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# On traces of functions in $\mathrm{W}^{2,p}(\Omega)$ for Lipschitz domains in $\mathbb{R}^3$

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**Abstract.** We consider the problem of the characterization of the range of the trace operator $(\gamma,\gamma_1):\mathrm{W}^{2,p}(\Omega)\to\mathcal{R}$, $p\in\,]1,\infty[$, defined by the mapping $u\mapsto(u_{|\Gamma},\partial_{\mathbf{n}}u)$, when $\Omega$ is a Lipschitz bounded subset of $\mathbb{R}^3$. $\mathcal{R}$ turns out to be a subspace of $\mathrm{W}^{1,p}(\Gamma)\times\mathrm{L}^p(\Gamma)$. To this aim we need to prove a suitable Hodge decomposition for vector fields belonging to $\mathbf{L}^p(\mathbf{curl},\Omega)$, and also to study some properties of the tangential gradient $\nabla_\Gamma$ on a Lipschitz orientable manifold. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

***Traces de*** $\mathrm{W}^{2,p}(\Omega)$ ***dans de domains lipschitziens en*** $\mathbb{R}^3$

**Résumé.** *On caractérise l'image $\mathcal{R}$ de l'application $u\mapsto(\gamma(u)=u_{|\Gamma},\gamma_1(u)=\partial_{\mathbf{n}}u)$ de $\mathrm{W}^{2,p}(\Omega)$ dans $\mathrm{W}^{1,p}(\Gamma)\times\mathrm{L}^p(\Gamma)$, $p\in\,]1,\infty[$, quand $\Omega$ est ouvert borné lipschitzien de $\mathbb{R}^3$. Pour cela on montre une decomposition de Hodge pour les champs vectoriels de $\mathbf{L}^p(\mathbf{curl},\Omega)$ et on étudie quelque proprietés du gradient tangentiel $\nabla_\Gamma$ sur la surface lipschitzienne.* © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

## *Version française abrégée*

(Les numéros d'équations renvoient à la version anglaise.)

Soit $\Omega$ un ouvert borné de $\mathbb{R}^N$ de frontière $\Gamma$ lipschitzienne et soit $\mathbf{n}$ la normale extérieure. L'application $u\mapsto(\gamma(u)=u_{|\Gamma},\gamma_1(u)=\partial_{\mathbf{n}}\,u)$ est linéare et continue de $\mathrm{W}^{2,p}(\Omega)$ dans $\mathrm{W}^{1,p}(\Gamma)\times\mathrm{L}^p(\Gamma)$ et son image $\mathcal{R}$ est dense dans cet espace [8]. Quand $N=2$ et $p=2$, Geymonat et Krasucki [6] ont caracterisé $\mathcal{R}$ à l'aide de la fonction d'Airy ; la même technique a été utilisée par Duràn et Muschietti [5] dans le cas $N=2$ et $p\in\,]1,\infty[$.

Cette caractérisation peut se reformuler de la façon équivalente suivante :

$$\mathcal{R}=\left\{(g_0,g_1)\in\mathrm{W}^{1,p}(\Gamma)\times\mathrm{L}^p(\Gamma):(\partial_t g_0)\mathbf{t}+g_1\,\mathbf{n}\in\left(\mathrm{W}^{1-1/p,p}(\Gamma)\right)^2\right\},$$

où $\partial_t$ désigne la dérivée tangentielle et $\mathbf{t}$ le vecteur tangent.

**Note présentée par Philippe G. CIARLET.**

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Dans cette Note nous montrons (théorème 5) qu'une caractérisation analogue est valable pour $N=3$ :

$$\mathcal{R}=\left\{(g_0,g_1)\in \mathrm{W}^{1,p}(\Gamma)\times \mathrm{L}^p(\Gamma):\nabla_\Gamma g_0+g_1\,\mathbf{n}\in \left(\mathrm{W}^{1-1/p,p}(\Gamma)\right)^3\right\},$$

où $\nabla_\Gamma$ est le gradient tangentiel défini localement par (1). Dans le cas d'un polyèdre, la condition $\nabla_\Gamma g_0+g_1\,\mathbf{n}\in \left(\mathrm{W}^{1-1/p,p}(\Gamma)\right)^3$ traduit les conditions de compatibilité aux arêtes et aux sommets [7,2].

