Lec19

① ~~(##)~~

"wave-packet" $b^\dagger$ and $d^\dagger$

$$b_1^\dagger \equiv \int d^3\vec{p}\, f_1(\vec{p})\, b_{s_1}^\dagger(\vec{p})$$

$$|i\rangle = \lim_{t\to-\infty} b_1^\dagger(t)\, b_2^\dagger(t)|0\rangle$$

$$|f\rangle = \lim_{t\to+\infty} b_{1'}^\dagger(t)\, b_{2'}^\dagger(t)|0\rangle$$

$\langle i|i\rangle = 1, \quad \langle f|f\rangle = 1.$

for $\langle f|i\rangle$, we note (again, same as in scalar theory)

$$b_1^\dagger(-\infty) - b_1^\dagger(+\infty) = -\int_{-\infty}^{+\infty} dt\, \partial_0 b_1^\dagger(t)$$

$$= -\int d^3p\, f_1(\vec{p}) \int d^4x\, \partial_0\left(e^{-ipx}\, \bar{\Psi}(x)\gamma^0 u_{s_1}(\vec{p})\right)$$

$$= -\int d^3p\, f_1(\vec{p}) \int d^4x\, \bar{\Psi}(x)(\gamma^0\overleftarrow{\partial_0} - i\gamma^0 p^0)\, u_s(\vec{p})\, e^{-ipx}$$

$$\downarrow$$

$$\gamma^0\overleftarrow{\partial_0} - i\gamma^i p^i - im$$

$$\downarrow$$

$$\gamma^0\overleftarrow{\partial_0} - \cancel{i}\gamma^i \overrightarrow{\partial_i} - im$$

$$\downarrow \text{ (IBP).}$$

$$\gamma^0\overleftarrow{\partial_0} + \gamma^i\overleftarrow{\partial_i} - im$$

$$= i\int d^3p\, f_1(\vec{p}) \int d^4x\, \bar{\Psi}(x)\, (+i\overleftarrow{\cancel{\partial}} + m)\, \cancel{u_s(\vec{p})}\, u_s(\vec{p})\, e^{-ipx}$$

Recall in scalar case

$$a_1^\dagger(-\infty) - a_1^\dagger(+\infty) = -i\int d^3k\, f_1(\vec{k}) \int d^4x\, e^{ikx}(-\partial^2 + m^2)\varphi(x)$$

In free field theory $\bar{\Psi}(+i\overleftarrow{\partial\!\!\!/}+m)=0$
so the particle propagation is only changed when interaction is on.

Similarly
$$b_1(+\infty)-b_1(-\infty)=i\int d^3p\, f_1(\vec{p})\int d^4x\, e^{-ipx}\,\bar{u}_s(\vec{p})(-i\partial\!\!\!/+m)\Psi(x)$$

$$d_1^\dagger(-\infty)-d_1^\dagger(+\infty)=-i\int d^3p\, f_1(\vec{p})\int d^4x\, e^{ipx}\,\bar{v}_{s_1}(\vec{p})(i\partial\!\!\!/+m)\Psi(x)$$

$$d_1(+\infty)-d_1(-\infty)=-i\int d^3p\, f_1(\vec{p})\int d^4x\, \bar{\Psi}(x)(+i\overleftarrow{\partial\!\!\!/}+m)v_1(\vec{p})e^{-ipx}.$$

Now (for 2→2 of + type)
$$\langle f|i\rangle=\langle 0|b_2(+\infty)\,b_1(+\infty)\,b_1^\dagger(-\infty)\,b_2^\dagger(-\infty)|0\rangle$$
$$=\langle 0|\underset{\uparrow}{T}\,b_2(+\infty)\,b_1(+\infty)\,b_1^\dagger(-\infty)\,b_2^\dagger(-\infty)|0\rangle$$
here change order gives extra minus sign for fermions for odd # of changes.

$$=i^{-4}\int d^4x_1\, d^4x_2\, d^4x_1'\, d^4x_2'$$
$$e^{-ip_1'x_1'}\,[\bar{u}_{s_1'}(\vec{p}_1')(-i\partial\!\!\!/_{1'}+m)]_{\alpha_1'}$$
$$e^{-ip_2'x_2'}\,[\bar{u}_{s_2'}(\vec{p}_2')(-i\partial\!\!\!/_{2'}+m)]_{\alpha_2'}$$
$$\langle 0|T\,\Psi_{\alpha_2'}(x_2')\,\Psi_{\alpha_1'}(x_1')\,\bar{\Psi}_{\alpha_1}(x_1)\,\bar{\Psi}_{\alpha_2}(x_2)|0\rangle$$
$$[(+i\overleftarrow{\partial\!\!\!/}_1+m)u_{s_1}(\vec{p}_1)]_{\alpha_1}\,e^{ip_1x_1}$$
$$[(+i\overleftarrow{\partial\!\!\!/}_2+m)u_{s_2}(\vec{p}_2)]_{\alpha_2}\,e^{ip_2x_2}$$

