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AN ULTRAWEAK FORMULATION OF THE KIRCHHOFF–LOVE PLATE BENDING MODEL AND DPG APPROXIMATION

THOMAS FÜHRER, NORBERT HEUER, AND ANTTI H. NIEMI

Abstract. We develop and analyze an ultraweak variational formulation for a variant of the Kirchhoff–Love plate bending model. Based on this formulation, we introduce a discretization of the discontinuous Petrov–Galerkin type with optimal test functions (DPG). We prove well-posedness of the ultraweak formulation and quasi-optimal convergence of the DPG scheme.

The variational formulation and its analysis require tools that control traces and jumps in H^2 (standard Sobolev space of scalar functions) and $H(\operatorname{div}\,\mathbf{div})$ (symmetric tensor functions with L_2-components whose twice iterated divergence is in L_2), and their dualities. These tools are developed in two and three spatial dimensions. One specific result concerns localized traces in a dense subspace of $H(\operatorname{div}\,\mathbf{div})$. They are essential to construct basis functions for an approximation of $H(\operatorname{div}\,\mathbf{div})$.

To illustrate the theory we construct basis functions of the lowest order and perform numerical experiments for a smooth and a singular model solution. They confirm the expected convergence behavior of the DPG method both for uniform and adaptively refined meshes.

1. Introduction

We develop an ultraweak variational formulation for a bending-moment variant of the Kirchhoff–Love plate model, and present a discontinuous Petrov–Galerkin method with optimal test functions (DPG method) that is based on this formulation. We prove well-posedness of the continuous formulation and quasi-optimal convergence of the discrete scheme. At the heart of the analysis is the space $H(\operatorname{div}\,\mathbf{div},\Omega)$ and its traces and jumps. This space consists of symmetric tensors with $L_2(\Omega)$-components whose twice iterated divergence is in $L_2(\Omega)$ (the notation $\mathbf{div}$ indicates the divergence operator that acts on the rows of tensors).

The Kirchhoff–Love model was introduced by Kirchhoff [32] in a form that is generally accepted today. Kirchhoff also applied the model to determine the free vibration frequencies and modes of circular plates. A historical account of the development of the model is incorporated in [33], where Love uses Kirchhoff's approach to study vibrations of initially curved shells. Nowadays, the model is widely used in structural engineering, e.g., to dimension reinforced concrete slabs under

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The second author is the corresponding author.

static loads [28] and to control disturbing vibrations of wooden floors and other lightweight plane structures.

Perhaps the most well-known mathematical representation of the Kirchhoff–Love model for linearly elastic and isotropic material is given by the biharmonic equation

$$D\Delta^2 u = f,$$

where $u : \Omega \to \mathbb{R}$ is the deflection of the plate mid-surface $\Omega \subset \mathbb{R}^2$, Δ is the Laplace operator, and $f : \Omega \to \mathbb{R}$ and $D > 0$ represent the external loading and bending rigidity of the plate, respectively.

It is evident that application of the model to complex geometries requires employment of numerical methods such as the finite element method. The literature on the numerical analysis of plate bending problems is vast due to the aforementioned practical relevance of the problems and respectable age of the structural models. It is not feasible to perform a thorough literature review here, but two points that motivate our work can be made. First, conventional methods based on the variational principle of virtual displacements produce as direct output only the deflection values. These, albeit needed values, are not sufficient for structural design purposes where stresses and their resultants are of utmost importance. Second, verification of numerical accuracy of finite element algorithms is at the heart of simulation governance; see [37]. This is a serious challenge in practical plate-bending problems, where both the geometry and applied loading can be very irregular so that many of the contemporary developments in the finite element modeling of plate problems are devoted to a posteriori error estimation and adaptivity; see, e.g., [2, 9, 27].

We develop the theoretical framework for a DPG discretization to address the above challenges and, perhaps more significant, to set a theoretical basis to develop and analyze DPG schemes for other structural models like the singularly perturbed Reissner–Mindlin plate model and different shell models. Our analysis includes the case of singular problems on non-convex plates in contrast to many publications that assume convexity or smooth boundaries. In this context, we mention the mixed formulation from Amara et al. [1], who specifically use the space $H(\operatorname{div}\mathbf{div},\Omega)$ (without symmetry), thus allowing for singularities. Their numerical scheme is based on a decomposition of $H(\operatorname{div}\mathbf{div},\Omega)$ resulting in a mixed formulation that can be discretized by standard finite elements. In [25], Gallistl proposes a similar splitting approach for polyharmonic problems with corresponding finite element scheme.

The DPG framework has been founded by Demkowicz and Gopalakrishnan in [15]. It is very flexible and can be used with various variational formulations. A posteriori error estimation is also built in; see [18]. DPG schemes have been applied previously to structural engineering problems in [8, 34] and to more general problems of elasticity in [3, 31]. The most closely related investigation to the present work is probably [8]. That investigation showed that an ultraweak variational formulation of the Reissner–Mindlin plate bending model is well posed and that the associated discretization is convergent. Rather accurate numerical results were observed despite the fact that the theoretically obtained stability constant is very weak and depends on the slenderness of the plate. In particular, the question of well-posedness of the ultraweak variational formulation of the asymptotic Kirchhoff–Love model was left open.

Essential motivation for the use of DPG schemes is their possible robustness for singularly perturbed problems. The intrinsic energy norm can bound in a robust way approximation errors in the sense that quasi-optimal error estimates (by the energy norm, which is accessible) are uniform with respect to perturbation parameters. To achieve this robustness in appropriate (selected) norms, it is paramount to have an appropriate variational formulation, and proving robustness is usually non-trivial. For an analysis of second-order elliptic problems with convection-dominated diffusion ("confusion") and reaction-dominated diffusion ("refusion") we refer to [5, 6, 12, 19], and [30], respectively. The DPG setting for refusion from [30] has been extended to transmission problems and the coupling with boundary elements [23] and to Signorini-type contact problems [24].

Whereas we do not consider a singularly perturbed problem in this paper, the development of a DPG scheme for the Kirchhoff–Love model is relevant in its own right as discussed before and will be essential to dealing with other models of plate problems. Since we expect our technical tools to be useful also for fourth-order problems in three dimensions, they are developed for both two and three space dimensions (they can be generalized to any space dimension). Discretizations of fourth-order problems usually avoid H^2-bilinear forms to employ simpler than H^2-conforming basis functions. In this respect, our choice of ultraweak variational formulation has the advantage that field variables are only in L_2-spaces, whereas appearing trace variables (traces of $H^2(\Omega)$ and $H(\operatorname{div}\,\mathbf{div},\Omega)$) are relatively straightforward to discretize.

Let us discuss the structure of our work. In the next section we introduce the model problem of a certain bending-moment formulation for the Kirchhoff–Love model. For simplicity we assume fully clamped plates, but this is not essential as our formulation gives access to all kinds of boundary conditions. In that section, we also start developing a variational formulation. Since DPG schemes use product spaces[1] with respect to subdivisions of Ω into elements, trace operations in the underlying Sobolev spaces appear naturally. For fourth-order problems this is a non-trivial issue. Therefore, in order to define a well-posed variational formulation in product spaces we need to develop trace and jump operations, in our case in $H^2(\Omega)$ and $H(\operatorname{div}\,\mathbf{div},\Omega)$. This is the subject of Section 3, whose contents are discussed in more detail below. Eventually, in Section 4, we are able to define our ultraweak variational formulation and state its well-posedness (Theorem 4.1). We then briefly define the DPG scheme and state its quasi-optimal convergence (Theorem 4.2). Proofs of Theorems 4.1 and 4.2 are given in Section 5. We do not dwell much on the discussion of DPG schemes and their analysis. It is known that an analysis of the underlying adjoint problem gives access to the well-posedness of the variational formulation and quasi-optimal convergence of the DPG method (references have been given above), though we do stress the fact that our analysis goes beyond standard techniques. Rather than splitting the adjoint problem into a homogeneous one in product spaces and an inhomogeneous one in global ("unbroken" or non-product) spaces (as, e.g., in [14, 19, 30]) or deducing stability of the adjoint problem in product spaces from the one of the global form [11], we consider the full adjoint problem as a whole. Section 5 starts with defining the adjoint problem. Its well-posedness is proved in §5.1. The key idea is to describe the primal unknown of the adjoint problem as the solution to a saddle point problem without Lagrange

[1] Often they are referred to as "broken" spaces. We prefer to call them product spaces since important Sobolev spaces, e.g., of negative order or order 1/2, cannot be localized but have to be defined as product spaces from the start; cf. [29].

multiplier. Specifically, the primal unknown stays in the original product space, and test functions are considered in the corresponding global space. Of course, this problem could be reformulated as a traditional saddle point problem. However, our technique is applicable to adjoint problems with data that require continuity,[2] that is, leaving the L_2 setting of ultraweak formulations. In this sense, our new technique of analyzing the adjoint problem is fundamental. Extensions to other problems will be the subject of future research.

Let us note that there is a recent abstract framework by Demkowicz et al. [17]. Under specific assumptions it yields the well-posedness of L_2-ultraweak formulations in product spaces without explicitly analyzing trace spaces. In [26], Gopalakrishnan and Sepúlveda applied this setting to acoustic wave problems. In both references, an essential density assumption is proved only for simple geometries. Furthermore, trace variables are discretized via their domain counterparts, whereas we only discretize the traces. It is also unknown whether the new framework gives robust control of variables in the case of singularly perturbed problems. In [20,21], Ernesti and Wieners presented a simplified DPG analysis based on the framework from [17]. They used the density results for simple geometries from [17,26]. Furthermore, the construction of their trace discretization is done without explicitly defining the domain parts, although they are needed for the stability and approximation analysis. In conclusion, in comparison with the current state of the framework from [17], our strategy has the advantages of giving access to singularly perturbed problems, being extendable to non-L_2 settings, avoiding domain contributions for trace discretizations, and not requiring density assumptions which can be hard to prove (though, see [1, Proposition 2.1] for the density of smooth tensor functions in $H(\mathrm{div}\,\mathbf{div},\Omega)$ defined by the graph norm without symmetry).

Now, to continue discussing the contents of our paper, having the analysis of the adjoint problem from §5.1 at hand, the proofs of Theorems 4.1 and 4.2 are straightforward. They are given in §5.2. Finally, in Section 6 we discuss the construction of discrete spaces for our DPG scheme and give some numerical examples. Subsection 6.1 is devoted to the construction of lowest-order basis functions. Whereas the field variables do not require any continuity across element interfaces, it is more technical to identify unknowns associated with trace variables. Specifically, the construction of basis functions for traces of $H(\mathrm{div}\,\mathbf{div},\Omega)$ requires us to identify *local* continuity constraints. It turns out that traces of $H(\mathrm{div}\,\mathbf{div},\Omega)$-functions cannot be split into natural components that allow for such a construction. This is analogous to $H(\mathrm{div},\Omega)$ where one uses a slightly more regular subspace of vector functions with normal (then localizable) traces in L_2. In the literature, this subspace is usually denoted by $\mathcal{H}(\mathrm{div},\Omega)$. In $H(\mathrm{div}\,\mathbf{div},\Omega)$ the situation is worse since the definition of traces requires us to integrate by parts *twice*. This generates two combined traces. We present lowest-order basis functions (for traces of $H(\mathrm{div}\,\mathbf{div},\Omega)$) that correspond to local unknowns associated with edges and nodes of triangular elements, plus jump constraints associated with interior nodes and neighboring elements. These constraints can be imposed by Lagrange multipliers. For sufficiently smooth solutions, our lowest order scheme converges with optimal order (Theorem 6.5). This result assumes the use of *optimal test functions*, whereas, obviously, our numerical implementation uses approximated optimal test

[2]Here we only note that such restrictions appear when considering first-order formulations of plate-bending models.

functions. We do not analyze the influence of this approximation here. In §6.2 we present numerical results for two examples, the case of a smooth solution and the case of a singular solution. Uniform mesh refinement yields optimal and suboptimal convergence, respectively, whereas an adaptive variant restores optimal convergence for the singular example. It is worth mentioning that the singular example solution generates a tensor of $H(\mathrm{div}\,\mathbf{div},\Omega)$ whose divergence is less than L_2-regular. This shows, in particular, that our analysis of traces and jumps in $H(\mathrm{div}\,\mathbf{div},\Omega)$ cannot be split into two steps/spaces (symmetric tensors in $\mathbf{H}(\mathbf{div},\Omega)$ whose divergence are elements of $H(\mathrm{div},\Omega)$).

To conclude, the central focus of this paper is on the analysis of traces and jumps in $H(\mathrm{div}\,\mathbf{div},\Omega)$, in Section 3. Despite considering a plate model, this analysis is done in two and three space dimensions. It is relevant for other fourth-order problems in three dimensions. Section 3 is split into several subsections. In the first two, §§3.1 and 3.2, we define and analyze trace operators in $H(\mathrm{div}\,\mathbf{div},\Omega)$ and $H^2_0(\Omega)$ (denoted by $\mathrm{tr}^{\mathrm{dDiv}}$ and $\mathrm{tr}^{\mathrm{Ggrad}}$, with local versions $\mathrm{tr}^{\mathrm{dDiv}}_T$ and $\mathrm{tr}^{\mathrm{Ggrad}}_T$, respectively) and corresponding trace spaces and norms. In §3.3, we consider the product variant $H(\mathrm{div}\,\mathbf{div},\mathcal{T})$ of $H(\mathrm{div}\,\mathbf{div},\Omega)$ and jumps of its elements. Specifically, we characterize the inclusion $H(\mathrm{div}\,\mathbf{div},\mathcal{T})\subset H(\mathrm{div}\,\mathbf{div},\Omega)$ through (vanishing) duality with $H^2_0(\Omega)$ (Proposition 3.4). In §3.4 we revisit (a subspace of) the product space $H(\mathrm{div}\,\mathbf{div},\mathcal{T})$ and study traces rather than jumps (of course, trace operators can be used to define and analyze jumps). We define a dense product subspace $\mathcal{H}(\mathrm{div}\,\mathbf{div},\mathcal{T})\subset H(\mathrm{div}\,\mathbf{div},\mathcal{T})$ and prove that our previous "local" trace operators $\mathrm{tr}^{\mathrm{dDiv}}_T$ (they act on boundaries of elements) can be further localized when acting on this subspace (Proposition 3.6). This is of utmost importance for the numerical scheme since it implies density of our discrete spaces in $H(\mathrm{div}\,\mathbf{div},\Omega)$, and thus convergence. Subsection 3.5 corresponds to §3.3, considering jumps of a product space $H^2(\mathcal{T})$ rather than of $H(\mathrm{div}\,\mathbf{div},\mathcal{T})$, with continuity characterization by duality with the trace space $\mathrm{tr}^{\mathrm{dDiv}}(H(\mathrm{div}\,\mathbf{div},\Omega))$ (Proposition 3.8).

The final subsection, §3.6, provides a Poincaré inequality in the product space $H^2(\mathcal{T})$. Recall that traditional stability proofs of adjoint problems separate the analysis into a global non-homogeneous problem and a homogeneous one in product spaces and with jump data. The non-homogeneous problem usually gives control of a seminorm of the primal variable so that a Poincaré inequality is required to bound the norm. Furthermore, proving stability of homogeneous adjoint problems with jump data is usually done via a Helmholtz decomposition. For details see, e.g., [14, Lemmas 4.2, 4.3]. In our case, the global adjoint problem also gives access only to a seminorm of the primal variable, and still the connection between jump data and the field variable is established by a Helmholtz decomposition. We combine both techniques and give a short proof of a Poincaré inequality in $H^2(\mathcal{T})$ which uses a Helmholtz decomposition only implicitly.

Throughout the paper, $a\lesssim b$ means that $a\le cb$ with a generic constant $c>0$ that is independent of the underlying mesh (except for possible general restrictions like shape-regularity of elements). Similarly, we use the notation $a\simeq b$ and $a\gtrsim b$.

2. Model problem

We start by recalling the Kirchhoff–Love model; cf. [38]. The static variables of the model are the shear force vector $\mathbf{Q}$ and the symmetric bending-moment tensor $\mathbf{M}$. These stand for stress resultants representing internal forces and moments per unit length along the coordinate lines on the plate mid-surface Ω. They are related to the external surface load f and to each other by the laws of static equilibrium (force and moment balance) as

$$\begin{aligned} -\operatorname{div} \mathbf{Q} &= f && \text{in} \quad \Omega, \\ \mathbf{Q} &= \mathbf{div}\,\mathbf{M} && \text{in} \quad \Omega. \end{aligned}$$

The operator div denotes the divergence of vector functions, and **div** is the divergence operator acting on rows of tensors. Denoting by $\boldsymbol{\varepsilon}$ the infinitesimal strain tensor, or symmetric gradient, we introduce the bending curvature $\boldsymbol{\kappa} = \boldsymbol{\varepsilon}(\nabla u) := \frac{1}{2}(\nabla(\nabla u) + \nabla(\nabla u)^T)$, the Hessian of u in our case. For linearly elastic isotropic material, the bending moments can be written in terms of $\boldsymbol{\kappa}$ as

$$\mathbf{M} = -\mathcal{C}\boldsymbol{\kappa} = -D[\nu \operatorname{tr}\boldsymbol{\kappa}\mathbf{I} + (1-\nu)\boldsymbol{\kappa}]$$

where

$$D = \frac{Et^3}{12(1-\nu^2)}$$

is the bending rigidity of the plate defined in terms of the Young modulus E and Poisson ratio ν of the material and the plate thickness t. The values of these parameters are not very critical concerning the numerical solution of the problem. D acts as scaling parameter, and the influence of the Poisson ratio on the solution is mild. We select fixed $\nu \in (-1, 1/2]$ and $t > 0$ so that $\mathcal{C}$ is positive definite.

Let us now assume that $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) is a bounded simply connected Lipschitz domain with boundary $\Gamma = \partial\Omega$. (Of course, for the plate-bending problem, only $d = 2$ is physically motivated.) For a given $f \in L_2(\Omega)$ our model problem is

$$-\operatorname{div} \mathbf{div}\,\mathbf{M} = f \quad \text{in} \quad \Omega, \tag{2.1a}$$

$$\mathcal{C}^{-1}\mathbf{M} + \boldsymbol{\varepsilon}\nabla u = 0 \quad \text{in} \quad \Omega, \tag{2.1b}$$

$$u = 0, \quad \mathbf{n}\cdot\nabla u = 0 \quad \text{on} \quad \Gamma. \tag{2.1c}$$

Here, $\mathbf{n}$ is the exterior unit normal vector on Γ. Later, $\mathbf{n}$ will be used generically for normal vectors. Before starting to develop a variational formulation, we introduce a mesh $\mathcal{T}$ that consists of general non-intersecting open Lipschitz elements. Only in §3.4 we will require that the mesh be conforming and consist of generalized (curved) polyhedra/polygons, and in the numerical section, Section 6, we restrict ourselves to two space dimensions and conforming triangular meshes of shape-regular elements. To the mesh $\mathcal{T} = \{T\}$ we associate the skeleton $\mathcal{S} = \{\partial T;\ T \in \mathcal{T}\}$. For $T \in \mathcal{T}$, scalar functions z and symmetric tensors $\boldsymbol{\Theta}$, let us define the norms

$$\|z\|_{2,T}^2 := \|z\|_T^2 + \|\boldsymbol{\varepsilon}\nabla z\|_T^2, \qquad \|\boldsymbol{\Theta}\|_{\operatorname{div}\mathbf{div},T}^2 := \|\boldsymbol{\Theta}\|_T^2 + \|\operatorname{div}\mathbf{div}\,\boldsymbol{\Theta}\|_T^2,$$

and induced spaces $H^2(T)$, $H(\operatorname{div}\mathbf{div}, T)$ as closures of $\mathcal{D}(\overline{T})$ and $\mathbb{D}^s(\overline{T})$ with respect to the corresponding norm. Here, $\mathcal{D}(\overline{T})$ and $\mathbb{D}^s(\overline{T})$ are the spaces of smooth functions and smooth symmetric tensors, respectively, on T. (The logic for the notation $H(\operatorname{div}\mathbf{div}, T)$ with plain letter H is that tensors are mapped to scalar functions by the operator div **div**. Similarly, below we introduce $\mathbf{H}(\mathbf{div}, T)$ with bold **H** as **div** maps tensor functions to vector functions.) Throughout the paper, $\|\cdot\|_\omega$ denotes the $L_2(\omega)$-norms for scalar, vector, and tensor functions on the indicated set ω. When $\omega = \Omega$ we drop the index and simply write $\|\cdot\|$ instead of $\|\cdot\|_\Omega$. The corresponding bilinear forms are $(\cdot,\cdot)_\omega$ and $(\cdot,\cdot)$. The spaces $\mathbb{L}_2^s(\Omega)$ and $\mathbb{L}_2^s(T)$ denote symmetric tensor functions on Ω and T, respectively.

Now, given a mesh $\mathcal{T}$, we define product spaces (tacitly identifying product spaces with their broken variants)

$$H^2(\mathcal{T}) := \{z \in L_2(\Omega);\ z|_T \in H^2(T)\ \forall T \in \mathcal{T}\},$$
$$H(\operatorname{div}\,\mathbf{div},\mathcal{T}) := \{\boldsymbol{\Theta} \in \mathbb{L}_2^s(\Omega);\ \boldsymbol{\Theta}|_T \in H(\operatorname{div}\,\mathbf{div},T)\}$$

with canonical product norms $\|\cdot\|_{2,\mathcal{T}}$ and $\|\cdot\|_{\operatorname{div}\,\mathbf{div},\mathcal{T}}$, respectively. We will also need the global spaces $H_0^2(\Omega)$ and $H(\operatorname{div}\,\mathbf{div},\Omega)$ which are the closures of $\mathcal{D}(\Omega)$ and $\mathbb{D}^s(\overline{\Omega})$, respectively, with corresponding norms $\|z\|_2^2 = \|z\|^2 + \|\boldsymbol{\varepsilon}\nabla z\|^2$ and $\|\boldsymbol{\Theta}\|_{\operatorname{div}\,\mathbf{div}}^2 = \|\boldsymbol{\Theta}\|^2 + \|\operatorname{div}\,\mathbf{div}\,\boldsymbol{\Theta}\|^2$, and similarly $H_0^2(T)$ for $T \in \mathcal{T}$.

