Brief intro to infrared divergence.

We just got

$$\sigma = \sigma_{tree}\left[1 - \frac{11}{12}\alpha\left(\ln(S/m^2) + O(m^0)\right) + O(\alpha^2)\right].$$

no large logs.

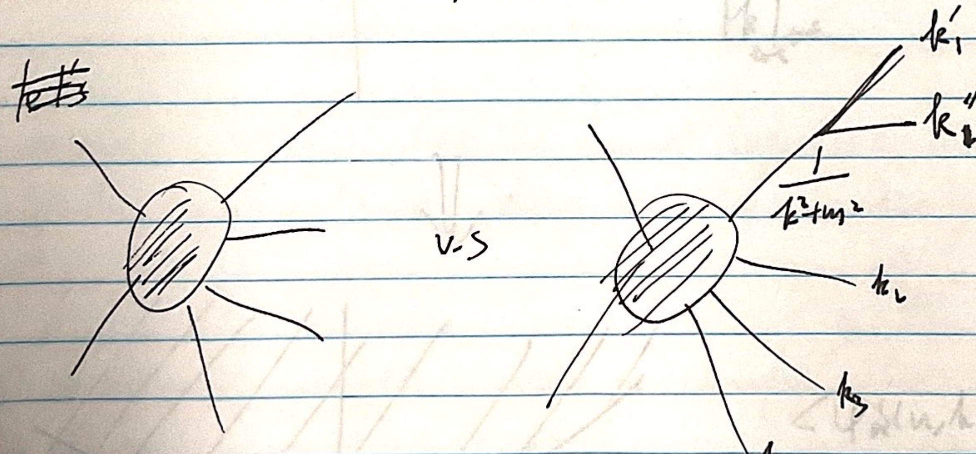
What if we are dealing with a theory with massless particles?

$m^2 = 0$.

The matrix amplitude would blow up., we encounter infinity again.

But this time, it is from IR. (as it is there no matter how small S is. or it is in the limit of $m \to 0$).

The cause are two folders.



If k_1' and k_2' are colinear or if k_2 is soft.,

the $\frac{1}{k^2+m^2} \to \frac{1}{k^2}$ can allow such configuration and ~~to~~ blow-up.

For real detectable physical process, minimal Energy & separation angle are needed to separate.

identical particles.

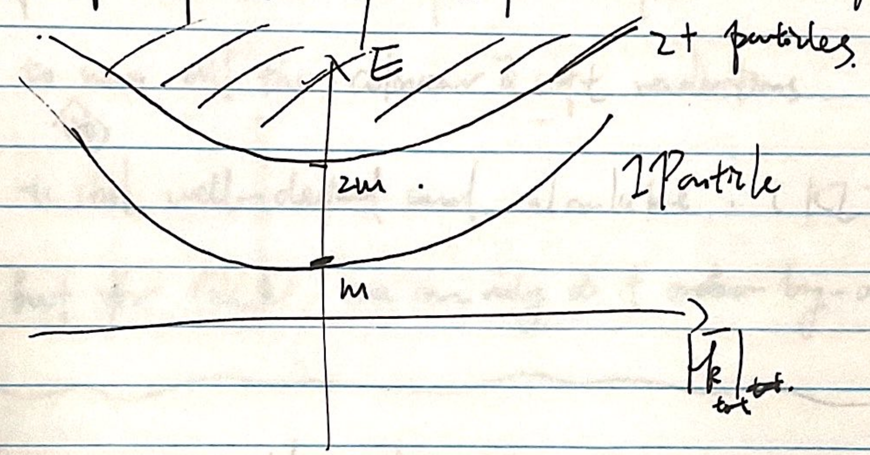
So detectability regulates the process.

