Loop corrections to the propagator (Sec 14)

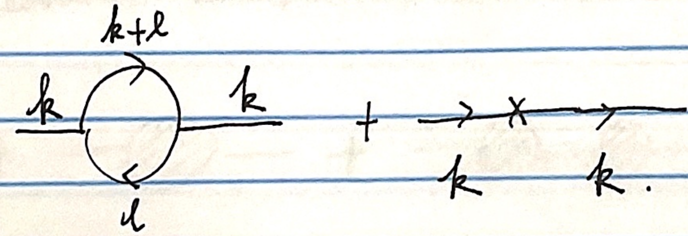
In path integral, we wrote the exact propagator as

$$\frac{1}{i}\Delta(x_1-x_2) \equiv \langle 0|T\,\varphi(x_1)\varphi(x_2)|0\rangle = \delta_1\delta_2\, iW(J)\Big|_{J=0}$$

$iW(J)$ is the sum of all connected diagrams

leading order (tree level) which is $\Delta(x_1-x_2)$

next-to leading order



using Feynman Rules

$$\frac{1}{i}\hat{\Delta}(k^2) = \frac{1}{i}\Delta(k^2) + \frac{1}{i}\hat{\Delta}(k^2)\left[i\Pi(k^2)\right]\frac{1}{i}\hat{\Delta}(k^2) + O(g^4)$$

where $\hat{\Delta}(k^2) = \dfrac{1}{k^2+m^2-i\epsilon}$

which is the free-field propagator.

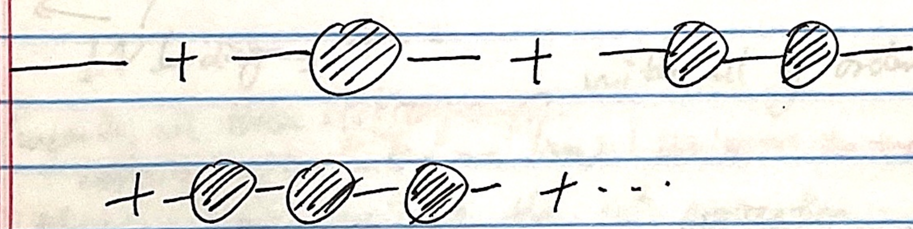
from the diagram we can read that

$$Z_g = 1 + O(g^2)$$

$$i\Pi(k^2) = \underbrace{\frac{1}{2}}_{\text{symmetry factor}} (ig)^2 \left(\frac{1}{i}\right)^2 \int \frac{d^d l}{(2\pi)^d} \tilde{\Delta}((l+k)^2)\, \tilde{\Delta}(l^2)$$

$$- i\,(\underbrace{A}_{(Z_\varphi - 1)} k^2 + \underbrace{B}_{(Z_m - 1).} m^2) + O(g^4)$$

but we can see these effect can be further summed via geometric series.



$$\frac{1}{i}\Delta(k^2) = \frac{1}{i}\tilde{\Delta}(k^2) + \frac{1}{i}\tilde{\Delta}(k^2)\left[i\Pi(k^2)\right]\frac{1}{i}\tilde{\Delta}(k^2)$$

$$+ \frac{1}{i}\tilde{\Delta}(k^2)\left[i\Pi(k^2)\right]\frac{1}{i}\tilde{\Delta}(k^2)\left[i\Pi(k^2)\right]\frac{1}{i}\Delta(k^2)$$

$$+ \cdots$$

$$\cancel{\frac{1}{i}}\Delta(k^2) = \cancel{\frac{1}{i}}\tilde{\Delta}(k^2)\, \frac{1}{1 - \Pi(k^2)\tilde{\Delta}(k^2)}$$

