Chapter 2: Brief QFT

2.1 Basic Group Theory

Here is a quick refresher on the group theory, which one could have learned from the mathematical physics class and the QFT class. The logic is the same: I hope you have a better understanding of where the "rules" in particle physics come from.

Group theory is used in many places in particle physics, in understanding global symmetries, (local) gauge symmetries, particle representations, etc. Here, we recap the basics.

A Group G is defined by a set of elements $g_i \in G$ and its multiplication rule, $g_1 \cdot g_2$, which satisfy the following,

- Closure: if $g_1 \in G$ and $g_2 \in G$, then $g_1 \cdot g_2 \in G$;
- Associativity: if $g_1, g_2, g_3 \in G$, then $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ (note the order is unchanged);
- Identity: there exists $I \in G$ such that $I \cdot g_i = g_i \cdot I = g_i$;
- Inverse: if $g_1 \in G$, then there exists $g_1^{-1} \in G$ such that $g_1^{-1} \cdot g_1 = g_1 \cdot g_1^{-1} = I$.

Here are a few simple examples:

- $\mathbb{Z}_2 = \{-1, 1\}$, with the multiplication rule defined as multiplication.
- $\{\mathbb{Z}\}$, the set of integers, with the multiplication rule defined as addition. One can see that if $n, m \in \mathbb{Z}$, $m \cdot n = m + n \in \mathbb{Z}$ and the identity is 0 and the inverse of each element m is $-m$.

> **Q & As**
>
> (ChimeIn) Can $\{x \in \mathbb{R} | x \neq 0\}$, the set of all real numbers with 0 removed, be a group?
> (ChimeIn) What is the identity?
> (ChimeIn) What is the multiplication?
> (ChimeIn) Can $\{\mathbb{R}\}$, the set of all real numbers, be a group?

The groups we will use in this class can be categorized differently. For instance, the groups can be **discrete** or **continuous**, which describes the group member's continuity behavior. Importantly, we also deal with the following types of groups,

- **Abelian group**: for all $g_1, g_2 \in G$, $g_1 \cdot g_2 = g_2 \cdot g_1$;
- **Non-abelian** group: there exists $g_1, g_2 \in G$, $g_1 \cdot g_2 \neq g_2 \cdot g_1$.

In particle physics, we often use groups (with n degrees) made up of elements of n-by-n matrices. Particularly,

- **Unitary Group**: with all group elements $g_i^\dagger = g_i$; e.g., U(1), U(3)
- **Special Unitary Group**: additionally $\det[g_i] = 1$; e.g., SU(2), SU(3), SU(5)
- **Orthogonal Group** group: with all group elements $g_i^T = g_i$; e.g., O(3)
- **Special Orthogonal Group**: additionally $\det[g_i] = 1$ e.g., SO(3)

Another critically important concept is representation. **A representation**[1] of group G is a mapping, D of elements of G onto a set of linear operators with the following properties

- $D(I) = 1$, where 1 is the identity operator in the space on which the linear operators act.
- $D(g_1)D(g_2) = D(g_1 \cdot g_2)$, in other words, the group multiplication law is mapped onto the natural multiplication in the linear space on which the linear operators act.

One can view the group multiplication rule as a mapping table where group elements (column) group-multiplied by group elements (row) gives another group element. For a group of n elements (this is to say the group G is of **order** n), this forms an $n \times n$ generalized matrix D whose matrix elements are group members g_i. Now define $D(g_i)$ as a $n \times n$ matrix where group element g_i in D is replaced by 1, and all others are set to zero. Then, we get a set (total numbers is n) of $n \times n$ matrices, which automatically satisfies the group multiplication rules with its natural matrix multiplication. This is called the **regular representation** of a group, which can be **reducible**. The regular representation $D(g_i)$ satisfy the following, treating the group elements g_i to form an orthonormal basis for vector space $|g_i\rangle$:

$$D(g_i)\,|g_j\rangle = |g_i \cdot g_j\rangle\,. \tag{2.1.1}$$

Clearly, one can obtain the **matrix elements** of representation for arbitrary group element g, $D(g)$ via,

$$D(g)_{ij} = \langle g_i|\, D(g)\,|g_j\rangle \tag{2.1.2}$$

One can see such a n-by-n matrix representation $D(g)$ constructed this way satisfies the group multiplication rule by the linear matrix natural multiplication,

$$[D(g_a)D(g_b)]_{ik} = \sum_j D(g_a)_{ij}D(g_b)_{jk} = \sum_j \langle g_i|\, D(g_a)\,|g_j\rangle \langle g_j|\, D(g_b)\,|g_k\rangle \tag{2.1.3}$$

$$= \langle g_i|\, D(g_a)D(g_b)\,|g_k\rangle = \langle g_i|\, D(g_a)\,|g_b \cdot g_k\rangle = \langle g_i|g_a \cdot g_b \cdot g_k\rangle$$

$$= \langle g_i|(g_a \cdot g_b) \cdot g_k\rangle = \langle g_i|\, D(g_a \cdot g_b)\,|g_k\rangle = D(g_a \cdot g_b)_{ik}. \tag{2.1.4}$$

[1] Here I directly quote the wording in Ref. [Geo99].

