Analysis 24, 147 – 169 (2004)

Analysis
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TRACES OF DIFFERENTIAL FORMS ON LIPSCHITZ BOUNDARIES

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Received : February 19, 2004

Abstract: Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary Γ. The "tangential trace" $\mathcal{T}\mathbf{E}$ of a differential form $\mathbf{E}$ on Ω is defined with the aid of the pullback ι^* where ι denotes the embedding of Γ into $\overline{\Omega}$. It is well known that the "trace theorem" for differential forms assures that $\mathcal{T}$ defines a topological isomorphism from $\mathbf{R}^q(\Omega)$ (the space of L^2–forms $\mathbf{E}$ with exterior derivative $\mathbf{d}\,\mathbf{E}$ in L^2) onto $R^{-1/2,q}(\Gamma)$ (the space of differential forms E on Γ such that both E and $\mathrm{d}\,E$ belong to a fractional order Sobolev space $H^{-1/2}$). We generalize and extend this and related results to domains Ω with a Lipschitz boundary Γ where even the definition of the spaces $H^{\pm 1/2}(\Gamma)$ is not obvious. (For the special case of classical vector analysis corresponding results may be found in [3] and its bibliography.)

MOS classifications: 35J55 , 35F15 , 35Q60

1 Notation and preliminaries

Throughout this paper Ω will denote a bounded open subset of $\mathbb{R}^N$ with a Lipschitz boundary Γ. (We remark in passing that $\mathbb{R}^N$ may as well be replaced by a smooth manifold of dimension N.) For $x \in \mathbb{R}^N$ let us denote by $\mathbf{A}^q(x)$ the $\binom{N}{q}$–dimensional space of alternating covariant tensors of rank q at x ("q–forms"). The space of all tensor fields of rank q on $S \subset \mathbb{R}^N$ ("full q–forms") will be denoted by

$$\mathbf{F}^q(S) := \{ \ F \ : \ S \longrightarrow \bigcup_{s\in S} \mathbf{A}^q(s) \quad : \ F(x) \in \mathbf{A}^q(x) \ \text{for all} \ x \in S\} \quad (1)$$

and subspaces of $\mathbf{F}^q(S)$ defined by regularity properties like L^2–spaces, Sobolev–spaces etc. will be written as

$$\mathbf{L}^{2,q}(S) \quad , \quad \mathbf{H}^{t,q}(S) \quad , \text{etc.} \tag{2}$$

For test fields let us write

$$\mathcal{D}^q := \mathcal{D}^q(\mathbb{R}^N) := \mathbf{C}_o^{\infty,q}(\mathbb{R}^N) \quad .$$

We shall regard Γ as a Lipschitz manifold embedded in $\mathbb{R}^N$. As pointed out in [6] we still have parts of the calculus for differential forms on Γ ("boundary q–forms"). The spaces corresponding to (1) and (2) on Γ shall be denoted by $A^q(y)$, $y \in \Gamma$, $F^q(\Gamma)$, $L^{2,q}(\Gamma)$ etc. (Note that $\mathbf{F}^q(\Gamma)$ etc. are <u>not</u> the same as $F^q(\Gamma)$ etc. because even $\mathbf{A}^q(y)$ and $A^q(y)$ are different vector spaces for $y \in \Gamma$ – the latter having dimension $\binom{N-1}{q}$. (We intend to denote "full" tensors, fields and spaces by boldface letters and "boundary" items by Roman letters.)

Remark 1 *Strictly speaking, the spaces $A^q(y)$ may not be defined for all $y \in \Gamma$. However, we shall keep this notation and interpret it in the following sense: If Γ is represented locally as a Lipschitz graph then the defining function will be differentiable almost everywhere (see [7, Thm.3.2.]). So almost everywhere, we may introduce a tangent space and the spaces $A^q(y)$ as well as a Riemannian bilinear form on Γ in the usual way. (This also gives rise to a canonical measure o on Γ.) By this construction, Γ becomes a Lipschitz manifold in the sense of [6] and $L^{2,q}(\Gamma)$ is well defined. Furthermore, we may construct an orthonormal basis*

$$\mathbf{B}(y) := \left\{\boldsymbol{\nu}(y), \mathbf{t}^2(y), \ldots, \mathbf{t}^N(y)\right\} \tag{3}$$

of $\mathbf{A}^1(y)$ for each $y \in \Gamma$ such that $\boldsymbol{\nu}$, $\mathbf{t}^n \in \mathbf{L}^{\infty,1}(\Gamma)$ and furthermore $\iota^*\boldsymbol{\nu} = 0$ (almost everywhere) if ι^* denotes the embedding of Γ into $\mathbb{R}^N$. This implies that

$$B(y) := \left\{t^2(y) := \iota^*\mathbf{t}^2(y), \ldots, t^N(y) := \iota^*\mathbf{t}^N(y)\right\} \tag{4}$$

is well defined and furnishes an orthonormal basis for $A^q(y)$ for almost every $y \in \Gamma$. We may also assume that $\mathbf{B}(y)$ and $B(y)$ are positively oriented.

On Γ, (using the arguments of [6]) we have a Hodge star operator $*$ as well as the exterior derivative d on the L^2–level and may define the Hilbert spaces

$$R^q(\Gamma) := \{E \in L^{2,q}(\Gamma) \quad : \quad \mathrm{d}\, E \in L^{2,q+1}(\Gamma)\}$$
$$D^q(\Gamma) := \{E \in L^{2,q}(\Gamma) \quad : \quad \mathrm{d} \, * \, E \in L^{2,N-q}(\Gamma)\} \quad .$$

Let us denote the counterparts of these on $\mathbb{R}^N$ by

$$\star \quad , \quad \mathbf{d} \quad , \quad \mathbf{R}^q \quad \text{and} \quad \mathbf{D}^q \quad .$$

On several occasions we shall need the operators

$$\begin{array}{rcccl} R := R_p & : & \mathbf{A}^q(y) & \longrightarrow & \mathbf{A}^{q+1}(y) \\ & & \mathbf{E} & \longmapsto & p \wedge \mathbf{E} \end{array}$$

$$\begin{array}{rcccl} T := T_p & : & \mathbf{A}^q(y) & \longrightarrow & \mathbf{A}^{q-1}(y) \\ & & \mathbf{E} & \longmapsto & (-1)^{(q-1)N} \star (p \wedge \star \mathbf{E}) = (-1)^{(q-1)N} \star R \star \mathbf{E} \end{array}$$

depending on $p \in \mathbf{A}^1(y)$ (cf. [9] or [11]). These satisfy

$$TR + RT = |p|^2 \,\mathrm{id} \tag{5}$$

and define the symbols of $\mathbf{d}$ and the co–derivative

$$\boldsymbol{\delta} := (-1)^{(q-1)N} \star \mathbf{d} \, \star \quad .$$

Namely, if F denotes the Fourier transform on q–forms (defined by application of the scalar Fourier transform to the scalar components in the Cartesian representation of, say, $\mathbf{E} \in \mathbf{L}^{2,q}(\mathbb{R}^N)$) then we have

$$F(\mathbf{d}\,\mathbf{E}) = iRFE \quad , \quad (RFE)(\xi) := R_p(\xi) FE(\xi) \tag{6}$$

$$F(\boldsymbol{\delta}\mathbf{E}) = iTFE \quad , \quad (TFE)(\xi) := T_p(\xi) FE(\xi) \tag{7}$$

where (with Cartesian coordinates x_n)

$$p(\xi) := \sum_{n=1}^{N} \xi_n dx_n \quad .$$

Despite the low regularity of Γ, the embedding $\iota \; : \; \Gamma \longrightarrow \mathbb{R}^N$ retains the familiar rules:

$$\iota^*(\mathbf{E} \wedge \mathbf{F}) = \iota^*\mathbf{E} \wedge \iota^*\mathbf{F} \tag{8}$$

$$\iota^*\mathbf{d}\,\mathbf{E} = \mathrm{d}\,\iota^*\mathbf{E} \tag{9}$$

if $\mathbf{E} \in \mathcal{D}^q$, $\mathbf{F} \in \mathcal{D}^p$ and

$$\int_\Omega \mathbf{d}\,\mathbf{\Phi} \wedge \mathbf{\Psi} + (-1)^q \int_\Omega \mathbf{\Phi} \wedge \mathbf{d}\,\mathbf{\Psi} = \int_\Gamma \iota^*(\mathbf{\Phi} \wedge \mathbf{\Psi}) \tag{10}$$

if $\mathbf{\Phi} \in \mathcal{D}^q$ and $\mathbf{\Psi} \in \mathcal{D}^{N-1-q}$.

