9.25 Dimensional Regularization on Different Divergencies

The Core Mechanism of Dimensional Regularization

In dimensional regularization (dim reg), we analytically continue the spacetime dimension from $d = 4$ to $d = 4 - \epsilon$. The fundamental quantity governing divergences is the Euler Gamma function, as seen in the master formula for Euclidean integrals:

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-d/2} \tag{9.25.1}$$

The Gamma function $\Gamma(z)$ has simple poles at non-positive integers $z = 0, -1, -2, \ldots$. Therefore, divergences (as poles in ϵ) appear when the argument $n - d/2$ approaches one of these integers.

The Gamma function poles are the *only* source of divergences in dim reg. If $\Gamma(n - d/2)$ is finite, the integral is finite in this scheme.

Odd versus Even Dimensions: A Topological Divide

The behavior of loop integrals differs profoundly between odd and even dimensions. The superficial degree of divergence (ω) for any 1-loop diagram ($L = 1$) with I internal propagators is:

$$\omega = dL - 2I = d - 2I$$

Let's analyze the parity (Even/Odd) of this equation:

- If d is Even: $\omega = (\text{Even}) - 2I = (\text{Even}) - (\text{Even}) = \text{Even}$. The result is even, so $\omega = 0$ (a logarithmic divergence) is topologically possible.
- If d is Odd: $\omega = (\text{Odd}) - 2I = (\text{Odd}) - (\text{Even}) = \text{Odd}$. The result is always odd.

It is topologically impossible for a 1-loop diagram in an odd dimension to be logarithmically divergent ($\omega = 0$). This directly explains the "special" nature of odd dimensions in dim reg:

- Even dimensions ($d = 2, 4, 6, \ldots$): $d/2$ is an integer. 1-loop diagrams can have $\omega = 0, 2, 4, \ldots$. The Gamma function argument $n - d/2$ can hit integer poles, producing $1/\epsilon$ divergences for *both* power-law and logarithmic cases.

- Odd dimensions ($d = 3, 5, 7, \ldots$): $d/2$ is a half-integer. 1-loop diagrams can only have $\omega = 1, 3, 5, \ldots$ (odd). The Gamma function argument $n - d/2$ is *always* a half-integer, which is *never* a pole. Thus, all one-loop diagrams in odd dimensions are finite in dim reg. **But we know divergence is a divergence; they do not disappear.** DimReg does something special.

Clarification: The above discussion only applies to 1-loop. At 2-loop level, DimReg will generate $1/\epsilon$ divergencies for both even and odd dimensions. Consider ϕ^6 theory in $d = 3$. At one-loop, diagrams are finite. However, at two loops, the degree of divergence becomes $\omega = 2d - 2I$, which can be zero. DimReg will see the $1/\epsilon$ pole in this case.

Power-Law versus Logarithmic Divergences: How Dim Reg "Evaluates" Them

Dim reg treats these two types of divergences very differently. It "evaluates" the power-law component (Λ^ω) of a divergence into a coefficient proportional to m^ω.
Let's see this in $d = 4 - \epsilon$:

- **Logarithmic ($\omega = 0$):** $I_{\log} = \int \frac{d^dk}{(k^2+D)^2} \propto \Gamma(2 - d/2) = \Gamma(\epsilon/2) \sim (\text{dimensionless}) \times \frac{2}{\epsilon}$. This is a "pure" $1/\epsilon$ pole.
- **Power-Law ($\omega = 2$):** $I_{quad} = \int \frac{d^dk}{k^2+D} \propto \Gamma(1 - d/2) = \Gamma(-1 + \epsilon/2) \sim \mathbf{m^2} \times \frac{1}{\epsilon}$. (The m^2 factor comes from the $(1/\Delta)^{n-d/2}$ term in Eq. 9.25.1, which becomes $D^{d/2-1} \approx D^1 = m^2$).

One might be tempted to claim: in even dimensions, dim reg "tags" the $1/\epsilon$ pole with the correct mass dimension ($m^2, m^4, \ldots$) of the power-law divergence it represents. In odd dimensions, this $1/\epsilon$ "tag" is absent for power-law divergences, which are simply evaluated to their finite m^ω values.

Dim Reg is subtle: **DimReg evaluates all powerlaw divergences to be finite!** For a scaleful integral (e.g., with m parameters in D), DimReg evaluates the powerlaw divergence level ω to be m^ω. For completely scaleless integrals like $\int d^dk\, k^2$ (no mass scale at all), dim reg **defines** them to be zero. This happens because introducing the renormalization scale μ in dim reg ($d^dk \to \mu^{4-d} d^dk$ to keep couplings dimensionless) cannot generate a mass scale from nothingthe integral remains scaleless and is set to zero. This automatic subtraction of pure power divergences is both a feature (computational simplicity) and a limitation (obscures physical thresholds).