In the second line, I used the fact that $\sum_k |g_k\rangle \langle g_k|$ is complete and in the third line the associativity of group G. Unitary representations ($D(g)^\dagger = D(g)$) are particularly important, and all representations of finite groups are equivalent to unitary representations.

This regular representation can form a new **equivalent representation**, by a similarity transformation S,

$$D'(g) = S^{-1} D(g) S, \tag{2.1.5}$$

and this new representation satisfies all group transformation rules.

An important concept in representation is its reducibility. A representation is reducible if it has an **invariant subspace**, that the multiplication of $D(g)$ on any of the elements in the subspace remains in the subspace. This will allow us to decompose a representation into a "sum" of irreducible representations. As the representations of the SM elementary fermions are found to live in irreducible representations called fundamental representations, and the gauge boson lives in the adjoint representations. We do not delve much further here into the representation theory. But if you work on model building, I highly recommend you to read more of Ref. [Geo99].

Q & As

Q: There is a jump between the group multiplication rule, $g_1 \cdot g_2$, to the claim that I can represent the group by matrices following natural multiplication in the linear space. Do representations exist for all groups?

A: Yes. See the discussion about the construction of a regular representation.

Q: How are we sure that we are able to align the group elements to form an orthonormal basis (also using the braket notation)?

A: One can declare rules of the basis by treating each group element as a label and define a new inner product rule that only the group element squared in the bracket notation yields one; otherwise, it yields zero. In other words, Kronecker Delta. This shall be separated from the group multiplication rules.

Back to one of the simplest group $\mathbb{Z}_2$. The group shall be more generally described as two elements g_1 and g_2, which satisfy $g_1 = I$ and $g_1 \cdot g_2 = g_2$, $g_2 \cdot g_2 = g_1$. One can then decide how to represent this group by various representations. In our first encounter of $\mathbb{Z}_2$, we've already chosen a one-dimensional representation $D(g_1) = 1$, $D(g_2) = -1$. One can also have a trivial representation of $D(g_1) = D(g_2) = 1$, or an identity matrix of arbitrary dimension. We can also construct the regular representation using the procedure mentioned above; we have a table of multiplication mapping

$$\begin{pmatrix} g_1 \cdot g_1 & g_1 \cdot g_2 \\ g_2 \cdot g_1 & g_2 \cdot g_2 \end{pmatrix} = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix}. \tag{2.1.6}$$

Then, the regular representation is

$$D(g_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, D(g_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.1.7}$$

and it satisfies

$$D(g_1)D(g_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = D(g_1 \cdot g_1) = D(g_1), \tag{2.1.8}$$

$$D(g_2)D(g_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = D(g_2 \cdot g_1) = D(g_2), \tag{2.1.9}$$

$$D(g_2)D(g_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = D(g_2 \cdot g_2) = D(g_1)... \tag{2.1.10}$$

We can take a peek at how the simplest discrete symmetry works in the SM. For instance, in the SM, we have a positively charged pion and a negatively charged pion, represented by $|\pi^{\pm}\rangle$. Defining a charged conjugation operator $\hat{C}$ such that $\hat{C}\,|\pi^{\pm}\rangle = |\pi^{\mp}\rangle$. In the basis of the two states $|\pi^{+}\rangle, |\pi^{-}\rangle$, $\hat{C}$ is simply $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Together with the identity matrix I, $\{I, \hat{C}\}$ is a representation of the $\mathbb{Z}_2$ group. If the commutator $[\hat{C}, \hat{H}] = 0$, then the Hamiltonian is invariant under this group, which means the system respects this symmetry. This means many of the properties of π^{+} are the same as that of π^{-}. One can also construct a linear combination of pions to be the eigenstate of $\hat{C}$ with eigenvalues ± 1 for $1/\sqrt{2}(|\pi^{+}\rangle \pm |\pi^{-}\rangle)$. As $\hat{C}^2 = I$, the eigenvalue can only be ± 1. Since the only non-trivial "operation" of $\mathbb{Z}_2$ is $\hat{C}$ (or g_2 that has $g_2 \cdot g_2 = g_1 = I$), we can call it a parity. One can claim these two eigenstate charge conjugations are even/odd. There are many other parity/conjugation symmetries in the SM.

2.2 Lie Algebra

Now, we turn to a somewhat more complex class of group, which is extremely powerful in helping us understand quantum field theory and particle physics, the **continuous group**. One can imagine a successful reapplication of all the previous tricks in the discrete group in the previous section, but you can immediately see some inconvenience that the linear vector space spanned by the group elements where I want to construct regular representation is infinite, and hence, the representations are infinite-dimensional. The continuity comes to the rescue.

