Section 11

Cross sections & decay rates

In most cases, we only deal with two kinds of cases:

① one particle initial state (Decay) Decay rate.

② two particle initial state (Scattering) Cross Section.

Cross Section is in units of Length^2 (motivate).

Previously, we got that, the matrix element.

$$\mathcal{T} = g^2 \left[\frac{1}{(k_1+k_2)^2+m^2} + \frac{1}{(k_1-k_1')^2+m^2} + \frac{1}{(k_2-k_2')^2+m^2} \right] + O(g^4).$$

k_1 and k_2 are the four-momenta of the two incoming particles

k_1' and k_2' are the four-momenta of the outgoing particles

Energy Momentum is conserved

$$k_1 + k_2 = k_1' + k_2'$$

We also set all particles on-shell.

$$k_i^2 = -m_i^2$$

for general use, we allow for different masses.

We work out some kinematics.

Define the center of mass frame (CM). $\vec{k}_1 + \vec{k}_2 = 0 = \vec{k}_1' + \vec{k}_2'$

choose $\vec{k}_1$ in the positive z direction, only $|\vec{k}_1| = k_z$ is left unspecified.

The total energy is

$$E_1 + E_2 = \sqrt{|\vec{k}_1|^2 + m_1^2} + \sqrt{|-\vec{k}_1|^2 + m_2^2}$$

In fact, it is more convenient to define a L.I. scalar product.

$$s = -(k_1 + k_2)^2 \quad \text{which is } (E_1 + E_2)^2 \text{ in the CM frame.}$$

s is call the center of mass energy squared.

$$|\vec{k}_1| = \frac{1}{2\sqrt{s}} \sqrt{s^2 - 2(m_1^2 + m_2^2)s + (m_1^2 - m_2^2)^2} \quad (\text{CM frame})$$

not L.I.

Similarly, we also have in the final state

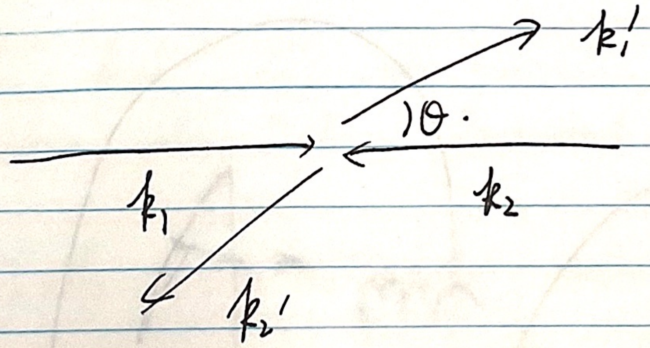
$$|\vec{k}_{1'}| = \frac{1}{2\sqrt{s}}\sqrt{s^2 - 2(m_{1'}^2 + m_{2'}^2)s + (m_{1'}^2 - m_{2'}^2)^2}$$

The total d.o.f is 6 for 2-unspecific on-shell final state particles,

$6 - \underbrace{4}_{\text{Energy-momentum conservation}} = 2$, which ~~are~~ can be identied as polar & Arthimutal angles

The arthimutal angle is typically irrelevant as the system would be rotationally (along +z axis) invariant.

So the remaining most physical d.o.f is the polar angle θ (in the CM frame).



beyond $s \equiv -(k_1 + k_2)^2$

there are two other. Lorentz Invariant Mandelstam variables

$$t = -(k_1 - k_1')^2 = -k_1^2 - k_{1'}^2 + 2k_1 \cdot k_2$$
$$= m_1^2 + m_{1'}^2 - 2E_1 E_{1'} + 2|\vec{k}_1||\vec{k}_{1'}|\cos\theta.$$

