9.16 Lorentz Invariance of Cross Sections

The cross-section quantifies the likelihood of a scattering process (most commonly used with two incoming particles). A property of cross sections is their **Lorentz invariance**: they remain unchanged under Lorentz transformations (boosts, rotations). This short note explains why cross sections are Lorentz invariant, despite their dimensional interpretation as "areas," and clarifies common misconceptions.

The Paradox of Cross Sections as Invariant "Areas"

Cross sections (σ) have dimensions of area ($[\text{Length}]^2$), leading to the apparent contradiction: *"How can an area, which contracts under boosts, be Lorentz invariant?"*

The resolution lies in distinguishing between:

- **Geometric areas**: Physical spatial regions subject to Lorentz contraction.
- **Cross sections**: Derived quantities in QFT constructed from Lorentz-invariant ingredients.

Cross sections are not geometric objects but **Lorentz scalars** built from invariant components:

$$\sigma = \frac{|\mathcal{M}|^2 \cdot (\text{LIPS})}{\text{Flux Factor}}, \tag{9.16.1}$$

where:

- $|\mathcal{M}|^2$: Lorentz-invariant scattering amplitude.
- LIPS: Lorentz-invariant phase space.
- Flux Factor: Lorentz-invariant collision flux.

The flux factor is manifestly invariant, as it is written as a product of Lorentz invariant quantities. The "tricky" point here is that we motivated the flux factor from a particular frame and then "dressed it up" with normalization factors from $\langle i|i\rangle/V$ (which, for two particles, scaled as V).

Cancellation of Frame Dependencies

Cross sections are defined as:

$$\sigma = \frac{\text{Interaction Rate per Target}}{\text{Incident Rate per Target per Incoming Particle}}. \tag{9.16.2}$$

- **Misconception**: "Flux is invariant because relative velocity is invariant." **More properly**: Relative velocity v_{rel} is frame-dependent. The invariant flux is its relativistic *generalization*.

Experimental Consistency (Convenience)

Cross section invariance ensures experimental results are frame-independent:

- **Example**: The same scattering process measured in a collider (center-of-mass frame) and fixed-target lab frame yields identical σ.
- **Importance**: Lorentz invariance is *required* for consistent predictions across all inertial frames.