Lec 23

Having understood QED, the Feynman Rules should be straightforward. just add the following to before.

Add, Each photon line (wavy lines)

Internal photons, we have

$$\frac{-i\,p^{\mu\nu}}{k^2-i\varepsilon}$$

[here $p^{\mu\nu}=g^{\mu\nu}$, typically most convenient. Feynman Gauge].

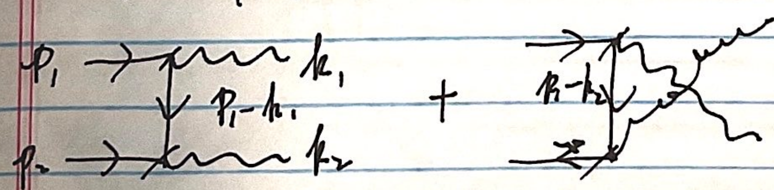
Incoming photons $\varepsilon^{\mu\,*}_{\lambda_i}(\vec{k}_i)$

Out going photons $\varepsilon^{\mu}_{\lambda_i}(\vec{k}_i)$.

And each ~~add~~ vertex is $ie\gamma^\mu$.

Note that typically one need to pay attention to a Fermion "order" as there are hidden indices that we didn't spell out. Photons can be moved around.

Let's calculate a simple process. $e^+e^-\to\gamma\gamma$.
(A pair of particle antiparticle annihilate into two photons)



$$iT = \bar{v}_{s_2}(p_2)\, ie\gamma_\nu (-i)\frac{-\not{p}_1+\not{k}_1+m}{(p_1-k_1)^2+m^2-i\varepsilon}\, ie\gamma_\mu u_{s_1}(p_1)\, \varepsilon^{*\mu}_{\lambda_1}(k_1)\, \varepsilon^{\nu}_{\lambda_2}(k_2)$$

$$+ \cdots$$

$$T = e^2 \varepsilon_1^\mu \varepsilon_2^\nu \bar{v}_{2'}\left[\gamma_\nu \frac{-\not{p}_1+\not{k}_1+m}{(p_1-k_1)^2+m^2-i\varepsilon}\gamma_\mu + \gamma_\mu \frac{-\not{p}_1+\not{k}_2+m}{(p_1-k_2)^2+m^2-i\varepsilon}\gamma_\nu\right] u_{1'}$$

as $t = -(p_1-k_1)^2$

$u = -(p_1-k_2)^2$

$$T = e^2 \varepsilon_1^\mu \varepsilon_2^\nu \bar{v}_{2'}\left[\frac{-\not{p}_1+\not{k}_1+m}{(m^2-t)} + \frac{-\not{p}_1+\not{k}_2+m}{(m^2-u)}\right] u_{1'}$$

$|T|^2 =$ four terms.

$$e^4 \varepsilon_1^\mu \varepsilon_1^{*\mu'} \varepsilon_2^\nu \varepsilon_2^{*\nu'}$$

$$\times \left\{ \frac{\#_{tt}}{(t-m^2)^2} + \frac{\#_{uu}}{(u-m^2)^2} + \frac{\#_{ut} + \#_{ut}}{(t-m^2)(u-m^2)} \right\}$$

Let me demonstrate one, say, as usual, we are interested in the spin & polarization ~~averaged~~ averaged summed cross section.

$$\frac{1}{4}\sum_{s_1,s_2=\pm}\ \sum_{\lambda_1,\lambda_2=\pm} |T|^2$$

for the $\#_{tt}$.

