Sec. 2.

Lorentz Invariance & Transformations.

(It is a non-compact continuous group, and non-Abelian, which creates a lot of additional works & notations)

A Lorentz transformation is linear, homogeneous change of coordinates

(see my lecture notes in the "Additional material" for an introduction to representations & group theory in particle physics)

$$\bar{x}^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

distance preserved:

$$x^2 \equiv x^{\mu} x_{\mu} = g_{\mu\nu} x^{\mu} x^{\nu}$$
$$= \vec{x}^2 - c^2 t^2$$

(Lie groups & their Exponential Representation).

and the metric obeys.

$$g_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma} \quad \ldots \boxed{\text{Eq } ①}$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$$ Minkowski metric

$$\text{for } \Lambda = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R_{ij} & \\ 0 & & & \end{pmatrix}$$ ← ordinary rotations

All Lorentz transformations form a $\boxed{\text{Group}}$.

Product is associative & closure: two consecutive L.T. still a L.T.

Identity exist: $\exists \Lambda^\mu{}_\nu = \delta^\mu{}_\nu$

Inverse exist: $\exists \Lambda^{-1}$ that $(\Lambda^{-1})^\mu{}_\nu \Lambda^\nu{}_\sigma = \delta^\mu{}_\sigma$

~~And closure~~ :

(Let's explore a few familiar properties in the vector Rep of Lorentz Group, sometimes we call it $\boxed{\text{defining representation}}$)

$$g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}$$

$$g^{\rho\alpha} \qquad g^{\rho\alpha}$$

$$\Lambda_{\nu\rho}\Lambda^\nu{}_\sigma g^{\rho\alpha} = g_{\rho\sigma} g^{\rho\alpha}$$

$$\Lambda_\nu{}^\alpha \Lambda^\nu{}_\sigma = \delta_\sigma{}^\alpha = \delta^\alpha{}_\sigma \quad \ldots \text{(A)}$$

we would also demand.

$$(\Lambda^{-1})^\alpha{}_\nu \Lambda^\nu{}_\sigma = \delta^\alpha{}_\sigma \quad \ldots \text{(B)}$$

such that we know (compare (A) and (B))

$$(\Lambda^{-1})^\alpha{}_\nu = \Lambda_\nu{}^\alpha$$

we also have

$$g^{\mu\nu}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu = g^{\rho\sigma}$$ .... prove it using the above

For infinitesimal transformation.

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu.$$

From Eq. ① we can see that

$$\delta\omega_{\rho\sigma} = -\delta\omega_{\sigma\rho}$$

$$\delta\omega^{\mu\nu} = -\delta\omega^{\mu\nu} \quad \cdots \text{(H.W.)}.$$

Hence in 4D, there are 6 independent L.T.s (why $\delta\omega$ real? for students) coordinates are real).

$$6 \text{ L.T.s} \begin{cases} 3 \text{ rotations.} & \delta\omega_{ij} = -\epsilon_{ijk}\hat{n}_k\,\delta\theta. \\ 3 \text{ boosts} & \delta\omega_{i0} = \hat{n}_i\,\delta\eta. \end{cases}$$

☆ Not All L.T.s can be done through compounding infinitesimal ones:

back to the basics of Λ

note that the core is about their product.

Hence sign is unfixed, e.g.,

$$(\Lambda^{-1})^\rho{}_\sigma = \Lambda_\sigma{}^\rho$$

$$\mathrm{Det}(\Lambda^{-1}) = \mathrm{Det}(\Lambda).$$
$$\quad\; \shortparallel \; (\mathrm{Det}\Lambda)^{-1}$$

it implies $\det\Lambda = \pm 1$.

$$\Lambda \begin{cases} \det\Lambda = 1 & \text{proper} \\ \det\Lambda = -1 & \text{improper.} \end{cases}$$

Note that the product of any two L.T.s are $\boxed{\text{proper}}$.

And any infinitesimal L.T.s are proper, as

$$\Lambda = 1 + \delta\omega.$$

Any L.T.s can be reached by compounding infinitesimal L.T.s are proper.

these forms a subgroup of L.T.s.

Yet anyother subgroup. is $\boxed{\text{orthochronous}}$ L.Ts.

those for which $\Lambda^0{}_0 \geq +1$.

