Sec 10. Scattering Amplitudes & Feynman Rules.

With $Z(J) = e^{iW(J)}$

we can calculate time-ordered amplitudes through functional derivatives.

$\frac{1}{i}\frac{\delta}{\delta J}$ 改 $\frac{1}{i}\frac{\delta}{\delta J_j}$

$$\frac{1}{i}\mathbf{\Delta}(x_1 - x_2) \equiv \langle 0 | T\varphi(x_1)\varphi(x_2)|0\rangle. \quad \text{with } \delta_j = \frac{1}{i}\frac{\delta}{\delta J(x_j)}$$

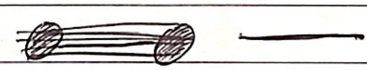
$$= \delta_1\delta_2 Z(J)\big|_{J=0} = \delta_1\delta_2 iW(J)\big|_{J=0} - \delta_1 iW(J)\big|_{J=0}\,\delta_2 iW(J)\big|_{J=0}$$

$$= \delta_1\delta_2 iW(J)\big|_{J=0}$$

as $\delta_j W(J)\big|_{J=0} = \langle 0|\varphi(x_j)|0\rangle = 0$

δ_1 removes a source and gives a label x_1

$\frac{1}{i}\Delta(x_1 - x_2)$ is given by all diagrams with two sources

at lowest order 

Feynman Propagator

$$\Delta(x - x') = \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-x')}}{k^2 + m^2 - i\varepsilon}$$

Thus we get

$$\frac{1}{i}\mathbf{\Delta}(x_1 - x_2) = \frac{1}{i}\Delta(x_1 - x_2) + O(g^2)$$

$$\langle 0|T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle = \delta_1\delta_2\delta_3\delta_4 Z(J)\Big|_{J=0}$$

$$= [\delta_1\delta_2\delta_3\delta_4 iW(J) + (\delta_1\delta_2 iW(J))(\delta_3\delta_4 iW(J))$$
$$+ (\delta_1\delta_3 iW(J))(\delta_2\delta_4 iW(J))$$
$$+ (\delta_1\delta_4 iW(J))(\delta_2\delta_3 iW(J))]_{J=0}$$

$$= \delta_1\delta_2\delta_3\delta_4 iW(J)\big|_{J=0} + \frac{1}{i}\mathbf{\Delta}(x_1 - x_2)\frac{1}{i}\mathbf{\Delta}(x_3 - x_4) + \cdots$$

We can insert these terms to the LSZ reduction for $2 \to 2$ process

$$\langle f|i\rangle = i^4 \int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\, e^{i(k_1x_1+k_2x_2-k_1'x_1'-k_2'x_2')}$$

$$(-\partial_1^2+m^2)(-\partial_2^2+m^2)(-\partial_3^2+m^2)(-\partial_4+m^2)$$

$$\langle 0|T\varphi(x_1)\varphi(x_2)\varphi(x_1')\varphi(x_2')|0\rangle.$$

There are several products of Green's functions contributing,

let's check $\frac{1}{i}\Delta(x_1-x_1')\,\frac{1}{i}(x_2-x_2')$

$$\int d^4x_1\, d^4x_2\, d^4x_1'\, d^4x_2'\, e^{i(k_1x_1+k_2x_2-k_1'x_1'-k_2'x_2')} \qquad F(x_{11'})F(\cancel{x}x_{22'})$$

where we defined $F(x_{ij}) = (-\partial_i^2+m^2)(-\partial_j^2+m^2)\Delta(x_{ij})$.

$$x_{ij} = x_i - x_j, \quad \bar{k}_{ij} = (k_i+k_j)/2.$$

$$= (2\pi)^4\delta^4(k_1-k_1')(2\pi)^4\delta^4(k_2-k_2')\,\tilde{F}(\bar{k}_{11'})\,\tilde{F}(\bar{k}_{22'})$$

$\tilde{F}(k)$ is ~~#~~ $F(x_{ij})$'s $\boxed{\text{Fourier Transform}}$. check

but clearly this represents no scattering as $k_1 = k_1'$, $k_2 = k_2'$

In general, we are only interested in diagrams where something happened (scattering), which is "suffiently" represented by $\boxed{\text{Fully Connected}}$ diagrams.

what if some connected, some not? They can be factorized.

