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The Cahn-Hilliard/Allen-Cahn equation with inertial and proliferation terms

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This paper is concerned with the initial-boundary value problem for the hyperbolic relaxation of the Cahn-Hilliard/Allen-Cahn equation with a proliferation term, in an arbitrary two dimensional smooth domain. With appropriate assumptions on the nonlinearities, we first prove the existence and uniqueness of an energy solution of the considered problem. Then, establishing the validity of the energy equality we show the regularity (in time) of the energy solution and its continuous dependence on the initial data. Under additional conditions on the nonlinearities, we also prove that the associated semigroup possesses a global attractor.

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1. Introduction

In materials science, rapidly quenching of a high-temperature initially homogeneous mixture of two components, such as binary alloys, glasses or polymer mixtures, to a certain lower critical temperature has a significant role in the design of a new material. Until reaching to its final two-phase state, the system evolves to a pattern formation where spatially extended domains occur, while one of the two mixture components gets enriched (e.g. see [46] for aluminium-zinc and [38–40] for iron-chromium). Based on the initial concentration and final temperature, the process may lead to either partial nucleation or total nucleation, so called spinodal decomposition. In the former case, isolated small regions appear with different concentrations. However, in the latter case, very finely dispersed microstructures quickly occur with distinctly different chemical compositions and then the microstructures coarsen slowly. This more uniform structure enhances the significant mechanical properties of the material, such as strength, hardness and fracture.

As a breakthrough among so many efforts to model the spinodal decomposition phenomena, Cahn and Hilliard [9,8] proposed the Cahn-Hilliard equation as a fundamental diffuse interface model for multi-phase systems. After scaling up all the relevant physical constants to one, it can be written as

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$$u_t - \Delta(-\Delta u + f(u)) = 0, \tag{1.1}$$

where the unknown function u denotes the phase-field order parameter and f is the derivative of a non-convex double-well potential F (e.g. $f(s) = s^3 - s$ where $F(s) = \frac{1}{4}(s^2 - 1)^2$). Due to its interesting features, many authors have widely concentrated on analyzing many mathematical aspects of the original Cahn-Hilliard equation such as the well-posedness, asymptotic behavior of solutions and existence of the attractors of (1.1). It has usually been studied in a bounded and regular domain occupied by the material, together with different types of boundary conditions (see [48,11,14,3,37,51,45,18] and references therein).

In the past years, it has been realized that the Cahn-Hilliard equation may find extensive applications in many various fields in physics, mechanics and biology such as dealloying (see [15]), lithium-ion batteries (see [57]), the rings of Saturn (see [56]), solid tumors (see [31]). For further references about other applications, we can mention [44]. Due to these wide practical usages, many extensions and generalizations of the classical Cahn-Hilliard equation have been studied by many authors. In particular, the generalized Cahn-Hilliard equation

$$u_t - \Delta(-\Delta u + f(u)) + g(x,u) = 0 \tag{1.2}$$

has been considerably analyzed. In the case when g is the linear function $g(s) = \alpha s, \alpha > 0$, the equation (1.2) is known as the Cahn-Hilliard-Oono equation proposed by [49] to take into account long-range interactions in the phase separation. As another possibility for g, the authors of [31] introduced the proliferation term which is the nonlinear function $g(s) = \alpha s(s-1)$, $\alpha > 0$, for biological applications such as wound healing, tumor growth and the clustering of mussels. For the studies on the equation (1.2), under various boundary conditions, in terms of the well-posedness, regularity of solutions and existence of finite dimensional attractors, we refer to [42,43,16,47,41,10,17] and references therein.

On the other hand, if we regard certain materials like a liquid glass after rapidly cooling below their freezing point, we observe that there happens non-equilibrium decomposition without solidification or crystallization (see [1,19–21]). However, the classical Cahn-Hilliard theory is not sufficient to describe this process despite its good agreement with experimental data of the systems with long-range interactions (see [4]). In this case, the hyperbolic relaxation of the Cahn-Hilliard equation

$$\varepsilon u_{tt} + u_t - \Delta(-\Delta u + f(u)) = 0, \tag{1.3}$$

where ε is a relaxation time, is proposed by Galenko et al. (see [19–21]). However, the term εu_{tt}, namely the inertial term, causes the parabolic equation to be hyperbolic, which yields the loss of smoothing (in finite time) property of the solutions. It is possible to gain the smoothing effect by adding a viscosity term $-\Delta u_t$ which is also known as strong damping. With the help of the additional regularity provided by this term, it is easier to handle the hyperbolic Cahn-Hilliard equation with the viscosity term than the equation (1.3). The mathematical studies on the viscous variant of (1.3) under different boundary conditions are very extensive. For the well-posedness and long-time dynamics, we refer to [5,24,28,7] and references therein. On the other hand, in the one dimensional case, the above obstacle can also be annihilated thanks to the Sobolev embedding $H^1 \subset L^\infty$. By this way, the well-posedness and long-time dynamics of (1.3) were studied in [58,59,22,23]. However, the situation yields difficulties in higher dimensions because of the lack of this embedding. In [25], the attractors for the equation (1.3) were studied in the three dimensional case, for very small ε and initial data from a ball whose radius depends on ε. The existence of the global attractors for the quasi-strong solutions and energy solutions of two dimensional (1.3) with the cubic nonlinearity was proved in [26]. Nevertheless, because of the lack of dissipativity, the existence of the global attractor for the weak solutions was left as an open question (see [26, Remark 6.8]). Afterwards, as a continuation of [25] and [26], the authors in [27] established that the attractors for energy and strong solutions in both two- and three-dimensional cases coincide. In addition, by giving a positive answer to the open question about the dissipativity of the weak solutions in the two dimensional case, with additional (in comparison with [26]) regularity conditions as

$$f \in C^4(\mathbb{R}) \text{ and } f''' \in L^\infty(\mathbb{R}), \tag{1.4}$$

they stated the existence of the global attractor for the weak solutions of (1.3). Later, the authors in [35] proved the existence of a global attractor for the weak solutions to (1.3) by imposing, instead of (1.4), the weaker condition on the sub-cubic nonlinearity like the local Hölder continuity of f'' at infinity. Recently, in [32] and [34], the existence of a global attractor was established for (1.3) and some of its variants with quartic nonlinearity.

As another important variant of (1.1), we state the Cahn-Hilliard/Allen-Cahn equation

$$u_t + (-\Delta + I)(-\Delta u + f(u)) = 0, \tag{1.5}$$

which was proposed in [29] as a simplified mesoscopic model describing a pattern formation mechanism for a prototypical model of surface processes that involves multiple microscopic mechanisms. The existence of the weak solutions of (1.5) under Neumann boundary conditions was studied in [30]. Recently, in [33], the existence of a global attractor for the weak solutions of

$$u_{tt} + u_t + (-\Delta + I)(-\Delta u + f(u)) = 0$$

was proved under the conditions

$$f \in C^2(\mathbb{R}), |f''(s)| \le C(1 + |s|), \ \forall s \in \mathbb{R}$$

and

$$\liminf_{|s| \to \infty} f(s)s > 0.$$

The above literature has been related to the Cahn-Hilliard equation and its variants in bounded domains. However, the case of the unbounded domain is more complicated due to the absence of the Sobolev compact embeddings. As a result, it is difficult to obtain an asymptotically compact and dissipative solution semigroup even for energy solutions. For the studies on the Cahn-Hilliard equation and its variants in unbounded domains, we can refer [6,12,13,53] and references therein. In particular, in [6], the author proved the existence of a global attractor with finite Haussdorf dimension for the viscous Cahn-Hilliard equation

$$\partial_t(u - \beta\Delta u) + \nu\Delta^2 u - \Delta(f(u) + \lambda_0 u + g) = 0,$$

where $\beta, \nu > 0, \lambda_0 \ge 0$, in an unbounded domain, by using the weighted Sobolev spaces due to the noncompactness of the operators. The authors of [12] modified the original viscous Cahn-Hilliard equation by adding ϵI to the operator $-\Delta$ as follows:

$$(1 - \nu)u_t = (-\Delta + \epsilon I)((\Delta - \epsilon I)u + f(x, u) - \nu u_t),$$

where $\nu \in [0, 1], \epsilon > 0$. By using the bounded operator $(-\Delta + \epsilon I)^{-1}$ in $L^2(\mathbb{R}^N)$, they proved the existence of global attractors in $H^1(\mathbb{R}^N)$ for the above equation. Afterwards, in [13], the viscous Cahn-Hilliard equation

$$(1 - \nu)u_t = -\Delta(\Delta u + f(x, u) - \nu u_t)$$

was considered, where $\nu \in [0,1)$. The authors of [13] established the global solvability and the existence of a bounded absorbing set, by using some properties of the unbounded operator $(-\Delta)^{-1}$. Furthermore, in [53], the authors proved that the hyperbolic relaxation of the Cahn-Hilliard-Oono equation

$$u_{tt} + u_t + \Delta(\Delta u - f(u) + g) + \alpha u = 0 \tag{1.6}$$

is globally well-posed assuming the sub-quintic nonlinearity

$$|f''(u)| \le C(1 + |u|^{3-\kappa}), \kappa > 0.$$

They also established the dissipativity of solutions and therefore the existence of a smooth global attractor by applying the Strichartz estimates for the linear Schrodinger equation in $\mathbb{R}^3$.

