lec 22.

We had in the last lecture that photon propagator

$$\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \tilde{\Delta}^{\mu\nu}(k)$$

$$\tilde{\Delta}^{\mu\nu}(k) = -\frac{1}{\vec{k}^2} \delta^{\mu 0} \delta^{\nu 0} + \frac{1}{k^2 - i\varepsilon} \sum_{\lambda=\pm} \varepsilon_\lambda^{\mu *}(\vec{k}) \varepsilon_\lambda^{\nu}(\vec{k})$$

We also had

$$\sum_{\lambda=\pm} \varepsilon_\lambda^{i*}(\vec{k}) \varepsilon_\lambda^{j}(\vec{k}) = \delta_{ij} - \frac{k_i k_j}{\vec{k}^2}$$

(recall $\varepsilon_\pm = \frac{1}{\sqrt{2}}(\mp 1, \pm i, 0)$
$\vec{k} = (0, 0, k)$)

Extending to μ. ($i \to \mu$, $j \to \nu$)

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{g^{\mu\nu}}{k^2 - i\varepsilon} \quad \boxed{\text{Feynman Gauge}}$$

$$\cancel{\tilde{\Delta}^{\mu\nu}(k) = \frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}}{k^2 - i\varepsilon}} \quad \cancel{\boxed{\text{Coulomb Gauge}}}$$

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but this is somewhat alarming already as we know photons are massless.

One also naturally asks, why these different choices yield a same result?

Let's Path Integral.

$$Z_0(J) = \int \mathcal{D}A\, e^{iS_0}$$

$$S_0 = \int d^4x \, [-\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu].$$

Into momentum space, we have

$$S_0 = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} [-\tilde{A}_\mu(k)(k^2 g^{\mu\nu} - k^\mu k^\nu)\tilde{A}_\nu(-k)$$

$$+ \tilde{J}^\mu(k)\tilde{A}_\mu(-k) + \tilde{J}^\mu(-k)\tilde{A}(k)]$$

secretely

$\int dx\, e^{i(xAx + Bx)}$

$= \int d(x + A^{-1}B/2)^2$

$e^{i(x - A^{-1}B/2)^2 A - A^{-1}B/4}$.

(Recall that

$$\tilde{A}_\mu(k) = \int \frac{d^4x}{(2\pi)^4} e^{ikx} A(x)$$

$$\tilde{J}_\mu(k) = \int \frac{d^4x}{(2\pi)^4} e^{ikx} J(x)$$

$$d^4x\, \cancel{d^4k}\, d^4k'\, e^{-i(k+k')x} \quad \text{set } k' = -k.)$$

Then we can shift the integration variable $\mathcal{D}A$ by $\tilde{J}$ to make it Gaussian and leave with the source terms. but with one problem that the inverse of

$$k^2 g^{\mu\nu} - k^\mu k^\nu \text{ doesn't exist.}$$

Let's write $k^2 g^{\mu\nu} - k^\mu k^\nu = k^2 P^{\mu\nu}(k)$.

$$P^{\mu\nu}(k) = \frac{k^2 g^{\mu\nu} - k^\mu k^\nu}{k^2} = \underline{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}}$$

this is again a projection matrix that is idempotent.

$$P^{\mu\nu}(k) P_\nu{}^\lambda(k) = P^{\mu\lambda}(k)$$

The only allowed eigenvalues are zero and one.

as

$\cancel{k_\mu P^{\mu\nu}(k) k_\nu = 0}$

$$P^{\mu\nu}(k) k_\nu = k^\mu - k^\mu = 0.$$

So a 4-vector proportional to k^μ will be an eigenvector with eigenvale 0.

Then $g_{\mu\nu} P^{\mu\nu}(k) = 4 - \frac{k^2}{k^2} = 3$

the "Trace" as $\sum_\mu P^\mu{}_\mu$

So the other 3 eigenvalues are 1, 1, 1.

To make the matrices invertible, we just need to focus on those with non-zero eigenvalues.

