Mathematical Model of Coupled Plates Meeting at an Angle $0 < \theta < \pi$ and Its Finite Element Method*

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A mathematical model of coupled plates meeting at an arbitrary angle $0 < \theta < \pi$ is established. The finite element approximation for the mathematical model is considered, and the optimal error bound is obtained, which shows that the convergence of the approximation is reciprocally proportional to the sine of the angle θ between the two plates. © 1993 John Wiley & Sons, Inc.

I. INTRODUCTION

In Feng [1, 2], the mechanical and mathematical models of composite structures have been proposed in general. Ciarlet [3, 4] and LeDret [5] have dealt with the field by the asymptotic analysis. In our recent paper [6], a composite elastic structure consisting of coupled plates meeting at a right angle and its finite element approximations are considered using the approach proposed by Feng [1, 2]. In this article, we consider an elastic structure consisting of coupled plates meeting at an arbitrary angle and its finite element method.

Consider a structure of coupled plates meeting at an angle $0 < \theta < \pi$ as shown in Fig. 1. The notations are as follows:

Ω, the middle surface of an elastic plate;
Ω', the elastic thin plate with thickness $2t$ $(0 < t \ll 1)$;
$\mathbf{f} = (f_i), f_i = f_i(x_1, x_2)$, $(i = 1, 2, 3)$, the applied surface force in Ω;
$\mathbf{u} = (u_i), u_i = u_i(x_1, x_2)$, $(i = 1, 2, 3)$, the displacement vector in Ω, with $u_\alpha = 0$, $\alpha = 1, 2$, $u_3 = \partial u_3/\partial x_1 = 0$ on Γ_0;
Ω', the middle surface of an elastic plate;
$(\Omega')'$, the elastic thin plate, with thickness $2t'(0 < t' \ll 1)$;
$\mathbf{f}' = (f_i'), f_i' = f_i'(x_2', x_3')$, $(i = 1, 2, 3)$, the applied surface force in Ω';

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$\mathbf{u}' = (u_i'),\ u_i' = u_i'(x_2', x_3'),\ (i = 1, 2, 3)$, the displacement vector in Ω';
$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, a system of coordinates with right hand in space;
$\{\mathbf{e}_1, \mathbf{e}_2\}$, a system of coordinates in the plane containing Ω;
$\{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$, another system of coordinates with right hand in space;
$\{\mathbf{e}_1', \mathbf{e}_2'\}$, a system of coordinates in the plane containing Ω';
θ, the angle between two plates Ω and Ω', or between $\mathbf{e}_1$ and $\mathbf{e}_3'$;
(x_1, x_2, x_3), Cartesian coordinate in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$;
(x_1', x_2', x_3'), Cartesian coordinate in $\{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$.

Then the following relations between the two systems of coordinates $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ hold:

$$\begin{cases} \mathbf{e}_1' = \sin\theta \cdot \mathbf{e}_1 - \cos\theta \cdot \mathbf{e}_3, \\ \mathbf{e}_2' = \mathbf{e}_2, \\ \mathbf{e}_3' = \cos\theta \cdot \mathbf{e}_1 + \sin\theta \cdot \mathbf{e}_3; \end{cases} \qquad \begin{cases} \mathbf{e}_1 = \sin\theta \cdot \mathbf{e}_1' + \cos\theta \cdot \mathbf{e}_3', \\ \mathbf{e}_2 = \mathbf{e}_2', \\ \mathbf{e}_3 = -\cos\theta \cdot \mathbf{e}_1' + \sin\theta \cdot \mathbf{e}_3'. \end{cases} \tag{1.1}$$

Assume that there exists the rigid junction on the part Γ between these two plates Ω and Ω', i.e., (cf. [2])

$$\mathbf{u} = \mathbf{u}' \quad \text{on } \Gamma, \tag{1.2}$$

$$-\,\partial u_3/\partial x_1 = \partial u_1'/\partial x_3' \quad \text{on } \Gamma. \tag{1.3}$$

By (1.1), the continuity (1.2) between the displacements $\mathbf{u}$ and $\mathbf{u}'$ of plates Ω and Ω' on Γ can be written as

$$\begin{cases} u_1 = u_1' \sin\theta + u_3' \cos\theta, \\ u_2 = u_2' & \text{in } \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}; \\ u_3 = -u_1' \cos\theta + u_3' \sin\theta, \end{cases}$$
$$\begin{cases} u_1' = u_1 \sin\theta + u_3 \cos\theta, \\ u_2' = u_2 & \text{in } \{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}. \\ u_3' = u_1 \cos\theta + u_3 \sin\theta, \end{cases} \tag{1.4}$$