Now, we test

$$\text{(2.1a) with } z \in H^2(\mathcal{T}) \quad\text{and}\quad \text{(2.1b) with } \boldsymbol{\Theta} \in H(\operatorname{div}\,\mathbf{div},\mathcal{T}).$$

Formally integrating by parts on every element $T \in \mathcal{T}$ and summing over the elements and summing the two equations, the testing results in

$$(\mathbf{M}, \boldsymbol{\varepsilon}_{\mathcal{T}}\nabla_{\mathcal{T}} z) + \sum_{T\in\mathcal{T}} \langle \mathbf{n}\cdot\mathbf{div}\,\mathbf{M}, z\rangle_{\partial T} - \sum_{T\in\mathcal{T}} \langle \mathbf{M}\mathbf{n}, \nabla z\rangle_{\partial T}$$
$$+(\mathcal{C}^{-1}\mathbf{M}, \boldsymbol{\Theta}) + (u, \operatorname{div}_{\mathcal{T}}\mathbf{div}_{\mathcal{T}}\boldsymbol{\Theta}) + \sum_{T\in\mathcal{T}} \langle \boldsymbol{\Theta}\mathbf{n}, \nabla u\rangle_{\partial T} - \sum_{T\in\mathcal{T}} \langle \mathbf{n}\cdot\mathbf{div}\,\boldsymbol{\Theta}, u\rangle_{\partial T} = -(f, z). \tag{2.2}$$

Here and in the following, a differential operator with index $\mathcal{T}$ means that it is taken piecewise with respect to the elements $T \in \mathcal{T}$. We will write equivalently, e.g., $(\mathbf{M}, \boldsymbol{\varepsilon}_{\mathcal{T}}\nabla_{\mathcal{T}} z) = (\mathbf{M}, \boldsymbol{\varepsilon}\nabla z)_{\mathcal{T}}$, and similarly for other differential operators taken in a piecewise form. Furthermore, we use the generic notation $\mathbf{n}$ for the unit normal vector on ∂T and Γ, pointing outside T and Ω, respectively. The notation $\langle\cdot,\cdot\rangle_\omega$, and later $\langle\cdot,\cdot\rangle_\Gamma$, indicates dualities on $\omega \subset \partial T$ and Γ, respectively, with L_2-pivot space.

At this point it is not clear whether the appearing normal components in (2.2) on the boundaries of elements are well defined. Indeed, an essential part of this paper is to study the relation between traces and jumps of the involved spaces $H^2(\mathcal{T})$, $H_0^2(\Omega)$, $H(\operatorname{div}\,\mathbf{div},\mathcal{T})$, and $H(\operatorname{div}\,\mathbf{div},\Omega)$. This will be done in the next section, before returning to a variational formulation of (2.1) in Section 4.

3. Traces, jumps, and a Poincaré inequality

In the following we introduce and analyze operators and norms that serve to give the terms $\mathbf{n}\cdot\mathbf{div}\,\mathbf{M}|_{\partial T}$, $\mathbf{M}\mathbf{n}|_{\partial T}$, $\mathbf{n}\cdot\mathbf{div}\,\boldsymbol{\Theta}|_{\partial T}$, and $\boldsymbol{\Theta}\mathbf{n}|_{\partial T}$ from (2.2) a meaning for $\mathbf{M} \in H(\operatorname{div}\,\mathbf{div},\Omega)$ and $\boldsymbol{\Theta} \in H(\operatorname{div}\,\mathbf{div},\mathcal{T})$.

3.1. Traces of $H(\operatorname{div}\,\mathbf{div},\Omega)$.

We start by introducing linear operators $\operatorname{tr}_T^{\mathrm{dDiv}} : H(\operatorname{div}\,\mathbf{div},T) \to H^2(T)'$ for $T \in \mathcal{T}$ by

$$\langle \operatorname{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\Theta}), z\rangle_{\partial T} := (\operatorname{div}\,\mathbf{div}\,\boldsymbol{\Theta}, z)_T - (\boldsymbol{\Theta}, \boldsymbol{\varepsilon}\nabla z)_T. \tag{3.1}$$

We note that this definition is consistent with the observation made by Amara et al. in [1, Theorem 2.2] (they consider the whole domain Ω instead of an element T). The range of the operator $\operatorname{tr}_T^{\mathrm{dDiv}}$ is denoted by

$$\mathbf{H}^{-3/2,-1/2}(\partial T) := \operatorname{tr}_T^{\mathrm{dDiv}}(H(\operatorname{div}\,\mathbf{div},T)), \quad T \in \mathcal{T}.$$

These traces are supported on the boundary of the respective element since

$$(\operatorname{div}\,\mathbf{div}\,\boldsymbol{\Theta}, z)_T - (\boldsymbol{\Theta}, \boldsymbol{\varepsilon}\nabla z)_T = 0 \qquad \forall\boldsymbol{\Theta} \in H(\operatorname{div}\,\mathbf{div},T),\ \forall z \in H_0^2(T);$$

cf. Proposition 3.4 below. It is therefore clear that, for given $\boldsymbol{\Theta}$, the duality $\langle \mathrm{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\Theta}), z\rangle_{\partial T}$ depends only on the traces of z and ∇z on ∂T. Analogously, $\mathrm{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\Theta}) = 0$ for any $\boldsymbol{\Theta} \in \mathbb{D}^s(T)$ (smooth symmetric tensors with support in T) since

$$(\mathrm{div}\,\mathbf{div}\,\boldsymbol{\Theta}, z)_T - (\boldsymbol{\Theta}, \boldsymbol{\varepsilon}\nabla z)_T = 0 \qquad \forall \boldsymbol{\Theta} \in \mathbb{D}^s(T),\ \forall z \in H^2(T).$$

Remark 3.1. Let us define $\mathbf{H}(\mathbf{div}, T)$ as the closure of $\mathbb{D}^s(\overline{T})$ with respect to the norm $(\|\cdot\|^2 + \|\mathbf{div}\,(\cdot)\|^2)^{1/2}$. It is clear that $\mathbf{H}(\mathbf{div}, T)$ is not a subspace of $H(\mathrm{div}\,\mathbf{div}, T)$, nor is $H(\mathrm{div}\,\mathbf{div}, T)$ a subspace of $\mathbf{H}(\mathbf{div}, T)$ (see the second example in §6.2). This is precisely the reason we have to consider the trace operator $\mathrm{tr}_T^{\mathrm{dDiv}}$ in the form (3.1). When restricting this operator as

$$\mathrm{tr}_T^{\mathrm{dDiv}} : \begin{cases} H(\mathrm{div}\,\mathbf{div}, T) \cap \mathbf{H}(\mathbf{div}, T) & \to \quad H^2(T)', \\ \qquad\qquad \boldsymbol{\Theta} & \mapsto \quad \langle \mathbf{n}\cdot\mathbf{div}\,\boldsymbol{\Theta}, z\rangle_{\partial T} - \langle \boldsymbol{\Theta}\mathbf{n}, \nabla z\rangle_{\partial T}, \end{cases}$$

it reduces to standard trace operations. In this case the two dualities are defined independently in the standard way,

$$\langle \mathbf{n}\cdot\mathbf{div}\,\boldsymbol{\Theta}, z\rangle_{\partial T} := (\mathbf{div}\,\boldsymbol{\Theta}, \nabla z)_T + (\mathrm{div}\,\mathbf{div}\,\boldsymbol{\Theta}, z)_T,$$
$$\langle \boldsymbol{\Theta}\mathbf{n}, \nabla z\rangle_{\partial T} := (\boldsymbol{\Theta}, \boldsymbol{\varepsilon}\nabla z)_T + (\mathbf{div}\,\boldsymbol{\Theta}, \nabla z)_T.$$

Now, in the three-dimensional case $d = 3$, defining the tangential trace $\pi_{\mathbf{t}}(\boldsymbol{\phi}) := \mathbf{n} \times (\boldsymbol{\phi} \times \mathbf{n})|_{\partial T}$ for $\boldsymbol{\phi} \in \boldsymbol{\mathcal{D}}(\overline{T})$ and the surface gradient $\nabla_{\partial T}(\cdot) := \pi_{\mathbf{t}}(\nabla\cdot)|_{\partial T}$, we formally write

$$\langle \boldsymbol{\Theta}\mathbf{n}, \nabla z\rangle_{\partial T} = \langle \pi_{\mathbf{t}}(\boldsymbol{\Theta}\mathbf{n}), \nabla_{\partial T} z\rangle_{\partial T} + \langle \mathbf{n}\cdot\boldsymbol{\Theta}\mathbf{n}, \mathbf{n}\cdot\nabla z\rangle_{\partial T}. \tag{3.2}$$

Correspondingly, in two dimensions ($d = 2$), we introduce the unit tangential vector $\mathbf{t}$ along ∂T in mathematically positive orientation and use the notation $\pi_{\mathbf{t}}(\boldsymbol{\phi}) := (\mathbf{t}\cdot\boldsymbol{\phi})\mathbf{t}|_{\partial T}$ for $\boldsymbol{\phi} \in \boldsymbol{\mathcal{D}}(\overline{T})$ with corresponding tangential derivative $\nabla_{\partial T}(\cdot) := \pi_{\mathbf{t}}(\nabla\cdot)|_{\partial T}$. Then, (3.2) applies as well. We also need the surface divergence $\mathrm{div}_{\partial T}(\cdot)$ defined by $\langle \mathrm{div}_{\partial T}(\boldsymbol{\phi}), z\rangle_{\partial T} := -\langle \boldsymbol{\phi}, \nabla_{\partial T} z\rangle_{\partial T}$ for sufficiently smooth vector functions $\boldsymbol{\phi}$ with $\pi_{\mathbf{t}}(\boldsymbol{\phi}) = \boldsymbol{\phi}$. For precise definitions and appropriate spaces we refer to [7]. With these definitions it is clear that we can define separate traces

$$\mathrm{tr}_{T,\mathbf{n}}^{\mathrm{dDiv}} : \begin{cases} H(\mathrm{div}\,\mathbf{div}, T) & \to \quad \big(H^2(T) \cap H_0^1(T)\big)' \\ \qquad \boldsymbol{\Theta} & \mapsto \quad \langle \mathbf{n}\cdot\boldsymbol{\Theta}\mathbf{n}, \mathbf{n}\cdot\nabla z\rangle_{\partial T} := -\langle \mathrm{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\Theta}), z\rangle_{\partial T} \end{cases} \tag{3.3}$$

and

$$\mathrm{tr}_{T,\mathbf{t}}^{\mathrm{dDiv}} : \begin{cases} H(\mathrm{div}\,\mathbf{div}, T) & \to \quad \{z \in H^2(T);\ \mathbf{n}\cdot\nabla z = 0 \text{ on } \partial T\}' \\ \qquad \boldsymbol{\Theta} & \mapsto \quad \langle \mathbf{n}\cdot\mathbf{div}\,\boldsymbol{\Theta} + \mathrm{div}_{\partial T}\,\pi_{\mathbf{t}}(\boldsymbol{\Theta}\mathbf{n}), z\rangle_{\partial T} := \langle \mathrm{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\Theta}), z\rangle_{\partial T} \end{cases} \tag{3.4}$$

that coincide with the corresponding trace terms for sufficiently smooth functions $\boldsymbol{\Theta}$; cf. the operators γ_0 and γ_1 in [1, p. 1635]. On the one hand, these traces are relevant to identify basis functions for the approximation of traces of $H(\mathrm{div}\,\mathbf{div}, \Omega)$-functions. On the other hand, specifying one of these traces, the other is well defined as a functional acting on traces of H^2-functions (without the trace conditions for z in (3.3) and (3.4)). Applying this on the boundary Γ of Ω, it is possible to specify any physically meaningful boundary condition based on the terms $\mathbf{n}\cdot\mathbf{M}\mathbf{n}$, $\mathbf{n}\cdot\mathbf{div}\,\mathbf{M} + \mathrm{div}_\Gamma\,\pi_{\mathbf{t}}(\mathbf{M}\mathbf{n})$, u, and $\mathbf{n}\cdot\nabla u$ on Γ. Note that div_Γ refers to the operator that is dual to the (negative) global surface gradient $-\nabla_\Gamma$, and integrating by parts piecewise on subsets of Γ generates a piecewise surface divergence plus jumps at the interfaces; cf. [35]. Indeed, these jumps will be essential for the approximation analysis, based on Proposition 3.6 below.

The collective trace operator is defined by

$$\mathrm{tr}^{\mathrm{dDiv}}: \begin{cases} H(\mathrm{div}\,\mathbf{div},\Omega) & \to \quad H^2(\mathcal{T})', \\ \quad\boldsymbol{\Theta} & \mapsto \quad \mathrm{tr}^{\mathrm{dDiv}}(\boldsymbol{\Theta}) := (\mathrm{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\Theta}))_T \end{cases}$$

with duality

$$\langle \mathrm{tr}^{\mathrm{dDiv}}(\boldsymbol{\Theta}), z\rangle_{\mathcal{S}} := \sum_{T\in\mathcal{T}} \langle \mathrm{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\Theta}), z\rangle_{\partial T} \tag{3.5}$$

and range

$$\mathbf{H}^{-3/2,-1/2}(\mathcal{S}) := \mathrm{tr}^{\mathrm{dDiv}}(H(\mathrm{div}\,\mathbf{div},\Omega)).$$

(Here, and in the following, considering dualities $\langle\cdot,\cdot\rangle_{\partial T}$ on the whole of ∂T, possibly involved traces onto ∂T are always taken from T without further notice.) These global and local traces are measured in the *minimum energy extension* norms,

$$\|\boldsymbol{q}\|_{\mathrm{dDiv},\mathcal{S}} := \inf\Big\{\|\boldsymbol{\Theta}\|_{\mathrm{div}\,\mathbf{div}};\ \boldsymbol{\Theta}\in H(\mathrm{div}\,\mathbf{div},\Omega),\ \mathrm{tr}^{\mathrm{dDiv}}(\boldsymbol{\Theta}) = \boldsymbol{q}\Big\},$$
$$\|\boldsymbol{q}\|_{\mathrm{dDiv},\partial T} := \inf\Big\{\|\boldsymbol{\Theta}\|_{\mathrm{div}\,\mathbf{div},T};\ \boldsymbol{\Theta}\in H(\mathrm{div}\,\mathbf{div},T),\ \mathrm{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\Theta}) = \boldsymbol{q}\Big\}.$$

Alternative norms are defined by duality as follows:

$$\|\boldsymbol{q}\|_{-3/2,-1/2,\partial T} := \sup_{0\neq z\in H^2(T)} \frac{\langle \boldsymbol{q}, z\rangle_{\partial T}}{\|z\|_{2,T}}, \qquad \boldsymbol{q}\in\mathbf{H}^{-3/2,-1/2}(\partial T),\ T\in\mathcal{T},$$

$$\|\boldsymbol{q}\|_{-3/2,-1/2,\mathcal{S}} := \sup_{0\neq z\in H^2(\mathcal{T})} \frac{\langle \boldsymbol{q}, z\rangle_{\mathcal{S}}}{\|z\|_{2,\mathcal{T}}}, \qquad \boldsymbol{q}\in\mathbf{H}^{-3/2,-1/2}(\mathcal{S}). \tag{3.6}$$

Here, for given $\boldsymbol{q}\in\mathbf{H}^{-3/2,-1/2}(\partial T)$, the duality with $z\in H^2(T)$ is defined by

$$\langle \boldsymbol{q}, z\rangle_{\partial T} := \langle \mathrm{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\Theta}), z\rangle_{\partial T} \quad \text{for } \boldsymbol{\Theta}\in H(\mathrm{div}\,\mathbf{div},T) \text{ with } \mathrm{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\Theta}) = \boldsymbol{q},$$

and, for $\boldsymbol{q} = (\boldsymbol{q}_T)_T\in\mathbf{H}^{-3/2,-1/2}(\mathcal{S})$ and $z\in H^2(\mathcal{T})$,

$$\langle \boldsymbol{q}, z\rangle_{\mathcal{S}} := \sum_{T\in\mathcal{T}} \langle \boldsymbol{q}_T, z\rangle_{\partial T}. \tag{3.7}$$

This is consistent with definitions (3.1) and (3.5).

Lemma 3.2. *The identity*

$$\|\boldsymbol{q}\|_{-3/2,-1/2,\partial T} = \|\boldsymbol{q}\|_{\mathrm{dDiv},\partial T} \qquad \forall \boldsymbol{q}\in\mathbf{H}^{-3/2,-1/2}(\partial T),\ T\in\mathcal{T},$$

holds so that

$$\mathrm{tr}_T^{\mathrm{dDiv}}:\ H(\mathrm{div}\,\mathbf{div},T)\to\mathbf{H}^{-3/2,-1/2}(\partial T)$$

has unit norm and $\mathbf{H}^{-3/2,-1/2}(\partial T)$ *is closed.*

Proof. The estimate $\|\boldsymbol{q}\|_{-3/2,-1/2,\partial T}\le\|\boldsymbol{q}\|_{\mathrm{dDiv},\partial T}$ is immediate by bounding

$$\langle \mathrm{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\Theta}), z\rangle_{\partial T} \le \|\boldsymbol{\Theta}\|_{\mathrm{div}\,\mathbf{div},T}\|z\|_{2,T} \qquad \forall\boldsymbol{\Theta}\in H(\mathrm{div}\,\mathbf{div},T),\ \forall z\in H^2(T),\ T\in\mathcal{T}.$$

Now let $T\in\mathcal{T}$ and $\boldsymbol{q}\in\mathbf{H}^{-3/2,-1/2}(\partial T)$ be given. We define $z\in H^2(T)$ as the solution to the problem

$$(\boldsymbol{\varepsilon}\nabla z, \boldsymbol{\varepsilon}\nabla\delta z)_T + (z, \delta z)_T = \langle \boldsymbol{q}, \delta z\rangle_{\partial T} \qquad \forall \delta z\in H^2(T). \tag{3.8}$$

Note that the right-hand side functional implies a natural boundary condition for z. Furthermore, since $\langle \boldsymbol{q}, \delta z\rangle_{\partial T} = 0$ for $\delta z \in H_0^2(T)$, z satisfies

$$\operatorname{div}\,\mathbf{div}\,\boldsymbol{\varepsilon}\nabla z + z = 0 \quad \text{in } T, \tag{3.9}$$

first in the distributional sense and, by the regularity of z, also in $L_2(T)$. Using the function z we continue to define $\boldsymbol{\Theta} \in H(\operatorname{div}\,\mathbf{div}, T)$ as the solution to

$$(\operatorname{div}\,\mathbf{div}\,\boldsymbol{\Theta}, \operatorname{div}\,\mathbf{div}\,\boldsymbol{\delta}\mathbf{Q})_T + (\boldsymbol{\Theta}, \boldsymbol{\delta}\mathbf{Q})_T = \langle \operatorname{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\delta}\mathbf{Q}), z\rangle_{\partial T} \quad \forall \boldsymbol{\delta}\mathbf{Q} \in H(\operatorname{div}\,\mathbf{div}, T). \tag{3.10}$$

Again, the right-hand side functional induces a natural boundary condition for $\boldsymbol{\Theta}$, and it holds that

$$\boldsymbol{\varepsilon}\nabla \operatorname{div}\,\mathbf{div}\,\boldsymbol{\Theta} + \boldsymbol{\Theta} = 0 \quad \text{in } \mathbb{L}_2^s(T). \tag{3.11}$$

We show that $\boldsymbol{\Theta} = -\boldsymbol{\varepsilon}\nabla z$. Indeed, defining $\boldsymbol{\Theta}^z := -\boldsymbol{\varepsilon}\nabla z$, we find with (3.9) that $\operatorname{div}\,\mathbf{div}\,\boldsymbol{\Theta}^z = z$ so that by definition of $\boldsymbol{\Theta}^z$ and definition (3.1) of $\operatorname{tr}_T^{\mathrm{dDiv}}$,

$$\begin{aligned}(\operatorname{div}\,\mathbf{div}\,\boldsymbol{\Theta}^z, \operatorname{div}\,\mathbf{div}\,\boldsymbol{\delta}\mathbf{Q})_T + (\boldsymbol{\Theta}^z, \boldsymbol{\delta}\mathbf{Q})_T &= (z, \operatorname{div}\,\mathbf{div}\,\boldsymbol{\delta}\mathbf{Q})_T - (\boldsymbol{\varepsilon}\nabla z, \boldsymbol{\delta}\mathbf{Q})_T \\ &= \langle \operatorname{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\delta}\mathbf{Q}), z\rangle_{\partial T}\end{aligned}$$

for any $\boldsymbol{\delta}\mathbf{Q} \in H(\operatorname{div}\,\mathbf{div}, T)$. This shows that $\boldsymbol{\Theta}^z$ solves (3.10) and by uniqueness, $\boldsymbol{\Theta} = \boldsymbol{\Theta}^z = -\boldsymbol{\varepsilon}\nabla z$. Using this relation and $\operatorname{div}\,\mathbf{div}\,\boldsymbol{\Theta}^z = z$, it follows by (3.8) that

$$\begin{aligned}\langle \operatorname{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\Theta}), \delta z\rangle_{\partial T} &= (\operatorname{div}\,\mathbf{div}\,\boldsymbol{\Theta}, \delta z)_T - (\boldsymbol{\Theta}, \boldsymbol{\varepsilon}\nabla\delta z)_T \\ &= (z, \delta z)_T + (\boldsymbol{\varepsilon}\nabla z, \boldsymbol{\varepsilon}\nabla\delta z)_T = \langle \boldsymbol{q}, \delta z\rangle_{\partial T} \qquad \forall \delta z \in H^2(T).\end{aligned}$$

In other words, $\operatorname{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\Theta}) = \boldsymbol{q}$. This relation, together with selecting $\delta z = z$ in (3.8) and $\boldsymbol{\delta}\mathbf{Q} = \boldsymbol{\Theta}$ in (3.10), shows that

$$\langle \boldsymbol{q}, z\rangle_{\partial T} = \|z\|_{2,T}^2 = \langle \operatorname{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\Theta}), z\rangle_{\partial T} = \|\boldsymbol{\Theta}\|_{\operatorname{div}\,\mathbf{div},T}^2. \tag{3.12}$$

Noting that

$$\|\boldsymbol{\Theta}\|_{\operatorname{div}\,\mathbf{div},T} = \inf\{\|\widetilde{\boldsymbol{\Theta}}\|_{\operatorname{div}\,\mathbf{div},T};\ \widetilde{\boldsymbol{\Theta}} \in H(\operatorname{div}\,\mathbf{div}, T),\ \operatorname{tr}_T^{\mathrm{dDiv}}(\widetilde{\boldsymbol{\Theta}}) = \boldsymbol{q}\} = \|\boldsymbol{q}\|_{\mathrm{dDiv},\partial T}$$

by (3.11), relation (3.12) finishes the proof of the norm identity. The space $\mathbf{H}^{-3/2,-1/2}(\partial T)$ is closed as the image of a bounded below operator. □

3.2. **Traces of $H_0^2(\Omega)$.** Let us study traces of $H_0^2(\Omega)$ similarly to $H(\operatorname{div}\,\mathbf{div}, \Omega)$ in the previous section.