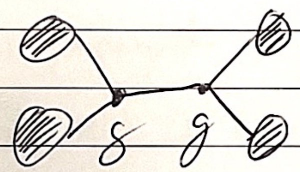
From here on, we restrict ourselves to those diagrams alone and define Connected correlation functions via.

$$\langle 0|T\varphi(x_1)\dots\varphi(x_E)|0\rangle_C \equiv \delta_1\dots\delta_E\, iW[J]\big|_{J=0}$$

Back to 2→2 process.

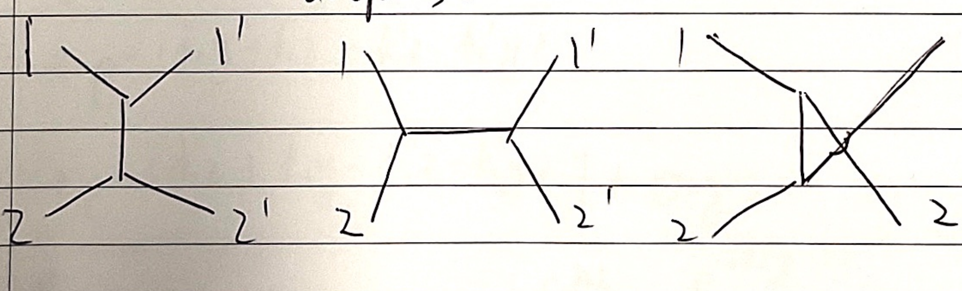
$$\langle 0|T\varphi(x_1)\varphi(x_2)\varphi(x_1')\varphi(x_2')|0\rangle_C = \delta_1\delta_2\delta_{1'}\delta_{2'}\, iW[J]\big|_{J=0}$$

at the lowest order in g, the contributing process is



with the 4 (sources) functional cleared away by δ.

There are 4! ways to $\delta_1\delta_2\delta_3\delta_4$.
which are 3 groups (A&B pairing, C&D pairing, and exchange of the g vertices/reflection) each with 8 identical diagrams.



Note ~~here~~ here that the ~~factor~~ 8 nicely cancels the symmetry factor $S=8$ of the original sourced diagram.

This is a general result for $\boxed{\text{TREE}}$-level diagrams (those without closed loops), that the symmetry factor got cancelled with explicit labelling. This is the first non-zero contribution to the amplitude.

each propagator is $\frac{1}{i}$ as $\frac{1}{i}\frac{\delta}{\delta J}$ $i\Delta \cdot \frac{1}{i}\frac{\delta}{\delta J}$.

Now.

$$\langle 0| T\varphi(x_1)\varphi(x_2)\varphi(x_1')\varphi(x_2')|0\rangle_C$$

$$= (ig)^2 \left(\frac{1}{i}\right)^5 \int d^4y\, d^4z\, \Delta(y-z) \times$$

$$[\Delta(x_1-y)\Delta(x_2-y)\Delta(x_1'-z)\Delta(x_2'-z)$$

$$+\Delta(x_1-y)\Delta(x_2-z)\Delta(x_1'-z)\Delta(x_2'-y)$$

$$+\Delta(x_1-z)\Delta(x_2-y)\Delta(x_1'-z)\Delta(x_2'-y)] + O(g^4)$$

Put in LSZ. Each Klein-Gordon wave operator acts on a propagator to give.

$$(-\partial_i^2 + m^2)\,\Delta(x_i - y) = \delta^4(x_i - y) \leftarrow$$

(note that Δ is free-particle propagator, as it is from $Z_0(J)$ part).

Then we get.

$$\langle f|i\rangle = (ig)^2 \frac{1}{i} \int d^4y\, d^4z\, \Delta(y-z)$$

$$[e^{i(k_1y + k_2y - k_1'z - k_2'z)}$$

$$+ e^{i(k_1y + k_2z - k_1'z - k_2'y)}$$

$$+ e^{i(k_1z + k_2y - k_1'z - k_2'y)}] + O(g^4).$$

knowing that $\Delta(y-z) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(y-z)}}{k^2+m^2-i\epsilon}$.

