9.19 (Perturbative) QFT as Asymptotic Series

One of a profound insights into the nature of perturbative Quantum Field Theory (pQFT) comes not from a complex calculation, but from a simple, elegant physical argument by Freeman Dyson. It explains *why* the power series we calculate in QFT cannot be a convergent series, but must instead be an asymptotic one.

The Premise: Convergence (of power series) implies Analyticity

Let's consider a physical observable $\mathcal{A}$ (like a scattering amplitude) that we calculate in pQFT. We compute it as a power series in the coupling constant, g. (In QED, we often expand in $\alpha \propto e^2$).

$$\mathcal{A}(g) = \sum_{n=0}^{\infty} c_n g^n$$

c_n obtainable from the path integral on the fields. Now, let's make an assumption for the sake of contradiction: **Assume this series converges.** If the series has a non-zero radius of convergence $R > 0$, it means the function $\mathcal{A}(g)$ is **analytic** in a disk of radius R in the complex g-plane. This is a very powerful mathematical constraint. It implies that the function $\mathcal{A}(g)$ must be well-behaved and well-defined for *all* complex values of g inside that disk. This includes small positive g, small imaginary g, and – crucially – small *negative* g.

The Physical Test: Dyson's "Unstable World" ($g < 0$)

Dyson's genius was to ask: what would the physics of our theory look like if the coupling constant g had the opposite sign?
Let's use QED as our example, where the expansion parameter is $\alpha \propto e^2 > 0$.

- **Our World ($\alpha > 0$):**
  - The potential between two electrons is $V(r) = +\frac{e^2}{4\pi r} > 0$. This is **repulsive**.
  - The potential between an electron and a positron is $V(r) = -\frac{e^2}{4\pi r} < 0$. This is **attractive**.
  - This configuration leads to a **stable vacuum**. The vacuum is the state of lowest energy. Creating an e^+e^- pair costs energy

($2m_ec^2$), and they will be attracted to each other and annihilate.

- **Dyson's World ($\alpha < 0$):** Now, let's see what happens if we flip the sign of α (or equivalently, let $e^2 \to -e^2$).
  - The potential between two electrons becomes $V(r) = -\frac{e^2}{4\pi r} < 0$. **Like charges now attract!**
  - The potential between an electron and a positron becomes $V(r) = +\frac{e^2}{4\pi r} > 0$. **Opposite charges now repel!**

The Catastrophe: Vacuum Instability

This new "Dyson's World" is physically pathological and catastrophically unstable.
Imagine the "empty" vacuum. A quantum fluctuation can create an e^+e^- pair, costing $2m_ec^2$ of energy.

1. **Creation:** An e^+e^- pair is created from the vacuum.
2. **Repulsion:** In Dyson's World, the e^+ and e^- *repel* each other and fly apart.
3. **Spontaneous Boiling:** The vacuum can now create a *second* pair (e_2^+, e_2^-). But now, e_2^- is *attracted* to e_1^-, and e_2^+ is *attracted* to e_1^+.
4. **Runaway Process:** The vacuum can continue to "boil," creating a vast number of pairs. All the electrons will clump together in one region, and all the positrons will clump together in another region, repelling the electron-clump and flying away.
5. **Infinite Negative Energy:** The potential energy of a clump of N attracting electrons is proportional to $-N^2$. This energy becomes infinitely negative, overwhelming the positive $N \times m_ec^2$ mass-energy cost.

The conclusion is stark: for $g < 0$, the vacuum is not the state of lowest energy. The theory is ill-defined and explodes. The "ground state" energy is $-\infty$.

Connecting Physics Back to Mathematics

Here is the contradiction:

1. Our initial assumption (that the series converges) implies $\mathcal{A}(g)$ must be **analytic** for small negative g.
2. Our physical analysis (Dyson's argument) shows that the theory is **catastrophically unstable** and nonsensical for *any* small negative g. A function describing a physical observable for an unstable theory cannot be well-defined or analytic.

Therefore, our initial assumption must be false. The pQFT series **cannot converge**.

The point $g = 0$ is a non-analytic point; it is an **essential singularity**. A Taylor series expansion (which is what the perturbation series is) of a function about an essential singularity *always* has a radius of convergence of zero.

What is an Essential Singularity?

In complex analysis, a singularity is a point where a function is not "well-behaved" (analytic). An essential singularity is the most "pathological" or "wild" kind of singularity. Its behavior is so chaotic that it doesn't approach any single limit.

We can classify singularities at $z = 0$ using their Laurent series (a Taylor series that allows negative powers):

- **1. Removable Singularity (Tame):** A "patchable" hole.
  - **Example:** $f(z) = \frac{\sin(z)}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$
  - **Series:** Has *no* negative-power terms.
  - **Behavior:** As $z \to 0$, $f(z) \to 1$. It approaches a finite limit.
- **2. Pole (Predictable):** The function goes to infinity.
  - **Example:** $f(z) = \frac{1}{z^2}$
  - **Series:** Has a *finite number* of negative-power terms (here, just z^{-2}).
  - **Behavior:** As $z \to 0$, $|f(z)| \to \infty$ predictably.
- **3. Essential Singularity (Wild):** The function is chaotic.
  - **Example:** $f(z) = e^{1/z}$
  - **Series:** $f(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$
  - **Definition:** Has an *infinite number* of negative-power terms.
  - **Behavior:** The limit as $z \to 0$ does not exist. It depends on the path:
    * Path 1 (Real axis, $z = x \to 0^+$): $e^{1/x} \to \infty$.
    * Path 2 (Real axis, $z = x \to 0^-$): $e^{1/x} \to e^{-\infty} \to 0$.
    * Path 3 (Imaginary axis, $z = iy \to 0$): $e^{1/(iy)} = e^{-i/y} = \cos(1/y) - i\sin(1/y)$, which oscillates infinitely fast and approaches no limit.