In this paper, we consider the following initial-boundary value problem for the hyperbolic relaxation of the Cahn-Hilliard/Allen-Cahn equation with a proliferation term:

$$\begin{cases} \varepsilon u_{tt} + u_t + (-\Delta + I)(-\Delta u + f(x,u)) + g(x,u) = 0, (t,x) \in (0,\infty) \times \Omega, \\ u(t,x) = \Delta u(t,x) = 0, \quad (t,x) \in (0,\infty) \times \partial\Omega, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \quad x \in \Omega, \end{cases} \tag{1.7}$$

where $\Omega \subset \mathbb{R}^2$ is an arbitrary regular domain and $\varepsilon > 0$.

Our aim is to prove the well-posedness and existence of a global attractor for the energy solutions of (1.7). The motivation of this paper is to fill a gap in the study of the hyperbolic Cahn-Hilliard/Allen-Cahn equation with a proliferation term especially in the case of unbounded domains. It is worth to note that although in [53] the authors studied the well-posedness and existence of a global attractor for the hyperbolic Cahn-Hilliard-Ono equation in unbounded domains (see also [53, Remark 6.1]), since they critically used the fact that the equation (1.6) contains only specific nonlinearity Δf, the method of that article is not applicable to the problem (1.7).

2. Assumptions and results

Let us set $H = L^2(\Omega)$ and define in H the linear operator $A = -\Delta + I$ with $D(A) = H^2(\Omega) \times H_0^1(\Omega)$. It is well known that $A : D(A) \subset H \to H$ is a self-adjoint and strictly positive operator with the bounded inverse, so that we can define, for $s \in \mathbb{R}$, its powers $A^s : D(A^s) \to H$. It is also known that $D(A^{1/2}) = H_0^1(\Omega)$ and $D(A^{-s}) = (D(A^s))'$, for $s \in \mathbb{R}$, where $(D(A^s))'$ is a dual space of $D(A^s)$. Throughout the paper, by the symbol $\langle \cdot, \cdot \rangle$, we will indicate the inner product in H and by $|| \cdot ||$ the induced norm. The symbol $|| \cdot ||_X$ will indicate the norm of X.

Regarding the nonlinear functions f and g, we assume the following conditions:

• $f \in C^1(\overline{\Omega} \times \mathbb{R})$ such that $f(x,0) = 0$, for every $x \in \Omega$, and

$$C(1 + u^2) \ge \frac{\partial f}{\partial u}(x,u) \ge Mu^2 - |K_1(x)|, \tag{2.1}$$

for every $(x,u) \in \Omega \times \mathbb{R}$, where $C > 0$, $M > 0$ and $K_1 \in L^\infty(\Omega)$;

• $g(\cdot,u) : \Omega \to \mathbb{R}$ is measurable for every $u \in \mathbb{R}$ and $g(x,\cdot) \in C^1(\mathbb{R})$, $g(x,0) = 0$, for almost every $x \in \Omega$, such that

$$\left| \frac{\partial g}{\partial u}(x,u) - \lambda \right| \le |K_2(x)|(1 + |u|), \tag{2.2}$$

for every $u \in \mathbb{R}$ and almost every $x \in \Omega$, where $\lambda \in \mathbb{R}$ and $K_2 \in L^\infty(\Omega)$.

Now, we can rewrite the problem (1.7) as the following Cauchy problem:

$$\begin{cases} \varepsilon u_{tt} + u_t + A(Au + F(u)) + G(u) = 0, \quad \forall t > 0, \\ u(0) = u_0, \ u_t(0) = u_1, \end{cases} \tag{2.3}$$

where $F(u) = f(\cdot, u) - u$ and $G(u) = g(\cdot, u)$.

Let us recall the definitions of the energy solution and global attractor.

Definition 2.1. A function $u \in L^\infty(0,T; D(A^{1/2})) \cap W^{1,\infty}(0,T; D(A^{-1/2})) \cap W^{2,\infty}(0,T; D(A^{-3/2}))$ possessing the properties $u(0) = u_0$ and $u_t(0) = u_1$ is called an energy solution of the problem (2.3) on $[0,T)$ if the equation $(2.3)_1$ holds in $D(A^{-3/2})$ for almost every $t \in (0,T)$, where T is an arbitrary positive number.

Definition 2.2. Let $\{V(t)\}_{t\geq 0}$ be a semigroup on a metric space (X,d). A set $\mathcal{A} \subset X$ is called a global attractor for the semigroup $\{V(t)\}_{t\geq 0}$ iff

- $\mathcal{A}$ is a compact set,
- $\mathcal{A}$ is invariant, i.e. $V(t)\mathcal{A} = \mathcal{A}, \forall t \geq 0$,
- $\lim\limits_{t\to\infty} \sup\limits_{v\in B} \inf\limits_{w\in\mathcal{A}} d(V(t)v, w) = 0$ for each bounded set $B \subset X$.

Now, we can state our main results which are as follows:

Theorem 2.1. *Assume that the conditions (2.1)-(2.2) hold. Then for every $(u_0, u_1) \in D(A^{1/2}) \times D(A^{-1/2})$ and $T > 0$, the problem (2.3) admits a unique energy solution $u \in C([0,T]; D(A^{1/2})) \cap C^1([0,T]; D(A^{-1/2})) \cap C^2([0,T]; D(A^{-3/2}))$ which continuously depends on initial data (u_0, u_1).*

Theorem 2.2. *Let, in addition to the conditions (2.1)-(2.2), $\lambda > 0$ and $K_i \in L^{p_i}(\Omega)$ for some $p_i \in [1,\infty)$, $i = 1,2$. Then there exists $\varepsilon_0 = \varepsilon_0(M, \lambda, ||K_1||_{L^{p_1}(\Omega)}, ||K_2||_{L^{p_2}(\Omega)}) > 0$ such that the semigroup generated by the problem (2.3) possesses a global attractor in $D(A^{1/2}) \times D(A^{-1/2})$, for every $\varepsilon \in (0, \varepsilon_0)$.*

3. Well-posedness

We begin with the following existence and uniqueness result.

Lemma 3.1. *Assume that the conditions (2.1)-(2.2) hold and $h \in L^1(0,T; D(A^{-1/2}))$. Then for every $(u_0, u_1) \in D(A^{1/2}) \times D(A^{-1/2})$ and $T > 0$, the initial value problem*

$$\begin{cases} \varepsilon u_{tt} + \alpha u_t + A(Au + F(u)) + G(u) = h(t), \quad t \in (0,T), \\ u(0) = u_0, \ u_t(0) = u_1, \end{cases} \tag{3.1}$$

has a unique energy solution such that

$$\|(u, u_t)\|_{L^\infty(0,T; D(A^{1/2}) \times D(A^{-1/2}))} \leq C(T, \|(u_0, u_1)\|_{D(A^{1/2}) \times D(A^{-1/2})}), \tag{3.2}$$

where $\alpha \in \{0, 1\}$ and $C : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function with respect to each variable.

Proof. By the density of $D(A) \times H$ in $D(A^{1/2}) \times D(A^{-1/2})$, there exists a sequence $\{(u_{0n}, u_{1n})\} \subset D(A) \times H$ such that

$$(u_{0n}, u_{1n}) \to (u_0, u_1) \text{ strongly in } D(A^{1/2}) \times D(A^{-1/2}),$$

as $n \to \infty$. Similarly, since $C_0^\infty((0,T)\times\Omega)$ is dense in $L^1(0,T;D(A^{-1/2}))$, there exists a sequence $\{h_n\} \subset C_0^\infty((0,T)\times\Omega)$ such that

$$h_n \to h \text{ strongly in } L^1(0,T;D(A^{-1/2})),$$

as $n \to \infty$. Let us consider the initial value problem

$$\begin{cases} \varepsilon u_{ntt} + \alpha u_{nt} + A(Au_n + F_n(u_n)) + G_n(u_n) = h_n(t), & t \in (0,T), \\ u_n(0) = u_{0n}, \ u_{nt}(0) = u_{1n}, \end{cases} \tag{3.3}$$

where $F_n(v) = f_n(\cdot,v)$, $G_n(v) = g_n(\cdot,v)$, $f_n(x,v) = \begin{cases} f(x,v) - v, \ |v| \le n, \\ f(x,\frac{v}{|v|}n) - \frac{v}{|v|}n, \ |v| > n \end{cases}$ and $g_n(x,v) = \begin{cases} g(x,v), & |v| \le n, \\ g(x,\frac{v}{|v|}n), & |v| > n \end{cases}$. Denoting $\Lambda(u,v) = (v, -\frac{1}{\varepsilon}A^2u - \frac{\alpha}{\varepsilon}v)$ and $\mathcal{F}_n(t,u,v) = (0, -\frac{1}{\varepsilon}AF_n(u) - \frac{1}{\varepsilon}G_n(u) + \frac{1}{\varepsilon}h_n(t))$, we can rewrite (3.3) as

$$\begin{cases} \frac{d}{dt}(u_n,u_{nt}) = \Lambda(u_n,u_{nt}) + \mathcal{F}_n(t,u_n,u_{nt}), & t \in (0,T), \\ (u_n(0),u_{nt}(0)) = (u_{0n},u_{1n}). \end{cases} \tag{3.4}$$

