Vector Field.

Recall E & M.

$$\vec{\nabla}\cdot\vec{E} = \rho$$

$$\vec{\nabla}\times\vec{B} - \dot{\vec{E}} = \vec{J}$$

$$\vec{\nabla}\times\vec{E} + \dot{\vec{B}} = 0$$

$$\nabla\cdot\vec{B} = 0$$

← can be solved by writing a scalar and vector potential.

$$\vec{E} = -\vec{\nabla}\varphi - \dot{\vec{A}}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

← clearly $\varphi \to \varphi + \dot{\Gamma}$, $\vec{A} \to \vec{A} - \vec{\nabla}\Gamma$ would make the solution unchanged and it is still a solution. ~~And there is no physical~~ const

In the relativistic version (E & M should have covered it). Vector field

$$A^{\mu} \equiv (\varphi, \vec{A}).$$

And the field strength

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}.$$

$$F^{\mu\nu} = -F^{\nu\mu}.$$

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By comparison, we know that

$$F^{0i} = E^i$$

$$F^{ij} = \varepsilon^{ijk} B^k.$$

The first two of Maxwell's equations are then.

$$\partial_\nu F^{\mu\nu} = J^\mu.$$

where $J^\mu \equiv (\rho, \vec{J})$.

And $\underbrace{\partial_\mu \partial_\nu F^{\mu\nu}}_{=0} = \partial_\mu J^\mu$

no "−" sign here as ∂ is the same metric $\rightarrow$ $\dot{\rho} + \vec{\nabla} \cdot \vec{J} = 0$ the continuity equation.

To make sure the lower two maxwell equations are satisfied, we need the following

$$\varepsilon_{\mu\nu\rho\sigma} \partial^\rho F^{\mu\nu} = 0$$ (And it is automatically satisfied)

Then the transformation that doesn't change the resulting field now is $A'^\mu = A^\mu - \partial^\mu \Gamma$.
redundant

$x^\mu = (t, \vec{x})$, $x_\mu = (-t, \vec{x})$

$F'^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu A'^\mu = F^{\mu\nu} - (\partial^\mu \partial^\nu - \partial^\nu \partial^\mu)\Gamma$.

$\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial (t, \vec{x})}$ so no minus sign.

as $\partial^\mu \partial^\nu = \partial^\nu \partial^\mu$

$F'^{\mu\nu} = F^{\mu\nu}$ and hence invariant under such gauge transformations.

The following Lagrangian density leads to the Maxwell equations.

$$\mathcal{L} = \underbrace{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}}_{\text{Gauge Invariant}} + \underline{J^\mu A_\mu}.$$

$$J^\mu A'_\mu - J^\mu A_\mu$$

$$= J^\mu \partial_\mu \Gamma$$

$$= \underbrace{(\partial_\mu J^\mu)\Gamma}_{\substack{0 \text{ because} \\ J^\mu \text{ is a conserved} \\ \text{current.}}} - \underbrace{\partial_\mu (J^\mu \Gamma)}_{\substack{\text{total divergence} \\ \text{which vanishes at infinite} \\ \text{under normal/reasonable} \\ \text{vanishing boundary conditions.}}}$$

So $\mathcal{L}$ is gauge transformation invariant.

$$\frac{\delta \mathcal{L}}{\delta A_\mu} = \cancel{+\partial^\mu A^\nu \partial_\nu \frac{\delta}{\delta A_\mu}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu\right)} \frac{\partial \mathcal{L}(x)}{\partial A_\mu(x)} - \partial_\rho \frac{\partial \mathcal{L}(x)}{\partial(\partial_\rho A_\mu(x))}$$

$$= (g^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu) A_\nu + J^\mu = 0$$

which is

$$\partial_\nu F^{\mu\nu} = +J^\mu \quad \leftarrow \text{hence the Maxwell Equation}$$

$$(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\nu = (g^{\mu\nu}\partial_\alpha\partial^\alpha - \partial^\mu\partial^\nu)A_\nu$$

$$= \partial_\alpha\partial^\alpha A^\mu - \partial^\mu\partial^\nu A_\nu = \partial_\alpha\partial^\alpha A^\mu - \partial^\mu\partial^\alpha A_\alpha = \partial_\alpha F^{\alpha\mu}$$

A_μ is identified as the Photon Field.

