega_c^{\mathcal{M}}$ from $\Omega_c^{\mathcal{M}}$ into $\Omega\backslash\Omega_c^{\mathcal{M}}$. This force will balance the force on the element due to the contact with the rigid body. If the arc of $\partial\Omega_c^{\mathcal{M}}$ lies on a smooth part of the obstacle then the force is in the normal direction, so that

$$[\mathbf{k}] = 0 \quad \text{and then also } [\nabla u] = 0, \tag{1.12}$$

and the membrane leaves the obstacle with a continuous gradient.

Finally note that the elastic energy of the membrane for any displacement v is

$$\mathscr{E} = \int_{\Omega^{\mathcal{M}}} \sigma\, dS = \sigma \int_{\Omega} (1 + |\nabla v|^2)^{1/2}\, dx. \tag{1.13}$$

In this paper we consider the linearized problem by replacing $(1 + |\nabla v|^2)$ by $1 + \frac{1}{2}|\nabla v|^2$, so that

$$J \approx \Delta u \qquad \text{in } \Omega \tag{1.14}$$

and

$$\mathscr{E} = \sigma \int_{\Omega} dx + \tfrac{1}{2}\sigma \int_{\Omega} |\nabla v|^2\, dx. \tag{1.15}$$

We are now in a position to describe the two problems to be studied in this paper.

Problem 1. Static equilibrium is achieved by pushing down the rigid body with a force, where its vertical component is a prescribed constant <0, in such a way that the horizontal position and orientation of the body remain fixed. The rigid body may be conceived to be a finger or punch. One can also imagine a sphere inside a vertical tube; the role of the tube being to preserve horizontal position. The lower surface of the obstacle is described by the set

$$\{\mathbf{r}; x_3 = \psi(x_1, x_2) + \mu, \qquad (x_1, x_2) \in \bar{E}\} \tag{1.16}$$

where E is an open set with $\bar{E} \subset \Omega$, ψ is a single valued, nonnegative continuous function on $\bar{E}$ which is not identically zero and has minimum value 0 on E, and μ is the height of the lowest point of the obstacle. Thus, the problem is to determine (u, λ) where u is the displacement of the membrane and λ is the lowest point of the obstacle. Bearing in mind (1.15), we see that the total energy of the system, up to an additive constant, may be written as $\mathscr{E}(v, \mu)$ where

$$\mathscr{E}(v, \mu) = \tfrac{1}{2}\sigma \int_{\Omega} |\nabla v|^2 \, dx + g\mu. \tag{1.17}$$

The first term on the right-hand side is that part of the elastic energy of the membrane which varies with the displacement, and the second term is the potential energy of the rigid body; g is a given positive constant.

The principle of energy being minimized leads to the following problem:

(P1) find (u, λ) such that

$$\begin{aligned} &u = 0 \quad \text{on } \partial\Omega, && u \leqslant \psi + \lambda \quad \text{in } E, \\ &\mathscr{E}(u, \lambda) = \inf_{(v,\mu)} \mathscr{E}(v, \mu), && v = 0 \quad \text{on } \partial\Omega, \\ & && v \leqslant \psi + \mu \quad \text{in } E. \end{aligned}$$

The earlier discussion of the equilibrium conditions leads to the following free boundary problem:

(F.B.P.) Find (u, λ) such that

$$\begin{aligned} &u = 0 \text{ on } \partial\Omega, \quad u \leqslant \psi + \lambda \text{ in } E, \quad \Omega_c \equiv \{x; u(x) = \psi(x) + \lambda\}, \\ &\Delta u = 0 \text{ in } \Omega \backslash \Omega_c, \quad \Delta u \geqslant 0 \text{ in } \Omega, \\ &\nabla u \text{ is continuous across } \partial\Omega_c, \end{aligned}$$

and

$$\sigma \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, ds = g,$$

where $\partial/\partial\nu$ denotes the derivative with respect to the outward normal to $\partial\Omega$ (with respect to Ω).

Problem 2. A homogeneous sphere placed on a membrane $\mathscr{M}$ will proceed to roll on the surface $\mathscr{M}$ under the action of its own weight and the tension in $\mathscr{M}$. Note that the couple exerted on the sphere by the tension in $\mathscr{M}$ is always zero. Indeed, by definition the couple $\mathbf{G}$ about the center of gravity $\mathbf{r}_0$ of the sphere is

$$\mathbf{G} = \sigma \int_{\partial\Omega_c^{\mathscr{M}}} (\mathbf{r} - \mathbf{r}_0) \times \mathbf{k} \, ds.$$

Since $\mathbf{r} - \mathbf{r}_0 = -\rho\mathbf{n}$ where ρ is the radius of the sphere,

$$\begin{aligned} \mathbf{G} &= \sigma \int_{\partial\Omega_c^{\mathscr{M}}} \mathbf{k} \times \mathbf{n} \, ds = -\rho\sigma \int_{\partial\Omega_c^{\mathscr{M}}} \mathbf{t} \, ds \\ &= -\rho\sigma \int_{\partial\Omega_c^{\mathscr{M}}} \frac{d\mathbf{r}}{ds} \, ds = \mathbf{0}. \end{aligned}$$

Thus the orientation of the sphere does not affect its motion. Let $\mathbf{r}_0 = (y_1, y_2, \mu)$. For any position of the sphere and any displacement v of the membrane, the energy of the system, up to an additive constant, is given by

$$\mathscr{E}(y_1, y_2, \mu, v) = \tfrac{1}{2}\sigma \int_\Omega |\nabla v|^2 \, dx + g\mu.$$

The equilibrium position of the sphere may be defined by $\mathbf{r}_0^* = (y_1^*, y_2^*, \lambda^*)$ and u^*, the displacement of $\mathscr{M}$, where

$$\begin{aligned} &\mathscr{E}(y_1^*, y_2^*, \lambda^*, u^*) \leq \mathscr{E}(y_1, y_2, \mu, v) \\ &\forall y = (y_1, y_2) \in \Omega,\ B_\rho(y) \subset \Omega,\ v \leq \psi_\rho + \mu \quad \text{on } B_\rho(y), \\ &v = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.18}$$

Here we used the notation

$$\begin{aligned} B_\rho(y) &= \{x; |x - y| \leq \rho\}, \\ \psi_\rho(x) &= \rho - (\rho^2 - |x|^2)^{1/2} \end{aligned} \tag{1.19}$$

Defining

$$\mathscr{E}_\rho(y) = \inf_{(v,\mu)} \mathscr{E}(v, \mu), \qquad \begin{aligned} &v \leq \psi_\rho(y) + \mu \quad \text{on } B_\rho(y), \\ &v = 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{1.20}$$

we observe that if

$$\mathscr{E}_\rho(y^*) = \inf_y \mathscr{E}_\rho(y), \qquad B_\rho(y) \subset \Omega \tag{1.21}$$

then

$$\mathscr{E}_\rho(y^*) = \mathscr{E}(y_1^*, y_2^*, \lambda^*, u^*) \qquad \text{satisfies (1.18)}$$

where λ^*, u^* provide the minimum in (1.20) for $y = y^*$.

Problem 2 is to study the location of the points y^* (in Ω) for which (1.21) holds. The free boundary problem associated with (1.21) is: find (y, u, λ) such that

$$\begin{aligned} &u = 0 \quad \text{on } \partial\Omega, \qquad u \leq \psi_\rho + \lambda \quad \text{on } B_\rho(y), \\ &\Omega_c = \{x; u(x) = \psi_\rho(x) + \lambda\}, \\ &\Delta u = 0 \quad \text{in } \Omega\backslash\Omega_c, \quad \Delta u \geq 0 \qquad \text{in } \Omega, \\ &\nabla u \text{ is continuous across } \partial\Omega_c, \\ &\sigma \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, ds = g, \quad \sigma \int_{\partial\Omega} \frac{u_y}{|\nabla u|} \, ds = \sigma \int \frac{u_x}{|\nabla u|} \, ds = 0, \end{aligned}$$

where the vanishing of the last two integrals is a consequence of the fact that y^* is a minimum point of $\mathscr{E}_\rho(y)$. For convenience and without loss of generality we take $\sigma = 1$ in the remainder of the paper.