Similar to scalar case, consistency requires that

$$\langle 0|0\rangle = 1$$

$$\langle 0|\Psi(x)|0\rangle = 0, \quad \langle 0|\bar{\Psi}(x)|0\rangle = 0$$

$$\langle p, s, +|\Psi(x)|0\rangle = 0$$

$$\langle p, s, -|\Psi(x)|0\rangle = \bar{v}_s(\vec{p})\, e^{-ipx}$$

$$\langle p, s, +|\bar{\Psi}(x)|0\rangle = \bar{u}_s(\vec{p})\, e^{-ipx}$$

$$\langle p, s, -|\bar{\Psi}(x)|0\rangle = 0$$

Recall $\Psi(x) = \ldots b_s(\vec{p}) \ldots d_s^{\dagger}(\vec{p}) \ldots$

One can further see that

$$S(x-y)_{\alpha\beta} \equiv i\langle 0|T\,\Psi_\alpha(x)\,\bar{\Psi}_\beta(y)|0\rangle$$

$$= \cancel{1} \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \frac{(-\not p + m)_{\alpha\beta}}{p^2+m^2-i\varepsilon}.$$

And $(-i\not\partial_x + m)_{\alpha\beta}\, S(x-y)_{\beta\gamma} =$

$$\int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \frac{(\not p + m)_{\alpha\beta}(-\not p + m)_{\beta\gamma}}{p^2+m^2-i\varepsilon}$$

$$= \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \frac{(p^2+m^2)\,\delta_{\alpha\gamma}}{p^2+m^2-i\varepsilon}$$

$$= \delta^4(x-y)\,\delta_{\alpha\gamma}.$$

$$S(x-y)_{\alpha\beta}\,(+i\overleftarrow{\not\partial}_y + m)_{\beta\gamma} = \delta^4(x-y)\,\delta_{\alpha\gamma}.$$

To demonstrate.

$$\langle 0|\Psi_\alpha \bar{\Psi}_\beta|0\rangle = \sum_{s,s'}\int \widetilde{dp}\,\widetilde{dp'}\, e^{ipx}e^{-ip'y}\, u_{s\alpha}(\vec{p})\,\bar{u}_{s'\beta}(\vec{p'})\,\underbrace{\langle 0|b_s(\vec{p})\,b^\dagger_{s'}(\vec{p'})|0\rangle}_{(2\pi)^3 2\omega\,\delta_{ss'}}$$

$$= \sum_{s,s'}\int \widetilde{dp}\,\cancel{\widetilde{dp'}}\, e^{-ip(x-y)}\, u_{s\alpha}(\vec{p})\,u^*_{s\beta}(\vec{p})$$

$$= \int \widetilde{dp}\, e^{ip(x-y)}\,(-\not{p}+m)_{\alpha\beta}$$

and also

$$\langle 0|\bar{\Psi}_\beta \Psi_\alpha|0\rangle = \quad \cdots \langle 0|d_{s'}(\vec{p'})\,d^\dagger_s(\vec{p})|0\rangle$$

$$= \int \widetilde{dp}\, e^{-ip(x-y)}\,(-\not{p}-m)_{\alpha\beta}$$

Now Finally we have everything:

Taking the example of $\underset{(b\text{-type})}{e^-}\varphi \to \underset{(b\text{-type})}{e^-}\varphi$.

$$\langle f|i\rangle = \langle 0|T\, a_{out}(\vec{k'})\, b_{s'\,out}(\vec{p'})\, b^\dagger_{s\,in}(\vec{p})\, a^\dagger_{in}(\vec{k})|0\rangle.$$

plug-in $b^\dagger_s(\vec{p})_{in} \to i\int d^4y\, \bar{\Psi}(y)\,(\cancel{+i\overleftarrow{\partial}} + i\overleftarrow{\not{\partial}} + m)\,u_s(\vec{p})\,e^{ipy}$

$b_{s'}(\vec{p'})_{out} \to i\int d^4x\, e^{-ip'x}\,\bar{u}_{s'}(\vec{p'})\,(-i\not{\partial}+m)\,\Psi(x)$

$$a^\dagger(\vec{k}) \to i\int d^4z_1\, e^{-ikz_1}(-\partial^2+m^2)\varphi(z_1)$$

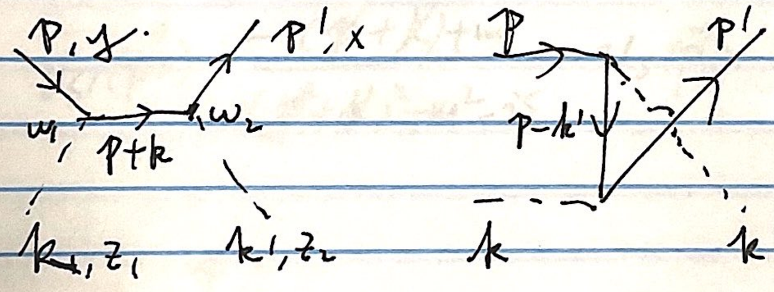
$$a(\vec{k}') \to i\int d^4z_2\, e^{-ik'z_2}(-\partial^2+m^2)\varphi(z_2)$$

