Chapter 5

L.S.Z Reduction

(Lehmann – Symanzik – Zimmermann)

We can get particle state

$$|k\rangle = a^\dagger(\vec{k})|0\rangle$$

$$a^\dagger(\vec{k}) = -i\int d^3x\, e^{ikx} \overleftrightarrow{\partial_0}\, \varphi(x)$$

$$a(\vec{k})|0\rangle = 0$$

We require the normalization, such that

$$\langle 0|0\rangle = 1$$

Then the one-particle state will have the $\boxed{\text{Lorentz invariant}}$ normalization

$$\langle k|k'\rangle = \langle 0|\cancel{k}\, a(\vec{k})\, a^\dagger(\vec{k}')|0\rangle$$

$$= (2\pi)^3\, 2\omega\, \delta^3(\vec{k}-\vec{k}')\,\langle 0|0\rangle$$

$$= (2\pi)^3\, 2\omega\, \delta^3(\vec{k}-\vec{k}')$$

Now we can define a time-independent operator that create a particle localized in $\vec{k}_1$ and near the origin $\vec{x}=0$.

(Recall "minimal wave packet" in QM free particle)

$$a_1^\dagger = \int d^3k \; f_1(\vec{k})\, a^\dagger(\vec{k}),$$

where $f_1(\vec{k}) \propto e^{-(\vec{k}-\vec{k}_1)^2/4\sigma^2}$

σ is the spread in momentum space.

If we time evolve this wave packet (in Schrödinger Picture), ~~both~~ the ~~two~~ localised particle in $\vec{x}=0$ will spread out at $t \to \pm\infty$

If we consider two particles

$a_1^\dagger a_2^\dagger |0\rangle$, at $t \to -\infty$, these two particles are both spreaded out the space, but as long as $\vec{k}_1 \neq \vec{k}_2$, they are wildly separated.

Now $\boxed{\text{Assume}}$ it still works for interacting theory.

we will be back to this point soonish.

$a^{\dagger}(\vec{k})$ can no longer be time-independent.

(as they will meet and interact).

A suitable initial state is still $(\vec{k}_1 \neq \vec{k}_2)$

$$|i\rangle = \lim_{t \to -\infty} a_1^{\dagger}(t)\, a_2^{\dagger}(t)|0\rangle$$

but proper normalization, we can make $\langle i|i\rangle = 1$

Similarly, final states

$$|f\rangle = \lim_{t \to +\infty} a_{1'}^{\dagger}(t)\, a_{2'}^{\dagger}(t)|0\rangle$$

$(\vec{k}_1' \neq \vec{k}_2')$

Now the scattering amplitude is given by

$\langle f|i\rangle$.

and we need to find a more useful expression to evaluate $\langle f|i\rangle$.

Note that

$$a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = \int_{-\infty}^{\infty} dt\, \partial_0 a_1^\dagger(t)$$

$$= -i\int d^3k\, f_1(\vec{k}) \int d^4x\, \partial_0 (e^{ikx} \overleftrightarrow{\partial_0} \varphi(x))$$

$$= -i\int d^3k\, f_1(\vec{k}) \int d^4x\, e^{ikx} (\partial_0^2 + \omega^2) \varphi(x)$$

$$= -i\int d^3k\, f_1(\vec{k}) \int d^4x\, e^{ikx} (\partial_0^2 + \vec{k}^2 + m^2) \varphi(x)$$

$$= -i\int d^3k\, f_1(\vec{k}) \int d^4x\, (e^{ikx} (\partial_0^2 + m^2) \varphi(x) + (\vec{\nabla}^2 e^{ikx})\varphi(x)$$

$$= -i\int d^3k\, f_1(\vec{k}) \int d^4x\, (e^{ikx} (\partial_0^2 - \vec{\nabla}^2 + m^2) \varphi(x) + \text{surface terms (dropped)}$$

$$= -i\int d^3k\, f_1(\vec{k}) \int d^4x\, e^{ikx} (-\partial^2 + m^2) \varphi(x)$$

In a free field theory $(-\partial^2+m^2)\varphi(x) = 0$ as $\varphi(x)$ satisfies the Klein-Gordon Equation.

