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Équations aux dérivées partielles/*Partial Differential Equations*

On the existence of the Airy function in Lipschitz domains. Application to the traces of H^2

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Abstract. We prove the existence of the Airy function w corresponding to the stress tensor $\mathbf{S}$ in a plane domain Ω connected, eventually not simply connected, with Lipschitz boundary Γ. In order to understand the relations between the boundary values of w and $\mathbf{S}\cdot\mathbf{n}$, we study the tangential derivative from $\mathrm{H}^{1/2}(\Gamma)$ into $\mathrm{H}^{-1/2}(\Gamma)$. With the help of these results we characterize the traces of a function in $\mathrm{H}^2(\Omega)$ extending the previous results of Nečas and Grisvard. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Existence de la fonction d'Airy dans un domaine plan lipschitzien. Application aux traces de H^2

Résumé. *On montre l'existence de la fonction w d'Airy associée au tenseur de contraintes $\mathbf{S}$ dans un domaine plan Ω connexe, éventuellement non simplement connexe, de frontière Γ lipschitzienne. Pour bien expliquer les relations entre les valeurs au bord de w et $\mathbf{S}\cdot\mathbf{n}$, on étudie l'application dérivée tangentielle entre $\mathrm{H}^{1/2}(\Gamma)$ et $\mathrm{H}^{-1/2}(\Gamma)$. Par application de ces résultats on caractérise les traces d'une fonction de $\mathrm{H}^2(\Omega)$ en généralisant les résultats antérieurs de Nečas et Grisvard.* © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

L'application $w \mapsto \{\gamma_0(w), \partial_{\mathbf{n}} w\}$ est linéaire et continue de $\mathrm{H}^2(\Omega)$ dans $\mathrm{H}^1(\Gamma)\times\mathrm{L}^2(\Gamma)$ (*voir* par exemple [6]). Dans le cas où l'ouvert plan Ω a une frontière Γ régulière on sait que l'image de cette application est $\mathrm{H}^{3/2}(\Gamma)\times\mathrm{H}^{1/2}(\Gamma)$. Quand la frontière Γ est polygonale Grisvard [5] a montré que l'image est un sous-espace de $\mathrm{H}^1(\Gamma)\times\mathrm{L}^2(\Gamma)$ caractérisé par des conditions de compatibilité dans les coins. Dans cette Note on donne la caractérisation de cette image dans le cas général où la frontière Γ est une courbe lipschitzienne (de composantes connexes Γ_i, $i=0,\ldots,q$).

Note présentée par Jacques-Louis LIONS.

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THÉORÈME 3. – *Soient* $g_0 \in \mathrm{H}^1(\Gamma)$ *et* $g_1 \in \mathrm{L}^2(\Gamma)$ *donnés. Il existe* $w \in \mathrm{H}^2(\Omega)$ *telle que*

$$\gamma_0(w) = g_0 \in \mathrm{H}^1(\Gamma) \quad \textit{et} \quad \partial_{\mathbf{n}} w = g_1 \in \mathrm{L}^2(\Gamma) \tag{17}$$

et si et seulement si

$$(\partial_{\mathbf{t}} g_0)\mathbf{n} - g_1 \mathbf{t} \in \left(\mathrm{H}^{1/2}(\Gamma)\right)^2. \tag{18}$$

Dans le cas d'une courbe régulière les vecteurs $\mathbf{n}$ et $\mathbf{t}$ sont partout linéairement indépendants et on retrouve la caractérisation classique. Dans le cas polygonal les limites de ces vecteurs aux sommets des polygones ne sont plus linéairement indépendantes et alors (18) traduit les conditions de compatibilité mises en évidence par Grisvard [5].