So in $\mathcal{D}A^\mu$, ~~wit~~ we only integrate over ~~sates~~ the states ~~with~~ three states ~~propor~~ orthogonal to k^μ direction.

So here we definiced the measure $\mathcal{D}A$ more properly.
Due to such orthogonality, we have

$$\cancel{k^\mu \tilde{A}_\mu(k) = 0}. \quad k^\mu \tilde{A}_\mu(k) = 0.$$

this is equivalent to the Lorentz Gauge.
For these states, therm proportional to k^μ do not contribute.
Further, with the shifting of momentum, $k^\mu J_\mu$ are not contributing to the path integral neither, as the current is conserved, $\partial_\mu J^\mu = 0$.

$$(\; k_\mu \tilde{J}^\mu(k) = 0 \;).$$

And also the matrix $P^{\mu\nu} = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$ is the projection operator to the subspace orthogonal to k^μ within the subspace, $P^{\mu\nu}$ is just the identity matrix.
~~Then~~ Hence the inverse of $k^2 P^{\mu\nu}$ is simply $P^{\mu\nu}/k^2$

Now we have

$$Z_0(J) = \exp\left[\frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \tilde{J}_\mu(k) \frac{P^{\mu\nu}}{k^2 - i\varepsilon} \tilde{J}_\nu(-k)\right]$$

$$= \exp\left[\frac{i}{2}\int d^4x\, d^4y\, J_\mu(k) \cancel{\frac{P^{\mu\nu}}{k^2 - i\varepsilon}} \Delta^{\mu\nu}(x-y) J_\nu(-k)\right]$$

where

$$\Delta^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{P^{\mu\nu}}{k^2 - i\varepsilon}$$

And of course, since $k^\mu J_\mu(k)$ is zero
we can choose

$$P^{\mu\nu} = \underbrace{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}}_{} \longrightarrow \underbrace{g^{\mu\nu}}_{\text{Feynman Gauge.}}$$

~~Feynman Gauge~~

Lorentz Gauge

(also known as) Landau Gauge

So here we have the free theory path integral
and understand that there is some freedom in the
propagator. $\tilde{\Delta}^{\mu\nu}(k)$

This motivates the R_ξ gauge. that

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{1}{k^2 - i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right).$$

So $\xi = 1$ or 0 gives use the
above gauge choices but other
choices are fine, too.

And physical result shall not depend on
the choice of ξ. This allows use to prove many useful aspects.
(this requires adding ξ-gauge term $-\frac{1}{2\xi}\partial^\mu A_\mu \partial^\nu A_\nu$
and a gauge choice is done by the gauge fixing function
in the path integral)

Continues Symmetries.

With a lagragian density $\mathcal{L}$, $S = \int d^4x\, \mathcal{L}(x)$

the variation would be, for d.o.f $\varphi_a(x)$

$$\frac{\delta S}{\delta \varphi_a(x)} = \frac{\delta \int d^4y\, \mathcal{L}(y)}{\delta \varphi_a(x)}$$

$$= \int d^4y\, \frac{\delta \mathcal{L}(y)}{\delta \varphi_a(x)}$$

$$= \int d^4y \left[\frac{\partial \mathcal{L}(y)}{\partial \varphi_b(y)} \frac{\delta \varphi_b(y)}{\delta \varphi_a(x)} + \frac{\partial \mathcal{L}(y)}{\partial(\partial_\mu \varphi_b(y))} \frac{\delta(\partial_\mu \varphi_b(y))}{\delta \varphi_a(x)} \right]$$

$$\cancel{\partial_\mu\left(\frac{\partial \mathcal{L}(y)}{\partial_\mu \varphi_b(y)} \frac{\delta \varphi_b(y)}{\delta \varphi_a(x)}\right) - \left(\partial_\mu \frac{\partial \mathcal{L}(y)}{\partial(\partial_\mu \varphi_b(y))}\right) \delta_{ab}\delta(x-y)}$$

$$= \int d^4y \left[\frac{\partial \mathcal{L}(y)}{\partial \varphi_b(y)} \delta_{ab}\delta(x-y) + \cancel{\frac{\partial \mathcal{L}(y)}{\partial(\partial_\mu \varphi_b(y))} \partial_\mu \delta_{ab}\delta(x-y)} \right]$$

$$= \frac{\partial \mathcal{L}(x)}{\partial \varphi_a(x)} - \partial_\mu \frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \varphi_a(x))}$$

Sign flips next line by IBP.