It is easy to see that the operator $\Lambda : D(A)\times H \subset H \times D(A^{-1}) \to H \times D(A^{-1})$ generates a linear continuous semigroup in $H \times D(A^{-1})$ and $\mathcal{F}_n : [0,T] \times H \times D(A^{-1}) \to H \times D(A^{-1})$ is Lipschitz continuous. Then due to [50, Theorem 1.6, p.189], the problem (3.4) has a unique solution $(u_n,u_{nt}) \in C([0,T];D(A)\times H) \cap C^1([0,T];H\times D(A^{-1}))$. Testing $(3.3)_1$ by $A^{-1}u_{nt}$, we get

$$\begin{aligned} E(u_n(t),u_{nt}(t)) + \int_\Omega \varphi_n(x,u_n(t,x))dx &\le E(u_n(0),u_{nt}(0)) + \int_\Omega \varphi_n(x,u_n(0,x))dx \\ &+ \frac{1}{4}\int_0^t \|u_{nt}(s)\|^2_{D(A^{-1/2})}ds + \int_0^t \|G_n(u_n(s))\|^2_{D(A^{-1/2})}ds \\ &+ \|h_n\|_{L^1(0,t;D(A^{-1/2}))}\|u_{nt}\|_{L^\infty(0,t;D(A^{-1/2}))}, \ \forall t \in [0,T], \end{aligned} \tag{3.5}$$

where $E(v,w) = \frac{\varepsilon}{2}\|w\|^2_{D(A^{-1/2})} + \frac{1}{2}\|v\|^2_{D(A^{1/2})}$ and $\varphi_n(x,v) = \int_0^v f_n(x,w)dw$. By (2.2), we have

$$\|G_n(u_n(s))\|^2_{D(A^{-1/2})} \le \mu\left(\int_\Omega \varphi_n(x,u_n(s,x))dx + \widehat{c}_2\|u_n(s)\|^2\right), \forall \mu \ge \widehat{c}_1, \ \forall n \in \mathbb{Z}^+,$$

for some $\widehat{c}_i > 0$, $i = 1,2$. Then, by (3.5), we find

$$\begin{aligned} H_n(t) \le H_n(0) + \mu\int_0^t &\left(\frac{1}{4\mu}\|u_{nt}(s)\|^2_{D(A^{-1/2})} + \int_\Omega \varphi_n(x,u_n(s,x))dx + \widehat{c}_2\|u_n(s)\|^2\right)ds \\ &+ \|h_n\|_{L^1(0,t;D(A^{-1/2}))}\|u_{nt}\|_{L^\infty(0,t;D(A^{-1/2}))} + \widehat{c}_2\left|\|u_n(t)\|^2 - \|u_n(0)\|^2\right|, \end{aligned}$$

where $H_n(t) = E(u_n(t),u_{nt}(t)) + \int_\Omega \varphi_n(x,u_n(t,x))dx + \widehat{c}_2\|u_n(t)\|^2$. Since,

$$\left|\|u_n(t)\|^2 - \|u_n(0)\|^2\right| \leq \int_0^t \left|\frac{d}{ds}\|u_n(s)\|^2\right| ds \leq \int_0^t \left(\|u_{nt}(s)\|^2_{D(A^{-1/2})} + \|u_n(s)\|^2_{D(A^{1/2})}\right) ds,$$

choosing $\mu = \max\left\{\widehat{c}_1, \frac{1}{\varepsilon}, \frac{4\widehat{c}_2}{\varepsilon}, 2\widehat{c}_2\right\}$, we infer

$$H_n(t) \leq H_n(0) + \mu \int_0^t H_n(s)ds + \|h_n\|_{L^1(0,t;D(A^{-1/2}))} \|H_n\|^{1/2}_{L^\infty(0,t)},$$

and consequently

$$L_n(t) \leq 2H_n(0) + 2\mu \int_0^t L_n(s)ds + \|h_n\|^2_{L^1(0,T;D(A^{-1/2}))}, \ \forall t \in [0,T], \forall n \in \mathbb{Z}^+,$$

where $L_n(t) = \|H_n\|_{L^\infty(0,t)}$. Applying Gronwall's lemma, we obtain

$$L_n(t) \leq \widehat{c}_3, \ \forall t \in [0,T], \forall n \in \mathbb{Z}^+,$$

which yields that

$$\|u_{nt}(t)\|_{D(A^{-1/2})} + \|u_n(t)\|_{D(A^{1/2})} \leq \widehat{c}_4, \ \forall t \in [0,T], \forall n \in \mathbb{Z}^+,$$

where the positive constants $\widehat{c}_3$ and $\widehat{c}_4$ depend on T and $\|(u_0,u_1)\|_{D(A^{1/2})\times D(A^{-1/2})}$. Coming back to $(3.3)_1$, with the help of the last estimate, we get

$$\|u_{ntt}(t)\|_{D(A^{-3/2})} \leq \widehat{c}_5, \ \forall t \in [0,T], \forall n \in \mathbb{Z}^+.$$

Hence, there exist a subsequence $\{u_{n_k}\}$ and a function $u \in L^\infty(0,T;D(A^{1/2})) \cap W^{1,\infty}(0,T;D(A^{-1/2})) \cap W^{2,\infty}(0,T;D(A^{-3/2}))$ such that

$$\begin{cases} u_{n_k} \to u \text{ weakly star in } L^\infty(0,T;D(A^{1/2})), \\ u_{n_k t} \to u_t \text{ weakly star in } L^\infty(0,T;D(A^{-1/2})), \\ u_{n_k tt} \to u_{tt} \text{ weakly star in } L^\infty(0,T;D(A^{-3/2})), \end{cases}$$

as $k \to \infty$. Thus, passing to the limit in (3.3), we conclude that $u(t)$ is the energy solution of (3.1). Also, using the lower semicontinuity of the norms in the penultimate inequality, we obtain (3.2).

Now, let us prove the uniqueness of an energy solution. Assume that $v(t)$ is also an energy solution of (3.1). Then, denoting $w = u - v$, we have

$$\begin{cases} \varepsilon w_{tt} + \alpha w_t + A(Aw + F(u) - F(v)) + G(u) - G(v) = 0, \quad t \in (0,T), \\ w(0) = 0, \ w_t(0) = 0. \end{cases} \tag{3.6}$$

Testing $(3.6)_1$ by $A^{-2}w_t$, we get

$$\widehat{E}(w(t), w_t(t)) \leq \widehat{c}_6 \int_0^t \widehat{E}(w(s), w_t(s))\, ds + \int_0^t \left|\left\langle F(u(s)) - F(v(s)), A^{-1}w_t(s)\right\rangle\right| ds, \tag{3.7}$$

where $\widehat{E}(u,v) = \frac{1}{2}\|u\|^2 + \frac{\varepsilon}{2}\|v\|^2_{D(A^{-1})}$. Let us set $\Omega_m(s) = \{x \in \Omega : |u(s,x)| + |v(s,x)| > m\}$ for $m > 0$. Then, with the help of (2.1), we obtain

$$\left|\left\langle F(u(s)) - F(v(s)), A^{-1}w_t(s)\right\rangle\right| = \left|\left\langle \left(\int\limits_0^1 \frac{\partial}{\partial u} f(\cdot, \tau w(s) + v(s))d\tau - 1\right) w(s), A^{-1}w_t(s)\right\rangle\right|$$
$$\leq \widehat{c}_7 meas^{1/2}(\Omega_m(s)) \left\|\left(1 + (|u(s)| + |v(s)|)^2\right) w(s)\right\|_{L^4(\Omega_m(s))} \left\|A^{-1}w_t(s)\right\|_{L^4(\Omega_m(s))}$$
$$+\widehat{c}_7 \left(1 + m^2\right) \|w(s)\| \|w_t(s)\|_{D(A^{-1})} \leq \widehat{c}_8 meas^{1/2}(\Omega_m(s)) \left(1 + \|u\|^3_{L^\infty(0,T;D(A^{1/2}))}\right.$$
$$\left. + \|v\|^3_{L^\infty(0,T;D(A^{1/2}))}\right) \left(\|u_t\|_{L^\infty(0,T;D(A^{-1/2}))} + \|v_t\|_{L^\infty(0,T;D(A^{-1/2}))}\right)$$
$$+\widehat{c}_7 \left(1 + m^2\right) \|w(s)\| \|w_t(s)\|_{D(A^{-1})}$$
$$\leq \widehat{c}_9 meas^{1/2}(\Omega_m(s)) + \widehat{c}_7 \left(1 + m^2\right) \|w(s)\| \|w_t(s)\|_{D(A^{-1})}, \ \forall m > 0, \tag{3.8}$$

where $\widehat{c}_9$ depends on $\max\left\{\|(u, u_t)\|_{L^\infty(0,T;D(A^{1/2})\times D(A^{-1/2}))}, \|(v, v_t)\|_{L^\infty(0,T;D(A^{1/2})\times D(A^{-1/2}))}\right\}$.