Pour démontrer le théorème 5, on introduit l'opérateur de projection $\pi_\tau^\Gamma:\mathbf{L}^p(\Gamma)\to \mathbf{L}_t^p(\Gamma)=\{\mathbf{u}\in \mathbf{L}^p(\Gamma):\mathbf{u}\cdot\mathbf{n}=0\}$ défini par $\mathbf{u}\mapsto \mathbf{n}\wedge(\mathbf{u}\wedge\mathbf{n})$ et l'opérateur de trace tangentiel $\pi_\tau$ défini pour $\mathbf{u}$ régulière dans $\Omega$ par $\pi_\tau(\mathbf{u})=\mathbf{n}\wedge(\mathbf{u}_{|\Gamma}\wedge\mathbf{n})$ dont on prolonge la définition à

$$\mathbf{u}\in \mathbf{L}^p(\mathbf{curl},\Omega)=\left\{\mathbf{u}\in \mathbf{L}^p(\Gamma);\ \mathbf{curl}\,\mathbf{u}\in \mathbf{L}^p(\Gamma)\right\}.$$

La démonstration du théorème utilise une décomposition de type Hodge pour les éléments de $\mathbf{L}_0^p(\mathbf{curl},\Omega)=\{\mathbf{u}\in \mathbf{L}^p(\Gamma);\ \mathbf{curl}\,\mathbf{u}\in \mathbf{L}^p(\Gamma),\ \pi_\tau(\mathbf{u})=0\}$ (lemme 6) et un résultat de relèvement pour $f\in \mathbf{W}^{1,p}(\Gamma)$ tel que $\nabla_\Gamma f\in V_\pi^p:=\pi_\tau^\Gamma\{\mathbf{W}^{1-1/p,p}(\Gamma)\}$ (lemme 7).

Nous conjecturons que la caractérisation de $\mathcal{R}$ obtenue est valable pour tout $N$.

## 1. Preliminaries

In [6], by means of the construction of a suitable Airy function, the range of the operator $(\gamma,\gamma_1)$ : $\mathrm{H}^2(\Omega)\to \mathrm{H}^1(\Gamma)\times \mathrm{L}^2(\Gamma)$ is characterized when $\Omega\subset\mathbb{R}^2$ is a Lipschitz domain ($\Gamma$ its boundary). The same technique and result is generalized in [5] for the Sobolev spaces $\mathrm{W}^{2,p}(\Omega)$, $p\in\,]1,\infty[$. Here we tackled the same problem in the three dimensional case (i.e., for the spaces $\mathrm{W}^{2,p}(\Omega)$, $\Omega\subset\mathbb{R}^3$, $p\in\,]1,\infty[$) and a different technique must be used. Related results for polyhedral domains can be found in [7] and [2].

Let $\Omega$ be a Lipschitz bounded subset of $\mathbb{R}^3$, we denote by $\Gamma$ its boundary; $\Gamma$ is orientable and the unit normal vector outward to $\Omega$ is denoted by $\mathbf{n}$. Following [8], $\mathbf{n}$ is defined almost everywhere and $\mathbf{n}\in \mathrm{L}^\infty(\Gamma)$. On $\Omega$, standard Sobolev spaces $\mathrm{W}^{s,p}(\Omega)$ and $\mathrm{W}_0^{s,p}(\Omega)$, $p\in\,]1,+\infty[$, $s>0$, are defined. We denote by $\|\cdot\|_{s,p,\Omega}$ the associated natural norm. Concerning the definition of Sobolev spaces on the boundary $\Gamma$, we follow Nečas [8]. Let $\Delta_r$ be the closed 2D unit square $\Delta=\{0\leqslant x_{r1},\ x_{r2}\leqslant 1\}$ associated to a system of coordinates $(x_{r1},x_{r2},x_{r3})$. There exist $M$ open, regular and connected subsets of $\Gamma$, say $\{\gamma_r\}_r$ such that $\bigcup_r\overline{\gamma_r}=\Gamma$, and $M$ Lipschitz functions $a_r:\Delta_r\to\mathbb{R}$ such that $\overline{\gamma_r}=\{\mathbf{x}=(x_{r1},x_{r2},a_r(x_{r1},x_{r2})),\ (x_{r1},x_{r2})\in\Delta_r\}$. Finally, we denote by $A_r:\mathbb{R}^2\to\mathbb{R}^3$ the application $(x_{r1},x_{r2})\mapsto(x_{r1},x_{r2},a_r(x_{r1},x_{r2}))$.

