# Sec. 9. Path Integral for Interacting theory

$$\mathcal{L} = -\frac{1}{2} Z_\varphi \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} Z_m m^2 \varphi^2 + \frac{1}{6} Z_g g \varphi^3 + Y\varphi$$

$\boxed{m}$ is identified as particle mass

$\boxed{g}$ is coupling strength whose meaning will be more clearly defined later.

The $Z$'s & $Y$ are defined such that

$\boxed{\langle 0|\varphi(x)|0\rangle = 0}$ and $\boxed{\langle k|\varphi(x)|0\rangle = e^{-ipx}}$

$\langle 0|0\rangle = 1$ and $|k\rangle$ one-particle state with four momentum $k^\mu$ $(k^2 = -m^2)$

such that

$$\langle k'|k\rangle = (2\pi)^3 2k^0 \delta^3(\vec{k} - \vec{k}')$$

These four conditions $\boxed{\phantom{x}}$ determines three $Z$'s and $Y$.

This theory is called $\varphi^3$ theory.

(this theory is not bounded from below, but we just use it to demonstrate perturbation theory, ignoring the tunneling process)

(more discussion can be done after class)

Also comment on $\varphi^4$ theory.

We want to evaluate

$$Z(J) \equiv \langle 0|0\rangle_J = \int \mathcal{D}\varphi \, e^{i\int d^4x\,[\mathcal{L}_0+\mathcal{L}_1+J\varphi]}$$

Similar to previously learned in the Path Integral formalism for Quantum Mechanics, we have

$$Z(J) = e^{i\int d^4x\,\mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} \int \mathcal{D}\varphi \, e^{i\int d^4x\,[\mathcal{L}_0+J\varphi]}$$

$$\propto e^{i\int d^4x\,\mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} Z_0(J) \quad \text{--- Eq(9.6)}$$

where $Z_0(J)$ is for the free theory

$$Z_0(J) = e^{\frac{i}{2}\int d^4x\, d^4x'\, J(x)\Delta(x-x')J(x')}$$

the $1-i\epsilon$ treatment gives additional factors so that we need to define / require that $Z(0)=1$ $\leftarrow$ normalization of probability

In the above equation, we implicitly assumed that

$$\mathcal{L}_0 = -\frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2$$

The rest needs to be included in $\mathcal{L}_1$.

$$\mathcal{L}_1 = \frac{1}{6}Z_g g\varphi^3 + \mathcal{L}_{CT}$$

$$\underset{\substack{\uparrow \\ \text{counter-term}}}{\mathcal{L}_{CT}} = -\frac{1}{2}(Z_\varphi-1)\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}(Z_m-1)m^2\varphi^2 + Y\varphi.$$

One anticipate $g \to 0$, then $\mathcal{L}_1 = 0$

so $Y \underset{g\to 0}{\to} 0$, $Z_i \underset{g \to 0}{\to} 1$

we shall see that $Y = O(g)$, $Z_i = 1 + O(g^2)$

To compute Eq (9.6), a lot of functional derivatives are needed $\mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)$.

Let's focus on the "leading-order" result

$$Z_1(J) \propto e^{\left[\frac{i}{6} Z_g g \int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right]} Z_0(J).$$

$\uparrow$

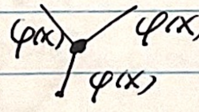
leading order

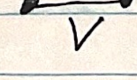
1st order perturbation [ $Z_1(0) = 1$ required ].

Expanding the exponential functions

$$Z_1(J) \propto \sum_{V=0}^{\infty} \frac{1}{V!}\left[\frac{i Z_g g}{6} \int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right]^V$$

$$\times \sum_{P=0}^{\infty} \frac{1}{P!}\left[\frac{i}{2}\int d^4y\, d^4z\, J(y)\Delta(y-z)J(z)\right]^P.$$

$V$ counts # of vertices 


$P$ counts # of propagators 

The number of external (surviving sources)

$E = 2P - 3V$

| each $P$ provide 2 $J$'s
| each $V$ get rid of 3 $J$'s

for a fixed order in $P$ and $V$ expansion.

