Section 8

The Path Integral for free field theory.

Our result for the one-dimensional simple harmonic oscillator can be ~~transform~~ transported into a free field theory with hamiltonian density

$$\mathcal{H}_0 = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2$$

$q(t) \rightarrow \varphi(\vec{x},t)$ (classical field)

$Q(t) \rightarrow \varphi(\vec{x},t)$ (operator field)

their distinction resides in the context.

$f(t) \rightarrow J(\vec{x},t)$ (classical source)

using the $1-i\epsilon$ trick, we need to shift $m^2 \rightarrow (1-i\epsilon)m^2$

From now on every m^2 means we multiplied by $1-i\epsilon$ already

Now we can write down the path integral:
(also called the functional integral of our free field theory)

$$Z_0(J) \equiv \langle 0|0\rangle_J = \int \mathcal{D}\varphi \, e^{i\int d^4x\,[\mathcal{L}_0 + J\varphi]}$$

with

$$\mathcal{L}_0 = -\frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2$$

and $\mathcal{D}\varphi \propto \prod_x d\varphi(x)$. the $\boxed{\text{Functional Measure}}$

Now every "Path" in the path integral mean a path in the space of field configuration.

We can evaluate $Z_0(J)$ by mimicking SHO.

$$\tilde{\varphi}(k) = \int d^4x\, e^{-ikx} \varphi(x)$$

$$\varphi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \tilde{\varphi}(k)$$

Starting with $S_0 = \int d^4x\, [\mathcal{L}_0 + J\varphi]$, we got

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \left[-\tilde{\varphi}(k)(k^2+m^2)\tilde{\varphi}(-k) + \tilde{J}(k)\tilde{\varphi}(-k) + \tilde{J}(-k)\tilde{\varphi}(k)\right]$$

Changing the path integration variable

$$\tilde{\chi}(k) = \tilde{\varphi}(k) - \frac{\tilde{J}(k)}{k^2+m^2}$$

Since $\frac{-\tilde{J}(k)}{k^2+m^2}$ is a constant in the functional space

$$\mathcal{D}\varphi = \mathcal{D}\chi.$$

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \left[\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2+m^2} - \tilde{\chi}(k)(k^2+m^2)\tilde{\chi}(-k)\right].$$

Similarly, $Z_0(0) = \langle 0|0\rangle_{J=0} = 1.$

Hence

$$Z_0(J) = e^{\frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \frac{\tilde{J}(k)\tilde{J}(-k)}{k^2+m^2-i\epsilon}}$$

$$= e^{\frac{i}{2}\int d^4x\, d^4x'\, J(x)\Delta(x-x')J(x')}$$

Here we have the Feynman propagator

$$\Delta(x-x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-x')}}{k^2+m^2-i\epsilon}$$

which is ~~the~~ a Green's function for the Klein-Gordon equation

$$(-\partial_x^2+m^2)\Delta(x-x') = \delta^4(x-x')$$

Again performing a contour integral, we get.

$$\Delta(x-x') = i\int \widetilde{dk}\, e^{i\vec{k}(\vec{x}-\vec{x}')-i\omega|t-t'|}$$ ← similar to previous example, the contour choice changes with time order

$$= i\theta(t-t')\int \widetilde{dk}\, e^{ik(x-x')} + i\theta(t'-t)\int \widetilde{dk}\, e^{-ik(x-x')}$$

$\theta(t)$ is the unit step function.

Now we should have

$$\langle 0|T\varphi(x_1)\ldots|0\rangle = \frac{1}{i}\frac{\delta}{\delta J(x_1)} \cdots Z_0(J)\Big|_{J=0}$$

with explicit $Z_0(J)$ expression, we have

$$\langle 0|T\varphi(x_1)\varphi(x_2)|0\rangle = \frac{1}{i}\frac{\delta}{\delta J(x_1)}\frac{1}{i}\frac{\delta}{\delta J(x_2)} Z_0(J)\Big|_{J=0}$$

$$= \frac{1}{i}\frac{\delta}{\delta J(x_1)}\left[\int d^4x'\,\Delta(x_2-x')J(x')\right] Z_0(J)\Big|_{J=0}$$

$$= \left[\frac{1}{i}\Delta(x_2-x_1) + \int \cdots J(x') \cdots J(x'')\right] Z_0(J)\Big|_{J=0}$$

$$= \frac{1}{i}\Delta(x_2-x_1)$$

Consequently

$$\langle 0|T\,\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle$$

$$= \frac{1}{i^2}\{\Delta(x_1-x_2)\Delta(x_3-x_4)+\Delta(x_1-x_3)\Delta(x_2-x_4)$$

$$+\Delta(x_1-x_4)\Delta(x_2-x_3)].$$

and generally

$$\langle 0|T\varphi(x_1)\cdots\varphi(x_{2n})|0\rangle$$

$$= \frac{1}{i^n}\sum_{\substack{\{i\ \}\\ \text{pairings}}}\Delta(x_{i_1}-x_{i_2})\cdots\Delta(x_{i_{2n-1}}-x_{i_{2n}}).$$

Also known as Wick's theorem.