Connection to QFT: Non-perturbative effects often have the form $f(g) = e^{-c/g^2}$. This function has an essential singularity at $g = 0$. If you try to calculate its Taylor series, you find that the function and all its derivatives are exactly zero at $g = 0$. Its Taylor series is just $0 + 0g + 0g^2 + \dots$, which clearly does not equal the function. This is why such terms are "invisible" to perturbation theory.

A Toy Model: 0-Dimensional ϕ^4 Theory

We can see this mathematical behavior in a simple integral, which we can think of as a 0-dimensional QFT. The "path integral" for a $g\phi^4$ theory is:

$$Z(g) = \int_{-\infty}^{\infty} d\phi \, \exp\left(-\frac{1}{2}\phi^2 - \frac{g}{4!}\phi^4\right)$$

This $Z(g)$ represents the "vacuum" of the theory.

- **Our World ($g > 0$):** The $e^{-g\phi^4}$ term helps the $e^{-\phi^2}$ term, making the integral *even more* convergent for large ϕ. The theory is stable.
- **Dyson's World ($g < 0$):** Let $g = -|\lambda|$ where $|\lambda| > 0$.

$$Z(-|\lambda|) = \int_{-\infty}^{\infty} d\phi \, \exp\left(-\frac{1}{2}\phi^2 + \frac{|\lambda|}{4!}\phi^4\right)$$

  As $\phi \to \pm\infty$, the $+\phi^4$ term in the exponent dominates. The $e^{+\phi^4}$ term explodes, and the integral **diverges** completely.

The function $Z(g)$ is well-defined for $g > 0$ but diverges for all $g < 0$. It is manifestly non-analytic at $g = 0$. If you were to generate the perturbation series for $Z(g)$ (by expanding the exponential in g), you would find that the coefficients c_n grow factorially ($c_n \sim n!$), which is the mathematical signature of an asymptotic series with zero radius of convergence.

Finding the $e^{-c/g}$ Term (An Instanton Calculation)

How do we know the non-analytic part has the form $e^{-c/g}$? This term doesn't just appear; it emerges from the path integral when we look beyond the "trivial" solution. We can find it using a **saddle-point approximation** (the 0-dimensional version of an "instanton" calculation). The integral is $Z(g) = \int d\phi \, e^{-S(\phi)}$, where the "action" is $S(\phi) = \frac{1}{2}\phi^2 + \frac{g}{24}\phi^4$. The dominant contributions to this integral come from points where the action is stationary, i.e., $S'(\phi) = 0$.

$$S'(\phi) = \frac{dS}{d\phi} = \phi + \frac{g}{6}\phi^3 = \phi\left(1 + \frac{g}{6}\phi^2\right) = 0$$

This equation has several solutions (saddle points):

- **The Trivial Saddle:** $\phi_0 = 0$. If we approximate the integral by expanding $S(\phi)$ around $\phi = 0$, we are just doing standard perturbation theory. This gives the asymptotic series $\sum c_n g^n$.
- **The Non-Trivial Saddles:** $1 + \frac{g}{6}\phi^2 = 0$. This gives $\phi_{inst}^2 = -6/g$. These are the "instanton" solutions.

- For $g > 0$ (our world), ϕ_{inst}^2 is negative, so these solutions $\phi_\pm = \pm i\sqrt{6/g}$ are in the complex plane. They aren't on our real integration path.
- For $g < 0$ (Dyson's world), let $g = -|\lambda|$. Then $\phi_{inst}^2 = -6/(-|\lambda|) = 6/|\lambda|$. The solutions $\phi_\pm = \pm\sqrt{6/|\lambda|}$ are **real**. These are the unstable vacua Dyson predicted, appearing as new real saddle points in the math.

The contribution of any saddle point to the integral is proportional to $e^{-S_{saddle}}$. Let's calculate the action at these new saddles:

$$S(\phi_{inst}) = \frac{1}{2}\phi_{inst}^2 + \frac{g}{24}\phi_{inst}^4$$

Substitute $\phi_{inst}^2 = -6/g$:

$$S(\phi_{inst}) = \frac{1}{2}\left(-\frac{6}{g}\right) + \frac{g}{24}\left(-\frac{6}{g}\right)^2$$

$$S(\phi_{inst}) = -\frac{3}{g} + \frac{g}{24}\left(\frac{36}{g^2}\right) = -\frac{3}{g} + \frac{3}{2g}$$

$$S(\phi_{inst}) = -\frac{3}{2g}$$

Therefore, the contribution to the integral from these saddles is:

$$Z_{non-pert} \sim e^{-S(\phi_{inst})} = e^{-(-3/(2g))} = e^{3/(2g)}$$

This is the non-analytic behavior. For $g > 0$, this is $e^{+3/(2g)}$ which blows up and is suppressed. But for $g = -|\lambda|$, as in Dyson's unstable world, this becomes:

$$Z_{non-pert} \sim e^{3/(2(-|\lambda|))} = e^{-3/(2|\lambda|)}$$

This is precisely the $e^{-c/|g|}$ form. It is a non-perturbative, non-analytic contribution to the full answer $Z(g)$ that is completely "invisible" to the Taylor series expansion around $g = 0$.