9.23 A Closer Look at the Spinor Representation

This discussion builds the structure of the Lorentz group and its representations by following a common train of thought, correcting key misconceptions along the way.

The Foundation: Rotations, $SO(3)$, and $SU(2)$

We begin with the non-relativistic case of 3D rotations.

- At the **group level**, we have $SO(3)$, the group of 3x3 real orthogonal matrices that act on vectors. We also have $SU(2)$, the group of 2x2 complex unitary matrices that act on spinors. $SU(2)$ is the **universal double-cover** of $SO(3)$.
- Both groups are 3-parameter groups, generated by 3 generators.
- At the **algebra level**, their Lie algebras are **isomorphic**: $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. They share the same commutation relations $[J_i, J_j] = i\epsilon_{ijk}J_k$.

This 2-to-1 group covering, backed by an algebraic isomorphism, is the template for the relativistic case.

The Lorentz Case: $SO(1,3)$ and $SL(2,\mathbb{C})$

We now extend this analogy to spacetime, which includes boosts.

- At the **group level**, we have $SO^+(1,3)$, the proper orthochronous Lorentz group, which acts on 4-vectors (x^μ). We also have $SL(2,\mathbb{C})$, the group of 2x2 complex matrices with determinant 1, which acts on 2-component Weyl spinors.
- $SL(2,\mathbb{C})$ is the **universal double-cover** of $SO^+(1,3)$.
- Both are 6-parameter groups, generated by 6 generators (3 rotations J_i, 3 boosts K_i).
- At the **algebra level**, their 6-dimensional *real* Lie algebras are **isomorphic**: $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$.

A Critical Distinction: $\mathfrak{so}(1,3)$ vs. $\mathfrak{so}(4)$

A very common point of confusion is to assume $\mathfrak{so}(1,3) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$. This is incorrect. The algebra $\mathfrak{su}(2) \times \mathfrak{su}(2)$ is isomorphic to the 4D *Euclidean* rotation algebra, $\mathfrak{so}(4)$, not the Lorentz algebra.
The difference lies in a single, crucial minus sign.

- **The $\mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$ Algebra:** Let the generators be $J_i = A_i + B_i$ and $K_i = A_i - B_i$, where $[A_i, B_j] = 0$. The boost-like commutator is:

$$[K_i, K_j] = [A_i - B_i, A_j - B_j] = [A_i, A_j] + [B_i, B_j] = i\epsilon_{ijk}A_k + i\epsilon_{ijk}B_k = +i\epsilon_{ijk}J_k$$

- **The $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$ Algebra:** The physical Lorentz algebra has the defining relation:
$$[K_i, K_j] = -i\epsilon_{ijk}J_k$$

Because their algebraic structure is fundamentally different, $\mathfrak{so}(1,3) \ncong \mathfrak{so}(4)$ and $\mathfrak{so}(1,3) \ncong \mathfrak{su}(2) \times \mathfrak{su}(2)$ as real algebras.

The Impossibility of Decomposing the Real Algebra

The fact that $\mathfrak{so}(1,3) \ncong \mathfrak{su}(2) \times \mathfrak{su}(2)$ is the entire problem. It means that the physical Lorentz algebra, as a *real* algebra, does not decompose into a direct product. The rotations (J_i) and boosts (K_i) are inextricably linked by $[J_i, K_j] = i\epsilon_{ijk}K_k$. We cannot find a simple set of commuting generators using only real numbers.

The Complexification Trick

Since we cannot decompose the *real* algebra, we employ a mathematical trick: we complexify it. We move from $\mathfrak{so}(1,3)$ to $\mathfrak{so}(1,3)_{\mathbb{C}}$ by allowing complex linear combinations of the generators.
This is done *purely* to find a basis where the algebra decouples. We define the complex generators:

$$N_i = \frac{1}{2}(J_i + iK_i) \quad \text{and} \quad \tilde{N}_i = \frac{1}{2}(J_i - iK_i)$$

The crucial minus sign in $[K_i, K_j] = -i\epsilon_{ijk}J_k$ is precisely what's needed to make these two new algebras commute:

$$[N_i, B_j] = 0$$

Thus, the *complexified* algebra **does** decompose:

$$\mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2)_{\mathbb{C}} \times \mathfrak{su}(2)_{\mathbb{C}}$$

This allows us to classify all representations by two labels, one for each $\mathfrak{su}(2)_{\mathbb{C}}$ factor.