These can be organized via the Feynman diagrams.

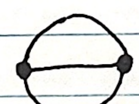
each line segment is $\frac{1}{i}\Delta(y-z)$

each filled circle is a source $i\int d^4x\, J(x)$

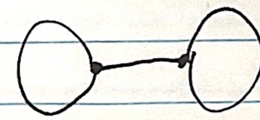
each vertex joining $\underbrace{\text{three lines}}_{\text{for } \varphi^3 \text{ theory}}$ is $i Z_g\, g \int d^4x$

We can draw various diagrams.

e.g. $E=0,\ V=2$
$(P=3)$

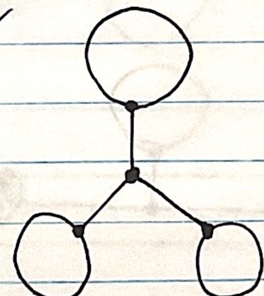


$S = 2\times 3!$

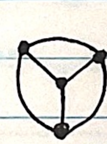


$S = 2\times 2\times 2$

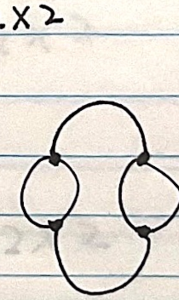
$E=0,\ V=4$
$(P=6)$



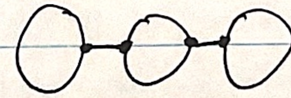
$S = 3!\times 2^3$



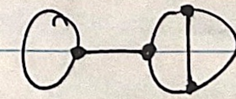
$S = 4!$



$S = 2^3 \times 2$



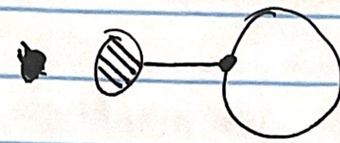
$S = 2^4$



$S = 2\times 2\times 2.$
0 1 –

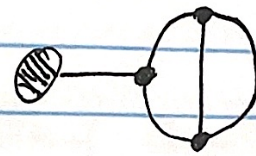
but calculate it, we only have one possibility! How can I count it $2!\times 3!$ times?

E=1 V=1
(P=2)

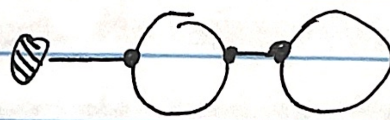


$S=2$

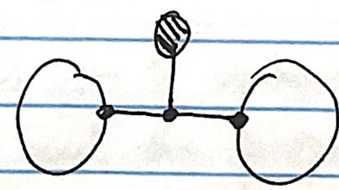
E=1 V=3
(P=5)



$S=2\times 2.$



$S=2\times 2$

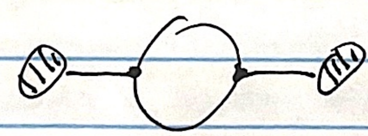


$S=2\times 2^2$

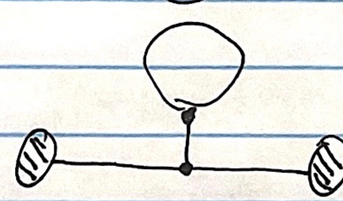
E=2, V=0
(P=1)

$S=2$

E=2, V=2
(P=4)



$S=2\times 2$



$S=2\times 2$

⋮

The essence is a diagrammatic counting of functional derivatives with the generating function $Z_0$

we do symmetry factor counting for a reason that will be transparent ~~a~~ later.

. . .

They represent Mathematical operations of functional derivatives and the Green's functions.

The three functional derivatives can be rearranged $3!$ without changing the diagrams.

We can rearrange the vertices themselves with $V!$

The two sources can be exchanged connecting the ~~by~~ propagators by $2!$, and the propagators themselves by $P!$

All together, they cancel explicitly the $\frac{1}{V!}\left(\frac{1}{6}\right)^V \cdot \frac{1}{P!}\left(\frac{1}{2}\right)^P$ factors in the dual Taylor expansion.

