We are very used to the vector-representation of the Lorentz group as space-time is a Lorentz 4-vector.
But other non-trivial representations exist and of particular importance, the representation naturally suitable for Fermions (~~Dir~~ Dirac's legendary contributions, together with many others).

Although practically I can simply show you the matrices & the Feynman rules, which will enable some calculation I want, instead show you where the spinors come from, and its transformation under Lorentz Groups. You need to at least see it once in your life, as a physicist.

Unitary operator transforms a scalar field $\varphi(x)$

$$\mathcal{U}(\Lambda)^{-1}\varphi(x)\,\mathcal{U}(\Lambda) = \varphi(\Lambda^{-1}x)$$

$$\mathcal{U}(\Lambda)^{-1}\partial^\mu\varphi(x)\,\mathcal{U}(\Lambda) = \Lambda^\mu{}_\rho\,\bar{\partial}^\rho\varphi(\Lambda^{-1}x).$$

$$\frac{\partial}{\partial\bar{x}_\rho}, \quad \bar{x} \equiv \Lambda^{-1}x.$$

Or for generic vector field $A^\mu(x)$, $B^{\mu\nu}(x)$

$$\mathcal{U}(\Lambda)^{-1}A^\mu(x)\,\mathcal{U}(\Lambda) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$$

$$\mathcal{U}(\Lambda)^{-1}B^{\mu\nu}(x)\,\mathcal{U}(\Lambda) = \Lambda^\mu{}_\rho\,\Lambda^\nu{}_\sigma\,B^{\rho\sigma}(\Lambda^{-1}x)$$

Generic tensor is decomposible to three component that do not mix under L.T.

$$B^{\mu\nu}(x) = \underset{\substack{\uparrow \\ A^{\mu\nu} = -A^{\nu\mu}}}{A^{\mu\nu}(x)} + \underset{\substack{\uparrow \\ S^{\mu\nu} = S^{\nu\mu}}}{S^{\mu\nu}(x)} + \frac{1}{4} g^{\mu\nu} \underset{\uparrow}{T(x)}$$

Scalar and ensures that $S^{\mu\nu}(x)$ is traceless.

$$\mathcal{U}(\Lambda)^{-1} \frac{1}{4} g^{\mu\nu} T(x)\, \mathcal{U}(\Lambda) = \frac{1}{4} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} g^{\rho\sigma} T(\Lambda^{-1}x)$$

$$= \frac{1}{4} g^{\mu\nu} T(\Lambda^{-1}x).$$

Clearly the Lorentz transformations of $A^{\mu\nu}(x)$ $S^{\mu\nu}(x)$, $T(x)$ do not mix.

As they correspond to different symmetries or antisymmetries of the matrices.

Can it be further decomposed into unmixed transformations?

Generically, for a general Lorentz (contravariant) covariant state with an abstract index (subscript A).

(note that we did not choose μ as it is not necessarily the vector representation)

(Secretly, we all know the existence of fermions).

(We will briefly mention a different perspective later: the Dirac's heuristic method) (this should be covered in advanced QM or particle physics I).

We would have

$$\mathcal{U}(\Lambda)^{+} \varphi_A(x)\, \mathcal{U}(\Lambda) = \underbrace{L_A{}^{B}(\Lambda)}_{\substack{\text{General}\\ \text{transformation}\\ \text{on } \Lambda.}} \varphi_B(\Lambda^{-1}x)$$

We hope for a finite dimensional rep (matrix), to ~~be~~ obey the group composition rules and hence forms a representation of the group.

$$L_A{}^{B}(\Lambda') L_B{}^{C}(\Lambda) = L_A{}^{C}(\Lambda'\Lambda)$$

For an infinitesimal transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu$ we can write down that

$$\mathcal{U}(1+\delta\omega) = I + \frac{i}{2}\delta\omega_{\mu\nu} M^{\mu\nu}$$

(which we introduced at the beginning of this class)

Generators of the Lorentz group.

They obey the $\boxed{\text{Lie Algebra}}$ of the Lorentz group:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\mu\rho} M^{\nu\sigma} - (\mu \leftrightarrow \nu)\right) - (\rho \leftrightarrow \sigma)$$

We've already identified various physical components.

