Summary Note: Lorentz Group Algebras and Representations

This note clarifies the key relationships between $SO(1,3)$, $SL(2,\mathbb{C})$, $SO(4)$, and $SU(2)$, and explains the strategy of complexification.

1. The Core Analogy: Rotations ($SO(3)$ and $SU(2)$)

- **Algebras (Local):** The Lie algebras are **isomorphic**: $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. They are the same 3-generator structure (the J_i's).
- **Groups (Global):** The groups are **not** isomorphic. $SU(2)$ is the **universal double-cover** of $SO(3)$. This is a 2-to-1 mapping (U and $-U$ in $SU(2)$ map to a single R in $SO(3)$).

2. The Lorentz Case ($SO(1,3)$ and $SL(2,\mathbb{C})$)

This follows the same pattern, but with 6 generators (3 rotations J_i, 3 boosts K_i).

- **Algebras (Local):** The 6-dimensional **real** Lie algebras are **isomorphic**: $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$.
- **Groups (Global):** $SL(2,\mathbb{C})$ is the **universal double-cover** of $SO^+(1,3)$. This is also a 2-to-1 mapping (M and $-M$ in $SL(2,\mathbb{C})$ map to a single Λ in $SO^+(1,3)$).

3. The Most Common Error: $\mathfrak{so}(1,3) \ncong \mathfrak{su}(2) \times \mathfrak{su}(2)$

- The algebra $\mathfrak{su}(2) \times \mathfrak{su}(2)$ is **isomorphic to** $\mathfrak{so}(4)$ (4D Euclidean rotations), not the Lorentz algebra.
- The difference is the crucial **minus sign** in the boost commutator.

$$
\begin{aligned}
\mathfrak{so}(1,3)\ \text{(Lorentz)}: &\quad [K_i, K_j] = -i\epsilon_{ijk} J_k \\
\mathfrak{so}(4)\ \text{(Euclidean)}: &\quad [K_i, K_j] = +i\epsilon_{ijk} J_k
\end{aligned}
$$

- Because this structure is different, the **real** algebra $\mathfrak{so}(1,3)$ **does not decompose** into a simple product.

4. The "Complexification Trick" – Why We Use It

This is the central strategy for classifying representations.

- **Problem:** Classifying the representations of the "messy," non-decomposed *real* algebra $\mathfrak{so}(1,3)$ is hard.

- **Key Theorem:** The set of (finite-dimensional) representations of a real algebra $\mathfrak{g}$ is in a **1-to-1 correspondence** with the representations of its complexification, $\mathfrak{g}_{\mathbb{C}}$.
- **The "Easy" Algebra:** We work with the complexification, $\mathfrak{so}(1,3)_{\mathbb{C}}$. This is a **6-dimensional *complex*** algebra (or 12D real). Its generators are the same $\{J_i, K_i\}$, but they can be multiplied by complex numbers.
- **The Payoff:** This *complex* algebra **does** decompose!

$$\mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2)_{\mathbb{C}} \times \mathfrak{su}(2)_{\mathbb{C}}$$

  (This works because $\dim_{\mathbb{C}}(\mathfrak{so}(1,3)_{\mathbb{C}}) = 6$, and $\dim_{\mathbb{C}}(\mathfrak{su}(2)_{\mathbb{C}} \times \mathfrak{su}(2)_{\mathbb{C}}) = 3 + 3 = 6$.)
- **Solution:** We classify the irreps of the "easy" product $\mathfrak{su}(2)_{\mathbb{C}} \times \mathfrak{su}(2)_{\mathbb{C}}$, which are just pairs of $SU(2)$ irreps, (j_A, j_B). By the theorem, this gives us the complete "menu" of representations for our "hard" physical algebra, $\mathfrak{so}(1,3)$.

5. Key Definitions

- **$\cong$: Isomorphic.** The two structures are mathematically identical (e.g., they have the same commutation relations).
- **Universal Double Cover** (e.g., $SL(2,\mathbb{C}) \to SO^+(1,3)$):
  - **Cover:** A projection map from a "parent" group to a "base" group.
  - **Double:** The map is exactly 2-to-1.
  - **Universal:** The "parent" group (e.g., $SL(2,\mathbb{C})$) is **simply connected** (it has no "topological holes"). It is the "top-level" parent cover.