Given that the group is continuous, we can claim it smoothly depends on some continuous parameter α, which itself can be multi-dimensional (we call it **degrees**). This implies the group elements can span in a number of independent directions.

$$g \rightarrow g(\alpha). \tag{2.2.1}$$

Because α now labeled the group elements, we can also use it to label the elements in a representation $D(\alpha)$. As a group, identity exist; we demand that identity group in a desirable (which we are constructing, or assuming the existence for now) representation is an identity (matrix) and the smooth parameters α taking value zero is the identity,

$$g(0) = I, \tag{2.2.2}$$

and hence for a representation $D(\alpha)$,

$$D(\alpha)|_{\alpha=0} = \mathbb{1}. \tag{2.2.3}$$

Here $\mathbb{1}$ is a n-by-n (the dimension of this representation $D(\alpha)$) identity matrix. Now, due to smoothness and continuity, we can reach group elements near the identity by a small "distance" $d\alpha$ via the expansion,

$$D(d\alpha) = \mathbb{1} + id\alpha_a X_a + ... \tag{2.2.4}$$

$$X_a \equiv -i\frac{\partial}{\partial \alpha_a} D(\alpha)|_{\alpha=0} \tag{2.2.5}$$

Here (and hereafter, unless otherwise specified), we use the "Einstein Summation" that all repeated indices are summed over. You can see a spans over all the degrees where identity is continuously connected, and X_a determines "how" the group elements span away from identity in each degree. X_as are called **generators** of the group. For the α parametrization that is sufficiently unique (i.e., being parsimonious, that one needs to specify all α_as to specify a group element), X_a will be independent. Again, you can already get a sense that a represents independent directions, the group element can span near the identity element, and X_a specifies how it spans. We include i here for general convenience; if the representation $D(\alpha)$ is unitary, the generator X_a will be Hermition and vice versa.

How about when we are somewhat far away from identity? And how about very far away from identity? We hope to arrive at some representation that is consistent and nice, which, hopefully, will also have some nice smoothness in the expansion as well. A natural choice is to assume the representation follows the exponential parametrization of the continuous group. One can reach group elements further away from identity by repeated small excursions,

$$D(d\alpha) = \mathbb{1} + i\ d\alpha_a X_a, \text{ such that} \tag{2.2.6}$$

$$D(\alpha) = \lim_{k\to\infty}\left(1 + i\frac{\alpha_a}{k}X_a\right)^k = \exp\left[i\alpha_a X_a\right]. \tag{2.2.7}$$

One can see that the generators here obey natural addition and multiplications, which are in the linear vector space.

We can immediately find usage of this representation in our familiar physics case. For simple free parameter wavefunction (let's not worry about normalization as we can either wave-packet it or normalize over infinity with proper counting), we have,

$$\langle x|p\rangle = e^{ipx} \tag{2.2.8}$$

Under the translation of $x \to x - a$, the coordinate system is shifted by $+a$. One can define a unitary operator (which is a group element of the translation group) in this representation,

$$\hat{U}(a) = e^{-ia\hat{P}} \tag{2.2.9}$$

and we have

$$\langle x|\hat{U}(a)|p\rangle = \langle x|e^{-ia\hat{P}}|p\rangle = \langle x|e^{-iap}|p\rangle = e^{ip(x-a)}. \tag{2.2.10}$$

Clearly, the translation a is the continuous parameter that the group elements smoothly (exponentially) depend on, and the momentum operator $\hat{P}$ is the generator of the spatial translation group.

In fact, for three spatial dimensions, there are three independent translation directions. Their generators commute with each other.

$$[\hat{P}_i, \hat{P}_j] = 0, \tag{2.2.11}$$

implying that

$$\hat{U}(a)\hat{U}(b) = \hat{U}(a+b), \tag{2.2.12}$$

satisfying simple addition rules on consecutive translations. The momentum operator $\hat{P}$ is Hermitian. Hence, the translation operator is indeed unitary. In the relativistic case, the four-dimensional energy momentum operator is similar.

How can we use it? Suppose a system with Hamiltonian $\hat{H}$, satisfying

$$[\hat{P}, \hat{H}] = 0, \tag{2.2.13}$$

we know the linear momentum is conserved. On the other hand, it is completely equivalent to say, in this case,

$$[\hat{U}(a), \hat{H}] = 0, \tag{2.2.14}$$

and the system is translational invariant. In fact, the commutation of the generator with Hamiltonian directly means a corresponding symmetry of the system exists. The system shall be invariant under the symmetry transformation. The above fact is closely tied to the Noether theorem, which we will revisit later in this note.