Angular momentum $J_i = \frac{1}{2}\varepsilon_{ijk} M^{jk}$

Boost operator $K_i \equiv M^{i0}$

with $[J_i, J_j] = i\varepsilon_{ijk} J_k$

$[J_i, K_j] = i\varepsilon_{ijk} K_k$

$[K_i, K_j] = -i\varepsilon_{ijk} J_k$

Now, we can write down the general index

$$L_A{}^B (1+\delta\omega) = \delta_A{}^B + \frac{i}{2}\omega_{\mu\nu} \underbrace{(S^{\mu\nu})_A{}^B}_{\text{another representation}}$$

To the ω order, the general transformation

$$[\varphi_A(x), M^{\mu\nu}] = \underbrace{\frac{1}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)}_{\substack{\parallel \\ \mathcal{L}^{\mu\nu}. \\ (\text{symbol})}}\varphi_A(x) + (S^{\mu\nu})_A{}^B \varphi_B(x)$$

They need to satisfy the Lie Algebra, do we have other choices?

(Recall your knowledge of spin is not invariant under rotation of 2π; there is a minus sign. As the spin is represented by $SU(2)$; not $SO(3)$, double-cover...)

(One can work out ~~the~~ such reps for arbitrary dim, but that's a separate topic. Here we show one trick.)

Note one can define:

$$N_i \equiv \frac{1}{2}(J_i - iK_i)$$

$$N_i^\dagger \equiv \frac{1}{2}(J_i + iK_i)$$

We find that

$$[N_i, N_j] = i\epsilon_{ijk} N_k.$$

$$[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk} N_k^\dagger$$

$$[N_i, N_j^\dagger] = 0.$$

But $[\quad\quad] = i\epsilon_{ijk}$ is for $SU(2)$.
so we found two "separat" $SU(2)$'s that are related by hermitian conjugation

$$N_i = (N_i^\dagger)^\dagger, \quad N_i^\dagger = (N_i)^\dagger$$

Note that $SU(2)$ representations are specifised by integers and half integers. (Recall the rotation group for angular momentum, we have J^2 and J_z.)

Since J_i and K_i are all the ~~separators~~ associate with the Lorentz group (non-zero generators) ($M^{\mu\nu}$ anti-symmetric Hermitian),

$$J_i = N_i + N_i^\dagger, \quad K_i = i(N_i - N_i^\dagger),$$

The two $SU(2)$'s also represent the Lorentz group.
We have another representation!
~~And~~ And
a representation of the Lorentz group in 4D is fixed by two integers n and n'

We label these representations as $(2n+1, 2n'+1)$ as it helps counting the # of components.
~~The~~ For a SU(2) with label n, the components are $2n+1$
So the total # of components for $(2n+1, 2n'+1)$ is $(2n+1)(2n'+1)$.
Different components within a representation can also be labled (grouped) by their angular momentum representations.
Since $J_i = N_i + N_i^\dagger$
then the allowed j are $|n-n'|, |n-n'|+1, \cdots, n+n'$
The most simple & often encountered representations are

$(1,1)$ = ~~Singlet~~ Lorentz Scalar/singlet
$(2,1)$ = left-handed spinor ← transfor under N_i
$(1,2)$ = right-handed spinor ← transfor under $N_i^\dagger$
$(2,2)$ = vector.

Lorentz vector has 4 components and are irreducible (they all mix under rotation + boost).

$(4,1)$ and $(1,4)$ contain 4 components but $j = \frac{3}{2}$

$(2,2)$ contain 4 components, with $j=1$ (3 components) and $j=0$ (1 component)
3 vector. time.

again ~~it~~ is all mixed as J_i, K_i transform both the ~~2~~ doublets.

Here we just ~~found some~~ found some interesting representations of the Lorentz group which contain "angular momentum"-like ~~reps~~ states with $j=1/2$. Without any surprise, they are fermions.

and they are special. that they are not labelled by vector index μ, but some abstract index A.

Let's study $(2,1)$ state, we call it left-handed Weyl field $\psi_a(x)$, back to our general expectation on their transformations.

$$U(\Lambda)^{-1}\psi_a(x)U(\Lambda) = \underbrace{L_a{}^b(\Lambda)}_{\text{a matrix in the } (2,1) \text{ rep.}}\psi_b(\Lambda^{-1}x)$$

following the group composition rule

$$L_a{}^b(\Lambda')L_b{}^c(\Lambda) = L_a{}^c(\Lambda'\Lambda)$$

For an infinitesimal transformation, we can write

$$L_a{}^b(1+\delta\omega) = \delta_a{}^b + \frac{i}{2}\delta\omega_{\mu\nu}(S_L^{\mu\nu})_a{}^b$$

left-handed (so far we are just calling it, no physical meaning yet. Don't worry too much!)