Remark 2 *For the convenience of the reader, let us indicate the argument leading to* (9) . *Let* M, N *be Riemannian manifolds and let* $\varphi \ : \ M \longrightarrow N$ *be a Lipschitz map. We may approximate* φ *by smooth maps* φ_k *such that*

$$\varphi_k^* F \to \varphi^* F \quad \text{in} \quad L^{2,\cdots}(M)$$

for $F \in L^{2,\cdots}(N)$. *Therefore the relation*

$$d\varphi_k^* E = \varphi_k^* dE$$

may be extended to φ *by approximation if* $E \in R^q(N)$. *(The approximating maps* φ_k *need not to be differentiated.)*

Gauß' Theorem remains valid on domains Ω with Lipschitz boundary (cf. [7, sec. 12.1]) which gives the following version of (10) :

$$\int_\Omega \mathbf{d}\,\mathbf{\Phi} \wedge \mathbf{\Psi} + (-1)^q \int_\Omega \mathbf{\Phi} \wedge \mathbf{d}\,\mathbf{\Psi} = \int_\Gamma \star(\nu(y) \wedge \mathbf{\Phi}(y) \wedge \mathbf{\Psi}(y)) d\mathbf{o}(y) \quad (11)$$

for $\mathbf{\Phi} \in \mathcal{D}^q$ and $\mathbf{\Psi} \in \mathcal{D}^{N-1-q}$. (Here $\mathbf{o}$ denotes the canonical volume measure on Γ .) Of course, (11) can be extended to more irregular fields by approximation arguments.

For $y \in \Gamma$ the space $\mathbf{A}^q(y)$ may be split into an orthogonal sum

$$\mathbf{A}^q(y) = \mathbf{A}_t^q(y) \oplus \mathbf{A}_\nu^q(y)$$

where (cf. (5))

$$\begin{aligned} \mathbf{A}_t^q(y) &:= \{\mathbf{E} \in \mathbf{A}^q(y) \ : \ T\mathbf{E} = 0\} \quad , \quad T := T_\nu \\ \mathbf{A}_\nu^q(y) &:= \{\mathbf{E} \in \mathbf{A}^q(y) \ : \ R\mathbf{E} = 0\} \quad , \quad R := R_\nu \quad . \end{aligned}$$

Standard calculations within the bases (3) and (4) show that there exists a canonical isometric isomorphism

$$J(y) \ : \ \mathbf{A}_t^q(y) \longrightarrow A^q(y) \quad .$$

Furthermore, the linear map

$$\begin{aligned} \boldsymbol{\rho}_q(y) \ : \ \mathbf{A}^q(y) &\longrightarrow \mathbf{A}^{N-1-q}(y) \\ \mathbf{E} &\longmapsto \star(\nu(y) \wedge \mathbf{E}) = \star R\mathbf{E} \end{aligned}$$

and the orthogonal projector $\boldsymbol{\pi}_q(y) = TR$ of $\mathbf{A}^q(y)$ onto $\mathbf{A}_t^q(y)$ satisfy the following rules (which may be read off from (5)):

$$\operatorname{im}(\boldsymbol{\rho}_q(y)) = \mathbf{A}_t^{N-1-q}(y) \quad (12)$$

$$\ker(\boldsymbol{\rho}_q(y)) = \mathbf{A}_\nu^q(y) \quad (13)$$

$$\boldsymbol{\rho}_{N-1-q}(y)\boldsymbol{\rho}_q(y) = (-1)^{qN}\boldsymbol{\pi}_q(y) \quad (14)$$

$$J(y)\boldsymbol{\rho}_q(y) = * J(y)\boldsymbol{\pi}_q(y) \quad . \quad (15)$$

Clearly, all this may be lifted to $\mathbf{L}^{2,q}(\Gamma)$. We obtain an orthogonal decomposition

$$\mathbf{L}^{2,q}(\Gamma) = \mathbf{L}_t^{2,q}(\Gamma) \oplus \mathbf{L}_\nu^{2,q}(\Gamma) \quad ,$$

an isometric isomorphism

$$J \ : \ \mathbf{L}_t^{2,q}(\Gamma) \ \longrightarrow \ L^{2,q}(\Gamma)$$

and operators $\boldsymbol{\rho}_q$, $\boldsymbol{\pi}_q$ such that

$$\operatorname{im}(\boldsymbol{\rho}_q) = \mathbf{L}_t^{2,N-1-q}(\Gamma) \tag{16}$$
$$\operatorname{im}(\boldsymbol{\pi}_q) = \mathbf{L}_t^{2,q}(\Gamma) \tag{17}$$
$$\ker(\boldsymbol{\rho}_q) = \ker(\boldsymbol{\pi}_q) = \mathbf{L}_\nu^{2,q}(\Gamma) =: \mathbf{N}_q \tag{18}$$
$$\boldsymbol{\rho}_{N-1-q}\boldsymbol{\rho}_q = (-1)^{qN}\boldsymbol{\pi}_q \tag{19}$$
$$\rho_q = \ * \ \pi_q \tag{20}$$

where we have introduced

$$\rho := J\boldsymbol{\rho} \quad , \quad \pi := J\boldsymbol{\pi} \quad .$$

2 Generalized trace spaces

In the case of a smooth boundary, the operator

$$\iota^* \ : \ \mathcal{D}^q \ \longrightarrow \ C^{\infty,q}(\Gamma)$$

can be extended by continuity to an operator $\mathcal{T}$ ("tangential trace") from $\mathbf{H}^{1,q}(\Omega)$ into $\mathbf{L}^{2,q}(\Gamma)$, say. Furthermore, one can define $H^{1/2,q}(\Gamma)$ in the usual way and one has (for "boundary" q–forms)

$$H^{1/2,q}(\Gamma) = \mathcal{T}\mathbf{H}^{1,q}(\Omega) \tag{21}$$

Also, by the scalar trace theorem (for "full" q–forms)

$$\mathbf{H}^{1/2,q}(\Gamma) = \boldsymbol{\tau}\mathbf{H}^{1,q}(\Omega) \tag{22}$$

where $\boldsymbol{\tau}$ is the "scalar" trace operator acting separately (as the trace operator on functions) on the components of $\mathbf{E}$ in its Cartesian representation. The relation (22) (for "full" q–forms) is well defined and true even for Lipschitz boundaries (cf. [12, Satz 8.7]) whereas (21) does not even make sense because $H^{1/2,q}(\Gamma)$ cannot be defined intrinsically on the Lipschitz manifold Γ. (Coordinate changes do not respect $H^{1/2}$–regularity.)

Recalling the bases (3) and (4) we find

$$\mathcal{T} = \pi\boldsymbol{\tau} \quad . \tag{23}$$

So a natural generalized definition of $H^{1/2,q}(\Gamma)$ is

$$H^{1/2,q}_{\pi}(\Gamma) := \pi\boldsymbol{\tau}\mathbf{H}^{1,q}(\Omega) \quad . \tag{24}$$

However, recalling (20) we might also define $H^{1/2,q}(\Gamma)$ by

$$H^{1/2,q}_{\rho}(\Gamma) := \rho\boldsymbol{\tau}\mathbf{H}^{1,N-1-q}(\Omega) \quad . \tag{25}$$

In the case of a smooth boundary, (24) and (25) define the same space (namely $H^{1/2,q}(\Gamma)$) because $*$ respects $H^{1/2}$–regularity. In our case, $H^{1/2,q}_{\pi}(\Gamma)$ and $H^{1/2,q}_{\rho}(\Gamma)$ will be different in general (cf. [2]) and we shall need both in order to give a generalized characterization of traces of $\mathbf{R}^q(\Omega)$. We introduce norms

$$|E|_{H^{1/2,q}_{\pi}(\Gamma)} := \inf\{\|\mathbf{E}\|_{\mathbf{H}^{1/2,q}(\Gamma)} \quad : \quad E = \pi\mathbf{E}\} \tag{26}$$

$$|E|_{H^{1/2,q}_{\rho}(\Gamma)} := \inf\{\|\mathbf{E}\|_{\mathbf{H}^{1/2,N-1-q}(\Gamma)} \quad : \quad E = \rho\mathbf{E}\} \tag{27}$$

into these spaces and denote their topological duals by $H^{-1/2,q}_{\pi}(\Gamma)$ and $H^{-1/2,q}_{\rho}(\Gamma)$. We could as well use the norms

$$\| E \|_{H^{1/2,q}_{\pi}(\Gamma)} := \inf\{\|\mathbf{E}\|_{\mathbf{H}^{1,q}(\Omega)} \quad : \quad E = \iota^*\mathbf{E}\} \tag{28}$$

$$\| E \|_{H^{1/2,q}_{\rho}(\Gamma)} := \inf\{\|\mathbf{E}\|_{\mathbf{H}^{1,N-1-q}(\Omega)} \quad : \quad E = *\,\iota^*\mathbf{E}\} \quad . \tag{29}$$

Namely, calculating within the bases (3) and (4) it may be easily seen that we have

$$\iota^* = \pi\boldsymbol{\tau} \tag{30}$$

$$*\,\iota^* = \rho\boldsymbol{\tau} \tag{31}$$

and (by the scalar trace theorem) the right hand sides of (26) and (27) are equivalent to the right hand sides of (28) and (29) , respectively. Thus $|\cdot|_{H^{1/2,q}_{\dots}}$ and $\| \cdot \|_{H^{1/2,q}_{\dots}}$ are equivalent norms.

Let us investigate the functional analytic properties of the spaces $H^{1/2,q}_{\dots}$ (generalizing part of the results in [3, sec. 2]).

It will be convenient to use the framework of "Gelfand triplets" (cf. [12, sec. 17.3]) which we need only for Hilbert spaces.