Ce théorème est conséquence de la caractérisation de la fonction d'Airy dans un domaine à frontière lipschitzienne donnée au théorème 2, qui généralise un résultat de [2]. Cette caractérisation est obtenue à l'aide d'un résultat de Girault–Raviart [3] complété par les propositions 1 et 2 qui peuvent se résumer de la façon suivante :

PROPOSITION 1. – *La dérivée tangentielle* $\partial_{\mathbf{t}}$*, bien définie dans* $\mathrm{H}^1(\Gamma)$*, peut être prolongée par continuité en une application linéaire et continue de* $\mathrm{H}^{1/2}(\Gamma)$ *dans* $\mathrm{H}^{-1/2}(\Gamma)$*, encore notée* $\partial_{\mathbf{t}}$*, dont le noyau est* $N = \{\varphi;\ \varphi \text{ est constante sur } \Gamma\} \equiv \mathbb{R}^{q+1}$ ($q+1$ *étant le nombre de composantes connexes de* Γ) *et dont l'image est*

$$\mathrm{Im}(\partial_{\mathbf{t}}) = \left\{\varphi^* \in \mathrm{H}^{-1/2}(\Gamma_0) \times \mathrm{H}^{-1/2}(\Gamma_1) \times \cdots \times \mathrm{H}^{-1/2}(\Gamma_q);\ \langle \varphi_i^*, 1\rangle_{\Gamma_i} = 0,\ i = 1, \ldots, q\right\}.$$

Ici $\langle \cdot, \cdot \rangle_{\Gamma_i}$ désigne la dualité entre $\mathrm{H}^{-1/2}(\Gamma_i)$ et $\mathrm{H}^{1/2}(\Gamma_i)$ extension du produit scalaire dans $\mathrm{L}^2(\Gamma_i)$.

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1. Let Ω be a bounded open connected (eventually not simply connected) subset of $\mathbb{R}^2$ with boundary Γ Lipschitz continuous. This means (*see* [4]) that for every $x \in \Gamma$ there exists a neighborhood of x where Ω is below the graph of some Lipschitz continuous function ϕ and Γ is represented by the graph of ϕ. Therefore Γ is a Lipschitz curve in $\mathbb{R}^2$ and we denote by Γ_0 the "exterior" component of Γ and by Γ_i, $i = 1, \ldots, q$, the "interior" components of Γ. A unit outward normal vector $\mathbf{n}(\mathbf{x}) = (n_1(x_1, x_2), n_2(x_1, x_2))$ is defined a.e.; the corresponding tangent vector is $\mathbf{t}(\mathbf{x}) = (-n_2(x_1, x_2), n_1(x_1, x_2))$.

When g is a (at least Lipschitzian) function defined on Γ its tangential derivative $\partial_{\mathbf{t}} g$ is defined a.e.; $\mathrm{H}^1(\Gamma) = \{\varphi \in \mathrm{L}^2(\Gamma);\ \partial_{\mathbf{t}}\varphi \in \mathrm{L}^2(\Gamma)\}$ is equipped with the natural graph norm; the space $\mathrm{C}^{0,1}(\Gamma)$ of Lipschitz functions is dense in $\mathrm{H}^1(\Gamma)$.

The Sobolev spaces $\mathrm{H}^m(\Omega)$ of real valued functions are defined for integer $m \geqslant 0$ in the usual way. The linear trace operator $u \mapsto \gamma_0(u) = u_{|\Gamma}$, well defined for u in the dense (in $\mathrm{H}^1(\Omega)$) subspace $\mathrm{C}^\infty(\overline{\Omega})$ admits a continuous extension from $\mathrm{H}^1(\Omega)$ onto $\mathrm{H}^{1/2}(\Gamma)$ where:

$$\mathrm{H}^{1/2}(\Gamma) = \left\{\varphi \in \mathrm{L}^2(\Gamma);\ \iint_{\Gamma\times\Gamma} \frac{|\varphi(\alpha) - \varphi(\beta)|^2}{|\alpha - \beta|^2}\,\mathrm{d}\alpha\,\mathrm{d}\beta < +\infty\right\}$$

equipped with the natural graph norm. When Ω is multiply connected then:

$$\mathrm{H}^{1/2}(\Gamma) = \mathrm{H}^{1/2}(\Gamma_0) \times \mathrm{H}^{1/2}(\Gamma_1) \times \cdots \times \mathrm{H}^{1/2}(\Gamma_q).$$

The kernel of γ_0 is the closure $\mathrm{H}_0^1(\Omega)$ of $\mathcal{D}(\Omega)$ into $\mathrm{H}^1(\Omega)$. Moreover, there exists a linear continuous lifting operator $\ell_\Omega : \varphi \mapsto \ell_\Omega(\varphi)$ from $\mathrm{H}^{1/2}(\Gamma)$ into $\mathrm{H}^1(\Omega)$ such that $\gamma_0 \ell_\Omega(\varphi) = \varphi$. We can therefore equip

