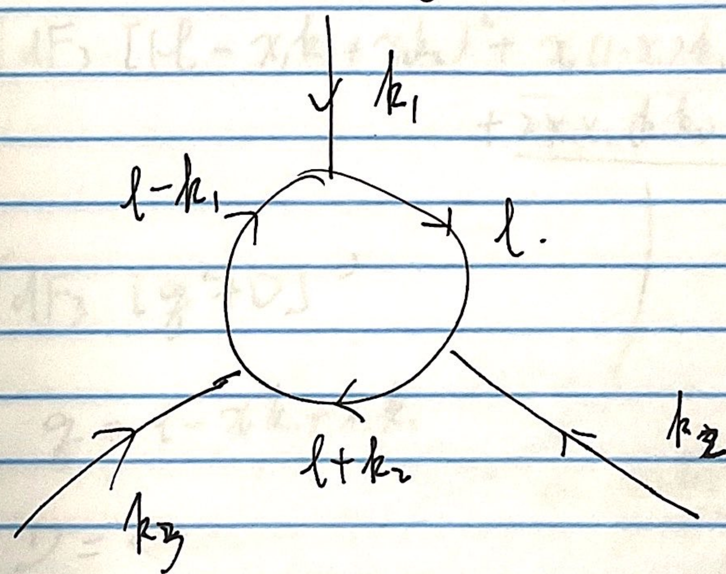
Other Loops & Renormalizability.

Z_φ, Z_m, Y, $\underset{\uparrow}{Z_g}$.

$$i V_3(k_1, k_2, k_3) = i Z_g g + (ig)^3 \left(\frac{1}{i}\right)^3 \int \frac{d^d l}{(2\pi)^d} \tilde{\Delta}((l-k_1)^2) \tilde{\Delta}((k_2+l)^2) \tilde{\Delta}(l^2) + O(g^5).$$



This time, I can't do geometric series for all ~~order~~.

Let's do it order by order.

$$\tilde{\Delta}((l-k_1)^2)\,\tilde{\Delta}((k_2+l)^2)\,\tilde{\Delta}(l^2)$$

$$= \int dF_3 \left[x_1 (l-k_1)^2 + x_2 (l+k_2)^2 + x_3 l^2 + m^2\right]^{-3}$$

$$\int dF_3 = 2\int_0^1 dx_1\, dx_2\, dx_3\, \delta(x_1+x_2+x_3-1)$$

$$= \int dF_3 \left[(l - x_1 k_1 + x_2 k_2)^2 + x_1(1-x_1)k_1^2 + x_2(1-x_2)k_2^2 + 2x_1x_2(k_1\cdot k_2) + m^2\right]^{-3}$$

$$= \int dF_3 \left[q^2 + D\right]^{-3}$$

$$q = l - x_1 k_1 + x_2 k_2$$

$D =$ ← using that $k_3 = -(k_1+k_2)$.

$$= x_3 x_1 k_1^2 + x_3 x_2 k_2^2 + x_1 x_2 k_3^2 + m^2.$$

Wick Rotate $q \to \bar{q}$.

$$V(k_1,k_2,k_3)/g = Z_g + g^2 \int dF_3 \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2+D)^3} + O(g^4)$$

It diverges for $d \geq 6$

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(q+D)^3} = \frac{\Gamma(3-\frac{1}{2}d)}{2(4\pi)^{d/2}} D^{-(3-d/2)}$$

for $\varepsilon > 0$, it converges.

$\rightarrow$ for $d = 6-\epsilon$, $g \to g\mu^{\epsilon/2}$

$$V_3(k_1, k_2, k_3)/g = Z_g + \frac{1}{2}\alpha\, \underbrace{\Gamma(\tfrac{\epsilon}{2})}_{(\frac{2}{\varepsilon}-\gamma+O(\varepsilon^2))} \int dF_3 \left(\frac{4\pi\tilde{\mu}^2}{D}\right)^{\epsilon/2} + O(\alpha^2)$$

$A^{\varepsilon/2} \simeq 1 + \frac{\varepsilon}{2}\ln A + O(\frac{\varepsilon}{2})^2$

Taking the $\epsilon \to 0$ limit.

$$\frac{2}{\varepsilon} + \ln\left(\frac{4\pi\tilde{\mu}^2}{D}\right) - \gamma + O(\varepsilon) \qquad 1 + \frac{\varepsilon}{2}\ln\left(\frac{4\pi\tilde{\mu}^2}{D}\right) + O\left(\left(\frac{\varepsilon}{2}\right)^2\right)$$

$$= Z_g + \frac{1}{2}\alpha\left[\frac{2}{\epsilon} + \int dF_3 \ln\left(\frac{4\pi\tilde{\mu}^2}{e^{\gamma} D}\right)\right] + O(\alpha^2)$$

Knowing $\int dF_n = 1$, letting $\mu^2 = 4\pi e^{-\gamma}\tilde{\mu}^2$,

setting $Z_g = 1 + C$, we get

$\ln\frac{\mu^2}{D} = 2\ln\frac{\mu}{m} - \ln\frac{D}{m^2}$

$\int dF_3 \ln\frac{\mu}{m} = \ln\frac{\mu}{m}$.