But life is not always as simple as this above case. As one is fully entitled to, when spanning further away from identity, higher order terms should enter, and the parameter dependence might not be as (conveniently) simple as $D(\alpha)D(\beta) = D(\alpha + \beta)$ for two consecutive operations. Let's examine it. In general, without loss of generality, we can say

$$D(\alpha)D(\beta) = D(\gamma), \tag{2.2.15}$$

that consecutive group multiplication gives me another group member. Then, we work out small operations first,

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\gamma_c X_c}, \tag{2.2.16}$$

$$i\gamma_c X_c = \ln\left[\mathbb{1} + e^{i\alpha_a X_a} e^{i\beta_b X_b} - \mathbb{1}\right], \tag{2.2.17}$$

$$\tag{2.2.18}$$

Denoting the $K \equiv e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1$, we have,

$$K = (\mathbb{1} + i\alpha_a X_a - \frac{1}{2}(\alpha_a X_a)^2 + ...)(\mathbb{1} + i\beta_b X_b - \frac{1}{2}(\beta_b X_b)^2 + ...) - \mathbb{1} \tag{2.2.19}$$

$$= i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}(\alpha_a X_a)^2 - \frac{1}{2}(\beta_b X_b)^2 - \alpha_a X_a \beta_b X_b + ... \tag{2.2.20}$$

On the other hand,

$$\ln\left[\mathbb{1} + K\right] = K - \frac{1}{2}K^2 + \frac{1}{3}K^3 ... \tag{2.2.21}$$

For infinitesimal operation, $K \simeq 0$, we can keep track of the leading terms. In particular, counting α and β in the same order and tracking up to the second order involves keeping track of the leading terms of $-\frac{1}{2}K^2$ term.

$$i\gamma_c X_c = i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}(\alpha_a X_a)^2 - \frac{1}{2}(\beta_b X_b)^2 - \alpha_a X_a \beta_b X_b - \frac{1}{2}(i\alpha_a X_a + i\beta_b X_b)^2 + ...$$

$$= i(\alpha + \beta)_a X_a - \frac{1}{2}[\alpha_a X_a, \beta_b X_b] + ... \tag{2.2.22}$$

$$= i(\alpha + \beta)_a X_a - \frac{1}{2}\alpha_a \beta_b [X_a, X_b] + ... \tag{2.2.23}$$

These somewhat non-surprising results (we already learned the exponential operator identities in quantum mechanics) have very nice features. First, one can see that if the generators commute with each other $[X_a, X_b] = 0$, then $\gamma = \alpha + \beta$, we get back to the simplest product rule. Second, since we are discussing the general arbitrary excursions from identities labeled by α and β, the key features of the group structure are encoded independently of them. They are embedded in the commutator of generators,

$$[X_a, X_b] = i f_{abc} X_c. \tag{2.2.24}$$

This is the **Lie Algebra** of the group. f_{abc} encodes the full information of the groups and is called **structure constant**. Clearly, $f_{abc} = -f_{bac}$. It captures the full information in the following sense: when we do excursions from the identity via the exponential parameters, it tells us what additional modifications we need to include to be consistent with the group multiplication. One might worry about higher order terms, but they are explicitly constructible as higher order commutators (non-commutator part cancels) are just repeated usage of f_{abc}. Once the group is specified, we know the structure constant. The rest is actually to study different representations (all in the exponential parameterization as done here), equivalently different sets of X_a, that are consistent with the Lie algebra.

Before I give the first example. I want to mention two other useful results. One can show or encounter Jacobi Identity in various forms, e.g.,

$$[X_a, [X_b, X_c]] + \text{cyclic permutation} = 0. \tag{2.2.25}$$

Implementing the structure constant, Jacobi Identity means,

$$f_{ade}f_{dbc} + f_{bde}f_{dca} + f_{cde}f_{dab} = 0. \tag{2.2.26}$$

One can further show that the structure constants themselves, as generators, form a representation of the group,

$$(A_a)_{bc} = -if_{abc} \tag{2.2.27}$$

automatically satisfies

$$[A_a, A_b] = if_{abc}A_c. \tag{2.2.28}$$

This representation is called **adjoint representation**. It is an n-by-n dimensional representation of the group, and n is the degree of the group.

Let's begin with one simple example: for the structure constant f_{abc} of ϵ_{ijk}, where $i, j, k \in 1, 2, 3$, what generators can satisfy this and what physical system can we use it for? They are the structure constants for the rotation! ϵ_{ijk} is the fully antisymmetric tensor, with $\epsilon_{123} = 1$.

One can construct an adjoint representation directly (let me call it J_i, and $(J_i)_{jk} = -i\epsilon_{ijk}$, such that),

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.2.29}$$

These are generators for rotations around three axes in the linear vector space by $(\hat{x}, \hat{y}, \hat{k})$, and the smooth parameters of these three rotations are the three Euler angles. These are the generators of the fundamental representation of the $SO(3)$ group, a three-degree special orthogonal group.