The spaces $\mathrm{W}^{s,p}(\Gamma)$, $s=[0,1]$, are Banach spaces endowed with the following norms:

$$\|u\|_{s,p,\Gamma}^2=\sum_{r=1}^M\|u\circ A_r\|_{s,p,\Delta_r}^2.$$

Different maps give rise to equivalent norms. The parameterizations $A_r$ induce, in a natural way, two tangent vectors on $\gamma_r$, namely $\mathbf{e}_1=(1,0,\partial_1 a_r(1,0))$, $\mathbf{e}_2=(0,1,\partial_2 a_r(0,1))$ which are not orthogonal, but are independent. We set $g_{ik}=\mathbf{e}_i\cdot\mathbf{e}_k$ for $i,k=1,2$, and $G=\{g_{ik}\}$ the corresponding invertible matrix. We set $G^{-1}=\{g^{ik}\}$ and $g=\det\{G\}$. As in the case of the regular domains, the dual base of tangential vectors reads $\mathbf{e}^i=\sum_{k=1}^2 g^{ik}\mathbf{e}_k$.

We use the boldface to denote the spaces of vector valued functions $\mathbf{v}:\Omega\to\mathbb{R}^3$, e.g., $\mathbf{W}^{s,p}(\Omega)=\left(\mathrm{W}^{s,p}(\Omega)\right)^3$. Moreover, we set:

$$\mathbf{L}^p(\mathbf{curl},\Omega):=\left\{\mathbf{u}\in \mathbf{L}^p(\Omega):\mathbf{curl}\,\mathbf{u}\in \mathbf{L}^p(\Omega)\right\};$$

$$\mathbf{L}_t^p(\Gamma) := \big\{\mathbf{u}\in\mathbf{L}^p(\Gamma) : \mathbf{u}\cdot\mathbf{n}=0\big\}.$$

We denote by $\gamma$ the standard trace operator both for scalar and vector functions, $\gamma : \mathrm{W}^{1,p}(\Omega) \to \mathrm{W}^{1-1/p,p}(\Gamma)$ and $\gamma : \mathbf{W}^{1,p}(\Omega) \to \mathbf{W}^{1-1/p,p}(\Gamma)$.

DEFINITION 1. – We define the operators $\pi_\tau : \mathbf{W}^{1,p}(\Omega) \to \mathbf{L}_t^p(\Gamma)$ and $\pi_\tau^\Gamma : \mathbf{L}^p(\Gamma) \to \mathbf{L}_t^p(\Gamma)$ as $\mathbf{v} \mapsto \mathbf{n}\wedge(\mathbf{v}_{|\Gamma}\wedge\mathbf{n})$, $\mathbf{v}\in\mathbf{W}^{1,p}(\Omega)$, and $\mathbf{u}\mapsto\mathbf{n}\wedge(\mathbf{u}\wedge\mathbf{n})$, $\mathbf{u}\in\mathbf{L}^p(\Gamma)$, respectively.

These operators are linear and continuous, and it holds $\pi_\tau(\mathbf{u}) = (\pi_\tau^\Gamma\circ\gamma)(\mathbf{u})$ for any $\mathbf{u}\in\mathbf{W}^{1,p}(\Omega)$. Moreover, since $\mathbf{L}_t^p(\Gamma)\subseteq\mathbf{L}^p(\Gamma)$, the operator $\pi_\tau^\Gamma$ is also surjective.