Generator Counting: The 12-Dimensional Real Space

Let's be precise about dimensions.

- $\mathfrak{su}(2)$ is a 3-dimensional *real* algebra (basis, e.g., $\{\sigma_1, \sigma_2, \sigma_3\}$).
- $\mathfrak{su}(2)_{\mathbb{C}}$ is its complexification. As a *complex* algebra, it is 3-dimensional (basis, e.g., $\{\sigma_1, \sigma_2, \sigma_3\}$ with complex coefficients in the "rotation angles"). As a *real* algebra, it is 6-dimensional (basis, e.g., $\{\sigma_i, i\sigma_i\}$).

- Therefore, the full decoupled space $\mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{su}(2)_{\mathbb{C}} \times \mathfrak{su}(2)_{\mathbb{C}}$ is a 12-dimensional *real* vector space. These are $\{\sigma_i, i\sigma_i\}$ for $\mathfrak{su}(2)_{\mathbb{C}}^L$ and $\{\sigma_i, i\sigma_i\}$ for $\mathfrak{su}(2)_{\mathbb{C}}^R$.

The Physical Subalgebra: Projecting back to $\mathfrak{so}(1,3)$

We are only interested in the physical world, which is described by the 6-dimensional *real* algebra $\mathfrak{so}(1,3)$, not the 12D complex space. Our physical algebra $\mathfrak{so}(1,3)$ must be a 6-dimensional "real slice" embedded within this 12D space.
We find this "slice" by inverting the definitions for N_i and B_i:

$$J_i = N_i + \tilde{N}_i$$

$$K_i = -i(N_i - \tilde{N}_i)$$

These are our 6 physical generators, built from the 12 generators of the complex space. Which 6? 3 out of the 6 hermitian ones $\sigma_i^L + \sigma_i^R$ and 3 out of the 6 anti-hermitian ones $-i\sigma_i^L + i\sigma_i^R$ while maintaining the "rotation/boost coefficients" being real.
The 6 physical generators are *not* just "two sets of Pauli matrices" (e.g., N_i and $\tilde{N}_i$). They are this specific combination:

- **Rotations** $J_i = N_i + \tilde{N}_i$**:** In a representation where N_i and B_i are Hermitian, the J_i are also **Hermitian**. This is required so that rotations $U_R = \exp(-i\theta_j J_j)$ are unitary.
- **Boosts** $K_i = -i(N_i - \tilde{N}_i)$**:** With N_i, B_i Hermitian, the K_i are **anti-Hermitian** (because of the i). This is also required, so that boosts $U_B = \exp(-i\beta_j K_j) = \exp(-\beta_j(N_j - \tilde{N}_j))$ are non-unitary.

Our 6D physical world $\mathfrak{so}(1,3)$ is this specific "slice" of the 12D complexified space that is spanned by 3 Hermitian generators (rotations) and 3 anti-Hermitian generators (boosts).

Summary Note: Lorentz Group Algebras and Representations

This note clarifies the key relationships between $SO(1,3)$, $SL(2,\mathbb{C})$, $SO(4)$, and $SU(2)$, and explains the strategy of complexification.

1. The Core Analogy: Rotations ($SO(3)$ and $SU(2)$)

- **Algebras (Local):** The Lie algebras are **isomorphic**: $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. They are the same 3-generator structure (the J_i's).
- **Groups (Global):** The groups are **not** isomorphic. $SU(2)$ is the **universal double-cover** of $SO(3)$. This is a 2-to-1 mapping (U and $-U$ in $SU(2)$ map to a single R in $SO(3)$).

2. The Lorentz Case ($SO(1,3)$ and $SL(2,\mathbb{C})$)

This follows the same pattern, but with 6 generators (3 rotations J_i, 3 boosts K_i).

- **Algebras (Local):** The 6-dimensional **real** Lie algebras are **isomorphic**: $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$.
- **Groups (Global):** $SL(2,\mathbb{C})$ is the **universal double-cover** of $SO^+(1,3)$. This is also a 2-to-1 mapping (M and $-M$ in $SL(2,\mathbb{C})$ map to a single Λ in $SO^+(1,3)$).