Eq (0) implies (take $\rho = \sigma = 0$)

$$(\Lambda^0{}_0)^2 - (\Lambda^i{}_0 \Lambda^i{}_0) = 1.$$

$\Rightarrow$ $\Lambda^0{}_0 \geq +1$ or $\Lambda^0{}_0 \leq -1$.

Clearly infinitesimal L.T.s are ~~chrono~~ orthochronous and the product of two L.Ts ~~are~~ orthochronous ones are orthochronous.

thus. Proper & Orthochronous L.T.s forms a subgroup via a series of infinitesimal L.T.s.

Two discrete operations can take us out of this subgroup: Parity & Time-Reversal.

$$\text{Parity}\quad P^\mu{}_\nu = (P^{-1})^\mu{}_\nu = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

proper $\leftrightarrow$ improper

remains ~~chron~~ orthochronous

Time Reversal:

$$T^{\mu}{}_{\nu} = (T^{-1})^{\mu}{}_{\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}.$$

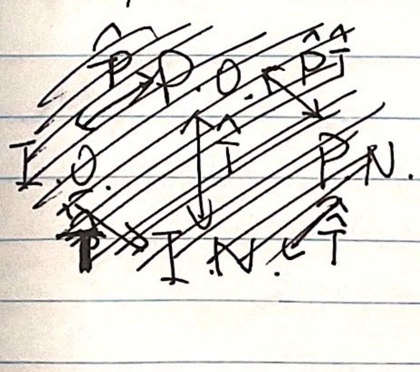
proper $\longleftrightarrow$ improper.

orthochronous $\longleftrightarrow$ non-orthochronous.

Note: Generally, when we say a theory is Lorentz invariant, we mean invariant under $\boxed{\text{proper orthochronous}}$ subgroup of Lorentz.

A theory can be not invariant under ~~P & T.~~ P/T.

Now let's focus on Proper Orthochronous subgroup. and revisit P/T in the future.



(We'll see in this language $\Lambda^\mu{}_\nu$ is nothing but a vector representation, and $\delta\omega_{\mu\nu}$ here means angle.) (There is a mix of notation in the textbook.)

In quantum theory, symmetries are representated

by unitary (anti-unitary) transformations.

(Now, ~~as~~ a $\boxed{\text{general representation}}$)

→ We can associate a unitary operator to

~~$U^\dagger(\Lambda)U(\Lambda) = 1$.~~ norm being one

each proper orthochronous L.T. Λ.

These operators $\boxed{\text{must obey the composition rule}}$

$$U(\Lambda'\Lambda) = U(\Lambda')U(\Lambda).$$

For an infinitesimal transformation, we can write.

$$U(1+\delta\omega) = I + \frac{i}{2\hbar}\delta\omega_{\mu\nu} M^{\mu\nu}$$

$\boxed{\text{symmetry factor}}$ $\boxed{\text{we will see the meaning later}}$.

where $M^{\mu\nu}$ is a set of hermitian operators

called the $\boxed{\text{Generators of the Lorentz Group}}$

$$M^{\mu\nu} = -M^{\nu\mu}$$

Now let's explore/motivate some properties of

generators $M^{\mu\nu}$

Try $\mathcal{U}(\Lambda^{-1})\mathcal{U}(\Lambda')\mathcal{U}(\Lambda) = \mathcal{U}(\Lambda^{-1}\Lambda'\Lambda)$.

let $\Lambda' = 1 + \delta\omega'$ and expand both side

by ~~$\delta\omega$~~ $\delta\omega'$ in <u>linear order</u>.

$$\mathcal{U}(\Lambda)^{-1}\delta\omega'_{\mu\nu} M^{\mu\nu}\mathcal{U}(\Lambda)$$

$$= \mathcal{U}((\Lambda^{-1})^{\mu}{}_{\nu}\Lambda'^{\nu}{}_{\rho}\Lambda^{\rho}{}_{\sigma})\Big|_{\text{linear}}$$

$$= \mathcal{U}(\Lambda_{\nu}{}^{\mu}(1+\delta\omega'^{\nu}{}_{\rho})\Lambda^{\rho}{}_{\sigma})\Big|_{\text{linear}}$$

$$= \mathcal{U}(1+\delta\omega'^{\nu}{}_{\rho}\Lambda_{\nu}{}^{\mu}\Lambda^{\rho}{}_{\sigma})\Big|_{\text{linear}}.$$