Similar to our 1st HW, we can find that.

$$[S_L^{\mu\nu}, S_L^{\rho\sigma}] = i(g^{\mu\rho}S_L^{\nu\sigma} - (\mu\leftrightarrow\nu)) - (\rho\leftrightarrow\sigma)$$

~~As~~

Again $[\psi_a(x), M^{\mu\nu}] = \underbrace{\underset{\downarrow}{\mathcal{L}^{\mu\nu}}}_{\frac{1}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)} \psi_a(x) + (S_L^{\mu\nu})_a{}^b \psi_b(x)$

we do not focus on this term and it disappears when we evaluate the fields at the origin. $(x^\mu = 0)$

Recalling that ~~$J_k = \frac{1}{2}\varepsilon_{ijk} M^{ij}$~~

$$J_i = \frac{1}{2}\varepsilon_{ijk} M^{jk}, \text{ then}$$

$$M^{ij} = \varepsilon^{ijk} J_k.$$ and J is the angular momentum operator.

then.

$$\varepsilon^{ijk}[\psi_a(0), J_k] = (S_L^{ij})_a{}^b \psi_b(0)$$

We also know that a ~~spin~~ angular momentum $-\frac{1}{2}$ object transforms under J_k.

to be $\frac{1}{2}\varepsilon^{ijk}\sigma_k$ (from QM class).

Here σ is a Pauli Matrix

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This leads to that

$$(S_L^{ij})_a{}^b = \underset{\uparrow}{\frac{1}{2}} \underset{\uparrow}{\underline{\varepsilon^{ijk}}} \sigma_k.$$

spin-$\frac{1}{2}$ more standard double-index notation. on rotation

For boost $K_i = \mathcal{M}^{i0}$ and

$$K_i = i(N_i - N_i^\dagger)$$

but $(2,1)$ is a singlet on $N_i^\dagger$ so

$$(S_L^{i0})_a{}^b = \frac{1}{2} i \sigma_i$$, so we get the full $(S_L^{\mu\nu})_a{}^b$

(Note here σ_i are just matrices, not about Lorentz vector indices)

As $(N_i)^\dagger$ is $N_i^\dagger$,

we anticipate to get all the properties on $(1,2)$ state, right-handed spinor from Hermitian conjugation.

to make clear that a state is the $(1,2)$, we use dotted index to note them.

$$[\psi_a(x)]^\dagger = \psi^\dagger_{\dot{a}}(x)$$

Under a Lorentz transformation, we have

$$\mathcal{U}(\Lambda)^{-1} \psi_{\dot{a}}^\dagger(x) \, \mathcal{U}(\Lambda) = R_{\dot{a}}{}^{\dot{b}} \psi^\dagger_{\dot{a}}(x)$$

Similarly

$$R_{\dot{a}}{}^{\dot{b}}(1+\delta\omega) = \delta_{\dot{a}}{}^{\dot{b}} + \frac{i}{2}\delta\omega_{\mu\nu}\underline{(S_R{}^{\mu\nu})_{\dot{a}}{}^{\dot{b}}}$$

~~and~~

anti-symmetric matrices obey the Lie Algebra, similar to $\mathcal{M}^{\mu\nu}$ and $S_L{}^{\mu\nu}$.

$$[\psi_{\dot{a}}^\dagger(0), \mathcal{M}^{\mu\nu}] = (S_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}} \psi^\dagger_{\dot{b}}(0)$$

Hermitian conjugate the above (Note $(S_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}}$ is a number) we get ($M^{\mu\nu}$ hermitian)

$$[M^{\mu\nu}, \psi_a(0)] = [(S_R{}^{\mu\nu})_{\dot{a}}{}^{\dot{b}}]^* \psi_b(0)$$

but we already know

$$[\psi_a(0), M^{\mu\nu}] = (S_L{}^{\mu\nu})_a{}^b \psi_b(0)$$

$$\text{so} \quad [(S_R{}^{\mu\nu})_{\dot{a}}{}^{\dot{b}}]^* = -(S_L{}^{\mu\nu})_a{}^b$$

component by component, we know $(S_R{}^{\mu\nu})$ and how right-handed Wyle field would transform under Lorentz.