But here is one problem:

Photon has two independent polarizations (as we measured the property of light) (it's essentially massless)

gauge symmetry preserved (not Stueckelberged)

Current limits on it
$m_\gamma < 10^{-15}$ eV
$= 1.8 \times 10^{-48}$ g

Let's see how we deal with it

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu$$

Since $F^{\mu\nu}$ is antisymmetric, there is no $\partial^0 A^0$ terms so no propagating (in time) for A^0 component and hence it is not dynamical

(no conjugate momentum for the ~~field's~~ field A^0)

Also there is the redundancy of gauge transformation that we need to eliminate.
(otherwise, imagine we do a path integral on all possible field configurations, there are infinite ways to capture the same physics!)
In principle, this doesn't matter so long as you known how to Module out the redundancy).

But we can also get rid of it by
"Gauge fixing". eg.

Require $\eta^\mu A_\mu(x)=0$
where η^μ is a Lorentz four vector.

If $\eta^2>0$, ~~time~~ space-like, it is axial gauge
$\eta^2=0$, light-cone gauge
$\eta^2<0$, time-like, temporal gauge.

Gauge Choices do not affect $\vec{E}, \vec{B}$ but remove dof.

or $\partial_\mu A^\mu=0$ ($k^\mu A_\mu(x)=0$) Lorentz gauge.

and $\vec{\nabla}\cdot\vec{A}(x)=0$, ~~Coloum~~ Coulomb Gauge.
(also known as radiation gauge or transverse gauge).

There is no need to uniquely define A_i's value (shifting by const allowed)

This corresponds to projection $A_i(x)\to(\delta_{ij}-\frac{\nabla_i\nabla_j}{\nabla^2})A_j(x)$

In this gauge, we can further the calculation

$$\mathcal{L} = \frac{1}{2}\dot{\vec{A}}\cdot\dot{\vec{A}} - \frac{1}{2}\nabla_j A_i \nabla_j A_i + J_i A_i$$

$$\boxed{+\frac{1}{2}\nabla_i A_j \nabla_j A_i + \dot{A}_i \nabla_i \varphi.}$$

$$+\frac{1}{2}\nabla_i\varphi\nabla_i\varphi - \rho\varphi.$$

IBP

$\nabla_i A_i = 0$

by the gauge choice

φ in

$J^\mu = (\varphi, \vec{A})$

vary φ, we get

$$-\nabla^2\varphi = \rho \rightarrow \varphi(x,t) = \int d^3y \frac{\rho(\vec{y},t)}{4\pi|\vec{x}-\vec{y}|}$$

this is unique solution for ρ & φ vanishing at infinity

so here φ has no dynamics, only given by the charge density.

$$\frac{1}{2}\nabla_i\varphi\nabla_i\varphi \rightarrow -\frac{1}{2}\varphi\nabla^2\varphi \rightarrow +\frac{1}{2}\varphi\rho$$

$$+(-\rho\varphi) = -\frac{1}{2}\rho\varphi.$$

$$\rightarrow \quad \underbrace{\mathcal{L}_{coul} = -\frac{1}{2}\int d^3y \frac{\rho(\vec{x},t)\rho(\vec{y},t)}{4\pi|x-y|}.}_{\text{Coulomb potential.}}$$

Note that this can only be done in Coulomb gauge. In other gauges, φ couples to $\nabla\dot{\vec{A}}$, which would be none zero.

$\mathcal{L}$

(In this gauge, for free field when $J_i = 0$.)

We can vary the $\mathcal{L}$ to find EOM.

$$-\partial^2 A_i(x) = \left(\delta_{ij} - \frac{\nabla_i \nabla_j}{\nabla^2}\right) J_j(x)$$

One can insert the projection to ensure the gauge choice

we have A_i obeys the massless Klein Gordon Equation (not surprising!!)

And the general solution
(Recall scalar and spinor case)

$$\vec{A}(x) = \sum_{\lambda=\pm} \int \widehat{dk} \left[\varepsilon_\lambda^*(\vec{k}) a_\lambda(\vec{k}) e^{ikx} + \varepsilon_\lambda(\vec{k}) a_\lambda^\dagger(\vec{k}) e^{-ikx} \right]$$

$\varepsilon_\pm(\vec{k})$ are the polarization vectors

With columb gauge, we can identify them as two circular polarizations. ε_+ right-handed

ε_- left-handed.

with wave vector $\vec{k} = (0, 0, k)$, chosen along z-direction

One can set $\varepsilon_+(\vec{k}) = \frac{1}{\sqrt{2}}(1, -i, 0)$

$$\varepsilon_-(\vec{k}) = \frac{1}{\sqrt{2}}(1, +i, 0)$$

but also more useful are the completeness relations, such that

$$\vec{k}\cdot\varepsilon_\lambda(\vec{k}) = 0$$

$$\varepsilon_\lambda(\vec{k})\cdot\varepsilon^*_{\lambda'}(\vec{k}) = \delta_{\lambda\lambda'}$$

$$\sum_{\lambda=\pm} \varepsilon^*_{i\lambda}(\vec{k})\,\varepsilon_{j\lambda}(\vec{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}$$