On the other hand, by Trudinger-Moser inequality (see [52]), there exists $\beta = \beta\left(\max\left\{\|u\|_{L^\infty(0,T;D(A^{1/2}))}, \|v\|_{L^\infty(0,T;D(A^{1/2}))}\right\}\right) > 0$ such that

$$meas(\Omega_m(s)) \leq (e^{\beta m^2} - 1)^{-1} \int\limits_{\Omega_m(s)} \left(e^{\beta(|u(s,x)|+|v(s,x)|)^2} - 1\right) dx$$
$$\leq (e^{\beta m^2} - 1)^{-1} \int\limits_{\Omega} \left(e^{\beta(|u(s,x)|+|v(s,x)|)^2} - 1\right) dx \leq \widehat{c}_{10}(e^{\beta m^2} - 1)^{-1}, \quad \forall m > 0, \tag{3.9}$$

for some $\widehat{c}_{10} > 0$, where $\beta : (0, \infty) \to (0, \infty)$ is a decreasing function. Taking into account (3.8) and (3.9) in (3.7), we find

$$\widehat{E}\left(w(t), w_t(t)\right) \leq \widehat{c}_{11}\left(1 + m^2\right) \int\limits_0^t \widehat{E}\left(w(s), w_t(s)\right) ds + \widehat{c}_{11}(e^{\beta m^2} - 1)^{-1/2}, \quad \forall m > 0.$$

Hence, applying Gronwall's lemma, we get

$$\left\|\widehat{E}\left(w, w_t\right)\right\|_{L^\infty(0,\widehat{t})} \leq \widehat{c}_{11} e^{\widehat{c}_{11}(1+m^2)\widehat{t}}(e^{\beta m^2} - 1)^{-1/2}, \ \forall m > 0.$$

Choosing $\widehat{t} = \min\left\{\frac{\beta}{3\widehat{c}_{11}}, T\right\}$ and passing to the limit as $m \to \infty$ in the last inequality, we obtain

$$\left\|\widehat{E}\left(w, w_t\right)\right\|_{L^\infty(0,\widehat{t})} = 0,$$

which implies the uniqueness of an energy solution. □

Lemma 3.2. *Let the condition (2.1) hold and* $u \in L^\infty(0, T; D(A^{1/2})) \cap W^{1,\infty}(0, T; D(A^{-1/2})) \cap W^{2,\infty}(0, T; D(A^{-3/2}))$ *be an energy solution of*

$$\varepsilon u_{tt} + A(Au + \widehat{F}(u)) = h(t), \quad t \in (0, T), \tag{3.10}$$

with $h \in L^1(0, T; D(A^{-1/2}))$, $\widehat{F}(u) = \widehat{f}(\cdot, u) - u$ *and* $\widehat{f}(x, u) = f(x, u) + (1 + ||K_1||_{L^\infty(\Omega)})u$. *Then for every* $\tau, t \in [0, T]$, *the equality*

$$E\left(u(t), u_t(t)\right) + \frac{1}{2}||K_1||_{L^\infty(\Omega)} \|u(t)\|^2 + \int\limits_\Omega \Phi(x, u(t, x))dx = E\left(u(\tau), u_t(\tau)\right)$$

$$+\frac{1}{2}||K_1||_{L^\infty(\Omega)}\left\|u(\tau)\right\|^2+\int\limits_\Omega \Phi(x,u(\tau,x))dx+\int\limits_\tau^t \left\langle A^{-1/2}h(s),A^{-1/2}u_t(s)\right\rangle ds \tag{3.11}$$

holds, where $\Phi(x,u)=\int\limits_0^u f(x,v)dv$.

Proof. By the embedding theorems (see, for example, [54, Corollary 4]) and Strauss lemma (see [55] and [36, Lemma 8.1, p.275]), we have $u\in C_s(0,T;D(A^{1/2}))$ and $u_t\in C_s(0,T;D(A^{-1/2}))$, where $C_s(0,T;D(A^\nu))=\{\varphi\in L^\infty(0,T;D(A^\nu)):\langle A^\nu\varphi,A^{-\nu}\psi\rangle\in C\left[0,T\right],\text{ for every }\psi\in D(A^{-\nu})\}$, with $\nu\in\left\{-\frac{1}{2},\frac{1}{2}\right\}$. Then $u(t)$ can be considered as the energy solution of (3.10) on (τ,T) with the initial data $(u(\tau),u_t(\tau))\in D(A^{1/2})\times D(A^{-1/2})$, for $\tau\in[0,T)$. It is easy to see that $\widehat{f}$, as f, satisfies (2.1). Hence, applying Lemma 3.1, with $\alpha=0$, $G=0$ and $\widehat{f}$ instead of f, we conclude that the energy solution $u(t)$ can be approximated by $u_n\in C([\tau,T];D(A))\cap C^1\left([\tau,T];H\right)\cap C^2\left([\tau,T];D(A^{-1})\right)$, which is the solution of

$$\begin{cases}\varepsilon u_{ntt}+A(Au_n+\widehat{F}_n(u_n))=h_n(t),\quad t\in(\tau,T),\\ \qquad u_n(\tau)=u_{0n},\ u_{nt}(\tau)=u_{1n},\end{cases} \tag{3.12}$$

where $h_n\in C_0^\infty((0,T)\times\Omega)$, $\widehat{F}_n(u)=\widehat{f}_n(\cdot,u)$, $\widehat{f}_n(x,u)=\begin{cases}f(x,v)+||K_1||_{L^\infty(\Omega)}v, & |v|\le n\\ f(x,\frac{v}{|v|}n)+||K_1||_{L^\infty(\Omega)}\frac{v}{|v|}n, & |v|>n\end{cases}$, $\tau\in[0,T)$ and

$$\begin{cases}h_n\to h\text{ strongly in }L^1(0,T;D(A^{-1/2})),\\ u_{0n}\to u(\tau)\text{ strongly in }D(A^{1/2}),\\ u_{1n}\to u_t(\tau)\text{ strongly in }D(A^{-1/2}),\end{cases} \tag{3.13}$$

as $n\to\infty$. As shown in the proof of Lemma 3.1, we have

$$\begin{cases}u_n\to u\text{ weakly star in }L^\infty(\tau,T;D(A^{1/2})),\\ u_{nt}\to u_t\text{ weakly star in }L^\infty(\tau,T;D(A^{-1/2})),\\ u_{ntt}\to u_{tt}\text{ weakly star in }L^\infty(\tau,T;D(A^{-3/2})),\end{cases} \tag{3.14}$$

as $n\to\infty$. From $(3.13)_2$, $(3.13)_3$ and (3.14), it also follows that

$$\begin{cases}u_n(t)\to u(t)\text{ weakly in }D(A^{1/2})\\ u_{nt}(t)\to u_t(t)\text{ weakly in }D(A^{-1/2})\end{cases},\ \forall t\in(\tau,T], \tag{3.15}$$

as $n\to\infty$. Testing $(3.12)_1$ by $A^{-1}u_{nt}$, we get

$$E\left(u_n(t),u_{nt}(t)\right)+\int\limits_\Omega \widehat{\varphi}_n(x,u_n(t,x))dx=E\left(u_{0n},u_{1n}\right)$$

$$+\int\limits_\Omega \widehat{\varphi}_n(x,u_n(\tau,x))dx+\int\limits_\tau^t \left\langle A^{-1/2}h_n(s),A^{-1/2}u_{nt}(s)\right\rangle ds,\ \forall t\in[\tau,T], \tag{3.16}$$

where $\widehat{\varphi}_n(x,u)=\int\limits_0^u \widehat{f}_n(x,v)dv$. From (2.1), it immediately follows that

$$\widehat{\varphi}_n(x,u)\ge 0,\ \text{for almost every }x\in\Omega\ \text{ and every }u\in\mathbb{R}.$$

Hence, applying Fatou's lemma, with the help of $(3.15)_1$, one can deduce that

$$\liminf_{n\to\infty}\int_\Omega \widehat{\varphi}_n(x,u_n(t,x))dx \geq \int_\Omega \Phi(x,u(t,x))dx + \frac{1}{2}||K_1||_{L^\infty(\Omega)}\left\|u(t)\right\|^2,\ \forall t\in(\tau,T]. \tag{3.17}$$

Also, with the help of $(3.13)_2$, we have

$$\lim_{n\to\infty}\int_\Omega \widehat{\varphi}_n(x,u_n(\tau,x))dx = \int_\Omega \Phi(x,u(\tau,x))dx + \frac{1}{2}||K_1||_{L^\infty(\Omega)}\left\|u(\tau)\right\|^2. \tag{3.18}$$

Thus, taking into account (3.13), $(3.14)_2$, (3.15), (3.17), (3.18) and passing to the limit in (3.16), we obtain

$$\begin{aligned}&E\left(u(t),u_t(t)\right)+\frac{1}{2}||K_1||_{L^\infty(\Omega)}\left\|u(t)\right\|^2+\int_\Omega \Phi(x,u(t,x))dx \leq E\left(u(\tau),u_t(\tau)\right)\\&+\frac{1}{2}||K_1||_{L^\infty(\Omega)}\left\|u(\tau)\right\|^2+\int_\Omega \Phi(x,u(\tau,x))dx+\int_\tau^t\left\langle A^{-1/2}h(s),A^{-1/2}u_t(s)\right\rangle ds,\ \forall t\in[\tau,T].\end{aligned} \tag{3.19}$$