Let's digest briefly here:

(A). (2, 1) and (1, 2) are the lowest point-~~like~~ like non-singlet spin states we have. With such building blocks, we can represent any higher spins. It is just often much more convient to use vector-reps, instead of spinor-reps for states with integer spins. (e.g. vector, tensor, etc.) But for fermions, ~~with~~ including spin-3/2, 5/2, ... we have no choices but to use spinor-rep.

Spinor-rep can also represent vectors, and in fact, shed some insights for us as well.

③. Typical operations - rotation J and boost K, now correspond to a combination of N_i and $N_i^\dagger$, which means, if I rotate J by $\delta\theta$, both the $(1,2)$ and ~~(2,1)~~ $(2,1)$ states will rotate in a correlated way

Now, back to discover more properties for such reps. Let's explore a field with two $(2,1)$ indices, $C_{ab}(x)$

$$\mathcal{U}(\Lambda)^{-1} C_{ab}(x)\, \mathcal{U}(\Lambda) = L_a{}^c(\Lambda)\, L_b{}^d(\Lambda)\, C_{cd}(\Lambda^{-1}x).$$

This 4 (2×2) component state can be further decomposed. We already know, they can make spin 0 (antisymmetric) or spin 1 (symmetric). $2 \otimes 2 = 1_A \oplus 3_S$.

For Lorentz group, we have

$$(2,1) \otimes (2,1) = (1,1)_A \oplus (3,1)_S.$$

We should be able to write

$$C_{ab}(x) = \varepsilon_{ab} D(x) + G_{ab}(x)$$

spinless

$G_{ab}(x) = + G_{ba}(x)$ as it is antisymmetric.

$\varepsilon_{ab} = -\varepsilon_{ba}$

Lorentz Scalar.

$\varepsilon_{21} = -\varepsilon_{12} = 1$

This implies that ε_{ab} is an invariant symbol of the Lorentz group.

$$L_a{}^c(\Lambda) L_b{}^d(\Lambda) \varepsilon_{cd} = \varepsilon_{ab}$$

We already have another invariant symbols, and we will find more.

$$\Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma g_{\rho\sigma} = g_{\mu\nu}$$

We use $g_{\mu\nu}$ and $g^{\mu\nu}$ to raise and lower indices, and we can do so as well for ε_{ab} and ε^{ab} for the left-handed indices.

$$\varepsilon^{12} = \varepsilon_{21} = +1, \quad \varepsilon^{21} = \varepsilon_{12} = -1 \quad \text{(definition)}$$

Hence we will have

$$\varepsilon_{ab}\varepsilon^{bc} = \delta_a{}^c, \quad \varepsilon^{ab}\varepsilon_{bc} = \delta^a{}_c$$

Then we have: $\psi^a(x) \equiv \varepsilon^{ab}\psi_b(x)$, it's consistently

$$\psi_a(x) = \varepsilon_{ab}\psi^b(x) = \varepsilon_{ab}\varepsilon^{bc}\psi_c(x) = \delta_a{}^c\psi_c(x) = \psi_a(x)$$

But ε^{ab} is anti-symmetric, we need to be careful with signs

$$\psi^a(x) = \underbrace{\varepsilon^{ab}}_{\uparrow}\psi_b(x) = -\varepsilon^{ba}\psi_b(x) = \underbrace{-\psi_b(x)\varepsilon^{ba}} = \psi_b(x)\varepsilon^{ab}$$

View it as an element in matrices and then summed over.

$$\psi^a\chi_a = \varepsilon^{ab}\psi_b\chi_a = -\varepsilon^{ba}\psi_b\chi_a = -\psi_b\chi^b$$

So even with contracted indices, "−" sign appears if one is not careful enough.

Again, we show these basics, which later on in more practical cases, there are simplifications.