$$= \frac{1}{k^2 + m^2 - i\epsilon - \Pi(k^2)}$$

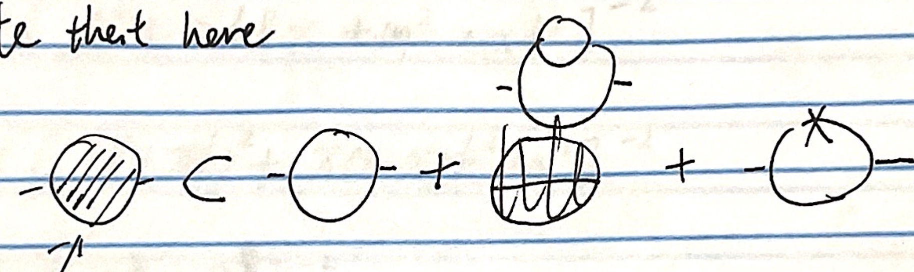
We know that the exact propagator has a pole at $k^2 = -m^2$ with residue of one, this means (only and only if)

$$\Pi(-m^2) = 0$$

$$\Pi'(-m^2) = 0 \quad \left(\left. \frac{d\Pi(k^2)}{dk^2} \right|_{k^2=-m^2} = 0 \right).$$

This is equivalent to canonical quantification $\langle k | k' \rangle = \ldots$

Note that here



one-particle irreducible

1PI diagrams + ... with all g^2 order corrections

(meaning all those still connected after cutting one line) (any cuts). [We ignore the two external propagators]

Now we can evaluate the g^2 correction

(let's do the integral) [This avoids double counting of the geometric series we did earlier].

To evaluate, we need to invoke

"Feynman's formula" that. (to combine denominators)

$$\frac{1}{A_1 A_2 \cdots A_n} = \int dF_n \, (x_1 A_1 + x_2 A_2 + \cdots + x_n A_n)^{-n}.$$

Here the integration is over Feynman parameter x_i

$$\int dF_n = (n-1)! \int_0^1 dx_1\, dx_2\, dx_3 \cdots dx_n\, \delta(x_1+x_2+x_3+\cdots+x_n-1).$$

Let's prove such an identity in HW.

Hence, knowing one more trick, we now can calculate.

$$\tilde{\Delta}((k+l)^2)\,\tilde{\Delta}(l^2) = \frac{1}{(l^2+m^2)((k+l)^2+m^2)}$$

$$= \int_0^1 dx \left[x((l+k)^2+m^2) + (1-x)(l^2+m^2)\right]^{-2}$$

$$= \int_0^1 dx \left[l^2 + 2klx + m^2 + xk^2\right]^{-2}$$

$$= \int_0^1 dx \left[(l+xk)^2 + x(1-x)k^2 + m^2\right]^{-2}$$

$$= \int_0^1 dx \left[q^2 + D\right]^{-2}$$

$q \equiv l + xk$, $D \equiv x(1-x)k^2 + m^2$.

The jacobian for changing variable from $d^dl \longrightarrow d^dq$ is trivially one.

Now as q^0 and q^i have different signs in the metric.

let's think about q^0. as ~~q^2~~ q appears in squares (why not? the L.I. quantity)

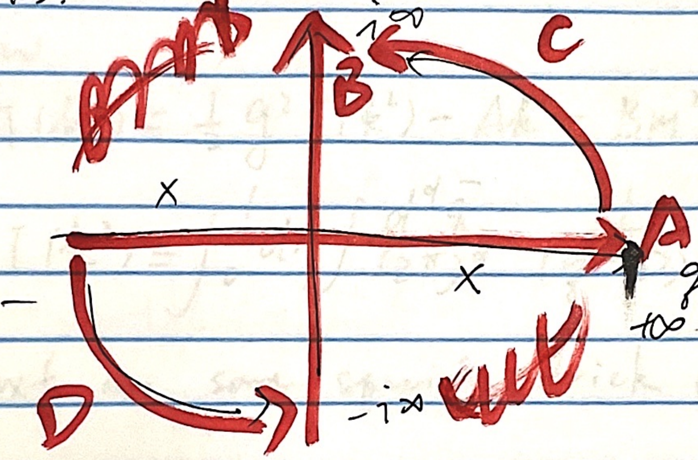
Let's perform the $\boxed{\text{Wick Rotation}}$

$$q^0 \longrightarrow iq^0.$$

If the integrand vanishes fast enough in $|q^0| \to \infty$

~~this~~ contour integral won't change value with such rotation

Here, $i\epsilon$ are super helpful in keeping track of the poles.



