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Supplement to

ERROR ESTIMATES WITH SMOOTH AND NONSMOOTH DATA FOR A FINITE ELEMENT METHOD FOR THE CAHN-HILLIARD EQUATION

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Appendix. In this appendix we present the proof of Theorem 4.1. The proof is based on estimating the right-hand side of the variation of constants formula (2.9) for solutions of (2.5), using the analyticity (4.2) of $E(t)$ together with certain bounds for the nonlinearity $\phi(u)$, and the *a priori* bound (2.8) for the H^1 norm of u. We begin with the required bounds for $\phi(u)$.

Lemma A.1. *Assume that* $u, v \in \dot{H}^1$ *with* $|u|_1, |v|_1 \le R$. *Then there is a constant* $C = C(R)$ *such that, under the appropriate regularity assumptions for* w *and* z, *we have*

$$|\phi'(u)w|_0 \le C\,|w|_1, \tag{A.1}$$

$$|\phi(u) - \phi(v)|_0 \le C\,|u - v|_1, \tag{A.2}$$

$$|\phi(u)|_0 \le C, \tag{A.3}$$

$$|AP\big(\phi'(u)w\big)|_0 \le C\,\big(|w|_3 + |u|_3|w|_1\big), \tag{A.4}$$

$$|AP\big(\phi(u) - \phi(v)\big)|_0 \le C\,\big(|u - v|_3 + (|u|_3 + |v|_3)|u - v|_1\big), \tag{A.5}$$

$$|AP\phi(u)|_0 \le C\,|u|_3, \tag{A.6}$$

$$|\phi''(u)wz|_0 \le C\,|w|_1|z|_1, \tag{A.7}$$

$$|AP\big(\phi''(u)wz\big)|_0 \le C\,\big(|w|_1|z|_3 + |w|_3|z|_1 + |u|_3|w|_1|z|_1\big), \tag{A.8}$$

$$\begin{aligned}|AP\big(\big[\phi'(u) - \phi'(v)\big]w\big)|_0 \le C\,\big(&|w|_1|u - v|_3 + |w|_3|u - v|_1\\ &+ (|u|_3 + |v|_3)|w|_1|u - v|_1\big).\end{aligned} \tag{A.9}$$

Proof. We only demonstrate (A.8) from which (A.9) readily follows. The bound (A.4) is proved in a similar way, and (A.5), (A.6) then follow. The same is true of (A.1)–(A.3) and (A.7). We present the proof for the case $d = 3$; the case $d \le 2$ is analogous.

For the proof of (A.8) we first note that

$$\|\Delta(fg)\|_{L_2} \le \|\Delta f\|_{L_6}\|g\|_{L_3} + 2\|\nabla f\|_{L_6}\|\nabla g\|_{L_3} + \|f\|_{L_6}\|\Delta g\|_{L_3}.$$

With the intention of applying this with $f = \phi''(u)$ and $g = wz$, we derive by Sobolev's inequality

$$\begin{aligned}
\|wz\|_{L_3} &\le \|w\|_{L_6}\|z\|_{L_6} \le C\|w\|_1\|z\|_1,\\
\|\nabla(wz)\|_{L_3} &\le \|\nabla w\|_{L_6}\|z\|_{L_6} + \|w\|_{L_6}\|\nabla z\|_{L_6} \le C\big(\|w\|_2\|z\|_1 + \|w\|_1\|z\|_2\big),\\
\|\Delta(wz)\|_{L_3} &\le \|\Delta w\|_{L_6}\|z\|_{L_6} + 2\|\nabla w\|_{L_6}\|\nabla z\|_{L_6} + \|w\|_{L_6}\|\Delta z\|_{L_6}\\
&\le C\big(\|w\|_3\|z\|_1 + \|w\|_2\|z\|_2 + \|w\|_1\|z\|_3\big),
\end{aligned}$$

and, using also the assumption (2.4) that ϕ is a cubic polynomial,

$$\begin{aligned}
\|\phi''(u)\|_{L_6} &\le C\left(1+\|u\|_{L_6}\right) \le C\left(1+\|u\|_1\right),\\
\|\nabla\phi''(u)\|_{L_6} &\le \|\phi'''(u)\|_{L_\infty}\|\nabla u\|_{L_6} \le C\|u\|_2,\\
\|\Delta\phi''(u)\|_{L_6} &\le \|\phi'''(u)\|_{L_\infty}\|\Delta u\|_{L_6} \le C\|u\|_3.
\end{aligned}$$