we have

$$\frac{1}{4} e^4 \cdot g^{\mu\mu'} g^{\nu\nu'} \bar{v}_{2'} \gamma_\nu \frac{-\not{p}_1 + \not{k}_1 + m}{-(t-m^2)} \gamma_\mu u_{1'}$$

$$\times \left(\bar{v}_{2'} \gamma_{\nu'} \frac{-\not{p}_1 + \not{k}_1 + m}{-(t-m^2)} \gamma_{\mu'} u_{1'}\right)^*$$

$$\cancel{= e^4 \bar{v}_{2'} \gamma_\nu}$$

$$= \frac{1}{4} \sum_{s_1, s_2} \frac{e^4}{(t-m^2)^2} \cdot \bar{v}_{2'} \gamma_\nu (-\not{p}_1 + \not{k}_1 + m) \gamma_\mu u_{1'} \bar{u}_{1'} \gamma^\mu (-\not{p}_1 + \not{k}_1 + m) \gamma^\nu v_{2'}$$

$$= \frac{1}{4} \frac{e^4}{(t-m^2)^2} \mathrm{Tr}\left[(-\not{p}_2 - m) \gamma_\nu (-\not{p}_1 + \not{k}_1 + m) \gamma_\mu (-\not{p}_1 + m) \gamma^\mu (-\not{p}_1 + \not{k}_1 + m) \gamma^\nu\right]$$

using $\gamma_\mu \not{a} \gamma^\mu = +2\not{a}$

$\gamma_\mu \gamma^\mu = -4.$

$$= \frac{e^4}{4(t-m^2)^2} \cdot \mathrm{Tr}\left[(-2\not{p}_2 + 4m)(-\not{p}_1 + \not{k}_1 + m)(-2\not{p}_1 - 4m)(-\not{p}_1 + \not{k}_1 + m)\right].$$

$$= \frac{e^4}{4(t-m^2)^2} \left[\mathrm{Tr}[4\gamma_s] + m^2 \mathrm{Tr}[2\gamma_s] + m^4 \cdot (-64) \right].$$

(Also notice that $\not{p}_1 \not{p}_1 = m^2$. one can reduce the # of 4 γ_s trace that needs to be done.)

eventually, we assemble everything and make it an observable cross section.

$$d\sigma = \frac{1}{\text{flux}} \cdot |T|^2 \cdot \underbrace{dPS_2}_{\substack{\text{two photons are identicle} \\ \text{particles, phrase space have } 1/2 \\ \text{factor in it.}}}$$

Now, let's work out the loop corrections.

We begin with

$\cancel{\mathcal{L}_1 = \mathcal{Z}}$

$\cancel{\mathcal{L}_0 = i\bar{\Psi}\not{\partial}\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - m\bar{\Psi}\Psi}$

~~wit~~

$$\mathcal{L}_0 = i\bar{\Psi}\not{\partial}\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - m\bar{\Psi}\Psi$$

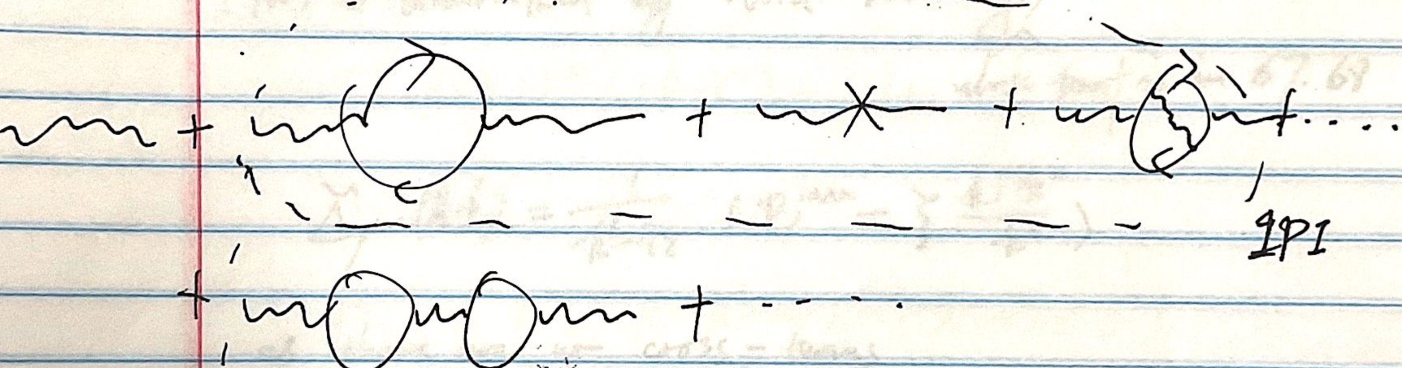
$$\mathcal{L}_1 = Z_1 e\bar{\Psi}\not{A}\Psi + \mathcal{L}_{CT}$$