2. PROBLEM (P1)

Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary, E be an open subset of Ω with $\bar{E} \subset \Omega$ and ψ a function satisfying:

$$\begin{aligned} &\psi \in C^\alpha(\bar{E}) \quad \text{for some } 0 < \alpha < 1, \\ &\psi \geq 0, \quad \min_{\bar{E}} \psi = 0, \quad \psi \not\equiv 0. \end{aligned} \tag{2.1}$$

We consider the functional

$$\mathscr{E}(v, \mu) = \tfrac{1}{2} \int_\Omega |\nabla v|^2 \, dx + g\mu \tag{2.2}$$

where g is a fixed positive constant, and the admissible class

$$K = \{(v, \mu); v \in H_0^1(\Omega), \quad \mu \in \mathbb{R}, \quad v \leq \psi + \mu \quad \text{a.e. in } E\}. \tag{2.3}$$

Problem (P1). Find $(u, \lambda) \in K$ such that

$$\mathscr{E}(u, \lambda) = \inf_{(v, \mu) \in K} \mathscr{E}(v, \mu).$$

We note that (u, λ) is a solution of this problem if and only if it satisfies the variational inequality

$$\int_\Omega \nabla u \cdot \nabla (v - u) \, dx + g(\mu - \lambda) \geq 0 \qquad \forall (v, \mu) \in K. \tag{2.4}$$

THEOREM 2.1. There exists a unique solution (u, λ) of (P1), and

$$\lambda < 0, \qquad u \leq 0 \quad \text{in } \Omega, \qquad u \not\equiv 0 \quad \text{in } \Omega. \tag{2.5}$$

Proof. We claim that $\mathscr{E}(v, \mu)$ is bounded from below on K. Indeed, if μ is negative and $|\mu|$ is large then, for some $c > 0$,

$$\int_\Omega |\nabla v|^2 \geq c \int_\Omega v^2 \geq c \int_E v^2 \geq c \int_E (\psi + \mu)^2 \geq \frac{c}{2} \mu^2 |E|.$$

If follows that

$$\mathscr{E}(v, \mu) \geq \frac{c}{2} \mu^2 |E| + g\mu \geq \frac{c}{3} \mu^2 |E|,$$

and consequently

$$\inf_K \mathscr{E}(v, \mu) > -\infty.$$

We now take a minimizing sequence (v_m, μ_m). By the above estimates we may assume that $\mu_m \to \lambda$ and $v_m \to u$ weakly in $H_0^1(\Omega)$. We easily deduce that $(u, \lambda) \in K$ and that

$$\mathscr{E}(u, \lambda) \leq \liminf \mathscr{E}(v_m, \mu_m),$$

and the proof of existence is complete.

If $(\bar{u}, \bar{\lambda})$ is another solution then taking $(v, \mu) = (\bar{u}, \bar{\lambda})$ in the variational inequality (2.4) for (u, λ) and similarly substituting (u, λ) in the variational inequality for $(\bar{u}, \bar{\lambda})$, we get upon adding,

$$\int_\Omega |\nabla(u - \bar{u})|^2 \leq 0, \qquad \text{so that } \bar{u} = u.$$

Since $\mathscr{E}(u, \lambda) = \mathscr{E}(\bar{u}, \bar{\lambda})$, it also follows that $\lambda = \bar{\lambda}$, since $g > 0$.

To prove (2.5) take $v = u - \varepsilon\phi$ for any $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$, $\varepsilon > 0$. Then $(v, \lambda) \in K$ and from (2.4) we obtain

$$-\int_\Omega \nabla u \cdot \nabla\phi \geq 0, \tag{2.6}$$

that is, $\Delta u \geq 0$ in the distribution sense. Since $u \in H_0^1(\Omega)$ it follows that $u \leq 0$ a.e. in Ω.

Suppose $u \equiv 0$ in Ω. Then the variational inequality (2.4) becomes

$$g(\mu - \lambda) \geq 0 \qquad \forall (v, \mu) \in K.$$

Choosing $v = \phi \in C_0^\infty(\Omega)$ with $\phi = \lambda - 1$ in E and $\mu = \lambda - 1$ we derive the contradiction $g \leq 0$. Hence $u \not\equiv 0$.

Noting that $\mathscr{E}(0, 0) = 0$ and $\mathscr{E}(v, \mu) \geq g\mu \ \forall (v, \mu) \in K$ we deduce that $\lambda \leq 0$. Suppose $\lambda = 0$. Then, taking $\mu = 0$ in (2.4) we find that u solves the obstacle problem

$$\int_\Omega \nabla u \cdot \nabla(v - u) \geq 0 \qquad \forall v \leq \psi \quad \text{in } E, \qquad v \in H_0^1(\Omega)$$

and $u \leq \psi$ in E, $u \in H_0^1(\Omega)$. Since this problem has a unique solution which is, in fact, $u \equiv 0$, we have derived a contradiction. Thus $\lambda < 0$, and (2.5) is proved.

We shall need a comparison lemma for an obstacle problem (Ω, E, ψ) with obstacle ψ in E only, where $\bar{E} \subset \Omega$:

$$\inf_{\substack{v \leq \psi \text{ on } E \\ v \in H_0^1(\Omega)}} \int_\Omega |\nabla v|^2 = \int_\Omega |\nabla u|^2, \qquad u \leq \psi \quad \text{on } E, \qquad u \in H_0^1(\Omega).$$

Here $\partial\Omega$ is Lipschitz and $\psi \in C^0(\bar{E})$.

LEMMA 2.2. Let u and v be the unique solutions of the obstacle problems (Ω_u, E_u, ψ_u) and (Ω_v, E_v, ψ_v) where

$$\Omega_v \subset \Omega_u, \qquad E_v \subset E_u, \qquad \psi_u \leq \psi_v.$$

Then $u \leq v$ in Ω_v.

Proof. Since $\partial\Omega_v$ is Lipschitz, the extension v_e by zero of v into $\Omega_u \backslash \Omega_v$ is in $H_0^1(\Omega_u)$. Hence $\xi = u - (u - v_e)^+$ is admissible for the problem (Ω_u, E_u, ψ_u), so that

$$\int_{\Omega_u} \nabla u \cdot \nabla(u - v_e)^+ \leq 0.$$

Similarly $\zeta = v + (u - v)^+$ is admissible for (Ω_v, E_v, ψ_v) (if $u \geq v$ then $\zeta = u \leq \psi_u$ in E_u and then $\zeta \leq \psi_v$ in E_v) so that

$$\int_{\Omega_v} \nabla v \cdot \nabla(u - v)^+ \geq 0.$$

By adding we obtain

$$\int_{\Omega_u} |\nabla(u - v_e)^+|^2 \leq 0$$

and thus $(u - v_e)^+ = 0$. It follows that $u \leq v_e$ in Ω_u, and $u \leq v$ in Ω_v.

We introduce the condition (S) for ψ and E:

(S) ∂E is in C^1 and for any $x_0 \in \partial E$ there is a ball $B_\rho(z_0)$ in E, with ρ independent of x_0, such that

(i) $\partial B_\rho(z_0)$ is tangent to E at x_0,
(ii) there is a function $\Psi(r) \in C^2(0, \rho)$ satisfying

$$\Psi'(0) > 0, \qquad (r^{n-1}\Psi'(r))' \geq 0, \qquad \Psi'(r) \to \infty \quad \text{if } r \to \rho, \tag{2.7}$$

$$\psi(z) \leq \Psi(r) \quad \text{in } B_\rho(z_\rho) \quad \text{where } r = |x - z_0|, \tag{2.8}$$

$$\psi(x) = \Psi(r) \quad \text{on the line segment } \overline{x_0 z_0}. \tag{2.9}$$

In the next theorem we assert that if (S) holds then the set

$$\Omega_c = \{x; u(x) = \psi(x) + \lambda\} \tag{2.10}$$

is a compact subset of E. (S) holds for example in the interesting case of the rigid body being a sphere. However if the obstacle is a sector strictly smaller than half a sphere then the condition (S) is not satisfied; the examples in Section 4 show that the assertion of theorem 2.3 is not true in this case.

THEOREM 2.3. If (S) holds then there exists an ε-neighborhood of Ω_c which is contained in E ($\varepsilon > 0$).

