Section #1. Class 1

Attempts in Relativistic QM.

Axioms: < Schrödinger Equation.

States are represented by vectors in Hilbert Space

Observables by Hermitian Operators

Measurements yield one of the eigenvalues

Identical particles, etc.

$$i\hbar \frac{\partial}{\partial t} |\psi, t\rangle = \mathcal{H} |\psi, t\rangle.$$

let's consider the simplest

$|\psi, t\rangle$ in Griffith was the general vector in Hilbert Space $|\mathcal{S}(t)\rangle$ and

$$\mathcal{H} = \frac{1}{2m} \vec{p}^{\,2}$$

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{x}, t) = -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{x}, t)$$

$\psi(x,t) = \langle x | \mathcal{S}(t)\rangle = \int d^3y\, \delta(x-y)\psi(y,t) = \psi(x,t)$

$\psi(x,t) = \langle x | \psi, t\rangle$ is the position space wave function.

We want to generalize into relativistic version

$$\mathcal{H} \sim E \longrightarrow \mathcal{H} = \sqrt{p^2c^2 + m^2c^4}$$

$$\mathcal{H} = \underbrace{mc^2}_{\text{constant}} + \underbrace{\frac{1}{2m}p^2}_{\text{QM.}} + \underbrace{\cdots}_{\text{corrections.}}$$

Now SE becomes.

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \underbrace{+\sqrt{m^2c^4 + \cancel{p^2c^2} - \hbar^2c^2\nabla^2}}\,\psi(x,t)$$

still treats space-time differently.
non-linear so solving it is problematic in general.

How about we "square" it?

$$-\hbar^2\frac{\partial^2}{\partial t^2}\psi(x,t) = (m^2c^4 - \hbar^2c^2\nabla^2)\psi(x,t)$$

~~$x = (x, t)$~~. $x^\mu = (ct, \vec{x})$.

and a contravariant

$$x_\mu = (-ct, \vec{x}) \qquad . \ \mu = 0, 1, 2, 3.$$

Minkowski metric.

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}.$$

$$x_\mu = g_{\mu\nu} x^\nu.$$

the inverse.

$$x^\mu = g^{\mu\nu} x_\nu.$$

requires $g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}.$

so we have

$$g^{\mu\nu} g_{\cancel{\mu\nu}\,\nu\rho} = \delta^\mu{}_\rho. \quad \text{the Kronecker Delta.}$$

$$= \begin{array}{cc} & \rho \\ \mu & \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \end{array}$$

Special Relativity tells us that physics looks the same in all inertial frames.

x^μ specify a inertial frame.

another coordinates $\bar{x}^\mu$ can be related to it by

$$\bar{x}^\mu = \underbrace{\Lambda^\mu{}_\nu}_{\text{const}} x^\nu + \underbrace{a^\mu}_{\text{const.}}$$

Lorentz transformation ... translation vector.

Furthermore, we need to require that.

$$g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}$$ (Recall proper time).

which keeps the length unchanged.
" interval* between two points "

as

$$(x-x')^2 = g_{\mu\nu}(x-x')^\mu (x-x')^\nu$$
$$= (\mathbf{x}-\mathbf{x}')^2 - c^2(t-t')^2$$

$$(\bar{x}-\bar{x}')^2 = g_{\mu\nu}(\bar{x}-\bar{x}')^\mu(\bar{x}-\bar{x}')^\nu$$

$$= g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma (x-x')^\rho (x-x')^\sigma$$

$$= g_{\rho\sigma}(x-x')^\rho (x-x')^\sigma$$

$$= (x-x')^2 \text{, as desired.}$$

Now Specialt relativity Equavalents demands

$$\psi(x) = \bar{\psi}(\bar{x})$$

and obey same equations in their own frame.

let's define

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right).$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right).$$

$$\underline{\partial^\mu x^\nu = g^{\mu\nu}}$$

since $\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$.

$$\bar{\partial}^\mu = \Lambda^\mu{}_\nu \partial^\nu .$$

we can easily check that.