Consider the total energy of the elastic structure under a virtual displacement $\omega = (\mathbf{v}, \mathbf{v}')$:

$$J(\omega) = \frac{1}{2} D(\omega, \omega) - F(\omega), \tag{1.5}$$

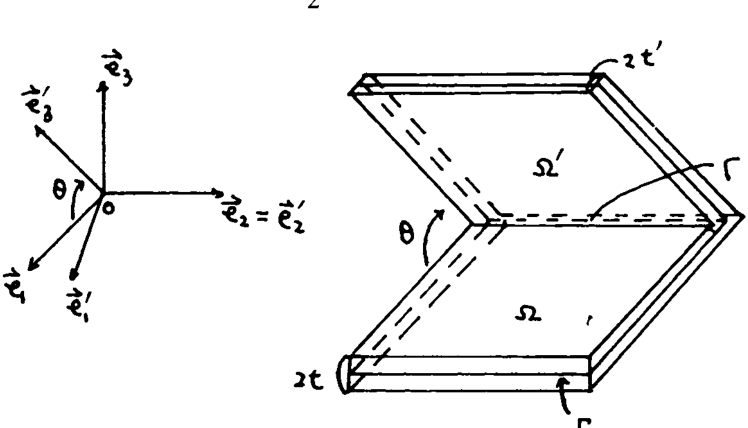

FIG. 1.

where [with $\eta = (\mathbf{u}, \mathbf{u}')$]

$$D(\eta,\omega) = \left\{\int_\Omega Q_{\alpha\beta}(\mathbf{u})\cdot\varepsilon_{\alpha\beta}(\mathbf{v})dx_1dx_2 + \int_\Omega M_{\alpha\beta}(u_3)K_{\alpha\beta}(v_3)dx_1dx_2\right\} + \left\{\int_{\Omega'} Q_{\alpha'\beta'}(\mathbf{u}')\varepsilon_{\alpha'\beta'}(\mathbf{v}')dx_2'dx_3' + \int_{\Omega'} M_{\alpha'\beta'}(u_1')K_{\alpha'\beta'}(v_1')dx_2'dx_3'\right\}, \tag{1.6}$$

$$\begin{cases}\varepsilon_{\alpha\beta}(\mathbf{v}) = \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha), \qquad \partial_\alpha v_\beta = \partial v_\beta/\partial x_\alpha, \\ Q_{\alpha\beta}(\mathbf{v}) = \dfrac{2Et}{1-\nu^2}\{1-\nu)\varepsilon_{\alpha\beta}(\mathbf{v}) + \nu\varepsilon_{\gamma\gamma}(\mathbf{v})\delta_{\alpha\beta}\},\end{cases} \tag{1.7}$$

$$\begin{cases}K_{\alpha\beta}(v_3) = -\partial_{\alpha\beta}v_3: = -\partial^2 v_3/\partial x_\alpha\partial x_\beta, \\ M_{\alpha\beta}(v_3) = \dfrac{2Et^3}{3(1-\nu^2)}\{(1-\nu)K_{\alpha\beta}(v_3) + \nu K_{\gamma\gamma}(v_3)\delta_{\alpha\beta}\}, \qquad (\alpha,\beta,\delta = 1,2);\end{cases} \tag{1.8}$$

$$\begin{cases}\varepsilon_{\alpha'\beta'}(\mathbf{v}') = \frac{1}{2}(\partial_{\alpha'}v_{\beta'}' + \partial_{\beta'}v_{\alpha'}'), \qquad \partial_{\alpha'}v_{\beta'}' = \partial v_{\beta'}'/\partial x_{\alpha'}', \\ Q_{\alpha'\beta'}(\mathbf{v}') = \dfrac{2E't'}{1-\beta'^2}\{(1-\nu')\varepsilon_{\alpha'\beta'}(\mathbf{v}') + \nu'\varepsilon_{\gamma'\gamma'}(\mathbf{v}')\delta_{\alpha'\beta'}\},\end{cases} \tag{1.9}$$

$$\begin{cases}K_{\alpha'\beta'}(v_1') = -\partial_{\alpha'\beta'}v_1': = -\partial^2 v_1'/\partial x_{\alpha'}'\partial x_{\beta'}', \\ M_{\alpha'\beta'}(v_1') = \dfrac{2E't'^3}{3(1-\nu'^2)}\{(1-\nu')K_{\alpha'\beta'}(v_1') + \nu' K_{\gamma'\gamma'}(v_1')\delta_{\alpha'\beta'}\},\end{cases} \tag{1.10}$$

$$(\alpha',\beta',\delta' = 2,3);$$

$$F(\omega) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx_1dx_2 + \int_{\Omega'}\mathbf{f}'\cdot\mathbf{v}'\,dx_2'dx_3', \tag{1.11}$$

and E and E' denote Young's moduli, ν, ν' denote Poisson's ratios of the plates Ω and Ω' respectively.

In here and what follows, Latin indices take their values in $\{1,2,3\}$, Greek and primed Greek indices take their values in $\{1,2\}$ and $\{2,3\}$, respectively; the repeated index convention for summation is systematically used.