$\cancel{u = -(k_1 - k_2')^2}$

$$t = -(k_1 - k_1')^2 = -(k_2 - k_2')^2$$

and $u \equiv -(k_1 - k_{2'})^2 = -(k_2 - k_{1'})^2$

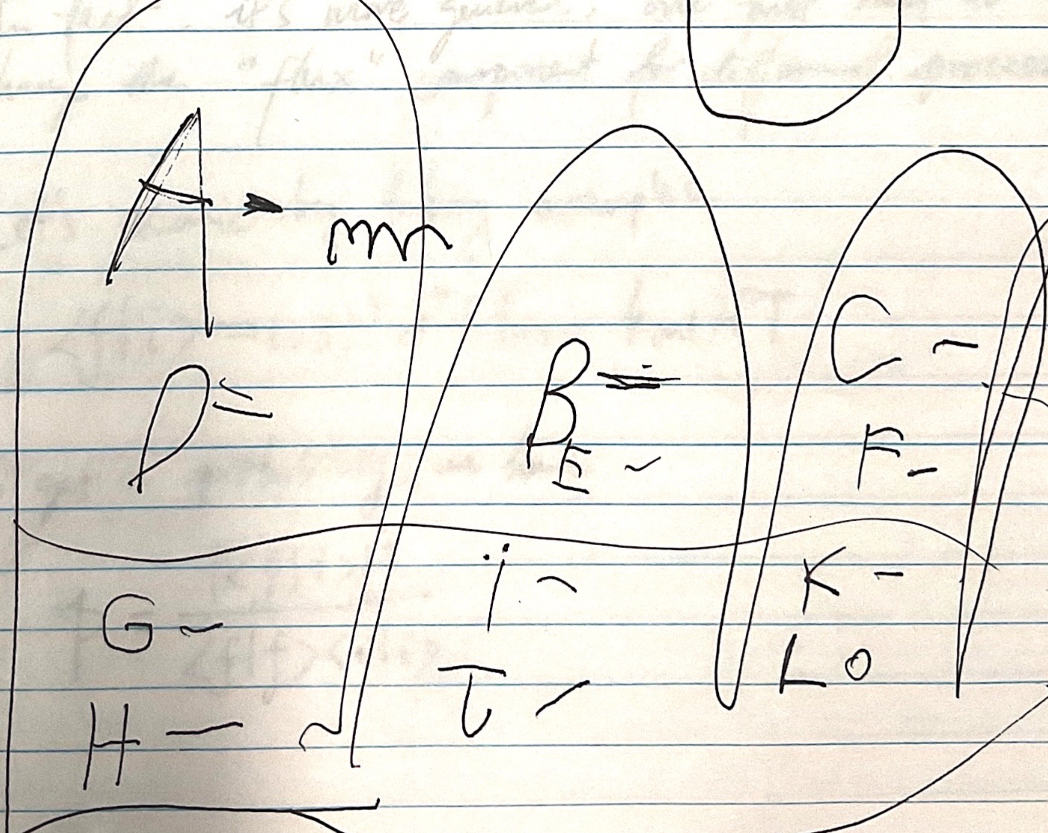
These three Mandelstam variables are not linearly independent, but they are quiet useful in many occassions.

They can make certain symmetry properties of the amplitudes manifest.

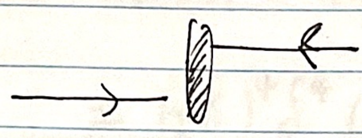
$$s + t + u = m_1^2 + m_2^2 + m_{1'}^2 + m_{2'}^2$$

so

$$\mathcal{T} = g^2\left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right] + O(g^4)$$



Now, introducing the concept of cross-section.



"geometric" size (If I throw two balls to each other, what's their chance to scatter?) (The ball spread $\propto$ De Broglie wavelength, 南)

normal to the incident direction

This is an observable by colliding two beams of particles and check out the scattering probability.

$$\boxed{d\sigma = \frac{1}{flux} |T|^2 \, dLIPS_{n'}}$$

→ for $2 \to n'$, $\frac{1}{\ }$

→ for $1 \to n'$. $\frac{1}{2E_1}$.

This is the relativistic version of the Fermi-Golden rule for scattering.

In fact, it's more general, one just need to change the "flux" component for different processes.