Taking these bounds together, we obtain

$$\|\Delta(\phi''(u)wz)\| \le C\left(1+\|u\|_1\right)\left(\|w\|_1\|z\|_3 + \|w\|_3\|z\|_1 + \|u\|_3\|w\|_1\|z\|_1\right),$$

where we have used the fact that

$$\|u\|_2\|v\|_2 \le C\left(\|u\|_1\|u\|_3\|v\|_1\|v\|_3\right)^{1/2} \le C\left(\|u\|_1\|v\|_3 + \|u\|_3\|v\|_1\right).$$

In view of the equivalence of the norms $\|\cdot\|_s$ and $|\cdot|_s$ on $\dot H^s$, this proves (A.8). ∎

Remark. If we replace the norm $|\cdot|_3$ by $|\cdot|_\gamma$ for some $\gamma \in (3,4)$ in the above result, then we may replace our assumption that ϕ is a cubic polynomial ($d \le 3$) by a more general polynomial bound: $|\phi^{(j)}(s)| \le C\,(1+|s|^{q-j})$ for $q \in [3,5)$, $j = 1,\dots,4$, cf. von Wahl [20]. Our regularity analysis then works in the same way, but we have chosen the present setup for the ease of presentation.

Before embarking on the proof of Theorem 4.1, it is convenient to formulate three technical lemmas concerning the following situation: Let τ, γ, K, T be fixed with $0 \le \tau < T$, $\gamma \in [0,1]$, $K \ge 0$, and assume that

$$w(t) = E(t-\tau)w(\tau) + \int_\tau^t E(t-s)F(s)\,ds, \qquad \tau \le t \le T, \tag{A.10}$$

$$|w(\tau)|_\gamma \le K. \tag{A.11}$$

Below we will encounter this situation with w replaced by u, u_t and u_{tt}.

Lemma A.2. *Let w satisfy (A.10) and (A.11). (a) If, in addition,*

$$|F(t)|_0 \le C|w(t)|_3 + K\,(t-\tau)^{-(3-\gamma)/4}, \qquad \tau < t < T, \tag{A.12}$$

then

$$|w(t)|_\rho \le C(T,\rho)K\,(t-\tau)^{-(\rho-\gamma)/4}, \qquad \rho \in [\gamma,4),\ \tau < t < T. \tag{A.13}$$

(b) If

$$|GF(t)|_0 \le C|w(t)|_1 + K\,(t-\tau)^{-(2-\gamma)/4}, \qquad \tau < t < T,$$

then

$$|w(t)|_1 \le C(T)K\,(t-\tau)^{-(1-\gamma)/4}, \qquad \tau < t < T.$$

Proof. Using (A.12) together with (A.11) and (4.2), we obtain

$$\begin{aligned}
|w(t)|_3 &\le |E(t-\tau)w(\tau)|_3 + \int_\tau^t |E(t-s)F(s)|_3\,ds\\
&\le C\,(t-\tau)^{-(3-\gamma)/4}|w(\tau)|_\gamma + C\int_\tau^t (t-s)^{-3/4}|F(s)|_0\,ds\\
&\le CK\,(t-\tau)^{-(3-\gamma)/4} + C\int_\tau^t (t-s)^{-3/4}|w(s)|_3\,ds\\
&\quad + CK\int_\tau^t (t-s)^{-3/4}(s-\tau)^{-(3-\gamma)/4}\,ds\\
&\le CK\,(t-\tau)^{-(3-\gamma)/4} + C\int_\tau^t (t-s)^{-3/4}|w(s)|_3\,ds, \qquad \tau < t < T,
\end{aligned}$$