Proof. Consider first the case $n = 2$ and introduce the obstacle problem

$$\{B_{\rho'}(z_0), B_\rho(z_0), \Psi + \lambda\}$$

where $\rho' > \rho$ and $B_{\rho'}(z_0) \subset \Omega$. We can construct the solution U as follows;

$$U = \Psi + \lambda \quad \text{if } 0 < r < \bar{\rho},$$

$$U = A \log \frac{r}{\rho'} \quad \text{if } \bar{\rho} < r < \rho'.$$

The condition $U' = \Psi'$ at $r = \bar{\rho}$ gives $A = \bar{\rho}\Psi'(\bar{\rho})$, and the condition $U = \Psi + \lambda$ at $r = \bar{\rho}$ then reduces to $G(\bar{\rho}) = \lambda$ where

$$G(s) \equiv s\Psi'(s) \log \frac{s}{\rho'} - \Psi(s).$$

Since $G(0) = -\Psi(0)$, $G'(s) \leq 0$ and $G(s) \to -\infty$ if $s \to \rho$, if $\Psi(0) + \lambda < 0$ then there exists a $\bar{\rho} \in (0, \rho)$ such that $G(\bar{\rho}) = \lambda$, and then U solves the obstacle problem. If, on the other hand, $\Psi(0) + \lambda > 0$ then we take $U \equiv 0$ in $B_\rho(z_0)$. In both cases we can apply lemma 2.2 to conclude that $u \leq U$ in $B_{\rho'}(z_0)$ ($\rho' = \rho$ in the second case) and consequently

$$u < \psi + \lambda \quad \text{if } x \in \overline{x_0 z_0}, \qquad \bar{\rho} < r < \rho.$$

Note finally that if $\Psi(0) + \lambda = 0$ then, since $\Psi'(0) > 0$, we can replace the center z_0 by any point $\bar{z}$ in $\overline{x_0 z_0}$ with $|\bar{z} - z_0|$ arbitrarily small and then arrive at the situation where $\Psi + \lambda > 0$ at $r = 0$. We have thus completed the proof with $\varepsilon = \rho - \bar{\rho}$.

If $n > 2$ then we take in the above proof, for $\bar{\rho} < r < \rho'$,

$$U = A(r^{2-n} - (\rho')^{2-n}) \quad \text{with } A = \frac{1}{2-n} \bar{\rho}^{n-1} \Psi'(\bar{\rho})$$

and then the corresponding function

$$G(s) = \frac{1}{2-n} s^{n-1} \Psi'(s)(s^{2-n} - \rho'^{2-n}) - \Psi(s)$$

again satisfies $G(0) = -\Psi(0)$, $G'(s) \leq 0$, $G(s) \to -\infty$ if $s \to \rho$.

In view of theorem 2.3, we can extend ψ into a C^α function $\tilde{\psi}$ in $\bar{\Omega}$ in such a way that u is a solution of the obstacle problem in Ω with obstacle $\tilde{\psi}$. Applying a standard regularity result for this obstacle problem (see, for instance, [8, p. 46]) we conclude the following corollary.

COROLLARY 2.4. If (S) holds, then the solution (u, λ) satisfies: $u \in C^\alpha(\Omega)$.

Similarly, if $\psi \in C^{1,1}(E)$ (i.e., $D\psi$ is locally Lipschitz in E) then we can choose $\tilde{\psi}$ in $C^{1,1}(\Omega)$ and deduce (using $C^{1,1}$ regularity for the obstacle problem; see, for instance [8, p. 41]) the next corollary.

COROLLARY 2.5. If (S) holds and $\psi \in C^{1,1}(E)$, then the solution (u, λ) satisfies: $u \in C^{1,1}(\Omega)$.

Remark 2.1. If, in addition, Ω is a convex domain in $\mathbb{R}^2$ and Ψ is a convex function, then the contact set $\{u = \psi + \lambda\}$ is connected and simply connected. Indeed, we can extend Ψ so that it remains convex in Ω, and then the assertion follows from a theorem of Lewy and Stampacchia [11].

COROLLARY 2.6. If $\partial\Omega \in C^{1+\beta}$, (S) holds and $\Psi \in C^{1,1}(E)$, then the solution (u, λ) satisfies:

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, ds = g. \tag{2.11}$$

Proof. Since u is harmonic in an Ω-neighborhood Ω_0 of $\partial\Omega$, it follows that u is in $C^{1+\beta}(\bar{\Omega}_0)$; hence the integral in (2.11) is well defined. Let $\tilde{\psi}$ be a function in $C^{1,1}(\bar{\Omega})$ which vanishes on $\partial\Omega$ and coincides with ψ in the set $\Omega_c = \{u = \psi + \lambda\}$. Then the pairs $(\tilde{\psi}, 0)$ and $(2u - \tilde{\psi}, 2\lambda)$

belong to K. Applying (2.4) we obtain

$$\int_\Omega \nabla u \cdot \nabla(\bar\psi - u) - g\lambda \geq 0,$$

$$\int_\Omega \nabla u \, \nabla(u - \bar\psi) + g\lambda \geq 0,$$

so that

$$\int_\Omega \nabla u \, \nabla(u - \bar\psi) + g\lambda = 0.$$

Since $u \in C^{1,1}(\Omega)$ and $u \in C^{1+\beta}(\bar\Omega_0)$, we obtain, by Green's formula,

$$\begin{aligned} g\lambda &= -\int_\Omega \nabla u \cdot \nabla(u - \bar\psi) = -\int_\Omega \nabla u \cdot \nabla(u - \bar\psi - \lambda) \\ &= -\int_{\partial\Omega} \frac{\partial u}{\partial \nu}(u - \bar\psi - \lambda) + \int_\Omega \Delta u(u - \bar\psi - \lambda). \end{aligned}$$

The last integral vanishes, since $\Delta u = 0$ in $\Omega \backslash E$ whereas

$$\Delta u(u - \bar\psi - \lambda) = \Delta u(u - \psi - \lambda) = 0 \qquad \text{a.e. in } E.$$

Therefore

$$\lambda \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \lambda g,$$

and (2.11) follows upon recalling that $\lambda < 0$.

Remark 2.2. Under the assumptions of corollary 2.6, $\int_\Omega \Delta u = g$. Since $\Delta u = 0$ in E,

$$\begin{aligned} u(x) + gG(x, y) &= -\int_\Omega G(x, x')\Delta_{x'} u \, dx' + \int_\Omega G(x, y)\Delta_{x'} u \, dx' \\ &= \int_E [G(x, y) - G(x, x')]\Delta_{x'} u \, dx'. \end{aligned}$$

Since $\Delta u \in L^\infty(\bar\Omega)$ we deduce, for any $1 \leq p < 2$,

$$\begin{aligned} \|u + gG(\cdot, y)\|_{W^{1,p}(\Omega)} &\leq C \sup_{\bar x \in E} \left\{ \int_\Omega |\nabla_x[G(x, \bar x) - G(x, y)]|^p \, dx \right\}^{1/p} \\ &\leq C \sup_{\bar x \in E} |\bar x - y|^{(2/p) - 1 - \delta} \qquad \forall \delta > 0, \end{aligned}$$

where the last inequality is due to Berger and Fraenkel [1]. Taking in particular $E = E_\rho$ and $\psi = \psi_\rho$ corresponding to a family of balls with center y and radius $\rho \to 0$, the corresponding solutions $u_\rho(x)$ satisfy:

$$\|u_\rho + gG(\cdot, y)\|_{W^{1,p}(\Omega)} \leq C\rho^{(2/p) - 1 - \delta} \to 0$$

if $\rho \to 0$.

Remark 2.3. Under the conditions of corollary 2.6, the solution (u, λ) satisfies (2.11) and is thus a solution of the free boundary problem (F.B.P.) introduced in Section 1. Conversely, any solution of the free boundary problem with "regular" free boundary is also a solution of (P1). Indeed, for any $(v, \mu) \in K$,

$$\int_\Omega \nabla u \cdot \nabla(v-u) = \int_{\Omega_c} \Delta\psi(\psi+\lambda-v) + \int_{\partial\Omega_c} \left[\frac{\partial u}{\partial \nu}\right](\lambda+\psi-v)$$
$$\geqslant \int_{\Omega_c} \Delta\psi(\lambda-\mu)$$

since $[\partial u/\partial\nu] = 0$ and $v \leqslant \psi + \mu$ in E. Noting that

$$\int_{\Omega_c} \Delta\psi = \int_{\partial\Omega_c} \frac{\partial\psi}{\partial\nu} = \int_{\partial\Omega_c} \frac{\partial u}{\partial\nu} = \int_{\partial\Omega} \frac{\partial u}{\partial\nu} = g,$$

we conclude that (u, λ) is a solution of (2.4).