(very cheap way to discuss this yet another profound topic)

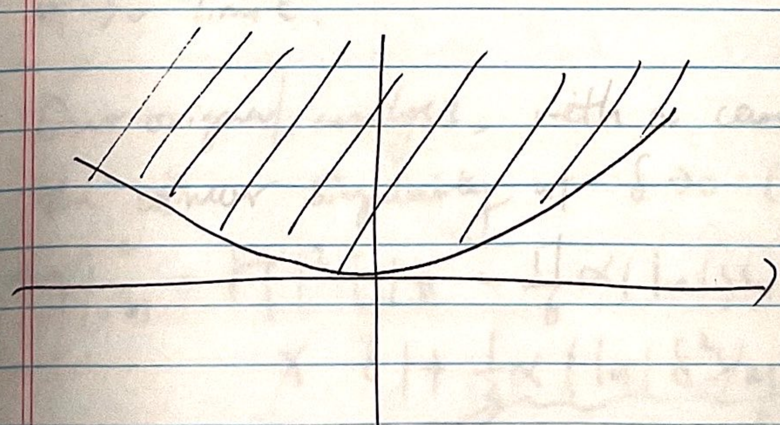
But slightly more mathematically.

massless particles make one thing hard:

the ~~sp~~ separation of one particle & multiparticle state.



$\Downarrow$



$\langle \psi_{\gamma s}(n,k) | \psi(k) | 0 \rangle \neq 0.$

Many other trouble occurs, the singular part & $\rho(s)$ part of the exact propagator hard to separate

1 particle state can funnel to multiparticle state, etc.

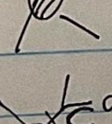
In real world, such situation happens.

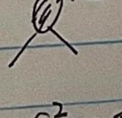
An electron can emit infinite # of soft massless photons and still hit the electron pole (and it is a stable particle). One needs to redefine the state of electron to sum all the colinear & soft radiations. The EM theory (the) is still well-defined and calculable. (KLT theorem).

But for QCD, one can only do it order-by-order.

However, still, when we cannot separate two particles in the massless limit, there is still a divergence in the $m^2 \to 0$ limit.

For instance

 T.

 T_{obs}

$m^2/|\vec k|^2 \ll \delta^2 \ll 1$.

Our original analysis, with a careful analysis to separate the colinear singularity at $\delta \to 0$ (δ ~~span~~ separation angle)

$$\frac{g^2}{(k^2+m^2)^2}(2\pi)^{d-1}2\omega\,\delta^{d-1}(\vec{k}-\vec{k}_1-\vec{k}_2)\,\frac{1}{2}d\hat{k}_1 d\hat{k}_2$$

$$|T|^2_{obs} = |T|^2\left[1 - \frac{1}{6}\alpha\left(\ln(s/m^2) + O(m^0)\right) + O(\alpha^2)\right]$$
$$\times \left[1 + \underbrace{\frac{1}{3}\alpha\left(\ln(\delta^2 s/m^2) + O(m^0)\right) + O(\alpha^2)}_{\text{colinear part from } n\to n'+1 \text{ process.}}\right]$$

$$= |T_0|^2\left[1 - \alpha\left(\frac{3}{2}\ln(s/m^2) + \frac{1}{3}\ln(1/\delta^2) + O(m^0)\right) + O(\alpha^2)\right]$$

Renormalization Schemes & RG flow.

We still didn't solve the case when we cannot resolve the particles, or in other ~~words~~ words, we still have the $\ln(s/m^2)$ problem, (this time, with a different coefficient). What did we do wrong?

We need to review our renormalization procedure;

We had:

$$\Pi(k^2) = -\left[A + \tfrac{1}{6}\alpha\left(\tfrac{1}{\epsilon}+\tfrac{1}{2}\right)\right]k^2 - \left[B + \alpha\left(\tfrac{1}{\epsilon}+\tfrac{1}{2}\right)\right]m^2 + \frac{1}{2}\alpha\int_0^1 dx\, D\ln(D/\mu^2) + O(\alpha^2)$$

and

$$\Pi'(k^2) = -\left[A + \tfrac{1}{6}\alpha\left(\tfrac{1}{\epsilon}+\tfrac{1}{2}\right)\right] + \frac{1}{2}\alpha\int_0^1 dx\,\left[\ln(D/\mu^2)+1\right]x(1-x) + O(\alpha^2).$$

Previously we set ~~$\Pi(k^2)$~~ $\Pi(-m^2)=0$ and $\Pi'(-m^2)=0$ to keep the one-particle state normalized properly.