$\mathrm{H}^{1/2}(\Gamma)$ with the equivalent norm:

$$|||\varphi|||_{1/2,\Gamma} = \sup\big\{\|v\|_{\mathrm{H}^1(\Omega)};\ v \in \mathrm{H}^1(\Omega) \text{ and } \gamma_0(v) = \varphi\big\};$$

then the trace theorem gives $|||\gamma_0(v)|||_{1/2,\Gamma} \leqslant \|v\|_{\mathrm{H}^1(\Omega)}$. The dual space $\mathrm{H}^{-1/2}(\Gamma)$ is equipped with the dual norm

$$|||\varphi^*|||_{-1/2,\Gamma} = \sup\big\{\langle\varphi^*,\varphi\rangle_\Gamma;\ \varphi \in \mathrm{H}^{1/2}(\Gamma) \text{ and } |||\varphi|||_{1/2,\Gamma} = 1\big\},$$

where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$ extension of the scalar product of $\mathrm{L}^2(\Gamma)$ in the sense that, when $\varphi^* \in \mathrm{L}^2(\Gamma)$, then

$$\langle\varphi^*,\varphi\rangle_\Gamma = \int_\Gamma \varphi^*(\alpha)\,\varphi(\alpha)\,\mathrm{d}\alpha.$$

When Ω is multiply connected, then $\mathrm{H}^{-1/2}(\Gamma) = \mathrm{H}^{-1/2}(\Gamma_0) \times \mathrm{H}^{-1/2}(\Gamma_1) \times \cdots \times \mathrm{H}^{-1/2}(\Gamma_q)$ and, with obvious notations:

$$\langle\varphi^*,\varphi\rangle_\Gamma = \sum_{i=0}^{i=q} \langle\varphi_i^*,\varphi_i\rangle_{\Gamma_i}.$$

2. In the case of a regular boundary the following result is well known (*see*, e.g., [2]).

PROPOSITION 1. – *The tangential derivative, well defined in* $\mathrm{H}^1(\Gamma)$, *can be extended to a linear and continuous operator from* $\mathrm{H}^{1/2}(\Gamma)$ *into* $\mathrm{H}^{-1/2}(\Gamma)$ *still denoted* $\partial_\mathbf{t}$, *whose kernel is* $N = \{\varphi;\ \varphi$ *is constant on* $\Gamma\} \equiv \mathbb{R}^{q+1}$, *where* $q+1$ *is the number of connected components of* Γ.

Proof. – Let $\varphi \in \mathrm{H}^{1/2}(\Gamma)$ and let $w = \ell_\Omega(\varphi) \in \mathrm{H}^1(\Omega)$. For every $\lambda \in \mathrm{H}^{1/2}(\Gamma)$ one defines:

$$\langle\partial_\mathbf{t}\varphi,\lambda\rangle_\Gamma = \int_\Omega \left(\frac{\partial w}{\partial x_2}\frac{\partial z}{\partial x_1} - \frac{\partial w}{\partial x_1}\frac{\partial z}{\partial x_2}\right)\mathrm{d}\mathbf{x}, \tag{1}$$

where $z = \ell_\Omega(\lambda) \in \mathrm{H}^1(\Omega)$. It then follows:

$$\big|\langle\partial_\mathbf{t}\varphi,\lambda\rangle_\Gamma\big| \leqslant \|w\|_{\mathrm{H}^1(\Omega)}\|z\|_{\mathrm{H}^1(\Omega)} = |||\varphi|||_{1/2,\Gamma}|||\lambda|||_{1/2,\Gamma}$$

and hence $|||\partial_\mathbf{t}\varphi|||_{-1/2,\Gamma} \leqslant |||\varphi|||_{1/2,\Gamma}$. Moreover, if φ and λ are in $\mathrm{H}^1(\Gamma)$ (and so w and z are in $\mathrm{H}^{3/2}(\Omega)$, *see* [5]) then in (1), $\partial_\mathbf{t}\varphi$ coincides with the usual tangential derivative. □

Since $\mathcal{D}(\Omega)$ is dense in $\mathrm{H}^1_0(\Omega)$ indeed $\partial_\mathbf{t}\varphi$ does not depend on the lifting in the sense that in (1), one may take any $w,\, z \in \mathrm{H}^1(\Omega)$ such that $\gamma_0(w) = \varphi$ and $\gamma_0(z) = \lambda$. From (1) it also follows that the operator $\partial_\mathbf{t}$ satisfies: $\langle\partial_\mathbf{t}\varphi,\lambda\rangle_\Gamma = -\langle\partial_\mathbf{t}\lambda,\varphi\rangle_\Gamma$.