3. AN ALTERNATIVE FORMULATION TO (P1)

For any $\mu \in \mathbb{R}$ introduce the admissible class

$$K_\mu = \{v \in H_0^1(\Omega); v \leqslant \psi + \mu \quad \text{a.e. in } E\}, \tag{3.1}$$

and consider the obstacle problem:

$$u_\mu \in K_\mu, \tfrac{1}{2}\int_\Omega |\nabla u_\mu|^2 = \inf_{v\in K_\mu} \tfrac{1}{2}\int_\Omega |\nabla v|^2. \tag{3.2}$$

Problem (P2). Find λ such that

$$\int_{\partial\Omega} \frac{\partial u_\lambda}{\partial\nu}\,ds = g. \tag{3.3}$$

Here we take g, Ω, E as in Section 2 and suppose that $\partial\Omega \in C^{1+\beta}$; since u_μ is harmonic in an Ω-neighborhood of $\partial\Omega$, it belongs to $C^{1+\beta}$ in an $\bar{\Omega}$-neighborhood of $\partial\Omega$, so that (3.3) is well defined.

THEOREM 3.1. There exists a solution of (P2), and

$$\lambda > -\frac{g}{\gamma|\partial\Omega|} - \sup_E \psi, \quad \text{where } \gamma = \gamma(\Omega, E) > 0. \tag{3.4}$$

Proof. Set

$$I(\mu) = \int_{\partial\Omega} \frac{\partial u_\mu}{\partial\nu}\,ds \tag{3.5}$$

for u_μ solving (3.2). If $\mu \geqslant 0$ then $u_\mu = 0$ and $I(\mu) = 0$. We next have, by comparison, that $u_\mu \geqslant u_{\mu'}$ if $\mu \geqslant \mu'$, and therefore

$$\frac{\partial}{\partial \nu}(u_\mu - u_{\mu'}) \geqslant 0 \quad \text{on } \partial\Omega,$$

so that $I(\mu)$ is monotone increasing in μ.

Take an Ω neighborhood Ω_0 of $\partial\Omega$. Since by [8, p. 17]

$$\|u_\mu - u_{\mu'}\|_{L^2(\Omega)} \to 0 \quad \text{if } \mu \to \mu',$$

we easily deduce that $u_\mu - u_{\mu'} \to 0$ uniformly on $\partial_1\Omega_0 \equiv \partial\Omega_0 \cap \Omega$. Hence, by the maximum principle and $C^{1+\beta}$ estimates for harmonic functions,

$$u_\mu - u_{\mu'} \to 0 \quad \text{in } C^{1+\beta}(\Omega_1)$$

where Ω_1 is an Ω-neighborhood of $\partial\Omega$, $\Omega_1 \subset \Omega_0$. It follows that $I(\mu) - I(\mu') \to 0$, i.e, $I(\mu)$ is continuous in μ.

There exists a positive constant $\gamma = \gamma(\Omega, E)$ such that for any function z satisfying

$$\Delta z = 0 \quad \text{in } \Omega \backslash E, \qquad z \leqslant -1 \quad \text{on } E, \qquad z = 0 \quad \text{on } \partial\Omega$$

the inequality

$$\frac{\partial z}{\partial \nu} \geqslant \gamma$$

holds on $\partial\Omega$. It follows that for μ negative and $|\mu|$ sufficiently large,

$$\frac{\partial u_\mu}{\partial \nu} \geqslant \gamma \left| \mu + \sup_E \psi \right|,$$

so that

$$I(\mu) \geqslant -\gamma|\partial\Omega|\mu - \gamma|\partial\Omega| \sup_E \psi.$$

Since $I(0) = 0$ and $I(\mu)$ is monotone and continuous, there exists a solution λ of $I(\lambda) = g$; furthermore, λ satisfies (3.4).

THEOREM 3.2. Let $\partial\Omega \in C^{1+\beta}$, $\psi \in C^{1,1}(E)$ and let (S) hold; then the solution (u, λ) of (P1) is also a solution of (P2). If, in addition, $\Delta\psi \geqslant 0$ then the solution of (P2) is unique and coincides with the solution of (P1).

Proof. The first part follows from corollary 2.6. To prove the second part we note that the proof of corollary 2.5 and theorem 2.3 (upon which it is based) extend without change to any solution (u_λ, λ) of (P2). Thus $u = u_\lambda$ is in $C^{1,1}$. Setting $\Omega_c = \{u = \psi + \lambda\}$ we have, for any $\zeta \in H_0^1(\Omega)$, $\zeta \leqslant \psi + \mu$,

$$\int_\Omega \nabla u \cdot \nabla(\zeta - u) = -\int_\Omega \Delta u(\zeta - u) = \int_{\Omega_C} \Delta\psi(\psi + \lambda - \zeta)$$

$$\geq \int_{\Omega_C} \Delta\psi(\lambda - \mu) \qquad (\text{since } \Delta\psi \geq 0, \quad \zeta \leq \psi + \mu)$$

$$= (\lambda - \mu)\int_{\Omega_C} \Delta u = (\lambda - \mu)\int_\Omega \Delta u$$

$$= (\lambda - \mu)\int_{\partial\Omega} \frac{\partial u}{\partial \nu} = (\lambda - \mu)g \qquad (\text{by } (3.3)).$$

Thus (2.4) holds and (u, λ) is then the unique solution of problem (P1).

4. A COMPARISON THEOREM: EXAMPLES

We wish to derive a comparison theorem for solutions of problem (P1) with different domains Ω but with the same ψ, E. We shall denote one solution by (u, λ_u) and the corresponding domain Ω by Ω_u. Similarly we denote another solution by (v, λ_v) with the corresponding domain by Ω_v.

We assume that $\partial\Omega_u$, $\partial\Omega_v$ are Lipschitz and that (2.1) holds. Recall that $\lambda_u < 0$, $\lambda_v < 0$ and both solutions have free boundary points.

THEOREM 4.1. If $\Omega_v \subset \Omega_u$ then

$$u \leq v \qquad \text{in } \Omega_v, \tag{4.1}$$

$$\lambda_u \leq \lambda_v. \tag{4.2}$$

Proof. Assume first the $\lambda_v \leq \lambda_u$. Denote by v_e the extension of v by zero into Ω_u. Then

$$\zeta \equiv u - (u - v_e)^+ \leq \psi + \lambda_v \quad \text{on } E, \qquad \zeta = 0 \quad \text{on } \partial\Omega_u$$

and thus (ζ, λ_v) is admissible for problem (P1) for Ω_u (since $\partial\Omega_v$ is Lipschitz, $v_e \in H_0^1(\Omega_u)$). Hence

$$-\int_{\Omega_u\setminus\Omega_v} |\nabla u^+|^2 + \int_{\Omega_v} \nabla u \cdot (-\nabla(u - v)^+ + g(\lambda_v - \lambda_u) \geq 0.$$

Next, $\zeta \equiv v + (u - v)^+ \leq \psi + \lambda_u$ on E (since $\lambda_v \leq \lambda_u$) and $\zeta = 0$ on $\partial\Omega_v$. Hence (ζ, λ_u) is admissible for problem (P1) in Ω_v. Thus,

$$\int_{\Omega_v} \nabla v \cdot \nabla(u - v)^+ + g(\lambda_u - \lambda_v) \geq 0.$$

Adding the inequalities we obtain

$$\int_{\Omega_u\setminus\Omega_v} |\nabla u^+|^2 + \int_{\Omega_v} \nabla(u - v) \cdot \nabla(u - v)^+ \leq 0.$$

It follows that $(u - v)^+ = \text{const.}$ in Ω_v; since $(u - v)^+ = 0$ on $\partial\Omega_v$, we conclude that $(u - v)^+ = 0$ in Ω_v, i.e. $u \leq v$.