We define linear operators $\operatorname{tr}_T^{\mathrm{Ggrad}} : H^2(T) \to H(\operatorname{div}\,\mathbf{div}, T)'$ for $T \in \mathcal{T}$ by

$$\langle \operatorname{tr}_T^{\mathrm{Ggrad}}(z), \boldsymbol{\Theta}\rangle_{\partial T} := (\operatorname{div}\,\mathbf{div}\,\boldsymbol{\Theta}, z)_T - (\boldsymbol{\Theta}, \boldsymbol{\varepsilon}\nabla z)_T \tag{3.13}$$

and observe that (cf. (3.1))

$$\langle \operatorname{tr}_T^{\mathrm{Ggrad}}(z), \boldsymbol{\Theta}\rangle_{\partial T} = \langle \operatorname{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\Theta}), z\rangle_{\partial T} \qquad \forall z \in H^2(T),\ \forall \boldsymbol{\Theta} \in H(\operatorname{div}\,\mathbf{div}, T). \tag{3.14}$$

The ranges are denoted by

$$\mathbf{H}^{3/2,1/2}(\partial T) := \operatorname{tr}_T^{\mathrm{Ggrad}}(H^2(T)), \qquad T \in \mathcal{T}.$$

It is immediate that $\operatorname{tr}_T^{\mathrm{Ggrad}}(z) = 0$ if and only if $z \in H_0^2(T)$. The collective trace operator is defined by

$$\operatorname{tr}^{\mathrm{Ggrad}} : \begin{cases} H_0^2(\Omega) & \to \quad H(\operatorname{div}\,\mathbf{div}, \mathcal{T})', \\ \quad z & \mapsto \quad \operatorname{tr}^{\mathrm{Ggrad}}(z) := (\operatorname{tr}_T^{\mathrm{Ggrad}}(z))_T \end{cases}$$

with duality

$$\langle \mathrm{tr}^{\mathrm{Ggrad}}(z)\,,\boldsymbol{\Theta}\rangle_{\mathcal{S}} := \sum_{T\in\mathcal{T}} \langle \mathrm{tr}_T^{\mathrm{Ggrad}}(z)\,,\boldsymbol{\Theta}\rangle_{\partial T} \tag{3.15}$$

and range

$$\mathbf{H}_{00}^{3/2,1/2}(\mathcal{S}) := \mathrm{tr}^{\mathrm{Ggrad}}(H_0^2(\Omega)).$$

These trace spaces are provided with canonical trace norms,

$$\|\boldsymbol{v}\|_{\mathrm{Ggrad},\partial T} = \inf\{\|v\|_{2,T};\ v\in H^2(T),\ \mathrm{tr}_T^{\mathrm{Ggrad}}(v) = \boldsymbol{v}\}, \qquad T\in\mathcal{T},$$
$$\|\boldsymbol{v}\|_{\mathrm{Ggrad},0,\mathcal{S}} = \inf\{\|v\|_2;\ v\in H_0^2(\Omega),\ \mathrm{tr}^{\mathrm{Ggrad}}(v) = \boldsymbol{v}\}.$$

Alternative norms are defined by duality as follows:

$$\|\boldsymbol{v}\|_{3/2,1/2,\partial T} := \sup_{0\neq\boldsymbol{\Theta}\in H(\mathrm{div}\,\mathbf{div},T)} \frac{\langle \boldsymbol{v}\,,\boldsymbol{\Theta}\rangle_{\partial T}}{\|\boldsymbol{\Theta}\|_{\mathrm{div}\,\mathbf{div},T}}, \qquad \boldsymbol{v}\in\mathbf{H}^{3/2,1/2}(\partial T), \qquad T\in\mathcal{T},$$
$$\|\boldsymbol{v}\|_{3/2,1/2,00,\mathcal{S}} := \sup_{0\neq\boldsymbol{\Theta}\in H(\mathrm{div}\,\mathbf{div},\mathcal{T})} \frac{\langle \boldsymbol{v}\,,\boldsymbol{\Theta}\rangle_{\mathcal{S}}}{\|\boldsymbol{\Theta}\|_{\mathrm{div}\,\mathbf{div},\mathcal{T}}}, \qquad \boldsymbol{v}\in\mathbf{H}_{00}^{3/2,1/2}(\mathcal{S}).$$

Here, for given $\boldsymbol{v}\in\mathbf{H}^{3/2,1/2}(\partial T)$, the duality with $\boldsymbol{\Theta}\in H(\mathrm{div}\,\mathbf{div},T)$ is defined by

$$\langle \boldsymbol{v}\,,\boldsymbol{\Theta}\rangle_{\partial T} := \langle \mathrm{tr}_T^{\mathrm{Ggrad}}(z)\,,\boldsymbol{\Theta}\rangle_{\partial T} \quad \text{for } z\in H^2(T) \quad \text{with } \mathrm{tr}_T^{\mathrm{Ggrad}}(z) = \boldsymbol{v},$$

and, for $\boldsymbol{v} = (\boldsymbol{v}_T)_T\in\mathbf{H}_{00}^{3/2,1/2}(\mathcal{S})$ and $\boldsymbol{\Theta}\in H(\mathrm{div}\,\mathbf{div},\mathcal{T})$,

$$\langle \boldsymbol{v}\,,\boldsymbol{\Theta}\rangle_{\mathcal{S}} := \langle \mathrm{tr}^{\mathrm{Ggrad}}(z)\,,\boldsymbol{\Theta}\rangle_{\mathcal{S}} \quad \text{for } z\in H_0^2(\Omega) \quad \text{with } \mathrm{tr}^{\mathrm{Ggrad}}(z) = \boldsymbol{v}. \tag{3.16}$$

This is consistent with definitions (3.13) and (3.15).

Lemma 3.3. *The identity*

$$\|\boldsymbol{v}\|_{3/2,1/2,\partial T} = \|\boldsymbol{v}\|_{\mathrm{Ggrad},\partial T} \qquad \forall\boldsymbol{v}\in\mathbf{H}^{3/2,1/2}(\partial T), \qquad T\in\mathcal{T},$$

holds so that

$$\mathrm{tr}_T^{\mathrm{Ggrad}}:\ H^2(T)\to\mathbf{H}^{3/2,1/2}(\partial T)$$

has unit norm and $\mathbf{H}^{3/2,1/2}(\partial T)$ *is closed.*

Proof. The proof is very similar to that of Lemma 3.2.

The estimate $\|\boldsymbol{v}\|_{3/2,1/2,\partial T}\le\|\boldsymbol{v}\|_{\mathrm{Ggrad},\partial T}$ follows by bounding

$$\langle \mathrm{tr}_T^{\mathrm{Ggrad}}(z)\,,\boldsymbol{\Theta}\rangle_{\partial T} \le \|z\|_{2,T}\|\boldsymbol{\Theta}\|_{\mathrm{div}\,\mathbf{div},T} \quad \forall z\in H^2(T),\ \forall\boldsymbol{\Theta}\in H(\mathrm{div}\,\mathbf{div},T), \quad T\in\mathcal{T}.$$

To show the other inequality, let $T\in\mathcal{T}$ and $\boldsymbol{v}\in\mathbf{H}^{3/2,1/2}(\partial T)$ be given. We define $\boldsymbol{\Theta}\in H(\mathrm{div}\,\mathbf{div},T)$ as the solution of

$$(\mathrm{div}\,\mathbf{div}\,\boldsymbol{\Theta}\,,\mathrm{div}\,\mathbf{div}\,\boldsymbol{\delta}\mathbf{Q})_T + (\boldsymbol{\Theta}\,,\boldsymbol{\delta}\mathbf{Q})_T = \langle \boldsymbol{v}\,,\boldsymbol{\delta}\mathbf{Q}\rangle_{\partial T} \qquad \forall\boldsymbol{\delta}\mathbf{Q}\in H(\mathrm{div}\,\mathbf{div},T). \tag{3.17}$$

It satisfies

$$\boldsymbol{\varepsilon}\nabla\,\mathrm{div}\,\mathbf{div}\,\boldsymbol{\Theta} + \boldsymbol{\Theta} = 0 \quad \text{in } \mathbb{L}_2^s(T). \tag{3.18}$$

We continue to define $z\in H^2(T)$ by

$$(\boldsymbol{\varepsilon}\nabla z\,,\boldsymbol{\varepsilon}\nabla\delta z)_T + (z\,,\delta z)_T = \langle \mathrm{tr}_T^{\mathrm{Ggrad}}(\delta z)\,,\boldsymbol{\Theta}\rangle_{\partial T} \qquad \forall\delta z\in H^2(T). \tag{3.19}$$

It satisfies

$$\mathrm{div}\,\mathbf{div}\,\boldsymbol{\varepsilon}\nabla z + z = 0 \quad \text{in } L_2(T), \tag{3.20}$$

and we conclude that $z = \operatorname{div}\mathbf{div}\,\mathbf{\Theta}$ as follows. Defining $z^{\mathbf{\Theta}} := \operatorname{div}\mathbf{div}\,\mathbf{\Theta}$, (3.18) shows that $\boldsymbol{\varepsilon}\nabla z^{\mathbf{\Theta}} = -\mathbf{\Theta}$ so that by definition of $z^{\mathbf{\Theta}}$ and $\operatorname{tr}_T^{\mathrm{Ggrad}}$ (cf. (3.13)),

$$(\boldsymbol{\varepsilon}\nabla z^{\mathbf{\Theta}}\,,\boldsymbol{\varepsilon}\nabla\delta z)_T + (z^{\mathbf{\Theta}}\,,\delta z)_T = -(\mathbf{\Theta}\,,\boldsymbol{\varepsilon}\nabla\delta z)_T + (\operatorname{div}\mathbf{div}\,\mathbf{\Theta}\,,\delta z)_T = \langle \operatorname{tr}_T^{\mathrm{Ggrad}}(\delta z)\,,\mathbf{\Theta}\rangle_{\partial T}$$

for any $\delta z \in H^2(T)$. Hence $z^{\mathbf{\Theta}} = z$ solves (3.19); that is, $z = \operatorname{div}\mathbf{div}\,\mathbf{\Theta}$. Using this relation and $\boldsymbol{\varepsilon}\nabla z = -\mathbf{\Theta}$, (3.17) shows that

$$\begin{aligned}\langle \operatorname{tr}_T^{\mathrm{Ggrad}}(z)\,,\boldsymbol{\delta}\mathbf{Q}\rangle_{\partial T} &= (\operatorname{div}\mathbf{div}\,\boldsymbol{\delta}\mathbf{Q}\,,z)_T - (\boldsymbol{\delta}\mathbf{Q}\,,\boldsymbol{\varepsilon}\nabla z)_T\\ &= (\operatorname{div}\mathbf{div}\,\boldsymbol{\delta}\mathbf{Q}\,,\operatorname{div}\mathbf{div}\,\mathbf{\Theta})_T + (\boldsymbol{\delta}\mathbf{Q}\,,\mathbf{\Theta})_T = \langle \boldsymbol{v}\,,\boldsymbol{\delta}\mathbf{Q}\rangle_{\partial T}\end{aligned}$$

for any $\boldsymbol{\delta}\mathbf{Q} \in H(\operatorname{div}\mathbf{div},T)$, so that $\operatorname{tr}_T^{\mathrm{Ggrad}}(z) = \boldsymbol{v}$. Then selecting $\boldsymbol{\delta}\mathbf{Q} = \mathbf{\Theta}$ in (3.17) and $\delta z = z$ in (3.19), we obtain

$$\|\mathbf{\Theta}\|^2_{\operatorname{div}\mathbf{div},T} = \langle \boldsymbol{v}\,,\mathbf{\Theta}\rangle_{\partial T} = \|z\|^2_{2,T}. \tag{3.21}$$

Since

$$\|z\|_{2,T} = \inf\{\|\widetilde{z}\|_{2,T};\ \widetilde{z}\in H^2(T),\ \operatorname{tr}_T^{\mathrm{Ggrad}}(\widetilde{z}) = \boldsymbol{v}\} = \|\boldsymbol{v}\|_{\mathrm{Ggrad},\partial T}$$

by (3.20), relation (3.21) finishes the proof of the norm identity. The space $\mathbf{H}^{3/2,1/2}(\partial T)$ is closed as the image of a bounded below operator. $\square$

3.3. Jumps of $H(\operatorname{div}\mathbf{div},\mathcal{T})$.

Proposition 3.4.
(i) *For* $\mathbf{\Theta}\in H(\operatorname{div}\mathbf{div},\mathcal{T})$ *it holds that*

$$\mathbf{\Theta}\in H(\operatorname{div}\mathbf{div},\Omega) \quad\Leftrightarrow\quad \langle \operatorname{tr}^{\mathrm{Ggrad}}(z)\,,\mathbf{\Theta}\rangle_{\mathcal{S}} = 0 \qquad \forall z\in H_0^2(\Omega).$$

(ii) *The identity*

$$\sum_{T\in\mathcal{T}} \|\boldsymbol{q}\|^2_{\mathrm{dDiv},\partial T} = \|\boldsymbol{q}\|^2_{\mathrm{dDiv},\mathcal{S}} \qquad \forall \boldsymbol{q}\in\mathbf{H}^{-3/2,-1/2}(\mathcal{S})$$

holds true.

Proof. The proof of (i) follows the standard procedure; cf. [11, Proof of Theorem 2.3]. For $\mathbf{\Theta}\in H(\operatorname{div}\mathbf{div},\Omega)$ and $z\in\mathcal{D}(\Omega)$,

$$\begin{aligned}\langle \operatorname{tr}^{\mathrm{Ggrad}}(z)\,,\mathbf{\Theta}\rangle_{\mathcal{S}} &\overset{\mathrm{def}}{=} \sum_{T\in\mathcal{T}} (\operatorname{div}\mathbf{div}\,\mathbf{\Theta}\,,z)_T - (\mathbf{\Theta}\,,\boldsymbol{\varepsilon}\nabla z)_T = (\operatorname{div}\mathbf{div}\,\mathbf{\Theta}\,,z) - (\mathbf{\Theta}\,,\boldsymbol{\varepsilon}\nabla z)\\ &= 0,\end{aligned}$$

showing the direction "$\Rightarrow$". Now, for given $\mathbf{\Theta}\in H(\operatorname{div}\mathbf{div},\mathcal{T})$ with $\langle \operatorname{tr}^{\mathrm{Ggrad}}(z)\,,\mathbf{\Theta}\rangle_{\mathcal{S}} = 0$ for any $z\in H_0^2(\Omega)$ we have in the distributional sense

$$\operatorname{div}\mathbf{div}\,\mathbf{\Theta}(z) = (\mathbf{\Theta}\,,\boldsymbol{\varepsilon}\nabla z) = (\operatorname{div}\mathbf{div}\,\mathbf{\Theta}\,,z)_{\mathcal{T}} - \langle \operatorname{tr}^{\mathrm{Ggrad}}(z)\,,\mathbf{\Theta}\rangle_{\mathcal{S}} = (\operatorname{div}\mathbf{div}\,\mathbf{\Theta}\,,z)_{\mathcal{T}}$$

for any $z\in\mathcal{D}(\Omega)$. Therefore, $\operatorname{div}\mathbf{div}\,\mathbf{\Theta}\in L_2(\Omega)$, that is, $\mathbf{\Theta}\in H(\operatorname{div}\mathbf{div},\Omega)$.

Next we show (ii). The inequality $\sum_{T\in\mathcal{T}}\|\boldsymbol{q}\|^2_{\mathrm{dDiv},\partial T} \le \|\boldsymbol{q}\|^2_{\mathrm{dDiv},\mathcal{S}}$ holds by definition of the norms. To show the other inequality let $\boldsymbol{q} = (\boldsymbol{q}_T)_T\in\mathbf{H}^{-3/2,-1/2}(\mathcal{S})$ be given. By definition of $\mathbf{H}^{-3/2,-1/2}(\mathcal{S})$ there is $\mathbf{\Theta}\in H(\operatorname{div}\mathbf{div},\Omega)$ such that $\operatorname{tr}^{\mathrm{dDiv}}(\mathbf{\Theta}) = \boldsymbol{q}$. Furthermore, for $T\in\mathcal{T}$, let $\widetilde{\mathbf{\Theta}}_T\in H(\operatorname{div}\mathbf{div},T)$ be such that $\operatorname{tr}_T^{\mathrm{dDiv}}(\widetilde{\mathbf{\Theta}}_T) = \boldsymbol{q}_T$ and

$$\|\boldsymbol{q}_T\|_{\mathrm{dDiv},\partial T} = \|\widetilde{\mathbf{\Theta}}_T\|_{\operatorname{div}\mathbf{div},T}.$$

Then, $\widetilde{\mathbf{\Theta}} \in H(\mathrm{div}\,\mathbf{div}, \mathcal{T})$ defined by $\widetilde{\mathbf{\Theta}}|_T := \widetilde{\mathbf{\Theta}}_T$ $(T \in \mathcal{T})$ satisfies (cf. (3.14))

$$\begin{aligned}\langle \mathrm{tr}^{\mathrm{Ggrad}}(z)\,, \widetilde{\mathbf{\Theta}}\rangle_{\mathcal{S}} &= \sum_{T\in\mathcal{T}} \langle \mathrm{tr}_T^{\mathrm{Ggrad}}(z)\,, \widetilde{\mathbf{\Theta}}_T\rangle_{\partial T} = \sum_{T\in\mathcal{T}} \langle \mathrm{tr}_T^{\mathrm{dDiv}}(\widetilde{\mathbf{\Theta}}_T)\,, z\rangle_{\partial T}\\ &= \sum_{T\in\mathcal{T}} \langle \mathrm{tr}_T^{\mathrm{dDiv}}(\mathbf{\Theta})\,, z\rangle_{\partial T} = \langle \mathrm{tr}^{\mathrm{Ggrad}}(z)\,, \mathbf{\Theta}\rangle_{\mathcal{S}} = 0 \qquad \forall z \in H_0^2(\Omega)\end{aligned}$$

by part (i). Again with (i) we conclude that $\widetilde{\mathbf{\Theta}} \in H(\mathrm{div}\,\mathbf{div}, \Omega)$. Therefore,

$$\sum_{T\in\mathcal{T}} \|\boldsymbol{q}\|^2_{\mathrm{dDiv},\partial T} = \sum_{T\in\mathcal{T}} \|\widetilde{\mathbf{\Theta}}_T\|^2_{\mathrm{div}\,\mathbf{div},T} = \|\widetilde{\mathbf{\Theta}}\|^2_{\mathrm{div}\,\mathbf{div}} \geq \|\boldsymbol{q}\|^2_{\mathrm{dDiv},\mathcal{S}}.$$

This finishes the proof. □

By Proposition 3.4, for given $\boldsymbol{v} \in \mathbf{H}_{00}^{3/2,1/2}(\mathcal{S})$, $\langle \boldsymbol{v}\,, \mathbf{\Theta}\rangle_{\mathcal{S}}$ defines a functional that only depends on the normal jumps of $\mathbf{\Theta}$ and $\mathbf{div}_{\mathcal{T}}\mathbf{\Theta}$ across the element interfaces. It will be denoted as

(3.22)

$$[(\cdot)\mathbf{n}, \mathbf{n}\cdot\mathbf{div}_{\mathcal{T}}(\cdot)] : \begin{cases} H(\mathrm{div}\,\mathbf{div}, \mathcal{T}) & \to \quad \left(\mathbf{H}_{00}^{3/2,1/2}(\mathcal{S})\right)' \\ \qquad \mathbf{\Theta} & \mapsto \quad [\mathbf{\Theta}\mathbf{n}, \mathbf{n}\cdot\mathbf{div}_{\mathcal{T}}\mathbf{\Theta}](\boldsymbol{v}) := \langle \boldsymbol{v}\,, \mathbf{\Theta}\rangle_{\mathcal{S}} \end{cases}$$

with duality pairing defined in (3.16). This functional defines a seminorm in $H(\mathrm{div}\,\mathbf{div}, \mathcal{T})$,

(3.23)

$$\|[\mathbf{\Theta}\mathbf{n}, \mathbf{n}\cdot\mathbf{div}_{\mathcal{T}}\mathbf{\Theta}]\|_{(3/2,1/2,00,\mathcal{S})'} := \sup_{0\neq\boldsymbol{v}\in\mathbf{H}_{00}^{3/2,1/2}(\mathcal{S})} \frac{\langle \boldsymbol{v}\,, \mathbf{\Theta}\rangle_{\mathcal{S}}}{\|\boldsymbol{v}\|_{3/2,1/2,00,\mathcal{S}}}, \quad \mathbf{\Theta} \in H(\mathrm{div}\,\mathbf{div}, \mathcal{T}).$$

Proposition 3.5. *The identity*

$$\|\boldsymbol{q}\|_{-3/2,-1/2,\mathcal{S}} = \|\boldsymbol{q}\|_{\mathrm{dDiv},\mathcal{S}} \qquad \forall \boldsymbol{q} \in \mathbf{H}^{-3/2,-1/2}(\mathcal{S})$$

holds. In particular,

$$\mathrm{tr}^{\mathrm{dDiv}} : H(\mathrm{div}\,\mathbf{div}, \Omega) \to \mathbf{H}^{-3/2,-1/2}(\mathcal{S})$$

has unit norm and $\mathbf{H}^{-3/2,-1/2}(\mathcal{S})$ *is closed.*

Proof. The norm identity is shown by standard duality arguments in product spaces; cf. [11, Theorem 2.3]. Specifically, for $\boldsymbol{q} = (\boldsymbol{q}_T)_T \in H^{-3/2,-1/2}(\mathcal{S})$ we calculate, by using Proposition (3.4)(ii) and Lemma 3.2,

$$\begin{aligned}\|\boldsymbol{q}\|^2_{-3/2,-1/2,\mathcal{S}} &= \Big(\sup_{0\neq z\in H^2(\mathcal{T})} \frac{\sum_{T\in\mathcal{T}} \langle \boldsymbol{q}_T\,, z\rangle_{\partial T}}{\|z\|_{2,\mathcal{T}}}\Big)^2 = \sum_{T\in\mathcal{T}} \sup_{0\neq z\in H^2(T)} \frac{\langle \boldsymbol{q}_T\,, z\rangle^2_{\partial T}}{\|z\|^2_{2,T}}\\ &= \sum_{T\in\mathcal{T}} \|\boldsymbol{q}_T\|^2_{-3/2,-1/2,\partial T} = \sum_{T\in\mathcal{T}} \|\boldsymbol{q}_T\|^2_{\mathrm{dDiv},\partial T} = \|\boldsymbol{q}\|^2_{\mathrm{dDiv},\mathcal{S}}.\end{aligned}$$

The space $\mathbf{H}^{-3/2,-1/2}(\mathcal{S})$ is closed as the image of a bounded below operator. □

3.4. **Traces of $H(\mathrm{div}\,\mathbf{div}, \mathcal{T})$.** For the discretization of $\mathrm{tr}^{\mathrm{dDiv}}(H(\mathrm{div}\,\mathbf{div}, \Omega))$ we need a characterization of continuity across the skeleton interfaces $\partial T \in \mathcal{S}$ that is based on local traces, rather than testing with $H_0^2(\Omega)$-functions as in Proposition 3.4. Therefore, in this section, we assume throughout that the mesh $\mathcal{T}$ consists of polyhedra ($d = 3$) or polygons ($d = 2$) with possibly curved faces/edges and that $\mathcal{T}$ is a conforming subdivision of Ω in the sense that the intersection of any two different (closed) elements is either empty, an entire face ($d = 3$), an entire edge ($d = 2, 3$), or a vertex ($d = 2, 3$) of both elements.