A+C−B−D = 0

C, D → 0

So A=B

With such rotation, we change the integration to Euclidean d-dimensional $\bar{q}$, with

$$q^0 = i\bar{q}_d \quad \text{and} \quad q_j = \bar{q}_j$$

$$q^2 = \bar{q}^2$$

with $\bar{q}^2 = \bar{q}_1^2 + \bar{q}_2^2 + \cdots + \bar{q}_d^2$

$$d^d q = i\, d^d \bar{q}$$

So $\int_{-\infty}^{\infty} dq^0 f(q^0)$

$= \int_{-\infty}^{+\infty} d\tilde{q}^0 f(i\tilde{q}^0)$

Wick Rotation avoid pole to make the calculation simple

Then

$$\int d^d q\, f(q^2 - i\epsilon) = i \int d^d \bar{q}\, f(\bar{q}^2)$$

as long as $f(\bar{q}^2) \to 0$ faster than $1/\bar{q}^d$ as $\bar{q} \to \infty$.

Now.

$$\Pi(k^2) = \frac{1}{2} g^2 I(k^2) - Ak^2 - Bm^2 + O(g^4).$$

$$I(k^2) \equiv \int_0^1 dx \int \frac{d^d \bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}+D)^2}.$$

Next are some specific trick ( ).

skip

$$\Pi''(k^2) = \frac{1}{2} g^2 I''(k^2) + O(g^4)$$

(this gets rid of A & B terms) (for our expression for φ^3 theory)

$$I''(k^2) = \int_0^1 dx\, 6x^2(1-x)^2 \int \frac{d^d \bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}+D)^4}.$$

We can perform this integral first and then $\int k^2$ twice (which allows us to match b.c.

$$\Pi(-m^2) = 0, \quad \Pi'(-m^2) = 0.)$$

Skip

This integral $\int d^d q \frac{1}{(q^2+D)^4}$

is convergent up to $d<8$

~~this~~ while the original $I(k^2)$ integration up to $d<4$

This new range contains $d=6$.

Let's view it again from Taylor expansion around $k^2=-m^2$ point of view.

$$\Pi(k^2) = \left[\tfrac{1}{2}g^2 I(-m^2) + (A-B)m^2\right] \quad \cdots \quad d\geq 4$$

$$+ \left[\tfrac{1}{2}g^2 I'(-m^2) - A\right](k^2+m^2) \quad \cdots \quad d\geq 6$$

$$+ \frac{1}{2!}\left[\tfrac{1}{2}g^2 I''(-m^2)\right](k^2+m^2)^2 + \cdots \quad \cdots \quad d\geq 8$$

$$+ O(g^4).$$

It's clear that $I^{(n)}(-m^2)$ is divergent for $d\geq 4+2n$.

~~so~~ but A can cancel the divergence at 2nd line from I'

and $A-B$ can cancel the divergence of 1st line from I.

no more free parameters and $\Pi(k^2)$ now is finite for

$d<8$.

For $d>8$, the procedure breaks down,

the theory is non-renormalizable.

(and we will see this through dim-analysis)

Now it is about how to handle (keep track of) infinities.

Method A. (mostly commonly used in earlier textbooks but also useful for clarifying many physics confusions) (we can have a brief discussion about it in the RGE part).

Pauli-Villars regularization.
(adding an effective cut-off scale to the theory)

$$\Delta(p^2) \longrightarrow \frac{1}{p^2+m^2-i\epsilon}\,\frac{\Lambda^2}{p^2+\Lambda^2-i\epsilon}$$

Λ is the ultra violate cutoff. this will make $d<8$ finite as it would be ~~$d^d q\, \Delta(q^2)\, \Delta($~~

$$d^d\bar{q}\,\frac{1}{(\bar{q}^2+D)^4}\,\cancel{(\bar{q}\ldots\Lambda^4}$$

Such a term also ad hoc violates gauge symmetry so one needs to be careful.

Still, one can calcurte and then take the $\Lambda\to\infty$ limit, that makes the $\propto\Lambda^2$, $\propto\Lambda^4$ parts cancel.

Λ tracks the divergencies.

Method B : Dimensional Regularization
(Dim - Reg).

Very popular & beautiful, ('t Hooft & Veltman).

One can directly evaluate from a dimension & analytically continue.