$$\mathcal{L}_{CT} = i(Z_2 - 1)\bar{\Psi}\not{\partial}\Psi - (Z_m - 1)\bar{\Psi}\Psi m$$

$$- \frac{1}{4}(Z_3 - 1)F^{\mu\nu}F_{\mu\nu}.$$

This is a renormalizable theory with finite # of counter-term needed.
other counter terms do not exist by various reasons (symmetries, etc).

Let's do one calculation for example.
What is the exact propagator for the photons?
(also known as "vacuum polarization" & "self energy correction")



the exact propagator (in momentum space)

$$\check{\Delta}_{\mu\nu}(k) = \check{\Delta}_{\mu\nu}(k) + \check{\Delta}_{\mu\rho}(\vec{k})\Pi^{\rho\sigma}(\vec{k})\check{\Delta}_{\sigma\nu}(\vec{k}) + \cdots$$

In the general R_ξ gauge

$$\Delta_{\mu\nu}(k) = \frac{1}{k^2 - i\varepsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right).$$

Physical result shall not depend on ξ, implying the result should be "transverse", e.g.

$$k_\mu \Pi^{\mu\nu} = k_\nu \Pi^{\mu\nu} = 0.$$

such that

$$\Pi^{\mu\nu}(k) = \Pi(k)\,(k^2 g^{\mu\nu} - k^\mu k^\nu)$$
$$= k^2\,\Pi(k)\,P^{\mu\nu}.$$

(this is Gauranteed by Word Identity)
whose proof is in 67.68

$$\tilde{\Delta}_{\mu\nu}(k) = \frac{1}{k^2 - i\varepsilon}\left(P^{\mu\nu} - \xi\frac{k^\mu k^\nu}{k^2}\right)$$

as there are no cross-terms

$$\tilde{\Delta}_{\mu\nu}(k) = \frac{P_{\mu\nu}(k)}{k^2[1-\Pi(k)] - i\varepsilon} + \xi\frac{k_\mu k_\nu / k^2}{k^2 - i\varepsilon}$$

Again the second term should be physically irrelevant, we can drop it or set $\xi = 0$ (Lorentz Gauge)

For ~~one-shell~~ renormalization scheme, we should have

$$\Pi(0) = 0 \quad \boxed{\cancel{\&}\ \Pi'(k^2)\big|_{k^2=0} = 0}$$

Let's calculate.

$$i\Pi^{\mu\nu}(k) = \underbrace{(-1)}_{\text{fermion loop}}\ \underbrace{(iZ_1 e)^2}_{\text{vertices}}\ \underbrace{\left(\frac{1}{i}\right)^2}_{\text{propagators}}\int\frac{d^4 l}{(2\pi)^4}$$

$$\mathrm{Tr}[\tilde{S}(\not{l}+\not{k})\gamma^\mu\tilde{S}(\not{l})\gamma^\nu]$$

$$-i(Z_3-1)(k^2 g^{\mu\nu} - k^\mu k^\nu) + O(e^4).$$

Now

$$\mathrm{Tr}[\tilde{S}(\not{l}+\not{k})\gamma^\mu\tilde{S}(\not{l})\gamma^\nu] = \int_0^1 dx\,\frac{4N^{\mu\nu}}{(q^2+D)^2}$$

$$D = x(1-x)k^2 + m^2 - i\varepsilon$$

$$4N^{\mu\nu} = \mathrm{Tr}[(-\not{l}-\not{k}+m)\gamma^\mu(-\not{l}+m)\gamma^\nu]$$

$$= 4[(l+k)^\mu l^\nu - (l+k)\cdot l\, g^{\mu\nu} - (l+k)^\nu l^\mu - m^2 g^{\mu\nu}]$$

Recall that we shift the momentum $l \to$ ~~$l+xk$~~ $q = l + xk$.

$$N^{\mu\nu} \to 2q^\mu q^\nu - 2x(1-x)k^\mu k^\nu - [q^2 - x(1-x)k^2 + m^2]g^{\mu\nu}$$

This integral diverges, we need to "regulate" it.