II. MATHEMATICAL MODEL

We now introduce a space:

$$|H| = \{\omega = (\mathbf{v},\mathbf{v}'); \qquad \mathbf{v}\in H^{112}(\Omega): = H^1(\Omega)\times H^1(\Omega)\times H^2(\Omega), \quad \mathbf{v}'\in H^{211}(\Omega'): = H^2(\Omega')\times H^1(\Omega')\times H^1(\Omega')\}, \tag{2.1}$$

with norm

$$\|\omega\|_{|H|}: = \left\{\sum_\alpha \|v_\alpha\|_{1,\Omega}^2 + \|v_3\|_{2,\Omega}^2 + \sum_{\alpha'}\|v_{\alpha'}'\|_{1,\Omega'}^2 + \|v_1'\|_{2,\Omega'}^2\right\}, \tag{2.2}$$

and a subspace

$$V = \{\omega = (\mathbf{v},\mathbf{v}')\in|H|; \qquad \mathbf{v} = 0, \quad \partial_1 v_3 = 0 \quad \text{on } \Gamma_0, \quad \text{and} \quad \mathbf{v} = \mathbf{v}', \quad -\partial v_3/\partial x_1 = \partial v_1'/\partial x_3' \quad \text{on } \Gamma\}. \tag{2.3}$$

Then in the weak form, the mathematical model of the elastic structure of coupled two plates with an angle θ is

$$\begin{cases} \text{to find } \eta = (\mathbf{u}, \mathbf{u}') \in V, \text{ such that} \\ D(\eta, \omega) = F(\omega) \quad \forall\, \omega \in V. \end{cases} \tag{2.4}$$

The following two lemmas hold as in [6]:

Lemma 2.1. *The subspace V is closed in $|H|$.*

Lemma 2.2. *The bilinear form $D(\eta, \omega)$ is continuous and coercive on $V \times V$.*

Then by Lax–Milgram's theorem, problem (2.4) has a unique solution.

By the same way as in [6], and taking into account the relation between the coordinate systems $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$, we can establish the boundary value problems and junction conditions on Γ as follows (cf. Fig. 2):

$$\begin{cases} -\partial_\beta Q_{\alpha\beta}(\mathbf{u}) = f_\alpha \quad \text{in } \Omega, \\ Q_{\alpha\beta}(\mathbf{u}) n_\beta = 0 \quad \text{on } \partial\Omega/(\Gamma_0 \cup \Gamma), \\ u_\alpha = 0 \quad \text{on } \Gamma_0, \qquad \alpha = 1,2; \end{cases} \tag{2.5}$$

$$\begin{cases} -\partial_{\alpha\beta} M_{\alpha\beta}(u_3) = f_3 \quad \text{in } \Omega, \\ M_{\alpha\beta}(u_3) n_\alpha n_\beta = 0 \quad \text{on } \partial\Omega/(\Gamma_0 \cup \Gamma), \\ \partial_\alpha M_{\alpha\beta}(u_3) n_\beta + \partial_s M_{\alpha\beta}(u_3) n_\alpha s_\beta = 0 \quad \text{on } \partial\Omega/(\Gamma_0 \cup \Gamma), \\ u_3 = \partial u_3/\partial x_1 = 0 \quad \text{on } \Gamma_0; \end{cases} \tag{2.6}$$

$$\begin{cases} -\partial_{\beta'} Q_{\alpha'\beta'}(\mathbf{u}') = f_\alpha' \quad \text{in } \Omega', \\ Q_{\alpha'\beta'}(\mathbf{u}') n'_{\beta'} = 0 \quad \text{on } \partial\Omega'/\Gamma, \qquad \alpha' = 2,3; \end{cases} \tag{2.7}$$

$$\begin{cases} -\partial_{\alpha'\beta'} M_{\alpha'\beta'}(u_1') = f_3' \quad \text{in } \Omega', \\ M_{\alpha'\beta'}(u_1') n_{\alpha'}' n_{\beta'}' = 0 \quad \text{on } \partial\Omega'/\Gamma, \\ \partial_{\alpha'} M_{\alpha'\beta'}(u_1') n_{\beta'}' + \partial_{s'} M_{\alpha'\beta'}(u_1') n_{\alpha'}' s_{\beta'}' = 0 \quad \text{on } \partial\Omega'/\Gamma, \\ M_{23}(u_1')(P_3') = M_{23}(u_1')(P_4') = 0, \end{cases} \tag{2.8}$$

and

$$\begin{cases} Q_{33}(\mathbf{u}') \cos\theta + \left(\dfrac{\partial M_{\alpha'3}(u_1')}{\partial x_{\alpha'}'} + \dfrac{\partial M_{32}(u_1')}{\partial x_2'} \right) \sin\theta + Q_{11}(\mathbf{u}) = 0, \\ Q_{23}(\mathbf{u}') + Q_{21}(\mathbf{u}) = 0, \\ Q_{33}(\mathbf{u}') \sin\theta - \left(\dfrac{\partial M_{\alpha'3}(u_1')}{\partial x_{\alpha'}'} + \dfrac{\partial M_{32}(u_1')}{\partial x_2'} \right) \cos\theta + \left(\dfrac{\partial M_{\alpha 1}(u_3)}{\partial x_\alpha} + \dfrac{\partial M_{12}(u_3)}{\partial x_2} \right) = 0, \\ M_{33}(u_1') - M_{11}(u_3) = 0, \end{cases} \quad \text{on } \Gamma; \tag{2.9}$$

$$\begin{cases} M_{23}(u_1')(P_\alpha) \sin\theta = 0, \\ M_{12}(u_3)(P_\alpha) - M_{23}(u_i')(P_\alpha) \cos\theta = 0, \end{cases} \qquad \alpha = 1,2. \tag{2.10}$$

Remark 2.1. The angle $\theta = \pi/2$ is the case in [6], and the angle $\theta = 0$ is the folded plate.