and the Gronwall Lemma 6.3 yields $|w(t)|_3 \le CK\,(t-\tau)^{-(3-\gamma)/4}$ for $\tau < t < T$. Substituting this back into (A.12) and repeating the above argument, we arrive at (A.13). Part (b) is proved in a similar way, using

$$|E(t-s)F(s)|_1 = |E(t-s)GF(s)|_3 \le C\,(t-s)^{-3/4}|GF(s)|_0. \quad ∎$$

Lemma A.3. *Let w satisfy (A.10) and (A.11), and assume that, in addition,*

$$|w(t)|_\rho \le K\,(t-\tau)^{-(\rho-\gamma)/4},\ |F(t)|_0 \le K\,(t-\tau)^{-(3-\gamma)/4},\ |GF(t)|_0 \le K\,(t-\tau)^{-(1-\gamma)/4},$$

for $\rho \in [\gamma,4)$, $\tau < t < T$. Then, for $\rho \in [1,3]$, $\epsilon \in [0, 4-\rho)$, $\tau < s < t < T$,

$$|w(t) - w(s)|_\rho \le C(T,\epsilon)K\,(t-s)^{\epsilon/4}(t-\tau)^{-(\rho+\epsilon-\gamma)/4}.$$

Proof. A simple calculation using (A.10) shows

$$w(t) - w(s) = \big(E(t-s) - I\big)w(s) + \int_s^t E(t-\sigma)F(\sigma)\,d\sigma \equiv I_1 + I_2.$$

Using our assumption, we have for the first term on the right

$$|I_1|_\rho \le C\,(t-s)^{\epsilon/4}|w(s)|_{\rho+\epsilon} \le CK\,(t-s)^{\epsilon/4}(s-\tau)^{-(\rho+\epsilon-\gamma)/4},$$

see, e.g., Pazy [16, Theorem 2.6.13]. For the second term we get

$$|I_2|_3 \le C\int_s^t (t-\sigma)^{-3/4}|F(\sigma)|_0\,d\sigma \le CK\,(t-s)^{1/4}(s-\tau)^{-(3-\gamma)/4},$$

and

$$|I_2|_1 \le C\int_s^t (t-\sigma)^{-3/4}|GF(\sigma)|_0\,d\sigma \le CK\,(t-s)^{1/4}(s-\tau)^{-(1-\gamma)/4},$$

and the desired result follows by interpolation using the moment inequality (6.20). ∎

Lemma A.4. *Let w satisfy (A.10) and (A.11). Assume that, in addition, for some $\epsilon \in (0,1)$ the following bounds hold:*

$$|F(t)|_0 \le K\,(t-\tau)^{-(3-\gamma)/4},\ |F(t)-F(s)|_0 \le K\,(t-s)^{\epsilon/4}(s-\tau)^{-(3+\epsilon-\gamma)/4},$$

for $\tau < s < t < T$. Then $w \in C^1((\tau,T),L_2)$ with $w(t) \in \dot H^4$, $w_t + A^2 w = F$, and

$$\text{(A.14)} \qquad |w_t(t)|_0 \le C(T)K\,(t-\tau)^{-1+\gamma/4}, \qquad \tau < t < T.$$

Moreover,

$$\text{(A.15)} \qquad w_t(t) = E(t-\tau_1)w_t(\tau_1) + \int_{\tau_1}^t E(t-s)F'(s)\,ds, \qquad \tau < \tau_1 < t < T.$$