Similarly $(1,2)\otimes(1,2) = (1,1)_A \oplus (1,3)_S$.
and we find the invariant symbol

$\varepsilon_{\dot{a}\dot{b}} = -\varepsilon_{\dot{b}\dot{a}}$, with a same convention to take

$$\varepsilon^{12} = \varepsilon_{21} = 1, \quad \varepsilon^{21} = \varepsilon_{12} = -1$$

Now turn to $(2,2)$, which is a vector. How?
Need a dictionary to translate / project
double spinor index (one left one right) to vector index μ.

$$A_{a\dot{a}}(x) = \underset{\text{should be}}{\sigma^{\mu}{}_{a\dot{a}}} A_\mu(x), \quad \text{here } \sigma^{\mu}{}_{a\dot{a}} = (I, \vec{\sigma})$$

as it is a general translation between $a\dot{a} \leftrightarrow$ to μ
One can construct it through another singlet, which is proven in the beginning part of $\boxed{\text{Sec 35.}}$ (I only highlight a few parts here).
If such an invariant symbol exists, there can be various invariant products.

$$\sigma^{\mu}{}_{a\dot{a}}\, \sigma_{\mu b\dot{b}} = -2\varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}} \leftarrow$$ exactly what you anticipate to form a singlet $(2,1)\otimes(1,2)\otimes(2,2)=(1,1)\oplus\ldots$ or $(2,2)\otimes(2,2)=(1,1)\oplus\ldots$

$$\varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}\,\sigma^{\mu}_{a\dot{a}}\,\sigma^{\nu}_{b\dot{b}} = -2g^{\mu\nu}$$

one can translate between different invariant symbols.

And one can find and verify that, the generators
$(S_L^{\mu\nu})_{ac}$, $(S_R^{\mu\nu})_{\dot{a}\dot{c}}$ and also be expressed as

$$(S_L^{\mu\nu})_{ac} = \frac{i}{4}\varepsilon^{\dot a\dot c}(\sigma^\mu_{a\dot a}\sigma^\nu_{c\dot c} - \sigma^\nu_{a\dot a}\sigma^\mu_{c\dot c})$$

$$(S_R^{\mu\nu})_{\dot a\dot c} = \frac{i}{4}\varepsilon^{ac}(\sigma^\mu_{a\dot a}\sigma^\nu_{c\dot c} - \sigma^\nu_{a\dot a}\sigma^\mu_{c\dot c})$$

Coming from using $\sigma^\mu_{a\dot a}$ being invariant symbol

$$\sigma^\mu_{a\dot a} = \Lambda^\mu{}_\nu L(\Lambda)_a{}^b R(\Lambda)_{\dot a}{}^{\dot b}\sigma^\nu_{b\dot b}$$

can be written as

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \frac{i}{2}\delta\omega_{\rho\sigma}(\frac{1}{i})(g^{\rho\mu}\delta^\sigma{}_\nu - g^{\sigma\mu}\delta^\rho{}_\nu)$$

~~Conventility~~, Conveniently, one can define another invariant symbol

$$\bar\sigma^{\mu\dot a a} \equiv \varepsilon^{ab}\varepsilon^{\dot a\dot b}\sigma^\mu_{b\dot b}$$

$$\bar\sigma^{\mu\dot a a} = (I, -\vec\sigma)$$

Now, more concisely

$\dot c b$ contracted (matrix multiplication)

$$(S_L^{\mu\nu})_a{}^b = +\frac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)_a{}^b$$

$c b$ contracted

$$(S_R^{\mu\nu})^{\dot a}{}_{\dot b} = -\frac{i}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)^{\dot a}{}_{\dot b}$$

Taking this contraction convention, dotted are contracted as ${}_{\dot c}{}^{\dot c}$ and undotted as ${}^c{}_c$, we get useful simplification that.

$$\chi\psi = \chi^a\psi_a \quad \text{and} \quad \chi^\dagger\psi^\dagger = \chi^\dagger_{\dot a}\psi^{\dagger\dot a}$$

left-handed fields

We also anticipate fermion fields are odd under exchange.

$$\psi_a(x)\chi_b(y) = -\chi_b(y)\psi_a(x)$$

Now we have

$$\chi\psi = \chi^a\psi_a = -\psi_a\chi^a = \psi^a\chi_a = \psi\chi$$

exchange get a minus

$a^{\,a}$ to ${}^a{}_a$ gets a minus sign as shown earlier.