III. FINITE ELEMENT APPROXIMATION

In this section, a finite element approximation to the problem (2.4) is considered.

Let $\mathcal{T}_h$ and $\mathcal{T}_h'$ be regular rectangular subdivisions of Ω and Ω', respectively, with the inverse hypotheses, which have the same nodes on Γ. Let

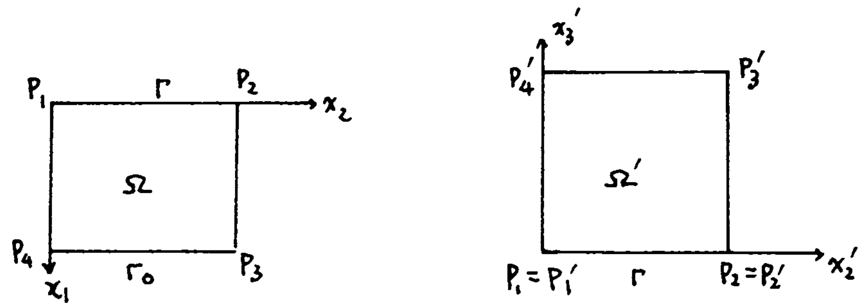

FIG. 2.

$$V_h^1(\Omega) \text{ and } V_h^1(\Omega')\text{—bilinear finite element spaces associated with the subdivisions } \mathcal{T}_h \text{ and } \mathcal{T}'_h\text{, respectively,} \tag{3.1}$$

$$V_h^A(\Omega) \text{ and } V_h^A(\Omega')\text{—Adini's finite element spaces associated with the subdivisions } \mathcal{T}_h \text{ and } \mathcal{T}'_h\text{, respectively,} \tag{3.2}$$

and

$$\begin{aligned} V_h = \{\omega_h = (\mathbf{v}_h, \mathbf{v}'_h);\quad & v_{\alpha, h} \in V_h^1(\Omega), \quad \alpha = 1,2, \quad v_{3h} \in V_h^A(\Omega),\\ & v'_{\alpha', h} \in V_h^1(\Omega'), \quad \alpha' = 2,3, \quad v'_{1,h} \in V_h^A(\Omega'),\\ & v_{ih}(Q) = 0, \quad i = 1,2,3, \quad (\partial v_{3h}/\partial x_1)(Q) = 0 \quad \forall \text{ nodes } Q \in \Gamma_0,\\ & \text{and} \quad v'_{1h}(P) = v_{1h}(P)\sin\theta - v_{3h}(P)\cos\theta, \quad v'_{2h}(P) = v_{2h}(P),\\ & v'_{3h}(P) = v_{1h}(P)\cos\theta + v_{3h}(P)\sin\theta, \quad -(\partial v_{3h}/\partial x_i)(P)\\ & = (\partial v'_{1h}/\partial x'_3)(P) \quad \forall \text{ nodes } P \in \Gamma\}. \end{aligned} \tag{3.3}$$

Then the finite element approximation of the problem (2.4) is

$$\begin{cases} \text{to find } \eta_h \in V_h, \text{ such that} \\ D_h(\eta_h, \omega_h) = F(\omega_h) \quad \forall\, \omega_h \in V_h, \end{cases} \tag{3.4}$$

where

$$\begin{aligned} D_h(\eta_h, \omega_h) = & \left\{\int_\Omega Q_{\alpha\beta}(\mathbf{u}_h)\varepsilon_{\alpha\beta}(\mathbf{v}_h)\,dx_1dx_2 + \sum_\tau \int_\tau M_{\alpha\beta}(u_{3h})K_{\alpha\beta}(v_{3h})\,dx_1dx_2\right\}\\ & + \left\{\int_{\Omega'} Q_{\alpha'\beta'}(\mathbf{u}'_h)\cdot\varepsilon_{\alpha'\beta'}(\mathbf{v}'_h)\,dx'_2dx'_3 + \sum_{\tau'}\int_{\tau'} M_{\alpha'\beta'}(u'_{1h})K_{\alpha'\beta'}(v'_{1h})\,dx'_2dx'_3\right\}, \end{aligned} \tag{3.5}$$

$$f(\omega_h) = \int_\Omega f_i v_{ih}\,dx_1dx_2 + \int_{e'} f'_i v'_{ih}\,dx'_2dx'_3. \tag{3.6}$$

Let

$$\omega = (\mathbf{v}, \mathbf{v}') \in V_h \longrightarrow \|\omega\|_h := \left\{\sum_\alpha \|v_\alpha\|^2_{1,\Omega} + \sum_\tau |v_3|^2_{2,\tau} + \sum_{\alpha'} \|v'_{\alpha'}\|^2_{1,\Omega'} + \sum_{\tau'} |v'_1|^2_{2,\tau'}\right\}^{1/2}, \tag{3.7}$$

which is a norm on V_h, and $D_h(\cdot,\cdot)$ is continuous and coercive on $V_h \times V_h$. The proof is easy.