Proof. The first claims follow from a standard regularity result for linear nonhomogeneous evolution equations, see, e.g., Pazy [16, Corollary 4.3.3]. We only need to verify (A.14) and (A.15). Differentiating (A.10), we obtain

$$\text{(A.16)} \qquad w_t(t) = D_tE(t-\tau)w(\tau) + F(t) + \int_\tau^t D_tE(t-s)F(s)\,ds,$$

or, using $D_tE(t-s) = -A^2E(t-s) = -D_sE(t-s)$,

$$w_t(t) = -A^2E(t-\tau)w(\tau) + E(t-\tau)F(t) + \int_\tau^t A^2E(t-s)\big(F(t)-F(s)\big)\,ds.$$

Hence,

$$\begin{aligned}|w_t(t)|_0 &\le C\,(t-\tau)^{-1+\gamma/4}|w(\tau)|_\gamma + C|F(t)|_0 + C\int_\tau^t (t-s)^{-1}|F(t)-F(s)|_0\,ds\\ &\le CK\,(t-\tau)^{-1+\gamma/4} + CK\,(t-\tau)^{-(3-\gamma)/4}\\ &\quad + CK\int_\tau^t (t-s)^{-1+\epsilon/4}(s-\tau)^{-(3+\epsilon-\gamma)/4}\,ds\\ &\le CK\,(t-\tau)^{-1+\gamma/4}, \qquad \tau < t < T.\end{aligned}$$

Finally, (A.15) now follows essentially by integration by parts in (A.16). ∎

Proof of Theorem 4.1. We first apply a standard argument based on the local Lipschitz condition (A.2) to obtain local existence: For any $R_1 \ge 0$ there is $T_1 > 0$ such that equation (2.9) has a unique solution $u \in C([0,T_1],\dot H^1)$, whenever $u_0 \in \dot H^1$ with $|u_0|_1 \le R_1$. Using the *a priori* bound (2.8), we may then conclude that the solution exists for all time.

It remains to show that u is a solution of (2.5) and that it has the regularity claimed in Theorem 4.1. For simplicity of exposition we present the proof for the special case $\alpha = 1$ only.

From (A.6) and (2.8) it follows that

$$|AP\phi(u(t))|_0 \le C(R)|u(t)|_3,$$

and we apply part (a) of Lemma A.2 with $\tau = 0$, $\gamma = 1$ and $K = R$, to obtain

$$\text{(A.17)} \qquad |u(t)|_\beta \le C(T,R,\beta)t^{-(\beta-1)/4}, \qquad 0 < t < T,\ \beta \in [1,4).$$

In view of the inequality (2.6), this proves the special case $j = l = 0$ of (4.1). (In the sequel we shall not indicate the dependence on T, R, β of various constants.) Substituted into (A.6) and (A.3), the estimate (A.17) also implies that

$$|AP\phi(u(t))|_0 \le Ct^{-1/2},\ |\phi(u(t))|_0 \le C,$$

and Lemma A.3 yields

$$\text{(A.18)} \qquad |u(t)-u(s)|_\rho \le C\,(t-s)^{\epsilon/4}(t-\tau)^{-(\rho+\epsilon-1)/4},$$

for $\rho \in [1,3]$, $\epsilon \in [0,4-\rho)$, $0 < s < t < T$. Substituting this and (A.17) into (A.5), we now have

$$\begin{aligned}|AP[\phi(u(t))-\phi(u(s))]|_0 &\le C\,\big(|u(t)-u(s)|_3 + (|u(t)|_3+|u(s)|_3)|u(t)-u(s)|_1\big)\\ &\le C\,(t-s)^{\epsilon/4}s^{-(2+\epsilon)/4}, \qquad 0 < s < t < T,\end{aligned}$$

and Lemma A.4 shows that u is a solution of (2.5) and that

$$\text{(A.19)} \qquad |u_t(t)|_0 \le Ct^{-3/4}, \qquad 0 < t < T.$$

Moreover, we have

$$\text{(A.20)} \qquad u_t(t) = E(t-\tau)u_t(\tau) - \int_\tau^t E(t-s)AP\big[\phi'(u(s))u_t(s)\big]\,ds, \qquad 0 < \tau < t < T.$$