But this is an overcounting, as different ways of moving vertices and propagators may yield a same diagram.

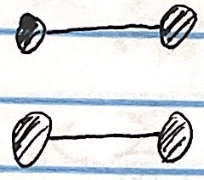
There ~~is~~ are some symmetry properties of the diagrams

We can evaluate the overcounting by dividing the "symmetry factors" $S$ for each diagram. (Automorphism)

All the abovementioned diagrams are "connected"

There are others disconnected contributions to $Z(J)$ as well

e.g., for 4E



also contributes

Let $C_I$ for a particular connected diagram

A general diagram can be expressed as

$$D = \frac{1}{S_D} \prod_I (C_I)^{n_I}$$

$n_I$ counts # of $C_I$ in $D$.

$S_D$ is additional symmetry in $D$.

$S_D = \prod_I \frac{1}{n_I!}$ as exchanging identical $C_I$s do not generate new diagrams

and each $C_I$ is different.

Now $Z_1(J)$ is given by summing all diagrams $D$, and each $D$ is labeled by the integers $n_I$,

$\uparrow$ ~~se~~ $I$ is index ~~of~~ ~~$C_I$~~ $\{n_I, C_I\}$

$$Z_1(J) \propto \sum_{\{n_I\}} D$$

$$\propto \sum_{\{n_I\}} \prod_I \frac{1}{n!} (C_I)^{n_I}$$

$$\propto \prod_I \sum_{n_I=0}^{\infty} \frac{1}{n!} (C_I)^{n_I}$$

$$\propto \prod_I e^{C_I}$$

$$\propto \exp[\Sigma_I C_I]$$

A remarkable result!

$Z_1(J)$ is given by the $\boxed{\text{exponential}}$ of all $\boxed{\text{connected diagrams}}$

And to impose the normalization condition $Z_1(0) = 1$ we only need to require all the vacuum diagrams being zero. $(E=0)$. hence $Z_1(0) = \langle 0|0\rangle_{J=0} = 1 = \exp[C_0] = 0.$

Now we have

$$Z_1(J) = \exp[iW_1(J)]$$

where

$$iW_1(J) = \sum_{I \neq \{0\}} C_I .$$

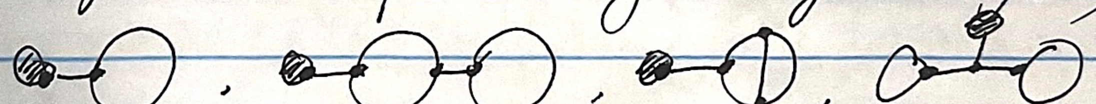
Recall that we wanted $Z(J)$ but we calculated $Z_1(J)$, which misses the counter terms that were introduced earlier for consistency.

Let's check the consequence.

$$\langle 0|\varphi(x)|0\rangle = \frac{1}{i}\frac{\delta}{\delta J(x)} Z_1(J)\Big|_{J=0}$$

$$= \underline{\frac{\delta}{\delta J(x)} W_1(J)\Big|_{J=0}}$$

This is the sum of all diagrams with a single source (after removing remaining sources by $J=0$)



$(+ \ldots$

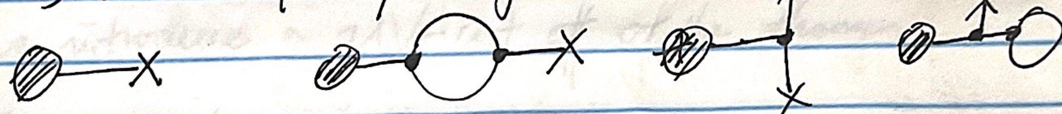
$$\langle 0|\varphi(x)|0\rangle = \frac{1}{2} ig\int d^4y \, \frac{1}{i}\Delta(x-y)\frac{1}{i}\Delta(y-y) + O(g^3).$$