3. In order to characterize the range of the map $\partial_\mathbf{t}$ let us recall at first that the linear map $\gamma_\mathbf{n} : \mathbf{u} \mapsto \gamma_\mathbf{n}(\mathbf{u}) = (\mathbf{u})_{|\Gamma} \cdot \mathbf{n} = (u_1)_{|\Gamma}n_1 + (u_2)_{|\Gamma}n_2$ is well defined for $\mathbf{u} \in \big(\mathrm{C}^\infty(\overline{\Omega})\big)^2$ with range in $\mathrm{L}^2(\Gamma)$ and can be extended by continuity to a linear continuous map, still denoted $\gamma_\mathbf{n}$, from $\mathrm{H}(\mathrm{div};\Omega) = \big\{\mathbf{u} = (u_1,u_2) \in \big(\mathrm{L}^2(\Gamma)\big)^2 = \mathbf{L}^2(\Omega);\ \mathrm{div}\,\mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \in \mathrm{L}^2(\Omega)\big\}$ into $\mathrm{H}^{-1/2}(\Gamma)$ (*see*, e.g., [1,3,7]). At last we recall the following theorem (*see*, e.g., [3]).

THEOREM 1. – (i) *A function* $\mathbf{v} \in \mathbf{L}^2(\Omega)$ *satisfies*:

$$\mathrm{div}\,\mathbf{v} = 0, \quad \langle\gamma_\mathbf{n}(\mathbf{v}),1\rangle_{\Gamma_i} \quad \textit{for } i = 0,\dots,q,$$

if and only if there exists a stream function $\phi \in \mathrm{H}^1(\Omega)$, *unique up to an additive constant, such that and* $v_1 = \frac{\partial \phi}{\partial x_2}$ *and* $v_2 = -\frac{\partial \phi}{\partial x_1}$.

(ii) *If, moreover,* $\mathbf{v} \in \left(\mathrm{H}^1(\Omega)\right)^2$, *then* $\phi \in \mathrm{H}^2(\Omega)$.

(iii) *Between* $\gamma_0(\phi)$ *and* $\gamma_{\mathbf{n}}(\mathbf{v})$ *the following relation holds*:

$$\gamma_{\mathbf{n}}(\mathbf{v}) = \partial_{\mathbf{t}} \gamma_0(\phi). \tag{3}$$

Using this theorem we characterize the range of $\partial_{\mathbf{t}}$, as in [2] for smooth Γ.

PROPOSITION 2. – *The map* $\partial_{\mathbf{t}}$ *has range*:

$$\operatorname{Im}(\partial_{\mathbf{t}}) = \left\{\varphi^* \in \mathrm{H}^{-1/2}(\Gamma_0) \times \mathrm{H}^{-1/2}(\Gamma_1) \times \cdots \times \mathrm{H}^{-1/2}(\Gamma_q);\ \langle \varphi_i^*, 1\rangle_{\Gamma_i} = 0,\ i = 1, \ldots, q\right\}.$$

Proof. – Given an element $\varphi^* \in \operatorname{Im}(\partial_{\mathbf{t}})$ there exists a $v \in \mathrm{H}^1(\Omega)$, unique up to an additive contant, such that:

$$\int_\Omega \left(\frac{\partial v}{\partial x_1}\frac{\partial w}{\partial x_1} + \frac{\partial v}{\partial x_2}\frac{\partial w}{\partial x_2}\right)\mathrm{d}\mathbf{x} = \langle \varphi^*, \gamma_0(w)\rangle_\Gamma \quad \text{for all } w \in \mathrm{H}^1(\Omega).$$