Consider to switch the notation m to be Λ or μ_0. μ was used before. Let's use μ_0. But for now use m for us to introduce IR divergence.

$$= 1 + \{\alpha[\tfrac{1}{\epsilon} + \ln(\mu/m)] + C\}$$

$$- \frac{1}{2}\alpha\int dF_3 \ln(D/m^2) + O(\alpha^2)$$

Let's choose $C = -\alpha[\frac{1}{\epsilon} + \ln(\mu/m) + K_c] + O(\alpha^2)$

Then

$$V_3(k_1, k_2, k_3)/g = 1 - \frac{1}{2}\alpha\int dF_3 \ln(D/m^2) - K_c\alpha + O(\alpha^2)$$

Sounds arbitrary, isn't it? ④

There is ~~not~~ no other B.C. to help me fix K_c.

so (now a choice $\boxed{\text{" scheme "}}$) can be made.

We'll return to this

for instance, we can set $K_c = 0 + O(\alpha^2)$, then

$$V_3(0,0,0)/g = 1 - \frac{1}{2}\alpha\int dF_3 \ln(m^2/m^2) + O(\alpha^2)$$
$$= 1.$$

This time, the $\int dF_3 \ln(D/m)$ cannot be done in a closed form, but it is clear

for $|k_i^2| \gg m^2$

$$V(k_1,k_2,k_3)/g \simeq 1 - \frac{1}{2}\alpha\underbrace{[\ln(k^2/m^2) + O(1)]}_{\text{vertex strength is changing with external momentum}} + O(\alpha^2)$$

Recall $D = x_1x_3k_1^2 + x_2x_3k_2^2 + x_1x_2k_3^2 + m^2$

Let $k_1^2 \gg k_2^2, k_3^2, m^2$

$$D \simeq x_1x_3k_1^2$$

$$\int dF_3 \ln(D/m^2) \simeq 2\int dx_1dx_2dx_3 \ln(x_1x_3k_1^2/m^2)\delta(1-x_1-x_2-x_3)$$
$$= 2\int_0^1 dx_1\int_0^{1-x_1} dx_3\, \cancel{x_1x_3}\ln(x_1x_3k_1^2/m^2)$$
$$\simeq \int dF_3 (\ln x_1 + \ln(x_3) + \ln(k_1^2/m^2))$$
$$\simeq \int dF_3 \ln(k_1^2/m^2)$$
$$= \ln(k_1^2/m^2).$$

Recall in the propagator case, we have

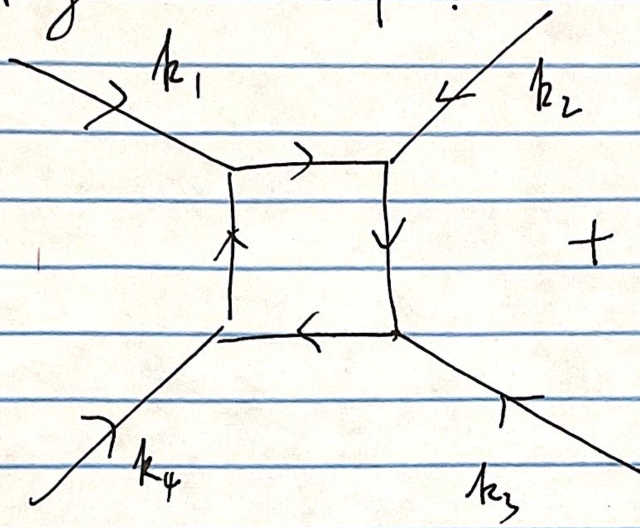
$$\Pi(k^2) = \frac{1}{12}\alpha[C_1k^2 + C_2m^2 + 2k^2 f(r)] + O(\alpha^2)$$

In the large $k^2 \gg m^2$ limit

$$\frac{\Pi(k^2)}{k^2+m^2} \simeq \frac{1}{12}\alpha[C_1 + \ln\frac{k^2}{m^2}]$$

A common feature is $[O(1) + \ln\frac{k^2}{m^2}]$ renormalization over the original (while Z's absorb the $\frac{1}{\varepsilon}$ parts)

Any more 1-loops?



$+ (k_3 \leftrightarrow k_2) + (k_3 \leftrightarrow k_4)$.

$$i V_4 = g^4 \int \frac{d^d l}{(2\pi)^d} \underbrace{\tilde{\Delta}\ \tilde{\Delta}\ \tilde{\Delta}\ \tilde{\Delta}}$$

$$+ (k_3 \leftrightarrow k_2) + (k_3 \leftrightarrow k_4).$$

$$\frac{1}{(q^2 + D_{1234})^4}.$$

$$V_4 = \frac{g^4}{6(4\pi)^3} \int dF_4 \left(\frac{1}{D_{1234}} + \frac{1}{D_{1324}} + \frac{1}{D_{1243}} \right)$$

$$+ O(g^6).$$

For $d=6$

$$\int \frac{d^6 l}{(2\pi)^6} \frac{1}{(q^2+D)^4}$$

$$= \frac{i}{6(4\pi)^3 D}.$$

Finite

We've done it!
(Or, have we done it?)
(See next)