Differential operators can be defined over the boundary $\Gamma$.

DEFINITION 2. – We define $\nabla_\Gamma : \mathrm{W}^{1,p}(\Gamma) \to \mathbf{L}_t^p(\Gamma)$ for any $\varphi\in\mathrm{W}^{1,p}(\Gamma)$:

$$(\nabla_\Gamma\varphi)_{|\gamma_r} = \partial_1(\varphi\circ A_r)\mathbf{e}^1 + \partial_2(\varphi\circ A_r)\mathbf{e}^2, \quad \forall r=1,\ldots,M. \tag{1}$$

The invariance of $\mathrm{W}^{1,p}(\Gamma)$ with respect to the choice of the local parameterization ensures that the definition (1) is independent of the choice of $\{A_r\}_r$ (*see* [8]).

Following [4], we set

$$V_\pi^p := \pi_\tau^\Gamma\big\{\mathbf{W}^{1-1/p,p}(\Gamma)\big\}. \tag{2}$$

$V_\pi^p$ is a Banach space (Hilbert for $p=2$) endowed with its natural norm:

$$\|\lambda\|_{V_\pi^p} := \inf\big\{\|\mathbf{u}\|_{1-1/p,p,\Gamma},\ \mathbf{u}\in\mathbf{W}^{1-1/p,p}(\Gamma),\ \pi_\tau^\Gamma(\mathbf{u})=\lambda\big\}.$$

Remark that for general Lipschitz domains, no intrinsic definition (by local maps) of the space $V_\pi^p$ is provided. In the case of smooth surfaces, several equivalent intrinsic definitions of $V_\pi^p$ can actually be given, but unfortunately they do not coincide with (2) in the case of non-smooth boundaries. See [4] for details in the case $p=2$. Using the well known continuous and dense injection $\mathbf{W}^{1-1/p,p}(\Gamma)\hookrightarrow\mathbf{L}^p(\Gamma)$, we immediately deduce that $V_\pi^p\hookrightarrow\mathbf{L}_t^p(\Gamma)$ is a continuous and dense injection.

We need now a preliminary result concerning the tangential trace operator for vector fields belonging to $\mathbf{L}^p(\mathbf{curl},\Omega)$.

PROPOSITION 3. – *Let $p\in]1,\infty[$, and $p'$ be its conjugate exponent $(1/p+1/p'=1)$. Let $\gamma_\tau^\Gamma : \mathbf{L}^p(\Gamma)\to\mathbf{L}_t^p(\Gamma)$, be the operator defined by the mapping $\mathbf{u}\mapsto\mathbf{u}\wedge\mathbf{n}$ for any $\mathbf{u}\in\mathbf{L}^p(\Gamma)$. We set $V_\gamma^p = \gamma_\tau^\Gamma\{\mathbf{W}^{1-1/p,p}(\Gamma)\}$. It is a Banach space endowed with its natural norm. The injection $V_\gamma^p\hookrightarrow\mathbf{L}_t^p(\Gamma)$ is continuous and dense. Let $\big(V_\gamma^p\big)'$ be the dual space of $V_\gamma^p$. It is a Banach space endowed with the induced norm and moreover $\mathbf{L}_t^{p'}(\Gamma)\hookrightarrow\big(V_\gamma^p\big)'$ is continuous and dense. We denote by $\langle\cdot,\cdot\rangle_{V_\gamma^p}$ the corresponding duality pairing, defined by density and by*:

$$\langle\mathbf{u},\mathbf{v}\rangle_{V_\gamma^p} := \int_\Gamma \mathbf{u}\cdot\mathbf{v}, \quad \forall\mathbf{u}\in\mathbf{L}_t^{p'}(\Gamma),\ \mathbf{v}\in\mathbf{L}_t^p(\Gamma).$$

*Then the operator $\pi_\tau$* (see *Definition* 1) *can be extended to a linear and continuous operator from $\mathbf{L}^p(\mathbf{curl},\Omega)$ to $\big(V_\gamma^{p'}\big)'$.*