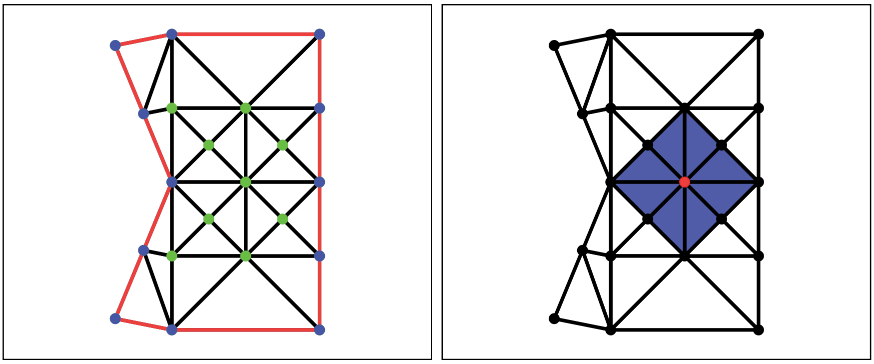

FIGURE 1. Notational convention for $d = 2$. Left: The dots visualize the set of all vertices $\mathcal{N}$. The dots highlighted in blue indicate boundary vertices. Consequently, green dots visualize the set of all interior vertices $\mathcal{N}_0$. Lines between two dots visualize the set of edges $\mathcal{E}$. Similarly, red lines indicate boundary edges, whereas black lines correspond to the set of interior edges $\mathcal{E}_0$. Right: The shaded elements (blue) indicate the patch $\omega(e)$ of an interior node e that is highlighted (red).

Let us introduce the set $\mathcal{E}_T$ of faces ($d = 3$) or edges ($d = 2$) of $T \in \mathcal{T}$, and define $\mathcal{E}_0$ to be the set of all faces/edges of $\mathcal{T}$ that are not subsets of Γ. We also need the set $\mathcal{N}_E$ of edges ($d = 3$) or nodes ($d = 2$) of $E \in \mathcal{E}_T$, $T \in \mathcal{T}$, and the set $\mathcal{N}_T = \bigcup_{E\in\mathcal{E}_T} \mathcal{N}_E$ of all edges/nodes of an element $T \in \mathcal{T}$. The set of all edges/nodes $e \in \mathcal{N} := \bigcup_{T\in\mathcal{T}} \mathcal{N}_T$ that are not subsets of Γ is denoted by $\mathcal{N}_0$; cf. the left side of Figure 1. For each $e \in \mathcal{N}_0$, let $\omega(e) \subset \mathcal{T}$ be the set (patch) of elements $T \in \mathcal{T}$ with $e \subset \overline{T}$; cf. the right side of Figure 1. The domain generated by a patch $\omega(e)$ will be denoted by ω_e. In three space dimensions, for a face $E \in \mathcal{E}_T$ ($T \in \mathcal{T}$), $\mathbf{n}_E$ denotes the unit normal vector along ∂E that is tangential to E. For an edge $E \in \mathcal{E}_T$ ($d = 2$), $\mathbf{n}_E$ indicates the orientation of E, with values $\mathbf{n}_E(e_1) = -1$ and $\mathbf{n}_E(e_2) = 1$, $e_1, e_2 \in \mathcal{N}_E$ being the starting and end points of E.

We also need the following trace spaces of $H^2(T)$ for $E \in \mathcal{E}_T$, $T \in \mathcal{T}$:

$$H^{3/2}(E) := \{z|_E;\ z \in H^2(T)\}, \qquad H^{1/2}(E) := \{(\mathbf{n}\cdot\nabla z)|_E;\ z \in H^2(T)\}$$

with canonical trace norms. Now, to localize the representation of the trace operators $\mathrm{tr}^{\mathrm{dDiv}}_{T,\mathbf{t}}$ (recall (3.4)), instead of the surface divergence operator $\mathrm{div}_{\partial T}$, we need the local surface divergence operator div_E defined, for a sufficiently smooth tangential function $\boldsymbol{\phi} = \pi_{\mathbf{t}}(\boldsymbol{\phi})$, by $\langle \mathrm{div}_E\, \boldsymbol{\phi}, \varphi\rangle_E := -\langle \boldsymbol{\phi}, \nabla_{\partial T}\varphi\rangle_E$ for any $\varphi \in H^1_0(E)$ (with obvious definition of this space). Defining

$$H(\mathrm{div}_E, E) := \{\boldsymbol{\phi} \in \mathbf{L}_2(E);\ \pi_{\mathbf{t}}(\boldsymbol{\phi}) = \boldsymbol{\phi},\ \mathrm{div}_E\, \boldsymbol{\phi} \in L_2(E)\} \qquad (E \in \mathcal{E}_T,\ T \in \mathcal{T}),$$

we then have that

$$\langle \boldsymbol{\phi}, \nabla_{\partial T} z\rangle_E = -\langle \mathrm{div}_E\, \boldsymbol{\phi}, z\rangle_E + \langle \mathbf{n}_E \cdot \boldsymbol{\phi}, z\rangle_{\partial E}$$
$$\forall \boldsymbol{\phi} \in H(\mathrm{div}_E, E), \qquad z \in H^1(E), \qquad E \in \mathcal{E}_T, \qquad T \in \mathcal{T}.$$

For an element $T \in \mathcal{T}$ and a sufficiently smooth function $\boldsymbol{\Theta} \in H(\operatorname{div}\,\mathbf{div},T)$, we introduce local trace operators (cf. (3.3) and (3.4))

(3.24)
$$\operatorname{tr}^{\mathrm{dDiv}}_{T,E,\mathbf{t}}(\boldsymbol{\Theta}) := \big(\mathbf{n}\cdot\mathbf{div}\,\boldsymbol{\Theta} + \operatorname{div}_E \pi_{\mathbf{t}}(\boldsymbol{\Theta}\mathbf{n})\big)|_E, \qquad \operatorname{tr}^{\mathrm{dDiv}}_{T,E,\mathbf{n}}(\boldsymbol{\Theta}) := \mathbf{n}\cdot\boldsymbol{\Theta}\mathbf{n}|_E \qquad (E\in\mathcal{E}_T)$$

and the jump functional

(3.25)
$$[\![\boldsymbol{\Theta}]\!]_{\partial T}(z) := \sum_{E\in\mathcal{E}_T}\Big(\langle\operatorname{tr}^{\mathrm{dDiv}}_{T,E,\mathbf{t}}(\boldsymbol{\Theta})\,,z\rangle_E - \langle\operatorname{tr}^{\mathrm{dDiv}}_{T,E,\mathbf{n}}(\boldsymbol{\Theta})\,,\mathbf{n}\cdot\nabla z\rangle_E\Big) - \langle\operatorname{tr}^{\mathrm{dDiv}}_{T}(\boldsymbol{\Theta})\,,z\rangle_{\partial T}$$

for $z\in H^2(T)$. Of course, for sufficiently smooth $\boldsymbol{\Theta}$ it holds that $[\![\boldsymbol{\Theta}]\!]_{\partial T}\in\big(H^2(T)\big)'$. Below we will require that the regularity of $\boldsymbol{\Theta}$ is such that the traces (3.24) are well defined. Assuming, again, sufficient regularity of $\boldsymbol{\Theta}$, integration by parts shows that the jump functional reduces to

$$[\![\boldsymbol{\Theta}]\!]_{\partial T}(z) = \langle\mathbf{n}_{E_2}\cdot\pi_{\mathbf{t}}(\boldsymbol{\Theta}\mathbf{n}|_{E_2}) + \mathbf{n}_{E_1}\cdot\pi_{\mathbf{t}}(\boldsymbol{\Theta}\mathbf{n}|_{E_1})\,,z\rangle_e, \tag{3.26}$$
$$\text{for } E_1,E_2\in\mathcal{E}_T:\ E_1\neq E_2,\quad e=\overline{E}_1\cap\overline{E}_2\quad\text{and}\quad z\in H^2(T):\ z|_{\tilde e}=0\quad\forall\tilde e\in\mathcal{N}_E\setminus\{e\}.$$

Here, in three dimensions, $\langle\varphi\,,\psi\rangle_e$ is the $L_2(e)$-bilinear form (and its extension by duality), and in two dimensions $\langle\varphi\,,\psi\rangle_e=\varphi(e)\psi(e)$ is the product of the point values of φ and ψ at the node e, and $z|_e=z(e)$.

In order to be able to localize the traces of a function $\boldsymbol{\Theta}\in H(\operatorname{div}\,\mathbf{div},T)$, according to (3.24), we need to assume the stronger regularity $\boldsymbol{\Theta}\in\mathcal{H}(\operatorname{div}\,\mathbf{div},T)$ where

$$\mathcal{H}(\operatorname{div}\,\mathbf{div},T)$$
$$:=\{\boldsymbol{\Theta}\in H(\operatorname{div}\,\mathbf{div},T);\ \operatorname{tr}^{\mathrm{dDiv}}_{T,E,\mathbf{t}}(\boldsymbol{\Theta})\in\big(H^{3/2}(E)\big)',\ \operatorname{tr}^{\mathrm{dDiv}}_{T,E,\mathbf{n}}(\boldsymbol{\Theta})\in\big(H^{1/2}(E)\big)'\ \forall E\in\mathcal{E}_T\}.$$

The corresponding product space is denoted by $\mathcal{H}(\operatorname{div}\,\mathbf{div},\mathcal{T})$. Since $\mathbb{D}^s(\overline{T})\subset\mathcal{H}(\operatorname{div}\,\mathbf{div},T)\subset H(\operatorname{div}\,\mathbf{div},T)$, the space $\mathcal{H}(\operatorname{div}\,\mathbf{div},T)$ is dense in $H(\operatorname{div}\,\mathbf{div},T)$.

Now we can formulate the main result of this subsection.

Proposition 3.6. *An element $\boldsymbol{\Theta}\in\mathcal{H}(\operatorname{div}\,\mathbf{div},\mathcal{T})$ satisfies $\boldsymbol{\Theta}\in H(\operatorname{div}\,\mathbf{div},\Omega)$ if and only if*

$$\begin{aligned}&\operatorname{tr}^{\mathrm{dDiv}}_{T_1,E,\mathbf{n}}(\boldsymbol{\Theta})+\operatorname{tr}^{\mathrm{dDiv}}_{T_2,E,\mathbf{n}}(\boldsymbol{\Theta})=0,\quad \operatorname{tr}^{\mathrm{dDiv}}_{T_1,E,\mathbf{t}}(\boldsymbol{\Theta})+\operatorname{tr}^{\mathrm{dDiv}}_{T_2,E,\mathbf{t}}(\boldsymbol{\Theta})=0\\ &\qquad\forall E\in\mathcal{E}_0\quad\textit{and}\quad T_1,T_2\in\mathcal{T}:\ T_1\neq T_2,\qquad\{E\}=\mathcal{E}_{T_1}\cap\mathcal{E}_{T_2},\\ &\textit{and}\quad\sum_{T\in\omega(e)}[\![\boldsymbol{\Theta}]\!]_{\partial T}(z)=0\qquad\forall z\in H^2_0(\omega_e),\qquad\forall e\in\mathcal{N}_0.\end{aligned}\tag{3.27}$$

Proof. For sufficiently smooth $\boldsymbol{\Theta}_T\in H(\operatorname{div}\,\mathbf{div},T)$ and $z\in H^2(T)$ $(T\in\mathcal{T})$ we find with (3.14) and (3.25) that

$$\begin{aligned}\langle\operatorname{tr}^{\mathrm{Ggrad}}_T(z)\,,\boldsymbol{\Theta}_T\rangle_{\partial T}&=\langle\operatorname{tr}^{\mathrm{dDiv}}_T(\boldsymbol{\Theta}_T)\,,z\rangle_{\partial T}\\&=\sum_{E\in\mathcal{E}_T}\Big(\langle\operatorname{tr}^{\mathrm{dDiv}}_{T,E,\mathbf{t}}(\boldsymbol{\Theta}_T)\,,z\rangle_E-\langle\operatorname{tr}^{\mathrm{dDiv}}_{T,E,\mathbf{n}}(\boldsymbol{\Theta}_T)\,,\mathbf{n}\cdot\nabla z\rangle_E\Big)-[\![\boldsymbol{\Theta}_T]\!]_{\partial T}(z).\end{aligned}\tag{3.28}$$

All terms can be interpreted as linear functionals depending on z and acting on $\boldsymbol{\Theta}_T$. Boundedness is guaranteed for $\boldsymbol{\Theta}_T\in\mathcal{H}(\operatorname{div}\,\mathbf{div},T)$. Therefore, relation (3.28) extends by continuity to $\boldsymbol{\Theta}\in\mathcal{H}(\operatorname{div}\,\mathbf{div},\mathcal{T})$. Considering $z\in H^2_0(\Omega)$ and summing over all elements $T \in \mathcal{T}$ yields

$$\langle \mathrm{tr}^{\mathrm{Ggrad}}(z), \boldsymbol{\Theta}\rangle_{\mathcal{S}} = \sum_{T\in\mathcal{T}} \sum_{E\in\mathcal{E}_T} \Big(\langle \mathrm{tr}^{\mathrm{dDiv}}_{T,E,\mathbf{t}}(\boldsymbol{\Theta}_T), z\rangle_E - \langle \mathrm{tr}^{\mathrm{dDiv}}_{T,E,\mathbf{n}}(\boldsymbol{\Theta}_T), \mathbf{n}\cdot\nabla z\rangle_E\Big) - \sum_{T\in\mathcal{T}} [\![\boldsymbol{\Theta}_T]\!]_{\partial T}(z).$$

One sees that the right-hand side vanishes for any $z \in H^2_0(\Omega)$ if and only if (3.27) is satisfied. Therefore, the statement follows by Proposition 3.4. $\square$

Remark 3.7. In two dimensions ($d = 2$) the trace operators (3.24) are, for $\boldsymbol{\Theta} \in \mathcal{H}(\mathrm{div}\,\mathbf{div}, T)$ and $T \in \mathcal{T}$,

(3.29)
$$\mathrm{tr}^{\mathrm{dDiv}}_{T,E,\mathbf{t}}(\boldsymbol{\Theta}) = \big(\mathbf{n}\cdot\mathbf{div}\,\boldsymbol{\Theta} + \partial_{\mathbf{t}}(\mathbf{t}\cdot\boldsymbol{\Theta}\mathbf{n})\big)|_E, \qquad \mathrm{tr}^{\mathrm{dDiv}}_{T,E,\mathbf{n}}(\boldsymbol{\Theta}) = \mathbf{n}\cdot\boldsymbol{\Theta}\mathbf{n}|_E \qquad (E \in \mathcal{E}_T).$$

Here, $\partial_{\mathbf{t}}$ indicates the positively oriented tangential derivative along E. The localized jump functional $[\![\boldsymbol{\Theta}]\!]_{\partial T}$ (cf. (3.26)) reduces to jump values at vertices,

(3.30)
$$\begin{aligned}[\![\boldsymbol{\Theta}]\!]_{\partial T}(e): z \mapsto [\![\boldsymbol{\Theta}]\!]_{\partial T}(z) &= \langle \mathbf{n}_{E_2}\cdot\pi_{\mathbf{t}}(\boldsymbol{\Theta}\mathbf{n}|_{E_2}) + \mathbf{n}_{E_1}\cdot\pi_{\mathbf{t}}(\boldsymbol{\Theta}\mathbf{n}|_{E_1}), z\rangle_e \\ &= \Big((\mathbf{t}\cdot\boldsymbol{\Theta}\mathbf{n}|_{E_2})(e) - (\mathbf{t}\cdot\boldsymbol{\Theta}\mathbf{n}|_{E_1})(e)\Big)z(e) \qquad \forall z \in H^2_0(\omega_e),\end{aligned}$$

where $E_1, E_2 \in \mathcal{E}_T$ are chosen in such a way that e is the end point of E_2 and starting point of E_1; that is, in our previous notation, $\mathbf{n}_{E_1}(e) = -1$, $\mathbf{n}_{E_2}(e) = 1$.

3.5. Jumps of $H^2(\mathcal{T})$.

Proposition 3.8.
(i) *For $z \in H^2(\mathcal{T})$ the following equivalence holds:*

$$z \in H^2_0(\Omega) \quad \Leftrightarrow \quad \langle \boldsymbol{q}, z\rangle_{\mathcal{S}} = 0 \qquad \forall \boldsymbol{q} \in \mathbf{H}^{-3/2,-1/2}(\mathcal{S}).$$

(ii) *The identity*

$$\sum_{T\in\mathcal{T}} \|\boldsymbol{v}\|^2_{\mathrm{Ggrad},\partial T} = \|\boldsymbol{v}\|^2_{\mathrm{Ggrad},0,\mathcal{S}} \qquad \forall \boldsymbol{v} \in \mathbf{H}^{3/2,1/2}_{00}(\mathcal{S})$$

holds true.

Proof. Some parts of this proof are almost identical to the proof of Proposition 3.4. However, in part (i) of the present case, we additionally have to guarantee boundary conditions in $H^2_0(\Omega)$, whereas previously they were not considered.

Let us start showing statement (i). For $z \in \mathcal{D}(\Omega)$, $\boldsymbol{q} \in \mathbf{H}^{-3/2,-1/2}(\mathcal{S})$, and $\boldsymbol{\Theta} \in H(\mathrm{div}\,\mathbf{div}, \Omega)$ with $\boldsymbol{q} = \mathrm{tr}^{\mathrm{dDiv}}(\boldsymbol{\Theta})$ we obtain

$$\begin{aligned}\langle \boldsymbol{q}, z\rangle_{\mathcal{S}} &\overset{\mathrm{def}}{=} \sum_{T\in\mathcal{T}} (\mathrm{div}\,\mathbf{div}\,\boldsymbol{\Theta}, z)_T - (\boldsymbol{\Theta}, \boldsymbol{\varepsilon}\nabla z)_T \\ &= (\mathrm{div}\,\mathbf{div}\,\boldsymbol{\Theta}, z) - (\boldsymbol{\Theta}, \boldsymbol{\varepsilon}\nabla z) = (\boldsymbol{\Theta}, \boldsymbol{\varepsilon}\nabla z) - (\boldsymbol{\Theta}, \boldsymbol{\varepsilon}\nabla z) = 0.\end{aligned}$$

By density, this holds for any $z \in H^2_0(\Omega)$.