We next consider the case $\lambda_u \leq \lambda_v$. Then

$$\zeta = u - (u - v_e)^- \leq u \leq \psi + \lambda_u \quad \text{on } E, \qquad \zeta = 0 \quad \text{on } \partial\Omega_u$$

so that (ζ, λ_u) is admissible for Ω_u. We get

$$-\int_{\Omega_u \setminus \Omega_v} |\nabla u^-|^2 - \int_{\Omega_v} \nabla u \cdot \nabla(u - v)^- \geq 0.$$

Next, $\zeta = v + (u - v)^- \leq \psi + \lambda_v$ on E (since $\lambda_u \leq \lambda_v$) and $\zeta = 0$ on $\partial\Omega_v$. Hence (ζ, λ_v) is admissible for Ω_v, and thus

$$\int_{\Omega_v} \nabla v \cdot \nabla(u - v)^- \geq 0.$$

Combining the inequalities we get

$$-\int_{\Omega_u \setminus \Omega_v} |\nabla u^-|^2 - \int_{\Omega_v} |\nabla(u - v)^-|^2 \geq 0$$

from which we again deduce that $u \leq v$.

We have thus proved that (4.1) holds regardless of whether $\lambda_u \gtrless \lambda_v$. Using (4.1) we can now in fact show that $\lambda_u \leq \lambda_v$. Indeed, since the coincidence set of u is nonempty, there is a point $x^* \in E$ such that

$$\psi(x^*) + \lambda_u = u(x^*) \leq v(x^*) \leq \psi(x^*) + \lambda_v,$$

and thus $\lambda_u \leq \lambda_v$.

Theorem 4.1 can be used to compare the solution of problem (P1) with radial solutions, in case ψ is a radial function. Thus we would like to write down the (explicit) solution in the radial case, taking for brevity, just the case $n = 2$.

Let $B_\sigma = \{(x_1, x_2);\ x_1^2 + x_2^2 < \sigma^2\}$, $r = (x_1^2 + x_2^2)^{1/2}$ and

$$\Omega = B_R, \qquad E = B_\rho \quad \text{for some } 0 < \rho < R, \qquad \psi(x_1, x_2) = \Psi(r). \tag{4.3}$$

THEOREM 4.2. Let $\Psi \in C^2(0, \rho) \cap C^\alpha[0, \rho]$, $\Psi(0) = 0$, $\Psi'(0) = 0$, $\Psi \not\equiv 0$ and $(r\Psi')' \geq 0$.

(i) If $\lim_{r \to \rho} \Psi'(r) = \infty$ then for any $g > 0$ the coincidence set is B_{s_g} where $s_g \in (0, \rho)$ is the unique solution of

$$s_g \Psi'(s_g) = \frac{g}{2\pi}, \tag{4.4}$$

and $s_g \to \rho$ if $g \to \infty$, and

$$\lambda_g = \frac{g}{2\pi} \log \frac{s_g}{R} - \Psi(s_g). \tag{4.5}$$

(ii) If $\Psi \in C^{1,\alpha}(\bar{B}_\rho)$ then setting

$$g_c = 2\pi\rho\Psi'(\rho)$$

the assertion (i) holds for all $0 < g < g_c$, whereas for $g > g_c$, $s_g = \rho$ and (4.5) holds.

Remark 4.1. In either case,

$$u(r) = \frac{g}{2\pi}\log\frac{r}{R} \qquad \text{in } s_g < r < R,$$
$$s_g \text{ is independent of } R, \tag{4.6}$$

and, in case (ii) for $g > g_c$ u' is not continuous at $r = s_g$. We also note that the constant $\gamma(\Omega, E)$ of theorem 3.1 is

$$(R \log R/\rho)^{-1}$$

which by (4.5) implies that the lower bound on λ given in theorem 3.1 cannot be bettered in general.

Proof. In both cases Ψ is convex, increasing and nonnegative. If the noncoincidence set $\{u < \psi + \lambda\}$ contains an interval I whose endpoints belong to the free boundary, then since Ψ is subharmonic we get a contradiction to the maximum principle. We thus conclude that the coincidence set consists of precisely one interval, say $0 < r < s_g$. Then the solution (u, λ) is given by

$$u = \Psi + \lambda g \qquad \text{if } 0 < r < s_g,$$

$$u = A \log\frac{r}{R} \qquad \text{if } s_g < r < R.$$

In view of corollary 2.6, $A = g/2\pi$. The condition that u is continuously differentiable at $r = s_g$ (which is valid by corollary 2.6, in case (i)) reduces to (4.5) and

$$\Psi'(s_g) = \frac{g}{2\pi s_\rho}. \tag{4.7}$$

We now observe that the function $F(s) = s\Psi'(s)$ satisfies: $F(0) = 0$, $F'(s) \geqslant 0$, $\lim_{s\to\rho} F(s) = \infty$. Hence there exists a unique solution $s = s_\rho$ of (4.7).

In case (ii) $F(\rho) < \infty$ so that (4.7) can be solved if and only if $0 < g < g_c$. For $g \geqslant g_c$ we must then have $s_g = \rho$.

Example 1. A finger pushing the membrane is capped by half a sphere, that is, $\Psi(r) = \rho - (\rho^2 - r^2)^{1/2}$. We then are in case (i) of theorem 4.2, with

$$F(s) = \frac{s^2}{(\rho^2 - s^2)^{1/2}}, \quad s_g^4 = \frac{g^2}{4\pi^2}(\rho^2 - s_g^2), \quad \text{and} \quad s_g^2 = \frac{g^2}{8\pi^2}\left\{\left(1 + 16\frac{\pi^2\rho^2}{g^2}\right)^{1/2} - 1\right\}.$$

Example 2. $\Psi(r) = \bar\rho - (\bar\rho^2 - r^2)^{1/2}$, $\rho < \bar\rho$. Now we are in case (ii) with

$$g_c = \frac{2\pi\rho^2}{(\bar\rho^2 - \rho^2)^{1/2}}.$$

For $g < g_c$,

$$s_g^2 = \frac{g^2}{8\pi^2}\left\{\left(1 + 16\frac{\pi^2\bar\rho^2}{g^2}\right)^{1/2} - 1\right\}.$$

Example 3. $\Psi(r) = \alpha/2r^2$, $0 < r < \rho$. Then

$$g_c = 2\pi\alpha\rho^2$$

and, for $g < g_c$,

$$s_g = \left(\frac{g}{2\pi\alpha}\right)^{1/2}.$$

5. PROBLEM 2.

In this section we consider problem 2 (for dimension $n = 2$) described in Section 1. It will suffice to treat it only in its first formation (1.18), or (1.21), i.e., we seek to characterize the points y^* in Ω such that

$$\mathscr{E}_\rho(y^*) = \min_{y\in\Omega} \mathscr{E}_\rho(y),$$

where $\mathscr{E}_\rho(y)$ is defined by (1.20). Recall that $\psi_{y,\rho}$ denotes the obstacle corresponding to a ball with center y and radius ρ; we denote by $(u_{y,\rho}, \lambda_{y,\rho})$ the corresponding solution of problem (P1).

We introduce Green's function $G(x, y)$ of $-\Delta$ in Ω. Thus,

$$G(x, y) = \frac{1}{2\pi}\log\frac{1}{|x-y|} - h(x, y)$$

where $h(x, y)$ is harmonic in $x \in \Omega$, $x \neq y$, and $h(x, y) = 0$ if $x \in \partial\Omega$.

Recall [8, pp. 547–548] that the function $h(x, x)$ is subharmonic, $h(x, x) \to +\infty$ if $x \to \partial\Omega$, and the set S of points $\bar{x}$ such that

$$h(\bar{x}, \bar{x}) = \min_{x\in\Omega} h(x, x) \tag{5.1}$$

consists of a finite number of points.

We fix a domain Ω_0 such that

$$S \subset \Omega_0, \qquad \bar{\Omega}_0 \subset \Omega, \tag{5.2}$$

and restrict ρ to be smaller than $\operatorname{dist}(S, \partial\Omega_0)$.

In this section we prove:

THEOREM 5.1. Let $\bar{y}_\rho$ be any point in $\bar{\Omega}_0$ such that

$$\min_{B_\rho(y)\subset\bar{\Omega}_0} \mathscr{E}_\rho(y) = \mathscr{E}_\rho(\bar{y}_\rho).$$

Then

$$\operatorname{dist}(\bar{y}_\rho, S) \leq C\left(\rho\log\frac{1}{\rho}\right)^\gamma \to 0 \qquad \text{as } \rho\to 0, \tag{5.3}$$

where C and γ are some positive constants.