We can use dim-reg. $d = 4 - \varepsilon$

("Recall" 14.5) $\int d^d q \; q^\mu q^\nu f(q^2) = \frac{1}{d} g^{\mu\nu} \int d^d q \; q^2 f(q^2)$

$$\int d^d q \; q^\mu f(q^2) = 0$$

. . .

$$N^{\mu\nu} \to -2x(1-x) k^\mu k^\nu + \left[\left(\frac{2}{d} - 1\right) q^2 + x(1-x) k^2 \cancel{+} - m^2 \right] g^{\mu\nu}$$

"before" we do the calculation, let's observe that.

$$\left(\frac{2}{d} - 1\right) \int \frac{d^d q}{(2\pi)^4} \frac{q^2}{(q^2 + D)^2} = D \int \frac{d^d q}{(2\pi)^4} \frac{1}{(q^2 + D)^2}$$

by comparing. the "master equation"

It is then clear that

$$N^{\mu\nu} \to 2x(1-x) (k^2 g^{\mu\nu} - k^\mu k^\nu)$$

telling us that $\Pi^{\mu\nu}(k)$ is transverse!

Now we can evaluate.

$$\tilde{\mu}^\varepsilon \int_0^1 \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + D)^2} = \frac{i}{16\pi^2} \Gamma(\varepsilon/2) \left(\frac{4\pi \tilde{\mu}^2}{D}\right)^{\varepsilon/2}$$

$$= \frac{i}{8\pi^2} \left[\frac{1}{\varepsilon} - \frac{1}{2} \ln(D/\mu^2) \right] + \dots$$

$$\mu^2 = 4\pi e^{-\gamma} \tilde{\mu}^2$$

So we know that

$$\Pi(k^2) = -\frac{e^2}{\pi^2}\int_0^1 dx\,(1-x)x\left[\frac{1}{\epsilon} - \frac{1}{2}\ln(D/\mu^2)\right] - (Z_3-1) + O(e^4)$$

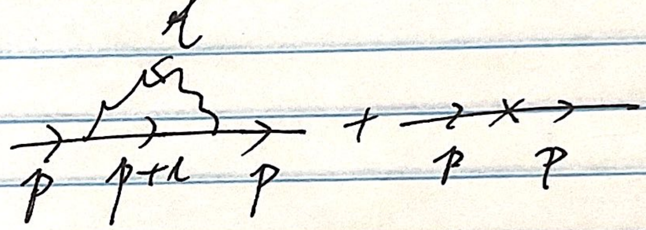
Imposing on-shell condition yields that

$$Z_3 = 1 - \frac{e^2}{6\pi^2}\left[\frac{1}{\epsilon} - \ln(m/\mu)\right] + O(e^4)$$

&

$$\Pi(k^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln(D/m^2) + O(e^4).$$

Similarly, we can do



can get Z_2, Z_m.

~~to~~ there are some IR subtitlies here but can be circumvented by giving γ a small mass and take $m_\gamma \to 0$ in the last step.

We can get

$$Z_2 = 1 - \frac{e^2}{6\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + O(e^4)$$

$$Z_m = 1 - \frac{e^2}{2\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + O(e^4).$$

and similarly

$$Z_3 = 1 - \frac{e^2}{8\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) + O(\epsilon^4).$$

We can also work out the beta function.

$$e_0 = Z_3^{-1/2} Z_2^{-1} Z_1 \tilde{\mu}^{\varepsilon/2} e.$$

$$\alpha_0 = Z_3^{-1} Z_2^{-2} Z_1^{2} \tilde{\mu}^{\varepsilon} \alpha.$$

In $\overline{MS}$ one can see that

$$\beta(\alpha) = \frac{2\alpha^2}{3\pi} + O(\alpha^3).$$