3. The Most Common Error: $\mathfrak{so}(1,3) \ncong \mathfrak{su}(2) \times \mathfrak{su}(2)$

- The algebra $\mathfrak{su}(2) \times \mathfrak{su}(2)$ is **isomorphic to** $\mathfrak{so}(4)$ (4D Euclidean rotations), not the Lorentz algebra.
- The difference is the crucial **minus sign** in the boost commutator.

$$\begin{aligned} \mathfrak{so}(1,3) \text{ (Lorentz)}: \quad & [K_i, K_j] = -i\epsilon_{ijk} J_k \\ \mathfrak{so}(4) \text{ (Euclidean)}: \quad & [K_i, K_j] = +i\epsilon_{ijk} J_k \end{aligned}$$

- Because this structure is different, the **real** algebra $\mathfrak{so}(1,3)$ **does not decompose** into a simple product.

4. The "Complexification Trick" – Why We Use It

This is the central strategy for classifying representations.

- **Problem:** Classifying the representations of the "messy," non-decomposed *real* algebra $\mathfrak{so}(1,3)$ is hard.

- **Key Theorem:** The set of (finite-dimensional) representations of a real algebra $\mathfrak{g}$ is in a **1-to-1 correspondence** with the representations of its complexification, $\mathfrak{g}_{\mathbb{C}}$.
- **The "Easy" Algebra:** We work with the complexification, $\mathfrak{so}(1,3)_{\mathbb{C}}$. This is a **6-dimensional *complex*** algebra (or 12D real). Its generators are the same $\{J_i, K_i\}$, but they can be multiplied by complex numbers.
- **The Payoff:** This *complex* algebra **does** decompose!

$$\mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2)_{\mathbb{C}} \times \mathfrak{su}(2)_{\mathbb{C}}$$

  (This works because $\dim_{\mathbb{C}}(\mathfrak{so}(1,3)_{\mathbb{C}}) = 6$, and $\dim_{\mathbb{C}}(\mathfrak{su}(2)_{\mathbb{C}} \times \mathfrak{su}(2)_{\mathbb{C}}) = 3 + 3 = 6$.)
- **Solution:** We classify the irreps of the "easy" product $\mathfrak{su}(2)_{\mathbb{C}} \times \mathfrak{su}(2)_{\mathbb{C}}$, which are just pairs of $SU(2)$ irreps, (j_A, j_B). By the theorem, this gives us the complete "menu" of representations for our "hard" physical algebra, $\mathfrak{so}(1,3)$.

5. Key Definitions

- **$\cong$: Isomorphic.** The two structures are mathematically identical (e.g., they have the same commutation relations).
- **Universal Double Cover** (e.g., $SL(2,\mathbb{C}) \to SO^+(1,3)$):
  - **Cover:** A projection map from a "parent" group to a "base" group.
  - **Double:** The map is exactly 2-to-1.
  - **Universal:** The "parent" group (e.g., $SL(2,\mathbb{C})$) is **simply connected** (it has no "topological holes"). It is the "top-level" parent cover.

9.24 Hermiticity Properties of Lorentz Group Generators Across Representations

The Lorentz group, being the fundamental symmetry of spacetime, admits multiple representations that play different roles in physics. A particularly subtle aspect concerns the hermiticity properties of the generators in these representations, which has profound implications for unitarity and physical interpretation.

Mathematical Preliminaries

The Lorentz algebra is generated by six operators J_i (rotations) and K_i (boosts) satisfying:

$$\begin{aligned}
[J_i, J_j] &= i\epsilon_{ijk} J_k \\
[J_i, K_j] &= i\epsilon_{ijk} K_k \\
[K_i, K_j] &= -i\epsilon_{ijk} J_k
\end{aligned}$$

A crucial observation is that the Lorentz group is **non-compact**, which immediately implies that it has **no non-trivial finite-dimensional unitary representations**. This fundamental constraint shapes the hermiticity properties across different representations.

The Defining (Vector) Representation

The vector representation acts on 4-vectors in Minkowski space $\mathbb{R}^{1,3}$ and provides the defining representation of the Lorentz group.