Since $\mathbf{v} = \mathbf{grad}\, v \in \mathbf{L}^2(\Omega) = \left(\mathrm{L}^2(\Omega)\right)^2$ verifies $\operatorname{div} \mathbf{v} = 0$, and $\gamma_{\mathbf{n}}(\mathbf{v}) = \varphi^*$, Theorem 1 implies that there exists a stream function $\phi \in \mathrm{H}^1(\Omega)$, unique up to an additive constant, such that $v_1 = \frac{\partial \phi}{\partial x_2}$ and $v_2 = -\frac{\partial \phi}{\partial x_1}$. Thanks to (3), $\gamma_0(\phi)$ is therefore a solution (up to an element of N) of $\partial_{\mathbf{t}}\varphi = \varphi^*$. □

The previous results implies that the (generalized) differential equation:

$$\partial_{\mathbf{t}}\varphi = \varphi^*$$

has a solution (unique up to an element of N) $\varphi \in \mathrm{H}^{1/2}(\Gamma)$ if and only if $\varphi^* \in \mathrm{H}^{-1/2}(\Gamma)$ satisfies the compatibility conditions: $\langle \varphi_i^*, 1\rangle_{\Gamma_i} = 0,\ i = 1, \ldots, q$.

4. We can now state and prove the following extension of a result known for a smooth domain (*see*, e.g., [2]).

THEOREM 2. – *A tensor* $\mathbf{S}(\sigma_{\alpha\beta})_{\alpha,\beta=1,2} \in \left(\mathbf{L}^2(\Omega)\right)^2$ *satisfies*:

$$\mathbf{S} = \mathbf{S}^{\mathrm{T}}, \tag{4}$$

$$\operatorname{div} \mathbf{S} = 0, \tag{5}$$

$$\langle \sigma_{\alpha 1} n_1 + \sigma_{\alpha 2} n_2, 1\rangle_{\Gamma_i} = 0 \quad \textit{for } \alpha = 1, 2 \textit{ and } i = 0, \ldots, q, \tag{6}$$

$$\langle \sigma_{11} n_1 + \sigma_{12} n_2, x_2\rangle_{\Gamma_i} = \langle \sigma_{21} n_1 + \sigma_{22} n_2, x_1\rangle_{\Gamma_i} \quad \textit{for } i = 0, \ldots, q, \tag{7}$$

if and only if there exists an Airy function $w \in \mathrm{H}^2(\Omega)$ *such that*:

$$\sigma_{11} = \frac{\partial^2 w}{\partial x_2^2}, \quad \sigma_{12} = \sigma_{21} = -\frac{\partial^2 w}{\partial x_1 \partial x_2}, \quad \sigma_{22} = \frac{\partial^2 w}{\partial x_1^2}. \tag{8}$$

Proof. – Let $\mathbf{S}$ satisfy (4)–(7). For $\alpha = 1, 2$, the vector fields $\mathbf{S}_\alpha = (\sigma_{\alpha 1}, \sigma_{\alpha 2}) \in \mathrm{H}(\operatorname{div}; \Omega)$. Thanks to (5) and (6) one can use Theorem 1 to find the stream functions $\phi_\alpha \in \mathrm{H}^1(\Omega)$; it is useful to remark that (3) reads now as follows:

$$\gamma_{\mathbf{n}}(\mathbf{S}_\alpha) = \partial_{\mathbf{t}} \gamma_0(\phi_\alpha). \tag{9}$$

Condition (4) implies: $\sigma_{12} = -\frac{\partial\phi_1}{\partial x_1} = \sigma_{21} = \frac{\partial\phi_2}{\partial x_2}$. Therefore the vector field $\mathbf{f} = (\phi_1, \phi_2)$ satisfies $\operatorname{div}\mathbf{f} = 0$. In order to apply once more Theorem 2, one has to verify the conditions:

$$\langle\gamma_{\mathbf{n}}(\mathbf{f}), 1\rangle_{\Gamma_i} = 0 \quad \text{for } i = 0, \ldots, q,$$

that now read

$$\int_{\Gamma_i} \big(\gamma_0(\phi_1)n_1 + \gamma_0(\phi_2)n_2\big)\,\mathrm{d}\Gamma_i = 0 \quad \text{for } i = 0, \ldots, q, \tag{10}$$

since $\mathbf{f} \in \big(\mathrm{H}^1(\Omega)\big)^2$.