The angular part of $d\bar{q}$ over whole ~~~~ range yields the area (angular volume) of a unit sphere.

$$\Omega_d = \frac{\cancel{2d}\ 2\pi^{d/2}}{\Gamma(\frac{1}{2}d)}.$$

(one can verify $\int d^d\bar{q}\, e^{-\bar{q}^2}$) $\boxed{\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt.}$

$\Gamma(x)$ is the Euler Gamma function.

$$\Gamma(n+1) = n!$$

$$\Gamma(n+\tfrac{1}{2}) = \frac{(2n)!}{n!\,2^{(2n)}}\sqrt{\pi}$$

$$\Gamma(-n+x) = \frac{(-1)^n}{n!}\left[\frac{1}{x} - \gamma + \sum_{k=1}^{n} k^{-1} + O(x)\right]$$

$\gamma = 0.5772\ldots$

Euler Mascheroni Constant.

$$\gamma = \lim_{n\to\infty}\left(-\log n + \sum_{k=1}^{n}\frac{1}{k}\right)$$

$$= \int_1^\infty \left(-\frac{1}{x} + \frac{1}{\lfloor x\rfloor}\right)dx,$$ $\lfloor\cdot\rfloor$ is the floor function.

The radial part is

$$\int \frac{d^d \bar{q}}{(2\pi)^d} \frac{(\bar{q}^2)^a}{(\bar{q}^2 + D)^b} = \frac{\Gamma(b-a-\frac{1}{2}d)\,\Gamma(a+\frac{1}{2}d)}{(4\pi)^{d/2}\,\Gamma(b)\,\Gamma(\frac{1}{2}d)} D^{-(b-a-d/2)}$$

Very useful formula (Eq. 14.27).

For the theory we are evaluating, $a=0$, $b=2$

(General case) For general d, the coupling g has a mass dimension $\epsilon/2$

$$\epsilon = 6-d.$$

To account for this, we introduce a new parameter $\tilde{\mu}$.

$$g \to g\tilde{\mu}^{\epsilon/2}$$ not a physical parameter. ~~expt?~~

(hence g remains dimensionless for general theory)

$\tilde{\mu}^{\cancel{\epsilon/2}}$ is unphysical so no observable can depend on it

(later we revisit and derived RGE).

Setting $d=6-\epsilon$, we get

$$I(k^2) = \frac{\Gamma(-1+\epsilon/2)}{(4\pi)^3}\int_0^1 dx\, D\left(\frac{4\pi}{D}\right)^{\epsilon/2}.$$

[We'll see this convenience later]

With the substitution of $g \to g\tilde{\mu}^{\epsilon/2}$

and defining $\alpha \equiv \dfrac{g^2}{(4\pi)^3}$

In normal HEP convention, it would be $\alpha \equiv \frac{g^2}{4\pi}$, and order-by-order $\frac{\alpha}{(4\pi)^2}$

(not the fine-structure constant, which is $\frac{e^2}{4\pi}$)

$$\Pi(k^2) = \frac{1}{2}\alpha\,\Gamma\left(-1+\frac{\epsilon}{2}\right)\int_0^1 dx\, D\left(\frac{4\pi\tilde{\mu}^2}{D}\right)^{\epsilon/2}$$
$$-Ak^2 - Bm^2 + O(\alpha^2)$$

Dimless.

Taking the $\epsilon \to 0$ limit.

$$y^{\epsilon/2} = 1 + \frac{\epsilon}{2}\ln y + O(\epsilon^2)$$

and $\Gamma(-n+x)$

$$\Pi(k^2) = -\frac{1}{2}\alpha\left[\left(\frac{2}{\epsilon}+1\right)\left(\frac{1}{6}k^2+m^2\right)\right.$$
$$\left.+\int_0^1 dx\, D\ln\left(\frac{4\pi\tilde{\mu}^2}{e^{\gamma}D}\right)\right] - Ak^2 - Bm^2 + O(\alpha^2)$$

