Sec 7. Path Integral for Harmonic Oscillator

Consider a harmonic oscillator with hamiltonian

$$H(P, Q) = \frac{1}{2m} P^2 + \frac{1}{2} m\omega^2 Q^2$$

$$\langle 0|0\rangle_f = \int \mathcal{D}p\,\mathcal{D}q\, e^{i\int_{-\infty}^{+\infty} dt\,[p\dot{q} - (1-i\epsilon)H + fq]}$$

the $1-i\epsilon$ can be traded as

$$\cancel{\frac{1}{2m} \to \frac{1}{2m(1+i\epsilon)}}, \quad \cancel{\frac{1}{2}m \to \frac{1}{2}(1-i\epsilon)m}$$

$$\dot{q} = \frac{\partial H}{\partial p},$$

Using the Lagrangian treatment:

$$\langle 0|0\rangle_f = \int \cancel{\mathcal{D}p}\,\mathcal{D}q\, e^{i\int_{-\infty}^{\infty} dt\,[\frac{1}{2}(1+i\epsilon)m\dot{q}^2 - \frac{1}{2}(1-i\epsilon)m\omega^2 q^2 + fq]}$$

from now on, we simply set $m=1$, this can be restored by dimensional ~~analysis~~ analysis.

Let use Fourier transformed variables

$$\tilde{q}(E) = \int_{-\infty}^{+\infty} dt\, e^{iEt} q(t), \quad q(t) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} e^{-iEt} \tilde{q}(E)$$

The exponential part then becomes

$$\int_{-\infty}^{+\infty} dt\, \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dE}{2\pi}\frac{dE'}{2\pi} e^{-i(E+E')t} \Big[[-(1+i\epsilon)EE' - (1-i\epsilon)\omega^2]\tilde{q}(E)\tilde{q}(E') + \tilde{f}(E)\tilde{q}(E') + \tilde{f}(E')\tilde{q}(E) \Big].$$

Integrating over dt:

$$S = \int_{-\infty}^{+\infty} dt \cdots \cdots \rightarrow E^2 - \omega^2 + i\overbrace{\boxed{(E^2+\omega^2)\epsilon}}^{\text{redefine } \epsilon}$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dE}{2\pi}\Big[\underbrace{\overbrace{\left((1+i\epsilon)E^2 - (1-i\epsilon)\omega^2\right)}\tilde{q}(E)\,\cancel{\tilde{q}(E)}\,\tilde{q}(-E)}_{\text{free theory}}$$

$$+ \hat{f}(E)\,\tilde{q}(-E) + \hat{f}(-E)\,\tilde{q}(E)\Big]$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dE}{2\pi}\left[(E^2-\omega^2+i\epsilon)\tilde{q}(E)\tilde{q}(-E) + \tilde{f}(E)\tilde{q}(-E) + \tilde{f}(-E)\tilde{q}(E)\right]$$

"symmetrize" the variables.

$$\tilde{x}(E) = \tilde{q}(E) + \frac{f(E)}{E^2-\omega^2+i\epsilon}, \quad \mathcal{D}q \rightarrow \mathcal{D}x.$$

$$S = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dE}{2\pi}\left[\tilde{x}(E)(E^2-\omega^2+i\epsilon)\tilde{x}(-E) - \frac{\hat{f}(E)\tilde{f}(-E)}{E^2-\omega^2+i\epsilon}\right]$$

As $\frac{\tilde{f}(E)}{E^2-\omega^2+i\epsilon}$ is only a constant ~~shift~~ function shift.