From the proof it follows that $0 < \gamma \leq 1/2$ and, under some conditions, $\gamma = 1/2$.

Remark 5.1. The unique minimizer (u_y, λ_y) of $\mathscr{E}_\rho(y)$ varies continuously in y, provided $B_\rho(y) \subset \bar{\Omega}_0$, where (u_y, λ_y) is equipped with the $W^{1,2}(\Omega) \times \mathbb{R}$ norm. Indeed, if (u_{y_m}, λ_{y_m}) is a sequence of minimizers with $y_m \to y$, then every subsequence of (u_{y_m}, λ_{y_m}) has a subsequence which converges to (u, λ), a minimizer of $\mathscr{E}_\rho(y)$, and by uniqueness $(u, \lambda) = (u_y, \lambda_y)$. This implies that (for the entire sequence) $(u_{y_m}, \lambda_{y_m}) \to (u_y, \lambda_y)$. It then easily follows also that $y \to \mathscr{E}_\rho(y)$ is continuous, which ensures the existence of a minimum $\bar{y}_\rho$ as in theorem 5.1.

Proof. We shall need the following fact: for any solution z of

$$\Delta z = 0 \quad \text{in } \Omega\backslash\bar{\Omega}_0, \qquad z = 0 \quad \text{on } \partial\Omega,$$
$$-1 \leqslant z < 0 \quad \text{on } \partial\Omega_0$$

there holds:

$$0 < \frac{\partial z}{\partial \nu} < \gamma^* \quad \text{on } \partial\Omega \tag{5.4}$$

where γ^* is a constant independent of the particular z.

Let w be defined by

$$\Delta w = 0 \quad \text{in } \Omega\backslash\Omega_{y,\rho}, \qquad w = 0 \quad \text{on } \partial\Omega, \qquad w = 1 \quad \text{on } \partial\Omega_{y,\rho}; \tag{5.5}$$

more precisely we define w by the variational formulation, as the minimizer of

$$\int_{\Omega\backslash\Omega_{y,\rho}} |\nabla v|^2$$

among functions satisfying the boundary conditions $v = 1$ on $\partial\Omega_{y,\rho}$, $v = 0$ on $\partial\Omega$ in the trace sense. The capacity of $\Omega_{y,\rho}$ with respect to Ω is defined by

$$\mathrm{Cap}_\Omega \Omega_{y,\rho} = \int_{\Omega\backslash\Omega_{y,\rho}} |\nabla w|^2 = -\int_{\partial\Omega} \frac{\partial w}{\partial \nu}\, ds. \tag{5.6}$$

We shall prove the estimate

$$-\frac{g}{\lambda_{y,\rho}} \leqslant \mathrm{Cap}_\Omega \Omega_{y,\rho} \leqslant -\frac{g}{\lambda_{y,\rho}} - \frac{C^*\rho}{\lambda_{y,\rho}} \tag{5.7}$$

where C^* is a constant independent of y, ρ. First we recall that

$$\psi = \psi_{y,\rho} = \rho - (\rho^2 - r^2)^{1/2} \qquad (r = |x - y|).$$

Now let $f = (u/\lambda_{y,\rho}) - w$, $u = u_{y,\rho}$. Then

$$\Delta f = 0 \quad \text{in } \Omega\backslash\Omega_{y,\rho}, \qquad f = 0 \quad \text{on } \partial\Omega, \qquad f = \frac{\psi}{\lambda_{y,\rho}} \quad \text{on } \partial\Omega_{y,\rho}.$$

Since $0 < \psi \leqslant \rho$ on $\partial\Omega_{y,\rho}$, the function

$$z = f\frac{|\lambda_{y,\rho}|}{\rho}$$

satisfies $-1 < z < 0$ on $\Omega\backslash\Omega_{y,\rho}$ and, in particular, $-1 \leqslant z < 0$ on $\partial\Omega_0$. It follows (cf. (5.4)) that

$$0 < \frac{\partial z}{\partial \nu} < \gamma^* \quad \text{on } \partial\Omega.$$

Hence

$$\operatorname{Cap}_\Omega \Omega_{y,\rho} = -\frac{1}{\lambda_{y,\rho}}\int_{\partial\Omega}\frac{\partial u}{\partial \nu}\,ds + \int_{\partial\Omega}\frac{\partial f}{\partial \nu}\,ds \in \left(-\frac{g}{\lambda_{y,\rho}}, -\frac{g}{\lambda_{y,\rho}} - C^*\frac{\rho}{\lambda_{y,\rho}}\right)$$

which is the assertion (5.7)

We shall next estimate the set $\Omega_{y,\rho}$. Let $R >$ diameter of Ω, $\rho < R_m < \operatorname{dist}(\Omega_0, \partial\Omega) + \rho$,

$$B_{R_M}(y) \supset \Omega \supset B_{R_m} \supset B_\rho(y)$$

and let U_M, U_m solve the obstacle problems

$$(B_{R_M}(y), B_\rho(y), \psi + \lambda_{y,\rho}), \qquad (B_{R_m}(y), B_\rho(y), \psi + \lambda_{y,\rho})$$

respectively. By the comparison lemma 2.2,

$$U_M \leqslant u \leqslant U_m.$$

Further, by (4.5) and example 1 in Section 4, the coincidence regions for U_M and U_m are discs with center y and radii s_M and s_m given by

$$F(\rho, s_m, R_m) = \lambda_{y,\rho} = F(\rho, s_M, R_M) \tag{5.8}$$

where

$$F(\rho, s, R) = s^2(\rho^2 - s^2)^{-1/2}\log\frac{s}{R} - \rho + (\rho^2 - s^2)^{1/2}. \tag{5.9}$$

Thus

$$B_{s_M}(y) \subset \Omega_{y,\rho} \subset B_{s_m}(y), \qquad s_M < s_m \leqslant \rho. \tag{5.10}$$

We proceed to estimate s_m, s_M and prove that

$$\rho(1 - \gamma\rho^2) \leqslant s_M < s_m \leqslant \rho(1 - \beta\rho^2) \tag{5.11}$$

for some positive constants γ, β, provided ρ is sufficiently small.

To prove (5.11), we begin by obtaining a bound for $-\lambda_{y,\rho}$ using capacities. By (5.7), (5.10)

$$\frac{2\pi}{\log R_m/s_m} = \operatorname{Cap}_{B_{R_m}(y)} B_{s_m}(y) \geqslant \operatorname{Cap}_\Omega \Omega_{y,\rho} \geqslant \frac{g}{-\lambda_{y,\rho}}.$$

Hence

$$-\lambda_{y,\rho} \geqslant \frac{g}{2\pi}\log\frac{R_m}{s_m} \geqslant \frac{g}{2\pi}\log\frac{R_m}{\rho}. \tag{5.12}$$

Similarly

$$\frac{2\pi}{\log R_M/s_M} = \operatorname{Cap}_{B_{R_m}} B_{s_M} \leqslant \operatorname{Cap}_\Omega \Omega_{y,\rho} \leqslant \frac{g + \rho C^*}{-\lambda_{y,\rho}}$$

so that

$$-\lambda_{y,\rho} \leqslant \frac{g+\rho C^*}{2\pi} \log \frac{R_M}{s_M}. \tag{5.13}$$

We now use the relation

$$F(\rho, s_M, R_M) = \lambda_{y,\rho}$$

together with (5.12), (5.13) and get

$$\begin{aligned} \frac{g}{2\pi} \log \frac{R_m}{\rho} &\leqslant s_M^2(\rho^2 - s_M^2)^{-1/2} \log \frac{R_M}{s_M} + \rho - (\rho^2 - s_M^2)^{1/2} \\ &\leqslant \frac{1}{2\pi}(g+\rho C^*) \log \frac{R_M}{s_M}. \end{aligned} \tag{5.14}$$

Noting that

$$\rho - (\rho^2 - s^2)^{1/2} = \frac{s^2}{\rho + (\rho^2 - s^2)^{1/2}} \leqslant \frac{s^2}{(\rho^2 - s^2)^{1/2}} \qquad (s = s_M)$$

we deduce from (5.14) that

$$\begin{aligned} \frac{g}{2\pi} \log \frac{R_m}{\rho} &\leqslant s_M^2(\rho^2 - s_M^2)^{-1/2}\left(\log \frac{R_M}{s_M} + 1\right). \\ s_M^2(\rho^2 - s_M^2)^{-1/2} \log \frac{R_M}{s_M} &\leqslant \frac{g+\rho C^*}{2\pi} \log \frac{R_M}{s_M}. \end{aligned} \tag{5.15}$$

We claim that

$$\frac{\log R_m/\rho}{1 + \log R_M/s_M} \geqslant c > 0 \qquad \text{as } \rho \to 0. \tag{5.16}$$

Indeed, if the left-hand side converges to zero for a sequence $\rho \to 0$, then we deduce that $s_M < \rho^N$ for any $N > 0$, provided ρ is small enough; therefore

$$s_M^2(\rho^2 - s_M^2)^{-1/2} \log \frac{R_M}{s_M} \to 0,$$

which contradicts the first inequality in (5.15).