(noting that $D = x(1-x)k^2 + m^2$

$$\int_0^1 dx\, D = \frac{1}{6}k^2 + m^2).$$

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It is convenient to define

$$\mu \equiv \sqrt{4\pi}\, e^{-\gamma/2}\tilde{\mu}$$

so we have now

$$\Pi(k^2) = \frac{1}{2}\alpha\int_0^1 dx\, D \ln(D/m^2)$$

$$- \left\{\frac{1}{6}\alpha\left[\frac{1}{\epsilon} + \ln(\mu/m) + \frac{1}{2}\right] + A\right\}k^2$$

$$- \left\{\alpha\left[\frac{1}{\epsilon} + \ln(\mu/m) + \frac{1}{2}\right] + B\right\}m^2 + O(\alpha^2)$$

If we demand

$$A = -\frac{1}{6}\alpha\left[\frac{1}{\epsilon} + \ln(\mu/m) + \frac{1}{2} + \kappa_A\right] + O(\alpha^2)$$

$$B = -\alpha\left[\frac{1}{\epsilon} + \ln(\mu/m) + 1/2 + \kappa_B\right] + O(\alpha^2)$$

$$\Pi(k^2) = \frac{1}{2}\alpha\int_0^1 dx\, D \ln(D/m^2)$$

$$+ \alpha\left(\frac{1}{6}\kappa_A k^2 + \kappa_B m^2\right) + O(\alpha).$$

now $\Pi(k^2)$ is finite! and independent of μ.

To fix κ_A and κ_B, we must still impose the condition.

$\Pi(-m^2) = 0$. & $\Pi'(-m^2) = 0$.

Noting that

Eq. 14.41 $\Pi(k^2) = \frac{1}{2}\alpha\int_0^1 dx\, D\ln(D) + \text{linear in } k^2 \text{ and } m^2 + O(\alpha^2)$

we can impose

$$\Pi(k^2) = \frac{1}{2}\alpha\int_0^1 dx\, D\ln(D/D_0) + \text{linear in } (k^2+m^2) + O(\alpha^2)$$

here $D_0 \equiv D|_{k^2=-m^2} = [1-x(1-x)]m^2$

We can then differentiate Eq 14.41 w.r.t. k^2

and find ~~...~~ $\Pi'(-m^2)$ vanishes for

$$\Pi(k^2) = \frac{1}{2}\alpha\int_0^1 dx\, \underbrace{D\ln(D/D_0)}_{} - \frac{1}{12}\alpha(k^2+m^2) + O(\alpha^2)$$

this integral can be done "numerically"

and there is a "branch point" at $k^2 = -4m^2$.

The exact propagator can now be written as

$$\tilde{\Delta}(k^2) = \left(\frac{1}{1-\Pi(k^2)/(k^2+m^2)}\right)\frac{1}{k^2+m^2-i\epsilon}.$$

Let's see the logic here,
we are essentially try to find solutions that matches the boundary conditions.
(Q for yourself: is the assignment unique?)

In any case, we found

$$\Pi(k^2) = \frac{1}{2}\frac{g^2}{(4\pi)^3}\underbrace{\int_0^1 dx\, D \ln(D/D_0)}_{\downarrow} - \frac{1}{12}\frac{g^2}{(4\pi)^3}(k^2+m^2) + O(g^4)$$

this can be done in a closed form.

$$\alpha \equiv \frac{g^2}{(4\pi)^3}$$

$$\Pi(k^2) = \frac{1}{12}\alpha\left[c_1 k^2 + c_2 m^2 + 2k^2 f(r)\right] + O(\alpha^2)$$

$$f(r) = r^3 \tanh^{-1}(1/r), \quad c_1 = 3 - \pi\sqrt{3}$$

$$\underbrace{r = (1 + 4m^2/k^2)^{1/2}}_{\downarrow}, \quad c_2 = 3 - 2\pi\sqrt{3}.$$