*Proof.* – Let $p\in]1,\infty[$, $p'$ be the conjugate exponent and $\mathbf{v}\in\mathbf{W}^{1,p'}(\Omega)$. By definition $\gamma_\tau^\Gamma(\gamma(\mathbf{v}))\in V_\gamma^{p'}$, and the following integration by parts holds true [4]:

$$\int_\Omega(\mathbf{curl}\,\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{curl}\,\mathbf{u}) = \int_\Gamma \gamma_\tau^\Gamma\big(\gamma(\mathbf{v})\big)\cdot\pi_\tau(\mathbf{u})$$

for any $\mathbf{u} \in \mathcal{D}(\overline{\Omega})^3$. Using the definition of the duality $\langle\cdot,\cdot\rangle_{V_\gamma^p}$, we have also:

$$\int_\Omega (\mathbf{curl}\,\mathbf{v}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{curl}\,\mathbf{u}) = \big\langle \pi_\tau(\mathbf{u}), \gamma_\tau^\Gamma\big(\gamma(\mathbf{v})\big)\big\rangle_{V_\gamma^{p'}}. \tag{3}$$

By density of $\mathcal{D}(\overline{\Omega})^3$ in $\mathbf{L}^p(\mathbf{curl},\Omega)$, the formula (3) holds true for any $\mathbf{u} \in \mathbf{L}^p(\mathbf{curl},\Omega)$. The statement is then straightforward. $\square$

This allows us to define the space

$$\mathbf{L}_0^p(\mathbf{curl},\Omega) := \big\{\mathbf{u} \in \mathbf{L}^p(\mathbf{curl},\Omega) : \pi_\tau(\mathbf{u}) = 0\big\}.$$

Finally, the following proposition is a consequence of Proposition 3:

PROPOSITION 4. – *Let $\phi \in \mathrm{W}^{2,p}(\Omega)$, then $\pi_\tau(\nabla\phi) \in V_\pi^p$ depends only on the trace of $\phi$ on $\Gamma$ and it holds*:

$$\pi_\tau(\nabla\phi) = \nabla_\Gamma(\phi_{|\Gamma}) \quad \textit{a.e. on } \Gamma. \tag{4}$$

*Moreover $\nabla_\Gamma$ can be extended from $\mathrm{W}^{1-1/p,p}(\Gamma)$ to $\big(V_\gamma^{p'}\big)'$ as a linear and continuous operator still denoted as $\nabla_\Gamma$.*

*Proof.* – We start proving (4). It is enough to prove that the quantity $\pi_\tau(\gamma(\nabla\phi))$ does depend only on the trace of $\phi$ on the boundary $\Gamma$. Then, let $\xi \in \mathrm{W}^{2,p}(\Omega) \cap \mathrm{W}_0^{1,p}(\Omega)$, we have to prove that $\pi_\tau(\nabla\xi) = 0$ almost everywhere on $\Gamma$. Using (3) and the standard integration by parts formula, we have:

$$\int_\Gamma \pi_\tau(\nabla\xi)\cdot\gamma_\tau^\Gamma\big(\gamma(\mathbf{v})\big) = \int_\Omega \mathbf{curl}\,\mathbf{v}\cdot\nabla\xi = \int_\Gamma \mathbf{curl}\,\mathbf{v}\cdot\mathbf{n}\,\xi, \quad \forall \mathbf{v} \in \mathcal{D}(\overline{\Omega})^3.$$

The identity (4) follows from the definition of $\pi_\tau$ and $\nabla_\Gamma$.

Let now $\phi \in \mathrm{W}^{1,p}(\Omega)$, we have that $\nabla\phi \in \mathrm{L}^p(\mathbf{curl},\Omega)$ and, using Proposition 3 we deduce that $\pi_\tau(\nabla\phi) \in \big(V_\gamma^{p'}\big)'$. By density of $\mathrm{W}^{2,p}(\Omega)$ in $\mathrm{W}^{1,p}(\Omega)$, the gradient operator can be extended using (4). $\square$

*Remark* 1. – Equality (4) is then, by definition of $\nabla_\Gamma$, valid for any $\phi \in \mathrm{W}^{1,p}(\Omega)$ and from (3) it holds $V_\pi^p \subseteq \big(V_\gamma^{p'}\big)'$.