Let's examine by taking examples.

$$\langle f | i \rangle = (2\pi)^4 \delta^4 (k_{in} - k_{out}) i T$$

To get a probability, we have

$$P = \frac{|\langle f | i \rangle|^2}{\langle f | f \rangle \langle i | i \rangle}$$

$$\langle f|i\rangle^2 = \left((2\pi)^4 \delta^4 (\cancel{\delta}\, k_{in} - k_{out})\right)^2 |T|^2$$

$$= (2\pi)^4 \delta^4 (k_{in} - k_{out}) \cdot (2\pi)^4 \delta^4(0)\, |T|^2$$

$$(2\pi)^4 \delta^4(0) = \int d^4x\, e^{i0x} = VT$$

volume of time.

volume of space

one-particle canonical normalization

$$\langle k|k\rangle = (2\pi)^3\, 2k^0\, \delta^3(\vec{0}) = \cancel{(2\pi)^3}\, 2k^0 V$$

so for two particle initial state

$$\langle i|i\rangle = 4E_1E_2V.$$

$$\langle f|f\rangle = \prod_{j=1}^{n'} 2k_j^0 V$$

but $\langle f|i\rangle$ is the amplitude for a fixed final state momentum, we need to account for measurable quantities (small window of the continuous variables).

Within a finite sized box, a particle's momentum is quantized

$$\vec{k}_j' = \left(\frac{2\pi}{L}\right)\vec{n}_j \quad , \quad \vec{n}_j \text{ is an integer vector.}$$

in limit of large $L = (V)^{1/3}$

$$\sum_{n_j} \to \frac{V}{(2\pi)^3}\int d^3\vec{k}_j'$$

The quantized momentum goes to the continuum limit.

$$\int \frac{V}{(2\pi)^3} d^3\vec{k}_j = \sum_{n_j}\left(\frac{2\pi}{L}\right)^3 \vec{n}_j \Rightarrow \cancel{d^3\vec{k}_j}$$

So for each outgoing particle, to get a probability per unit time

$$\cancel{d(\text{P.S.})_n = \frac{V}{(2\pi)^3}}$$

$$\text{P.S.} \to \cancel{(2\pi)^3}\frac{V}{(2\pi)^3} d^3\vec{k}_j'$$

$$\dot{P} = \frac{(2\pi)^4 \delta^4(k_{in} - k_{out})}{4E_1E_2V} |T|^2 \prod_{j=1}^{n'} \widetilde{dk_j'}$$

$$\widetilde{dk_j'} = \frac{d^3k}{(2\pi)^3 2k^0}$$

note that the VT from $\langle f|i\rangle^2$ was cancelled by V^2 from $\langle i|i\rangle$, T from $\dot{P}$

and $V^{n'}$ is cancelled by $\langle f|f\rangle$'s $V^{n'}$

Typically and in fact the initial particles are also in packs (not infinitely precise)

They come in flux.

The incident flux is the # of particles striking the other per unit volume.

~~We have~~ If we have 1 particle in volum V with a speed $v = |\vec{k}_1|/E_1$ that is striking particle 2 at rest.

We will have $\cancel{\varnothing}\ |\vec{k}_1|/E_1/V$ particle 1 trying to strike particle 2 per unit time. per unit volume.

I can pack more particles in volume V so a more meaning full quantity is

quantifies how frequent two particles "meet"

Rate of meeting per unit time per unit volume

$[\dot{P}] = T^{-1}$

$[\text{flux}] = \cancel{L}T^{-1}V^{-1} = L^{-2}T^{-1}$.

$$\frac{\dot{P}}{\text{flux}}$$, which we call cross section.

$$d\sigma = \frac{\dot{P}}{\cancel{V}\ |\vec{k}_1|/E_1/V} = \frac{(2\pi)^4\,\delta^4(k_{in}-k_{out})}{4|\vec{k}_1|E_2}|T|^2\prod_{j=1}^{n'} d\tilde{k}_j'$$

note that the above discussion was held in 2's rest frame

$$|\vec{k}_1|\overset{FT}{\cancel{\ldots}}E_2 \underset{FT}{=} |\vec{k}_1|^{FT}\cdot M_2$$ it is in L.I. form

~~flux~~ $\cancel{\frac{1}{\vec{\tau}^2}}$

$$4|\vec{k}_1^{FT}|m_2 = 2\sqrt{s^2 - 2(m_1^2+m_2^2)s + (m_1-m_2)^2}$$

And ~~for~~ typical $s \gg m_1^2, m_2^2$

$$4|\vec{k}_1^{FT}|m_2 \simeq 2s.$$

by comparing $|\vec{k}_1^{FT}|$ and $|\vec{k}_1^{cm}|$

we can see

$$4|\vec{k}_1^{FT}|m_2 = 4|\vec{k}_1^{cm}|\sqrt{s}$$

one gets a hint of L.I. for the flux factor here.