The $\overline{\text{MS}}$ Scheme: What Are We Really Doing?

In the $\overline{\text{MS}}$ scheme, we:

1. Subtract **only** the $1/\epsilon$ poles (plus associated constants like $\gamma_E - \log(4\pi)$).
2. We do *not* "ignore" power-law divergences. Rather, we **implicitly accept DimReg's finite evaluation of them.**

This works because in a renormalizable QFT, all power-law divergences can be absorbed by the same counterterms that absorb the logarithmic ones. The $\overline{\text{MS}}$ scheme appears "minimal" only because the regularization scheme (DimReg) has already done the heavy lifting of taming $\Lambda^2, \Lambda^3, \ldots$ into finite, mass-dependent terms.

Contrast with Pauli-Villars regularization:

$$\Pi_{\text{PV}}(p^2) \sim c_1\Lambda^2 + c_2 m^2 \log\left(\frac{\Lambda^2}{\mu^2}\right) + \text{finite} \tag{9.25.2}$$

Here we explicitly see both quadratic (Λ^2) and logarithmic ($\log \Lambda^2$) divergences. A counterterm must be "brute-forced" to cancel the Λ^2 term explicitly. In $\overline{\text{MS}}$, this Λ^2 term has been "evaluated" by DimReg and is simply part of the finite result we keep.

Physical Implications and the Limits of Mathematical Elegance

While dim reg + $\overline{\text{MS}}$ is computationally optimal, it obscures important physical puzzles by "hiding" power-law divergences.

The Hierarchy Problem

In cutoff regularization, Higgs mass corrections are:

$$\delta m_H^2 \sim y_t^2 \Lambda^2 + y_t^2 m_t^2 \log\left(\frac{\Lambda^2}{\mu^2}\right) \tag{9.25.3}$$

The quadratic term Λ^2 represents a dramatic fine-tuning problem if $\Lambda \sim M_{\text{Pl}}$.
In dim reg:

$$\delta m_H^2 \sim y_t^2 m_t^2 \left(\frac{1}{\epsilon} + \text{finite}\right) \tag{9.25.4}$$

The Λ^2 divergence is gone, "evaluated" into the finite m_t^2 coefficient. The fine-tuning problem becomes mathematically invisible but physically remains.

The Cosmological Constant Problem

The vacuum energy scales as:

$$\rho_\Lambda \sim \int d^4k \sqrt{k^2} \sim \Lambda^4 \quad \text{(cutoff)} \tag{9.25.5}$$

In dim reg, scaleless integrals like $\int d^d k$ are defined to be zero. This regularization *deletes* the problem from the calculation, obscuring the enormous discrepancy between QFT's expected vacuum energy and the observed value.

The Decoupling Problem: MSbar's Blindness to Thresholds

A significant limitation of $\overline{\text{MS}}$ is its failure to automatically account for decouplingthe physical phenomenon where heavy particles (mass M) stop affecting low-energy physics ($E \ll M$).

In physical regularization schemes (like Pauli-Villars or momentum cutoff), when $M \to \infty$, power-law divergences $\sim M^2$ or $\sim M^4$ appear, clearly signaling the heavy particle's effect. In $\overline{\text{MS}}$, these are set to zero, making heavy and light particles appear equally "active" at all scales.

Example: In the Standard Model, the top quark ($m_t \approx 173$ GeV) should decouple from Higgs physics at scales $\mu \ll m_t$. In $\overline{\text{MS}}$, this doesn't happen automaticallyone must manually implement "decoupling relations" by matching theories above and below m_t.

The Lesson

Final echo: (As I emphasized in class) We must distinguish between mathematical convenience and physical insight. Different (reasonable) regularization schemes make different problems/properties of the underlying theory apparent. Dim reg's mathematical elegance comes at the cost of physical transparency for fine-tuning problems and decoupling phenomena. Dim reg gives the correct answers for "renormalizable" observable quantities (like scattering amplitudes) but may obscure the deeper physical questions about naturalness, UV sensitivity, and decoupling that are made manifest in cutoff-based schemes. The choice of regularization scheme is not just computationalit embodies a philosophy about what constitutes "real" physics. For a bottom-up effective field theorist, a renormalizable theory is self-consistent to arbitrary high scales, and won't see any problem with the underlying theory. For a top-down UV theorists who want to understand the ultimate Lagrangian of nature, consistency is only one aspect; calculability (and hence predictivity of the bare parameter) is crucial, as long as one does not want to admit that we live in a special and randomly tuned universe. Misinterpretations of MSbar scheme has caused tremendous amount of confusion in the field, even today. I hope you learned something from the above note.