## 2. Trace theorem for $\mathbf{W}^{2,p}(\Omega)$

THEOREM 5. – *Let $(\gamma,\gamma_1) : \mathrm{W}^{2,p}(\Omega) \to \mathrm{W}^{1,p}(\Gamma) \times \mathrm{L}^p(\Gamma)$ be the standard trace operator*

$$u \mapsto \big(\gamma(u) = u_{|\Gamma}, \gamma_1(u) = \partial_\mathbf{n} u\big),$$

*where $\partial_\mathbf{n} u = (\nabla u)_{|\Gamma}\cdot\mathbf{n}$. The range of $(\gamma,\gamma_1)$ is characterized as follows*:

$$\mathcal{R} = \big\{(g_0,g_1) \in \mathrm{W}^{1,p}(\Gamma) \times \mathrm{L}^p(\Gamma) \textit{ such that } \nabla_\Gamma g_0 + g_1\,\mathbf{n} \in \mathbf{W}^{1-1/p,p}(\Gamma)\big\}.$$

The proof of this theorem requires few lemmas.

LEMMA 6. – *The following decomposition holds*:

$$\mathbf{L}_0^p(\mathbf{curl},\Omega) = \mathbf{W}_0^{1,p}(\Omega) + \nabla\big(\mathrm{W}_0^{1,p}(\Omega)\big),$$

*where the sum in the previous decomposition is not direct.*

*Proof.* – The proof follows the same steps as the proof of Proposition 4.1 in [3]. Let $\mathcal{O}$ be a regular connected and simply connected open subset of $\mathbb{R}^3$ such that $\overline{\Omega} \subset \mathcal{O}$. We call $\tilde{\cdot}$ the extension by zero outside $\Omega$, $\tilde{\cdot} : \mathrm{L}^p(\Omega) \to \mathrm{L}^p(\mathcal{O})$. Since $\mathbf{u} \in \mathbf{L}^p_0(\mathbf{curl}, \Omega)$ using (3), we easily deduce that $\tilde{\mathbf{u}} \in \mathbf{L}^p(\mathbf{curl}, \mathcal{O})$. Using now standard Hodge decomposition in $\mathcal{O}$:

$$\exists!\, \Psi \in \mathbf{L}^p(\mathbf{curl}, \mathcal{O}),\ \varphi \in \mathrm{W}^{1,p}(\mathcal{O})/\mathbb{R} \text{ such that } \tilde{\mathbf{u}} = \Psi + \nabla\varphi, \quad \mathrm{div}(\Psi) = 0, \quad \Psi \cdot \mathbf{n} = 0 \quad \text{on } \partial\mathcal{O}.$$

The set $\mathcal{O}$ can be chosen as regular as we want and then, by standard regularity theory, we know that $\Psi \in \mathbf{W}^{1,p}(\mathcal{O})$ (*see* [1]). Now, since $\tilde{\mathbf{u}} = 0$ in $\mathcal{O} \setminus \overline{\Omega}$, we deduce $\Psi = -\nabla\varphi$ in $\mathcal{O} \setminus \overline{\Omega}$ and, as a consequence, $\varphi_{|\mathcal{O}\setminus\overline{\Omega}} \in \mathrm{W}^{2,p}(\mathcal{O} \setminus \overline{\Omega})$. The function $\varphi_{|\mathcal{O}\setminus\overline{\Omega}}$ can now be extended in $\Omega$ preserving its regularity according to [8] and we denote by $\varphi_R$ this extension. Then, $\mathbf{u} = (\Psi + \nabla\varphi_R) + \nabla(\varphi - \varphi_R)$ where $\Psi + \nabla\varphi_R \in \mathbf{W}^{1,p}_0(\Omega)$ and $\varphi - \varphi_R \in \mathrm{W}^{1,p}_0(\Omega)$. $\square$