Let $v(s)=u(t+\tau-s)$, $s\in[\tau,t]$. It is easy to see that $v(s)$ is an energy solution of (3.10) on (τ,t), with $h(t+\tau-s)$ instead of $h(s)$. Then, by (3.19), we get

$$\begin{aligned}&E\left(v(t),v_s(t)\right)+\frac{1}{2}||K_1||_{L^\infty(\Omega)}\left\|v(t)\right\|^2+\int_\Omega \Phi(x,v(t,x))dx \leq E\left(v(\tau),v_s(\tau)\right)\\&+\frac{1}{2}||K_1||_{L^\infty(\Omega)}\left\|v(\tau)\right\|^2+\int_\Omega \Phi(x,v(\tau,x))dx+\int_\tau^t\left\langle A^{-1/2}h(t+\tau-\sigma),A^{-1/2}v_s(\sigma)\right\rangle d\sigma,\end{aligned}$$

and consequently

$$\begin{aligned}&E\left(u(\tau),u_t(\tau)\right)+\frac{1}{2}||K_1||_{L^\infty(\Omega)}\left\|u(\tau)\right\|^2+\int_\Omega \Phi(x,u(\tau,x))dx \leq E\left(u(t),u_t(t)\right)\\&+\frac{1}{2}||K_1||_{L^\infty(\Omega)}\left\|u(t)\right\|^2+\int_\Omega \Phi(x,u(t,x))dx-\int_\tau^t\left\langle A^{-1/2}h(s),A^{-1/2}u_t(s)\right\rangle ds.\end{aligned}$$

The last inequality, together with (3.19), yields (3.11). □

Lemma 3.3. *Under the conditions (2.1) and (2.2), the problem (2.3) admits a unique energy solution* $u\in C([0,T];D(A^{1/2}))\cap C^1\left([0,T];D(A^{-1/2})\right)\cap C^2([0,T];D(A^{-3/2}))$, *for every* $T>0$.

Proof. Due to Lemma 3.1, the problem (2.3) has a unique energy solution $u\in L^\infty(0,T;D(A^{1/2}))\cap W^{1,\infty}(0,T;D(A^{-1/2}))\cap W^{2,\infty}(0,T;D(A^{-3/2}))$. Denoting $h(t)=-u_t(t)-G(u(t))+||K_1||_{L^\infty(\Omega)}Au(t)$ $+Au(t)$ and applying Lemma 3.2, we obtain

$$\begin{aligned}&E\left(u(t),u_t(t)\right)+\frac{1}{2}||K_1||_{L^\infty(\Omega)}\left\|u(t)\right\|^2+\int_\Omega \Phi(x,u(t,x))dx = E\left(u(\tau),u_t(\tau)\right)\\&+\frac{1}{2}||K_1||_{L^\infty(\Omega)}\left\|u(\tau)\right\|^2+\int_\Omega \Phi(x,u(\tau,x))dx+\int_\tau^t\left\langle A^{-1/2}h(s),A^{-1/2}u_t(s)\right\rangle ds,\end{aligned}$$

and consequently

$$E\left(u(t),u_t(t)\right)-\frac{1}{2}\left\|u(t)\right\|^2+\int\limits_{\Omega}\Phi(x,u(t,x))dx$$
$$+\int\limits_{\tau}^{t}\|u_t(s)\|^2_{D(A^{-1/2})}\,ds+\int\limits_{\tau}^{t}\left\langle A^{-1/2}G(u(s)),A^{-1/2}u_t(s)\right\rangle ds$$
$$=E\left(u(\tau),u_t(\tau)\right)-\frac{1}{2}\left\|u(\tau)\right\|^2+\int\limits_{\Omega}\Phi(x,u(\tau,x))dx,\ \forall t,\tau\in[0,T]. \tag{3.20}$$

Let $\{t_n\}\subset[0,T]$ and $t_n\to t_0$. Since, by the definition of energy solutions and embedding theorems, $u\in C_s(0,T;D(A^{1/2}))\cap C([0,T];H)$ and $u_t\in C_s(0,T;D(A^{-1/2}))$, we have

$$\begin{cases} u(t_n)\to u(t_0) \text{ weakly in } D(A^{1/2}),\\ u(t_n)\to u(t_0) \text{ strongly in } H,\\ u_t(t_n)\to u_t(t_0) \text{ weakly in } D(A^{-1/2}),\end{cases} \tag{3.21}$$

as $n\to\infty$. By (2.1), $(3.21)_1$ and $(3.21)_2$, we infer

$$\lim_{n\to\infty}\int\limits_{\Omega}\Phi(x,u(t_n,x))dx=\int\limits_{\Omega}\Phi(x,u(t_0,x))dx.$$

Hence, putting t_n and t_0 instead of t and τ in (3.20), respectively, and passing to the limit, we get

$$\lim_{n\to\infty}E\left(u(t_n),u_t(t_n)\right)=E\left(u(t_0),u_t(t_0)\right),$$

which, together with (3.21), yields

$$\begin{cases} u(t_n)\to u(t_0) \text{ strongly in } D(A^{1/2}),\\ u_t(t_n)\to u_t(t_0) \text{ strongly in } D(A^{-1/2}),\end{cases}$$

as $n\to\infty$. Thus, we have $u\in C([0,T];D(A^{1/2}))\cap C^1\left([0,T];D(A^{-1/2})\right)$. Taking it into account in $(2.3)_1$, we also obtain that $u\in C^2([0,T];D(A^{-3/2}))$. $\square$

By Lemma 3.3, the energy solutions of the problem (2.3) generate a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$, by the formula $S(t)(u_0,u_1)=(u(t),u_t(t))$.

Now, to complete the proof of Theorem 2.1, we need the following lemma.

Lemma 3.4. *Let the conditions (2.1) and (2.2) hold. Also, assume that*

$$(u_{0n},u_{1n})\to(u_0,u_1) \textit{ strongly in } D(A^{1/2})\times D(A^{-1/2}), \tag{3.22}$$

as $n\to\infty$. Then for every $T>0$, we have

$$S(\cdot)(u_{0n},u_{1n})\to S(\cdot)(u_0,u_1) \textit{ strongly in } C([0,T];D(A^{1/2})\times D(A^{-1/2})), \tag{3.23}$$

as $n\to\infty$.

Proof. Let $(u_n(t), u_{nt}(t)) = S(t)(u_{0n}, u_{1n})$ and $(u(t), u_t(t)) = S(t)(u_0, u_1)$. By (3.22), there exists a subsequence $\{(u_{0n_m}, u_{1n_m})\} \subset \{(u_{0n}, u_{1n})\}$ such that

$$\|u_{0n_m} - u_0\|^2_{D(A^{1/2})} + \|u_{1n_m} - u_1\|^2_{D(A^{-1/2})} \le e^{-m^3}, \forall m \in \mathbb{Z}^+. \tag{3.24}$$

Setting $w_m = u_{n_m}(t) - u(t)$, by (2.3), we have

$$\begin{cases} \varepsilon w_{mtt} + w_{mt} + A(Aw_m + F(u_{n_m}) - F(u)) + G(u_{n_m}) - G(u) = 0, \\ w_m(0) = u_{0n_m} - u_0, \ w_{mt}(0) = u_{1n_m} - u_1. \end{cases} \tag{3.25}$$

Testing $(3.25)_1$ by $A^{-2}w_{mt}$ and arguing as in the proof of the uniqueness part of Lemma 3.1, we conclude that

$$\widehat{E}\left(w_m(t), w_{mt}(t)\right) \le \widehat{E}\left(w_m(0), w_{mt}(0)\right) + \widetilde{c}(1+m^2)\int_0^t \widehat{E}\left(w_m(\tau), w_{mt}(\tau)\right) d\tau$$

$$+\widetilde{c}(e^{\beta m^2} - 1)^{-1/2}, \ \forall t \in [0,T], \ \forall m \in \mathbb{Z}^+,$$

for some $\widetilde{c} > 0$ and $\beta > 0$. Applying Gronwall's lemma, we get

$$\widehat{E}\left(w_m(t), w_{mt}(t)\right) \le \widehat{E}\left(w_m(0), w_{mt}(0)\right) e^{\widetilde{c}(1+m^2)t} + \frac{\widetilde{c}e^{\widetilde{c}(1+m^2)t}}{(e^{\beta m^2} - 1)^{1/2}},$$

which, together with (3.24), implies that

$$\lim_{m\to\infty} \max_{0\le t\le t^*} \widehat{E}\left(w_m(t), w_{mt}(t)\right) = 0, \tag{3.26}$$

where $t^* = \min\left\{\frac{\beta}{3\widetilde{c}}, T\right\}$. From (3.26), it follows that

$$S(\cdot)(u_{0n_m}, u_{1n_m}) \to S(\cdot)(u_0, u_1) \text{ strongly in } C([0,t^*]; H \times D(A^{-1})),$$

as $m \to \infty$. By the same way, one can show that every subsequence of $\{S(\cdot)(u_{0n}, u_{1n})\}$ has a subsequence strongly convergent to $S(\cdot)(u_0, u_1)$ in $C([0,t^*]; H \times D(A^{-1}))$. Hence,