In an interacting theory, (say) $\mathcal{L}_I = \frac{1}{6} g \varphi^3$

now the E.O.M. gives

$$(-\partial^2+m^2)\varphi(x) = \frac{1}{2} g \varphi^2$$

then $a_1^\dagger(-\infty) = a_1^\dagger(+\infty) + i\int d^3k\, f_1(\vec{k}) \int d^4x\, e^{ikx} (-\partial^2+m^2)\varphi(x)$ ... (5.11)

h.c. of the above yields.

$$a_1(+\infty) = a_1(-\infty) + i\int d^3k\, f_1(\vec{k}_1) \int d^4x\, e^{-ikx} (-\partial^2+m^2)\varphi(x) \quad \text{... (5.12)}$$

~~so~~ back to

$$\langle f|i\rangle = \langle 0|\, a_{1'}(+\infty)\, a_{2'}(+\infty)\, a_2^\dagger(-\infty)\, a_1^\dagger(-\infty)\, |0\rangle$$

we introduce a "Time order" T.

$$\langle f|i\rangle = \langle 0|\, T\, a_{1'}(+\infty)\, a_{2'}(+\infty)\, a_2^\dagger(-\infty)\, a_1^\dagger(-\infty)\, |0\rangle$$

T means anything to the right of it are ordered (rearranged) in time, regardless of how written. (Left is later time and right is earlier time).

With (5.11) and (5.12)

$$\langle f|i\rangle = \langle 0|\, T\{(a_{1'}(-\infty) + i\int \quad)(a_{2'}(-\infty) + i\int \quad)$$

$$\cancel{a_1^\dagger(-\infty)\, a_2^\dagger(-\infty)}$$

$$(a_1^\dagger(+\infty) + i\int \quad)(a_2^\dagger(+\infty) + i\int \quad)\}|0\rangle$$

(any a, $a^\dagger$ in above is zero when acting on $|0\rangle$, or $\langle 0|$)

$$= i^{2+2} \int d^4x_1'\, d^4x_2'\, d^4x_1\, d^4x_2$$

$$e^{-ik_1'x_1'}\, e^{-ik_2'x_2'}\, e^{ik_1x_1}\, e^{ik_2x_2}$$

$$(-\partial_1^2+m^2)(-\partial_{1'}^2+m^2)(-\partial_2^2+m^2)(-\partial_{2'}^2+m^2)$$

$$\langle 0|\, T\varphi(x_1)\,\varphi(x_2)\,\varphi(x_1')\,\varphi(x_2')\, |0\rangle$$

$$ {}_{n'}\langle f|i\rangle_n = i^{n+n'} \int d^4x_1\, e^{ik_1x_1}(-\partial_1^2+m^2)\cdots \int d^4x_1'\, e^{-ik_1'x_1'}(-\partial_{1'}^2+m^2)\cdots $$

$$ \times \langle 0|T\, \underbrace{\varphi(x_1)\cdots}_{n}\ \underbrace{\varphi(x_1')\cdots}_{n'}\ |0\rangle $$

This is the $\boxed{\text{LSZ reduction}}$ formula.

one of the key equations of QFT.

Important additional considerations (Canonical quantization makes these transparent)

But here we are assuming that $\langle f|$ and $|i\rangle$ are described by free particles while $\langle f|i\rangle$ to due to interaction. $\boxed{\text{Is it a valid assumption?}}$

The rest is Only ~~brodly~~ mentioned during class

(probably skip this following part, if time is insufficient)

$\langle 0|0\rangle = 1$ should still be valid.

~~$a^\dagger|0\rangle = 0$~~? $\qquad \varphi(x) = \int d^3x\, e^{ikx} a(\vec{k}) + e^{-ikx} a^\dagger(\vec{k})$.

$\langle 0|\varphi(x)|0\rangle$ should be zero.

$$\varphi(x) = \exp(-iP^\mu x_\mu)\,\varphi(0)\,\exp(+iP^\mu x_\mu).$$

$\langle 0|\varphi(x)|0\rangle = \langle 0|\varphi(0)|0\rangle$ as $P^\mu|0\rangle = 0$.

$\quad = \text{const}$ (L.I.).

~~wh~~ we want $a^\dagger(\vec{k})|0\rangle$ to create a single particle state, not a linear combination of 0, single and multiple particle state at $t = \pm\infty$.

\# To satisfy $\langle 0|\varphi(x)|0\rangle = 0$, if we have $\langle 0|\varphi(x)|0\rangle = v$

we redefine $\varphi(x) \to \varphi(x) - v$.

So the new field $\varphi(x)$ has.

$\langle 0|\varphi(x)|0\rangle = 0$. ~~as a scalar, we can shift it.~~ ~~Lorentz.~~

Then $\underbrace{\langle p|}_{\text{a one-particle state.}}\varphi(x)|0\rangle = \langle p|e^{-iPx}\varphi(0)e^{+iPx}|0\rangle$

$= e^{-ipx}\underbrace{\langle p|\varphi(0)|0\rangle}_{\text{this is a number.}}$

we want $\langle p|\varphi(x)|0\rangle = 1$ but ~~they~~ it may not.

it is a number depends on L.I. function of p^μ, which is $p^2 = -m^2$ and other parameters. ($x^\mu = 0$ here).

for $\langle p|\varphi(0)|0\rangle = Z$, we can again

rescale $\varphi(0) \to \varphi(0)/Z$.