But in the $m^2 \to 0$ limit, $D = x(1-x)k^2$

$\Pi(-m^2) = 0$ automatically $\boxed{0\ln(0)=0}$

$\Pi'(-m^2)$ is $\ln(0/\mu^2) \to$ blows up. (ill defined)

as the one-particle pole overlaps with the multi-particle branch, so the pole of the exact propagator is no longer a simple pole.

The (only) way out is to change renormalization scheme.
[I hope you realize that ever after introducing path integral, the rest is just how to evaluate things in physics. (physically). This requires some consistent modifications in the Lagrangian density.]

Instead of setting A & B satisfying the b.c. $\Pi(-m^2) = 0$, $\Pi'(-m^2) = 0$, we set A and B just to cancel the divergencies.

$$A = -\frac{1}{6}\alpha\frac{1}{\epsilon} + O(\alpha^2)$$

$$B = -\cancel{\tfrac{1}{2}}\alpha\frac{1}{\epsilon} + O(\alpha^2)$$

This is called $\overline{MS}$ scheme., one of the most commonly used scheme.

Em-Es-bar

Modified (as we set $\mu = \tilde{\mu}\sqrt{4\pi}\,e^{-\gamma/2}$)
Minimal Subtraction
Scheme.

~~So~~ and our previous one was actually the On-Shell scheme.

So

$$\Pi_{\overline{MS}}(k^2) = -\frac{1}{12}\alpha(k^2+6m^2) + \frac{1}{2}\alpha\int_0^1 dx\, D\ln(D/\mu^2) + O(\alpha^2)$$

$$\Pi_{OS}(k^2) = -\frac{1}{12}\alpha(k^2+m^2) + \frac{1}{2}\alpha\int_0^1 dx\, D\ln(D/D_0) + O(\alpha^2)$$

$$D_0 = D|_{k^2=-m^2} = [x(1-x)k^2+m^2]|_{k^2=-m^2}$$
$$= [-x(1-x)+1]m^2.$$

Here $\Pi_{\overline{MS}}(k^2)$ has a well-defined $m \to 0$ limit.

but $\Pi_{\overline{MS}}(-m^2) \neq 0$, $\Pi'_{\overline{MS}}(-m_{ph}^2)$ will be evaluated separately.

~~so here.~~

Note. The physical quantity in the LSZ, scattering (measurable) is the pole mass: m_{ph}.

And LSZ requires proper normalization, here φ is to be rescaled by $R^{-1/2}$ (if the residue is R) to keep LSZ normalized.

Let's see the consequence.

$$\tilde{\Delta}_{\overline{MS}}(k^2) = \frac{1}{k^2+m^2-\Pi_{\overline{MS}}(k^2)}$$

and by definition $k^2 = -m_{ph}^2 \longrightarrow$

$$-m_{ph}^2 + m^2 - \Pi_{\overline{MS}}(-m_{ph}^2) = 0.$$

One can try to solve the intergal differential equation (typically very very hard), but we can do perturbation.

Because $\Pi(k^2)$ is an integral differential because perturbation are around $-m^2$ not $-m_{ph}^2$

Since we think/expect

$$m_{ph}^2 = m^2 - \Pi_{\overline{MS}}(-m_{ph}^2) + O(\alpha^2)$$

$\downarrow$

α. order. so $m_{ph}^2 = m^2 + O(\alpha)$

$$= m^2 - \Pi_{\overline{MS}}(-m^2) + O(\alpha^2)$$

now we have

$$m_{ph}^2 = m^2 - \frac{1}{2}\alpha\left[\frac{1}{6}m^2 - m^2 + \int_0^1 dx\, D_0 \ln(D_0/\mu^2)\right] + O(\alpha^2)$$

$$= m^2\left[1 + \frac{5}{12}\alpha\left(\ln(\mu^2/m^2) + c'\right) + O(\alpha^2)\right]$$

still have μ $\quad c' = (34 - 3\pi\sqrt{3})/15 = 1.18$

(unphysical μ dependence & m.)