In view of (A.1), we have here $|\phi'(u(t))u_t(t)|_0 \le C|u_t(t)|_1$ and, by (A.19), we may take $\gamma = 0$, $K = C\tau^{-3/4}$, and apply Lemma A.2 (b). Hence,

$$|u_t(t)|_1 \le C\tau^{-3/4}(t-\tau)^{-1/4}, \qquad 0 < \tau < t < T.$$

Using this result together with (A.17) in (A.4), we get

$$|AP[\phi'(u(t))u_t(t)]|_0 \le C\left(|u_t(t)|_3 + |u(t)|_3|u_t(t)|_1\right) \le C|u_t(t)|_3 + C\tau^{-3/4}(t-\tau)^{-3/4},$$

and Lemma A.2 (a) yields

$$\text{(A.21)} \qquad |u_t(t)|_\beta \le C\tau^{-3/4}(t-\tau)^{-\beta/4}, \qquad 0<\tau<t<T,\ \beta\in[0,4),$$

or, with $\tau = t/2$,

$$\text{(A.22)} \qquad |u_t(t)|_\beta \le Ct^{-(3+\beta)/4}, \qquad 0<t<T,\ \beta\in[0,4),$$

which implies the special case $j=1,\ l=0$ of (4.1). The bound (A.21) also implies that

$$|AP[\phi'(u(t))u_t(t)]|_0 \le C\tau^{-3/4}(t-\tau)^{-3/4},\ |\phi'(u(t))u_t(t)|_0 \le C\tau^{-3/4}(t-\tau)^{-1/4},$$

so that, by Lemma A.3,

$$\text{(A.23)} \qquad |u_t(t)-u_t(s)|_\rho \le C\tau^{-3/4}(t-s)^{\epsilon/4}(s-\tau)^{-(\rho+\epsilon)/4},$$

for $0<\tau<s<t<T,\ \rho\in[1,3],\ \epsilon\in[0,4-\rho)$. Writing next

$$\begin{aligned} AP\big[\phi'(u(t))u_t(t)-\phi'(u(s))u_t(s)\big] &= AP\big(\phi'(u(t))\big[u_t(t)-u_t(s)\big]\big) \\ &\quad + AP\big(\big[\phi'(u(t))-\phi'(u(s))\big]u_t(s)\big) \equiv I_1 + I_2, \end{aligned}$$

we have, by (A.4), (A.17) and (A.23),

$$|I_1|_0 \le C\left(|u_t(t)-u_t(s)|_3 + |u(t)|_3|u_t(t)-u_t(s)|_1\right) \le C\tau^{-3/4}(t-s)^{\epsilon/4}(s-\tau)^{-(3+\epsilon)/4}.$$

Similarly, by (A.9) and (A.18),

$$\begin{aligned} |I_2|_0 \le C\Big(&\big[|u_t(s)|_3 + (|u(t)|_3+|u(s)|_3)\,|u_t(s)|_1\big]|u(t)-u(s)|_1 \\ &+ |u_t(s)|_1|u(t)-u(s)|_3\Big) \le C\tau^{-3/4}(t-s)^{\epsilon/4}(s-\tau)^{-(3+\epsilon)/4}. \end{aligned}$$

We conclude that

$$|AP[\phi'(u(t))u_t(t)-\phi'(u(s))u_t(s)]|_0 \le C\tau^{-3/4}(t-s)^{\epsilon/4}(s-\tau)^{-(3+\epsilon)/4},$$

for $0<\tau<s<t<T$. Hence, by Lemma A.4, $|u_{tt}(t)|_0 \le C\tau^{-3/4}(t-\tau)^{-1} \le Ct^{-7/4}$, and further estimates of u_{tt} can be based on

$$u_{tt}(t) = E(t-\tau)u_{tt}(\tau) - \int_\tau^t E(t-s)AP\big[\phi'(u(s))u_{tt}(s) + \phi''(u(s))u_t(s)^2\big]\,ds.$$