We have the following error estimate

Theorem 3.1. *Assume that $\eta = (\mathbf{u}, \mathbf{u}')$ is the solution of the problem (2.4), with $u_\alpha \in H^2(\Omega)$, $\alpha = 1,2$, $u_3 \in H^3(\Omega)$, and $u'_{\alpha'} \in H^2(\Omega')$, $\alpha' = 2,3$, $u'_1 \in H^3(\Omega')$. Let*

$\eta_h = (\mathbf{u}_h, \mathbf{u}'_h)$ be the solution of the problem (3.4). Then the following error estimate holds

$$\|\eta - \eta_h\|_h \leq \mathrm{ch}\left\{\sum_\alpha |u_\alpha|_{2,\Omega} + \sum_{\alpha'} |u'_{\alpha'}|_{2,\Omega'} + \|u_3\|_{3,\Omega} + \|u'_1\|_{3,\Omega'} + |\sin^{-1}\theta|\left(\sum_\alpha \|u_\alpha\|_{2,\Omega} + \sum_{\alpha'} \|u'_{\alpha'}\|_{2,\Omega'}\right)\right\}. \tag{3.8}$$

Proof. By the abstract error estimate for the nonconforming finite element approximation (cf. [7]),

$$\|\eta - \eta_h\|_h \leq c\left\{\inf_{\omega_h\in V_h} \|\eta - \omega_h\|_h + \sup_{\zeta_h\in V_h} \frac{D_h(\eta,\zeta_h) - F(\zeta_h)}{\|\zeta_h\|_h}\right\}. \tag{3.9}$$

By using the interpolate error estimates (cf. [7]), the first term on the right-hand side of (3.9) can be estimated as

$$\inf_{\omega_h\in V_h} \|\eta - \omega_h\|_h \leq \mathrm{ch}\left\{\sum_\alpha |u_\alpha|_{2,\Omega} + |u_3|_{3,\Omega} + \sum_{\alpha'} |u'_{\alpha'}|_{2,\Omega'} + |u'_1|_{3,\Omega'}\right\}. \tag{3.10}$$

We now estimate the second term on the right-hand side of (3.9). By Green's formula and taking into account the boundary value problems and the junction conditions in section 2, as in [6], we obtain

$$\begin{aligned}
E_h(\eta,\zeta_h) &= D_h(\eta,\zeta_h) - F(\zeta_h) \\
&= \left\{\int_\Omega Q_{\alpha\beta}(\mathbf{u}) \cdot \varepsilon_{\alpha\beta}(\mathbf{w}_h)\, dx_1 dx_2 + \sum_\tau \int_\tau M_{\alpha\beta}(u_3) K_{\alpha\beta}(w_{3h})\, dx_1 dx_2 \right. \\
&\quad \left. - \int_\Omega \mathbf{f} \cdot \mathbf{w}_h\, dx_1 dx_2\right\} \\
&\quad + \left\{\int_{\Omega'} Q_{\alpha'\beta'}(\mathbf{u}') \cdot \varepsilon_{\alpha'\beta'}(\mathbf{w}'_h)\, dx'_2 dx'_3 + \sum_{\tau'} \int_{\tau'} M_{\alpha'\beta'}(u'_1) K_{\alpha'\beta'}(w'_{1h})\, dx'_2 dx'_3 \right. \\
&\quad \left. - \int_{\Omega'} \mathbf{f} \cdot \mathbf{w}'_h\, dx'_2 dx'_3\right\} \\
&= -\int_\Gamma Q_{11}(\mathbf{u}) w_{1h}\, dx_2 - \int_\Gamma Q_{21}(\mathbf{u}) w_{2h}\, dx_2 - \int_\Gamma Q_{23}(\mathbf{u}') w'_{2h}\, dx_2 - \int_\Gamma Q_{33}(\mathbf{u}') w'_{3h}\, dx_2 \\
&\quad - \int_\Gamma \{\partial_\alpha M_{\alpha 1}(u_3) + \partial_2 M_{12}(u_3)\} w_{3h}\, dx_2 - \int_\Gamma \{\partial_{\alpha'} M_{\alpha' 3}(u'_1) + \partial_2 M_{32}(u'_1)\} w'_{1h}\, dx_2 \\
&\quad - \sum_\tau \int_{2\tau} M_{\alpha\beta}(u_3) n_a \cdot n_\beta \cdot \partial_n w_{3h}\, ds - \sum_{\tau'} \int_{\partial\tau'} M_{\alpha'\beta'}(u'_1) n'_{\alpha'} \cdot n'_{\beta'} \cdot \partial_{n'} w'_{1h}\, ds'.
\end{aligned} \tag{3.11}$$