Here we set $Z_g = 1$ for consistent counting of $g$.
(perturbation implied) as $Z_g = 1 + \underbrace{O(g^2)}_{\text{to be proven later.}}$

We see
$\langle 0|\varphi(x)|0\rangle \neq 0$ at order $g$.

we must introduce the counter term to enable the LSZ

such a counter term $Y\varphi(x)$
introduces a new class of diagrams



$Y$ $\qquad$ $Yg^2$ $\qquad$ $Y^2g$ $\qquad$ $Yg^2$

assuming/recall $Y$ is $O(g)$, we see the leading diagram is  at order $g$.

so

$$\langle 0|\varphi(x)|0\rangle = \left(iY + \tfrac{1}{2}(ig)\tfrac{1}{i}\Delta(0)\right)\int d^4y\, \tfrac{1}{i}\Delta(x-y) + O(g^3)$$

so to have $\langle 0|\varphi(x)|0\rangle = 0$, we need to choose

$$Y = \tfrac{1}{2} i g \Delta(0) + O(g^3).$$

$\Upsilon$ is supposed to be real, let's check

$\Delta(0)$ needs to be imaginary

$$\Delta(0) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2-i\epsilon}$$

this is also a divergent ~~di~~ integral

that there are $k^2 = -m^2$ and infinite length of $|\vec{k}|$.

Again we encountered $\boxed{\text{Ultravidote Divergence}}$

UV-divergence.

so we introduce a UV-cut off of the theory.

($\Lambda >> m$ but $\Lambda$ ~~&~~ finite).

and other scales we can conduct the ~~x~~experiment)

The UV-cut off of the propagator is introduced

in the following subtitle way to respect Lorentz Invariance

$$\Delta(x-y) \to \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2+m^2-i\epsilon} \left(\frac{\Lambda^2}{k^2+\Lambda^2-i\epsilon}\right)^2$$

for $\Lambda^2 >> m^2$ ~~ne~~ for $k^2 << \Lambda^2$

it is domianted by the original pole but for larger ~~A~~ $k^2$, it's $\Lambda^2$

now

$$\Delta(0) = \frac{i}{16\pi^2}\Lambda^2 .$$

we could formally take $\Lambda \to \infty$ with

$\langle 0|\varphi(x)|0\rangle = 0$.

[ A view point can be that
$Z_*(J)$ contains infinity part
but physical observables are the properly
normalized probability $\left( \frac{\langle f|i\rangle^2}{Z(J)}\Big|_{J=0} \right)$, although
both numerator and denominator contains infinity, but
their ratio is finite. ].

This is a remarkable simplification, meaning
the sum of all connected diagrams with a single source is
zero. (No matter which order)

We only need to calculate diagrams with more sources.

We will discuss the renormalizing counter terms
later.

Note that

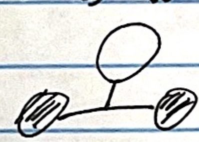
● — Anything (no source) is canceled by × — same thing.

Then Any diagram that contains

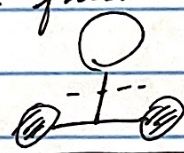
Any diagram contains ⋮ Anything (no source) would be cancelled.

~~this~~ this Any thing no source with one external line is called Tadpole terms.

e.g. for two ~~external~~ sources, ~~there is~~ there is no contribution from



because



cancels it

$\boxed{\text{Tadpole terms do not contribute}}$

Then the remaining $\mathcal{L}_{CT}$ are $Z_\varphi$ and $Z_m$ terms

$$Z(J) = e^{-\frac{i}{2}\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(-A\,\partial_x^2 + B\,m^2\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} Z_1(J)$$

$$Z_\varphi \partial_\mu \varphi\, \partial^\mu \varphi \xrightarrow{\text{G.O.M.}} -Z_\varphi\, \varphi\, \partial^2 \varphi.$$

We can proceed and calculate to get the full $Z(J)$