Definition 1 *The relation*

$$\mathbf{V} \subset \mathbf{H} \subset \mathbf{V}'$$

is called a "Gelfand triplet" if

i) $\mathbf{V}$ *and* $\mathbf{H}$ *are Hilbert spaces;*

ii) $\mathbf{V}$ *is densely and continuously embedded in* $\mathbf{H}$ *;*

iii) *the second inclusion is defined by*

$$f(v) := \langle f , v \rangle_{\mathbf{H}} \quad , \quad v \in \mathbf{V} \quad , \quad f \in \mathbf{H}$$

(which defines a continuous and dense embedding of $\mathbf{H}$ *in* $\mathbf{V}'$ *).*

The following result may then be obtained by basic arguments of functional analysis. (For $S \subset X$ we denote the annihilator of S in X' by S^0.)

Theorem 1 *Let* $\mathbf{V} \subset \mathbf{H} \subset \mathbf{V}'$ *be a Gelfand triplet. Assume furthermore*

i) $\boldsymbol{\pi} : \mathbf{H} \longrightarrow \mathbf{H}$ *is an orthogonal projector onto* $\mathbf{H}_t$ *and* $\mathbf{H}_\nu := \mathbf{H}_t^\perp$ *;*

ii) $J : \mathbf{H}_t \longrightarrow H$ *is an isometric isomorphism;*

iii) $V := \pi\mathbf{V}$ *(where* $\pi := J\boldsymbol{\pi}$*) is furnished with the norm*

$$|E|_V := \inf\{\|\mathbf{E}\|_{\mathbf{V}} \quad : \quad E = \pi\mathbf{E}\} \quad .$$

Then we have

iv) $V \subset H \subset V'$ *is a Gelfand triplet;*

v) $\pi' V' = (\mathbf{V}_\nu)^0 = \overline{\mathbf{H}_t}$ *(closure in* $\mathbf{V}'$ *;* $\mathbf{V}_\nu := \mathbf{V} \cap \mathbf{H}_\nu$*).*

An application of this theorem gives the following (cf. [3, Lemma 2.3]).

Theorem 2 *The inclusions*

$$H_\pi^{1/2,q}(\Gamma) \subset L^{2,q}(\Gamma) \subset H_\pi^{-1/2,q}(\Gamma)$$
$$H_\rho^{1/2,q}(\Gamma) \subset L^{2,q}(\Gamma) \subset H_\rho^{-1/2,q}(\Gamma)$$

are Gelfand triplets. Furthermore π' *defines an isomorphism between* $H_\pi^{-1/2,q}(\Gamma)$ *and*

$$(\mathbf{H}^{1/2,q}(\Gamma) \cap \mathbf{N}_q)^0 = \overline{\mathbf{L}_t^{2,q}(\Gamma)} \qquad \text{(closure in } \mathbf{H}^{-1/2,q}(\Gamma)\text{)}$$

and so does ρ' *between* $H_\rho^{-1/2,q}(\Gamma)$ *and*

$$(\mathbf{H}^{1/2,N-1-q}(\Gamma) \cap \mathbf{N}_{N-1-q})^0 = \overline{\mathbf{L}_t^{2,N-1-q}(\Gamma)} \qquad \text{(closure in } \mathbf{H}^{-1/2,N-1-q}(\Gamma)\text{)} \quad .$$

3 Generalized trace theorems

In this section we want to investigate the tangential traces of $\mathbf{R}^q(\Omega)$–fields. First, we supply an approximation argument.

Lemma 1 *The space $\mathcal{D}^q$ of test fields is dense in $\mathbf{H}^{1,q}(\Omega)$, in $\mathbf{R}^q(\Omega)$ and in $\mathbf{D}^q(\Omega)$.*

The first assertion is well known (see [12, Satz 3.6] e. g.). Second, consider $\mathbf{E} \in \mathbf{R}^q(\Omega)$. By localization, we may assume $\operatorname{supp} \mathbf{E} \subset V$ and that there exists a Lipschitz isomorphism

$$\varphi \ : \ U \longrightarrow V \quad , \quad U := \{x \in \mathbb{R}^N \ : \ |x| < 1\}$$

such that

$$\varphi(U_-) = V \cap \Omega \quad , \quad \varphi(U_0) = V \cap \Gamma$$

where $U_- := \{x \in U \ : \ x_1 < 0\}$, $U_0 := \{x \in U \ : \ x_1 = 0\}$. By Remark 2 we have $\varphi^* \mathbf{E} \in \mathbf{R}^q(U_-)$ and (using a reflection operator; see [4]) may extend $\varphi^*\mathbf{E}$ to $\widetilde{\mathbf{E}} \in \mathbf{R}^q(U)$ such that $\operatorname{supp} \widetilde{\mathbf{E}} \subset U$. Transforming back yields an extension $\hat{\mathbf{E}} \in \mathbf{R}^q(V)$ which may be approximated by $\hat{\mathbf{\Phi}} \in \mathbf{C}_0^{\infty,q}(V)$ via a smoothing operator. This proves the second assertion. The third follows by duality because $\star\mathcal{D}^q = \mathcal{D}^{N-q}$. **q.e.d.**

Second, we want to extend the distributional notion of $\mathrm{d}\, E$ from the $L^{2,q}$–level (as introduced in [6]) to our spaces $H^{\pm 1/2,q}_{\dots}$. In order to motivate our definition, let us look upon the case of a smooth boundary. For $\mathbf{\Phi} \in \mathcal{D}^q$ and $\mathbf{\Psi} \in \mathcal{D}^{N-2-q}$ we compute:

$$\begin{aligned}
\langle \mathrm{d}\, \iota^* \mathbf{\Phi} \, , \ * \iota^* \mathbf{\Psi} \rangle_{L^{2,q+1}(\Gamma)} &= (-1)^{(N-2-q)(N-1-[N-2-q])} \int_\Gamma \mathrm{d}\, \iota^* \mathbf{\Phi} \wedge \iota^* \mathbf{\Psi} \\
&= (-1)^{N(q+1)} \int_\Gamma \iota^* (\mathbf{d}\, \mathbf{\Phi} \wedge \mathbf{\Psi}) \\
&= (-1)^{(N-1)(q+1)} \int_\Omega \mathbf{d}\, \mathbf{\Phi} \wedge \mathbf{d}\, \mathbf{\Psi}
\end{aligned}$$

Lemma 2 *The operator*

$$\begin{array}{rccc} \mathrm{d} \ : & H_\pi^{1/2,q}(\Gamma) & \longrightarrow & H_\rho^{-1/2,q+1}(\Gamma) \\ & E & \longmapsto & \mathrm{d}\, E \end{array} \tag{32}$$

is well defined by

$$\mathrm{d}\, E(\Psi) := (-1)^{(N-1)(q+1)} \int_\Omega \mathbf{d}\, \mathbf{E} \wedge \mathbf{d}\, \mathbf{\Psi} \tag{33}$$

for

$$E = \iota^* \mathbf{E} \quad , \quad \mathbf{E} \in \mathbf{H}^{1,q}(\Omega) \quad , \quad \Psi = *\,\iota^* \mathbf{\Psi} \quad , \quad \mathbf{\Psi} \in \mathbf{H}^{1,N-2-q}(\Omega) \quad .$$

Furthermore, it is linear and continuous.

Proof: All we have to show is that (33) is independent of the choices of $\mathbf{E}$ and $\mathbf{\Psi}$ because the other assertions are obvious in view of the norms (28) and (29).

So suppose that

$$E = \iota^* \mathbf{E}_1 = \iota^* \mathbf{E}_2$$

and hence

$$\iota^* \mathbf{E} = 0 \quad , \quad \mathbf{E} := \mathbf{E}_1 - \mathbf{E}_2 \quad .$$

We may extend (10) by continuity to $\mathbf{\Phi} := \mathbf{E} \in \mathbf{H}^{1,q}(\Omega)$. Replacing $\mathbf{\Psi}$ by $\mathbf{d}\,\mathbf{\Psi}$ in (10) gives

$$\int_\Omega \mathbf{d}\,\mathbf{E} \wedge \mathbf{d}\,\mathbf{\Psi} = 0 \tag{34}$$

for $\mathbf{\Psi} \in \mathcal{D}^{N-2-q}$. Another approximation argument shows that (34) holds for $\mathbf{\Psi} \in \mathbf{H}^{1,N-2-q}(\Omega)$, too. Hence (33) is independent of the choice of $\mathbf{E}$. An analogous argument applies to $\mathbf{\Psi}$. **q.e.d.**

Guided by the above calculations in the smooth case, let us also try to generalize the notion of d acting on $F \in H_\rho^{-1/2,q}(\Gamma)$. We may assume that F is approximated by a sequence $\iota^* \mathbf{\Phi}_k$, $\mathbf{\Phi}_k \in \mathcal{D}^q$. We compute for $\mathbf{\Psi} \in \mathcal{D}^{N-2-q}$

$$\begin{aligned}
\langle \mathrm{d}\, \iota^* \mathbf{\Phi}_k \,,\ *\,\iota^* \mathbf{\Psi} \rangle_{L^{2,q+1}(\Gamma)} &= (-1)^{N(q+1)} \int_\Gamma \mathrm{d}\, \iota^* \mathbf{\Phi}_k \wedge \iota^* \mathbf{\Psi} \\
&= (-1)^{N(q+1)} (-1)^{q+1} \int_\Gamma \iota^* \mathbf{\Phi}_k \wedge \iota^* \mathbf{d}\,\mathbf{\Psi} \\
&= (-1)^{q+N-1} \langle \iota^* \mathbf{\Phi}_k \,,\ \rho\tau \mathbf{d}\,\mathbf{\Psi} \rangle_{\mathbf{L}^{2,q}(\Gamma)} \quad .
\end{aligned}$$