We show the other direction "$\Leftarrow$". For given $z \in H^2(\mathcal{T})$ with $\langle \boldsymbol{q}, z\rangle_{\mathcal{S}} = 0$ for any $\boldsymbol{q} \in \mathbf{H}^{-3/2,-1/2}(\mathcal{S})$, the distribution $\boldsymbol{\varepsilon}\nabla z$ satisfies

$$\boldsymbol{\varepsilon}\nabla z(\boldsymbol{\Phi}) = (z, \mathrm{div}\,\mathbf{div}\,\boldsymbol{\Phi}) = (\boldsymbol{\varepsilon}\nabla z, \boldsymbol{\Phi})_{\mathcal{T}} + \langle \mathrm{tr}^{\mathrm{dDiv}}(\boldsymbol{\Phi}), z\rangle_{\mathcal{S}} = (\boldsymbol{\varepsilon}\nabla z, \boldsymbol{\Phi})_{\mathcal{T}} \quad \forall \boldsymbol{\Phi} \in \mathbb{D}^s(\Omega),$$

that is, $\boldsymbol{\varepsilon}\nabla z \in \mathbb{L}^s_2(\Omega)$. Therefore, $z \in H^2(\Omega)$. Furthermore,

(3.31)
$$0 = \langle \mathrm{tr}^{\mathrm{dDiv}}(\boldsymbol{\Phi}), z\rangle_{\mathcal{S}} = (\mathrm{div}\,\mathbf{div}\,\boldsymbol{\Phi}, z) - (\boldsymbol{\Phi}, \boldsymbol{\varepsilon}_{\mathcal{T}}\nabla_{\mathcal{T}} z) = (\mathrm{div}\,\mathbf{div}\,\boldsymbol{\Phi}, z) - (\boldsymbol{\Phi}, \boldsymbol{\varepsilon}\nabla z)$$

for any $\mathbf{\Phi} \in H(\operatorname{div} \mathbf{div}, \Omega)$ shows that $z \in H_0^2(\Omega)$, as can be seen as follows. To show $z \in H_0^1(\Omega)$ we select $\mathbf{\Phi} := \boldsymbol{\varepsilon}\nabla w$ with $w \in H^2(\Omega)$ solving, for given $\varphi \in H^2(\Omega)|_\Gamma$,

$$(\boldsymbol{\varepsilon}\nabla w\,, \boldsymbol{\varepsilon}\nabla\delta z) + (w\,, \delta z) = -\langle\varphi\,, \delta z\rangle_\Gamma \qquad \forall \delta z \in H^2(\Omega).$$

Then, $\operatorname{div} \mathbf{div}\, \mathbf{\Phi} = -w$ and (3.31) shows that

$$(\operatorname{div} \mathbf{div}\, \mathbf{\Phi}\,, z) - (\mathbf{\Phi}\,, \boldsymbol{\varepsilon}\nabla z) = \langle\varphi\,, z\rangle_\Gamma = 0.$$

Since $\varphi \in H^2(\Omega)|_\Gamma$ was arbitrary, this yields $z|_\Gamma = 0$. Analogously, $\nabla z|_\Gamma = 0$ follows by selecting $\mathbf{\Phi} := \boldsymbol{\varepsilon}\nabla w$ with

$$(\boldsymbol{\varepsilon}\nabla w\,, \boldsymbol{\varepsilon}\nabla\delta z) + (w\,, \delta z) = -\langle\boldsymbol{\varphi}\,, \nabla\delta z\rangle_\Gamma \qquad \forall \delta z \in H^2(\Omega)$$

for arbitrary $\boldsymbol{\varphi} \in \mathbf{H}^1(\Omega)|_\Gamma$.

Next we show (ii). The bound $\sum_{T\in\mathcal{T}} \|\boldsymbol{v}\|^2_{\mathrm{Ggrad},\partial T} \le \|\boldsymbol{v}\|^2_{\mathrm{Ggrad},0,\mathcal{S}}$ holds by definition of the norms. To show the other inequality let $\boldsymbol{v} = (\boldsymbol{v}_T)_T \in \mathbf{H}_{00}^{3/2,1/2}(\mathcal{S})$ be given. There is $z \in H_0^2(\Omega)$ such that $\operatorname{tr}^{\mathrm{Ggrad}}(z) = \boldsymbol{v}$. Furthermore, for $T \in \mathcal{T}$, let $\widetilde{z}_T \in H^2(T)$ be such that $\operatorname{tr}_T^{\mathrm{Ggrad}}(\widetilde{z}_T) = \boldsymbol{v}_T$ and $\|\boldsymbol{v}_T\|_{\mathrm{Ggrad},\partial T} = \|\widetilde{z}_T\|_{2,T}$. Then, $\widetilde{z}$ defined by $\widetilde{z}|_T := \widetilde{z}_T$ $(T \in \mathcal{T})$ satisfies $\widetilde{z} \in H^2(\mathcal{T})$, and for any $\mathbf{\Theta} \in H(\operatorname{div} \mathbf{div}, \Omega)$ it holds (cf. (3.14)) that

$$\begin{aligned}\langle \operatorname{tr}^{\mathrm{dDiv}}(\mathbf{\Theta})\,, \widetilde{z}\rangle_{\mathcal{S}} &= \sum_{T\in\mathcal{T}} \langle \operatorname{tr}_T^{\mathrm{dDiv}}(\mathbf{\Theta})\,, \widetilde{z}_T\rangle_{\partial T} = \sum_{T\in\mathcal{T}} \langle \operatorname{tr}_T^{\mathrm{Ggrad}}(\widetilde{z}_T)\,, \mathbf{\Theta}\rangle_{\partial T}\\ &= \sum_{T\in\mathcal{T}} \langle \operatorname{tr}_T^{\mathrm{Ggrad}}(z)\,, \mathbf{\Theta}\rangle_{\partial T} = \langle \operatorname{tr}^{\mathrm{dDiv}}(\mathbf{\Theta})\,, z\rangle_{\mathcal{S}} = 0\end{aligned}$$

by part (i). Again with (i) we conclude that $\widetilde{z} \in H_0^2(\Omega)$. Therefore,

$$\sum_{T\in\mathcal{T}} \|\boldsymbol{v}|_{\partial T}\|^2_{\mathrm{Ggrad},\partial T} = \sum_{T\in\mathcal{T}} \|\widetilde{z}_T\|^2_{2,T} = \|\widetilde{z}\|_2^2 \ge \|\boldsymbol{v}\|^2_{\mathrm{Ggrad},0,\mathcal{S}}.$$

This finishes the proof. □

By Proposition 3.8(i), for given $\boldsymbol{q} \in \mathbf{H}^{-3/2,-1/2}(\mathcal{S})$, $\langle \boldsymbol{q}\,, z\rangle_{\mathcal{S}}$ defines a functional that depends only on the jumps of z and $\nabla_{\mathcal{T}} z$ across the element interfaces and their traces on Γ. It will be denoted as

$$[\cdot, \nabla_{\mathcal{T}}\,\cdot] : \begin{cases} H^2(\mathcal{T}) & \to \quad \left(\mathbf{H}^{-3/2,-1/2}(\mathcal{S})\right)' \\ \quad z & \mapsto \quad [z, \nabla_{\mathcal{T}} z](\boldsymbol{q}) := \langle \boldsymbol{q}\,, z\rangle_{\mathcal{S}} \end{cases} \tag{3.32}$$

with duality pairing defined in (3.7). As before, this functional defines a seminorm in $H^2(\mathcal{T})$,

$$\|[z, \nabla_{\mathcal{T}} z]\|_{(-3/2,-1/2,\mathcal{S})'} := \sup_{0\ne\boldsymbol{q}\in\mathbf{H}^{-3/2,-1/2}(\mathcal{S})} \frac{\langle \boldsymbol{q}\,, z\rangle_{\mathcal{S}}}{\|\boldsymbol{q}\|_{-3/2,-1/2,\mathcal{S}}}, \qquad z \in H^2(\mathcal{T}). \tag{3.33}$$

Proposition 3.9. *The identity*

$$\|\boldsymbol{v}\|_{3/2,1/2,00,\mathcal{S}} = \|\boldsymbol{v}\|_{\mathrm{Ggrad},0,\mathcal{S}} \qquad \forall \boldsymbol{v} \in \mathbf{H}_{00}^{3/2,1/2}(\mathcal{S})$$

holds. In particular,

$$\operatorname{tr}^{\mathrm{Ggrad}} : \; H_0^2(\Omega) \to \mathbf{H}_{00}^{3/2,1/2}(\mathcal{S})$$

has unit norm and $\mathbf{H}_{00}^{3/2,1/2}(\mathcal{S})$ *is closed.*

Proof. The norm identity is obtained as in the proof of Proposition 3.5. For $\boldsymbol{v} = (\boldsymbol{v}_T)_T \in \mathbf{H}_{00}^{3/2,1/2}(\mathcal{S})$ we use Proposition (3.8)(ii) and Lemma 3.3 to calculate

$$\begin{aligned}
\|\boldsymbol{v}\|_{3/2,1/2,00,\mathcal{S}}^2 &= \Big(\sup_{0\neq\boldsymbol{\Theta}\in H(\operatorname{div}\mathbf{div},\mathcal{T})} \frac{\sum_{T\in\mathcal{T}} \langle \boldsymbol{v}_T\,,\boldsymbol{\Theta}\rangle_{\partial T}}{\|\boldsymbol{\Theta}\|_{\operatorname{div}\mathbf{div},\mathcal{T}}} \Big)^2 \\
&= \sum_{T\in\mathcal{T}} \sup_{0\neq\boldsymbol{\Theta}\in H(\operatorname{div}\mathbf{div},T)} \frac{\langle \boldsymbol{v}_T\,,\boldsymbol{\Theta}\rangle_{\partial T}^2}{\|\boldsymbol{\Theta}\|_{\operatorname{div}\mathbf{div},T}^2} \\
&= \sum_{T\in\mathcal{T}} \|\boldsymbol{v}_T\|_{3/2,1/2,00,\partial T}^2 \\
&= \sum_{T\in\mathcal{T}} \|\boldsymbol{v}_T\|_{\mathrm{Ggrad},\partial T}^2 = \|\boldsymbol{v}\|_{\mathrm{Ggrad},0,\mathcal{S}}^2.
\end{aligned}$$

The space $\mathbf{H}_{00}^{3/2,1/2}(\mathcal{S})$ is closed as the image of a bounded below operator. ☐

3.6. A Poincaré inequality in $H^2(\mathcal{T})$.

The definition of $\mathcal{C}$ implies that it induces a self-adjoint isomorphism $\mathbb{L}_2^s(\Omega) \to \mathbb{L}_2^s(\Omega)$. This fact will be used in the following.

Let us define a projection operator $\mathbf{P} : \mathbb{L}_2^s(\Omega) \to \mathcal{C}\boldsymbol{\varepsilon}\nabla H_0^2(\Omega)$ by

$$(\mathbf{P}(\boldsymbol{\Theta})\,,\mathcal{C}\boldsymbol{\varepsilon}\nabla\delta z) = (\boldsymbol{\Theta}\,,\mathcal{C}\boldsymbol{\varepsilon}\nabla\delta z) \qquad \forall \delta z \in H_0^2(\Omega). \tag{3.34}$$

There is a mapping $\mathbb{L}_2^s(\Omega) \ni \boldsymbol{\Theta} \mapsto \xi = \xi(\boldsymbol{\Theta}) \in H_0^2(\Omega)$ with

$$(\boldsymbol{\varepsilon}\nabla\xi\,,\mathcal{C}\boldsymbol{\varepsilon}\nabla\delta z) = (\boldsymbol{\Theta}\,,\mathcal{C}\boldsymbol{\varepsilon}\nabla\delta z) \qquad \forall \delta z \in H_0^2(\Omega). \tag{3.35}$$

In other words,

(3.36)
$$\mathbf{P}(\boldsymbol{\Theta}) = \boldsymbol{\varepsilon}\nabla\xi(\boldsymbol{\Theta}), \quad \operatorname{div}\mathbf{div}\,\mathcal{C}(\boldsymbol{\varepsilon}\nabla\xi - \boldsymbol{\Theta}) = 0, \quad \text{and} \quad \mathcal{C}\big(\boldsymbol{\varepsilon}\nabla\xi - \boldsymbol{\Theta}\big) \in H(\operatorname{div}\mathbf{div},\Omega).$$

In the next proposition we present a Poincaré inequality to bound the $\|\cdot\|_{2,\mathcal{T}}$-norm of a function from $H_0^2(\mathcal{T})$ by its jumps and the projected piecewise iterated gradients. In the case of the Laplacian, such an estimate is provided by [14, Lemma 4.2] and [16, Lemma 3.3]. Our bound for $\|z\|$ can be proved almost identically to [14, Lemma 4.2], switching from the Laplacian to the fourth-order operator and by using the trace operator $\mathrm{tr}^{\mathrm{dDiv}}$.

The second bound, for $\|\boldsymbol{\varepsilon}_{\mathcal{T}}\nabla_{\mathcal{T}} z\|$, corresponds to [16, Lemma 3.3] for the Laplacian. The proof of [16, Lemma 3.3] uses a projection operator like $\mathbf{P}$, together with the technical lemma [14, Lemma 4.3]. This lemma provides an estimate by norms of jumps of natural and essential traces (traces that correspond to natural and essential boundary conditions on elements) and, moreover, uses a Helmholtz decomposition for its proof. Whereas there is a Helmholtz decomposition for $H(\operatorname{div}\mathbf{div},\Omega)$ (cf. [1, §4.1] and [35, Theorem 4.2]), the use of jumps of natural traces in $H(\operatorname{div}\mathbf{div},\mathcal{T})$ appears to be too complicated in our case. We present a shorter technique that is based on the projection operator $\mathbf{P}$ without using jumps of natural traces.

Proposition 3.10. *The following Poincaré inequality holds:*

$$\|z\|_{2,\mathcal{T}} \lesssim \|\mathbf{P}(\boldsymbol{\varepsilon}_{\mathcal{T}}\nabla_{\mathcal{T}} z)\| + \|[z,\nabla_{\mathcal{T}} z]\|_{(-3/2,-1/2,\mathcal{S})'} \qquad \forall z \in H^2(\mathcal{T}).$$

Here, the implicit constant is independent of the underlying mesh $\mathcal{T}$; it depends only on Ω and $\mathcal{C}$.

Proof. We start by proving

$$\|z\| \lesssim \|\mathbf{P}(\varepsilon_\mathcal{T}\nabla_\mathcal{T} z)\| + \|[z,\nabla_\mathcal{T} z]\|_{(-3/2,-1/2,\mathcal{S})'} \qquad \forall z \in H^2(\mathcal{T}).$$

For given $z \in H^2(\mathcal{T})$ let $\phi \in H^2_0(\Omega)$ be the solution to

$$\operatorname{div}\, \mathbf{div}\, \mathcal{C}\varepsilon\nabla\phi = z \quad \text{in} \quad \Omega.$$

Then $\|\mathcal{C}\varepsilon\nabla\phi\| \le C\|z\|$ for a constant $C > 0$ that depends only on Ω and $\mathcal{C}$, and we obtain, by using definition (3.33) and Proposition 3.5,

$$\begin{aligned}\|z\|^2 &= (\operatorname{div}\, \mathbf{div}\, \mathcal{C}\varepsilon\nabla\phi\,, z) = (\mathcal{C}\varepsilon\nabla\phi\,, \varepsilon_\mathcal{T}\nabla_\mathcal{T} z) + \langle \operatorname{tr}^{\mathrm{dDiv}}(\mathcal{C}\varepsilon\nabla\phi)\,, z\rangle_\mathcal{S}\\ &\le C\|z\|\|\mathbf{P}(\varepsilon_\mathcal{T}\nabla_\mathcal{T} z)\| + \|[z,\nabla_\mathcal{T} z]\|_{(-3/2,-1/2,\mathcal{S})'}\Big(\|\mathcal{C}\varepsilon\nabla\phi\|^2 + \|\operatorname{div}\, \mathbf{div}\, \mathcal{C}\varepsilon\nabla\phi\|^2\Big)^{1/2}\\ &\lesssim \|z\|\Big(\|\mathbf{P}(\varepsilon_\mathcal{T}\nabla_\mathcal{T} z)\| + \|[z,\nabla_\mathcal{T} z]\|_{(-3/2,-1/2,\mathcal{S})'}\Big).\end{aligned}$$

This proves the bound for $\|z\|$. It remains to show that

$$\|\varepsilon_\mathcal{T}\nabla_\mathcal{T} z\| \lesssim \|\mathbf{P}(\varepsilon_\mathcal{T}\nabla_\mathcal{T} z)\| + \|[z,\nabla_\mathcal{T} z]\|_{(-3/2,-1/2,\mathcal{S})'} \qquad \forall z \in H^2(\mathcal{T}).$$

For given $z \in H^2(\mathcal{T})$ let $\xi = \xi(\varepsilon_\mathcal{T}\nabla_\mathcal{T} z)$; cf. (3.35). By definition (3.34) and relation (3.36) it holds that

$$(\varepsilon_\mathcal{T}\nabla_\mathcal{T} z\,, \mathcal{C}\varepsilon\nabla\xi) = (\varepsilon\nabla\xi\,, \mathcal{C}\varepsilon\nabla\xi) \eqsim \|\varepsilon\nabla\xi\|^2 = \|\mathbf{P}(\varepsilon_\mathcal{T}\nabla_\mathcal{T} z)\|^2. \tag{3.37}$$

By (3.36) we have $\operatorname{div}\, \mathbf{div}\, \mathcal{C}\varepsilon_\mathcal{T}\nabla_\mathcal{T}(z-\xi) = 0$ and $\mathcal{C}\varepsilon_\mathcal{T}\nabla_\mathcal{T}(z-\xi) \in H(\operatorname{div}\, \mathbf{div}, \Omega)$. Recalling (3.1), (3.5) with $\mathbf{\Theta} = \mathcal{C}\varepsilon_\mathcal{T}\nabla_\mathcal{T}(z-\xi)$ we conclude that

$$\begin{aligned}(\varepsilon_\mathcal{T}\nabla_\mathcal{T} z\,, \mathcal{C}\varepsilon_\mathcal{T}\nabla_\mathcal{T}(z-\xi)) &= (z\,, \operatorname{div}\, \mathbf{div}\, \mathcal{C}\varepsilon_\mathcal{T}\nabla_\mathcal{T}(z-\xi)) - \langle \operatorname{tr}^{\mathrm{dDiv}}(\mathcal{C}\varepsilon_\mathcal{T}\nabla_\mathcal{T}(z-\xi))\,, z\rangle_\mathcal{S}\\ &= -\langle \operatorname{tr}^{\mathrm{dDiv}}(\mathcal{C}\varepsilon_\mathcal{T}\nabla_\mathcal{T}(z-\xi))\,, z\rangle_\mathcal{S}. \end{aligned}\tag{3.38}$$

The combination of (3.37) and (3.38) yields

$$\begin{aligned}\|\varepsilon_\mathcal{T}\nabla_\mathcal{T} z\|^2 &\eqsim (\varepsilon_\mathcal{T}\nabla_\mathcal{T} z\,, \mathcal{C}\varepsilon_\mathcal{T}\nabla_\mathcal{T} z) = (\varepsilon_\mathcal{T}\nabla_\mathcal{T} z\,, \mathcal{C}\varepsilon\nabla\xi) + (\varepsilon_\mathcal{T}\nabla_\mathcal{T} z\,, \mathcal{C}\varepsilon_\mathcal{T}\nabla_\mathcal{T}(z-\xi))\\ &\lesssim \|\mathbf{P}(\varepsilon_\mathcal{T}\nabla_\mathcal{T} z)\|\,\|\varepsilon\nabla\xi\| - \langle \operatorname{tr}^{\mathrm{dDiv}}(\mathcal{C}\varepsilon_\mathcal{T}\nabla_\mathcal{T}(z-\xi))\,, z\rangle_\mathcal{S}.\end{aligned}$$

We finish the proof by bounding $\|\varepsilon\nabla\xi\| \lesssim \|\varepsilon_\mathcal{T}\nabla_\mathcal{T} z\|$ by stability of problem (3.35) and by applying as before Proposition 3.5 in combination with definition (3.33). This gives

$$\begin{aligned}\langle \operatorname{tr}^{\mathrm{dDiv}}(\mathcal{C}\varepsilon_\mathcal{T}\nabla_\mathcal{T}(z-\xi))\,, z\rangle_\mathcal{S} &\le \|\mathcal{C}\varepsilon_\mathcal{T}\nabla_\mathcal{T}(z-\xi)\|_{\operatorname{div}\,\mathbf{div}}\,\|[z,\nabla_\mathcal{T} z]\|_{(-3/2,-1/2,\mathcal{S})'}\\ &= \|\mathcal{C}\varepsilon_\mathcal{T}\nabla_\mathcal{T}(z-\xi)\|\|[z,\nabla_\mathcal{T} z]\|_{(-3/2,-1/2,\mathcal{S})'} \lesssim \|\varepsilon_\mathcal{T}\nabla_\mathcal{T} z\|\|[z,\nabla_\mathcal{T} z]\|_{(-3/2,-1/2,\mathcal{S})'}.\end{aligned}$$

∎

4. Variational formulation and DPG method

Let us return to our preliminary formulation (2.2). We now know that we have to interpret the interface terms as

$$\sum_{T\in\mathcal{T}} \langle \mathbf{n}\cdot\mathbf{div}\,\mathbf{M}\,, z\rangle_{\partial T} - \sum_{T\in\mathcal{T}} \langle \mathbf{M}\mathbf{n}\,, \nabla z\rangle_{\partial T} = \langle \operatorname{tr}^{\mathrm{dDiv}}(\mathbf{M})\,, z\rangle_\mathcal{S}$$

and

$$\sum_{T\in\mathcal{T}} \langle \mathbf{n}\cdot\mathbf{div}\,\mathbf{\Theta}\,, u\rangle_{\partial T} - \sum_{T\in\mathcal{T}} \langle \mathbf{\Theta}\mathbf{n}\,, \nabla u\rangle_{\partial T} = \langle \operatorname{tr}^{\mathrm{Ggrad}}(u)\,, \mathbf{\Theta}\rangle_\mathcal{S}.$$

Introducing the independent trace variables $\widehat{\boldsymbol{q}} := \operatorname{tr}^{\mathrm{dDiv}}(\mathbf{M})$, $\widehat{\boldsymbol{u}} := \operatorname{tr}^{\mathrm{Ggrad}}(u)$, and spaces

$$\mathcal{U} := L_2(\Omega) \times \mathbb{L}_2^s(\Omega) \times \mathbf{H}_{00}^{3/2,1/2}(\mathcal{S}) \times \mathbf{H}^{-3/2,-1/2}(\mathcal{S}),$$
$$\mathcal{V} := H^2(\mathcal{T}) \times H(\operatorname{div} \mathbf{div}, \mathcal{T})$$

with respective norms

$$\|(u, \mathbf{M}, \widehat{\boldsymbol{u}}, \widehat{\boldsymbol{q}})\|_{\mathcal{U}}^2 := \|u\|^2 + \|\mathbf{M}\|^2 + \|\widehat{\boldsymbol{u}}\|_{3/2,1/2,00,\mathcal{S}}^2 + \|\widehat{\boldsymbol{q}}\|_{-3/2,-1/2,\mathcal{S}}^2,$$
$$\|(z, \mathbf{\Theta})\|_{\mathcal{V}}^2 := \|z\|_{2,\mathcal{T}}^2 + \|\mathbf{\Theta}\|_{\operatorname{div} \mathbf{div},\mathcal{T}}^2,$$

our ultraweak variational formulation of (2.1) is: *Find* $(u, \mathbf{M}, \widehat{\boldsymbol{u}}, \widehat{\boldsymbol{q}}) \in \mathcal{U}$ *such that*

$$b(u, \mathbf{M}, \widehat{\boldsymbol{u}}, \widehat{\boldsymbol{q}}; z, \mathbf{\Theta}) = L(z, \mathbf{\Theta}) \qquad \forall (z, \mathbf{\Theta}) \in \mathcal{V}. \tag{4.1}$$

Here,

$$b(u, \mathbf{M}, \widehat{\boldsymbol{u}}, \widehat{\boldsymbol{q}}; z, \mathbf{\Theta}) := (\mathbf{M}\,, \varepsilon_{\mathcal{T}} \nabla_{\mathcal{T}} z + \mathcal{C}^{-1}\mathbf{\Theta}) + (u\,, \operatorname{div}_{\mathcal{T}} \mathbf{div}_{\mathcal{T}} \mathbf{\Theta}) + \langle \widehat{\boldsymbol{q}}\,, z\rangle_{\mathcal{S}} - \langle \widehat{\boldsymbol{u}}\,, \mathbf{\Theta}\rangle_{\mathcal{S}} \tag{4.2}$$

and

$$L(z, \mathbf{\Theta}) := -(f\,, z).$$

Note that the skeleton dualities in (4.2) are defined by (3.7) and (3.16). One of our main results is the following theorem.

