Renormalizability.

Now we are armed with loop insights,

The superfacial divergencies of any loop diagrams

$$D = d \cdot L - 2I.$$

dim   Loops   internal propagators.

If $D \geq 0$, it is divergent.

But this doesn't tell ~~use~~ us much about the theory.

Now. $\cancel{i g_E Z_E}$. for n External legs.

skip this counting

Next page is better

$$[\text{diagram}] = [g_E]. = dL - 2I + \sum_{n=3}^{\infty} V_n [g_n].$$

$$D = [g_E] - \sum_{n=3}^{\infty} V_n [g_n].$$

For whatever $[g_E]$, there will always exist a large enough. V_n to make D semi positive

if $[g_n]$ is negative

So $[g_n]$ negative means the theory is non-renormalizable

A given # of external legs fixes the dimension as

$$iT = \text{leading-order} + \text{Next-leading order} + \text{higher } \overset{\text{order}}{\cancel{\text{od}}}$$

(typically)

$$iT = \underline{\text{tree}} + \underline{\text{one-loop}} + \underline{\text{two-loops}} + \cdots$$

units has to be the same.

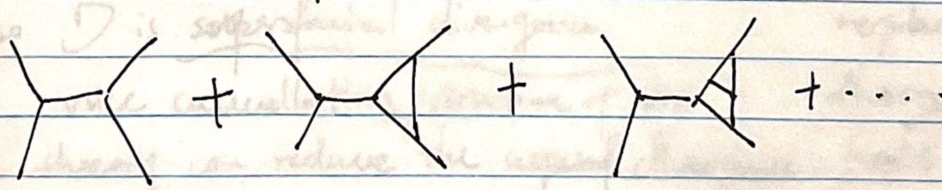
So for a fixed # of external legs

$$[\text{diagrams}] = \overset{\cancel{\text{tree}}\ \text{fixed by \# of}}{[\text{external legs}]} = dL - 2I + \sum_{\{n\}} V_n [g_n]$$

Note that n can be any consistent # of vertices.

For the example of three point interaction,

for the $2 \to 2$ process, we have



On the other hand $D = dL - 2I$.

so

$$D = [\text{external legs}] - \sum_{\{n\}} V_n [g_n]$$

where V_n can keep growing.

Clearly, if $[g_n]$ is negative, there will always exist a large enough V_n to make $D \geq 0$ and hence divergent.

So if any $[g] < 0$, the theory need to be renormalized infinite amount of times

Hence "Non-renormalizable".

↓

but this not that terrible thing ~~at~~

Effective theory

The theory can be replaced by another theory at some high scale.

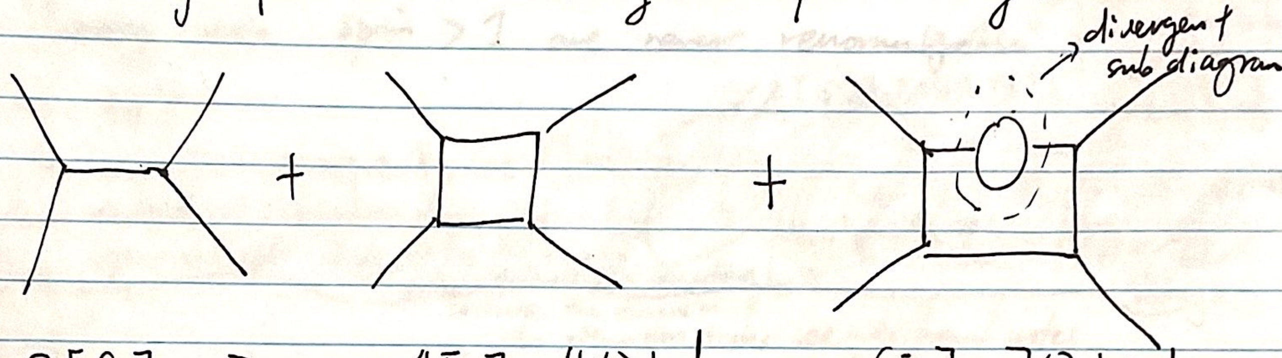
Also D is superficial divergence.

nice cancellations/structure of the theory can reduce the ~~superf~~ divergence.