Thus we are led to the following definition:

Lemma 3 *Let*

$$\mathbf{Y} := \{ \mathbf{\Psi} \in \mathbf{H}^{1,N-2-q}(\Omega) \quad : \quad \mathbf{d}\,\mathbf{\Psi} \in \mathbf{H}^{1,N-1-q}(\Omega) \}$$

be supplied with its natural norm and supply

$$Y := \{ *\,\iota^* \mathbf{\Psi} \quad : \quad \mathbf{\Psi} \in \mathbf{Y} \}$$

with the norm

$$\|\psi\|_Y := \inf\{|\Psi|_{\mathbf{Y}} \quad : \quad *\,\iota^*\boldsymbol{\Psi} = \Psi\} \quad .$$

The operator

$$\begin{array}{rcccc} \mathrm{d} & : & H_\rho^{-1/2,q}(\Gamma) & \longrightarrow & Y' \\ & & F & \longmapsto & \mathrm{d}\,F \end{array}$$

is well defined by

$$\mathrm{d}\,F(\Psi) := (-1)^{q+N-1}F(\,*\,\iota^*\mathbf{d}\,\boldsymbol{\Psi}) \quad \text{if} \quad \Psi = \,*\,\iota^*\boldsymbol{\Psi} \quad . \tag{35}$$

Furthermore, d *is linear and continuous.*

Proof: Again, all we have to show is that $*\,\iota^*\boldsymbol{\Psi} = 0$ implies $F(\,*\,\iota^*\mathbf{d}\,\boldsymbol{\Psi}) = 0$. But this is trivial because

$$*\,\iota^*\boldsymbol{\Psi} = 0 \Rightarrow \iota^*\boldsymbol{\Psi} = 0 \Rightarrow \iota^*\mathbf{d}\,\boldsymbol{\Psi} = \mathrm{d}\,\iota^*\boldsymbol{\Psi} = 0 \quad .$$

q.e.d.

Now we can prove (noting that $H_\rho^{-1/2,q+1}(\Gamma)$ is a subspace of Y')

Theorem 3 *Let*

$$R^{-1/2,q}(\Gamma) := \{E \in H_\rho^{-1/2,q}(\Gamma) \quad : \quad \mathrm{d}\,E \in H_\rho^{-1/2,q+1}(\Gamma)\}$$

be supplied with its natural norm. Then the tangential trace operator

$$\begin{array}{rcccc} \mathcal{T} & : & \mathbf{R}^q(\Omega) & \longrightarrow & R^{-1/2,q}(\Gamma) \\ & & \mathbf{E} & \longmapsto & \iota^*\mathbf{E} \end{array}$$

is well defined, linear and continuous.

Proof: Let $\boldsymbol{\Phi} \in \mathcal{D}^q$. Then we have

$$\langle\,\iota^*\boldsymbol{\Phi}\,,\ *\,\iota^*\boldsymbol{\Psi}\,\rangle_{L^{2,q}(\Gamma)} = (-1)^{qN}\int_\Gamma \iota^*\boldsymbol{\Phi}\wedge\iota^*\boldsymbol{\Psi} = (-1)^{qN}\int_\Omega \mathbf{d}\,(\boldsymbol{\Phi}\wedge\boldsymbol{\Psi}) \tag{36}$$

for $\boldsymbol{\Psi} \in \mathcal{D}^{N-1-q}$. Similarly, for $\boldsymbol{\Psi} \in \mathcal{D}^{N-2-q}$

$$\begin{aligned} \langle\,\mathrm{d}\,\iota^*\boldsymbol{\Phi}\,,\ *\,\iota^*\boldsymbol{\Psi}\,\rangle_{L^{2,q+1}(\Gamma)} &= (-1)^{q+N-1}\langle\,\iota^*\boldsymbol{\Phi}\,,\ *\,\iota^*\mathbf{d}\,\boldsymbol{\Psi}\,\rangle_{L^{2,q+1}(\Gamma)} \\ &= (-1)^{(q+1)(N-1)}\int_\Omega (\mathbf{d}\,\boldsymbol{\Phi}\wedge\mathbf{d}\,\boldsymbol{\Psi}) \quad . \end{aligned} \tag{37}$$

From (36) and (37) it is clear that

$$\iota^* \quad : \quad \mathcal{D}^q \quad \longrightarrow \quad R^{-1/2,q}(\Gamma)$$

is continuous if $\mathcal{D}^q$ is furnished with the $\mathbf{R}^q(\Omega)$–norm and hence can be extended by continuity to $\mathbf{R}^q(\Omega)$ (using Lemma 1). This is the usual interpretation of trace theorems. **q.e.d.**

Remark 3 *If* $\mathbf{E}_0 \in \mathbf{H}^{1,q}(\Omega)$ *and* $\mathbf{E}_1 \in \mathbf{H}^{1,q-1}(\Omega)$ *then* $\mathbf{E} := \mathbf{E}_0 + \mathbf{d}\,\mathbf{E}_1 \in \mathbf{R}^q(\Omega)$ *because* $\mathbf{d}\,\mathbf{d} = 0$. *In this case, we may express* (36) *and* (37) *in terms of* $\mathbf{E}_0$ *and* $\mathbf{E}_1$. *Namely let* $\mathbf{\Phi}_{0,k} \in \mathcal{D}^q$ *and* $\mathbf{\Phi}_{1,k} \in \mathcal{D}^{q-1}$ *approximate* $\mathbf{E}_0$ *resp.* $\mathbf{E}_1$ *in* $\mathbf{H}^{1,\cdots}(\Omega)$. *We compute*

$$\begin{aligned}\mathcal{T}\mathbf{E}(\,*\,\iota^*\mathbf{\Psi}) &= (-1)^{qN}\lim\int_\Omega \mathbf{d}\;[(\mathbf{\Phi}_{0,k} + \mathbf{d}\,\mathbf{\Phi}_{1,k}) \wedge \mathbf{\Psi}] \\ &= (-1)^{qN}\lim\int_\Omega [\mathbf{d}\,(\mathbf{\Phi}_{0,k} \wedge \mathbf{\Psi}) + (-1)^q \mathbf{d}\,\mathbf{\Phi}_{1,k} \wedge \mathbf{d}\,\mathbf{\Psi}] \\ &= (-1)^{qN}\int_\Omega [\mathbf{d}\,(\mathbf{E}_0 \wedge \mathbf{\Psi}) + (-1)^q \mathbf{d}\,\mathbf{E}_1 \wedge \mathbf{d}\,\mathbf{\Psi}]\end{aligned}$$

Thus we have

$$\mathcal{T}(\mathbf{E}_0 + \mathbf{d}\,\mathbf{E}_1)(\rho\boldsymbol{\tau}\mathbf{\Psi}) = (-1)^{qN}\int_\Omega [\mathbf{d}\,(\mathbf{E}_0 \wedge \mathbf{\Psi}) + (-1)^q \mathbf{d}\,\mathbf{E}_1 \wedge \mathbf{d}\,\mathbf{\Psi}] \quad . \tag{38}$$

A similar argument yields

$$\mathrm{d}\,\mathcal{T}(\mathbf{E}_0 + \mathbf{d}\,\mathbf{E}_1)(\rho\boldsymbol{\tau}\mathbf{\Psi}) = (-1)^{(q+1)(N-1)}\int_\Omega \mathbf{d}\,\mathbf{E}_0 \wedge \mathbf{d}\,\mathbf{\Psi} \quad . \tag{39}$$

Lemma 4 *The spaces* $H := H^{\pm 1/2,q}_{\rho/\pi}(\Gamma)$ *and the operators* π, ρ *and their adjoints* π' *and* ρ' *as well as the operator* d *are "local", i. e.* $E \in H$ *is equivalent to* $\varphi E \in H$ *for all* $\varphi \in C_0^\infty(\mathbb{R}^N)$ *and*

$$\operatorname{supp}(SE) \subset \operatorname{supp} E$$

if S *is one of the operators mentioned above.*

Proof: Both ρ and π do not extend supports by construction and this property carries over to ρ' and π' by duality. Therefore, $H^{\pm 1/2,q}_{\rho/\pi}(\Gamma)$ are local spaces because $\rho, \ldots$ commute with the multiplication by φ. Concerning d we note that it was defined using the representation of E by $\mathbf{E} \in \mathbf{H}^{1/2,\cdots}$ which are local spaces, too.

q.e.d.

Theorem 4 *The tangential trace operator*

$$\mathcal{T} \;:\; \mathbf{R}^q(\Omega) \;\longrightarrow\; R^{-1/2,q}(\Gamma)$$

(as defined in the previous theorem) is surjective and hence has a continuous right inverse $\mathcal{T}^{-1}$. *The latter may be chosen such that its range lies in*

$$\mathbf{H}^{1,q}(\Omega) + \mathbf{d}\,\mathbf{H}^{1,q-1}(\Omega) \quad .$$

Proof: Our proof extends arguments due to L. Tartar (as exhibited in [3]) to our more general situation.