So

$$d\sigma = \frac{1}{4|\vec{k}_1^{cm}|\sqrt{s}}|T|^2 dLIPS_{n'}(k_1+k_2)$$

here

$$dLIPS_{n'}(k_1+k_2) = (2\pi)^4\delta^4(k_1+k_2-k_{out})\prod_{j=1}^{n'} d\tilde{k}_j$$

Comments on the L.I. of σ.

~~So $2 \to n'$ pro~~

~~Now let's study a bit the behaviour of~~

~~$dLIPS_{n'}(k_1+k_2)$.~~

the most useful one is $n'=2$

In a similar fashion, we can see the $1 \to n'$ process would be.

$$\underbrace{\Gamma}_{\substack{\uparrow \\ \text{decay rate, decay width}}} = \frac{1}{2E_1} |T|^2 \, dLIPS_{n'}(k_1).$$

$d\sigma$ is L.I.

is Γ L.I.?

for proper decay width

$$\Gamma_{1\to n'} = \frac{1}{2M_1} |T|^2 dLIPS_{n'}$$

Now let's see a bit of useful properties for

$$dLIPS_{n'}(k_{in})$$

for $n'=2$

$$dLIPS_2 = (2\pi)^4 \delta^4(k_{in} - k_1' - k_2')\, \widehat{dk_1'}\, \widehat{dk_2'}$$

It is Lorentz invariant, so we can work in any frame we want, let's choose c.m. frame.

$$dLIPS_2 = \frac{(2\pi)^4 \delta(\sqrt{s} - E_1' - E_2')\, d^3k'}{4(2\pi)^6 E_1' E_2'}$$

$$E_1' = \sqrt{\vec{k}_1'^2 + m_1'^2}, \quad E_2' = \sqrt{\vec{k}_2'^2 + m_2'^2} = \sqrt{\vec{k}_1'^2 + m_2'^2}$$

$$d^3\vec{k}_1 = |\vec{k}_1^{cm}|^2 \, d|\vec{k}_1^{cm}| \, d\Omega_{cm}$$

$$= |\vec{k}_1^{cm}|^2 \, d|\vec{k}_1^{cm}| \, d\cos\theta_{cm} \, d\phi_{cm}.$$

Noting that $\int dx\, \delta(f(x)) = \sum_i |f'(x_i)|^{-1}$

$$\frac{\partial \delta(E_1' + E_2' - \sqrt{s})}{\partial |\vec{k}_1'^{cm}|} = \frac{|\vec{k}_1'|}{E_1} + \frac{|\vec{k}_1'^{cm}|}{E_2}$$

$$= |\vec{k}_1'^{cm}| \left(\frac{E_1' + E_2'}{E_1' E_2'} \right) = \frac{|\vec{k}_1'^{cm}| \sqrt{s}}{E_1 E_2}.$$

$$dLIPS_2(k) = \frac{|\vec{k}_1'^{cm}|}{16\pi^2 \cancel{E_1 E_2} \sqrt{s}}\, d\Omega_{cm}.$$

$$d\sigma = \frac{1}{64\pi^2 s} \cancel{4} \frac{|\vec{k}_1'^{cm}|}{|\vec{k}_1^{cm}|} |T|^2 d\Omega_{cm}.$$

another useful formula is

$$dt = 2 |\vec{k}_1^{cm}| |\vec{k}_1'^{cm}|\, d\cos\theta = 2 |\vec{k}_1| |\vec{k}_2| \frac{d\Omega_{cm}}{2\pi}$$

$$d\sigma = \frac{1}{64\pi s |\vec{k}_1^{cm}|^2} |T|^2 dt.$$ (This is L.I but $d\sigma/d\Omega$ is not since $d\Omega$ depends on frame)

L.I. with expressions earlier.

Finally, note that when population the final states, identical particles ~~wit~~ well "share" the phase space, so there ~~with~~ well be a separate "symmetry factor" for $dLIPS_n$, e.g. for two identical ~~sta~~ scalar, $\cancel{S = 1/2}$ $S = 2$. that one needs to divide

Added page 2025.

We've talked about the origin of unit Barn.

