9.10 Discarding Surface Terms in the Variational Principle

In quantum field theory, when deriving equations of motion such as the Klein-Gordon equation via the principle of least action, boundary terms arising from integration by parts are discarded under two justified assumptions:

1. Vanishing Field Variations at Boundaries: The variational principle considers small perturbations $\delta\phi(x)$ around the classical field configuration. These variations are required to have *compact support*, meaning $\delta\phi(x) \to 0$ at spacetime infinity. For example, when varying the action $S = \int d^4x\, \mathcal{L}(\phi, \partial_\mu\phi)$, integration by parts generates boundary terms of the form:

$$\int d^4x\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \delta\phi \right) = \oint_{\text{boundary}} d\Sigma^\mu \, \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \delta\phi,$$

which vanish because $\delta\phi = 0$ on the boundary. This ensures surface terms do not contribute to the equations of motion, regardless of the original field $\phi(x)$'s behavior. The vanishing of $\delta\phi$ at boundaries is a formal requirement of the variational principle, not a physical assumption about $\phi(x)$.

2. Physical Boundary Conditions for Fields: While the field $\phi(x)$ itself might oscillate rapidly at infinity, physical consistency demands fields decay sufficiently or behave "nicely" at spacetime boundaries. For quantum fields, this is often enforced by:

- *Square-integrability*: $\phi(x)$ vanishes at infinity (e.g., fields in L^2 space).
- *Tempered distributions*: Fields are assumed to not blow up uncontrollably (standard in axiomatic QFT).
- *Schwartz functions*: Fields and derivatives decay rapidly, ensuring boundary terms involving $\phi(x)$ are well-controlled.

Rapid oscillations of $\phi(x)$ are further suppressed in the path integral formalism. Configurations with wildly oscillating phases e^{iS} destructively interfere due to the *stationary phase approximation*, leaving only contributions near classical paths (where the action is stationary). Mathematically, this justifies ignoring ill-defined boundary terms even if $\phi(x)$ oscillates.

The Klein-Gordon equation arises as the condition for the action to be stationary under these variations. Crucially, boundary terms depend on $\delta\phi$ (not ϕ), and $\delta\phi$ vanishes at infinity by construction. Thus, discarding surface terms is rigorously valid in the variational framework, even if $\phi(x)$'s behavior at infinity appears ambiguous.