Let $E \in R^{-1/2,q}(\Gamma)$. We want to exhibit $\mathbf{E} \in \mathbf{R}^q(\Omega)$ such that $\mathcal{T}\mathbf{E} = E$. By Lemma 4 we may assume

i) $\operatorname{supp} E \subset\subset S := I \times \Omega'$, $I := (\alpha, \beta)$, Ω' (open) $\subset\subset \mathbb{R}^{N-1}$;

ii) $\Gamma \cap S = \{g(y) := (F(y), y) \ : \ y \in \Omega'\}$ where $F \ : \ \Omega' \longrightarrow \mathbb{R}$ is uniformly Lipschitz.

Let us denote Cartesian coordinates by (t, y) on $\mathbb{R} \times \Omega'$. We have

$$E \in H_\rho^{-1/2,q}(\Gamma) \quad , \quad G := \mathrm{d}\, E \in H_\rho^{-1/2,q+1}(\Gamma)$$

and therefore

$$\rho' E \in \mathbf{H}^{-1/2,N-1-q}(\Gamma)$$
$$\rho' G \in \mathbf{H}^{-1/2,N-2-q}(\Gamma)$$

and the relation $\mathrm{d}\, E = G$ is equivalent to

$$\rho' E(\tau \mathbf{d}\, \Phi) = (-1)^{q+N-1} \rho' G(\tau \Phi) \quad \text{for} \quad \Phi \in \mathcal{D}^{N-2-q} \quad . \tag{40}$$

For simplicity of notation, we may arrange matters such that $\Omega' = \mathbb{R}^{N-1}$ and $\operatorname{supp} F \subset\subset \mathbb{R}^{N-1}$.

By (5) with $R := R_{dt}$ and $T := T_{dt}$, each $W \in \mathbf{A}^q(x)$, $x = (t, y)$, may be written as

$$W = dt \wedge W' + W'' \quad , \quad W' := TW \in \mathbf{A}^{q-1}(x) \quad , \quad W'' := TRW \in \mathbf{A}^q(x) \tag{41}$$

and this pointwise orthogonal decomposition may be lifted to $W \in \mathbf{H}^{\pm s,q}(\Gamma)$. So we have orthogonal decompositions

$$\mathbf{H}^{\pm s,q}(\Gamma) = \mathbf{H}_v^{\pm s,q}(\Gamma) \oplus \mathbf{H}_h^{\pm s,q}(\Gamma) \quad , \quad \mathbf{H}_v^{\pm s,q}(\Gamma) = dt \wedge \mathbf{H}_h^{\pm s,q-1}(\Gamma) \quad . \tag{42}$$

The scalar pullbacks

$$\begin{array}{rcccc} \mu & : & H^{+s}(\Gamma) & \longrightarrow & H^{+s}(\mathbb{R}^{N-1}) \\ & & u & \longmapsto & u \circ g \end{array}$$

$$\begin{array}{rcccc} \lambda & : & H^{+s}(\mathbb{R}^{N-1}) & \longrightarrow & H^{+s}(\Gamma) \\ & & u & \longmapsto & u \circ g^{-1} \end{array}$$

are topological isomorphisms and inverses of each other (cf. [12, Satz 4.1]). Hence the same is true for their adjoints

$$\mu' \ : \ H^{-s}(\mathbb{R}^{N-1}) \ \longrightarrow \ H^{-s}(\Gamma)$$

$$\lambda' \ : \ H^{-s}(\Gamma) \ \longrightarrow \ H^{-s}(\mathbb{R}^{N-1}) \quad .$$

These operators may be extended to differential forms $\mathbf{E}$ by letting them act on their components in a Cartesian representation separately. So for $W \in \mathbf{H}^{+s,q}(\Gamma)$ decomposed as in (41) we define

$$\boldsymbol{\mu} W := (\boldsymbol{\mu} W' \, , \, \boldsymbol{\mu} W'')$$

and similarly

$$\boldsymbol{\lambda}' W := (\boldsymbol{\lambda}' W' \, , \, \boldsymbol{\lambda}' W'')$$

for $W \in \mathbf{H}^{-s,q}(\Gamma)$. Clearly, $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}'$ define topological isomorphisms from $\mathbf{H}^{\pm s,q}(\Gamma)$ onto $H^{\pm s,q-1}(\mathbb{R}^{N-1}) \times H^{\pm s,q}(\mathbb{R}^{N-1})$ and their inverses may be defined using $\boldsymbol{\mu}'$ and $\boldsymbol{\lambda}$ in an obvious way.

After these preparations let us decompose $\rho' E$ and $\rho' G$ according to (41):

$$\begin{aligned} \rho' E &= dt \wedge U' + U'' \quad , \quad U' \in \mathbf{H}_h^{-1/2,N-2-q}(\Gamma) \quad , \quad U'' \in \mathbf{H}_h^{-1/2,N-1-q}(\Gamma) \\ \rho' G &= dt \wedge V' + V'' \quad , \quad V' \in \mathbf{H}_h^{-1/2,N-q-3}(\Gamma) \quad , \quad V'' \in \mathbf{H}_h^{-1/2,N-2-q}(\Gamma) \end{aligned}$$

Pick $e \in C_0^\infty(\mathbb{R})$ such that $e(t) = 1$ for $t \in I = (\alpha \, , \, \beta)$. For $\hat{\Phi} \in C_0^{\infty,N-2-q}(\mathbb{R}^{N-1})$ we want to test (40) with

$$\boldsymbol{\Phi}(t,y) := e(t) \cdot \hat{\boldsymbol{\Phi}}(y) \quad . \tag{43}$$

We note

$$\boldsymbol{\tau} \boldsymbol{\Phi} = \boldsymbol{\lambda}(0 \, , \, \hat{\boldsymbol{\Phi}}) \in \mathbf{H}_h^{1/2,N-2-q} \tag{44}$$

$$\boldsymbol{\tau} \mathbf{d} \, \boldsymbol{\Phi} = \boldsymbol{\lambda}(0 \, , \, \mathrm{d} \, \hat{\boldsymbol{\Phi}}) \in \mathbf{H}_h^{1/2,N-1-q} \quad . \tag{45}$$

Therefore (40) implies

$$\boldsymbol{\lambda}' U''(\mathrm{d} \, \hat{\boldsymbol{\Phi}}) = \rho' E(\boldsymbol{\tau} \mathbf{d} \boldsymbol{\Phi}) = (-1)^{q+N-1} \rho' G(\boldsymbol{\tau} \boldsymbol{\Phi}) = (-1)^{q+N-1} \boldsymbol{\lambda}' V''(\hat{\boldsymbol{\Phi}}) \quad . \tag{46}$$

We introduce

$$\begin{aligned} D^{-1/2,p}(\mathbb{R}^{N-1}) &:= \{W \in H^{-1/2,p}(\mathbb{R}^{N-1}) \ : \ \delta \, W \in H^{-1/2,p-1}(\mathbb{R}^{N-1})\} \\ R^{-1/2,p}(\mathbb{R}^{N-1}) &:= \{W \in H^{-1/2,p}(\mathbb{R}^{N-1}) \ : \ \mathrm{d} \, W \in H^{-1/2,p+1}(\mathbb{R}^{N-1})\} \end{aligned}$$

and infer from (46)

$$\boldsymbol{\lambda}' U'' \in D^{-1/2,N-1-q}(\mathbb{R}^{N-1})$$

and therefore

$$E'' := * \, \boldsymbol{\lambda}' U'' \in R^{-1/2,q}(\mathbb{R}^{N-1}) \quad . \tag{47}$$

Pick $\chi \in C_0^\infty(\mathbb{R}^{N-1})$ such that $\chi(\xi) = 1$ for $|\xi| \leq 1$. Recalling (5)–(7) and using the Fourier transform F and its inverse F^{-1} we find

$$\begin{aligned}
E'' &= e_0'' + \mathrm{d}\, e_1'' \\
e_0'' &:= F^{-1}(\chi F E'') + F^{-1}\left((1-\chi) m^{-2} T R F E''\right) \\
e_1'' &:= -i F^{-1}\left((1-\chi) m^{-2} T F E''\right)
\end{aligned}$$

(where $R := R_\xi$, $T := T_\xi$ and m denotes the operator of multiplication by $|\xi|$). From (47) and (6) we find

$$\begin{aligned}
(1-\chi) m^{-2} T R F E'' &\in \hat{H}{}^{1/2,q}(\mathbb{R}^{N-1}) \\
(1-\chi) m^{-2} T F E'' &\in \hat{H}{}^{1/2,q-1}(\mathbb{R}^{N-1})
\end{aligned}$$

where

$$\hat{H}{}^{s,\cdots}(\mathbb{R}^{N-1}) := \{\hat{E} \quad : \quad \int_{\mathbb{R}^{N-1}} (1+|\xi|)^{2s} \| \hat{E} \|^2 < \infty\}$$

and thus

$$e_0'' \in H^{+1/2,q}(\mathbb{R}^{N-1}) \quad , \quad e_1'' \in H^{+1/2,q-1}(\mathbb{R}^{N-1}) \quad .$$

By the scalar trace theorem we may choose

$$\mathbf{e}_0 \in H^{1,q}(\Omega) \quad , \quad \mathbf{e}_1 \in H^{1,q-1}(\Omega)$$

such that

$$\boldsymbol{\tau} \mathbf{e}_0 = \boldsymbol{\lambda} e_0'' \quad , \quad \boldsymbol{\tau} \mathbf{e}_1 = \boldsymbol{\lambda} e_1'' \quad .$$