$$\cancel{= \delta\omega'^{\nu}{}_{\rho}\Lambda_{\nu}{}^{\mu}\Lambda^{\rho}{}_{\sigma} M^{\mu\sigma} g^{\alpha\sigma}\delta^{\mu}{}_{\sigma} M}$$

$$= \mathcal{U}(1+(\delta\omega'\Lambda\Lambda)^{\mu}{}_{\sigma})\Big|_{\text{linear}}$$

$$= \mathcal{U}(1+(\delta\omega'\Lambda\Lambda)_{\rho\sigma}g^{\mu\rho})\Big|_{\text{linear}}.$$

$$= I + \delta\omega'_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} \quad \cdots \text{ R.H.S.}$$

so $\delta\omega'_{\mu\nu}$ is arbitrary.

$$\underbrace{\mathcal{U}(\Lambda^{-1})M^{\mu\nu}\mathcal{U}(\Lambda)}_{(\text{L.H.S}-I)/\delta\omega'_{\mu\nu}} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma}$$

clearly $\underline{\mathcal{U}(\Lambda)^{-1} M^{\mu\nu} \mathcal{U}(\Lambda)}$

has each vector index μ, ν

goes through its own Lorentz transformation.

This is general; any operator carries Lorentz index should transform accordingly.

For instance $\hat{P}^\mu = (\hat{H}, \hat{P}^i)$

$$c\hat{P}^0 = \hat{H}$$

$$\mathcal{U}(\Lambda)^{-1} \hat{P}^\mu \mathcal{U}(\Lambda) = \Lambda^\mu{}_\nu \hat{P}^\nu \quad \text{.... (2.15)}$$

Now if we let $\Lambda = 1 + \delta\omega$, expand to linear order in $\delta\omega$, we get the commutation relation, "Lie Algebra" of the Lorentz group.

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar(g^{\mu\rho}M^{\nu\sigma} - (\mu \leftrightarrow \nu)) - (\rho \leftrightarrow \sigma).$$

$$= i\hbar[g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}]$$

We can identity (to a representation) that.

components of the angular momentum operator

$\vec{J}$ as $J_i \equiv \frac{1}{2} \epsilon_{ijk} M^{jk}$.

components of the boost operator

$\vec{K}$ as $K_i \equiv M^{i0}$

Then from the Lie Algebra, we have

(This also is a reflection of the non-Abelian nature)

$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$. ← commutation of angular momentum

$[J_i, K_j] = i\hbar \epsilon_{ijk} K_k$. ← K transforms as a 3-vector under rotation

$[K_i, K_j] = -i\hbar \epsilon_{ijk} J_k$. ← a series of boost can be releted to rotation.

(Note that translation generators do not commute with Lorentz generators, so Poincare group is not simple outter product of translation ⊗ Lorentz)

Similarly, we can show that. (from Eq 2.15)

$[P^\mu, M^{\rho\sigma}] = i\hbar (g^{\mu\sigma} P^\rho - (\rho \leftrightarrow \sigma))$.

which gives.

$[J_i, H] = 0$.

$[J_i, P_j] = i\hbar \epsilon_{ijk} P_k$.

$[K_i, H] = i\hbar c P_i$

$[K_i, P_j] = i\hbar \delta_{ij} H/c$.

also $[P_i, P_j] = 0$

$[P_i, \mathcal{H}] = 0.$

Together, they formed the poincaré group

(with ~~t~~ translation).

Let's now consider what happens for a scalar field $\varphi(x)$.

under a Lorentz transformation:

back to the Heisenberg picture.

$$e^{+i\mathcal{H}t/\hbar} \varphi(x,0)\, e^{-i\mathcal{H}t/\hbar} = \varphi(x,t).$$

$$\Big\downarrow \text{ Relativistic}$$

$$e^{-iP\cdot x/\hbar}\, \cancel{x}\, \varphi(0)\, e^{iPx/\hbar} = \varphi(x).$$

$$Px = P^\mu x_\mu = \vec{P}\cdot\vec{x} - \mathcal{H}t.$$

We can make it a bit fancier by defining

unitary space-time translation operator

$$T(a) \equiv e^{-iP^\mu a_\mu/\hbar}.$$

Then we have

new coordinate

$$T(a)^{-1}\varphi(x)T(a) = \varphi(x-a) \quad \ldots \quad (2.24)$$

relates to the old one.