$$S(\cdot)(u_{0n}, u_{1n}) \to S(\cdot)(u_0, u_1) \text{ strongly in } C([0,t^*]; H \times D(A^{-1})), \tag{3.27}$$

as $n \to \infty$. On the other hand, by (3.2), the sequence $\{S(\cdot)(u_{0n}, u_{1n})\}$ is bounded in $L^\infty(0,T; D(A^{1/2}) \times D(A^{-1/2}))$. Then, with the help of (3.27), we get

$$\begin{cases} u_n \to u \text{ weakly star in } L^\infty(0,t^*; D(A^{1/2})), \\ u_{nt} \to u_t \text{ weakly star in } L^\infty(0,t^*; D(A^{-1/2})), \\ u_n(t) \to u(t) \text{ weakly in } D(A^{1/2}), \forall t \in [0,t^*], \\ u_n(t) \to u(t) \text{ strongly in } H, \ \forall t \in [0,t^*], \\ u_{nt}(t) \to u_t(t) \text{ weakly in } D(A^{-1/2}), \forall t \in [0,t^*], \end{cases} \tag{3.28}$$

as $n \to \infty$. Therefore, putting u_n in place of u in (3.20) with $\tau = 0$ and passing to the limit, with the help of (3.22) and (3.28), we obtain

$$\limsup_{n\to\infty} E\left(u_n(t), u_{nt}(t)\right) \le E\left(u(t), u_t(t)\right), \ \forall t \in [0,t^*],$$

which, together with $(3.28)_3$ and $(3.28)_5$, yields that

$$S(t)(u_{0n}, u_{1n}) \to S(t)(u_0, u_1) \text{ strongly in } D(A^{1/2}) \times D(A^{-1/2}),\ \forall t \in [0, t^*], \tag{3.29}$$

as $n \to \infty$. Now, let us show that

$$S(\cdot)(u_{0n}, u_{1n}) \to S(\cdot)(u_0, u_1) \text{ strongly in } C([0, t^*]; D(A^{1/2}) \times D(A^{-1/2})), \tag{3.30}$$

as $n \to \infty$. Suppose the opposite. Then there exist $\delta > 0$, a subsequence $\{(u_{0n_m}, u_{1n_m})\}$, a sequence $\{t_m\} \subset [0, t^*]$ and $t_0 \subset [0, t^*]$ such that

$$t_m \to t_0$$

and

$$\|S(t_m)(u_{0n_m}, u_{1n_m}) - S(t_m)(u_0, u_1)\|_{D(A^{1/2}) \times D(A^{-1/2})} \geq \delta.$$

Since, $S(t)(u_0, u_1)$ is a continuous function in $D(A^{1/2}) \times D(A^{-1/2})$, passing to the limit, we find

$$\liminf_{m\to\infty} \|S(t_m)(u_{0n_m}, u_{1n_m}) - S(t_0)(u_0, u_1)\|_{D(A^{1/2}) \times D(A^{-1/2})} \geq \delta. \tag{3.31}$$

On the other hand, thanks to (3.27), we have

$$S(t_m)(u_{0n_m}, u_{1n_m}) \to S(t_0)(u_0, u_1) \text{ strongly in } H \times D(A^{-1}), \tag{3.32}$$

which, together with the boundedness of $\{S(t_m)(u_{0n_m}, u_{1n_m})\}$ in $D(A^{1/2}) \times D(A^{-1/2})$, yields

$$S(t_m)(u_{0n_m}, u_{1n_m}) \to S(t_0)(u_0, u_1) \text{ weakly in } D(A^{1/2}) \times D(A^{-1/2}), \tag{3.33}$$

as $m \to \infty$. Hence, putting u_{n_m}, t_m and t_0 instead of u, t and τ in (3.20), respectively, and passing to the limit, with the help of (3.29), (3.32) and (3.33), we obtain

$$\lim_{m\to\infty} E\left(u_{n_m}(t_m), u_{n_m t}(t_m)\right) = E\left(u(t_0), u_t(t_0)\right).$$

The last equality and (3.33) give us

$$S(t_m)(u_{0n_m}, u_{1n_m}) \to S(t_0)(u_0, u_1) \text{ strongly in } D(A^{1/2}) \times D(A^{-1/2}),$$

which contradicts (3.31).

Now, taking t^* in place of 0 and repeating the procedure leading to (3.30), we conclude that

$$S(\cdot)(u_{0n}, u_{1n}) \to S(\cdot)(u_0, u_1) \text{ strongly in } C([t^*, t^* + t^{**}]; D(A^{1/2}) \times D(A^{-1/2})),$$

which, together with (3.30), yields

$$S(\cdot)(u_{0n}, u_{1n}) \to S(\cdot)(u_0, u_1) \text{ strongly in } C([0, t^* + t^{**}]; D(A^{1/2}) \times D(A^{-1/2})),$$

as $n \to \infty$, where $t^{**} = \min\{t^*, T - t^*\}$. Therefore, by repeating this procedure a finite number of times, we get (3.23). □

Thus, Lemma 3.3 and Lemma 3.4 complete the proof of Theorem 2.1.

4. Existence of the global attractor

To prove the dissipativity of the energy solutions, we need the following lemma.

Lemma 4.1. *In addition to the conditions (2.1) and (2.2), assume that $K_i \in L^{p_i}(\Omega)$, for some $p_i \in [1,\infty)$, $i = 1, 2$. Then for every $\delta > 0$ there exists $C_\delta > 0$ such that*

$$\left\langle A^{-1/2}G(u), A^{-1/2}u \right\rangle \geq \lambda \|u\|^2_{D(A^{-1/2})} - \delta \left(\|u\|^2 + \|u\|^4_{L^4(\Omega)} \right) - C_\delta, \tag{4.1}$$

$$\langle F(u) + u, u \rangle \geq \frac{M}{3} \|u\|^4_{L^4(\Omega)} - \delta \left(\|u\|^2 + \|u\|^4_{L^4(\Omega)} \right) - C_\delta \tag{4.2}$$

and

$$\int_\Omega \Phi(x, u(x))dx \geq \frac{M}{12} \|u\|^4_{L^4(\Omega)} - \delta \left(\|u\|^2 + \|u\|^4_{L^4(\Omega)} \right) - C_\delta, \tag{4.3}$$

for every $u \in L^2(\Omega) \cap L^4(\Omega)$.

Proof. By (2.2), we have

$$\left\langle A^{-1/2}G(u), A^{-1/2}u \right\rangle \geq \lambda \|u\|^2_{D(A^{1/2})} - \left| \left\langle K_2(|u| + \frac{1}{2}u^2), A^{-1}u \right\rangle \right|. \tag{4.4}$$

If $p_2 \in [1, 4]$, then from $L^{p_2}(\Omega) \cap L^\infty(\Omega) \subset L^4(\Omega)$, it follows that $K_2 \in L^4(\Omega)$ and consequently for every $\delta > 0$ there exists $C^{(1)}_\delta > 0$ such that

$$\begin{aligned} &\left| \left\langle K_2(|u| + \frac{1}{2}u^2), A^{-1}u \right\rangle \right| \leq \|K_2\|_{L^4(\Omega)} \left\| |u| + \frac{1}{2}u^2 \right\| \left\| A^{-1}u \right\|_{L^4(\Omega)} \\ &\leq \widetilde{c}_1 \left(\|u\| \|u\|_{L^4(\Omega)} + \|u\|^3_{L^4(\Omega)} \right) \leq \delta \left(\|u\|^2 + \|u\|^4_{L^4(\Omega)} \right) + C^{(1)}_\delta, \ \forall \delta > 0. \end{aligned} \tag{4.5}$$

If $p_2 \in (4, \infty)$, then by using the interpolation inequality in $L^p(\Omega)$, we conclude that for every $\delta > 0$ there exists $C^{(2)}_\delta > 0$ such that

$$\begin{aligned} &\left| \left\langle K_2(|u| + \frac{1}{2}u^2), A^{-1}u \right\rangle \right| \leq \|K_2\|_{L^{p_2}(\Omega)} \left\| |u| + \frac{1}{2}u^2 \right\| \left\| A^{-1}u \right\|_{L^{2p_2/(p_2-2)}(\Omega)} \\ &\leq \widetilde{c}_2 \left(\|u\| + \|u\|^2_{L^4(\Omega)} \right) \left\| A^{-1}u \right\|^{(p_2-4)/(p_2-2)} \left\| A^{-1}u \right\|^{2/(p_2-2)}_{L^4(\Omega)} \\ &\leq \widetilde{c}_2 \left(\|u\| + \|u\|^2_{L^4(\Omega)} \right) \|u\|^{(p_2-4)/(p_2-2)} \|u\|^{2/(p_2-2)}_{L^4(\Omega)} \leq \delta \left(\|u\|^2 + \|u\|^4_{L^4(\Omega)} \right) + C^{(2)}_\delta, \ \forall \delta > 0. \end{aligned} \tag{4.6}$$

Taking into account (4.5) and (4.6) in (4.4), we get (4.1).