Theorem 4.1. *For any function $f \in L_2(\Omega)$ (or any functional $L \in \mathcal{V}'$) there exists a unique and stable solution $(u, \mathbf{M}, \widehat{\boldsymbol{u}}, \widehat{\boldsymbol{q}}) \in \mathcal{U}$ to* (4.1),

$$\|u\| + \|\mathbf{M}\| + \|\widehat{\boldsymbol{u}}\|_{3/2,1/2,00,\mathcal{S}} + \|\widehat{\boldsymbol{q}}\|_{-3/2,-1/2,\mathcal{S}} \lesssim \|f\| \quad (\text{or } \|L\|_{\mathcal{V}'})$$

with a hidden constant that is independent of f (or L) and $\mathcal{T}$.

A proof of this theorem is given in §5.2.

Now, the DPG method with optimal test functions consists of solving (4.1) within discrete spaces $\mathcal{U}_h \subset \mathcal{U}$ and $\mathrm{T}(\mathcal{U}_h) \subset \mathcal{V}$. Here, $\mathrm{T} : \mathcal{U} \to \mathcal{V}$ is the *trial-to-test operator*, defined by

$$\langle\!\langle \mathrm{T}(\mathbf{u})\,, \mathbf{v}\rangle\!\rangle_{\mathcal{V}} = b(\mathbf{u}, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathcal{V}$$

with inner product $\langle\!\langle \cdot\,, \cdot\rangle\!\rangle$ in $\mathcal{V}$.

Then, for given finite-dimensional space $\mathcal{U}_h \subset \mathcal{U}$, the discrete method is: *Find* $\mathbf{u}_h \in \mathcal{U}_h$ *such that*

$$b(\mathbf{u}_h, \mathrm{T}\,\boldsymbol{\delta}\mathbf{u}) = L(\mathrm{T}\,\boldsymbol{\delta}\mathbf{u}) \qquad \forall \boldsymbol{\delta}\mathbf{u} \in \mathcal{U}_h. \tag{4.3}$$

It is a minimum residual method that delivers the best approximation in the *energy norm* (or residual norm) $\|\cdot\|_{\mathbb{E}} := \|B(\cdot)\|_{\mathcal{V}'}$; cf., e.g., [14]. Here, $B : \mathcal{U} \to \mathcal{V}'$ is the operator induced by the bilinear form $b(\cdot, \cdot)$.

Our second main result is the quasi-optimal convergence of the DPG scheme (4.3).

Theorem 4.2. *Let $f \in L_2(\Omega)$ be given. For any finite-dimensional subspace $\mathcal{U}_h \subset \mathcal{U}$ there exists a unique solution $\mathbf{u}_h \in \mathcal{U}_h$ to* (4.3)*. It satisfies the quasi-optimal error estimate*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{U}} \lesssim \|\mathbf{u} - \mathbf{w}\|_{\mathcal{U}} \qquad \forall \mathbf{w} \in \mathcal{U}_h$$

with a hidden constant that is independent of f, $\mathcal{T}$, and $\mathcal{U}_h$.

A proof of this theorem is given in §5.2.

5. Adjoint problem and proofs of Theorems 4.1, 4.2

As discussed in the introduction, a key step to showing well-posedness of the variational formulation (4.1) is to show stability of its adjoint problem, which we formulate next.

Find $z \in H^2(\mathcal{T})$ *and* $\boldsymbol{\Theta} \in H(\operatorname{div}\,\mathbf{div},\mathcal{T})$ *such that*

$$\operatorname{div}_{\mathcal{T}}\mathbf{div}_{\mathcal{T}}\boldsymbol{\Theta} = g \quad \in L_2(\Omega), \tag{5.1a}$$

$$\mathcal{C}^{-1}\boldsymbol{\Theta} + \boldsymbol{\varepsilon}_{\mathcal{T}}\nabla_{\mathcal{T}} z = \mathbf{H} \quad \in \mathbb{L}_2^s(\Omega), \tag{5.1b}$$

$$[\boldsymbol{\Theta}\mathbf{n}, \mathbf{n}\cdot\mathbf{div}_{\mathcal{T}}\boldsymbol{\Theta}] = \mathbf{r} \quad \in \left(\mathbf{H}_{00}^{3/2,1/2}(\mathcal{S})\right)', \tag{5.1c}$$

$$[z, \nabla_{\mathcal{T}} z] = \mathbf{j} \quad \in \left(\mathbf{H}^{-3/2,-1/2}(\mathcal{S})\right)'. \tag{5.1d}$$

Here, initially, the data g, $\mathbf{H}$, $\mathbf{r}$, and $\mathbf{j}$ are obtained as indicated from the given (arbitrary) function $(z,\boldsymbol{\Theta}) \in \mathcal{V}$. Recall (3.22) and (3.32) for the definition of the jumps.

Proving well-posedness of (5.1) means that we separate the data from the particular test functions z, $\boldsymbol{\Theta}$. Then, the functionals on the right-hand sides of (5.1) are arbitrary elements of the corresponding spaces as indicated. Specifically, by definition of the dual spaces in (5.1c), (5.1d), the functionals $\mathbf{r}$ and $\mathbf{j}$ stem from corresponding functions (now using different symbols) $\boldsymbol{\Theta}_{\mathbf{r}} \in H(\operatorname{div}\,\mathbf{div},\mathcal{T})$ and $z_{\mathbf{j}} \in H^2(\mathcal{T})$, respectively, so that the following definitions apply:

$$\begin{aligned} &\text{Given } \boldsymbol{v} \in \mathbf{H}_{00}^{3/2,1/2}(\mathcal{S}), && \mathbf{r}(\boldsymbol{v}) := \langle \boldsymbol{v}\,,\boldsymbol{\Theta}_{\mathbf{r}}\rangle_{\mathcal{S}} && \text{(according to (3.22))},\\ &\text{and given } \boldsymbol{q} \in \mathbf{H}^{-3/2,-1/2}(\mathcal{S}), && \mathbf{j}(\boldsymbol{q}) := \langle \boldsymbol{q}\,,z_{\mathbf{j}}\rangle_{\mathcal{S}} && \text{(according to (3.32))}.\end{aligned}$$

Of course, the functions $\boldsymbol{\Theta}_{\mathbf{r}}$, $z_{\mathbf{j}}$ are not unique, but the induced functionals are. As indicated in (5.1c), (5.1d), the functionals $\mathbf{r}$ and $\mathbf{j}$ are measured in dual norms $\|\cdot\|_{(3/2,1/2,00,\mathcal{S})'}$ and $\|\cdot\|_{(-3/2,-1/2,\mathcal{S})'}$, respectively; see (3.23), (3.33).

5.1. Well-posedness of the adjoint problem.

In the following we again use that $\mathcal{C}$ induces a self-adjoint isomorphism $\mathbb{L}_2^s(\Omega) \to \mathbb{L}_2^s(\Omega)$.

Combining (5.1a) and (5.1b) we obtain, in distributional form,

$$\operatorname{div}_{\mathcal{T}}\mathbf{div}_{\mathcal{T}}\big(\mathcal{C}\boldsymbol{\varepsilon}_{\mathcal{T}}\nabla_{\mathcal{T}} z\big) = \operatorname{div}_{\mathcal{T}}\mathbf{div}_{\mathcal{T}}\big(\mathcal{C}\mathbf{H}\big) - g. \tag{5.2}$$

Testing with $\delta z \in H_0^2(\Omega)$ and twice integrating piecewise by parts give

$$\begin{aligned}(\mathcal{C}\boldsymbol{\varepsilon}_{\mathcal{T}}\nabla_{\mathcal{T}} z\,,\boldsymbol{\varepsilon}\nabla\delta z) + \langle \operatorname{tr}^{\mathrm{Ggrad}}(\delta z)\,,\mathcal{C}\boldsymbol{\varepsilon}_{\mathcal{T}}\nabla_{\mathcal{T}} z\rangle_{\mathcal{S}}&\\ = (\mathcal{C}\mathbf{H}\,,\boldsymbol{\varepsilon}\nabla\delta z) + \langle \operatorname{tr}^{\mathrm{Ggrad}}(\delta z)\,,\mathcal{C}\mathbf{H}\rangle_{\mathcal{S}} - (g\,,\delta z)&\end{aligned}$$

(recall the trace operator $\operatorname{tr}^{\mathrm{Ggrad}}$ from (3.15)). Now, by (5.1b), $\mathcal{C}\big(\mathbf{H} - \boldsymbol{\varepsilon}_{\mathcal{T}}\nabla_{\mathcal{T}} z\big) = \boldsymbol{\Theta} \in H(\operatorname{div}\,\mathbf{div},\mathcal{T})$ so that the combined interface terms are well defined via (3.15) and coincide with the jumps associated to $\boldsymbol{\Theta}$,

$$\langle \operatorname{tr}^{\mathrm{Ggrad}}(\delta z)\,,\mathcal{C}\big(\mathbf{H} - \boldsymbol{\varepsilon}_{\mathcal{T}}\nabla_{\mathcal{T}} z\big)\rangle_{\mathcal{S}} = \langle \operatorname{tr}^{\mathrm{Ggrad}}(\delta z)\,,\boldsymbol{\Theta}\rangle_{\mathcal{S}} = [\boldsymbol{\Theta}\mathbf{n}, \mathbf{n}\cdot\mathbf{div}_{\mathcal{T}}\boldsymbol{\Theta}](\operatorname{tr}^{\mathrm{Ggrad}}(\delta z));$$

cf. (3.22). Taking into account (5.1c) and (5.1d), the z-component of the solution to (5.1) satisfies the following reduced adjoint problem.

Given $g \in L_2(\Omega)$, $\mathbf{H} \in \mathbb{L}_2^s(\Omega)$, $\mathbf{r} \in \left(\mathbf{H}_{00}^{3/2,1/2}(\mathcal{S})\right)'$, *and* $\mathbf{j} \in \left(\mathbf{H}^{-3/2,-1/2}(\mathcal{S})\right)'$, *find* $z \in H^2(\mathcal{T})$ *such that*

(5.3a)
$$(\mathcal{C}\boldsymbol{\varepsilon}_{\mathcal{T}}\nabla_{\mathcal{T}} z\,,\boldsymbol{\varepsilon}\nabla\delta z) = (\mathcal{C}\mathbf{H}\,,\boldsymbol{\varepsilon}\nabla\delta z) - (g\,,\delta z) + \mathbf{r}(\mathrm{tr}^{\mathrm{Ggrad}}(\delta z)) \qquad \forall \delta z \in H_0^2(\Omega),$$

(5.3b) $$[z,\nabla_{\mathcal{T}} z](\boldsymbol{\delta q}) = \mathbf{j}(\boldsymbol{\delta q}) \qquad \forall \boldsymbol{\delta q} \in \mathbf{H}^{-3/2,-1/2}(\mathcal{S}).$$

Lemma 5.1. *Problem* (5.3) *has a unique solution* $z \in H^2(\mathcal{T})$. *It satisfies*

$$\|z\|_{2,\mathcal{T}} \lesssim \|g\| + \|\mathbf{H}\| + \|\mathbf{r}\|_{(3/2,1/2,00,\mathcal{S})'} + \|\mathbf{j}\|_{(-3/2,-1/2,\mathcal{S})'}.$$

Proof. Adding relations (5.3a), (5.3b) we represent (5.3) with the notation

$$a(z;\delta z,\boldsymbol{\delta q}) = l(\delta z,\boldsymbol{\delta q}).$$

We show that $a(\cdot;\cdot)$ and $l(\cdot)$ are bounded and that $a(\cdot;\cdot)$ satisfies the required inf-sup conditions.

The boundedness of $l(\cdot)$ is immediate by duality of involved norms,

$$\begin{aligned} l(\delta z,\boldsymbol{\delta q}) &= (\mathcal{C}\mathbf{H}\,,\boldsymbol{\varepsilon}\nabla\delta z) - (g\,,\delta z) + \mathbf{r}(\mathrm{tr}^{\mathrm{Ggrad}}(\delta z)) + \mathbf{j}(\boldsymbol{\delta q}) \\ &\le \|\mathcal{C}\mathbf{H}\|\|\boldsymbol{\varepsilon}\nabla\delta z\| + \|g\|\|\delta z\| + \|\mathbf{r}\|_{(3/2,1/2,00,\mathcal{S})'}\left(\|\delta z\|^2 + \|\boldsymbol{\varepsilon}\nabla\delta z\|^2\right)^{1/2} \\ &\quad + \|\mathbf{j}\|_{(-3/2,-1/2,\mathcal{S})'}\|\boldsymbol{\delta q}\|_{-3/2,-1/2,\mathcal{S}} \\ &\lesssim \left(\|\mathbf{H}\| + \|g\| + \|\mathbf{r}\|_{(3/2,1/2,00,\mathcal{S})'} + \|\mathbf{j}\|_{(-3/2,-1/2,\mathcal{S})'}\right)\left(\|\delta z\|_2 + \|\boldsymbol{\delta q}\|_{-3/2,-1/2,\mathcal{S}}\right). \end{aligned}$$

The boundedness of $a(\cdot;\cdot)$ is also immediate by using definition (3.32) and duality norm (3.6),

$$[z,\nabla_{\mathcal{T}} z](\boldsymbol{\delta q}) \le \|\boldsymbol{\delta q}\|_{-3/2,-1/2,\mathcal{S}}\|z\|_{2,\mathcal{T}}.$$

It remains to show the inf-sup conditions.

Let $\delta z \in H_0^2(\Omega)$ and $\boldsymbol{\delta q} \in H^{-3/2,-1/2}(\mathcal{S})$ with $a(z;\delta z,\boldsymbol{\delta q}) = 0$ for any $z \in H^2(\mathcal{T})$. Selecting $z := \delta z \in H_0^2(\Omega)$, Proposition 3.8(i) shows that $[z,\nabla_{\mathcal{T}} z](\boldsymbol{\delta q}) = \langle \boldsymbol{\delta q}\,, z\rangle_{\mathcal{S}} = 0$ (recall (3.7) for the definition of the duality) so that

$$a(z;\delta z,\boldsymbol{\delta q}) = (\mathcal{C}\boldsymbol{\varepsilon}\nabla\delta z\,,\boldsymbol{\varepsilon}\nabla\delta z) + [z,\nabla_{\mathcal{T}} z](\boldsymbol{\delta q}) = (\mathcal{C}\boldsymbol{\varepsilon}\nabla\delta z\,,\boldsymbol{\varepsilon}\nabla\delta z) \gtrsim \|\boldsymbol{\varepsilon}\nabla\delta z\|^2 = 0,$$

that is, $\delta z = 0$. Using the observed relation for the jump of z, it follows that

$$a(z;\delta z,\boldsymbol{\delta q}) = \langle \boldsymbol{\delta q}\,, z\rangle_{\mathcal{S}} = 0 \qquad \forall z \in H^2(\mathcal{T}),$$

i.e.,

$$\|\boldsymbol{\delta q}\|_{-3/2,-1/2,\mathcal{S}} = \|\boldsymbol{\delta q}\|_{\mathrm{dDiv},\mathcal{S}} = 0$$

by (3.6) and Proposition 3.5. Therefore, $\boldsymbol{\delta q} = 0$.

Finally we check the inf-sup condition,

$$\begin{aligned} \sup_{0\ne(\delta z,\boldsymbol{\delta q})\in H_0^2(\Omega)\times H^{-3/2,-1/2}(\mathcal{S})} & \frac{\left((\mathcal{C}\boldsymbol{\varepsilon}_{\mathcal{T}}\nabla_{\mathcal{T}} z\,,\boldsymbol{\varepsilon}\nabla\delta z) + [z,\nabla_{\mathcal{T}} z](\boldsymbol{\delta q})\right)^2}{\|\mathcal{C}\boldsymbol{\varepsilon}\nabla\delta z\|^2 + \|\boldsymbol{\delta q}\|_{-3/2,-1/2,\mathcal{S}}^2} \\ &= \|\mathbf{P}(\boldsymbol{\varepsilon}_{\mathcal{T}}\nabla_{\mathcal{T}} z)\|^2 + \|[z,\nabla_{\mathcal{T}} z]\|_{(-3/2,-1/2,\mathcal{S})'}^2 \qquad \forall z \in H^2(\mathcal{T}); \end{aligned}$$

cf. (3.34). The result follows by the equivalence of the norms $\|\delta z\|_2$ and $\|\mathcal{C}\boldsymbol{\varepsilon}\nabla\delta z\|$ for $\delta z \in H_0^2(\Omega)$, and an application of the Poincaré inequality (Proposition 3.10). $\square$

Having analyzed the reduced adjoint problem (5.3), we are ready to prove the well-posedness of the full adjoint problem (5.1).

Proposition 5.2. *For arbitrary* $g \in L_2(\Omega)$, $\mathbf{H} \in \mathbb{L}_2^s(\Omega)$, $\mathbf{r} \in \big(\mathbf{H}_{00}^{3/2,1/2}(\mathcal{S})\big)'$, *and* $\mathbf{j} \in \big(\mathbf{H}^{-3/2,-1/2}(\mathcal{S})\big)'$, *the adjoint problem* (5.1) *has a unique solution* $(z, \boldsymbol{\Theta}) \in \mathcal{V}$. *It satisfies*

$$\|z\|_{2,\mathcal{T}} + \|\boldsymbol{\Theta}\|_{\mathrm{div}\,\mathbf{div},\mathcal{T}} \lesssim \|g\| + \|\mathbf{H}\| + \|\mathbf{r}\|_{(3/2,1/2,00,\mathcal{S})'} + \|\mathbf{j}\|_{(-3/2,-1/2,\mathcal{S})'}.$$

Proof. By construction, the z-component of any solution $(z, \boldsymbol{\Theta}) \in \mathcal{V}$ of (5.1) satisfies (5.3), which is uniquely solvable by Lemma 5.1. Therefore, the z-component of (5.1) is unique. Starting with the solution $z \in H^2(\mathcal{T})$ to (5.3), we show that this leads to a unique solution $(z, \boldsymbol{\Theta}) \in \mathcal{V}$ of (5.1), satisfying the stated bound. By relation (5.3b), z satisfies (5.1d). According to Lemma 5.1, z also satisfies the required bound.

It remains to construct $\boldsymbol{\Theta}$ and to bound its norm. We define $\boldsymbol{\Theta} := \mathcal{C}\big(\mathbf{H} - \boldsymbol{\varepsilon}_{\mathcal{T}} \nabla_{\mathcal{T}} z\big) \in \mathbb{L}_2^s(\Omega)$, thus satisfying (uniquely) (5.1b). Using the bound for $\|\boldsymbol{\varepsilon}_{\mathcal{T}} \nabla_{\mathcal{T}} z\|$, we also see that $\|\boldsymbol{\Theta}\| \lesssim \|g\| + \|\mathbf{H}\| + \|\mathbf{r}\|_{(3/2,1/2,00,\mathcal{S})'} + \|\mathbf{j}\|_{(-3/2,-1/2,\mathcal{S})'}$.

Now, (5.2) shows that

$$\mathrm{div}_{\mathcal{T}}\, \mathbf{div}_{\mathcal{T}} \boldsymbol{\Theta} = \mathrm{div}_{\mathcal{T}}\, \mathbf{div}_{\mathcal{T}} \mathcal{C}\big(\mathbf{H} - \boldsymbol{\varepsilon}_{\mathcal{T}} \nabla_{\mathcal{T}} z\big) = g$$

holds first in the distributional sense, and then in $L_2(\Omega)$ by the regularity of g. This is (5.1a) and also concludes the proof of the bound for $\|\boldsymbol{\Theta}\|_{\mathrm{div}\,\mathbf{div},\mathcal{T}}$.

It remains to show (5.1c). Let $\boldsymbol{v} \in \mathbf{H}_{00}^{3/2,1/2}(\mathcal{S})$ with $\boldsymbol{v} = \mathrm{tr}^{\mathrm{Ggrad}}(v)$ for $v \in H_0^2(\Omega)$. Recalling the definitions (3.22), (3.16), (3.15), and (3.13), we calculate with the previous relations for $\boldsymbol{\Theta}$ and (5.3a),

$$\begin{aligned}[\boldsymbol{\Theta}\mathbf{n}, \mathbf{n}\cdot \mathbf{div}_{\mathcal{T}} \boldsymbol{\Theta}](\boldsymbol{v}) &= \langle \boldsymbol{v}\,, \boldsymbol{\Theta}\rangle_{\mathcal{S}} = \langle \mathrm{tr}^{\mathrm{Ggrad}}(v)\,, \boldsymbol{\Theta}\rangle_{\mathcal{S}} = (\mathrm{div}_{\mathcal{T}}\, \mathbf{div}_{\mathcal{T}} \boldsymbol{\Theta}\,, v) - (\boldsymbol{\Theta}\,, \boldsymbol{\varepsilon}\nabla v)\\ &= (g\,, v) - (\mathcal{C}(\mathbf{H} - \boldsymbol{\varepsilon}_{\mathcal{T}} \nabla_{\mathcal{T}} z)\,, \boldsymbol{\varepsilon}\nabla v) = \mathbf{r}(\mathrm{tr}^{\mathrm{Ggrad}}(v)) = \mathbf{r}(\boldsymbol{v}).\end{aligned}$$

This shows (5.1c) and finishes the proof. ☐

5.2. **Proofs of Theorems 4.1, 4.2.** We are ready to prove our main results. To show Theorem 4.1, it is enough to check the standard properties.