But μ is not a physical parameter.

How can a physical observable depends on ~~up~~ unphysical quantities?

Something else is consistently varying m, α is!

We then know how α varies with μ.

$$\ln m_{ph} = \ln m + \underbrace{\frac{5}{12}\alpha\left(\ln(\mu/m) + \frac{1}{2}c'\right)} + O(\alpha^2)$$

here we view $(1+\delta) = e^{\delta}$

$\ln(1+x) = x + \cdots$

here we would assume $\frac{d\alpha}{d\ln\mu} = O(\alpha^2)$ for next page

A leap in our understanding is that
unphysical parameters could depend on μ
so $m(\mu)$, $\alpha(\mu)$

To keep physical parameter unchanged, by observing
the form of the dependence, we
require (as $\ln\mu$ and $\ln m$ contains infinite powers negative)

$$\frac{d\ln M_{ph}}{d\ln\mu} = 0$$

$$= \frac{1}{m}\frac{dm}{d\ln\mu} + \frac{5}{12}\alpha + O(\alpha^2)$$

here $\alpha \frac{d\alpha}{d\ln\mu}$ is $O(\alpha^2)$

$$\longrightarrow \quad \frac{d\ln m}{d\ln\mu} = \underbrace{-\frac{5}{12}\alpha + O(\alpha^2)}$$

called the anomalous dimension
of mass parameter m, often dubbed
$\gamma_m(\alpha)$.

$$\frac{dm}{d\ln\mu} = (-\frac{5}{12}\alpha + O(\alpha^2))m.$$

If we try, e.g. $d\ln M_{ph}/d\ln\mu$ or $dM_{ph}/d\mu$, there won't
be clean expressions to get rid of the $\ln\mu$ dependance

Now get the quantity R. (residue)

$$R^{-1} = \frac{d}{dk^2}\left[\Delta_{\overline{MS}}(k^2)^{-1}\right]\Big|_{k^2=-m_{ph}^2}$$

$$= 1 - \Pi'_{\overline{MS}}(-m_{ph}^2)$$

$$= 1 - \Pi'_{\overline{MS}}(-m^2) + O(\alpha^2)$$

$$= 1 + \frac{1}{12}\alpha\left(\ln(\mu^2/m^2) + c''\right) + O(\alpha^2)$$

$(17 - 3\pi\sqrt{3})/3 = 0.23$

$$-\frac{1}{12}\alpha + \frac{1}{12}\alpha + \frac{1}{12}\alpha \ln \mu^2/m^2 \quad \text{by } \Pi'(k^2)\big|_{k^2=-m^2}$$

Also the vertex function using $\overline{MS}$:

$$C = -\alpha\frac{1}{\epsilon} + O(\alpha^2)$$

$$\rightarrow \quad V_{3\,\overline{MS}}(k_1, k_2, k_3) = g\left[1 - \frac{1}{2}\alpha\int dF_3 \ln(D/\mu^2) + O(\alpha^2)\right]$$

$$D_3 = \cancel{x_1 x_2 k_1^2}\; xyk_1^2 + yzk_2^2 + zxk_3^2 + m^2$$

Now we can assemble everything.

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In $\overline{MS}$ scheme.

$V_{3\overline{MS}}$ ~~V_{4}~~, $\Delta_{\overline{MS}}$

$$T = R^2 T_0 \left[1 - \frac{11}{12} \alpha \left(\ln s/\mu^2\right) + O(m^0)\right] + O(\alpha^2)\right]$$

normalization, tree amplitude $T_0 = -g^2\left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right)$

Plug in R, we get

$$T = T_0 \left[1 - \alpha\left(\frac{11}{12} \ln\frac{s}{\mu^2} + \frac{1}{6}\ln\frac{\mu^2}{m^2} + O(m^0)\right) + O(\alpha^2)\right]$$

$$|T_{obs}| = |T|^2 \left(1 + \frac{1}{3}\alpha\left(\ln(\delta^2 s/m^2) + O(m^0)\right) + O(\alpha^2)\right]$$

$$= |T_0|^2 \left[1 - \alpha\left(\frac{3}{2}\ln\frac{s}{\mu^2} + \frac{1}{3}\ln\frac{1}{\delta^2} + O(m^0)\right) + O(\alpha^2)\right]$$