LEMMA 7. – *Let $f \in \mathbf{W}^{1,p}(\Gamma)$ such that $\nabla_\Gamma f \in V^p_\pi$. Then there exists a function $F \in \mathrm{W}^{2,p}(\Omega)$ such that $F_{|\Gamma} \equiv f$.*

*Proof.* – Let $u \in \mathrm{W}^{1,p}(\Omega)$ be any continuous lifting of $f$ on $\Omega$. By Proposition 4, we have $\pi_\tau(\nabla u) = \nabla_\Gamma(u_{|\Gamma}) = \nabla_\Gamma f$. On the other hand, by definition of the space $V^p_\pi$, we know that there exists a vector $\xi \in \mathrm{W}^{1,p}(\Omega)$ such that $\pi_\tau(\xi) = \nabla_\Gamma f$. Immediately we have that $\xi - \nabla u \in \mathbf{L}^p_0(\mathbf{curl}, \Omega)$. Using Lemma 6, we know that $\xi - \nabla u = \Psi - \nabla p$, with $\Psi \in \mathbf{W}^{1,p}_0(\Omega)$ and $p \in \mathrm{W}^{1,p}_0(\Omega)$. Now, let $F := u - p$. It verifies $F_{|\Gamma} = f$ and $F \in \mathrm{W}^{2,p}(\Omega)$. $\square$

*Proof of Theorem* 5. – Given a function $\phi \in \mathrm{W}^{2,p}(\Omega)$. Then, we have $(\nabla\phi)_{|\Gamma} \in \mathbf{W}^{1-1/p,p}(\Gamma)$ and thanks to Proposition 4, it also holds $\nabla_\Gamma(\phi_{|\Gamma}) \in V^p_\pi$. By definition of $\nabla_\Gamma$ and using (4), we have:

$$\nabla_\Gamma \phi_{|\Gamma} + \partial_{\mathbf{n}}\phi\mathbf{n} \equiv \gamma(\nabla\phi) \in \mathbf{W}^{1-1/p,p}(\Gamma).$$

We are now given with $(g_0, g_1)$ belonging to $\mathcal{R}$ and we have to construct a function $\varphi \in \mathrm{W}^{2,p}(\Omega)$ such that $(\gamma(\varphi), \gamma_1(\varphi)) = (g_0, g_1)$. Applying the operator $\pi_\tau$ to the quantity $\nabla_\Gamma g_0 + g_1\mathbf{n}$, we have that $\nabla_\Gamma g_0 \in V^p_\pi$. By means of Lemma 7, we know that there exists a function $G_0 \in \mathrm{W}^{2,p}(\Omega)$ such that $G_{0|\Gamma} = g_0$.

Thanks to the first part of proof, we have now that $\nabla_\Gamma G_0 + \partial_{\mathbf{n}} G_0\mathbf{n} \in \mathbf{W}^{1-1/p,p}(\Gamma)$. By difference we obtain, then

$$\partial_{\mathbf{n}} G_0\mathbf{n} - g_1\mathbf{n} \in \mathbf{W}^{1-1/p,p}(\Gamma).$$

By standard trace theorem, there exists a function $\Psi \in \mathbf{W}^{1,p}(\Omega)$ such that $\Psi_{|\Gamma} = \partial_{\mathbf{n}} G_0\mathbf{n} - g_1\mathbf{n}$. Moreover, by construction $\pi_\tau(\Psi) = 0$ almost everywhere on $\Gamma$. By means of Lemma 6, we know that $\Psi$ can be decomposed in the following way:

$$\Psi = \Xi + \nabla p, \quad \Xi \in \mathbf{W}^{1,p}_0(\Omega), \quad p \in \mathrm{W}^{2,p}(\Omega) \cap \mathrm{W}^{1,p}_0(\Omega).$$

Now, if we take $\varphi = G_0 - p$, it belongs to $\mathrm{W}^{2,p}(\Omega)$ by construction and it is not hard to see that $(\gamma(\varphi), \gamma_1(\varphi)) = (g_0, g_1)$. $\square$

*Remark* 2. – We conjecture that Theorem 5 is still true in $\mathbb{R}^n$, for $n > 3$.

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