9.23 A Closer Look at the Spinor Representation

This discussion builds the structure of the Lorentz group and its representations by following a common train of thought, correcting key misconceptions along the way.

The Foundation: Rotations, $SO(3)$, and $SU(2)$

We begin with the non-relativistic case of 3D rotations.

- At the **group level**, we have $SO(3)$, the group of 3x3 real orthogonal matrices that act on vectors. We also have $SU(2)$, the group of 2x2 complex unitary matrices that act on spinors. $SU(2)$ is the **universal double-cover** of $SO(3)$.
- Both groups are 3-parameter groups, generated by 3 generators.
- At the **algebra level**, their Lie algebras are **isomorphic**: $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. They share the same commutation relations $[J_i, J_j] = i\epsilon_{ijk}J_k$.

This 2-to-1 group covering, backed by an algebraic isomorphism, is the template for the relativistic case.

The Lorentz Case: $SO(1,3)$ and $SL(2,\mathbb{C})$

We now extend this analogy to spacetime, which includes boosts.

- At the **group level**, we have $SO^+(1,3)$, the proper orthochronous Lorentz group, which acts on 4-vectors (x^μ). We also have $SL(2,\mathbb{C})$, the group of 2x2 complex matrices with determinant 1, which acts on 2-component Weyl spinors.
- $SL(2,\mathbb{C})$ is the **universal double-cover** of $SO^+(1,3)$.
- Both are 6-parameter groups, generated by 6 generators (3 rotations J_i, 3 boosts K_i).
- At the **algebra level**, their 6-dimensional *real* Lie algebras are **isomorphic**: $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$.

A Critical Distinction: $\mathfrak{so}(1,3)$ vs. $\mathfrak{so}(4)$

A very common point of confusion is to assume $\mathfrak{so}(1,3) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$. This is incorrect. The algebra $\mathfrak{su}(2) \times \mathfrak{su}(2)$ is isomorphic to the 4D *Euclidean* rotation algebra, $\mathfrak{so}(4)$, not the Lorentz algebra.
The difference lies in a single, crucial minus sign.

- **The $\mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$ Algebra:** Let the generators be $J_i = A_i + B_i$ and $K_i = A_i - B_i$, where $[A_i, B_j] = 0$. The boost-like commutator is:

$$[K_i, K_j] = [A_i - B_i, A_j - B_j] = [A_i, A_j] + [B_i, B_j] = i\epsilon_{ijk}A_k + i\epsilon_{ijk}B_k = +i\epsilon_{ijk}J_k$$

- **The $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$ Algebra:** The physical Lorentz algebra has the defining relation:

$$[K_i, K_j] = -i\epsilon_{ijk}J_k$$

Because their algebraic structure is fundamentally different, $\mathfrak{so}(1,3) \ncong \mathfrak{so}(4)$ and $\mathfrak{so}(1,3) \ncong \mathfrak{su}(2) \times \mathfrak{su}(2)$ as real algebras.

The Impossibility of Decomposing the Real Algebra

The fact that $\mathfrak{so}(1,3) \ncong \mathfrak{su}(2) \times \mathfrak{su}(2)$ is the entire problem. It means that the physical Lorentz algebra, as a *real* algebra, does not decompose into a direct product. The rotations (J_i) and boosts (K_i) are inextricably linked by $[J_i, K_j] = i\epsilon_{ijk}K_k$. We cannot find a simple set of commuting generators using only real numbers.

The Complexification Trick

Since we cannot decompose the *real* algebra, we employ a mathematical trick: we complexify it. We move from $\mathfrak{so}(1,3)$ to $\mathfrak{so}(1,3)_{\mathbb{C}}$ by allowing complex linear combinations of the generators.
This is done *purely* to find a basis where the algebra decouples. We define the complex generators:

$$N_i = \frac{1}{2}(J_i + iK_i) \quad \text{and} \quad \tilde{N}_i = \frac{1}{2}(J_i - iK_i)$$

The crucial minus sign in $[K_i, K_j] = -i\epsilon_{ijk}J_k$ is precisely what's needed to make these two new algebras commute:

$$[N_i, B_j] = 0$$

Thus, the *complexified* algebra **does** decompose:

$$\mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2)_{\mathbb{C}} \times \mathfrak{su}(2)_{\mathbb{C}}$$

This allows us to classify all representations by two labels, one for each $\mathfrak{su}(2)_{\mathbb{C}}$ factor.

Generator Counting: The 12-Dimensional Real Space

Let's be precise about dimensions.

- $\mathfrak{su}(2)$ is a 3-dimensional *real* algebra (basis, e.g., $\{\sigma_1, \sigma_2, \sigma_3\}$).
- $\mathfrak{su}(2)_{\mathbb{C}}$ is its complexification. As a *complex* algebra, it is 3-dimensional (basis, e.g., $\{\sigma_1, \sigma_2, \sigma_3\}$ with complex coefficients in the "rotation angles"). As a *real* algebra, it is 6-dimensional (basis, e.g., $\{\sigma_i, i\sigma_i\}$).

- Therefore, the full decoupled space $\mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{su}(2)_{\mathbb{C}} \times \mathfrak{su}(2)_{\mathbb{C}}$ is a 12-dimensional *real* vector space. These are $\{\sigma_i, i\sigma_i\}$ for $\mathfrak{su}(2)_{\mathbb{C}}^L$ and $\{\sigma_i, i\sigma_i\}$ for $\mathfrak{su}(2)_{\mathbb{C}}^R$.

The Physical Subalgebra: Projecting back to $\mathfrak{so}(1,3)$

We are only interested in the physical world, which is described by the 6-dimensional *real* algebra $\mathfrak{so}(1,3)$, not the 12D complex space. Our physical algebra $\mathfrak{so}(1,3)$ must be a 6-dimensional “real slice” embedded within this 12D space.
We find this "slice" by inverting the definitions for N_i and B_i:

$$J_i = N_i + \tilde{N}_i$$

$$K_i = -i(N_i - \tilde{N}_i)$$

These are our 6 physical generators, built from the 12 generators of the complex space. Which 6? 3 out of the 6 hermitian ones $\sigma_i^L + \sigma_i^R$ and 3 out of the 6 anti-hermitian ones $-i\sigma_i^L + i\sigma_i^R$ while maintaining the “rotation/boost coefficients” being real.
The 6 physical generators are *not* just "two sets of Pauli matrices" (e.g., N_i and $\tilde{N}_i$). They are this specific combination:

- **Rotations** $J_i = N_i + \tilde{N}_i$**:** In a representation where N_i and B_i are Hermitian, the J_i are also **Hermitian**. This is required so that rotations $U_R = \exp(-i\theta_j J_j)$ are unitary.
- **Boosts** $K_i = -i(N_i - \tilde{N}_i)$**:** With $N_i, \tilde{N}_i$ Hermitian, the K_i are **anti-Hermitian** (because of the i). This is also required, so that boosts $U_{\text{boost}} = \exp(-i\beta_j K_j) = \exp(-\beta_j(N_j - \tilde{N}_j))$ are non-unitary.

Our 6D physical world $\mathfrak{so}(1,3)$ is this specific “slice” of the 12D complexified space that is spanned by 3 Hermitian generators (rotations) and 3 anti-Hermitian generators (boosts).