~~not~~ $\boxed{\text{no } \ln m \text{ dependence!}}$

a good theory in the $m \to 0$ limit.

$$-\left(\frac{1}{6}\ln\frac{\mu^2}{m^2}\right)\times 2 + \frac{1}{3}\ln\frac{s}{m^2} = 0 \ln m^2 + c\ln\frac{s}{\mu^2}$$

but the observable cross section $|T_{obs}|^2$ still depends on fake parameter μ. , it can only bee physical if we also get rid of it $\ln\mu$ dependence.

$\alpha(\mu)$ is varying w.r.t. some good ~~rules~~ relations.
(again observing the structures & attempt to find simple relations)

$$\ln|T|^2_{obs} = C_1 + \underset{\uparrow \atop |T_0|^2}{2\ln\alpha} + 3\alpha(\ln\mu + C_2) + \mathcal{O}(\alpha^2).$$

$$0 = \frac{d\ln|T|^2_{obs}}{d\ln\mu} = \frac{2}{\alpha}\frac{d\alpha}{d\ln\mu} + 3\alpha + \mathcal{O}(\alpha^2).$$

$$\rightarrow \frac{d\alpha}{d\ln\mu} = \underbrace{-\frac{3}{2}\alpha^2 + \mathcal{O}(\alpha^3)}_{\text{beta function}}$$

$$\rightarrow \alpha(\mu_2) = \frac{\alpha(\mu_1)}{1+\frac{3}{2}\alpha(\mu_1)\ln\mu_2/\mu_1}$$

with b.c. that $\mu_2 = \mu_1 \to \alpha(\mu_2) = \alpha(\mu_1)$

Physically, if one determines α at scale μ_1
(typically ~~tho~~ we take $\mu = s$)
the scale at which we do the experiment.

we know how it should be at another scale μ_2.
Physics becomes predictable!

Here for beta function negative,
the theory is more and more weakly coupled at higher and higher energy.

We call it asympotic freedom.
(QCD is this kind).

On the contrary, if beta is positive, the theory is more and more strongly coupled and we have to deal with it differently at high energy.
It is infrared free, but UV-strongly coupled.
($U(1)_{EM}$ is such kind, as well $SU(2)$ weak).

but for lower and lower energies, the theory becomes strongly coupled and our perturbative description fails. Luckily, we know the effective theory at low energy for QCD., that is mesons & baryons (of particular relavence to us, protons & neutrons)

Remember that all
$m(\mu)$, $\alpha(\mu)$ varies.
so $m_{\overline{MS}}$ is a parameter of the theory but not physical.
One has to be careful we discussing quantities.

What if $\gamma_m = 0$, $\beta_\alpha = 0$.? if they are strickly zero.
we are at fixed point of a theory. Lovely core of CFT Conformal Theory

The discussion is reasonably physical but very complex. Now let's view it from a different angle: RG

We have been dealing with two Lagrangians.

Renormalized $\mathcal{L} = -\frac{1}{2} Z_\varphi \partial^\mu\varphi\, \partial_\mu\varphi - \frac{1}{2} Z_m m^2 \varphi^2 + \frac{1}{3!} Z_g\, g \tilde{\mu}^{\epsilon/2} \varphi^3 + Y\varphi$
($\overline{MS}$)

and

Bare Lagrangian $\mathcal{L} = -\frac{1}{2}\partial^\mu\varphi_0\, \partial_\mu\varphi_0 - \frac{1}{2} m_0^2 \varphi_0^2 + \frac{1}{3!} g_0 \varphi_0^3 + Y_0 \varphi_0$

By comparison, ~~if we~~ if

$$\varphi_0(x) = Z_\varphi^{1/2} \varphi(x)$$

$$m_0 = Z_\varphi^{-1/2} Z_m^{1/2} m$$

$$g_0 = Z_\varphi^{-3/2} Z_g\, g \tilde{\mu}^{\epsilon/2}$$

$$Y_0 = Z_\varphi^{-1/2} Y.$$

These two would be equivalent, which means so long as any bare Lagrangian, which is physical if parameters are written properly, would be fine and predictive.