Thanks to (9) we have:

$$\langle\sigma_{11}n_1 + \sigma_{12}n_2, x_2\rangle_{\Gamma_i} = \langle\gamma_{\mathbf{n}}(\mathbf{S}_1), x_2\rangle_{\Gamma_i} = \langle\partial_{\mathbf{t}}\gamma_0(\phi_1), x_2\rangle_{\Gamma_i} = -\int_{\Gamma_i}\gamma_0(\phi_1)\partial_{\mathbf{t}}x_2\,\mathrm{d}\Gamma_i, \tag{11}$$

$$\langle\sigma_{21}n_1 + \sigma_{22}n_2, x_1\rangle_{\Gamma_i} = \langle\gamma_{\mathbf{n}}(\mathbf{S}_2), x_1\rangle_{\Gamma_i} = \langle\partial_{\mathbf{t}}\gamma_0(\phi_2), x_1\rangle_{\Gamma_i} = -\int_{\Gamma_i}\gamma_0(\phi_2)\partial_{\mathbf{t}}x_1\,\mathrm{d}\Gamma_i, \tag{12}$$

and since $\partial_{\mathbf{t}}x_2 = n_1$ and $\partial_{\mathbf{t}}x_1 = -n_2$ it follows from (7) that (10) holds true and one can apply Theorem 1 to the vector field $\mathbf{f}$. Hence there exists a function $w \in \mathrm{H}^2(\Omega)$ such that:

$$\phi_1 = \frac{\partial w}{\partial x_2} \quad \text{and} \quad \phi_2 = -\frac{\partial w}{\partial x_1}, \tag{13}$$

and so it satisfies (8).

Let now be given $w \in \mathrm{H}^2(\Omega)$ and define $\mathbf{S} = (\sigma_{\alpha\beta})_{\alpha,\beta=1,2} \in \big(\mathbf{L}^2(\Omega)\big)^2$ with (8). It follows immediately that $\mathbf{S} = \mathbf{S}^{\mathrm{T}}$ and $\mathbf{Div}\,\mathbf{S} = \mathbf{0}$. If one takes $w \in \mathrm{C}^\infty(\overline{\Omega})$, which is dense in $\mathrm{H}^2(\Omega)$, then (6) and (7) are immediately satisfied since: $\sigma_{11}n_1 + \sigma_{12}n_2 = \partial_{\mathbf{t}}\big(\frac{\partial w}{\partial x_2}\big)$ and $\sigma_{12}n_1 + \sigma_{22}n_2 = \partial_{\mathbf{t}}\big(-\frac{\partial w}{\partial x_1}\big)$. $\square$

From the relations (9), (11) and (12) one can find as are linked the boundary values of $w \in \mathrm{H}^2(\Omega)$ to the traction at the boundary $\mathbf{S}\cdot\mathbf{n} = (s_1, s_2) \in \big(\mathrm{H}^{-1/2}(\Gamma)\big)^2$ satisfying (6) and (7). Indeed, (9) implies that $\gamma_0(\mathbf{f}) \in \big(\mathrm{H}^{1/2}(\Gamma)\big)^2$ is a solution of

$$\partial_{\mathbf{t}}\gamma_0(\mathbf{f}) = \mathbf{S}\cdot\mathbf{n}. \tag{14}$$

From (13) it follows then that:

$$\partial_{\mathbf{n}}w = \left(\frac{\partial w}{\partial x_1}\right)_{|\Gamma} n_1 + \left(\frac{\partial w}{\partial x_2}\right)_{|\Gamma} n_2 = -\gamma_0(\phi_2)n_1 + \gamma_0(\phi_1)n_2 = -\gamma_0(\mathbf{f})\cdot\mathbf{t} \in \mathrm{L}^2(\Gamma). \tag{15}$$

At last one solves

$$\partial_{\mathbf{t}}\gamma_0(w) = \gamma_0(\phi_1)n_1 + \gamma_0(\phi_2)n_2 = \gamma_{\mathbf{n}}(\mathbf{f}) = \gamma_0(\mathbf{f})\cdot\mathbf{n} \in \mathrm{L}^2(\Gamma). \tag{16}$$

5. We can now obtain our main result: the characterization of the traces of a function $w \in \mathrm{H}^2(\Omega)$. Let us recall that (*see*, e.g., [6]) the map $w \mapsto \{\gamma_0(w), \partial_{\mathbf{n}}w\}$ is linear and continuous from $\mathrm{H}^2(\Omega)$ into $\mathrm{H}^1(\Gamma) \times \mathrm{L}^2(\Gamma)$. When Γ is regular the range of this map is $\mathrm{H}^{3/2}(\Gamma) \times \mathrm{H}^{1/2}(\Gamma)$; Grisvard [4] has characterized the range when Γ is a curvilinear polygon.