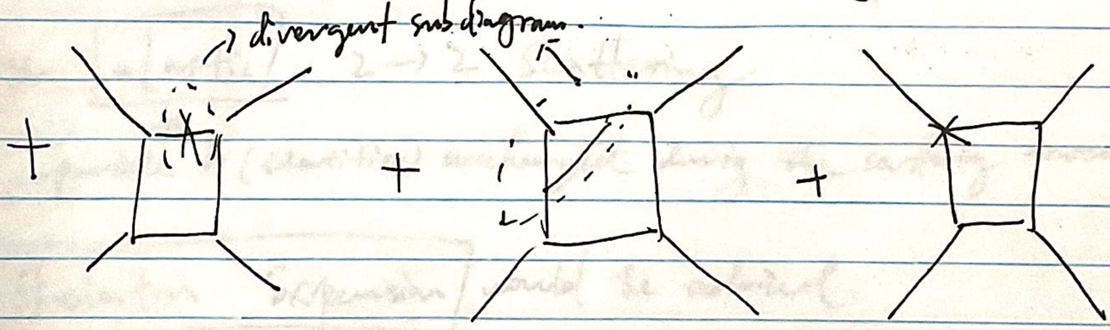
For $[g] \leq 0$ $\begin{cases} \text{super-renormalizable} \\ \text{renormalizable.} \end{cases}$

Let's "see" the meaning:

(e.g. for $2 \to 2$ scattering in φ^3 theory)



$2[g]-2$ $\quad\quad$ $4[g]-4\times 2+d.$ $\quad\quad$ $6[g]-7\times 2+2d$



but these divergences are exactly removed by

Z_φ, Z_m, Z_g

So we only need $\boxed{\text{finite } \#}$ of counter terms to renormalize the theory.

The theorem on renormalizability is true for spin-0 and spin-$\frac{1}{2}$, and for the spin-1 field with $\boxed{\text{gauge symmetry}}$

to be introduced last part of this class.

Theory with spin > 1 are never renormalizable in $d \geq 4$.

This why GR is not part of QFT

Also why hadron theories are to be replaced by

η is number of ~~[illegible]~~ hadrons

One can Dim analyze it, as $[\mathcal{L}] =$ Kinetic + interaction.

some details of interaction forms need.

e.g. spinor and vector indices makes the minimal interactions involve derivatives

increase the dimension of the interaction terms. see additional notes.

Now let's take a look at the ~~asymptotically~~.

For the $\boxed{\text{elastic}}$ 2 → 2 Scattering

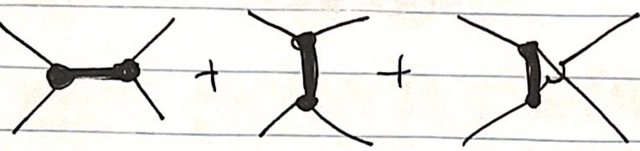
particle # (identities) unchanged during the scattering process.

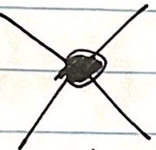
A $\boxed{\text{Skeleton Expansion}}$ would be advised.

↳ draw all 1PI diagrams that does not contain propagator correction or vertice 3-point correction (for φ^3 theory) for subdiagrams.

Then sum all tree-level diagram including the above (constructed) one for V_n.

Then we use the exact $\hat{\Delta}(k^2)$ and $V_3(k_1, k_2, k_3)$.
Such that we are consistent in ordering & counting.





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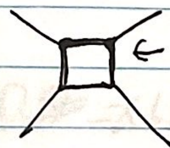


← not including corrections.

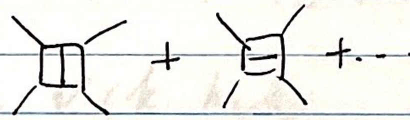
Counting.

tree (g^2)

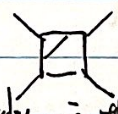
loop $\left(g^2 \cdot \frac{g^2}{(4\pi)^3}\right) + O(g^6)$
and it is finite.