Gauge Dependent.

gauge

Just as before, one can Fourier Transform and project out the photon field's creation & anihilation operators using the polarization vectors

$$a_\lambda(\vec{k}) = +i\,\varepsilon_\lambda(\vec{k})\cdot\int d^3x\, e^{-ikx}\overleftrightarrow{\partial_0}\vec{A}(x)$$

$$a^\dagger_\lambda(\vec{k}) = -i\,\varepsilon^*_\lambda(\vec{k})\cdot\int d^3x\, e^{ikx}\overleftrightarrow{\partial_0}\vec{A}(x).$$

here $f\overleftrightarrow{\partial_\mu}g = f(\partial_\mu g) - (\partial_\mu f)g$.

we can also see that

$$[a_\lambda(\vec{k}), a_{\lambda'}(\vec{k}')] = 0$$
$$[a^\dagger_\lambda(\vec{k}), a^\dagger_{\lambda'}(\vec{k}')] = 0$$
$$[a_\lambda(\vec{k}), a^\dagger_{\lambda'}(\vec{k}')] = (2\pi)^3 2\omega \delta^3(\vec{k}-\vec{k}')\delta_{\lambda\lambda'}$$

LSZ reduction

Good old trick

$$a_\lambda(+\infty) - a_\lambda(-\infty) = \int_{-\infty}^{+\infty} dt\, \partial_0 a_\lambda(t)$$
$$\rightarrow \vec{\varepsilon}^{\,*}_\lambda(\vec{k}) \cdot \int \vec{A}\, e^{-ikx} \overleftrightarrow{\partial_0} A(x)$$

...

$$= i\varepsilon^\mu_\lambda(\vec{k}) \int d^4x\, e^{-ikx} (-\partial^2) A_\mu(x)$$

here we promote $\vec{A}(x) \rightarrow A^\mu(x)$

by adding $\varepsilon^0_\lambda(\vec{k}) \equiv 0$. ~~for this gauge.~~

more later

Then we have the usual

$$\langle 0 | A^i(x) | 0 \rangle = 0$$
$$\langle k, \lambda | A^i(x) | 0 \rangle = \varepsilon^i_\lambda(\vec{k}) e^{-ikx}$$

$$\langle k, \lambda | k', \lambda' \rangle = (2\pi)^3 2\omega \delta^3(\vec{k}-\vec{k}')\delta_{\lambda\lambda'}$$

This implies that we need to renormalize the theory

$$\mathcal{L} = -\frac{1}{4} Z_3 F^{\mu\nu} F_{\mu\nu} + Z_1 J^\mu A_\mu.$$

We can continue and evaluate the propagator

$$\frac{1}{i}\Delta^{ij}(x-y) \equiv \langle 0| T A^i(x) A^j(y) |0\rangle$$

$$= \frac{1}{i}\int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - i\varepsilon} \sum_{\lambda=\pm} \varepsilon_\lambda^{i*}(\vec{k})\varepsilon_\lambda^j(\vec{k})$$

We now have to evaluate the path integral

$$Z_0(J) \equiv \langle 0|0\rangle_J = \int \mathcal{D}A\, e^{i\int d^4x [-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \underbrace{J^\mu A_\mu}_{\text{viewed as external sources}}]}.$$

again A^0 is not dynamical as we can get

$$S_{coul} = -\frac{1}{2}\int d^4x\, d^4y\, \delta(x^0 - y^0) \frac{J^0(x)J^0(y)}{4\pi |x-y|}.$$

$$Z_0(J) = \exp[i S_{coul} + \frac{i}{2}\int d^4x\, d^4y\, J_i(x)\Delta(x-y)J_j(y)].$$

$$\Delta^{\mu\nu}(x-y) \equiv \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \tilde{\Delta}^{\mu\nu}(k)$$

$$\tilde{\Delta}^{\mu\nu}(k) \equiv -\frac{1}{\vec{k}^2}\delta^{\mu 0}\delta^{\nu 0} + \frac{1}{k^2 - i\varepsilon}\sum_{\lambda=\pm}\varepsilon_\lambda^{\mu*}(\vec{k})\varepsilon_\lambda^\nu(\vec{k}).$$

$$\int_{-\infty}^{+\infty}\frac{dk^0}{(2\pi)} e^{-ik(x^0-y^0)} = \delta(x^0-y^0)$$

$$\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}(\vec{x}-\vec{y})}}{\vec{k}^2} = \frac{1}{4\pi|x-y|}.$$