Generator Structure:. The generators are 4×4 matrices:

$$(M^{\mu\nu})_{\alpha\beta} = i(\eta^{\mu\alpha}\delta^{\nu}_{\beta} - \eta^{\nu\alpha}\delta^{\mu}_{\beta})$$

For example, the rotation generator $J_z = M^{12}$ and boost generator $K_x = M^{01}$ are:

$$J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_x = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hermiticity Properties:.

- Rotation generators: $J_i^\dagger = J_i$ (Hermitian)
- Boost generators: $K_i^\dagger = -K_i$ (Anti-Hermitian)
- Complex combinations: $N_i = J_i + iK_i$, $\tilde{N}_i = J_i - iK_i$ are both **Hermitian**

Physical Significance:. This representation describes how 4-vectors (position, momentum, electromagnetic potential) transform under Lorentz transformations. The non-unitarity is acceptable since 4-vectors are classical objects (on the embedding manifold), not quantum states (vector field discussed separately).

The Spinor Representation

Spinor representations act on complex spinor spaces and are essential for describing fermionic fields.

Generator Structure:. The Dirac representation uses 4×4 gamma matrices:

$$J_i = \frac{1}{2}\epsilon_{ijk}\gamma^j\gamma^k, \quad K_i = \frac{i}{2}\gamma^0\gamma^i$$

The complex combinations reveal the underlying $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ structure:

$$N_i = J_i + iK_i, \quad \tilde{N}_i = J_i - iK_i$$

Hermiticity Properties:.

- Rotation generators: $J_i^\dagger = J_i$ (Hermitian)
- Boost generators: $K_i^\dagger = -K_i$ (Anti-Hermitian)
- Complex combinations: Both N_i and $\tilde{N}_i$ are **Hermitian**
- The $\tilde{}$ symbol in $\tilde{N}_i$ denotes **left-right conjugation**, not hermitian conjugation

Physical Significance:. This representation describes how fermion fields (electrons, quarks) transform. The Weyl representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ correspond to left-handed and right-handed fermions, respectively.

The Orbital Angular Momentum Representation

This representation acts on the Hilbert space of physical states and is crucial for quantum mechanics.

Generator Structure:. The generators are differential operators:

$$L^{\mu\nu} = \frac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)$$

This gives explicit forms:

$$J_i = \frac{\hbar}{i}\epsilon_{ijk}x^j\partial^k, \quad K_i = \frac{\hbar}{i}(x^0\partial^i - x^i\partial^0)$$

Hermiticity Properties:.

- Both rotation and boost generators are **Hermitian**: $J_i^\dagger = J_i$, $K_i^\dagger = K_i$
- The complex combinations satisfy: $N_i^\dagger = \tilde{N}_i$, $\tilde{N}_i^\dagger = N_i$
- This ensures that Lorentz transformations $U(\Lambda) = \exp\left(\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\right)$ are **unitary**

Physical Significance:. This representation acts on physical states in the Hilbert space $L^2(\mathbb{R}^4)$, where unitarity is mandatory for probability conservation.

Comparative Analysis

Property	**Vector Rep**	**Spinor Rep**	**Orbital Rep**
Space	$\mathbb{R}^{1,3}$	$\mathbb{C}^4$	$L^2(\mathbb{R}^4)$
Dimension	4 (finite)	4 (finite)	Infinite
J_i hermiticity	Hermitian	Hermitian	Hermitian
K_i hermiticity	Anti-Hermitian	Anti-Hermitian	Hermitian
$N_i, \tilde{N}_i$	Both Hermitian	Both Hermitian	Mutual conjugates
Unitary?	No	No	Yes
Purpose	4-vectors	Fermion fields	Physical states

Comparison of Lorentz group representations

Physical Interpretation and Conclusions

The pattern of hermiticity properties reveals deep physical insights:

- **Finite-dimensional representations** (vector and spinor) are necessarily **non-unitary** due to the non-compact nature of the Lorentz group.
- **Infinite-dimensional representations** can be unitary for non compact group and we are deadling with such one.
- The **complex combinations** N_i, $\tilde{N}_i$ reveal the underlying $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ structure, but their hermiticity properties depend on the specific representation.

This comprehensive understanding is essential for constructing consistent quantum field theories where fields transform under finite-dimensional non-unitary representations while physical states transform under infinite-dimensional unitary representations, ensuring both covariance and probability conservation.