We have

$$\langle f|i\rangle = (i)^4\int d^4x\, d^4y\, d^4z_1\, d^4z_2\, e^{i(py-p'x+kz_1-k'z_2)}$$

$$(-\partial_{z_1}^2+m_\varphi^2)(-\partial_{z_2}^2+m_\varphi^2)$$

$$\bar{u}_{s'}(\vec{p}')(-i\not\partial_x+m)\langle 0|T\varphi(z_2)\Psi(x)\bar{\Psi}(y)\varphi(z_1)|0\rangle\overleftarrow{(+i\not\partial_y+m)}\,u_s(\vec{p})$$



$\downarrow$

$$i\,i^2\left(\frac{1}{i}\right)^4(ig)^2\underbrace{\int d^4w_1\,d^4w_2}\; S(x-w_2)\,\Delta(z_2-w_2)\,S(w_2-w_1)\,\Delta(z_1-w_1)\,S(w_1-y)$$

$(-i\not\partial_x+m)$ $\to \delta(x-w_2)$ $\quad$ $(-\partial_{z_2}^2+m_\varphi^2)$ $\to \delta(z_2-w_2)$ $\quad$ $S(w_2-w_1)$ $\quad$ $\delta(z_1-w_1)$ $\quad$ $\delta(w_1-y)$

Then $\int d^4x\, d^4y\, d^4z_1\, d^4z_2\, e^{-i\cdots}$

$\longrightarrow$

Knowing $S(w_2-w_1) = \int \frac{d^4p''}{(2\pi)^4} e^{ip''(w_2-w_1)}$

$$\frac{(-\not p''+m)}{p''^2+m^2-i\varepsilon}$$

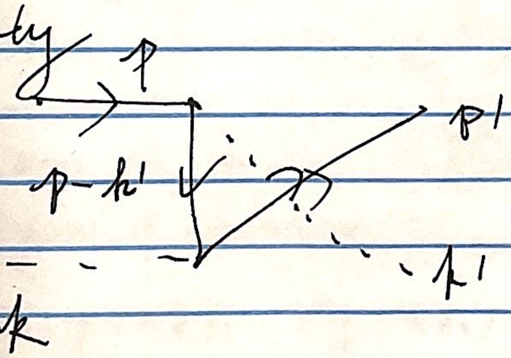
$$\to \langle f|i\rangle \ni \frac{1}{i}(ig)^2 \int d^4w_1\, d^4w_2\, \cancel{\bar{u}_{s'}(\vec{p}')} \int \frac{d^4p''}{(2\pi)^4} e^{ip''(w_2-w_1)}$$

$$e^{i(p+k)w_1-(p'+k')w_2}\ \bar{u}_{s'}(\vec{p}')\frac{-\not p''+m}{p''^2+m^2-i\varepsilon}u_s(\vec{p})$$

$$= \frac{1}{i}(ig)^2 (2\pi)^4 \delta^4(p+k-p'-k')$$

$$\bar{u}_{s'}(\vec{p}')\frac{-(\not p+\not k)+m}{(p+k)^2+m^2-i\varepsilon}u_s(\vec{p})$$

Similarly



has

$$\frac{1}{i}(ig)^2\ \bar{u}_{s'}(\vec{p}')\frac{-(\not p-\not k')+m}{(p-k)^2+m^2-i\varepsilon}u_s(\vec{p}).$$

We can see the "Rules" here.

But it is slightly more subtitle than that as.

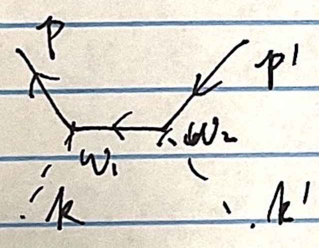
For $e^+\varphi \to e^+\varphi$. (there are 4-type)

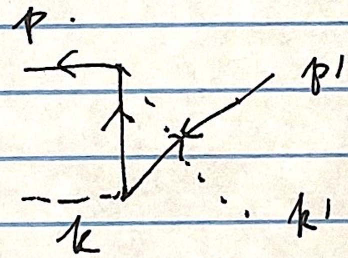
$$\langle f|i\rangle = \langle 0| T\, a(\vec{k}')\, d_{s'}(\vec{p}')\, d_s(\vec{p})\, a(\vec{k})|0\rangle.$$

$$d_s^\dagger(\vec{p})_{in} \to -i\int d^4x\, e^{+ipx}\, \bar{v}_s(\vec{p})(-i\not{\partial} + m)\Psi(x)$$

$$d_s(\vec{p}')_{out} \to -i\int d^4y\, \bar{\Psi}(y)(+i\overleftarrow{\not{\partial}} + m)\, v_{s'}(\vec{p}')\, e^{-ip'y}$$

$\vdots$





The "sign in p" is flipped and hence the "flow of momentum"

$$e^{i(px + kz_1 - p'y - k'z_2)}$$

$$\downarrow$$

$$e^{i(p+k)w_1 - i(p'+k')w_2}\, e^{ip''(w_2 - w_1)}$$