With the cut-off function e as above and $\overset{\wedge}{\Psi} \in C_0^{\infty, N-1-q}(\mathbb{R}^{N-1})$ we put

$$\boldsymbol{\Psi}(t,y) := e(t) \overset{\wedge}{\Psi}(y)$$

(which has properties analogous to (44) , (45)) and compute (using Remark 3)

$$\begin{aligned}
\rho'(\mathcal{T}(\mathbf{e}_o + \mathbf{d}\,\mathbf{e}_1))(\boldsymbol{\tau}\boldsymbol{\Psi}) &= (-1)^{qN} \int_\Gamma \star \left[\nu \wedge \boldsymbol{\lambda} e_0'' \wedge \boldsymbol{\lambda} \overset{\wedge}{\Psi}\right] do \\
&\quad + (-1)^{q(N-1)} \int_\Gamma \star \left[\nu \wedge \boldsymbol{\lambda} e_1'' \wedge \boldsymbol{\lambda} \mathrm{d} \overset{\wedge}{\Psi}\right] do \\
&= \int_\Gamma \boldsymbol{\lambda}(\langle e_0'', \star \overset{\wedge}{\Psi} \rangle) \nu_1 do \\
&\quad + (-1)^{N-q} \int_\Gamma \boldsymbol{\lambda}(\langle e_1'', \star \mathrm{d} \overset{\wedge}{\Psi} \rangle) \nu_1 do
\end{aligned}$$

because the forms $\boldsymbol{\lambda} e_0''$, $\boldsymbol{\lambda} \overset{\wedge}{\Psi}$, ... belong to $\mathbf{H}_h^{\cdot\cdot}(\Gamma)$. Finally,

$$\nu_1 = \left(1 + |\nabla F(y)|^2\right)^{-1/2} \quad , \quad d\mathbf{o} = \left(1 + |\nabla F(y)|^2\right)^{+1/2}$$

imply

$$\begin{aligned}\rho'(\mathcal{T}(\mathbf{e}_o + \mathbf{d}\,\mathbf{e}_1))(\boldsymbol{\tau}\boldsymbol{\Psi}) &= \langle\, e_0'' \,,\ *\ \overset{\wedge}{\Psi}\,\rangle - \langle\, e_1'' \,,\, \delta\ *\ \overset{\wedge}{\Psi}\,\rangle \\ &= (\,*\,e_0'' + \ *\,\mathrm{d}\,e_1'')(\overset{\wedge}{\Psi}) = \langle\, U'' \,,\, \overset{\wedge}{\Psi}\,\rangle = \rho' E(\boldsymbol{\tau}\boldsymbol{\Psi}) \quad .\end{aligned}$$

The fields $\boldsymbol{\tau}\boldsymbol{\Psi}$ being dense in $\mathbf{H}_h^{+1/2,N-1-q}$ this implies that

$$\rho'(\widetilde{E}) \in \mathbf{H}_v^{+1/2,N-1-q} \quad .$$

for $\widetilde{E} := E - \mathcal{T}(\mathbf{e}_o + \mathbf{d}\,\mathbf{e}_1)$. Furthermore, we still have

$$\mathrm{d}\ \widetilde{E} =: \widetilde{G} \in H^{-1/2,q+1} \quad . \tag{48}$$

Let us multiply the test field (43) by $t - \gamma$, $\gamma \in \mathbb{R}$, i. e.

$$\boldsymbol{\Phi} := e(t) \cdot (t-\gamma) \cdot \overset{\wedge}{\Phi}\,(y) \quad . \tag{49}$$

We have

$$\begin{aligned}\boldsymbol{\tau}\boldsymbol{\Phi} &= (t-\gamma)\boldsymbol{\lambda}(0\,,\ \overset{\wedge}{\Phi}) \in \mathbf{H}_h^{1/2,N-2-q}(\Gamma) \\ \boldsymbol{\tau}\mathbf{d}\ \boldsymbol{\Phi} &= dt \wedge W' + W''\end{aligned}$$

$$W' := \boldsymbol{\lambda}(0\,,\ \overset{\wedge}{\Phi}) \in \mathbf{H}_h^{1/2,N-1-q}(\Gamma) \quad , \quad W'' := (t-\gamma)\boldsymbol{\lambda}(0\,,\,\mathrm{d}\ \overset{\wedge}{\Phi}) \in \mathbf{H}_h^{1/2,N-q}(\Gamma)$$

Thus using (49) for testing the relation (40) corresponding to (48) gives

$$\rho'\ \widetilde{E}\ (W'') = (-1)^{q+N-1} \rho'\ \widetilde{G}\ ((t-\gamma)\boldsymbol{\lambda}\ \overset{\wedge}{\Phi})$$

for <u>all</u> $\gamma \in \mathbb{R}$. But this implies $\rho'\ \widetilde{G} = 0$ hence $\rho'\ \widetilde{E} = 0$ and finally $\widetilde{E} = 0$ as desired. **q.e.d.**

The usual homogeneous boundary value problem for the generalized Maxwell system in domains without any regularity properties (cf. [9]) may be formulated with the aid of

$$\overset{\circ}{R}{}^q(\Omega) := \overline{\mathbf{C}_0^{\infty,q}(\Omega)} \quad (\text{closure in } \mathbf{R}^q(\Omega)). \tag{50}$$

However, $\mathbf{E} \in \overset{\circ}{R}{}^q(\Omega)$ is equivalent to

$$\langle\, \mathbf{d}\,\mathbf{E}\,,\ \boldsymbol{\Phi}\,\rangle_{\mathbf{L}^{2,q+1}(\Omega)} + \langle\, \mathbf{E}\,,\ \boldsymbol{\delta\Phi}\,\rangle_{\mathbf{L}^{2,q}(\Omega)} = 0 \tag{51}$$

(by an analogous argument as given for [5, Thm. 2.4]). In this connection we have the following result (cf. [10, Rem. 1 and Thm. 5*]).

Theorem 5 *The tangential trace operator $\mathcal{T}$ is a topological isomorphism from*

$$\mathbf{R}^q(\Omega)/\overset{\circ}{R}{}^q(\Omega) \simeq \mathbf{R}^q(\Omega) \ominus \overset{\circ}{R}{}^q(\Omega)$$

onto $R^{-1/2,q}(\Gamma)$.

Proof: The preceding two theorems show that $\mathcal{T}$ is continuous from $\mathbf{R}^q(\Omega)$ onto $R^{-1/2,q}(\Gamma)$. So by basic results of functional analysis our assertion is equivalent to

$$\ker \mathcal{T} = \overset{\circ}{R}{}^q(\Omega) \quad .$$

But from (51) it is clear that

$$\overset{\circ}{R}{}^q(\Omega) \subset \ker \boldsymbol{\tau} \quad .$$

On the other hand, if $\mathbf{E} \in \ker \boldsymbol{\tau}$ then by (36) we get

$$\int_\Omega \mathbf{d}\,(\mathbf{E} \wedge \boldsymbol{\Psi}) = 0 \quad .$$

Replacing $\boldsymbol{\Psi}$ by $\star\boldsymbol{\Phi}$ yields (51) (see Lemma 1) and hence the other inclusion.
q.e.d.

On Γ , we have three different notions of d :

i) $\mathrm{d}_+ \; : \; H_\pi^{1/2,q}(\Gamma) \longrightarrow H_\rho^{-1/2,q+1}(\Gamma)$ as defined in Lemma 2 ;

ii) $\mathrm{d}_0 \; : \; R^q(\Gamma) \longrightarrow L^{2,q+1}(\Gamma)$ as defined in [6];

iii) $\mathrm{d}_- \; : \; H_\rho^{-1/2,q}(\Gamma) \longrightarrow Y'$ as defined in Lemma 4 .

The following result shows that they are compatible.

Theorem 6 *The operators* d_+ *and* d_0 *coincide on the intersection*

$$R^q(\Gamma) \cap H_\pi^{1/2,q}(\Gamma)$$

of their domains of definition and d_- *is an extension of both* d_+ *and* d_0 .

Proof: Let $E \in R^q(\Gamma) \cap H_\pi^{1/2,q}(\Gamma)$ and put

$$G_+ := \mathrm{d}_+ E \in H_\rho^{-1/2,q+1}(\Gamma)$$
$$G_0 := \mathrm{d}_0 E \in L^{2,q+1}(\Gamma) \quad .$$

There exists $\mathbf{E} \in \mathbf{H}^{1,q}(\Omega)$ such that $E = \pi\boldsymbol{\tau}\mathbf{E} = \iota^*\mathbf{E}$. As test device, pick $\boldsymbol{\Psi} \in \mathcal{D}^{N-2-q}$ and put

$$\Psi := \rho\boldsymbol{\tau}\boldsymbol{\Psi} = *\,\iota^*\boldsymbol{\Psi} \in H_\rho^{1/2,q+1}(\Gamma) \hookrightarrow L^{2,q+1}(\Gamma) \quad .$$

We have

$$\begin{aligned}\delta\,\Psi &= (-1)^{q(N-1)} * \mathrm{d} * \Psi \\ &= (-1)^{q(N-1)} * \mathrm{d} * *\,\iota^*\boldsymbol{\Psi} \\ &= (-1)^{q+N)}\iota^*\mathbf{d}\,\boldsymbol{\Psi} \in \mathbf{L}^{2,q}(\Omega) \hookrightarrow H_\rho^{-1/2,q}(\Gamma) \quad .\end{aligned}$$

So we can compute

$$\begin{aligned}G_0(\Psi) &= \langle \mathrm{d}_0 E\,,\, \Psi \rangle_{L^{2,q+1}(\Gamma)} \\ &= -\langle E\,,\, (-1)^{q+N} * \iota^*\mathbf{d}\,\boldsymbol{\Psi} \rangle_{L^{2,q}(\Gamma)} \\ &= (-1)^{q+N-1}\int_\Gamma \iota^*\mathbf{E} \wedge * * \iota^*\mathbf{d}\,\boldsymbol{\Psi} \\ &= (-1)^{(N-1)(q+1)}\int_\Omega \mathbf{d}\,\mathbf{E} \wedge \boldsymbol{\Psi} = G_+(\Psi) \quad .\end{aligned}$$

which proves the first assertion.