(1) **Boundedness of the functional.** This is immediate since, for $f \in L_2(\Omega)$, it holds that $L(z) \le \|f\|\,\|z\| \le \|f\|\,\big(\|z\|_{2,\mathcal{T}} + \|\boldsymbol{\Theta}\|_{\mathrm{div}\,\mathbf{div},\mathcal{T}}\big)$ for any $(z, \boldsymbol{\Theta}) \in \mathcal{V}$.

(2) **Boundedness of the bilinear form.** The bound $b(\mathbf{u}, \mathbf{v}) \lesssim \|\mathbf{u}\|_{\mathcal{U}} \|\mathbf{v}\|_{\mathcal{V}}$ for all $\mathbf{u} \in \mathcal{U}$ and $\mathbf{v} \in \mathcal{V}$ is also immediate by definition of the norms in $\mathcal{U}$ and $\mathcal{V}$; cf. the corresponding functional spaces in (5.1a)–(5.1d).

(3) **Injectivity.** If $\mathbf{u} \in \mathcal{U}$ with $b(\mathbf{u}, \mathbf{v}) = 0\ \forall \mathbf{v} \in \mathcal{V}$, then $\mathbf{u} = 0$, as can be seen as follows. For given $\mathbf{u} = (u, \mathbf{M}, \widehat{\boldsymbol{u}}, \widehat{\boldsymbol{q}}) \in \mathcal{U}$ we select $g = u$, $\mathbf{H} = \mathbf{M}$, and let $\mathbf{j} \in (\mathbf{H}^{-3/2,-1/2}(\mathcal{S}))'$ and $\mathbf{r} \in (\mathbf{H}_{00}^{3/2,1/2}(\mathcal{S}))'$ be the Riesz representatives of $\widehat{\boldsymbol{q}} \in \mathbf{H}^{-3/2,-1/2}(\mathcal{S})$ and $-\widehat{\boldsymbol{u}} \in \mathbf{H}_{00}^{3/2,1/2}(\mathcal{S})$, respectively. According to Proposition 5.2, there exists $\mathbf{v} \in \mathcal{V}$ that satisfies the adjoint problem (5.1) with these functionals. It also yields

$$b(\mathbf{u}, \mathbf{v}) = \|u\|^2 + \|\mathbf{M}\|^2 + \|\widehat{\boldsymbol{u}}\|^2_{3/2,1/2,00,\mathcal{S}} + \|\widehat{\boldsymbol{q}}\|^2_{-3/2,-1/2,\mathcal{S}} = 0,$$

which proves that $\mathbf{u} = 0$.

(4) **Inf-sup condition.** For given $\mathbf{v} = (z, \boldsymbol{\Theta}) \in \mathcal{V}$ let g, $\mathbf{H}$, $\mathbf{j}$, and $\mathbf{r}$ be defined by (5.1). Then, by Proposition 5.2,

$$\sup_{0\neq\mathbf{u}\in\mathcal{U}} \frac{b(\mathbf{u},\mathbf{v})}{\|\mathbf{u}\|_{\mathcal{U}}} = \sup_{0\neq\mathbf{u}\in\mathcal{U}} \frac{(u\,,g) + (\mathbf{M}\,,\mathbf{H}) - \mathbf{r}(\widehat{\boldsymbol{u}}) + \mathbf{j}(\widehat{\boldsymbol{q}})}{\left(\|u\|^2 + \|\mathbf{M}\|^2 + \|\widehat{\boldsymbol{u}}\|^2_{3/2,1/2,00,\mathcal{S}} + \|\widehat{\boldsymbol{q}}\|^2_{-3/2,-1/2,\mathcal{S}}\right)^{1/2}}$$
$$= \left(\|g\|^2 + \|\mathbf{H}\|^2 + \|\mathbf{r}\|^2_{(3/2,1/2,00,\mathcal{S})'} + \|\mathbf{j}\|^2_{(-3/2,-1/2,\mathcal{S})'}\right)^{1/2} \gtrsim \|\mathbf{v}\|_{\mathcal{V}}$$

with an implicit constant that is independent of $\mathbf{v}$ and $\mathcal{T}$.

This proves Theorem 4.1.

Recall that the DPG method delivers the best approximation in the energy norm $\|\cdot\|_{\mathbb{E}}$,

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{E}} = \min\{\|\mathbf{u} - \mathbf{w}\|_{\mathbb{E}};\ \mathbf{w} \in \mathcal{U}_h\}.$$

Therefore, to show Theorem 4.2, it is enough to prove the equivalence of the energy norm and the norm $\|\cdot\|_{\mathcal{U}}$. The bound $\|\mathbf{u}\|_{\mathbb{E}} \lesssim \|\mathbf{u}\|_{\mathcal{U}}$ is equivalent to the boundedness of $b(\cdot,\cdot)$, which we have just checked. By definition of $\|\cdot\|_{\mathbb{E}} = \|B(\cdot)\|_{\mathcal{V}'}$, the other inequality, $\|\mathbf{u}\|_{\mathcal{U}} \lesssim \|\mathbf{u}\|_{\mathbb{E}}$ for all $\mathbf{u} \in \mathcal{U}$, is equivalent to the stability of the adjoint problem (5.1), which has been shown by Proposition 5.2. We have thus shown Theorem 4.2.

6. Discretization and numerical examples

In this section we discuss the construction of low-order discrete spaces, some implementational aspects, and present numerical tests. Throughout, we consider $d = 2$ and use regular triangular meshes $\mathcal{T}$ of shape-regular elements,

$$\sup_{T\in\mathcal{T}} \frac{\operatorname{diam}(T)^2}{|T|} \le C_{\text{shape}}.$$

As usual we denote by $h := h_{\mathcal{T}} := \max_{T\in\mathcal{T}} \operatorname{diam}(T)$ the discretization parameter.

6.1. **Discrete spaces.** For $T \in \mathcal{T}$, let $P^p(T)$ denote the space of polynomials on T of order less than or equal to $p \in \mathbb{N}_0$ and define

$$P^p(\mathcal{T}) := \{v \in L_2(\Omega);\ v|_T \in P^p(T)\ \forall T \in \mathcal{T}\}.$$

We set $\mathbb{P}^p(T) := P^p(T)^{2\times2}$ and $\mathbb{P}^p(\mathcal{T}) := P^p(\mathcal{T})^{2\times2}$. We seek approximations of the field variables $(u, \mathbf{M}) \in L_2(\Omega) \times \mathbb{L}_2^s(\Omega)$ in piecewise polynomial spaces,

$$(u_h, \mathbf{M}_h) \in P^0(\mathcal{T}) \times (\mathbb{P}^0(\mathcal{T}) \cap \mathbb{L}_2^s(\Omega)).$$

In the following we use the notation for edges, nodes, and their sets as introduced at the beginning of §3.4. Specifically, $\mathcal{E}_T$ denotes the set of edges of T and $\mathcal{E} := \bigcup_{T\in\mathcal{T}} \mathcal{E}_T$. Let $P^p(E)$ denote the space of polynomials on $E \in \mathcal{E}$ and define

$$P^p(\mathcal{E}_T) := \{v \in L_2(\partial T);\ v|_E \in P^p(E)\ \forall E \in \mathcal{E}_T\}, \qquad T \in \mathcal{T}.$$

The definition of conforming discrete spaces for the skeleton variables $(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{q}})$ is a little more involved. For a simpler representation we only consider lowest-order spaces. We start by defining, for $T \in \mathcal{T}$, the local space

$$\widehat{U}_{\text{Ggrad},\partial T} := \operatorname{tr}_T^{\text{Ggrad}}\left(\{v \in H^2(T);\ \Delta^2 v + v = 0,\ v|_{\partial T} \in P^3(\mathcal{E}_T),\ \mathbf{n}\cdot\nabla v|_{\partial T} \in P^1(\mathcal{E}_T)\}\right).$$

Let $\mathcal{N}_T$ denote the vertex set of $T \in \mathcal{T}$ and set $\mathcal{N} := \bigcup_{T\in\mathcal{T}} \mathcal{N}_T$. We associate the following degrees of freedom to a triangle T and the space $\widehat{U}_{\text{Ggrad},\partial T}$:

$$\{(v(e), \nabla v(e));\ e \in \mathcal{N}_T\}. \tag{6.1}$$

Observe that these degrees of freedom define a unique function in $\widehat{U}_{\mathrm{Ggrad},\partial T}$. The corresponding global discrete space is then defined by

$$\widehat{U}_{\mathrm{Ggrad},\mathcal{S}} := \{\widehat{\boldsymbol{v}} \in \mathbf{H}^{3/2,1/2}(\mathcal{S});\ \widehat{\boldsymbol{v}}|_{\partial T} \in \widehat{U}_{\mathrm{Ggrad},\partial T} \qquad \forall T \in \mathcal{T}\}$$

with associated global degrees $\{(v(e), \nabla v(e)); \quad e \in \mathcal{N}\}$. To get a subspace of $\mathbf{H}^{3/2,1/2}_{00}(\mathcal{S})$ we set the degrees of freedom corresponding to boundary vertices to zero, leading to the space

$$\widehat{U}_{\mathcal{S}} := \mathrm{tr}^{\mathrm{Ggrad}}(H^2_0(\Omega)) \cap \widehat{U}_{\mathrm{Ggrad},\mathcal{S}}$$

with dimension $3\#\mathcal{N}_0$.

Remark 6.1. Our definition of the skeleton spaces is closely related to the traces of spaces used in virtual element methods. In fact, for the present case the trace of the space defined in [4, §4.2] is the same as $\widehat{U}_{\mathcal{S}}$. In particular, we get the approximation property (cf. [4, Remark 4.6])

$$\min_{\widehat{\boldsymbol{u}}_h \in \widehat{U}_{\mathcal{S}}} \|\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_h\|_{3/2,1/2,00,\mathcal{S}} \le Ch|u|_{H^3(\Omega)},$$

where $\widehat{\boldsymbol{u}} = \mathrm{tr}^{\mathrm{Ggrad}}(u)$ and $C > 0$ is a generic $\mathcal{T}$-independent constant. Let us note that $\widehat{U}_{\mathcal{S}}$ coincides also with the trace of the (reduced) Hsieh–Clough–Tocher composite finite element space; cf. [13].

It remains to construct a finite-dimensional subspace for the approximation of $\widehat{\boldsymbol{q}} \in \mathbf{H}^{-3/2,-1/2}(\mathcal{S})$. For $T \in \mathcal{T}$ we define the local (volume) space

(6.2)

$$U_{\mathrm{dDiv},T} := \{\boldsymbol{\Theta} \in \mathcal{H}(\mathrm{div}\,\mathbf{div}, T);\ \boldsymbol{\varepsilon}\nabla\,\mathrm{div}\,\mathbf{div}\,\boldsymbol{\Theta} + \boldsymbol{\Theta} = 0,$$
$$\big(\mathbf{n}\cdot\mathbf{div}\,\boldsymbol{\Theta} + \partial_{\mathbf{t},\mathcal{E}_T}(\mathbf{t}\cdot\boldsymbol{\Theta}\mathbf{n})\big)|_{\partial T} \in P^0(\mathcal{E}_T), \qquad \mathbf{n}\cdot\boldsymbol{\Theta}\mathbf{n}|_{\partial T} \in P^0(\mathcal{E}_T)\}.$$

Here, $\partial_{\mathbf{t},\mathcal{E}_T}$ denotes the tangential derivative operator that is taken piecewise on the edges of ∂T; cf. Remark 3.7. To this space we associate the moments and point values

$$\alpha_E := \langle \mathbf{n}\cdot\mathbf{div}\,\boldsymbol{\Theta} + \partial_{\mathbf{t}}(\mathbf{t}\cdot\boldsymbol{\Theta}\mathbf{n})\,, 1\rangle_E \qquad (E \in \mathcal{E}_T), \tag{6.3a}$$

$$\beta_E := \langle \mathbf{n}\cdot\boldsymbol{\Theta}\mathbf{n}\,, 1\rangle_E \qquad (E \in \mathcal{E}_T), \tag{6.3b}$$

$$\gamma_e := [\![\boldsymbol{\Theta}]\!]_{\partial T}(e) \qquad (e \in \mathcal{N}_T); \tag{6.3c}$$

cf. (3.29), (3.30).

Lemma 6.2. *The degrees of freedom* (6.3) *define a unique element in* $U_{\mathrm{dDiv},T}$ *and vice versa.*

Proof. We prove that (6.3) define a unique functional $\ell(\cdot)$ on $H^2(T)$ that vanishes for $z \in H^2_0(T)$. Then, the proof of Lemma 3.2 shows that this functional can be uniquely identified with the trace of a function $\boldsymbol{\Theta} \in \mathcal{H}(\mathrm{div}\,\mathbf{div}, T)$ with $\boldsymbol{\varepsilon}\nabla\,\mathrm{div}\,\mathbf{div}\,\boldsymbol{\Theta} + \boldsymbol{\Theta} = 0$. Let $\alpha, \beta \in P^0(\mathcal{E}_T)$ be the functions associated to (6.3a) and (6.3b), that is, $\alpha|_E := \alpha_E|E|^{-1}$ and $\beta|_E := \beta_E|E|^{-1}$ $(E \in \mathcal{E}_T)$. For $z \in H^2(T)$ we set

$$\ell(z) := \langle \alpha\,, z\rangle_{\partial T} - \langle \beta\,, \mathbf{n}\cdot\nabla z\rangle_{\partial T} - \sum_{e\in\mathcal{N}_T} \gamma_e z(e) \tag{6.4}$$

with γ_e $(e \in \mathcal{N}_T)$ as in (6.3c). Note that $\ell(z) = 0$ if $z \in H^2_0(T)$ and that $\ell(\cdot)$ is indeed a bounded functional. By selecting appropriate test functions $z \in H^2(T)$ we obtain

$\ell(\cdot) = \mathrm{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\Theta})$. Furthermore, we prove that

$$\ell(z) = 0 \quad \text{for all } z \in H^2(T) \Leftrightarrow (\alpha,\beta,\gamma) = 0.$$

The direction "$\Leftarrow$" is trivial and "$\Rightarrow$" follows by selecting appropriate test functions. Note that $\ell(\cdot) = 0$ implies $\mathrm{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\Theta}) = 0$ with $\boldsymbol{\varepsilon}\nabla \operatorname{div} \mathbf{div}\,\boldsymbol{\Theta} + \boldsymbol{\Theta} = 0$. Since $\|\boldsymbol{\Theta}\|_{\operatorname{div}\mathbf{div},T} = \|\mathrm{tr}_T^{\mathrm{dDiv}}(\boldsymbol{\Theta})\|_{-3/2,-1/2,\partial T}$ it follows that $\boldsymbol{\Theta} = 0$.

Finally, to see the other direction, let $\boldsymbol{\Theta} \in U_{\mathrm{dDiv},T}$ be given. Note that $\boldsymbol{\Theta}$ depends only on its trace values, and by the localization of traces from §3.4 we conclude that $\dim(U_{\mathrm{dDiv},T}) = 9$, which is the number of degrees of freedom (6.3) and thus finishes the proof. ☐

The corresponding global (volume) space is defined by

$$U_{\mathrm{dDiv},\mathcal{T}} := \{\boldsymbol{\Theta} \in \mathcal{H}(\operatorname{div}\mathbf{div},\Omega);\ \boldsymbol{\Theta}|_T \in U_{\mathrm{dDiv},T}\ \forall T \in \mathcal{T}\}$$

with associated degrees of freedom

$$\langle \mathbf{n}\cdot\mathbf{div}\,\boldsymbol{\Theta} + \partial_{\mathbf{t}}(\mathbf{t}\cdot\boldsymbol{\Theta}\mathbf{n}), 1\rangle_E \qquad (E \in \mathcal{E}), \tag{6.5a}$$

$$\langle \mathbf{n}\cdot\boldsymbol{\Theta}\mathbf{n}, 1\rangle_E \qquad (E \in \mathcal{E}), \tag{6.5b}$$

$$[\![\boldsymbol{\Theta}]\!]_{\partial T}(e) \qquad (e \in \mathcal{N}_T,\ T \in \mathcal{T}), \tag{6.5c}$$

$$\text{subject to} \sum_{T\in\omega(e)} [\![\boldsymbol{\Theta}]\!]_{\partial T}(e) = 0 \qquad \forall e \in \mathcal{N}_0. \tag{6.5d}$$

Analogously to (6.3), (6.4), these variables define a functional acting on $z \in H^2(\mathcal{T})$.

Lemma 6.3. *The degrees of freedom* (6.5) *uniquely define an element in* $U_{\mathrm{dDiv},\mathcal{T}}$.

Proof. Note that by Lemma 6.2, (6.5a)–(6.5c) define a unique function $\boldsymbol{\Theta} \in H(\operatorname{div}\mathbf{div},\mathcal{T})$. Proposition 3.6 and (6.5d) conclude the proof. ☐

Summing up, $U_{\mathrm{dDiv},\mathcal{T}}$ has $\#\mathcal{E} + \#\mathcal{E} + 3\#\mathcal{T} - \#\mathcal{N}_0$ degrees of freedom. In the implementation we take care of the constraints (6.5d) by using Lagrange multipliers. Now, for the approximation of $\widehat{\boldsymbol{q}} \in \mathbf{H}^{-3/2,-1/2}(\mathcal{S})$, we use the discrete space

$$\widehat{Q}_{\mathcal{S}} := \mathrm{tr}^{\mathrm{dDiv}}(U_{\mathrm{dDiv},\mathcal{T}}).$$

By Proposition 3.5 there is an isomorphism between the volume space $U_{\mathrm{dDiv},\mathcal{T}}$ and its trace $\widehat{Q}_{\mathcal{S}}$ (note the PDE-constraint in (6.2)). Therefore, the trace space $\widehat{Q}_{\mathcal{S}}$ has the same degrees of freedom (6.5).

Lemma 6.4. *Let* $u \in H^4(\Omega)$ *and set* $\widehat{\boldsymbol{q}} := \mathrm{tr}^{\mathrm{dDiv}}(\mathcal{C}\boldsymbol{\varepsilon}\nabla u)$. *Then,*

$$\min_{\widehat{\boldsymbol{q}}_h\in\widehat{Q}_{\mathcal{S}}} \|\widehat{\boldsymbol{q}} - \widehat{\boldsymbol{q}}_h\|_{-3/2,-1/2,\mathcal{S}} \le Ch\|u\|_{H^4(\Omega)},$$

where the generic constant $C > 0$ *depends only on the shape-regularity constant* C_{shape} *of* $\mathcal{T}$, *and* $\mathcal{C}$.

Proof. Set $\boldsymbol{\Theta} := \mathcal{C}\boldsymbol{\varepsilon}\nabla u$. We start with defining an element $\widehat{\boldsymbol{q}}_h \in \widehat{Q}_{\mathcal{S}}$. Let $T \in \mathcal{T}$ be given and let $\Pi^p : L_2(\partial T) \to P^p(\mathcal{E}_T)$ denote the L_2-projection. Below, this projection operator will also be used component-wise for vector functions. For $E \in \mathcal{E}_T$ we set (cf. (3.24)) $\phi_E := \mathrm{tr}_{T,E,\mathbf{t}}^{\mathrm{dDiv}}(\boldsymbol{\Theta}) = \big(\mathbf{n}\cdot\mathbf{div}\,\boldsymbol{\Theta}|_T + \partial_{\mathbf{t}}(\mathbf{t}\cdot\boldsymbol{\Theta}|_T\mathbf{n})\big)|_E$, $\psi_E := \mathrm{tr}_{T,E,\mathbf{n}}^{\mathrm{dDiv}}(\boldsymbol{\Theta}) = \big(\mathbf{n}\cdot\boldsymbol{\Theta}|_T\mathbf{n}\big)|_E$ and define $\phi,\psi \in L_2(\partial T)$ by $\phi|_E := \phi_E$ and $\psi|_E := \psi_E$ for $E \in \mathcal{E}_T$.

By the regularity assumption we even have $\phi_E \in H^1(T)|_E$. Thus, there exists (a more regular) antiderivative g_E, that is, $\partial_{\mathbf{t}} g_E = \phi_E$, and it satisfies

$$\langle \phi_E \,, z\rangle_E = -\langle g_E \,, \partial_{\mathbf{t}} z\rangle_E + g_E(e_+)z(e_+) - g_E(e_-)z(e_-),$$

where E is the edge with vertices $e_\pm$. Define $g \in L_2(\partial T)$ by $g|_E = g_E$ with jumps $[\![g]\!]_{\partial T}(e) := g|_{E_2}(e) - g|_{E_1}(e)$ for $e \in \mathcal{N}_T$. Here, $E_1, E_2 \in \mathcal{E}_T$ are the unique edges with $\overline{E}_1 \cap \overline{E}_2 = \{e\}$, and the sign is chosen to be consistent with the definition of $[\![\mathbf{\Theta}]\!]_{\partial T}(e)$; cf. (3.30). We set $\gamma_e := [\![\mathbf{\Theta}]\!]_{\partial T}(e) - [\![g - \Pi^1 g]\!]_{\partial T}(e)$ for $e \in \mathcal{N}_T$. Prescribing the values of the degrees of freedom (6.3) as $\partial_{\mathbf{t},\mathcal{E}_T}(\Pi^1 g) \in P^0(\mathcal{E}_T)$, $\Pi^0\psi \in P^0(\mathcal{E}_T)$, and $(\gamma_e)_{e\in\mathcal{N}_T} \in \mathbb{R}^3$, this defines a unique element of $U_{\mathrm{dDiv},T}$. Doing this for all elements $T \in \mathcal{T}$ we obtain a unique element of $U_{\mathrm{dDiv},\mathcal{T}}$ since

$$\sum_{T\in\omega(e)} [\![\mathbf{\Theta}]\!]_{\partial T}(e) - [\![g - \Pi^1 g]\!]_{\partial T}(e) = 0 \qquad \forall e \in \mathcal{N}_0.$$

This also defines an element in $\widehat{Q}_{\mathcal{S}}$ which we denote by $\widehat{\boldsymbol{q}}_h$.