Same for $(\chi\psi)^\dagger = \psi^\dagger\chi^\dagger = \chi^\dagger\psi^\dagger$ as per our convention

Further, one can adopt a convention that ~~the~~ a right handed field always written as the Hermitian conjugate of a left-handed field. (and left-handed fields are written without a dagger)

for instance.

$$\psi^\dagger_{\dot a}\bar\sigma^{\mu\dot a a}\chi_a \;\overset{\text{now can be written as}}{=\!=\!=}\; \psi^\dagger\bar\sigma^\mu\chi$$

$\dot a$ and a are independent indices

All the spinor index contracted, and now with only a bare vector index μ, it will transform like a vector.

Hence

$$\mathcal{U}(\Lambda)^{-1}[\underset{\downarrow\, \psi^{\dagger}(x)}{\psi^{\dagger}}\bar{\sigma}^{\mu}\underset{\downarrow\, \chi(x)}{\chi}]\,\mathcal{U}(\Lambda) = \Lambda^{\mu}{}_{\nu}[\underset{\downarrow\, \psi^{\dagger}(\Lambda^{-1}x)}{\psi^{\dagger}}\bar{\sigma}^{\nu}\underset{\downarrow\, \chi(\Lambda^{-1}x)}{\chi}]$$

And

$$[\psi^{\dagger}\bar{\sigma}^{\mu}\chi]^{\dagger} = [\psi^{\dagger}_{\dot{a}}\bar{\sigma}^{\mu\dot{a}a}\chi_a]^{\dagger}$$

$$= \chi^{\dagger}_{\dot{a}}(\bar{\sigma}^{\mu\dot{a}a})^{\dagger}\psi_a$$

$$= \chi^{\dagger}_{\dot{a}}\bar{\sigma}^{\mu\dot{a}a}\psi_a \quad \text{as } \bar{\sigma}^{\mu} = (I, -\vec{\sigma}) \text{ is hermitian.}$$

$$= \chi^{\dagger}\bar{\sigma}^{\mu}\psi$$

We are going to build ~~Ror~~ spinor Lagrangian with these properties and further notational simplifications. (by adopting conventions)

Before we get to QFT for ~~spin~~ fermions, we also solved some ~~few~~ small ~~problem~~ questions: can $B^{\mu\nu}$ be further decomposed into different pieces that transform separately under Lorentz group?

$$B^{\mu\nu} = S^{\mu\nu} + \underbrace{A^{\mu\nu}}_{\text{can be further decomposed.}} + \frac{1}{4}g^{\mu\nu}T$$

$B^{\mu\nu}$ under spinor representation is

$$\underbrace{(2,2)}_{\mu}\otimes\underbrace{(2,2)}_{\nu} = \underbrace{(\underset{\substack{A\\ \downarrow\\ a a}}{1},\ \underset{\substack{A\\ \downarrow\\ \dot{a}\dot{a}}}{1})_S}_{\frac{1}{4}g^{\mu\nu}T} \oplus \underbrace{(1_A, 3_S)_A \oplus (3_S, 1_A)_A}_{A^{\mu\nu}} \oplus \underbrace{(3_S, 3_S)_S}_{S^{\mu\nu}}$$

The projection operator (covariant)

$$(S_L{}^{\mu\nu})_a{}^b = -\frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}(S_{L\rho\sigma})_a{}^b$$

$$(S_R{}^{\mu\nu})_{\dot{a}}{}^{\dot{b}} = -\frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}(S_{R\rho\sigma})_{\dot{a}}{}^{\dot{b}}$$

$$A^{\mu\nu}(x) = \underbrace{G^{\mu\nu}(x)}_{\substack{\text{Self-dual}\\ \text{antisymmetric tensor}}} + \underbrace{G^{+\mu\nu}}_{\substack{\text{anti-self-dual}\\ \text{antisymmetric tensor.}}}$$

$$G^{\mu\nu}(x) \equiv (S_L{}^{\mu\nu})^{ab} G_{ab}(x)$$
$$= -\frac{i}{2}\varepsilon^{\mu\nu\rho\sigma} G^{\dagger}_{\rho\sigma}(x)$$

$$G^{+\mu\nu}(x)$$
$$= -(S_R{}^{\mu\nu})^{\dot{a}\dot{b}} G_{\dot{a}\dot{b}}(x)$$
$$= +\frac{i}{2}\varepsilon^{\mu\nu\rho\sigma} G^{+}_{\rho\sigma}(x)$$