Second, let

$$E \in H_\pi^{1/2,q}(\Gamma) \hookrightarrow L^{2,q}(\Gamma) \hookrightarrow H_\rho^{-1/2,q}(\Gamma)$$

and put

$$\begin{aligned}G_+ &:= \mathrm{d}_+ E \in H_\rho^{-1/2,q}(\Gamma) \\ G_- &:= \mathrm{d}_- E \in Y' \quad .\end{aligned}$$

Again, there exists $\mathbf{E} \in \mathbf{H}^{1,q}(\Omega)$ such that $E = \pi\boldsymbol{\tau}\mathbf{E} = \iota^*\mathbf{E}$. Pick $\boldsymbol{\Psi} \in \mathcal{D}^{N-2-q}$ and put $\Psi := \rho\boldsymbol{\tau}\boldsymbol{\Psi} = *\,\iota^*\boldsymbol{\Psi}$. We compute

$$\begin{aligned}G_-(\Psi) &= (-1)^{q+N-1}E(*\,\iota^*\mathbf{d}\,\boldsymbol{\Psi}) \\ &= (-1)^{q+N-1}\int_\Gamma \iota^*\mathbf{E} \wedge * * \iota^*\mathbf{d}\,\boldsymbol{\Psi} \\ &= (-1)^{q+N+qN-1}\int_\Omega \mathbf{d}\,\mathbf{E} \wedge \mathbf{d}\,\boldsymbol{\Psi} = G_+(\Psi)\end{aligned}$$

thus proving the second assertion.

Third, let $E \in R^q(\Gamma) \hookrightarrow L^{2,q}(\Gamma) \hookrightarrow H_\rho^{-1/2,q}(\Gamma)$ and

$$G_0 := \mathrm{d}_0 E \in H_\rho^{-1/2,q}(\Gamma) \quad , \quad G_- := \mathrm{d}_- E \in Y' \quad .$$

Pick $\boldsymbol{\Psi} \in \mathcal{D}^{N-2-q}$ and put $\Psi := \rho\boldsymbol{\tau}\boldsymbol{\Psi} \ * \ \iota^*\boldsymbol{\Psi}$. As computed above, we have

$$\delta\,\Psi = (-1)^{q+N} \ * \ \iota^*\mathbf{d}\,\boldsymbol{\Psi} \in H_\rho^{1/2,q}(\Gamma) \hookrightarrow L^{2,q}(\Gamma) \quad .$$

Therefore

$$\begin{aligned} G_-(\boldsymbol{\Psi}) &= (-1)^{q+N-1} E(\, * \,\iota^*\mathbf{d}\,\boldsymbol{\Psi}) \\ &= -E(\delta\,\Psi) = G_0(\Psi) \quad . \end{aligned}$$

q.e.d.

4 Dual results and Hodge–Helmholtz–decomposition

The calculus of alternating differential forms has a convenient built–in duality device. Let us apply it to the preceding results.

Lemma 5 *The operator*

$$* \ : \ L^{2,q}(\Gamma) \ \longrightarrow \ L^{2,N-1-q}(\Gamma)$$

may be restricted to $H_{\rho/\pi}^{+1/2,q}(\Gamma)$ *and extended by continuity to* $H_{\rho/\pi}^{-1/2,q}(\Gamma)$ *. It has the following properties.*

i) *The maps*

$$* \ : \ H_\rho^{\pm 1/2,q} \ \longrightarrow \ H_\pi^{\pm 1/2,N-1-q}$$

and

$$* \ : \ H_\pi^{\pm 1/2,q} \ \longrightarrow \ H_\rho^{\pm 1/2,N-1-q}$$

are isometric isomorphisms.

ii) *If* $E \in H_\pi^{1/2,q}(\Gamma)$ *is represented by* $\mathbf{E} \in \mathbf{H}^{1,q}(\Omega)$ *(i. e.* $E = \pi\boldsymbol{\tau}\mathbf{E} = \iota^*\mathbf{E}$*) then* $F := \, * \, E$ *is represented by* $\mathbf{E}$ *, too, i. e.*

$$F = \rho\boldsymbol{\tau}\mathbf{E} = \, * \,\iota^*\mathbf{E} \quad .$$

On the other hand, if $F \in H_\rho^{1/2,q}(\Gamma)$ *is represented by* $\mathbf{F} \in \mathbf{H}^{1,N-1-q}(\Omega)$ *then* $E := \, * \, F$ *is represented by*

$$\mathbf{E} := (-1)^{qN}\mathbf{F} \quad .$$

iii) *We have*

$$* \, E(\, * \, F) = E(F)$$

if $E \in H_{\rho/\pi}^{-1/2,q}(\Gamma)$ *and* $F \in H_{\rho/\pi}^{+1/2,q}(\Gamma)$ *.*

Proof: i) and ii) may be read off from (20) and iii) follows by continuity from

$$\langle E, F\rangle_{L^{2,q}(\Gamma)} = \langle *E, *F\rangle_{L^{2,N-1-q}(\Gamma)} \quad .$$

q.e.d.

The operator $\delta := (-1)^{(q-1)(N-1)} * \mathrm{d} *$ is the formal adjoint of d . An application of Lemma 5 gives the following dual version of Lemmas 2 and 4 .

Lemma 6 *The operators*

$$\begin{array}{llcl} \delta & : H_\rho^{1/2,q}(\Gamma) & \longrightarrow & H_\pi^{-1/2,q-1}(\Gamma) \\ & \quad E & \longmapsto & \delta E := (-1)^{(q-1)(N-1)} * \mathrm{d} * E \end{array}$$

and

$$\begin{array}{llcl} \delta & : H_\pi^{-1/2,q}(\Gamma) & \longrightarrow & Z' \\ & \quad F & \longmapsto & \delta F := (-1)^{(q-1)(N-1)} * \mathrm{d} * F \end{array}$$

are well defined by

$$\delta E(\Phi) := (-1)^{(q-1)(N-1)} \int_\Omega \mathbf{d\, E} \wedge \mathbf{\Phi}$$

if

$$\begin{array}{ll} E = \rho\tau\mathbf{E} \quad , & \mathbf{E} \in \mathbf{H}^{1,N-1-q}(\Omega) \\ \Phi = \pi\boldsymbol{\tau}\mathbf{\Phi} \quad , & \mathbf{\Phi} \in \mathbf{H}^{1,q-1}(\Omega) \end{array}$$

resp. by

$$\delta F(\Phi) := -F(\iota^* \mathbf{d} \star \mathbf{\Phi})$$

if

$$\Phi = \iota^* \star \mathbf{\Phi} \quad , \quad \mathbf{\Phi} \in \mathbf{Z}_{N-1-q}$$

where

$$\mathbf{Z}_p := \{\mathbf{\Phi} \in \mathbf{H}^{1,p}(\Omega) \quad : \quad \boldsymbol{\delta}\mathbf{\Phi} \in \mathbf{H}^{1,p-1}(\Omega)\}$$

carries its natural norm and

$$Z := \{\iota^* \star \mathbf{\Phi} \quad : \quad \mathbf{\Phi} \in \mathbf{Z}_{N-1-q}\}$$

carries the norm

$$|\Phi|_Z := \inf\{\|\mathbf{\Phi}\| \quad : \quad \Phi = \iota^* \star \mathbf{\Phi} \quad , \quad \mathbf{\Phi} \in \mathbf{Z}_{N-1-q}\} \quad .$$

The counterparts of Theorems 3–5 are collected in the following theorem (cf. [4, (2.33)]).

Theorem 7 *Let*

$$D^{-1/2,q-1}(\Gamma) := \{E \in H_\pi^{-1/2,q-1}(\Gamma) \quad : \quad \delta\, E \in H_\pi^{-1/2,q-2}(\Gamma)\}$$

be supplied with its natural norm. Then the "normal trace operator"

$$\begin{array}{rcll} \mathcal{N} & : & \mathbf{D}^q(\Omega) & \longrightarrow & D^{-1/2,q-1}(\Gamma) \\ & & \mathbf{E} & \longmapsto & (-1)^{(q-1)N} * \iota^* \star \mathbf{E} \end{array}$$

is well defined, linear and continuous.