we can get

$$\cancel{\langle f|i\rangle = ig^2\int d^4y \, e}$$

$$\langle f|i\rangle = ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2-i\epsilon}$$

$$\times\left[(2\pi)^4\delta^4(k_1+k_2+k)\,(2\pi)^4\delta^4(k_1'+k_2'+k)\right.$$

$$+(2\pi)^4\delta^4(k_1-k_1'+k)\,(2\pi)^4\delta^4(k_2-k_2'+k)$$

$$\left.+(2\pi)^4\delta^4(k_1-k_2'+k)\,(2\pi)^4\delta^4(k_2-k_1'+k)\right]+O(g^4)$$

$$= ig^2\cdot(2\pi)^4\cdot\delta^4(k_1+k_2-k_1'-k_2')$$

$$\left[\frac{1}{(k_1+k_2)^2+m^2-i\epsilon}+\frac{1}{(k_1-k_1')^2+m^2-i\epsilon}+\frac{1}{(k_1-k_2')^2+m^2-i\epsilon}\right]+O(g^4)$$

The overall Dirac delta shows the energy momentum conservation. So

$$\langle f|i\rangle = (2\pi)^4\delta^4(k_{in}^{total}-k_{out}^{total})\,\underbrace{i\mathcal{T}}_{\text{The scattering }\boxed{\text{Matrix Element}}}$$

Bring QFT to dummies

Now we can finally arrive at the $\boxed{\text{Feynman Rules}}$ that we learned from the above process for calculating $i\mathcal{T}$.

The Feynman Rules are:

① Draw lines (external lines) for each incoming or outgoing particles

② leave one end of each external line free and connect the other end to ~~a~~ a vertex where exactly three lines meet.
for φ^3 theory.

Include extra ~~external~~ internal lines to do this if necessary.

Draw all diagrams that are Topologically Inequivalent.

③ On each incoming line, draw arrow towards the vertex
On each outgoing line, draw arrow ~~to~~ away from the vertex
On each internal line, draw it arbitrarily.

④. Assign each line its own momentum. The four momentum of external lines are their own momentum.

⑤. Thinking of the four-momenta as following along the arrows, and conserve four-momenta at each vertex.
(For a tree diagram, this fix all ~~momentum~~ momenta)

⑥. The value of each diagram contains the following factors
each external line has, 1;
each internal line with momentum k, $\frac{-i}{k^2+M^2-i\epsilon}$.
each vertex, $iZ_g g$.

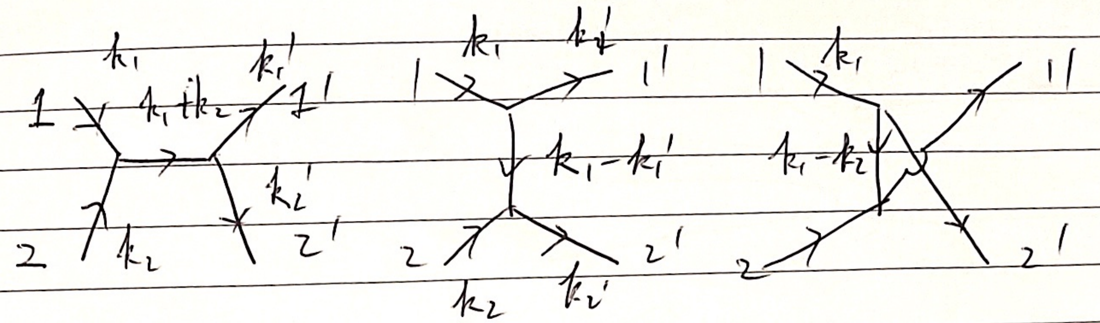
7 (loop). A diagram with L ~~loop~~ closed loops will leave L internal momenta unfixed by rule #5. Integrate over these 4-momenta with measure

$$\frac{d^4 l_i}{(2\pi)^4}.$$

8. ~~A~~ (loop) A loop diagram may have left over symmetry factors if there are exchange of $\boxed{\text{internal}}$ propagators and vertices that leave the diagram unchanged. Divide the value by such ~~sym~~ leftover symmetry factors.

9. (counter terms) Include diagrams with the counterterm vertex that connects two propagators, each with the same four momentum k. The value of this vertex is $-i(Ak^2 + Bm^2)$, where $A \equiv Z_\varphi - 1$ and $B = Z_m - 1$, each are expected to be $O(g^2)$.

10. The value of iT is given by a sum over the value of all these diagrams.



At ~~the~~ tree level, we have.

$$iT = (-iZ_g g)^2\left(+ \frac{-i}{(k_1+k_2)^2+m^2-i\epsilon} + \frac{-i}{(k_1-k_1')^2+m^2-i\epsilon} + \frac{-i}{(k_1-k_2')^2+m^2-i\epsilon}\right) + O(g^4).$$

Now we can proceed to get observable cross section from the scattering amplitudes from iT.