THEOREM 3. – *Let be* $g_0 \in \mathrm{H}^1(\Gamma)$ *and* $g_1 \in \mathrm{L}^2(\Gamma)$. *Then there exists* $w \in \mathrm{H}^2(\Omega)$ *such that*:

$$\gamma_0(w) = g_0 \in \mathrm{H}^1(\Gamma) \quad \textit{and} \quad \partial_{\mathbf{n}}w = g_1 \in \mathrm{L}^2(\Gamma) \tag{17}$$

if and only if

$$(\partial_{\mathbf{t}} g_0)\mathbf{n} - g_1\mathbf{t} \in \left(\mathrm{H}^{1/2}(\Gamma)\right)^2. \tag{18}$$

Proof. – (i) Let be given $w \in \mathrm{H}^2(\Omega)$ and define $\mathbf{S} = (\sigma_{\alpha\beta})_{\alpha,\,\beta=1,\,2} \in \left(\mathbf{L}^2(\Omega)\right)^2$ with (8). From (15) and (16) it follows that $(\partial_{\mathbf{t}}\gamma_0(w))\mathbf{n} - (\partial_{\mathbf{n}} w)\mathbf{t} = \mathbf{f}$ and so (14) implies (18).

(ii) Let be $g_0 \in \mathrm{H}^1(\Gamma)$ and $g_1 \in \mathrm{L}^2(\Gamma)$ satisfying (18) and define:

$$\mathbf{f} = (f_1, f_2) \cong (\partial_{\mathbf{t}} g_0)\mathbf{n} - g_1\mathbf{t} \in \left(\mathrm{H}^{1/2}(\Gamma)\right)^2. \tag{19}$$

Let us remark that

$$\langle \partial_{\mathbf{t}} f_\alpha, 1\rangle_\Gamma = 0 \quad \text{for } \alpha = 1,\, 2,$$

and

$$\langle \partial_{\mathbf{t}} f_1, x_2\rangle_\Gamma - \langle \partial_{\mathbf{t}} f_2, x_1\rangle_\Gamma = -\int_\Gamma \mathbf{f}\cdot\mathbf{n} = -\int_\Gamma \partial_{\mathbf{t}} g_0 = 0.$$

We can therefore solve the variational problem:
find $\mathbf{u} \in \left(\mathrm{H}^1(\Omega)\right)^2$ *such that, for all* $\mathbf{z} \in \left(\mathrm{H}^1(\Omega)\right)^2$:

$$\int_\Omega \varepsilon(\mathbf{u})\,\varepsilon(\mathbf{z})\,\mathrm{d}\Omega = \langle \partial_{\mathbf{t}}\mathbf{f}, \gamma_0(\mathbf{z})\rangle_\Gamma.$$

We then define $\mathbf{S} = \varepsilon(\mathbf{u}) \in \mathbf{L}^2(\Omega)$ which satisfies the assumptions of Theorem 2 with $\mathbf{S}\cdot\mathbf{n} = \partial_{\mathbf{t}}\mathbf{f}$. Using (19) it is then easy to verify that the function $w \in \mathrm{H}^2(\Omega)$ obtained by Theorem 2 satisfies $\partial_{\mathbf{n}} w = -\mathbf{f}\cdot\mathbf{t} = g_1$ and $\partial_{\mathbf{t}} w = \mathbf{f}\cdot\mathbf{n} = \partial_{\mathbf{t}} g_0$. $\square$

Remark 1. – When Γ is a curvilinear polygon, then relation (15) exactly contains the compatibility conditions of Grisvard; when Γ is a smooth curve then (15) exactly gives the classical conditions: $g_0 \in \mathrm{H}^{3/2}(\Gamma)$ and $g_1 \in \mathrm{H}^{1/2}(\Gamma)$.

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