# 9.20 The Renormalization Scale $\mu$ in the $\overline{\text{MS}}$ Scheme

In quantum field theories such as Quantum Chromodynamics (QCD), calculations often encounter infinities from quantum loop corrections in Feynman diagrams. To obtain finite, physical predictions, we employ renormalization, which absorbs these divergences into redefinitions of parameters like masses and coupling constants (e.g., the strong coupling $\alpha_s$).

The $\overline{\text{MS}}$ (Modified Minimal Subtraction) scheme is a widely used renormalization method. It subtracts only the divergent parts (poles in dimensional regularization) and certain constants. In this scheme, we introduce the renormalization scale $\mu$, which at first appears unphysical since physical observables should not depend on our calculational choices. However, $\mu$ plays a vital role in capturing how parameters evolve with energy scales.

Consider $\mu$ as a "ruler" for measuring interactions at different energy "zoom levels." In QCD, the strong coupling $\alpha_s$ is not constant but "runs" due to quantum effects, exhibiting asymptotic freedom: it is strong at low energies and weak at high energies. The renormalization group equation (RGE) describes this evolution:

$$\frac{d\alpha_s}{d\ln(\mu^2)} = -\beta(\alpha_s),$$

The solution is approximately

$$\alpha_s(\mu) \approx \frac{\alpha_s(\mu_0)}{1 + \beta_0 \alpha_s(\mu_0)\ln(\mu^2/\mu_0^2)}.$$

Here, $\alpha_s$ explicitly depends on $\mu$. Measuring $\alpha_s$ at one scale (e.g., $\mu = M_Z \approx 91$ GeV) allows prediction at others via the RGE. This running is physical and experimentally verified.

While $\mu$ is a technical parameter, it is dynamical, reflecting the physical scale of the process. In perturbative calculations, terms like $\ln(q^2/\mu^2)$ appear, where $q$ is a momentum. In idealized processes with a single hard scale, such as the center-of-mass energy $\sqrt{s}$ in $e^+e^- \to$ hadrons, we set $\mu^2 = s$ for several reasons:

1. **Minimizing Logarithms**: This makes $\ln(s/\mu^2) \approx 0$, reducing higher-order corrections and improving perturbation theory reliability.
2. **Resumming Effects**: It incorporates some higher-order contributions into the running coupling.

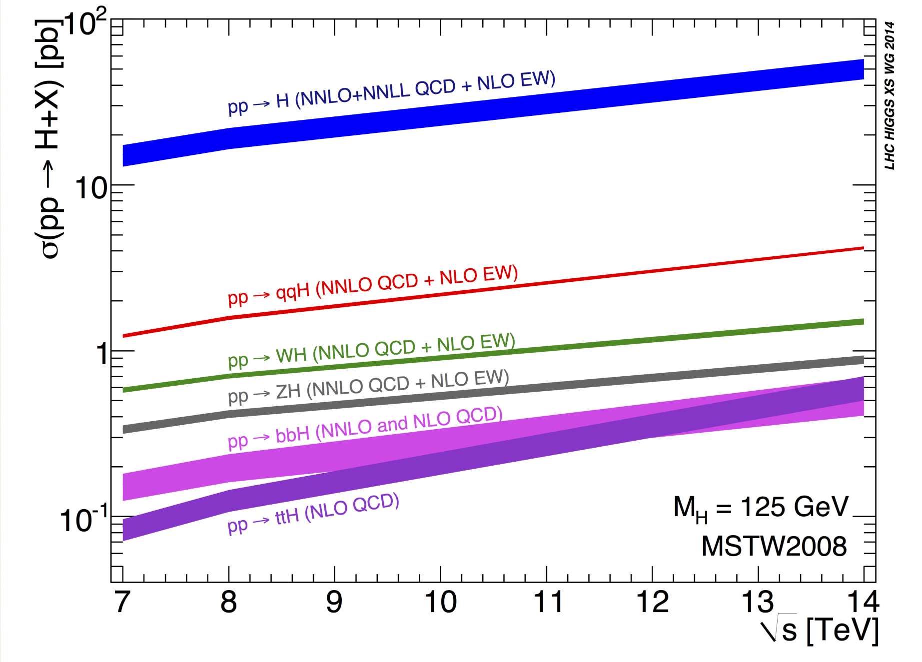


Figure 9.2: Higgs cross section as a function of center of mass energy of the collision. Part of the “band” is from uncertainty estimation via varying the renormalization (and factorization) scales.

3. **Physical Matching**: $\mu$ separates short- and long-distance physics; aligning it with the dominant scale captures the dynamics effectively.
4. **Scheme Independence**: Good choices minimize truncation errors in finite-order calculations.

Choosing $\mu$ inappropriately (e.g., too small for high-energy processes) can lead to large $\alpha_s$ or missed effects, breaking perturbation theory.

## However, $\mu^2 = s$ is a merely a “best-guess” choice.

Real physics calculations often involve multiple dynamical scales, such as relative momenta, particle masses, or kinematic variables. In these cases, setting $\mu^2 = s$ may not be optimal, as it could introduce large logarithms from other scales. Instead, we use physics intuition to select a “best-guessed” central scale $\mu_0$:

- Identify key scales (e.g., momentum transfer $Q$, jet $p_T$, masses $m_q$).
- Common choices: Geometric means like $\sqrt{Q_1 Q_2}$ for two scales, or dynamical options like $H_T/2$ (half the scalar sum of transverse momenta) in multi-jet events.

- In effective theories like SCET, multiple $\mu$'s handle different sectors (hard, collinear, soft).

To validate, compute at higher orders (e.g., NLO or NNLO), where $\mu$ dependence should decrease as more terms cancel. Additionally, estimate uncertainties via scale variation: Vary $\mu$ around $\mu_0$ (typically by factors of 1/2 to 2), and independently vary related scales like the factorization scale $\mu_F$. The resulting band provides a conservative theoretical error estimate. Stability under variation indicates reliability; otherwise, higher orders or resummation may be needed.
**This approach is iterative:** Start with a reasonable $\mu_0$, compute, vary scales, compare orders, and refine based on the physics.