For an infinitesimal transformation.

~~now depends on metric choice~~

$$T(\delta a) = I - \frac{i}{\hbar}\delta a_\mu P^\mu.$$

~~we do want to keep~~ ~~$\frac{i}{\hbar}\delta t\omega$.~~

Comparing with Eq. 2.12 $\;\mathcal{U}(1+\delta\omega) = I + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}$

we see $\;\mathcal{U}(\Lambda)^{-1}\varphi(x)\mathcal{U}(\Lambda)$ in (2.24) means.

$$\mathcal{U}(\Lambda)^{-1}\varphi(x)\,\mathcal{U}(\Lambda) = \varphi(\Lambda^{-1}x).$$

Derivatives behave following the rules.

$$\mathcal{U}(\Lambda)^{-1}\partial^\mu\varphi(x)\,\mathcal{U}(\Lambda) = \Lambda^\mu{}_\nu\,\bar{\partial}^\nu\varphi(\Lambda^{-1}x).$$

and hence

$$\mathcal{U}(\Lambda)^{-1}\partial^2\varphi(x)\,\mathcal{U}(\Lambda) = \bar{\partial}^2\varphi(\Lambda^{-1}x).$$

So that

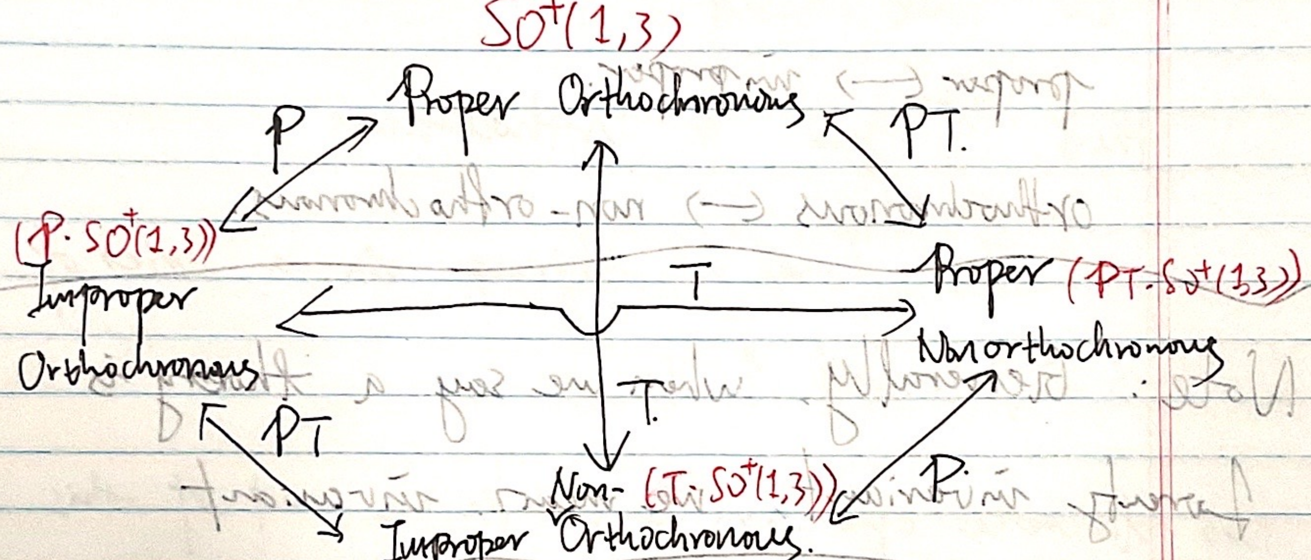
$$\mathcal{U}(\Lambda)^{-1}\left(-\partial^2\varphi(x) + m^2/\hbar^2 c^2\,\varphi(x) = 0\right)\mathcal{U}(\Lambda)$$

is L.T. invariant. in the sense

$$(-\bar{\partial} + m^2/\hbar^2c^2)\,\underbrace{\varphi(\Lambda^{-1}x)}_{\text{new field}} = 0.$$

Comment on passive & active transformations

(possibily also Tangent Manifold & Embedding manifold)



These're the four
"Connected Component"
(topological concept)