Certainly, we can ask if a two-dimensional irreducible representation exists. The answer is yes, and we know that well, the Pauli matrices. So, a two-dimensional representation of the rotational group can be written as,

$$J_1 = \frac{\sigma_1}{2} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, J_2 = \frac{\sigma_2}{2} = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, J_3 = \frac{\sigma_3}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2.30}$$

They also satisfy the Lie algebra with the structure constant ϵ_{ijk}. Once again, we are already familiar with this in quantum mechanics. These are the spin operators for spin-1/2 fermions, which obey the same commutation relations of the angular momentum operators and are part of the total angular momentum. From the group theory point of view, these are the generators in the fundamental representation of $SU(2)$, the two-degree special unitary group. One can see these generators in different representations are closely related. In fact, $SO(3)$ is isomorphic to $SU(2)/\{I, -I\}$[2], which means there is one-to-one mapping between these two.

What about one-dimensional representation? One can find the only way to satisfy the Lie algebra is the trivial representation, where all group elements are represented by identity (here in 1-D, just the number 1). The generators are nothing but

$$J_1 = J_2 = J_3 = 0. \tag{2.2.31}$$

What's so useful about this knowledge? In fact, most directly, we understand how to write down the spin projection operators for different systems. For a spin-0 particle, it is a spin singlet, and it transforms in the 1-d representation under rotation. For a spin-1/2 particle, it is a spin doublet (two-level system of s_i eigenvalue $\pm 1/2$), and it transforms in the 2-d representation under rotation. For a spin-1 particle, we call it a vector, and it is a spin triplet (three-level system of s_i eigenvalue of $0, \pm 1$. We can understand and show the symmetries and conserved currents when expressing the system in such basis and representations. Without any surprise, one can develop a method to write down a $(2s + 1)$ by $(2s + 1)$ representation for a spin-s system. Note that this is applying the representation to one of the quantum properties of a state, which we call spin. We can also *separately* apply these generators and exponential representations to other properties of a state, such as its orbital angular momentum. While the generators can be of the same form, they act on different subspaces of the Hilbert space of the wave function.

A curious reader may ask what happens to a possible generator of the identity matrix. In the exponential representation, the trial generator is no longer the identity matrix but zero. The exponential of zero is identity, which is a trivial representation. The inclusion of the identity matrix in the generator will actually change the group from special to non-special, so $SU(2)$ becomes $U(2)$ and $SO(3)$ becomes $O(3)$.

While one can see that $SU(2)$ and $SO(3)$ have generators follow the angular momentum operator rule, $SU(N)$ and $SO(N)$ have other usages. For instance, $SO(N)$ group is a set of continuous (real) linear transformations that leaves the inner product of N real numbers invariant. The group elements in the fundamental, irreducible representation would

[2] a quotient group of $SU(2)$

be a real N by N matrix D,

$$\begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_N \end{pmatrix} \to \begin{pmatrix} x'_1 \\ x'_2 \\ \cdots \\ x'_N \end{pmatrix} = D \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_N, \end{pmatrix} \tag{2.2.32}$$

The invariance of the inner product means

$$\vec{x} \cdot \vec{x} = \vec{x}' \cdot \vec{x}', \tag{2.2.33}$$

which requires that

$$D^T . D = \mathbb{1}. \tag{2.2.34}$$

Here, you can see why they are called orthogonal groups. The new vectors x'_i are orthogonal to each other if the original basis is orthonormal. For $SO(N)$, there are N^2 real parameters in the matrices, and the above condition removes N parameters (from equating the diagonal elements r.h.s.) and $N(N-1)/2$ parameters (from equating the off-diagonal elements of the r.h.s., and it is symmetric). Hence, the total number of free parameters is $N^2 - N - N(N-1)/2 = N(N-1)/2$. This is the degree of this group that a linear transformation leaves the inner product invariant, and hence the number of generators. If we apply this to $SO(3)$, we know there should be $3 \times (3-1)/2 = 3$ generators. If we, instead, only require the inner product to multiply proportional to the original inner product for arbitrary vectors, one boundary condition is removed, and hence, there will be one more generator, and that would be the identity matrix, which changes the length of the inner product.

Similarly, for $SU(N)$, the linear mapping keeps the inner product of $\vec{x}^* \cdot \vec{x}$ invariant. It would be $2N^2$ degree of freedom, and after subtracting the conditions from $D^\dagger D = \mathbb{1}$ [3] one gets $N(N+1)/2$ degrees. For the case of $SU(2)$, one gets $2 \times (2+1)/2$ generators.

2.3 Poincaré Group and Lorentz Group

Poincaré group is a semi-direct product of the translation group and the Lorentz group. Its importance in relativistic physics is self-evident. However, it is not a simple group, and we will have to deal with some richer structures and features than the abovementioned groups. The Lorentz group is a transformation that preserves distance with a metric

[3] $D^\dagger D$ being real removes $(N^2 - N)$ d.o.f., as the diagonal entries are automatically diagonal. Then, the rest of the counting is the same as the $SO(N)$ case.

containing opposite signs between the distance in time and the distance in space.

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu, \text{such that } x^2 \equiv x \cdot x \equiv g_{\mu\nu} x^\mu x^\nu = x'^{\,2} \tag{2.3.1}$$

Here $g_{\mu\nu}$ is the metric (in particle physics, most of us take the most-minus metric)

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2.3.2}$$

and the covariant form $g^{\mu\nu}$ is a matrix of exactly the same form. We can use them to raise and lower Lorentz indices.