Using (A.1) and (A.7), we first get

$$\begin{aligned} |\phi'(u(t))u_{tt}(t) + \phi''(u(t))u_t(t)^2|_0 &\le C\left(|u_{tt}(t)|_1 + |u_t(t)|_1^2\right) \\ &\le C|u_{tt}(t)|_1 + C\tau^{-7/4}(t-\tau)^{-1/2}, \end{aligned}$$

and Lemma A.2 (b) may be applied with $\gamma=0,\ K=C\tau^{-7/4}$ to yield

$$|u_{tt}(t)|_1 \le C\tau^{-7/4}(t-\tau)^{-1/4}.$$

Hence, by (A.4) and (A.8),

$$\begin{aligned} &|AP[\phi'(u(t))u_{tt}(t) + \phi''(u(t))u_t(t)^2]|_0 \\ &\quad\le C\left(|u_{tt}(t)|_3 + |u|_3|u_{tt}(t)|_1 + (|u_t(t)|_3 + |u(t)|_3|u_t(t)|_1)|u_t(t)|_1\right) \\ &\quad\le C|u_{tt}(t)|_3 + C\tau^{-7/4}(t-\tau)^{-3/4}, \end{aligned}$$

and Lemma A.2 (a) yields $|u_{tt}(t)|_\beta \le C\tau^{-7/4}(t-\tau)^{-\beta/4},\ \beta\in[0,4)$, which proves the special case $j=2,\ l=0$ of (4.1).

We now turn to the cases $l=1,2$. From (A.22) we have

$$|Gu_t(t)|_4 = |u_t(t)|_2 \le Ct^{-5/4},$$

and, using equation (2.5) and (A.3), (A.17),

$$|Gu_t(t)|_0 \le |Au(t)|_0 + |\phi(u(t))|_0 \le |u(t)|_2 + C \le Ct^{-1/4}.$$

Interpolating between these results by means of the moment inequality (6.20), we obtain

$$|Gu_t(t)|_\beta \le Ct^{-(\beta+1)/4}, \qquad \beta\in[0,4],$$

which is the desired result when $j=l=1$. In a similar way we find

$$|G^2u_t(t)|_4 = |u_t(t)|_0 \le Ct^{-3/4},$$

$$|G^2u_t(t)|_1 \le |u(t)|_1 + |G\phi(u(t))|_1 \le C,$$

since $|G\phi(u(t))|_1 \le C|\phi(u(t))|_0 \le C$, and interpolation yields the desired result when $j=1,\ l=2,\ \beta\in[1,4]$. Next

$$|Gu_{tt}(t)|_4 = |u_{tt}(t)|_2 \le Ct^{-9/4},$$

$$|Gu_{tt}(t)|_0 \le |Au_t(t)|_0 + |\phi'(u(t))u_t(t)|_0 \le |u_t(t)|_2 + C|u_t(t)|_1 \le Ct^{-5/4},$$

which covers the case $j=2,\ l=1,\ \beta\in[0,4]$. Finally,

$$|G^2u_{tt}(t)|_4 = |u_{tt}(t)|_0 \le Ct^{-7/4},$$

$$|G^2u_{tt}(t)|_0 \le |u_t(t)|_0 + |G\big(\phi'(u(t))u_t(t)\big)|_0 \le C|u_t(t)|_0 \le Ct^{-3/4},$$

since, by Sobolev's inequality,

$$|Gf|_0 = \sup_{\chi\in\dot H^2}\frac{|(f,\chi)|}{|\chi|_2} \le C\|f\|_{L_1}, \quad \|\phi'(u)u_t\|_{L_1} \le |\phi'(u)|_0|u_t|_0 \le C|u_t|_0.$$

This proves the remaining case $j=l=2,\ \beta\in[0,4]$ of (4.1). ∎