By (2.1), we have

$$\langle F(u) + u, u \rangle \geq \frac{M}{3} \|u\|^4_{L^4(\Omega)} - \left\| K_1 u^2 \right\|_{L^1(\Omega)}. \tag{4.7}$$

If $p_1 \in [1, 2]$, then from $L^{p_1}(\Omega) \cap L^\infty(\Omega) \subset L^2(\Omega)$, it follows that $K_1 \in L^2(\Omega)$ and consequently for every $\delta > 0$ there exists $C^{(3)}_\delta > 0$ such that

$$\left\| K_1 u^2 \right\|_{L^1(\Omega)} \leq \|K_1\|_{L^2(\Omega)} \|u\|^2_{L^4(\Omega)} \leq \delta \left(\|u\|^2 + \|u\|^4_{L^4(\Omega)} \right) + C^{(3)}_\delta, \ \forall \delta > 0. \tag{4.8}$$

If $p_1 \in (2,\infty)$, then by using the interpolation inequality in $L^p(\Omega)$, we deduce that for every $\delta > 0$ there exists $C_\delta^{(4)} > 0$ such that

$$\begin{aligned}\left\|K_1 u^2\right\|_{L^1(\Omega)} &\le \|K_1\|_{L^{p_1}(\Omega)} \|u\|^2_{L^{2p_1/(p_1-1)}(\Omega)} \le \widetilde{c}_3 \|u\|^{2(p_1-2)/p_1} \|u\|^{4/p_1}_{L^4(\Omega)} \\ &\le \delta\left(\|u\|^2 + \|u\|^4_{L^4(\Omega)}\right) + C_\delta^{(4)},\ \forall \delta > 0. \end{aligned} \tag{4.9}$$

The inequalities (4.7)-(4.9) yield (4.2). By the same way, one can prove (4.3). $\square$

Now, we can prove the dissipativity of $\{S(t)\}_{t\ge 0}$.

Theorem 4.1. *Let, in addition to the conditions (2.1)-(2.2), $\lambda > 0$ and $K_i \in L^{p_i}(\Omega)$ for some $p_i \in [1,\infty)$, $i = 1, 2$. Then there exists $\varepsilon_0 = \varepsilon_0(M, \lambda, ||K_1||_{L^{p_1}(\Omega)}, ||K_2||_{L^{p_2}(\Omega)}) > 0$ such that the semigroup $\{S(t)\}_{t\ge 0}$ generated by the energy solutions of (2.3) possesses a bounded absorbing set in $D(A^{1/2}) \times D(A^{-1/2})$, for every $\varepsilon \in (0,\varepsilon_0)$.*

Proof. Let $B \subset D(A^{1/2}) \times D(A^{-1/2})$ be a bounded set, $(u_0,u_1) \in B$ and $(u(t),u_t(t)) = S(t)(u_0,u_1)$. Since, by the definition of A,

$$\|v\|^2_{D(A^{1/2})} - \|v\|^2 = \|\nabla v\|^2,\ \forall v \in D(A^{1/2}),$$

from (3.20), it follows that

$$\begin{aligned}\mathcal{E}\left(u(t),u_t(t)\right) + \int_\tau^t \|u_t(s)\|^2_{D(A^{-1/2})}\,ds + \int_\tau^t \left\langle A^{-1/2}G(u(s)), A^{-1/2}u_t(s)\right\rangle ds \\ = \mathcal{E}\left(u(\tau),u_t(\tau)\right),\ \forall t,\tau \ge 0,\end{aligned}$$

where $\mathcal{E}(v,w) = \frac{\varepsilon}{2}\|w\|^2_{D(A^{-1/2})} + \frac12 \|\nabla v\|^2 + \int_\Omega \Phi(x,v(x))dx$. Hence, $\mathcal{E}\left(u(t),u_t(t)\right)$ is an absolutely continuous function and

$$\frac{d}{dt}\mathcal{E}\left(u(t),u_t(t)\right) + \|u_t(t)\|^2_{D(A^{-1/2})} + \left\langle A^{-1/2}G(u(t)), A^{-1/2}u_t(t)\right\rangle = 0, \text{ a.e. in } \mathbb{R}^+. \tag{4.10}$$

Coming back to (2.2), we have

$$\begin{aligned}\left\langle A^{-1/2}G(u(t)), A^{-1/2}u_t(t)\right\rangle \ge \frac{\lambda}{2}\frac{d}{dt}\|u(t)\|^2_{D(A^{-1/2})} \\ -\bar{c}_1\left(\|u(t)\| + \|u(t)\|^2_{L^4(\Omega)}\right)\|u_t(t)\|_{D(A^{-1/2})},\ \forall t \ge 0.\end{aligned}$$

Taking into account the last inequality in (4.10), we get

$$\begin{aligned}&\frac{d}{dt}\left(\mathcal{E}\left(u(t),u_t(t)\right) + \frac{\lambda}{2}\|u(t)\|^2_{D(A^{-1/2})}\right) + \|u_t(t)\|^2_{D(A^{-1/2})} \\ &\le \bar{c}_1\left(\|u(t)\| + \|u(t)\|^2_{L^4(\Omega)}\right)\|u_t(t)\|_{D(A^{-1/2})},\ \text{ a.e. in } \mathbb{R}^+.\end{aligned} \tag{4.11}$$

On the other hand, testing $(2.3)_1$ by $\gamma A^{-1}u$, we find

$$\frac{d}{dt}\left(\gamma\varepsilon\left\langle A^{-1/2}u_t(t), A^{-1/2}u(t)\right\rangle + \frac{\gamma}{2}\|u(t)\|_{D(A^{-1/2})}\right) - \gamma\varepsilon\|u_t(t)\|_{D(A^{-1/2})}$$

$$+\gamma \|\nabla u(t)\|^2 + \gamma \langle F(u(t)) + u(t), u(t)\rangle + \gamma \left\langle A^{-1/2}G(u(t)), A^{-1/2}u(t)\right\rangle = 0, \tag{4.12}$$

which, with the help of (4.1) and (4.2), gives

$$\frac{d}{dt}\left(\gamma\varepsilon \left\langle A^{-1/2}u_t(t), A^{-1/2}u(t)\right\rangle + \frac{\gamma}{2}\|u(t)\|_{D(A^{-1/2})}\right) + \gamma \|\nabla u(t)\|^2$$
$$+\gamma\lambda \|u(t)\|^2_{D(A^{-1/2})} + \frac{\gamma M}{3}\|u(t)\|^2_{L^4(\Omega)} \le \gamma\varepsilon \|u_t(t)\|_{D(A^{-1/2})}$$
$$+2\gamma\delta\left(\|u(t)\|^2 + \|u(t)\|^4_{L^4(\Omega)}\right) + 2\gamma C_\delta,\ \forall t \ge 0,\ \forall \delta > 0,$$

where γ is an arbitrary positive number. Summing this inequality with (4.11), we obtain

$$\frac{d}{dt}\widehat{\mathcal{E}}(u(t), u_t(t)) + \left(\frac{1}{2} - \gamma\varepsilon\right)\|u_t(t)\|^2_{D(A^{-1/2})} + \gamma\|\nabla u(t)\|^2$$
$$+\gamma\lambda \|u(t)\|^2_{D(A^{-1/2})} + \frac{\gamma M}{3}\|u(t)\|^2_{L^4(\Omega)}$$
$$\le \left(\overline{c}_1^2 + 2\gamma\delta\right)\left(\|u(t)\|^2 + \|u(t)\|^4_{L^4(\Omega)}\right) + 2\gamma C_\delta,\ \text{a.e. in } \mathbb{R}^+,\ \forall\delta > 0,$$

where $\widehat{\mathcal{E}}(u(t), u_t(t)) = \mathcal{E}(u(t), u_t(t)) + \frac{1}{2}(\gamma + \lambda)\|u(t)\|^2_{D(A^{-1/2})} + \gamma\varepsilon\left\langle A^{-1/2}u_t(t), A^{-1/2}u(t)\right\rangle$. Choosing first γ large enough and then δ and ε small enough, we infer

$$\frac{d}{dt}\widehat{\mathcal{E}}(u(t), u_t(t)) + \overline{c}_2\|u_t(t)\|^2_{D(A^{-1/2})}$$
$$+\overline{c}_3\left(\|\nabla u(t)\|^2 + \|u(t)\|^2_{D(A^{-1/2})} - \frac{1}{2}\|u(t)\|^2 + \|u(t)\|^2_{L^4(\Omega)}\right) \le \overline{c}_4,\ \text{a.e. in } \mathbb{R}^+. \tag{4.13}$$

Again, by the definition of A, we have

$$\|\nabla u(t)\|^2 + \|u(t)\|^2_{D(A^{-1/2})} = \left\|A^{1/2}u(t)\right\|^2 - \|u(t)\|^2 + \left\|A^{-1/2}u(t)\right\|^2$$
$$\ge 2\left\langle A^{1/2}u(t), A^{-1/2}u(t)\right\rangle - \|u(t)\|^2 = \|u(t)\|^2,\ \forall t \ge 0, \tag{4.14}$$

which, together with (4.13), yields

$$2\frac{d}{dt}\widehat{\mathcal{E}}(u(t), u_t(t)) + 2\overline{c}_2\|u_t(t)\|^2_{D(A^{-1/2})} + \overline{c}_3\left(\|u(t)\|^2 + 2\|u(t)\|^2_{L^4(\Omega)}\right) \le 2\overline{c}_4,\ \text{a.e. in } \mathbb{R}^+.$$