The Poincaré group further is translational invariance, which preserves separations between points.

$$\begin{aligned} x^\mu \to & \quad x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu, && (2.3.3) \\ y^\mu \to & \quad y'^\mu = \Lambda^\mu_\nu y^\nu + a^\mu, \text{such that} && (2.3.4) \\ & \quad (x-y)^2 = (x'-y')^2 && (2.3.5) \end{aligned}$$

One and perform two Poincaré transformations consecutively, and it should be equal to one single Poincaré transformation with some parametric correspondence. We have

$$\begin{aligned} & x^\mu \to x'^{\,\mu} \to x''^\mu && (2.3.6) \\ x''^\mu &= \Lambda''^\mu{}_\nu x'^\nu + a'^\mu \\ &= \Lambda'^\mu{}_\nu (\Lambda^\nu{}_\rho x^\rho + a^\nu) + a'^\mu \\ &= (\Lambda'\Lambda)^\mu{}_\rho x^\rho + (\Lambda^\mu{}_\nu a^\nu + a'^\mu), && (2.3.7) \end{aligned}$$

implying

$$x''^\mu = \Lambda''^\mu{}_\nu x^\nu + a''^\mu, \tag{2.3.8}$$

with

$$\Lambda'' = \Lambda'\Lambda, \;\; a''^\mu = \Lambda^\mu{}_\nu a^\nu + a'^\mu. \tag{2.3.9}$$

You can see here why we say the Poincaré group is a semi-direct product group of the translation group and Lorentz group.

We are already somewhat familiar with the translation group and its generator from the previous section. Let's focus on the Lorentz part and then put them together.

The Lorentz part could be an $SO(4)$ group if the metric would be an identity matrix. However, due to the opposite sign in the space and time part in the metric, it becomes a non-compact group. For instance, we can elongate the spatial part arbitrarily so long that we enlarge the time part correspondingly; the separation is unchanged between points, hence satisfying the Poincaré invariance. It demands some more complex treatment that we discuss in this section.

Note that we are working in a representation of Lorentz and Poincaré group that we are most familiar with; that is how the 3+1-D coordinates transform under these symmetries. This representation is called the defining representation of these groups. What we want to do in this section is to understand the important properties of these symmetries in this representation and then derive the structure constants of these groups. This will allow us to start understanding different representations of these groups, and they play important roles in defining fundamental particles.

The Lorentz symmetry require $g_{\mu\nu}x^\mu x^\nu = g_{\mu\nu}x'^\mu x'^\nu$ and hence that

$$g_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = g_{\alpha\beta}, \tag{2.3.10}$$

and the identity group element is

$$\delta^\mu{}_\nu = \mathbb{1}_{4\times4}. \tag{2.3.11}$$

Now multiply Eq. (2.3.10) by $g^{\alpha\rho}$ and contract the indices that get raised and lowered, we get,

$$\Lambda_\nu{}^\rho\Lambda^\nu{}_\beta = \delta^\rho{}_\beta. \tag{2.3.12}$$

Noting that[4],

$$\Lambda_\nu{}^\rho = (\Lambda^T)^\rho{}_\nu = (\Lambda^{-1})^\rho{}_\nu, \tag{2.3.13}$$

and hence

$$\Lambda^T\Lambda = \mathbb{1}_{4\times4}, \tag{2.3.14}$$

implying

$$\det[\Lambda^T\Lambda] = \det[\Lambda^T]\det[\Lambda] = \det[\Lambda]^2 = 1,\ \det[\Lambda] = \pm1. \tag{2.3.15}$$

Clearly, there is an identity group element in this representation, $\delta^\mu{}_\nu$, and it has a determinant of $+1$. We call the subset of Lorentz transformation with determinant $+1$ *proper* and those with determinant -1 *improper*.

Furthermore, a subsect of Lorentz transformation has $\Lambda^0{}_0$ component greater or equal to unity[5], $\Lambda^0{}_0 \geq 1$, which preserves the time direction[6], are called *orthochronous*.

The Proper Orthochronous (P.O.) subset of Lorentz transformation forms a subgroup of Lorentz transformation; it satisfies all the group properties

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[4]here the notation is a bit confusing, but I don't have a better way. $\Lambda_\nu{}^\rho = (\Lambda^\rho{}_\nu)^T$ (Here I am transposing the matrix, not only the elements). One transposes Λ to get Λ^T, exchanging the corresponding labels.

[5]From Eq. (2.3.10), we have $(\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2 = 1$

[6]For Lorentz four vectors with time-like separation $x^2 \geq 0$, the positive time vectors will remain positive time for arbitrary Lorentz transformation satisfying $\Lambda^0{}_0 \geq 1$.

and, in addition, can reach every group element by successive infinitesimal transformations. The P.O. subgroup of the Lorentz group is very powerful and convenient for us since we can use Lie Algebra to fully describe it. It is denoted as $SO^{+}(1,3)$, and for most of the time of this lecture, when we refer to Lorentz symmetry and invariance, in particular when talking about generators, we meant this subgroup. This subgroup is also "central" in the sense that it is continuously connected to the identity group element $\delta^{\mu}{}_{\nu}$, as the identity is P.O. as well.