Since $\zeta_h = (\mathbf{w}_h, \mathbf{w}'_h) \in V_h$, then on Γ,

$$\begin{cases}
w_{1h} = w^I_{3h} \cos\theta \cdot \sin^{-1}\theta + (w'_{1h})^I \cdot \sin^{-1}\theta, \\
w_{2h} = w'_{2h}, \\
w'_{3h} = w^I_{3h}\, sin^{-1}\theta + (w'_{1h})^I \cdot \cos\theta \cdot \sin^{-1}\theta,
\end{cases} \tag{3.12}$$

where v^I denotes the piecewise linear interpolation of v or Γ. From (3.12) and (2.9), it can be seen that

$$\begin{aligned}
\Delta_h := & -\int_\Gamma Q_{11}(\mathbf{u})w_{1h}\,dx_2 - \int_\Gamma Q_{21}(\mathbf{u})w_{2h}\,dx_2 - \int_\Gamma Q_{23}(\mathbf{u}')w'_{2h}\,dx_2 - \int_\Gamma Q_{33}(\mathbf{u}')w'_{3h}\,dx_2 \\
& -\int_\Gamma \{\partial_\alpha M_{\alpha 1}(u_3) + \partial_2 M_{12}(u_3)\}w_{3h}\,dx_2 - \int_\Gamma \{\partial_{\alpha'} M_{\alpha' 3}(u'_1) + \partial_2 M_{32}(u'_1)\}w'_{1h}\,dx_2 \\
= & -\int_\Gamma Q_{11}(\mathbf{u})\{(w'_{1h})^I \cdot \sin^{-1}\theta + w^I_{3h}\cos\theta \cdot \sin^{-1}\theta\}\,dx_2 \\
& -\int_\Gamma \{Q_{21}(\mathbf{u}) + Q_{23}(\mathbf{u}')\}w_{2h}\,dx_2 \\
& -\int_\Gamma Q_{33}(\mathbf{u}')\{(w'_{1h})^I\cos\theta \cdot \sin^{-1}\theta + w^I_{3h}\sin^{-1}\theta\}\,dx_2 \\
& -\int_\Gamma \{\partial_\alpha M_{\alpha 1}(u_3) + \partial_2 M_{12}(u_3)\} \cdot w_{3h}\,dx_2 - \int_\Gamma \{\partial_{\alpha'} M_{\alpha' 3}(u'_1) + \partial_2 M_{32}(u'_1)\} \cdot w'_{1h}\,dx_2 \\
= & -\int_\Gamma Q_{11}(\mathbf{u})\sin^{-1}\theta \cdot (w'_{1h})^I + Q_{33}(\mathbf{u}')\cos\theta \cdot \sin^{-1}\theta \cdot (w'_{1h})^I \\
& + (\partial_{\alpha'} M_{\alpha' 3}(u'_1) + \partial_2 M_{32}(u'_1))w'_{1h}\}\,dx_2 \\
& -\int_\Gamma \{Q_{11}(\mathbf{u})\cos\theta \cdot \sin^{-1}\theta (w_{3h})^I + Q_{33}(\mathbf{u}')\sin^{-1}\theta \cdot w^I_{3h} \\
& + (\partial_\alpha M_{\alpha 1}(u_3) + \partial_2 M_{12}(u_3)) \cdot w_{3h}\}\,dx_2 \\
& -\int_\Gamma \{Q_{21}(\mathbf{u}) + Q_{23}(\mathbf{u}')\}w_{2h}\,dx_2 \\
= & -\int_\Gamma \{Q_{11}(\mathbf{u}) + Q_{33}(\mathbf{u})\cos\theta + (\partial_{\alpha'} M_{\alpha' 3}(u'_1) + \partial_2 M_{32}(u'_1))\sin\theta\}\sin^{-1}\theta \cdot w'_{1h}\,dx_2 \\
& -\int_\Gamma \{Q_{11}(\mathbf{u})\cos\theta \cdot \sin^{-1}\theta + Q_{33}(\mathbf{u}')\sin^{-1}\theta + (\partial_\alpha M_{\alpha 1}(u_3) + \partial_2 M_{12}(u_3))\} \cdot w_{3h}\,dx_2 \\
& -\int_\Gamma \{Q_{21}(\mathbf{u}) + Q_{23}(\mathbf{u}')\}w_{2h}\,dx_2 \\
& +\int_\Gamma \{(Q_{11}(\mathbf{u}) + Q_{33}(\mathbf{u}')\cos\theta) \cdot \sin^{-1}\theta \cdot (w'_{1h} - (w'_{1h})^I) \\
& + (Q_{11}(\mathbf{u})\cos\theta + Q_{33}(\mathbf{u}')) \cdot \sin^{-1}\theta \cdot (w_{3h} - w^I_{3h})\}\,dx_2 \\
= & \ \Delta_{1h} + \Delta_{2h} + \Delta_{3h} + \Delta_{4h}\,.
\end{aligned} \tag{3.13}$$