Stationary & "minimal" action has

$\delta S = 0$ so. the above is zero.

~~the~~ So

$$\frac{\partial \mathcal{L}(x)}{\partial \varphi_a(x)} \to \partial_\mu \frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \varphi_a(x))} + \frac{\delta S}{\delta \varphi_a(x)}$$

Note the difference between partial derivative ∂ and functional variation.

On the otherhand, with $\mathcal{L}(x) = \mathcal{L}(\varphi_a(x), \partial_\mu \varphi_a(x))$.

An infinitesimal transformation of field

$$\varphi_a(x) \to \varphi_a(x) + \delta\varphi_a(x)$$

$$\mathcal{L}(x) \xrightarrow[\text{goes to}]{\text{then}} \mathcal{L}(x) + \delta\mathcal{L}(x)$$

$$\delta\mathcal{L}(x) = \frac{\partial\mathcal{L}}{\partial\varphi_a(x)}\delta\varphi_a(x) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a(x))}\partial_\mu\delta\varphi_a(x)$$

Combining with the previous page, we get.

$$\delta\mathcal{L}(x) = \underbrace{\partial_\mu\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\varphi_a(x))}\delta\varphi_a(x)\right)}_{\text{combined two terms.}} + \frac{\delta S}{\delta\varphi_a(x)}\delta\varphi_a(x)$$

$$\partial_\mu(\quad)\delta\varphi_a(x)$$
$$+$$
$$(\quad)\partial_\mu\delta\varphi_a(x)$$

Defining current $\boxed{j^\mu(x) = \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\varphi_a(x))}\delta\varphi_a(x),}$ we have.

$$\partial_\mu j^\mu(x) = \underbrace{\delta\mathcal{L}(x)}_{} - \underbrace{\frac{\delta S}{\delta\varphi_a(x)}\delta\varphi_a(x)}_{0 \text{ for stationary action.}}$$

If $\delta\mathcal{L}(x) = 0$ (meaning a transformation doesn't change $\mathcal{L}(x)$ hence $\mathcal{L}(x)$ is symmetric for such a change).

We then have

$$\partial_\mu j^\mu(x) = 0.$$

Noether current is conserved., and in components

$$\frac{\partial}{\partial t} \underset{\substack{\uparrow \\ \text{charge density}}}{j^0(x)} + \vec{\nabla} \cdot \underset{\substack{\uparrow \\ \text{current density}}}{\vec{j}(x)} = 0.$$

(non-compressible fluid dynamics)

for this transformation.

Applying to E & M.

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu .$$

$$= -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + J^\mu A_\mu .$$

$$\cancel{\frac{\delta \mathcal{L}}{\delta}} \quad \frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = 0.$$

We can see.

$$J^\mu - \partial_\nu \left[-\frac{1}{\cancel{2}_{1}} \times \cancel{2} (\partial^\nu A^\mu - \partial^\mu A^\nu) \right] = 0.$$

$(\quad)^2$

$-\frac{1}{4} \times 2$

$$J^\mu + \partial^2 A^\mu - \partial_\nu \partial^\mu A^\nu = 0$$

$$J^\mu + \cancel{\partial^2 A^\mu - g^{\mu\nu}} \; g^{\mu\nu} \partial^2 A_\nu - \partial^\nu \partial^\mu A_\nu = 0$$

this is the Maxwell Equation

$$\partial_\nu F^{\mu\nu} = J^\mu .$$

Now add the real interactions back.

$j^\mu(x)$ is actually in E&M to

Noether current for the $U(1)$ symmetry of the

Dirac field.