Recall that in $\overline{MS}$ we just cancel the divergencies labelled by $1/\epsilon$, generally,

$$Z_\varphi = 1 + \sum_{n=1}^{\infty} \frac{a_n(\alpha)}{\epsilon^n}$$

$$Z_m = 1 + \sum_{n=1}^{\infty} \frac{b_n(\alpha)}{\epsilon^n}$$

$$Z_g = 1 + \sum_{n=1}^{\infty} \frac{c_n(\alpha)}{\epsilon^n} \quad , \quad \alpha = \frac{g^2}{(4\pi)^3}$$

Our calculations of $\Pi(k^2)$ and $V_3(k_1, k_2, k_3)$ in perturbation theory in the $\overline{MS}$ scheme, give us Taylor series in α for $a_n(\alpha)$, $b_n(\alpha)$ and $c_n(\alpha)$.

And we found

$$a_1 = -\frac{1}{6}\alpha + O(\alpha^2)$$

$$b_1 = -\alpha + O(\alpha^2)$$

$$c_1 = -\alpha + O(\alpha^2)$$

Here is the core physical condition. Bare parameters has to be independent from μ.

$$\alpha_0 \equiv g_0^2/(4\pi)^3 = \underline{Z_g^2 Z_\varphi^{-3}} \mu^{\epsilon} \alpha .$$

$$\ln(Z_g^2 Z_\varphi^{-3}) \equiv G(\alpha, \epsilon) \equiv \sum_{n=1}^{\infty} \frac{G_n(\alpha)}{\epsilon^n}$$

$$G_1(\alpha) = 2G(\alpha) - 3a_1(\alpha) = -\frac{3}{2}\alpha + O(\alpha^2).$$

Now we know

$$\ln\alpha_0 = G(\alpha,\epsilon) + \ln\alpha + \epsilon\ln\tilde{\mu}$$

$\epsilon\ln\mu + \epsilon C.$

$$0 = \frac{d\ln\alpha_0}{d\ln\mu} = \frac{\partial G(\alpha,\epsilon)}{\partial\alpha}\frac{d\alpha}{d\ln\mu} + \frac{1}{\alpha}\frac{d\alpha}{d\ln\mu} + \epsilon.$$

$$\Rightarrow \quad 0 = \left(1 + \frac{\alpha G_1'(\alpha)}{\epsilon} + \frac{\alpha G_2'(\alpha)}{\epsilon^2} + \dots\right)\frac{d\alpha}{d\ln\mu} + \epsilon\alpha$$

In the $\epsilon \to 0$ limit, α should move slowly

$$\frac{d\alpha}{d\ln\mu} = -\epsilon\alpha + \beta(\alpha)$$

$$\cancel{\beta(\alpha) = \alpha^2 G_1'(\alpha)}$$

they should order by order in $\frac{1}{\epsilon}$ cancel. hence

$$\beta(\alpha) = \alpha^2 G_1'(\alpha)$$

and $\quad G_2'(\alpha) = \alpha G_1'(\alpha)^2 \ldots$

$$\beta(\alpha) = -\frac{3}{2}\alpha^2 + O(\alpha^3).$$

Similarly, one can derive various $d\ln \cancel{\Delta}\, m_0$

$$d\ln \tilde{\Delta}_0(k^2)$$

$$\Rightarrow \left(\frac{\partial}{\partial \ln\mu} + \beta(\alpha)\frac{\partial}{\partial\alpha} + \gamma_m(\alpha)\, m\frac{\partial}{\partial m} + 2\gamma_\varphi(\alpha)\right)\tilde{\Delta}(k^2) = 0.$$

Callan-Symanzik-Equation.