

9.18 THE $i\epsilon$ PRESCRIPTION AND LOGARITHMS IN LOOP INTEGRALS

INTRODUCTION: THE PROBLEM OF THE IMAGINARY PART

When calculating loop integrals in Quantum Field Theory, one frequently encounters logarithmic terms that depend on Mandelstam variables, such as $\log(-s/m^2)$ or $\log(t/m^2)$. A critical question arises when these kinematic invariants are in their physical regions (e.g., $s > 0$ and $t < 0$), as the argument of the logarithm becomes negative. This introduces an imaginary part of the form $\pm i\pi$. Note that such integrals come from the integrand of $D(x_1, x_2\dots; k_1, k_2\dots)$ where we can restore the $-i\epsilon$ easily since it always contains a $+m^2$ term (following our metric choice).²

The sign is not arbitrary; it is unambiguously fixed by the fundamental principles of causality. This physical requirement is mathematically encoded in the Feynman $i\epsilon$ prescription, see earlier notes [Section 9.12](#). This section explains how this one fundamental rule determines the correct imaginary parts for all resulting functions.

THE FUNDAMENTAL RULE: CAUSALITY AND THE FEYNMAN PROPAGATOR

The origin of all $i\epsilon$ rules in a scattering amplitude can be traced back to a single source: the Feynman propagator. For a scalar particle of mass m , the propagator in momentum space is:

$$\frac{i}{p^2 + m^2 - i\epsilon} \quad \text{where } \epsilon \rightarrow 0^+ \tag{9.18.1}$$

This $+i\epsilon$ (which can be thought of as giving the mass a small negative imaginary part, $m^2 \rightarrow m^2 - i\epsilon$) is the precise mathematical statement of causality. It ensures that positive-energy states propagate forward in time. Every other prescription for handling singularities in the final, integrated amplitude is a **consequence** of applying this fundamental rule to all propagators in the loop *before* integration.

²The m^2 in the denominator is nothing but a dimensionful quantity to make the argument dimensionless, so it doesn't come with a $-i\epsilon$.

1098 THE CONSEQUENCE: A "SIMPLE RULE" FOR 1099 KINEMATIC INVARIANTS

1100 When we perform the loop integration (e.g., using Feynman parameters),
1101 the $i\epsilon$ terms from all internal propagators combine. They work together to
1102 define the analytic properties of the final integral, which we can call
1103 $I(s, t, u, \dots)$. The result of this process is that the final amplitude I is an
1104 analytic function of its kinematic variables.

1105 This leads to a simple, practical shortcut for evaluating the final
1106 expressions, which we can call the "Simple Rule":

1107 To find the correct imaginary part of a final amplitude, replace
1108 all independent kinematic invariants x (like s , t , and u) with
1109 $x + i\epsilon$.

1110 This rule is the *result* of the fundamental $m^2 \rightarrow m^2 - i\epsilon$ prescription,
1111 combined with the knowledge that all products of x_i 's are positive, not a
1112 new or independent assumption.

1113 APPLYING THE SIMPLE RULE TO LOGARITHMS

1114 We use the standard "principal branch" convention for the complex
1115 logarithm, where the argument $\text{Arg}(z)$ is defined to be in the range $(-\pi, \pi]$.
1116 The imaginary part of the log is determined by the quadrant of its
1117 argument.

1118 CASE 1: $\log(t/m^2)$

1119 In this region, t is the (spacelike) momentum transfer, so $t < 0$. Let $t = -|t|$.

- 1120 • **Apply Rule:** We replace $t \rightarrow t + i\epsilon = -|t| + i\epsilon$.
- 1121 • **Evaluate Log:** $\log\left(\frac{t+i\epsilon}{m^2}\right) = \log\left(\frac{-|t|+i\epsilon}{m^2}\right)$.
- 1122 • **Analysis:** The argument $z = (-|t| + i\epsilon)/m^2$ has a negative real part
1123 and a small positive imaginary part. It is in the **second quadrant**.
- 1124 • **Result:** $\text{Arg}(z) = +\pi$. Therefore:

$$\log(t/m^2) \rightarrow \log\left(\frac{|t|}{m^2}\right) + i\pi$$

1124 CASE 2: $\log(-s/m^2)$

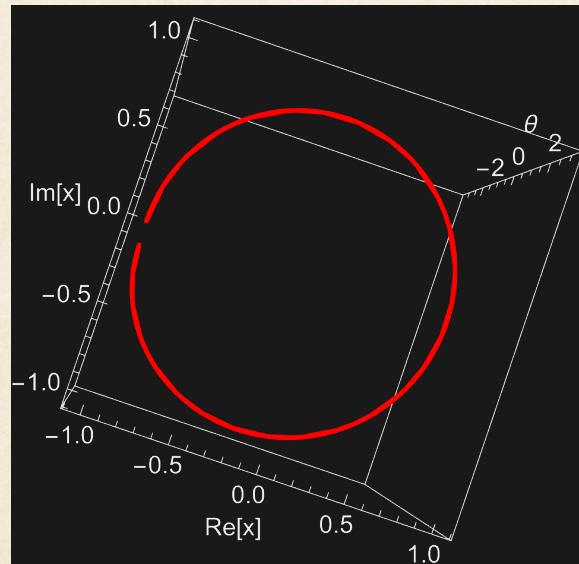
1125 In this region, s is the (timelike) center-of-mass energy squared, so $s > 0$.

- 1126 • **Apply Rule:** We replace $s \rightarrow s + i\epsilon$.
- 1127 • **Evaluate Log:** $\log\left(\frac{-(s+i\epsilon)}{m^2}\right) = \log\left(\frac{-s-i\epsilon}{m^2}\right)$.
- 1128 • **Analysis:** The argument $z = (-s - i\epsilon)/m^2$ has a negative real part
1129 and a small negative imaginary part. It is in the **third quadrant**.

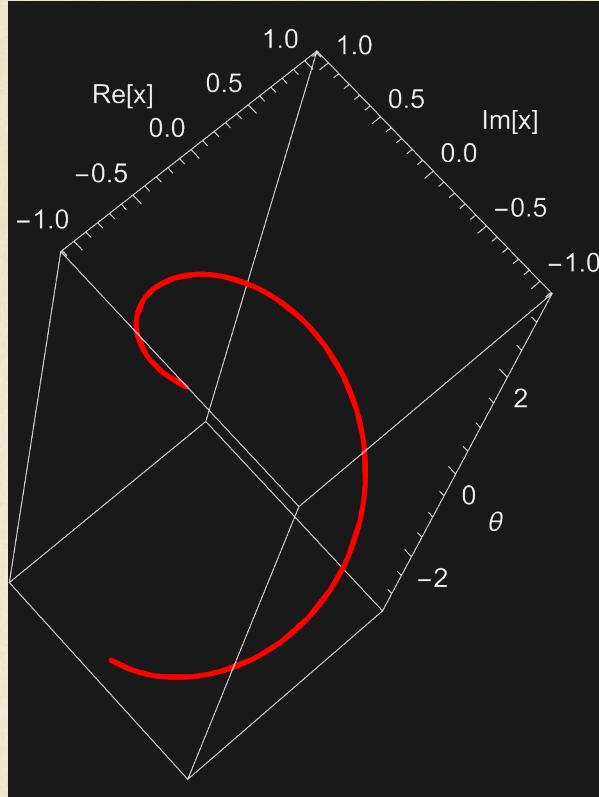
- **Result:** $\text{Arg}(z) = -\pi$. Therefore:

$$\log(-s/m^2) \rightarrow \log\left(\frac{s}{m^2}\right) - i\pi$$

¹¹³⁰ This imaginary part is physically crucial. It is related via the Optical
¹¹³¹ Theorem to the discontinuity of the amplitude, which corresponds to the
¹¹³² probability of the intermediate particles going on-shell.



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