Recall

$\mathcal{L}_\Psi = \cancel{i\bar{\Psi}\partial\Psi}\ \cancel{i\bar{\Psi}\gamma^\mu}\ i\bar{\Psi}\partial\!\!\!/\Psi - m\bar{\Psi}\Psi$

The Noether current

$$j^\mu = \frac{\partial \mathcal{L}_\Psi(x)}{\partial(\partial_\mu \Psi(x))}\delta\Psi(x)$$

here $\Psi(x) \to e^{-i\alpha}\Psi(x)$ | $\delta\Psi(x) = (-i\alpha)\Psi(x)$

we can $\alpha \to e$.

$$j^\mu = e\bar{\Psi}(x)\gamma^\mu\Psi(x).$$

$e = -0.302$ in Natural Units.

(Heaviside-Lorentz).

fine structure constant is $\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$.

(and it is running).

And we know $J^\mu = (\rho, \vec{J})$

this is exactly, e.g.,

$$\int d^3x\, \bar{\Psi}\gamma^0\Psi = \int d^3x\, \underbrace{\chi^\dagger \chi}_{\#\text{of } e^-} - \underbrace{\xi^\dagger \xi}_{\#\text{of } e^+}$$

So Naturally, we shall have the interaction term

$$e\bar{\Psi}\gamma^\mu\Psi A_\mu.$$

but even before renormalizing the theory, we have one problem:

this term is not $\underline{\text{gauge transformation}}$ invariant

$$(A_\mu \to A_\mu - \partial_\mu \Gamma(x)).$$

$$\mathcal{L} = \underline{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}} + \underline{i\bar{\Psi}\not{\partial}\Psi} - \underline{m\bar{\Psi}\Psi} + \underline{e\bar{\Psi}\gamma^\mu\Psi A_\mu}.$$

We can extend the definition of gauge transformation to be a set of transformations, such that

Gauge transformation:

$$A_\mu(x) \to A_\mu(x) \mp \partial_\mu \Gamma(x)$$

$$\Psi(x) \to e^{-ie\Gamma(x)}\Psi(x)$$

$$\bar{\Psi}(x) \to e^{+ie\Gamma(x)}\bar{\Psi}(x)$$

Then

$$i\bar{\Psi}\not{\partial}\Psi \Longrightarrow i\bar{\Psi}e^{ie\Gamma(x)}\gamma^\mu\partial_\mu e^{-ie\Gamma(x)}\Psi(x)$$

$$= i\bar{\Psi}\not{\partial}\Psi + \underline{e\bar{\Psi}\gamma^\mu\Psi(x)\Gamma(x)}$$

cancelling

$$e\bar{\Psi}(x)\gamma^\mu\Psi(x)A_\mu \to e\bar{\Psi}(x)\gamma^\mu\Psi(x)A_\mu - e\bar{\Psi}(x)\gamma^\mu\Psi(x)P(x)$$

leaving the $\mathcal{L}$ invariant under such gauge transformation

One can simplify the $\mathcal{L}$ by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\Psi}\not{D}\Psi \cancel{*}\, m\bar{\Psi}\Psi$$

here $\not{D}$ is the covariant derivative.

(gauge)

$$D_\mu \cancel{\Psi} = \partial_\mu + ieQA_\mu$$

Q for election is "−1"

We then see that

$$D_\mu\Psi \to (\partial_\mu - ie(A_\mu - \partial_\mu P))\cancel{\Psi}\, e^{-ieP(x)}\Psi(x)$$

$$= \exp[-ieP]\cancel{\Psi}\, D_\mu\Psi$$

also $D_\mu \to e^{-ieP} D_\mu e^{ieP}$

and see $F^{\mu\nu} \cancel{*} = \frac{i}{e}[D^\mu, D^\nu]$

In reflection, we have promoted the global $U(1)$ symmetry of Ψ to a local (gauged) $U(1)$ symmetry.

the introduction of $A_\mu\bar{\Psi}\gamma^\mu\Psi$ is to make the kinetic term invariant.