It is surjective and thus has a continuous right inverse which may be chosen such that its range lies in

$$\mathbf{H}^{1,q}(\Omega) + \boldsymbol{\delta}\mathbf{H}^{1,q+1}(\Omega) \quad .$$

Furthermore

$$\ker \mathcal{N} = \overset{\circ}{D}{}^q(\Omega) := \overline{\mathbf{C}_0^{\infty,q}(\Omega)} \quad (\text{closure in } \mathbf{D}^q(\Omega)\,)$$

and hence $\mathcal{N}$ gives rise to a topological isomorphism from

$$\mathbf{D}^q(\Omega)/\overset{\circ}{D}{}^q(\Omega) \simeq \mathbf{D}^q(\Omega) \ominus \overset{\circ}{D}{}^q(\Omega)$$

onto

$$D^{-1/2,q-1}(\Gamma) \quad .$$

Proof: The map $\star \ : \ \mathbf{D}^q(\Omega) \longrightarrow R^{N-q}(\Omega)$ is a topological isomorphism and by Lemmas 5 and 6 the same is true for $* \ : \ R^{-1/2,N-q}(\Gamma) \longrightarrow D^{-1/2,q-1}(\Gamma)$ So Theorem 7 is a direct consequence of Theorems 3–5. **q.e.d.**

There exists a duality between the "tangential trace space" $R^{-1/2,q}(\Gamma)$ and the "normal trace space" $D^{-1/2,q}(\Gamma)$ as exhibited in the following result (cf. [3, Lemma 5.6.]).

Theorem 8 *The $L^{2,q}$–scalar–product can be extended as a continuous bilinear form to $R^{-1/2,q}(\Gamma) \times D^{-1/2,q}(\Gamma)$.*

Proof: For $\mathbf{E} \in \mathcal{D}^q$ and $\mathbf{F} \in \mathcal{D}^{q+1}$ we get

$$\langle\, \mathcal{T}\mathbf{E}\,,\, \mathcal{N}\mathbf{F}\,\rangle_{L^{2,q}(\Gamma)} = \langle\, \mathbf{d}\,\mathbf{E}\,,\, \mathbf{F}\,\rangle_{\mathbf{L}^{2,q+1}(\Omega)} + \langle\, \mathbf{E}\,,\, \boldsymbol{\delta}\mathbf{F}\,\rangle_{\mathbf{L}^{2,q}(\Omega)}$$

by inserting our previous definitions and applying (10). Writing $E \in R^{-1/2,q}(\Gamma)$ as $\mathcal{T}\mathbf{E}$, $\mathbf{E} \in \mathbf{R}^q(\Omega)$ and $F \in D^{-1/2,q}(\Gamma)$ as $\mathcal{N}\mathbf{F}$, $\mathbf{F} \in \mathbf{D}^{q+1}(\Omega)$ (as we may by

Theorems 4 and 7) and approximating $\mathbf{E}$ and $\mathbf{F}$ with the aid of Lemma 1 proves our assertion. **q.e.d.**

Hodge–Helmholtz–decompositions on the $H^{-1/2}$–level may be based on well-known results for Lipschitz manifolds on the L^2–level (cf. [6, Thm. 2],[8]).

Theorem 9 *(K. McLeod, R. Picard) With a finite-dimensional space $\mathcal{H}$ of "harmonic q-forms" we have the orthogonal decomposition*

$$L^{2,q}(\Gamma) = \mathrm{d}\, R^{q-1}(\Gamma) \oplus \delta\, D^{q+1}(\Gamma) \oplus \mathcal{H} \quad .$$

In the case of a smooth manifold Γ, the preceding theorem can easily be sharpened. Namely, if $E \in L^{2,q}(\Gamma)$ is decomposed as

$$E = \mathrm{d}\, F + \delta\, G + h \quad , \quad F \in R^{q-1}(\Gamma) \quad , \quad G \in D^{q+1}(\Gamma) \quad , \quad h \in \mathcal{H} \tag{52}$$

then we may decompose F according to Theorem 9 . This argument and an analogous one for G leads to the decomposition

$$E = \mathrm{d}\, F_0 + \delta\, G_0 + h \quad , \quad F_0 \in R^{q-1}(\Gamma) \quad , \quad G_0 \in D^{q+1}(\Gamma) \quad , \quad h \in \mathcal{H} \tag{53}$$

where additionally

$$\delta\, F_0 = 0 \quad , \quad \mathrm{d}\, G_0 = 0 \quad .$$

But this implies (applying Gaffney's inequality as we may in the case of a smooth boundary)

$$F_0 \in H^{1,q-1}(\Gamma) \quad , \quad G_0 \in H^{1,q+1}(\Gamma)$$

and therefore

$$L^{2,q}(\Gamma) = \mathrm{d}\, H^{1,q-1}(\Gamma) \oplus \delta\, H^{1,q+1}(\Gamma) \oplus \mathcal{H} \quad .$$

In the case of a Lipschitz manifold, Gaffney's inequality is no longer available in general. But we can show that we have at least our generalized $H^{1/2}$–regularity. Namely, from Theorem 6 as well as from Theorems 4 and 7 we infer

$$R^q(\Gamma) \hookrightarrow R^{-1/2,q}(\Gamma) = \mathcal{T}\mathbf{H}^{1,q}(\Omega) + \mathcal{T}\mathbf{d}\,\mathbf{H}^{1,q-1}(\Omega) \tag{54}$$

$$D^q(\Gamma) \hookrightarrow D^{-1/2,q}(\Gamma) = \mathcal{N}\mathbf{H}^{1,q}(\Omega) + \mathcal{N}\boldsymbol{\delta}\mathbf{H}^{1,q+1}(\Omega) \quad . \tag{55}$$

Decompose $E \in L^{2,q}(\Gamma)$ according to Theorem 9

$$E = \mathrm{d}\, F + \delta\, G + h \quad , \quad F \in R^{q-1}(\Gamma) \quad , \quad G \in D^{q+1}(\Gamma) \quad , \quad h \in \mathcal{H}$$

and write F , G according to (54) , (55) as

$$F = \mathcal{T}\mathbf{F}_0 + \mathcal{T}\mathbf{d}\,\mathbf{F}_1 \quad , \quad \mathbf{F}_0 \in \mathbf{H}^{1,q-1}(\Omega) \quad , \quad \mathbf{F}_1 \in \mathbf{H}^{1,q-2}(\Omega)$$
$$G = \mathcal{N}\mathbf{G}_0 + \mathcal{N}\boldsymbol{\delta}\mathbf{G}_1 \quad , \quad \mathbf{G}_0 \in \mathbf{H}^{1,q+1}(\Omega) \quad , \quad \mathbf{G}_1 \in \mathbf{H}^{1,q+2}(\Omega) \quad .$$

From Remark 3 we have

$$\mathrm{d}\,\mathcal{T}\mathbf{d}\,\mathbf{F}_1 = 0 \quad , \quad \delta\,\mathcal{N}\boldsymbol{\delta}\mathbf{G}_1 = 0$$

and therefore

$$E = \mathrm{d}\,F_0 + \delta\,G_0 + h$$
$$F_0 := \mathcal{T}\mathbf{F}_0 \in H_\pi^{1/2,q-1}(\Gamma)$$
$$G_0 := \mathcal{N}\mathbf{G}_0 \in H_\pi^{1/2,q-1}(\Gamma) \quad .$$

Thus we get the following improvement of Theorem 9 .

Theorem 10

$$L^{2,q}(\Gamma) = \mathrm{d}\ \left(H_\pi^{1/2,q-1}(\Gamma) \cap R^{q-1}(\Gamma)\right) \oplus \delta\ \left(H_\rho^{1/2,q+1}(\Gamma) \cap D^{q+1}(\Gamma)\right) \oplus \mathcal{H}$$

But we can also show the following extension to the $H^{-1/2}$–level.

Theorem 11 *The spaces $R^{-1/2,q}(\Gamma)$ and $D^{-1/2,q}(\Gamma)$ may be decomposed as*

$$R^{-1/2,q}(\Gamma) = \mathrm{d}\ \left(H_\pi^{1/2,q-1}(\Gamma)\right) + \delta\ \left(H_\rho^{1/2,q+1}(\Gamma) \cap D^{q+1}(\Gamma)\right) + \mathcal{H}$$
$$D^{-1/2,q}(\Gamma) = \mathrm{d}\ \left(H_\pi^{1/2,q-1}(\Gamma) \cap R^{q-1}(\Gamma)\right) + \delta\ \left(H_\rho^{1/2,q+1}(\Gamma)\right) + \mathcal{H}$$

with direct sums.

Proof: The directness follows from Theorem 10 . In order to construct the decomposition, write $E \in R^{-1/2,q}(\Gamma)$ as

$$E = \mathcal{T}\mathbf{E}_0 + \mathcal{T}\mathbf{d}\,\mathbf{E}_1 \quad , \quad \mathbf{E}_0 \in \mathbf{H}^{1,q}(\Omega) \quad , \quad \mathbf{E}_1 \in \mathbf{H}^{1,q-1}(\Omega)$$

according to Theorem 4 . From the preceding theorem, we get

$$\mathcal{T}\mathbf{E}_0 = \mathrm{d}\,F_0 + \delta\,G_0 + h \tag{56}$$

where F_0 , G_0 and h are in the appropriate spaces. Furthermore, from Remark 3 ,

$$\mathcal{T}\mathbf{d}\,\mathbf{E}_1 = \mathrm{d}\,E_1 \quad , \quad E_1 := \mathcal{T}\mathbf{E}_1 \in H_\pi^{1/2,q-1}(\Gamma) \quad . \tag{57}$$

Thus combining (56) and (57) yields the first assertion and the second can be proved analogously. **q.e.d.**

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