Before we dive into the representations. It is useful to comment on the other parts of the Lorentz group. They are connected to the P.O. subgroup by two meaningful operators (and their product), which are the time-reversal operator $\mathcal{T}$ and the parity operator $\mathcal{P}$,

$$\mathcal{T}^{\mu}{}_{\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \mathcal{P}^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2.3.16}$$

be mindful of the Lorentz indices. Clearly, one can reach all four domains of the Lorentz transformation by applying $\mathbb{1}$, $\mathcal{T}$, $\mathcal{P}$ and $\mathcal{PT}$ on the P.O. subgroup.

Now, let's talk about the exponential representation of the P.O. subgroup of the Lorentz group in its defining representation. For short, I will call this a representation of the Lorentz group, instead of calling it a representation of the P.O. subgroup of the Lorentz group.

We can play the game of infinitesimal transformations, and let's express them (later on, I will add translation),

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \alpha^{\mu}{}_{\nu}. \tag{2.3.17}$$

Then, given Eq. (2.3.12), we have

$$(\delta_{\nu}{}^{\alpha} + \alpha_{\nu}{}^{\alpha} + ...)(\delta^{\nu}{}_{\beta} + \alpha^{\nu}{}_{\beta} + ...) = \delta^{\alpha}{}_{\beta}, \tag{2.3.18}$$

and tracking to order α (the continuous parameter that a representation of the Lorentz group smoothly depends on), we get

$$\alpha_{\beta}{}^{\alpha} + \alpha^{\alpha}{}_{\beta} = 0. \tag{2.3.19}$$

Hence, this excursion parameter is antisymmetric,

$$\alpha_{\alpha\beta} = -\alpha_{\beta\alpha}. \tag{2.3.20}$$

Q & As

Q: (Chimeln) How many generators does the Lorentz group have?
Q: (Chimeln) How many generators does the Poincaré group have?
Q: (Chimeln) Is this representation unitary? Are all the generators in this representation hermitian?
Q: Eq. (2.3.17) is a Taylor expansion near the identity; how do I ensure there are no new directions (d.o.f.) when I have a large expansion?
A: By assuming the existence of exponential representation and the continuity as well as smoothness of the dependence, I can cover all the group elements smoothly connected to identity by repeated infinitesimal transformations. In this sense, for this fully connected group space, there are no new directions. In other words, the emergence of a new d.o.f., which is linearly independent from the original ones, means discontinuity in the group element. However, for the full Lorentz group, there are large expansions, discontinuously connected to the P.O. subgroup, by the group elements of $\mathcal{T}$, $\mathcal{P}$, and $\mathcal{PT}$.

Since it is real-valued, as an anti-symmetry 4-by-4 number, there are six d.o.f, and hence six generators $M^{\mu\nu}$, who are also anti-symmetric. We already know what these six d.o.f. are: three rotations and three boosts. Of course, one can also combine these directions linearly into new directions, but it is merely a linear transformation of the generators and expansion parameters.

We can denote the Lorentz generators as $\hat{M}^{\mu\nu}$, such that,

$$\hat{U}(\alpha) = e^{\frac{i}{2}\alpha_{\mu\nu}\hat{M}^{\mu\nu}}, \tag{2.3.21}$$

For an appropriate choice of generators, one can identify the three rotations and three boosts as

$$\hat{K}_i = \hat{M}_{i0} \tag{2.3.22}$$

$$\hat{J}_i = -\frac{1}{2}\epsilon_{ijk}\hat{M}^{jk}. \tag{2.3.23}$$

Correspondingly, the three boost parameters α_{0i} are the rapidity in each direction, and the three rotation parameters $n_i \equiv \epsilon_{ijk}\alpha_{jk}$ are the three Euler angles.

We already know the translation and its generator from Eq. (2.2.9). Hence, we can write down a representation (the defining representation in 3+1D, in the exponential form, for the P.O. subgroup) for the Poincaré transformation,

$$\hat{U}(a,\Lambda) = \hat{U}(a)\hat{U}(\Lambda). \tag{2.3.24}$$

Note *the order matters*; as usual, we apply the transformations from the rightmost operator to the left. There is no ambiguity in this notation that,

$$\begin{aligned}\hat{U}(a) &= \hat{U}(a,\mathbb{1}),\\ \hat{U}(\Lambda) &= \hat{U}(0,\Lambda).\end{aligned} \tag{2.3.25}$$

In the above equations, we did a small "trick" that we replaced the expansion parameter from $\alpha^{\mu}{}_{\nu}$ to $\Lambda^{\mu}{}_{\nu}$, and there shouldn't be any ambiguities as there is one (set of parameters) to one correspondence between α and Λ for P.O. subset of Lorentz transformations. We are simply too used to the Lorentz transformations in their defining representation, and the space-time transformation is quite a rudimentary building block in particle physics.