From (5.15) and (5.16) it follows that there exist positive constants c_1, c_2 such that

$$c_1 < s_M^2(\rho^2 - s_M^2)^{-1/2} \leqslant c_2;$$

similarly we deduce that

$$c_3 < s_m^2(\rho^2 - s_m^2)^{-1/2} \leqslant c_4 \qquad (c_3 > 0, c_4 > 0).$$

From these inequalities we easily deduce the assertions in (5.11).

Using (5.11) we shall now prove that

$$|\mathscr{E}_\rho(y) - \tfrac{1}{2}g\lambda_{y,\rho}| \leqslant C\rho \tag{5.17}$$

if $y \in \Omega_0$, where C is a constant independent of y. Indeed,

$$\begin{aligned}
\mathscr{E}_\rho(y) &= \tfrac{1}{2}\int_\Omega |\nabla u|^2 + \lambda_{y,\rho} g \\
&= \tfrac{1}{2}\int_\Omega \nabla u \cdot \nabla(u-\lambda) + \lambda_{y,\rho} g \\
&= -\tfrac{1}{2}\int_\Omega \Delta u(u-\lambda) + \tfrac{1}{2}\int_{\partial\Omega} \frac{\partial u}{\partial \nu}(u-\lambda) + \lambda_{y,\rho} g \\
&= -\tfrac{1}{2}\int_{\Omega_{y,\rho}} \psi\Delta\psi + \tfrac{1}{2}\lambda_{y,\rho} g.
\end{aligned}$$

Next, since $\Omega_{y,\rho} \subset B_{s_m}(y)$ and $\psi \leqslant \rho$,

$$\left|\int_{\Omega_{y,\rho}} \psi\Delta\psi\right| \leqslant 2\pi\rho\int_0^{s_m} (r\psi')'\,\mathrm{d}r = \frac{2\pi\rho s_m^2}{(\rho^2 - s_m^2)^{1/2}} \leqslant C\rho,$$

and (5.17) follows.

From the estimates leading to (5.11), (5.12) we also have

$$\frac{g}{\mathrm{Cap}_\Omega \Omega_{y,\rho}} \leqslant -\lambda_{y,\rho} \leqslant \frac{g + \rho C^*}{\mathrm{Cap}_\Omega \Omega_{y,\rho}}.$$

Combining this with (5.17) we find that

$$\mathscr{E}_\rho(y) + \frac{g^2}{2}\frac{1 + O(\rho)}{\mathrm{Cap}_\Omega \Omega_{y,\rho}} = O(\rho). \tag{5.18}$$

We shall finally express $\mathrm{Cap}_\Omega \Omega_{y,\rho}$ in terms of $h(y, y)$.

Let z be the solution of

$$\Delta z = 0 \quad \text{in } \Omega\backslash\Omega_{y,\rho}, \qquad \begin{aligned} z &= h(x,y) - h(y,y) \quad \text{on } \partial\Omega_{y,\rho}, \\ z &= 0 \ \text{ on } \partial\Omega \end{aligned}$$

and let ϕ be the solution of

$$\Delta\phi = 0 \quad \text{in } \Omega\backslash\Omega_{y,\rho}, \qquad \begin{aligned} \phi &= \frac{\log(|x-y|/\rho)}{\log \rho} \quad \text{on } \partial\Omega_{y,\rho}, \\ \phi &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Using (5.11) we see that

$$\phi = \frac{\log(1 + O(\rho^2))}{\log \rho} = O\left(\frac{\rho^2}{\log 1/\rho}\right) \quad \text{on } \partial\Omega_{y,\rho}$$

and consequently, by the maximum principle

$$-\frac{\gamma^*\rho^2}{|\log \rho|} < \frac{\partial\phi}{\partial\nu} \leqslant 0 \qquad \text{on } \partial\Omega. \tag{5.19}$$

Since $z = 0(\rho)$ on $\partial\Omega_{y,\rho}$, we also have

$$\frac{\partial z}{\partial \nu} = 0(\rho) \quad \text{on } \partial\Omega. \tag{5.20}$$

Observe now that the harmonic functions in $\Omega\backslash\Omega_{y,\rho}$

$$G(x, y) + h(y, y)w + z(x) \qquad (w \quad \text{as in (5.5)})$$

and

$$\frac{1}{2\pi}\log\frac{1}{\rho}\cdot(\phi + w)$$

agree on $\partial\Omega_{y,\rho}$ as well as on $\partial\Omega$; hence they coincide. It follows that

$$\int_{\partial\Omega}\frac{\partial G}{\partial \nu} + h(y, y)\int_{\partial\Omega}\frac{\partial w}{\partial \nu} + \int_{\partial\Omega}\frac{\partial z}{\partial \nu} = \frac{1}{2\pi}\log\frac{1}{\rho}\int_{\partial\Omega}\frac{\partial w}{\partial \nu} + \frac{1}{2\pi}\log\frac{1}{\rho}\int_{\partial\Omega}\frac{\partial \phi}{\partial \nu}.$$

Recalling (5.6) and (5.19), (5.20), we find that

$$\left(h(y, y) + \frac{1}{2\pi}\log\rho\right)\mathrm{Cap}_{\Omega}\Omega_{y,\rho} = -1 + O(\rho).$$

Hence, by (5.18),

$$\mathscr{E}_\rho(y) + \frac{g^2}{4\pi}\log\frac{1}{\rho} = \frac{g^2}{2}h(y, y) + O\left(\rho\log\frac{1}{\rho}\right). \tag{5.21}$$

If $y_0 \in S$ then (cf. [2])

$$h(y, y) - h(y_0, y_0) \geqslant c|y - y_0|^\alpha$$

for some $c > 0$ and $\alpha \geqslant 2$ as $y \to y_0$ (the case $\alpha = 2$ holds under some assumptions; see [2, (7.28)]). From this and (5.21) we deduce that if

$$\mathrm{dist}(\bar{y}_\rho, S) = |\bar{y}_\rho - y_0| \qquad (y_0 \in S)$$

then

$$\begin{aligned} 0 \leqslant \mathscr{E}_\rho(y_0) - \mathscr{E}_\rho(\bar{y}_\rho) &\leqslant \frac{g^2}{2}(h(y_0, y_0) - h(\bar{y}_\rho, \bar{y}_\rho)) + C\rho\log\frac{1}{\rho} \\ &\leqslant -\frac{g^2}{2}c|\bar{y}_\rho - y_0|^\alpha - C\rho\log\frac{1}{\rho}, \end{aligned}$$

which establishes (5.3).

Remark 5.2. The set S is contained in the set of equilibrium positions for a vortex free to move in an irrotational flow of an ideal fluid in Ω; in case Ω is convex, S consists of just one point (see Gustafson [9]) and $\alpha = 2$ (by [2, (7.28)] and [14]).