From the junction conditions (2.9), it can be seen that

$$\Delta_{1h} = 0, \qquad \Delta_{3h} = 0\,, \tag{3.14}$$

and that the integrand of Δ_{2h} can be written as

$$\begin{aligned}
& \{-\{Q_{33}(\mathbf{u}')\cos\theta + [\partial_{\alpha'} M_{\alpha' 3}(u'_1) + \partial_2 M_{32}(u'_1)]\sin\theta\}\cos\theta \cdot \sin^{-1}\theta \\
& + Q_{33}(\mathbf{u}')\sin^{-1}\theta + [\partial_\alpha M_{\alpha 1}(u_3) + \partial_2 M_{12}(u_3)]\}w_{3h} \\
& = \{Q_{33}(\mathbf{u}')\sin\theta - [\partial_{\alpha'} M_{\alpha' 3}(u'_1) + \partial_2 M_{32}(u'_1)]\cos\theta + [\partial_\alpha M_{\alpha 1}(u_3) + \partial_2 M_{12}(u_3)]\}w_{3h} = 0\,,
\end{aligned}$$

then

$$\Delta_{2h} = 0\,. \tag{3.15}$$

Thus

$$\Delta_h = \int_\Gamma \{[Q_{11}(\mathbf{u}) + Q_{33}(\mathbf{u}')\cos\theta]\sin^{-1}\theta \cdot [w'_{1h} - (w'_{1h})^I] + [Q_{11}(\mathbf{u})\cos\theta + Q_{33}(\mathbf{u}')]\sin^{-1}\theta \cdot (w_{3h} - w^I_{3h})\}\,dx_2\,. \tag{3.16}$$

From (3.11), (3.13), and (3.16), it can be seen that

$$\begin{aligned} E_h(\eta,\zeta_h) &= \Delta_h - \sum_\tau \int_{\partial\tau} M_{\alpha\beta}(u_3)n_\alpha \cdot n_\beta \cdot \partial_n w_{3h}\,ds \\ &\quad - \sum_{\tau'} \int_{\partial\tau'} M_{\alpha'\beta'}(u'_1)n'_{\alpha'} \cdot n'_{\beta'} \cdot \partial_{n'} w'_{1h}\,ds' \\ &= \Delta_h + \delta_{1h} + \delta_{2h}\,. \end{aligned} \tag{3.17}$$

By the interpolate error estimates (cf. [7, 8]), it can be seen that

$$|\Delta_h| \le ch\left(\sum_\alpha \|u_\alpha\|_{2,\Omega} + \sum_{\alpha'} \|u'_{\alpha'}\|_{2,\Omega'}\right)|\sin^{-1}\theta|\,\|\zeta_h\|_h\,. \tag{3.18}$$

Since $-(\partial w_{3h}/\partial x_1)^I(P) = (\partial w'_{1h}/\partial x'_3)(P)$ $\forall$ nodes $P \in \Gamma$, which imply that $-(\partial w_{3h}/\partial x_1)^I = (\partial w'_{1h}/\partial x'_3)^I$ on Γ, and taking into account the junction condition $M_{33}(u'_1) - M_{11}(u_3) = 0$ on Γ, we can see that (cf. [8])

$$\begin{aligned} \delta_{1h} + \delta_{2h} &= \sum_{F\subset\Gamma}\int_F M_{11}(u_3)(\partial w_{3h}/\partial x_1)dx_2 + \sum_{F'\subset\Gamma'}\int_{F'} M_{33}(u'_1)(\partial w'_{1h}/\partial x'_3)dx_2 \\ &\quad - \sum_\tau \sum_{\substack{F\in\partial\tau\\ F\notin\partial\Omega}} \int_F M_{\alpha\beta}(u_3)n_\alpha n_\beta \partial_n w_{3h}\,ds - \sum_{\tau'} \sum_{\substack{F'\in\partial t'\\ F'\notin\partial\Omega'}} \int_{F'} M_{\alpha'\beta'}(u'_1)n'_{\alpha'}n'_{\beta'}\partial_{n'} w'_{1h}\,ds' \\ &= \sum_{F\subset\Gamma}\int_F M_{11}(u_3)R_1(\partial w_{3h}/\partial x_1)dx_2 + \sum_{F'\subset\Gamma'}\int_{F'} M_{33}(u'_1)R_1(\partial w'_{1h}/\partial x'_3)dx_2 \\ &\quad - \sum_\tau \sum_{\substack{F\in\partial\tau\\ F\notin\partial\Omega}} \int_F M_{\alpha\beta}(u_3)n_\alpha \cdot n_\beta \cdot R_1(\partial_n w_{3h})ds \\ &\quad - \sum_{\tau'} \sum_{\substack{F'\in\partial\tau'\\ F'\notin\partial\Omega'}} \int_{F'} M_{\alpha'\beta'}(u'_1)n'_{\alpha'} \cdot n'_{\beta'} \cdot R_1(\partial_{n'} w'_{1h})ds' \\ &= -\sum_\tau \sum_{F\in\partial\tau} \int_F M_{\alpha\beta}(u_3)n_\alpha \cdot n_\beta \cdot R_1(\partial_n w_{3h})ds \\ &\quad - \sum_{\tau'} \sum_{F'\in\partial\tau'} \int_{F'} M_{\alpha'\beta'}(u'_1) \cdot n'_{\alpha'} \cdot n'_{\beta'} \cdot R_1(\partial_{n'} w'_{1h})ds\,, \end{aligned} \tag{3.19}$$