(renormalize).

so we have so far

$$\langle p|\varphi(0)|0\rangle = 1, \quad \langle 0|\varphi(0)|0\rangle = 1.$$

What about $\langle \underbrace{p,n}_{\text{n-particle state with total momentum } p} |\varphi(x)|0\rangle$?

we want it to be zero.

$$\langle p,n|\varphi(x)|0\rangle = e^{-ipx}\langle p,n|\varphi(0)|0\rangle$$

$$= e^{-ipx} A_n(p).$$

$A_n(p)$ is a function of 2.I. of various particle momentum product.

Actually, we really only need

$\langle p,n|a^{\dagger}(\pm\infty)|0\rangle = 0$; such that our assumption on the initial/final states are reasonable.

Let's be slightly more serious.

We define the $|p,n\rangle$ state more properly.

$$|\psi\rangle = \sum_n \underbrace{\int d^3p\, \psi_n(\vec{p})}_{\text{total wave packet.}} |p,n\rangle.$$

We want ($t \to \pm\infty$ to be taken later)

$\langle\psi| a_1^\dagger(t) |0\rangle$

$$= -i \sum_n \int d^3p\, \psi_n^*(\vec{p}) \int d^3k\, f_1(\vec{k}) \int d^3x\, e^{ikx} \overleftrightarrow{\partial_0} \underbrace{\text{}}_{\langle p,n|\varphi(x)|0\rangle}$$

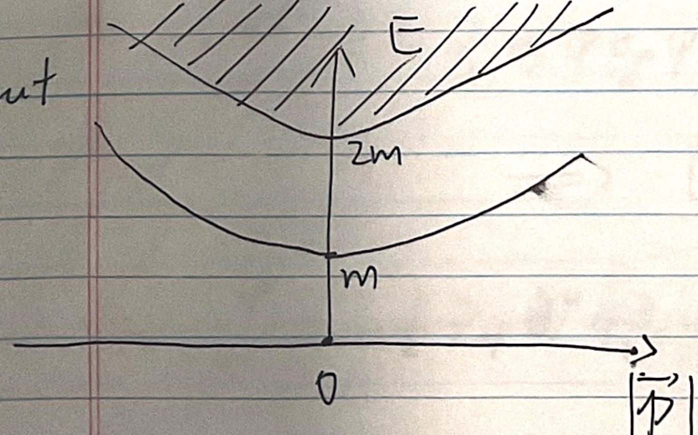
$$= -i \sum_n \int d^3p\, \psi_n^*(\vec{p}) \int d^3k\, f_1(\vec{k}) \int d^3x\, e^{ikx} \overleftrightarrow{\partial_0} e^{-ipx} A_n(p)$$

$$= \sum_n \int d^3p\, \psi_n^*(\vec{p}) \int d^3k\, f_1(\vec{k}) \int d^3x\, (k^0+p^0)\, e^{i(k-p)x} A_n(p)$$

$$= \sum_n \int d^3p\, \psi_n^*(\vec{p})\, f_1(\vec{p})\, (2\pi)^3 (k^0+p^0)\, e^{i(k^0-p^0)t} A_n(p) \quad \cdots (5.23)$$

here $p^0 = (\vec{p}^{\,2} + M^2)^{1/2}$, $k^0 = (\vec{p}^{\,2} + m^2)^{1/2}$

but



we know then

$M \geq 2m > m$

thus (5.23) osillates fast at $t \to \pm\infty$.

by Riemann-Lebesgue lemma.,

the right-hand side vanishes as $t \to \pm\infty$

Physically: one particle wave packet $a^\dagger|0\rangle$

spread out differently than $|p,n\rangle$

~~there~~ they are unlikely to overlap via infinite propagation (at least by constructing one-particle wave packet carefully).

LSZ formula is good so long as

$\langle 0|\varphi(x)|0\rangle = 0$ and $\langle k|\varphi(x)|0\rangle = e^{-ikx}$.

this means shift & rescale an original Lagrangian to a new one. e.g.

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 + \frac{1}{6}g\varphi^3$$

$\Longrightarrow$ (Will ~~truth~~ formally introduce $\boxed{\text{renormalization}}$ later).

$$\mathcal{L} = -\frac{1}{2}Z_\varphi\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}Z_m m^2\varphi^2 + \frac{1}{6}g Z_g\varphi^3 + Y\varphi.$$

we can identify $Z_m m^2 / Z_\varphi$ with the physical mass ~~treat~~ and other "physically"