6. RELATED PROBLEMS

We consider an obstacle problem arising from a fluid flow with a moving boundary. The evolution of an expanding blob of viscous fluid in a Hele–Shaw cell of cross-section $\Omega \subset \mathbb{R}^2$ can be studied as a time parametrized variational inequality (see [4, 7] and the references given there). Let Ω be a region lying between two smooth curves, $\partial_i\Omega$ the inner boundary, and $\partial_0\Omega$ the outer boundary. Fluid is injected into the cell across $\partial_i\Omega$ causing the expansion of the fluid blob $\Omega^+(t) \subset \Omega$, and $\partial_0\Omega$ is a rigid wall along which the flow is tangential (at the points of $\partial\Omega^+(t) \cap \partial_0\Omega$). Thus if $p(x, t)$ is the pressure and

$$u(x, t) = \int_0^t p(x, \tau)\,\mathrm{d}\tau,$$

one obtains at any time $t > 0$

$$-\Delta u + 1 \geqslant 0, \quad u \geqslant 0, \quad (-\Delta u + 1)u = 0 \quad \text{a.e. in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial_0\Omega. \tag{6.1}$$

The fluid blob is

$$\Omega^+(t) = \{x;\, u(x, t) > 0\}$$

and

$$\frac{\mathrm{d}|\Omega^+(t)|}{\mathrm{d}t} = \int_{\partial_i\Omega} \frac{\partial p}{\partial \nu}\,\mathrm{d}s, \qquad |\Omega^+(0)| = 0, \tag{6.2}$$

or

$$|\Omega^+(t)| = \int_{\partial_i\Omega} \frac{\partial u}{\partial \nu}\,\mathrm{d}s. \tag{6.3}$$

The problem studied in [7] is that of specified flux on $\partial_i\Omega$. It is interesting to ask the question: can $P(t)$ be chosen so that

$$p = P(t) \quad \text{on } \partial_i\Omega \quad \text{and} \quad |\Omega^+(t)| = g(t) \tag{6.4}$$

where $g(t)$ is a prescribed increasing function of time.

Let

$$\lambda(t) = \int_0^t P(\tau)\,\mathrm{d}\tau.$$

It is easy to see that the problem for $u(x, t)$ becomes:

(P3) find $(u, \lambda) \in K$ such that

$$\int_\Omega \nabla u \cdot \nabla(v - u)\,\mathrm{d}x + g(\lambda - \mu) \geqslant \int_\Omega -(v - u)\,\mathrm{d}x \qquad \forall (v, \mu) \in K; \tag{6.5}$$

here

$$K = \{(v, \mu);\, v \in H^1(\Omega), \mu \in \mathbb{R}, v \geqslant 0 \quad \text{in } \Omega, v = \mu \text{ on } \partial_i\Omega\}.$$

THEOREM 6.1. There exists a solution to (P3) if and only if $g \leqslant |\Omega|$, and the solution is unique if $g < |\Omega|$.

Proof. The variational inequality (6.5) is equivalent to

$$J(u, \lambda) = \inf_{(v, \mu) \in K} J(v, \mu)$$

where

$$J(v, \mu) = \int_\Omega (\tfrac{1}{2}|\nabla v|^2 + v)\, \mathrm{d}x - g\mu.$$

Since for $(v, \mu) \in K$

$$J(v, \mu) = \int_\Omega [\tfrac{1}{2}|\nabla(v - \mu)|^2 + (v - \mu)]\, \mathrm{d}x + (|\Omega| - g)\mu,$$

the functional is bounded from below if and only if $g \leqslant |\Omega|$, and the assertion about existence then easily follows.

Suppose $(\hat{u}, \hat{\lambda})$ and $(\bar{u}, \bar{\lambda})$ are two solutions. The usual substitution into (6.5) gives

$$\int_\Omega |\nabla(\hat{u} - \bar{u})|^2\, \mathrm{d}x = 0$$

so that $\hat{u} - \bar{u} = \text{const.} = \hat{\lambda} - \bar{\lambda}$. From the equality $J(\hat{u}, \hat{\lambda}) = J(\bar{u}, \bar{\lambda})$ we then get

$$(\hat{\lambda} - \bar{\lambda})|\Omega| = \int_\Omega (\hat{u} - \bar{u})\, \mathrm{d}x = g(\hat{\lambda} - \bar{\lambda}),$$

and for $g < |\Omega|$ we obtain uniqueness.

Remark 6.1. Let v be the solution of

$$\Delta v = 1 \quad \text{in } \Omega,$$

$$v = 0 \quad \text{on } \partial_i\Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial_0\Omega$$

and denote the negative minimum of v in Ω by $-\lambda_c$. Let w_λ be the solution of

$$\Delta w_\lambda = 1 \quad \text{in } \Omega,$$

$$w_\lambda = \lambda \quad \text{on } \partial_i\Omega, \qquad \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial_0\Omega.$$

If $g = |\Omega|$ then (w_λ, λ) is a solution of (P3) for any $\lambda \geqslant \lambda_c$.

LEMMA 6.2. Let (U_i, Λ_i) solve (P3) for $g = G_i$ $(i = 1, 2)$. If $G_2 > G_1$ then there holds:

$$U_2 \geqslant U_1 \quad \text{and} \quad 0 \leqslant \Lambda_2 - \Lambda_1 \leqslant \gamma(\Omega)(G_2 - G_1) \tag{6.6}$$

where $\gamma(\Omega)$ is a constant.

Proof. (U_i, Λ_i) are admissible in each other variational inequality. Hence

$$\int_\Omega |\nabla(U_1 - U_2)|^2 \leqslant (G_2 - G_1)(\Lambda_2 - \Lambda_1) \tag{6.7}$$

which implies $\Lambda_2 \geq \Lambda_1$. Next, $(U_1 - (U_1 - U_2)^-, \Lambda_1)$ is admissible for the inequality for U_1 and $(U_2 + (U_1 - U_2)^-, \Lambda_2)$ is admissible for the inequality for U_2. Hence

$$\int_\Omega |\nabla(U_1 - U_2)^+|^2 \leq 0,$$

which implies $(U_1 - U_2)^- = 0$ in Ω, since $(U_1 - U_2)^- = 0$ on $\partial_i\Omega$.

The inequality (6.7) also implies that

$$c(\Omega)(\Lambda_2 - \Lambda_1)^2 \leq (G_2 - G_1)(\Lambda_2 - \Lambda_1),$$

yielding the second inequality in (6.6).

We are now in a position to answer the question concerning the existence of $P(t) = \lambda'(t)$ satisfying (6.4).

THEOREM 6.3. Let $g(t)$ be absolutely continuous function on $[0, T]$, $g(0) = 0$, $g(T) = |\Omega|$ and $g'(t) \geq 0$ a.e. For each $t \in [0, T]$ let $(u, \lambda(t))$ be the solution of (P3) corresponding to $g = g(t)$. Defining $\lambda(T) = \lambda_c$, we have that $\lambda(t)$ is absolutely continuous on $[0, T]$, and thus $\lambda'(t)$ is measurable on $[0, T]$; if $g' \in L^p[0, T]$ then $\lambda' \in L^p[0, T]$.

Proof. For any $t \in [0, T - \delta)$, lemma 6.2 implies that

$$0 \leq \lambda(t + \delta) - \lambda(t) \leq \gamma(\Omega)(g(t + \delta) - g(t))$$

and the theorem immediately follows from this inequality.

Remark 6.1. The results of Sections 2–4 extend to the situation of several obstacles lying above or below u. Here the admissible class is

$$K = \{(v, \mu);\ v \in H_0^1(\Omega), \mu \in \mathbb{R}^d, v \leq \psi_j + \mu_j \quad \text{on } E_j$$
$$\text{for } j \in J^+, v \geq \psi_j + \mu_j \quad \text{on } E_j \quad \text{for } j \in J^-\}$$

where J^-, J^+ are two finite disjoint sets of positive integers, $\psi_j \in C^\alpha(\bar{E}_j)$ and the $\bar{E}_j$ are mutually disjoint and contained in Ω, with

$$\inf_{E_j} \psi_j = 0 \quad \text{if } j \in J^+, \qquad \sup_{E_j} \psi_j = 0 \quad \text{if } j \in J^-.$$

We introduce positive constants g_j for $j \in J^+$ and negative constants g_j for $j \in J^-$, and consider the functional

$$J(v, \mu) = \tfrac{1}{2}\int_\Omega |\nabla v|^2 \,dx + \sum_{J^+ \cup J^-} g_j\mu_j.$$

As in theorem 2.1 one can show that there exists a unique minimizer (u, λ) of J in the set K. The other results of Sections 2–4 also extend with minor changes.

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