$$\mathcal{D}q = \mathcal{D}x$$

$$\langle 0|0\rangle_f = \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\frac{\hat{f}(E)\tilde{f}(-E)}{-E^2+\omega^2-i\epsilon}\right] \times$$

$$\underbrace{\boxed{\int \mathcal{D}x \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\tilde{x}(E)(E^2-\omega^2+i\epsilon)\tilde{x}(-E)\right]}}_{\text{Already there for free theory}}$$

and free theory has $\langle 0|0\rangle_{f=0} = 1$

so

$$\langle 0|0\rangle_f = \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\frac{\hat{f}(E)\hat{f}(-E)}{-E^2+\cancel{\omega^2}\omega^2-i\epsilon}\right].$$

rewrite in time-domain variables:

$$\langle 0|0\rangle_f = \exp\left[\frac{i}{2}\int_{-\infty}^{+\infty} dt\, dt'\, f(t) G(t-t') f(t')\right]$$

with

$$G(t-t') = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{-E^2+\omega^2-i\epsilon}$$

as $f(t) = \int dE\, e^{-iEt} \tilde{f}(E)$

$f(t') = \int dE\, e^{-i(-E)t'} \tilde{f}(-E)$

Now $G(t-t')$ is a Green's function for the oscillator equation of motion:

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$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right) G(t-t') = \delta(t-t')$$

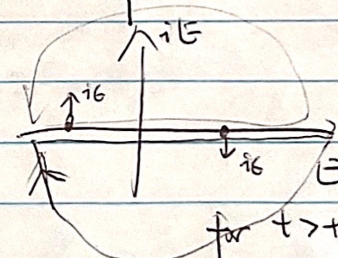
as

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right) G(t-t') = \lim_{\epsilon\to 0}\int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{(-E^2+\omega^2)e^{-iE(t-t')}}{-E^2+\omega^2-i\epsilon}$$

$$= \delta(t-t')$$

note $\delta(t-t') = \delta(t'-t)$
$= \delta(|t-t'|)$

one can also find that, through ~~residue~~ residue theorem

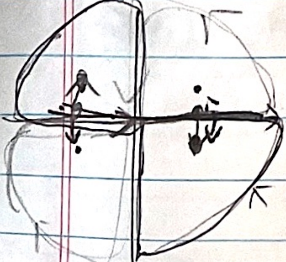


$$G(t-t') = \frac{i}{2\omega} \exp(-i\omega\underline{|t-t'|})$$

for $t>t'$
or $t<t'$

note it is symmetric function of t and t'

(you can also see time ordering come in here).



ensure vanishing $(t>t')$

Recall

$$\langle 0|TQ(t_1)\cdots|0\rangle = \frac{1}{i}\frac{\delta}{\delta f(t_1)}\cdots\langle 0|0\rangle_f\Big|_{f=0}$$

with our new formula. here

$$\langle 0|TQ(t_1)Q(t_2)|0\rangle = \frac{1}{i}\frac{\delta}{\delta f(t_1)}\frac{1}{i}\frac{\delta}{\delta f(t_2)}\langle 0|0\rangle_f\Big|_{f=0}$$

$$= \frac{1}{i}\frac{\delta}{\delta f(t_1)}\left[\int_{-\infty}^{+\infty}dt'\,G(t_2-t')f(t')\right]\langle 0|0\rangle_f\Big|_{f=0}$$

$$= \left[\frac{1}{i}G(t_2-t_1)+\int_{-\infty}^{+\infty}dt'\,G(t_2-t')f(t')\int_{-\infty}^{+\infty}dt''\,G(t''-t_1)f(t'')\right]\langle 0|0\rangle_f\Big|_{f=0}$$

$$= \frac{1}{i}G(t_2-t_1).$$

So it is clear we need even numbers of Q so the transition amplitude is not zero as $f=0$.

$$\langle 0|TQ(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = \frac{1}{i^2}\big[G(t_1-t_2)G(t_3-t_4) + G(t_1-t_3)G(t_2-t_4) + G(t_1-t_4)G(t_2-t_3)\big]$$

and.

$$\langle 0|TQ(t_1)\cdots Q(t_{2n})|0\rangle = \frac{1}{i^n}\sum_{\substack{\{\text{all pairings}\}\\ \{i_1,i_2,\cdots i_{2n}\}}} G(t_{i_1}-t_{i_2})\cdots G(t_{i_{2n-1}}-t_{i_{2n}})$$