9.17 Green's Functions, Propagators, and Causality

The Green's function is a powerful mathematical tool for solving inhomogeneous linear differential equations, representing the fundamental response of a system to an idealized point-like impulse. Its physical interpretation evolves significantly from classical mechanics to quantum field theory, intricately tied to the concept of causality.

9.17.1 Classical Impulse Response

In a classical context, such as the simple harmonic oscillator described by $\ddot{x}(t) + \omega_0^2 x(t) = F(t)/m$, the Green's function $G(t, t')$ represents the position of the oscillator at time t resulting from a unit impulse applied at time t'. The **retarded Green's function** strictly enforces causality, ensuring the effect does not precede the cause. This is mathematically expressed as:

$$G_R(t, t') = \begin{cases} \frac{1}{m\omega_0} \sin(\omega_0(t - t')) & \text{if } t \geq t' \\ 0 & \text{if } t < t' \end{cases}$$

The solution for an arbitrary forcing function $F(t)$ is then found by convoluting it with the Green's function, effectively summing the responses to a continuous series of impulses.

9.17.2 The Quantum Propagator

In quantum mechanics, the Green's function acquires a more profound meaning: it becomes the **propagator**. It is the solution to the Schrödinger equation with a delta function source and represents the probability amplitude for a particle to propagate from a spacetime point (x', t') to another (x, t). The propagator $G(x, t; x', t')$ allows for the time evolution of any initial wavefunction $\Psi(x', t')$ to be determined via the integral:

$$\Psi(x, t) = \int G(x, t; x', t')\Psi(x', t')dx'$$

This interpretation forms the basis for path integral formulations of quantum mechanics and is a cornerstone of many-body theory.

9.17.3 Causality in Quantum Field Theory (QFT)

The treatment of causality becomes more nuanced in relativistic QFT, which must account for the existence of antiparticles. This necessitates a distinction between the retarded Green's function and the **Feynman propagator** (G_F).

- **The Retarded Propagator (G_R):** As in the classical case, it describes the amplitude for a particle to propagate strictly forward in time ($t > t'$). It embodies the classical notion of cause and effect.
- **The Feynman Propagator (G_F):** This is the crucial tool for QFT. It correctly describes two physical processes under a single mathematical formalism:
    1. A particle propagating forward in time ($t > t'$).
    2. An antiparticle propagating forward in time. This is mathematically equivalent to a particle propagating *backward* in time ($t < t'$).

The Feynman "prescription" provides a specific rule for integrating in the complex energy plane that yields this result. It is essential for calculating scattering amplitudes using Feynman diagrams, where internal lines represent virtual particles that can be either particles or antiparticles. Thus, the Feynman propagator abandons the strict, classical time-ordering of the retarded function to correctly model a quantum reality that includes particle creation and annihilation.