← loop $\left(g^4 \cdot \frac{1}{(4\pi)^3}\right)$ finite.
$+ O(g^6)$

What about next order?
top-line automatic
second-line ~~into~~ includes



but not



(as it is already in the first line)

Let's see the amusing result.

$$iT_{tree\,(2\to2)} = \frac{1}{i}(ig)^2\left[\hat{\Delta}(-s) + \hat{\Delta}(-t) + \hat{\Delta}(-u)\right]$$

$$iT_{1\text{-loop}\,(2\to2)} = \frac{1}{i}\Big([iV_3(s)]^2\hat{\Delta}(-s) + [iV_3(t)]^2\hat{\Delta}(-t) + [iV_3(u)]^2\hat{\Delta}(-u)\Big) + iV_4(s,t,u)$$

You can observe the cyclic property of this amplitude here.

Here we have already calculated every piece to the given order:

$$\tilde{\Delta}(-s) = \frac{1}{-s^2+m^2-\Pi(-s)}$$

$$\Pi(s) = \frac{1}{2}\alpha\int_0^1 dx\, D_2(s)\ln[D_2(s)/D_0] - \frac{1}{12}\alpha(-s+m^2).$$

$$V_3(s)/g = 1 - \frac{1}{2}\int dF_3 \ln(D_3(s)/m^2)$$

$$V_4(s,t,u) = \frac{1}{6}g^2\alpha\int dF_4\left[\frac{1}{D_4(s,t)}+\frac{1}{D_4(t,u)}+\frac{1}{D_4(u,s)}\right]$$

it is cyclic, not exchange.

$$D_2(s) = -x(1-x)s+m^2, \quad D_0 = x(1-x)m^2 + m^2$$

$$D_3(s) = -x_1x_2s + [1-(x_1+x_2)x_3]m^2$$

$$D_4(s,t) = \underbrace{-x_1x_2s}_{\text{symmetric}} - x_3x_4t + [1-(x_1+x_2)(x_3+x_4)]m^2.$$

Here we get $V_3(s)$ from $V(k_1,k_2,k_3)$ by letting $k_1^2=-m^2$, $k_2^2=-m^2$, $k_3=(k_1+k_2)^2=-s$.

That's the result. One can further evaluate them using various special functions or numericals.

To gain a bit more intuition, we can do the high energy limit at a fixed scattering angle, such that

$$s \gg m^2, \quad |t| \geq m^2, \quad |u| \geq m^2,$$

$$(\text{recall } s+t+u = 4m^2)$$

$$t, u < 0.$$

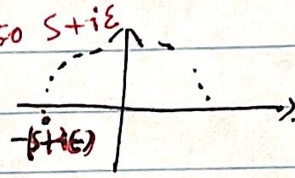
In such limit, we have.

$$\Pi(-s) = -\frac{1}{12}\alpha s\left[\ln(-s/m^2)+3-\pi\sqrt{3}\right].$$

$$\hat{\Delta}(-s) = \frac{1}{-s-\Pi(s)} = -\frac{1}{s}\left(1+\frac{1}{12}\alpha\left[\ln(-s/m^2)+3-\pi\sqrt{3}\right]\right)+O(\alpha^2)$$

↑ recall $+m^2-i\varepsilon$. so $s+i\varepsilon$

$$\ln(-s) = \ln s - i\pi$$



$$\ln(-t) = \ln|t| , \quad \ln(t) = \ln|t| + i\pi .$$

$$V_3(s)/g = 1-\frac{1}{2}\alpha\int dF_3\left[\ln(-s/m^2)+\ln(x_1x_2)\right]$$
$$= 1-\frac{1}{2}\alpha\left[\ln(-s/m^2)-3\right]$$

$$\int\frac{dF_4}{D_4(s,t)} = -\frac{3}{s+t}\left(\pi^2+[\ln(s/t)]^2\right) = \frac{3}{u}\left(\pi^2+\ln^2[s/t]\right)$$

We get that in the high energy limit that.

$$T_{1\text{-loop}} = g^2\left[F(s,t,u)+F(t,u,s)+F(u,s,t)\right]$$

↑ cyclic property.

$$F(s,t,u) = -\frac{1}{s}\Big(1-\underbrace{\frac{11}{12}\alpha\left[\ln(-s/m^2)+c\right]}-\frac{1}{2}\alpha\ln^2(s/u)\Big).$$

$6\pi^2+\pi\sqrt{3}-39 = \boxed{2.33}$

If α is large this correction term is bigger than the leading tree level result. (a warning sign of perturbative interpretation)

$-\frac{11}{12}\alpha\times 2.33 \simeq -2\alpha$

tree