$$\bar{\partial}^\rho \bar{x}^\sigma = g^{\rho\sigma}$$

as

$$\bar{\partial}^\rho \bar{x}^\sigma = \Lambda^\rho{}_\nu \partial^\nu (\Lambda^\sigma{}_\mu x^\mu + a^\sigma)$$

$$= \Lambda^\rho{}_\nu \partial^\nu x^\mu \Lambda^\sigma{}_\mu$$

$$= \Lambda^\rho{}_\nu \Lambda^\sigma{}_\mu g^{\mu\nu} = g^{\rho\sigma}.$$

Now $\partial^2 \equiv \partial^\mu \partial_\mu = -\frac{1}{c^2}\frac{\partial}{\partial t^2} + \vec{\nabla}^2$.

K.G. $(-\partial^2 + m^2c^2/\hbar^2)\psi(x) = 0$.

And in $\bar{x}$ inertial frame

$$(-\bar{\partial}^2 + m^2c^2/\hbar^2)\psi(\bar{x}) = 0$$

Now

we need to show that the two K.G in two frames are the same,

$$\bar{\partial}^2 = g_{\mu\nu}\bar{\partial}^\mu\bar{\partial}^\nu = g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\,\partial^\rho\partial^\sigma$$

$$= g_{\rho\sigma}\partial^\rho\partial^\sigma = \partial^2$$

so these are two identical equations.

But K.G Equation doesn't obey SE.

and the equation changes from first order to second order.

then the normalization of states

$$\langle\psi,t|\psi,t\rangle = \int d^3x \,\cancel{\langle\psi,t|\psi,t\rangle}\,\langle\psi,t|\vec{x}\rangle\langle\vec{x}|\psi,t\rangle$$

$$= \int d^3\,\psi^*(x)\,\psi(x)$$

is not time independent.

thus probability is not conserved

one can see more details by working out $\frac{d}{dt}\langle\quad\rangle$ (K.G doesn't help here since it is 2nd order derivative)

$$\mathcal{H}_{ab} = c P_j (\alpha^j)_{ab} + mc^2 (\beta)_{ab}$$

Skipping Dirac's effort. ~~as~~ we'll be introducing it with a fresh eye of symmetries & Representations later this ~~sp~~ course.

linear & conserves probability.

found anti particle states and Dirac Sea.

negative Energy.

And its already about infinite # of particles.

but still we can't describible bosons.

we need a pause: and now we ~~realise~~ that realise.

Time is not yet an ~~ope~~ Hermitian operator.

so we never treated it properly.

stopped Jan. 22. 25

Two operations

$\begin{cases} \text{demote } \vec{x} \text{ to a label} \\ \\ \text{promote } t \text{ to an operator.} \end{cases}$

When promote $t \to T$ operator.

one can use the proper time. τ in SE.

coordinate time, in Heisenberg Picture.

$\hat{x}^\mu(\tau)$. has $\hat{x}^0 = T$.

many complications, any monotonic function of τ

is good here so infinite redundancy.

Also one can consider more parameters

$\hat{x}^\mu(\sigma, \tau)$ ~~now~~ classically this gives

a family of world lines that we call world sheets.

$\hat{x}^\mu(\sigma,\tau)$ is a propagating String

beyond ~~my~~ our scope here.

Another option is to demote x., as labels on operators.

consider an operator $\varphi(\vec{x})$.

A set of such operators are call quantum fields.

$\varphi(\vec{x}) \longrightarrow$

Heisenberg Picture

$$\varphi(\vec{x},t) = e^{iHt/\hbar}\,\varphi(\vec{x},0)\,e^{-iHt/\hbar}.$$

so now $\vec{x}, t$ are labels on operators $\varphi(\vec{x}, t)$.

none is an eigenvalue of an operator.

Quantum Field.

These two options are equivalent.

but $\varphi(\vec{x}, t)$ corresponds to Quantum Field Theory.

is much more convenient for most problems, which

is the subject of this course.

For example of classical field theory, see, e.g. Tong, Schwartz, etc.

There is another desired property,

Normal fixed # non-~~relativistic~~ QM can be written as QFT.

