9.22 Further Discussion on the Spinor Representation

This discussion unifies the proper orthochronous Lorentz group $SO^+(1,3)$, its Lie algebra $\mathfrak{so}(1,3)$, its universal cover $SL(2,\mathbb{C})$, and the classification of its fundamental representations.

The Physical Algebra $\mathfrak{so}(1,3)$

The proper orthochronous Lorentz group, $SO^+(1,3)$, is the group of 4x4 real matrices Λ that preserve the Minkowski metric $g = \text{diag}(-1,+1,+1,+1)$ and satisfy $\det(\Lambda) = 1$ and $\Lambda^0_0 \geq 1$.
Its Lie algebra, $\mathfrak{so}(1,3)$, is a 6-dimensional real algebra spanned by the generators of rotations, $\mathbf{J} = (J_1, J_2, J_3)$, and boosts, $\mathbf{K} = (K_1, K_2, K_3)$. These obey the commutation relations:

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k$$

$$[K_i, K_j] = -i\epsilon_{ijk}J_k$$

The minus sign in the $[K_i, K_j]$ commutator is the defining feature of this Minkowskian algebra. It is the mathematical expression of the fact that two successive boosts do not equal a single boost, but also involve a rotation (Thomas precession). Crucially, because $[J_i, K_j] \neq 0$, the algebra is not a direct sum (product) and does not trivially decompose. Boost generators K_i are anti-Hermitian (for finite dimensional represenation of this non-compact group).

The Non-Relativistic Analogy: $SU(2)$ and $SO(3)$

Before solving the full algebra, we examine its rotation subgroup. The generators J form a closed subalgebra, $\mathfrak{su}(2) \cong \mathfrak{so}(3)$.

- The group of 3D rotations is $SO(3)$, which acts on 3-component vectors.
- The group $SU(2)$ is the group of 2x2 complex unitary matrices with determinant 1, which acts on 2-component spinors.

$SU(2)$ is the **universal double-cover** of $SO(3)$. This means there is a 2-to-1 homomorphism from $SU(2)$ to $SO(3)$. For any rotation $R \in SO(3)$, there are two matrices, U and $-U$, in $SU(2)$ that map to it.

This is evident in the transformation of a vector x mapped to a matrix $X = \mathrm{x} \cdot \boldsymbol{\sigma}$ (where $\boldsymbol{\sigma}$ are the Pauli matrices):

$$X \to X' = UXU^\dagger$$

Replacing U with $-U$ yields the same X':

$$X' = (-U)X(-U)^\dagger = (-1)(-1)UXU^\dagger = UXU^\dagger$$

Physically, a 2π rotation returns a (3-)vector to its original state, but a spinor by an additional phase of -1. A 4π rotation is required to return a spinor to its original state. This $SU(2) \to SO(3)$ relationship is the template for the full Lorentz group.

The Universal Cover $SL(2, \mathbb{C})$

The group $SL(2, \mathbb{C})$ is the group of 2x2 complex matrices M with $\det(M) = 1$. This group is the **universal double-cover** of the Lorentz group, $SO^+(1, 3)$. This is the direct relativistic analogue of $SU(2)$ covering $SO(3)$. For every Lorentz transformation $\Lambda \in SO^+(1, 3)$, there are two matrices, M and $-M$, in $SL(2, \mathbb{C})$ that map to it. The fundamental representations of $SL(2, \mathbb{C})$ are the 2-component Weyl spinors.

Complexification and the $N, N^\dagger$ Algebras

To simplify the $\mathfrak{so}(1, 3)$ algebra, we move to its complexification, $\mathfrak{so}(1, 3)_{\mathbb{C}}$, by allowing complex coefficients. We define two new sets of generators. Following the textbook notation, we call them N and N†[3]:

$$N_i = \frac{1}{2}(J_i + iK_i)$$

$$B_i \equiv (N_i^\dagger)_{\text{symbolic}} = \frac{1}{2}(J_i - iK_i)$$

By computing the commutators, we find the algebra completely decouples into two independent $SU(2)$-like algebras:

$$[N_i, N_j] = i\epsilon_{ijk}N_k$$

$$[B_i, B_j] = i\epsilon_{ijk}B_k \quad (\text{or } [N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk}N_k^\dagger)$$

$$[N_i, B_j] = 0 \quad (\text{or } [N_i, N_j^\dagger] = 0)$$

The complexified Lorentz algebra is therefore isomorphic to two copies of $\mathfrak{sl}(2, \mathbb{C})$, which is the complexification of $\mathfrak{su}(2)$.

$$\mathfrak{so}(1, 3)_{\mathbb{C}} \cong \mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{su}(2)_{\mathbb{C}} \times \mathfrak{su}(2)_{\mathbb{C}}$$

[3] Note that $N^\dagger$ here is merely symbolic. A critical point of confusion is the $N_i^\dagger$ notation for the B_i generators. This $\dagger$ is **symbolic** and does *not* represent Hermitian conjugation. The $N_i^\dagger$ notation is a mnemonic used to recall the relationship between the *finite group elements* for the left- and right-handed representations (L and R), which are related by $R = (L^\dagger)^{-1}$.

Classifying the Representations

We can now classify all irreducible representations by a pair of labels $(2n+1, 2n'+1)$, where $2n+1$ is the dimension of the representation under the corresponding algebra (N or $B \equiv N^\dagger_{\text{symbolic}}$).

(2, 1) Representation: The Left-Handed Spinor

- **Generators:** This representation is built from $J_i = \frac{\sigma_i}{2}$ and $K_i = -\frac{i\sigma_i}{2}$.
- **Check:** $N_i = \frac{1}{2}(J_i + iK_i) = \frac{\sigma_i}{2}$ (non-trivial, dim 2) and $B_i = \frac{1}{2}(J_i - iK_i) = 0$ (trivial, dim 1).
- **Transformation:** A left-handed spinor ψ_L transforms as: $\psi_L \to L\psi_L$.

(1, 2) Representation: The Right-Handed Spinor

- **Generators:** This representation is built from $J_i = \frac{\sigma_i}{2}$ and $K_i = +\frac{i\sigma_i}{2}$.
- **Check:** $N_i = \frac{1}{2}(J_i + iK_i) = 0$ (trivial, dim 1) and $B_i = \frac{1}{2}(J_i - iK_i) = \frac{\sigma_i}{2}$ (non-trivial, dim 2).
- **Transformation:** A right-handed spinor ψ_R transforms as: $\psi_R \to R\psi_R = (L^\dagger)^{-1}\psi_R$.

(2, 2) Representation: The 4-Vector

- **Interpretation:** A 4-vector x^μ is an object that transforms non-trivially under both algebras.
- **The Map:** We map x^μ to a 2x2 Hermitian matrix X using $\sigma_\mu = (\mathbb{I}, \boldsymbol{\sigma})$:

$$X = x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}$$

- **The Invariant:** The Lorentz interval is the determinant: $\det(X) = (x^0)^2 - |\mathbf{x}|^2 = x_\mu x^\mu$.
- **Transformation:** The transformation $X \to X' = LXR$ applies the (2,1) rule from the left and the (1,2) rule from the right. This transformation preserves the determinant This proves that $X \to LXR^\dagger$ is a Lorentz transformation $\Lambda(M)$ on the components x^μ.