To analyze the convergence order it suffices to do so for one element $T \in \mathcal{T}$. Let $\widetilde{\mathbf{\Theta}} \in \mathcal{H}(\operatorname{div}\mathbf{div}, T)$ be the unique element with $\operatorname{tr}_T^{\mathrm{dDiv}}(\widetilde{\mathbf{\Theta}}) = (\widehat{\boldsymbol{q}} - \widehat{\boldsymbol{q}}_h)|_{\partial T}$ and $\boldsymbol{\varepsilon}\nabla \operatorname{div}\mathbf{div}\,\widetilde{\mathbf{\Theta}} + \widetilde{\mathbf{\Theta}} = 0$ on T. The proof of Lemma 3.2 shows that

$$\|(\widehat{\boldsymbol{q}} - \widehat{\boldsymbol{q}}_h)|_{\partial T}\|^2_{-3/2,-1/2,\partial T} = \|\widetilde{\mathbf{\Theta}}\|_T^2 + \|\operatorname{div}\mathbf{div}\,\widetilde{\mathbf{\Theta}}\|_T^2 = \langle \operatorname{tr}_T^{\mathrm{dDiv}}(\widetilde{\mathbf{\Theta}})\,, z\rangle_{\partial T},$$

where $z := -\operatorname{div}\mathbf{div}\,\widetilde{\mathbf{\Theta}} \in H^2(T)$ and $\|z\|^2_{2,T} = \|\widetilde{\mathbf{\Theta}}\|_T^2 + \|\operatorname{div}\mathbf{div}\,\widetilde{\mathbf{\Theta}}\|_T^2$. Note that (cf. (3.28) and Remark 3.7)

$$\begin{aligned}(6.6)\quad &\langle \operatorname{tr}_T^{\mathrm{dDiv}}(\widetilde{\mathbf{\Theta}})\,, z\rangle_{\partial T}\\ &= \langle \partial_{\mathbf{t},\mathcal{E}_T}(1-\Pi^1)g\,, z\rangle_{\partial T} - \sum_{e\in\mathcal{N}_T} [\![g - \Pi^1 g]\!]_{\partial T}(e) z(e) - \langle (1-\Pi^0)\psi\,, \mathbf{n}\cdot\nabla z\rangle_{\partial T}\end{aligned}$$

by the definition of γ_e. The last term in (6.6) is estimated by

$$\begin{aligned}&|\langle (1-\Pi^0)\psi\,, \mathbf{n}\cdot\nabla z\rangle_{\partial T}| = |\langle (1-\Pi^0)\psi\,, (1-\Pi^0)\mathbf{n}\cdot\nabla z\rangle_{\partial T}|\\ &\le \|(1-\Pi^0)\psi\|_{\partial T}\|(1-\Pi^0)\nabla z\|_{\partial T} \lesssim h\|u\|_{H^3(T)}\|\boldsymbol{\varepsilon}\nabla z\|_T \le h\|u\|_{H^3(T)}\|\widetilde{\mathbf{\Theta}}\|_{\operatorname{div}\mathbf{div},T}.\end{aligned}$$

Here we have used the trace inequality $\|(1-\Pi^0)v|_E\|_E \lesssim h^{1/2}\|\nabla v\|_T$ for $E \in \mathcal{E}_T$ and $v \in H^1(T)$. The involved constants depend only on the shape-regularity of $\mathcal{T}$. To bound the two remaining terms in (6.6) we integrate by parts and use properties of the L_2 projection Π^1 and the trace inequality $\|(1-\Pi^1)\nabla z\|_{\partial T} \lesssim h^{1/2}\|\boldsymbol{\varepsilon}\nabla z\|_T$. This yields

$$\begin{aligned}&\langle \partial_{\mathbf{t},\mathcal{E}_T}(1-\Pi^1)g\,, z\rangle_{\partial T} - \sum_{e\in\mathcal{N}_T} [\![g - \Pi^1 g]\!]_{\partial T}(e) z(e) = -\langle (1-\Pi^1)g\,, \partial_{\mathbf{t}} z\rangle_{\partial T}\\ &= -\langle (1-\Pi^1)g\,, (1-\Pi^1)\partial_{\mathbf{t}} z\rangle_{\partial T} \le \|(1-\Pi^1)g\|_{\partial T}\|(1-\Pi^1)\nabla z\|_{\partial T}\\ &\lesssim h^{3/2}\|\partial_{\mathbf{t},\mathcal{E}_T} g\|_{\partial T}\|\boldsymbol{\varepsilon}\nabla z\|_T.\end{aligned}$$

With the trace inequality $\|v|_E\|_E \lesssim h^{-1/2}\|v\|_{H^1(T)}$ for $E \in \mathcal{E}_T$ and $v \in H^1(T)$ we have that

$$h\|\partial_{\mathbf{t},\mathcal{E}_T} g\|^2_{\partial T} = h\|\phi\|^2_{\partial T} = h\sum_{E\in\mathcal{E}_T} \|\operatorname{tr}^{\mathrm{dDiv}}_{T,E,\mathbf{t}}(\mathbf{\Theta})\|_E^2 \lesssim \|u\|_{H^4(T)}.$$

Therefore, eventually we obtain the desired bound for the remaining terms in (6.6),

$$\Big|\langle \partial_{\mathbf{t},\mathcal{E}_T}(1-\Pi^1)g\,, z\rangle_{\partial T} - \sum_{e\in\mathcal{N}_T} [\![g - \Pi^1 g]\!]_{\partial T}(e) z(e)\Big| \lesssim h\|u\|_{H^4(T)}\|z\|_{2,T}.$$

Altogether we have thus shown that

$$\|(\widehat{\boldsymbol{q}}-\widehat{\boldsymbol{q}}_h)|_{\partial T}\|^2_{-3/2,-1/2,\partial T} = \|\widetilde{\boldsymbol{\Theta}}\|^2_{\mathrm{div}\,\mathbf{div},T} \lesssim h\|u\|_{H^4(T)}\|\widetilde{\boldsymbol{\Theta}}\|_{\mathrm{div}\,\mathbf{div},T},$$

that is, $\|(\widehat{\boldsymbol{q}}-\widehat{\boldsymbol{q}}_h)|_{\partial T}\|^2_{-3/2,-1/2,\partial T} \lesssim h^2\|u\|^2_{H^4(T)}$. Summation over all $T \in \mathcal{T}$ finishes the proof. $\square$

Our final discrete subspace of $\mathcal{U}$ for the DPG approximation is

$$\mathcal{U}_h := P^0(\mathcal{T}) \times (\mathbb{P}^0(\mathcal{T}) \cap \mathbb{L}_2^s(\Omega)) \times \widehat{U}_{\mathcal{S}} \times \widehat{Q}_{\mathcal{S}}.$$

By standard results on the approximation properties of $P^0(T)$ in $L_2(T)$, Remark 6.1, and Lemma 6.4 we get the following result:

Theorem 6.5. *Let $u \in H^4(\Omega)$ and set $\mathbf{u} := (u, \mathcal{C}\boldsymbol{\varepsilon}\nabla u, \mathrm{tr}^{\mathrm{Ggrad}}(u), \mathrm{tr}^{\mathrm{dDiv}}(\mathcal{C}\boldsymbol{\varepsilon}\nabla u)) \in \mathcal{U}$. Then, it holds that*

$$\min_{\mathbf{w}_h \in \mathcal{U}_h} \|\mathbf{u}-\mathbf{w}_h\|_{\mathcal{U}} \le Ch\|u\|_{H^4(\Omega)}.$$

The constant $C > 0$ depends on the shape-regularity of $\mathcal{T}$ and $\mathcal{C}$, but is otherwise independent of $\mathcal{T}$. $\square$

Remark 6.6. Let us note that the regularity assumption $u \in H^4(\Omega)$ in Theorem 6.5 may be reduced to $u \in H^3(\Omega)$ subject to $\mathrm{div}\,\mathbf{div}\,\mathcal{C}\boldsymbol{\varepsilon}\nabla u \in L_2(\Omega)$ with a refined analysis of Lemma 6.4. Such a reduction in the regularity assumption was observed in the recent work [22] for ultra-weak formulations of second-order elliptic problems.

Since the optimal test functions cannot be computed exactly, we approximate them in the enlarged space

$$\mathcal{V}_h = P^3(\mathcal{T}) \times (\mathbb{P}^2(\mathcal{T}) \cap \mathbb{L}_2^s(\Omega)) \subset \mathcal{V}.$$

That is, we replace $\mathrm{T} : \mathcal{U} \to \mathcal{V}$ by $\mathrm{T}_h : \mathcal{U} \to \mathcal{V}_h$, which is defined by

$$\langle\!\langle \mathrm{T}_h\,\mathbf{u}, \mathbf{v}\rangle\!\rangle_{\mathcal{V}} = b(\mathbf{u},\mathbf{v}) \qquad \forall \mathbf{v} \in \mathcal{V}_h.$$

Particularly, the space of approximated discrete optimal test functions is given by $\mathrm{T}_h(\mathcal{U}_h) \subseteq \mathcal{V}_h$.

6.2. **Examples.** In the following two examples, refinements are obtained by using the newest vertex bisection (NVB). It maintains shape-regularity of the triangulation, i.e.,

$$\sup_{T\in\mathcal{T}} \frac{\mathrm{diam}(T)^2}{|T|} \le C \sup_{T\in\mathcal{T}_0} \frac{\mathrm{diam}(T)^2}{|T|},$$

where $C > 0$ is independent of $\mathcal{T}$, and $\mathcal{T}$ is an arbitrary refinement of the initial mesh $\mathcal{T}_0$. Uniform refinement means that each triangle is divided into four son triangles with the same area; i.e., it corresponds to two bisections of the father element. In the second example we use a simple adaptive loop of the form

$$\boxed{\texttt{SOLVE}} \longrightarrow \boxed{\texttt{ESTIMATE}} \longrightarrow \boxed{\texttt{MARK}} \longrightarrow \boxed{\texttt{REFINE}}\ .$$

The estimation step is done with the error estimator that is automatically provided by the DPG method, $\eta := \|B(\mathbf{u}-\mathbf{u}_h)\|_{\mathcal{V}_h'}$. We refer to [10] for an abstract analysis of the DPG error estimator. Let us note that η can be written as the sum of local contributions

$$\eta^2 = \sum_{T\in\mathcal{T}} \eta(T)^2.$$

The marking step is done using the bulk criterion ($\theta \in (0,1)$),

$$\theta\eta^2 \le \sum_{T\in\mathcal{M}} \eta(T)^2,$$

where $\mathcal{M} \subseteq \mathcal{T}$ is the set of marked elements. It is the set of (up to a constant) minimal cardinality that satisfies the above relation. In §6.2.2 we use the parameter $\theta = \frac{1}{2}$.

6.2.1. *Square domain.* Let $\Omega = (0,1)^2$. We use the constant load $f = 1$, the identity $\mathcal{C} = I$, and the boundary conditions

$$u|_{\partial\Omega} = 0, \qquad \mathbf{n}\cdot\mathbf{M}\mathbf{n}|_{\partial\Omega} = 0.$$

It is known that the exact solution can be expressed by the double Fourier series

$$u(x,y) = \frac{16}{\pi^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\sin((2n+1)\pi x)\sin((2m+1)\pi y)}{(2n+1)(2m+1)((2n+1)^2+(2m+1)^2)^2}.$$

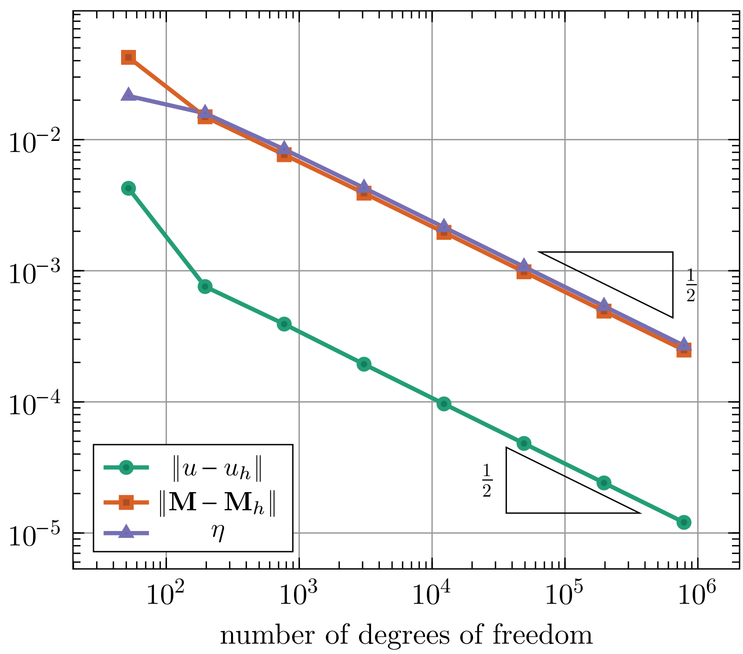

FIGURE 2. L_2 error for the field variables u, $\mathbf{M}$, and the DPG error estimator η with respect to the degrees of freedom (§6.2.1).

In particular, the solution is smooth, and we therefore expect a convergence of order $\mathcal{O}(h)$. In order to compute the $L_2(\Omega)$ errors $\|u-u_h\|$ and $\|\mathbf{M}-\mathbf{M}_h\|$ we replace the Fourier series by finite sums,

$$u(x,y) \approx \frac{16}{\pi^2}\sum_{n=0}^{15}\sum_{m=0}^{15} \frac{\sin((2n+1)\pi x)\sin((2m+1)\pi y)}{(2n+1)(2m+1)((2n+1)^2+(2m+1)^2)^2}.$$

Figure 2 shows the convergence behavior of the L_2 errors and the DPG error estimator η with respect to the number of degrees of freedom ($= \dim(\mathcal{U}_{\mathcal{T}})$) for a sequence of uniformly refined meshes. The number $\alpha > 0$ besides the triangle in the plots indicates its negative slope; i.e., the hypotenuse is parallel to $\dim(\mathcal{U}_\mathcal{T})^{-\alpha}$. We observe that all the plotted quantities have the same order of convergence $\alpha = 1/2$. Note that by §6.1 we have $\dim(\mathcal{U}_\mathcal{T}) \simeq \#\mathcal{T} \simeq h^{-2}$. Hence, we see the optimal convergence behavior $\mathcal{O}(h)$ as stated in Theorem 6.5.

6.2.2. *Domain with reentrant corner.* We consider the non-convex domain with reentrant corner at $(x,y) = (0,0)$ visualized in Figure 3 with angle $\frac{3}{4}\pi$ between the two edges that meet at $(x,y) = (0,0)$. We use the singularity function

$$u(r,\varphi) = r^{1+\alpha}(\cos((\alpha+1)\varphi) + C\cos((\alpha-1)\varphi))$$

with polar coordinates (r,φ) centered at the origin. A straightforward calculation yields

$$\operatorname{div} \mathbf{div}\, \boldsymbol{\varepsilon}\nabla u = 0 =: f.$$

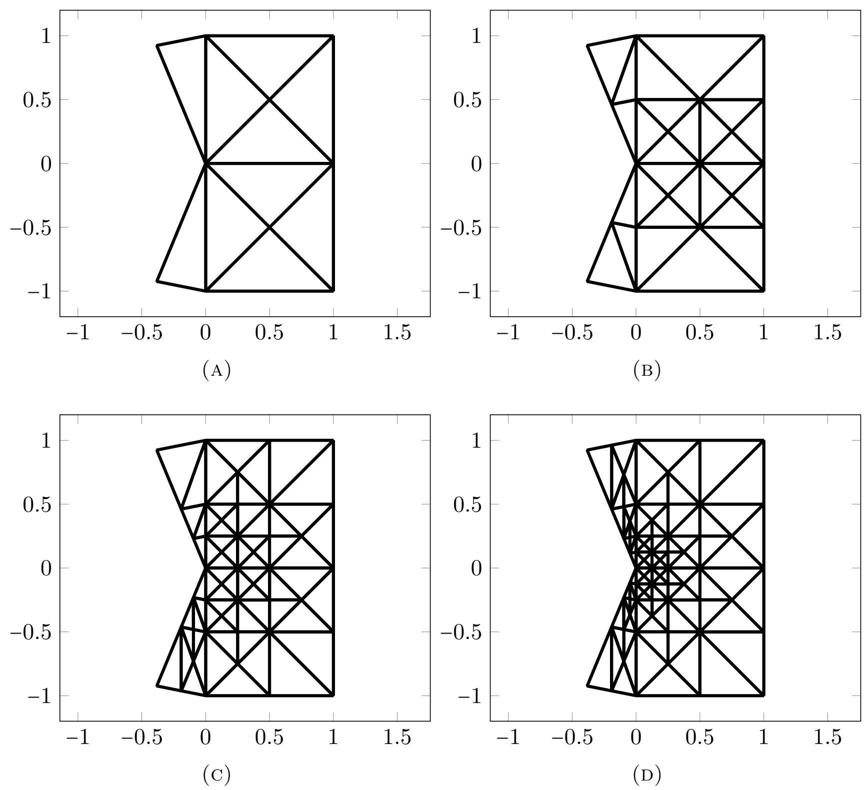

FIGURE 3. Meshes obtained from the adaptive algorithm (iterations $j = 1,2,3,4$) with $\#\mathcal{T} = 10, 28, 67, 108$ elements.

For the boundary conditions we prescribe the values of $u|_\Gamma$ and $\nabla u|_\Gamma$. The parameters α and C are chosen such that u and its normal derivative vanish on the boundary edges that meet at the origin. Here, we have $\alpha \approx 0.673583432147380$ and $C \approx 1.234587795273723$. Note that $u \in H^{2+\alpha-\varepsilon}(\Omega)$ and, selecting $\mathcal{C} = I$,

$$\mathbf{M} = \varepsilon\nabla u \in (H^{\alpha-\varepsilon}(\Omega))^{2\times 2}$$

for $\varepsilon > 0$. Furthermore, one verifies that $|\mathbf{div}\,\mathbf{M}(r,\varphi)| \simeq r^{\alpha-2} \notin L_2(\Omega)$. Therefore, $\mathbf{M} \in H(\mathrm{div}\,\mathbf{div},\Omega)$ and $\mathbf{M} \notin \mathbf{H}(\mathbf{div},\Omega)$ (recall our discussion in Remark 3.1). However, $\mathbf{M} \in \mathcal{H}(\mathrm{div}\,\mathbf{div},\Omega)$ as can be seen as follows. Let E denote one of the boundary edges with end point $(0,0)$. Then, $\mathbf{n}\cdot\mathbf{Mn}|_E \simeq r^{\alpha-1} \in L_2(E)$ and $\mathbf{t}\cdot\mathbf{Mn}|_E = 0$. Moreover, $\mathbf{n}\cdot\mathbf{div}\,\mathbf{M}|_E \simeq r^{\alpha-2} \in (H^1(E))'$. To see the last claim we note that $r^{\alpha-2} \simeq (r^{\alpha-1})'$, where $(\cdot)'$ denotes the generalized derivative operator. In particular, the latter is bounded as a mapping from $L_2(E) \to (H^1(E))'$ (see, e.g., [36, Proof of Lemma 3.5]), and therefore $\|\mathbf{n}\cdot\mathbf{div}\,\mathbf{M}\|_{(H^1(E))'} \lesssim \|r^{\alpha-1}\|_{L_2(E)} < \infty$.

Due to the reduced regularity of $\mathbf{M}$, uniform mesh refinements will lead to a suboptimal convergence order $\mathcal{O}(h^\alpha) = \mathcal{O}(\dim(\mathcal{U}_h)^{-\alpha/2})$. In Figure 4 we plot the DPG error estimator η and the L_2 errors of the field variables in the case of uniform and adaptive mesh refinements. We observe that uniform refinements lead indeed to a suboptimal convergence rate, whereas with our adaptive algorithm the optimal rates $\mathcal{O}(\dim(\mathcal{U}_h)^{-1/2})$ are restored for the error estimator and $\|\mathbf{M}-\mathbf{M}_h\|$.

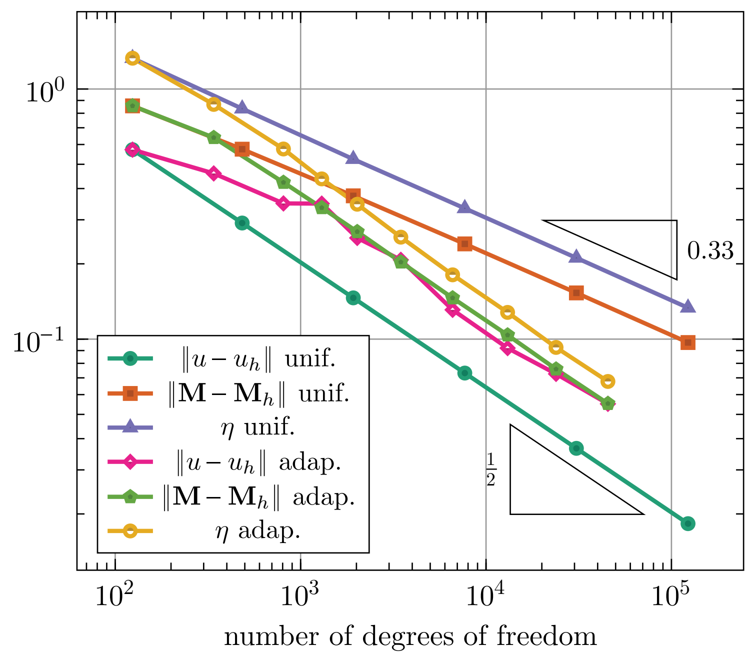

FIGURE 4. Convergence rates of the DPG error estimator for uniformly and adaptively refined meshes

Figure 3 shows meshes obtained from the adaptive algorithm in the iterations $j = 1,2,3,4$. We observe a strong refinement towards the reentrant corner where the (higher order) derivatives of u are singular.

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Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Avenida Vicuña Mackenna 4860, Santiago, Chile
Email address: tofuhrer@mat.uc.cl

Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Avenida Vicuña Mackenna 4860, Santiago, Chile
Email address: nheuer@mat.uc.cl

Structures and Construction Technology Research Unit, Faculty of Technology, University of Oulu, Erkki Koiso-Kanttilan katu 5, Linnanmaa, 90570 Oulu, Finland
Email address: antti.niemi@oulu.fi