Let's begin with a position space S.E. for n particles.

$$\cancel{i\hbar} \; i\hbar \frac{\partial}{\partial t} \cancel{\psi} \psi = \left[\sum_{j=1}^{n} \left(-\frac{\hbar^2}{2m} \vec{\nabla}_j^2 + U(\vec{x}_j) \right) + \sum_{j=1}^{n} \sum_{k=1}^{j-1} V(x_j - x_k) \right] \psi$$

Here $\cancel{\psi = \psi}$ $\psi = \psi(\vec{x}_1, \vec{x}_2, \vec{x}_3, \ldots \vec{x}_n ; t)$

The QM can be rewritten by introducing a quantum field $a(\vec{x})$ and $a^+(\vec{x})$ to have

$$[a(\vec{x}), a(\vec{x}')] = 0$$

$$[a^+(\vec{x}), a^+(\vec{x}')] = 0$$

$$[a(\vec{x}), a^+(\vec{x}')] = \delta^3(\vec{x} - \vec{x}')$$

$a^+(x)$ and $a(x)$ is like SHO's creation & annihilation operators.

We can introduce a new $\mathcal{H}$ for our QFT.

$$\mathcal{H}_{QFT} = \int d^3x \; a^+(\vec{x}) \left(-\frac{\hbar^2}{2m} \vec{\nabla}^2 + U(\vec{x}) \right) a(\vec{x})$$

$$+ \frac{1}{2} \int d^3x \, d^3y \; V(\vec{x} - \vec{y}) \; \cancel{a a^+} \; a^+(\vec{x}) a^+(\vec{y}) a(\vec{y}) a(\vec{x})$$

Now consider a state

$$|\psi, t\rangle = \int d^3x_1 \, d^3x_2 \cdots d^3x_n \; \psi(\vec{x}_1, \vec{x}_2, \ldots \vec{x}_n ; t) \; a^+(\vec{x}_1) a^+(\vec{x}_2) \cdots a^+(\vec{x}_n) |0\rangle$$

α

here $|0\rangle$ is the vacuum state that

$$a(x)|0\rangle = 0.$$

We can show that. SE of $\mathcal{H}^{QFT}$ is satisfied if and not if

$$\mathcal{H}\ \psi(\vec{x}_1, \vec{x}_2, \dots \vec{x}_n; t)$$

satisfy SE of $\mathcal{H}$.

so they are equivalent

and. $a^\dagger(\vec{x}_1)|0\rangle$ can be interpreted as create a particle ~~of~~ at position $\vec{x}_1$

the operator $N = \int d^3x\ a^\dagger(\vec{x}) a(\vec{x})$ counts the total # of particles.

$[N, \mathcal{H}] = 0$ so particle # is conserved.

e.g.

$$[a(x), a^\dagger(y) a(y)] = \delta^3(x-y)$$

$$[a^\dagger(x), a^\dagger(y) a(y)] = -\delta^3(x-y)$$

so even powers cancel & commutes

And $\mathcal{H}$ acts on ∞

but for a general theory, e.g.

$$\mathcal{H} \supset \int d^3x\,[a^{\dagger 2}(\vec{x}) a(\vec{x}) + h.c.]$$

QM will fail

and QFT prevail.

There are also different requirements for bosons & fermions,

that we will study in near future classes.

$$[V, N]$$

$$= \left[\iint d^3x\, d^3y\, V(\vec{x}-\vec{y})\, a^\dagger(x)\, a^\dagger(y)\, a(y)\, a(x),\ \int d^3z\, a^\dagger(z)\, a(z) \right]$$

$$\int d^3x\, d^3y\, d^3z \Big[V(\vec{x}-\vec{y})\, a^\dagger(x)\, a^\dagger(y)\, a(y)\, a(x)\, a^\dagger(z)\, a(z)$$

$$- a^\dagger(z)\, a(z)\, V(\vec{x}-\vec{y})\, a^\dagger(x)\, a^\dagger(y)\, a(y)\, a(x) \Big]$$