Summing the last inequality with (4.13), we obtain

$$\frac{d}{dt}\widehat{\mathcal{E}}(u(t), u_t(t)) + \overline{c}_5\left(\|u_t(t)\|^2_{D(A^{-1/2})} + \|u(t)\|^2_{D(A^{1/2})} + \|u(t)\|^2_{L^4(\Omega)}\right) \le \overline{c}_4,\ \text{a.e. in } \mathbb{R}^+. \tag{4.15}$$

Since, by (2.1),

$$\widehat{\mathcal{E}}(u(t), u_t(t)) \le \overline{c}_6\left(\|u_t(t)\|^2_{D(A^{-1/2})} + \|u(t)\|^2_{D(A^{1/2})} + \|u(t)\|^2_{L^4(\Omega)}\right),\ \forall t \ge 0,$$

from (4.15), it follows that

$$\frac{d}{dt}\widehat{\mathcal{E}}(u(t), u_t(t)) + \frac{\overline{c}_5}{\overline{c}_6}\widehat{\mathcal{E}}(u(t), u_t(t)) \le \overline{c}_4,\ \text{a.e. in } \mathbb{R}^+,$$

and consequently

$$\widehat{\mathcal{E}}\left(u(t), u_t(t)\right) \leq e^{-\frac{\overline{c}_5}{\overline{c}_6}t}\widehat{\mathcal{E}}\left(u_0, u_1\right) + \frac{\overline{c}_4\overline{c}_6}{\overline{c}_5}, \ \forall t \geq 0.$$

The last inequality, with the help of (2.1), (4.3) and (4.14), shows that $\{S(t)\}_{t\geq 0}$ possesses a bounded absorbing set in $D(A^{1/2}) \times D(A^{-1/2})$, for small enough $\varepsilon > 0$. $\square$

Now, to complete the proof of Theorem 2.2, it is enough to show the asymptotical compactness of $\{S(t)\}_{t\geq 0}$.

Theorem 4.2. *Assume that the conditions of Theorem 4.1 hold and $\varepsilon \in (0, \varepsilon_0)$. Then the semigroup $\{S\,(t)\}_{t\geq 0}$ is asymptotically compact in $D(A^{1/2}) \times D(A^{-1/2})$, i.e. every sequence of the form $\{S(t_n)(u_{0n}, u_{1n})\}_{n=1}^{\infty}$, where $t_n \to \infty$ and $\{(u_{0n}, u_{1n})\}_{n=1}^{\infty} \subset B$, has a convergent subsequence in $D(A^{1/2}) \times D(A^{-1/2})$, for every bounded $B \subset D(A^{1/2}) \times D(A^{-1/2})$.*

Proof. By (4.10) and (4.12), we have

$$\begin{aligned} &\frac{d}{dt}\widehat{\mathcal{E}}\left(u(t), u_t(t)\right) + (1-\gamma\varepsilon)\left\|u_t(t)\right\|^2_{D(A^{-1/2})} + \gamma\left\|\nabla u(t)\right\|^2 \\ &\quad + \gamma\lambda\left\|u(t)\right\|^2_{D(A^{-1/2})} + \gamma\left\langle F(u(t)), u(t)\right\rangle \\ &\quad + \left\langle A^{-1/2}(G(u(t)) - \lambda u(t)), A^{-1/2}(u_t(t) + \gamma u(t))\right\rangle = 0, \text{ a.e. in } \mathbb{R}^+, \end{aligned} \tag{4.16}$$

where $\gamma > 0$. A straightforward computation shows that

$$\begin{aligned} &(1-\gamma\varepsilon)\left\|u_t(t)\right\|^2_{D(A^{-1/2})} + \gamma\left\|\nabla u(t)\right\|^2 + \gamma\lambda\left\|u(t)\right\|^2_{D(A^{-1/2})} \\ &= \gamma\widehat{\mathcal{E}}\left(u(t), u_t(t)\right) - \gamma\int_{\Omega}\Phi(x, u(t,x))dx \\ &+ \frac{\gamma}{2}\left\|\nabla u(t)\right\|^2 + \left(1 - \frac{3\gamma\varepsilon}{2} - \frac{\gamma\varepsilon^2}{2(\lambda-\gamma)}\right)\left\|u_t(t)\right\|^2_{D(A^{-1/2})} \\ &+ \frac{\gamma}{2}\left\|\frac{\varepsilon}{\sqrt{\lambda-\gamma}}u_t(t) - \sqrt{\lambda-\gamma}u(t)\right\|^2_{D(A^{-1/2})}, \ \forall t \geq 0 \text{ and } \forall \gamma \in (0, \lambda), \end{aligned}$$

which, together with (4.16), gives

$$\frac{d}{dt}\widehat{\mathcal{E}}\left(u(t), u_t(t)\right) + \gamma\widehat{\mathcal{E}}\left(u(t), u_t(t)\right) = \mathcal{K}\left(u(t), u_t(t)\right), \text{ a.e. in } \mathbb{R}^+, \tag{4.17}$$

for every $\gamma \in (0, \lambda)$, where $\mathcal{K}\left(u(t), u_t(t)\right) = \left\langle A^{-1/2}(\lambda u(t) - G(u(t))), A^{-1/2}(u_t(t) + \gamma u(t))\right\rangle$ $-\frac{\gamma}{2}\left\|\nabla u(t)\right\|^2 - \gamma\left(\left\langle F(u(t)), u(t)\right\rangle - \int_{\Omega}\Phi(x, u(t,x))dx\right) - \frac{\gamma}{2}\left\|\frac{\varepsilon}{\sqrt{\lambda-\gamma}}u_t(t) - \sqrt{\lambda-\gamma}u(t)\right\|^2_{D(A^{-1/2})}$ $-C(\gamma,\varepsilon)\left\|u_t(t)\right\|^2_{D(A^{-1/2})}$ and $C(\gamma,\varepsilon) = 1 - \frac{3\gamma\varepsilon}{2} - \frac{\gamma\varepsilon^2}{2(\lambda-\gamma)}$. Applying Gronwall's lemma to (4.17), we obtain

$$\widehat{\mathcal{E}}\left(u(t), u_t(t)\right) \leq e^{-\gamma t}\widehat{\mathcal{E}}\left(u_0, u_1\right) + \int_0^t e^{-\gamma(t-\tau)}\mathcal{K}\left(u(\tau), u_t(\tau)\right)d\tau, \ \forall t \geq 0. \tag{4.18}$$

Choosing γ small enough, we have $C(\gamma,\varepsilon) > 0$. Hence, by using (2.1), (2.2), Fatou's lemma and the semi-continuity of the norms, it is not difficult to show that

$$\limsup_{n\to\infty}\int_0^t e^{-\gamma(t-\tau)}\mathcal{K}\left(u_n(\tau),u_{nt}(\tau)\right)d\tau \le \int_0^t e^{-\gamma(t-\tau)}\mathcal{K}\left(u(\tau),u_t(\tau)\right)d\tau,\ \forall t\ge 0, \tag{4.19}$$

if

$$\begin{cases} u_n \to u \text{ weakly star in } L^\infty(0,t;D(A^{1/2})), \\ u_{nt}\to u_t \text{ weakly star in } L^\infty(0,t;D(A^{-1/2})), \end{cases}$$

as $n\to\infty$. Thus, applying the energy method developed in [2], with the help of (2.1), (4.14), (4.18), (4.19) and Fatou's lemma, one can deduce that the semigroup $\{S(t)\}_{t\ge 0}$ is asymptotically compact in $D(A^{1/2})\times D(A^{-1/2})$. $\square$

Remark 4.1. It is easy to see that the operator $-\Delta+I$ with the domain $\left\{u\in H^2(\Omega): \frac{\partial}{\partial\nu}u\mid_{\partial\Omega}=0\right\}$ is, like the operator A defined at the beginning of Section 2, a self-adjoint and strictly positive operator with a bounded inverse in $L^2(\Omega)$. Hence, the results of the paper remain true if we replace the Dirichlet boundary condition $(1.7)_2$ with the Neumann boundary condition

$$\frac{\partial}{\partial\nu}u(t,x)=\frac{\partial}{\partial\nu}\Delta u(t,x)=0,\quad (t,x)\in(0,\infty)\times\partial\Omega,$$

and suppose, in addition to (2.1), $\frac{\partial}{\partial\nu}f(\cdot,u)\mid_{\partial\Omega}=0$, for every $u\in\mathbb{R}$.

Remark 4.2. As mentioned above, in [26], [35], [32], [34] and [33], the global attractors were studied for the weak and quasi-strong solutions of 2D Cahn-Hilliard and Cahn-Hilliard/Allen-Cahn equations with the inertial term. To establish the dissipativity of the weak and quasi-strong solutions, the authors of the papers mentioned above critically used the estimate

$$\int_0^\infty \|u_t(\tau)\|^2_{D(A^{-1/2})}\,d\tau \le C\left(\|(u_0,u_1)\|_{D(A^{1/2})\times D(A^{-1/2})}\right),$$

which, unfortunately, is no longer true for (2.3). Therefore, the existence of the global attractors for the weak and quasi-strong solutions of (2.3) remains an unsolved problem.

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