where $R_1(v) = v - v^I$, and by the error estimate for the Adini's element [8], we have

$$|\delta_{1h}| + |\delta_{2h}| \le ch(\|u_3\|_{3,\Omega} + \|u'_1\|_{3,\Omega'})\left(\sum_\tau |w_{3h}|^2_{2,\tau} + \sum_{\tau'} |w'_{1h}|^2_{2,\tau'}\right)^{1/2}. \tag{3.20}$$

Finally, we have

$$|E_h(\eta,\zeta_h)| \le \mathrm{ch}\left\{\left(\sum_{\alpha}\|u_\alpha\|_{2,\Omega} + \sum_{\alpha'}\|u'_{\alpha'}\|_{2,\Omega'}\right)|\sin^{-1}\theta| + \|u_3\|_{3,\Omega} + \|u'_1\|_{3,\Omega'}\right\}\|\zeta_h\|_h, \tag{3.21}$$

and the proof is completed. ■

Remark 3.1. It may appear from the error estimate (3.8), that the convergence of the approximation η_h is reciprocally proportional to $|\sin\theta|$, where θ is the angle between the plates Ω and Ω'. If $\theta = 0$, any error bound can not be obtained from the estimate (3.8). This case we consider in Remark 3.2. It is an open question whether the dependence of the angle $0 < \theta < \pi$ is essential or technical for the convergence of the approximation η_h.

Remark 3.2. In this remark, the case of $\theta = 0$ is considered. if $\theta = 0$, then the relations (1.1) and (1.4) will be reduced into simple relations:

$$\mathbf{e}'_1 = -\mathbf{e}_3,\quad \mathbf{e}'_2 = \mathbf{e}_2,\quad \mathbf{e}'_3 = \mathbf{e}_1, \tag{3.22}$$

and

$$u_1 = u'_3,\quad u_2 = u'_2,\quad u_3 = -u'_1\quad \text{on } \Gamma. \tag{3.23}$$

And the junction conditions on Γ, (2.9) will be reduced to the following conditions

$$\begin{cases} Q_{33}(\mathbf{u}') + Q_{11}(\mathbf{u}) = 0,\quad Q_{23}(\mathbf{u}') + Q_{21}(\mathbf{u}) = 0, \\ -\{\partial_{\alpha'}M_{\alpha'3}(u'_1) + \partial_2 M_{32}(u'_1)\} + \{\partial_\alpha M_{\alpha 1}(u_3) + \partial_2 M_{12}(u_3)\} = 0, \\ M_{33}(u'_1) - M_{11}(u_3) = 0,\quad \text{on } \Gamma. \end{cases} \tag{3.24}$$

We now add such conditions at the nodes $P \in \Gamma$, for the element $\omega_h = (\mathbf{v}_h, \mathbf{v}'_h)$ of V_h such that

$$-(\partial v_{3h}/\partial x_2)(P) = (\partial v'_{1h}/\partial x'_2)(P),$$

then the element $\omega_h = (\mathbf{v}_h, \mathbf{v}'_h)$ of V_h satisfies the following relations at nodes $P \in \Gamma$:

$$\begin{cases} v'_{1h}(P) = -v_{3h}(P),\quad -(\partial v'_{1h}/\partial x'_2)(P) = -(\partial v_{3h}/\partial x_2)(P), \\ -(\partial v_{3h}/\partial x_1)(P) = (\partial v'_{1h}/\partial x'_3)(P), \\ v'_{2h}(P) = v_{2h}(P),\quad v'_{3h}(P) = v_{1h}(P)\quad \forall \text{ nodes } P \in \Gamma, \end{cases} \tag{3.25}$$

which means that

$$v'_{1h} = -v_{3h},\quad v'_{2h} = v_{2h},\quad v'_{3h} = v_{1h}\quad \text{on } \Gamma. \tag{3.26}$$

Thus it can be seen that the following equality holds exactly (c.f. (3.16))

$$\Delta_h = 0, \tag{3.27}$$

and we have an error estimate for this case:

$$\|\eta - \eta_h\|_h \le \mathrm{ch}\left\{\sum_{\alpha}|u_\alpha|_{2,\Omega} + \sum_{\alpha'}|u'_{\alpha'}|_{2,\Omega'} + \|u_3\|_{3,\Omega} + \|u'_1\|_{3,\Omega'}\right\}. \tag{3.28}$$

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