there is a square root branch point at $k^2 < -4m^2$

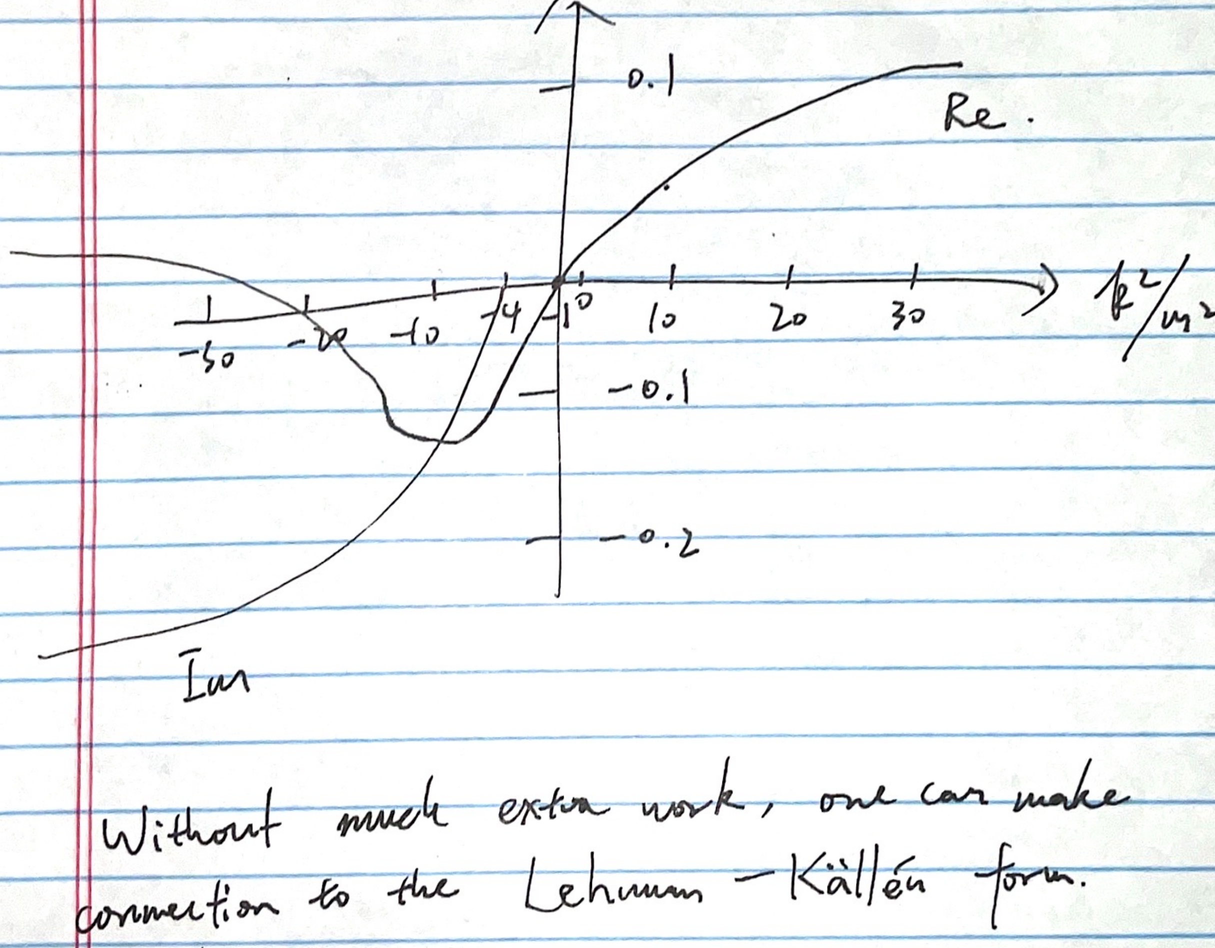
Recall that

$$\tilde{\mathbf{\Delta}}(k^2) = \tilde{\Delta}(k^2)\frac{1}{1 - \Pi(k^2)\tilde{\Delta}(k^2)}$$

For large values of $|k^2|$

$$\frac{\Pi(k^2)}{k^2+m^2} \simeq \frac{1}{12}\alpha\left[C_1 + \ln\frac{k^2}{m^2}\right]$$

k^2 being negative & $\ln(k^2 - i\epsilon) = \ln|k^2| - i\pi$



Without much extra work, one can make connection to the Lehmann - Källén form.

Sees $$\pi\rho(s) = \frac{\mathrm{Im}\,\Pi(-s)}{(-s+m^2-\mathrm{Re}\,\Pi((-s))^2 + (\mathrm{Im}\,\Pi(-s))^2}.$$

spectra density, which starts to grow and ~~negat~~ $k^2 \leq -4m^2$