1 Barn = 100 fm^2

Proton size? $O(fm) \rightarrow p\text{-}p \ \sigma \sim O(10mb)$.

neutron $U^{235} \sim 1$ barn
for fast neutrons
$E > 100 keV$.

$\sqrt{s} \simeq 10 TeV$. $\left\{ \begin{array}{l} \text{inelastic} \sim 60mb \\ \text{single diffractive} \sim 12mb \\ \text{elastic} \sim 40mb \end{array} \right.$

The particle physics searches, e.g. Higgs discovery

$\sigma_{pp\to H\to\gamma\gamma} \simeq 50pb \times 10^{-3} \simeq 50 \cancel{f}b = 50\times 10^{-15}$ barn

compare with
100×10^{-3} barn

We discovery 10^{-12} rare rates
from the background.
(A needle in a haystack)

We actually measure stuff to 10^{-18} level.

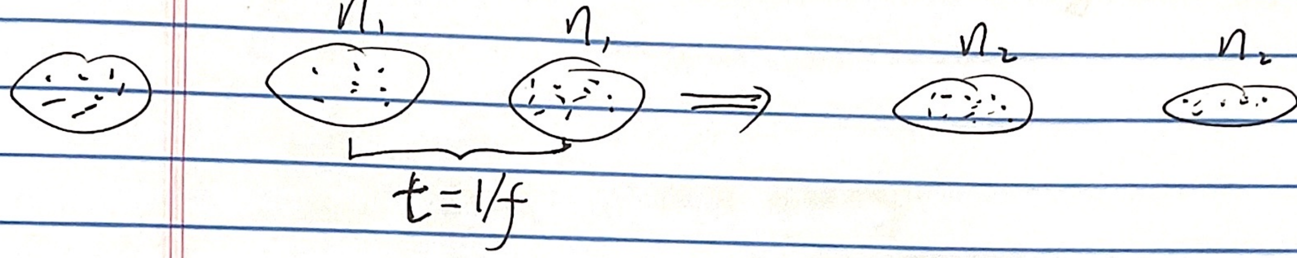
Cross section ~~is~~ 1 on 1 rate
characterises.

So what's rate?

A pairing concept is called "Luminosity"

Luminosity:

How intense the two spill of particles passing by each other.



$$\mathcal{L} = f\, n_1 n_2 / a$$

particles / cm² / sec.

For instance:

LHC:

$$n_1 = n_2 = 10^{11}$$

$$a \simeq (64\,\mu m)^2$$

$$f = 1/(25 \text{ nano sec})$$

$$\mathcal{L} = 40 \times 10^6 \times 10^{22} \times \frac{1}{4000 \times (10^{-4})^2\ cm^2}$$

$$= 10^{34} / cm^2/sec.$$

$$= 10\ nb^{-1}\ sec^{-1}$$

rate $= \sigma \cdot \mathcal{L}$: p-p $\quad 100\,mb \times 10\,nb^{-1}\,sec^{-1}$

$$= 100 \times 10^6\ nb \times 10\ nb^{-1}\ sec^{-1}$$

$$= 10^9 / sec.$$

(We actually record about 10^3 event/sec. that already provides 10 PB data /yr)

"Big data"

One can see that

Cross Section & Luminosity is a "Pair"

Cross Section ↓ 1 on 1 ↓ Hence one can compare different experiments directly.

Luminosity ↓ How I prepare my experiment.

Phase space:

$$d\Phi_n(P; p_1 \ldots p_n) = (2\pi)^4 \delta^{(4)}\left(P - \sum_{i=1}^{n} p_i\right) \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 \, 2E_i}$$

and a recursive relation

$$d\Phi_n(P; p_1, \ldots, p_n) = d\Phi_j(q; p_1 \cdots p_j)$$
$$\times \, d\Phi_{n-j+1}(P; q, p_{j+1} \ldots p_n) \, (2\pi)^{3} \, dq^2$$

~~And "An in the~~ What's the phase space volumn of n identical particles (that are massless)

$$\int dLISP_n = \frac{\hat{S}^{n-2} \, \cancel{(2\pi)^4} \, (2\pi)^{-4}}{8\pi (16\pi^2)^{n-2} \, n! \, (n-1)!}$$

The red one is our textbook's dLISP convention.
The black one is PDG review convention.