We know exactly how Poincaré transformation should be from Eqs. (2.3.6) to (2.3.9), and hence we can derive the commutation relations, the Lie Algebra, between the generators.

$$\hat{U}(a)\hat{U}(a') = \hat{U}(a + a') \tag{2.3.26}$$

$$\hat{U}^{-1}(\Lambda)\hat{U}(\Lambda')\hat{U}(\Lambda) = \hat{U}(\Lambda^{-1}\Lambda'\Lambda) \tag{2.3.27}$$

$$\hat{U}(\Lambda)\hat{U}(a) = \hat{U}(\Lambda a)\hat{U}(\Lambda) \tag{2.3.28}$$

Expanding the above with infinitesimal transformations labeled by a and $\alpha_{\mu\nu}$, with their corresponding generators $\hat{P}$ and $\hat{M}^{\mu\nu}$, at linear order on the expansion parameters, allows us to find the consistent commutation relations.

We can use Eq. (2.3.26) to find that,

$$[\hat{P}_{\mu}, \hat{P}_{\nu}] = 0. \tag{2.3.29}$$

We have already done this from a different angle in the earlier section in Eqs. (2.2.11) and (2.2.12).

We can use Eq. (2.3.27) to find that[7],

$$[\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}] = i\left(\hat{M}^{\mu\rho}g^{\nu\sigma} + \hat{M}^{\nu\sigma}g^{\mu\rho} - \hat{M}^{\mu\sigma}g^{\nu\rho} - \hat{M}^{\nu\rho}g^{\mu\sigma}\right). \tag{2.3.30}$$

So far, the above contains the commutator between generators within translation and Lorentz. What about the commutator between the translation generators and Lorentz generators? We can use Eq. (2.3.28). Here, I will take a slower approach and show it step by step to set an example. In fact, I will derive Eq. (2.3.28) first from the defining property in Eq. (2.3.7) and then derive the commutation relations between $\hat{P}$ and $\hat{M}$.

[7] Note that the argument of the right-hand side $\Lambda^{-1}\Lambda'\Lambda$ is still a ground member of the Lorentz Group in the defining representation. Hence, it can be written as $\Lambda'' \equiv \Lambda^{-1}\Lambda'\Lambda = \delta + \alpha''$ and hence the exponential form in $\hat{U}(\Lambda'')$.

$$\hat{U}(a', \Lambda')\hat{U}(a, \Lambda) = \hat{U}(\Lambda' a + a', \Lambda'\Lambda) \tag{2.3.31}$$

$$\begin{aligned}\hat{U}(a')\hat{U}(\Lambda')\hat{U}(a)\hat{U}(\Lambda) &= \hat{U}(\Lambda' a + a')U(\Lambda'\Lambda)\\ &= \hat{U}(a')\hat{U}(\Lambda' a)\hat{U}(\Lambda')\hat{U}(\Lambda)\end{aligned} \tag{2.3.32}$$

$$\hat{U}(\Lambda')\hat{U}(a) = \hat{U}(\Lambda' a)\hat{U}(\Lambda') \tag{2.3.33}$$

$$\hat{U}(\Lambda)\hat{U}(a)\hat{U}^{-1}(\Lambda) = \hat{U}(\Lambda a), \tag{2.3.34}$$

We derive Eq. (2.3.28) here in Eq. (2.3.33) after left-multiplying $\hat{U}^{-1}(a')$ and right-multiplying $\hat{U}^{-1}(\Lambda)$ on both side of the equation. Since Λ no longer appears in Eq. (2.3.33) and below (arrived by right-multiplying it with $\hat{U}^{-1}(\Lambda')$), we can unprime Λ' and rewrite it as Λ.

Now, we can expand according to the plan and the expansion parameters[8],

$$\hat{U}(\Lambda)e^{i\hat{P}^{\alpha}a_{\alpha}}\hat{U}(\Lambda^{-1}) = e^{i\hat{P}^{\mu}\Lambda_{\mu}{}^{\alpha}a_{\alpha}} \tag{2.3.35}$$

$$\begin{aligned}(1+\frac{i}{2}\alpha_{\mu\nu}\hat{M}^{\mu\nu}+...)\hat{P}^{\alpha}(1-\frac{i}{2}\alpha_{\mu\nu}\hat{M}^{\mu\nu}+...) &= \hat{P}^{\mu}\Lambda_{\mu}{}^{\alpha} = \hat{P}^{\mu}(\delta_{\mu}{}^{\alpha}+\alpha_{\mu}{}^{\alpha})\\ \frac{i}{2}\alpha_{\mu\nu}[\hat{M}^{\mu\nu},\hat{P}^{\alpha}] &= \frac{1}{2}(\hat{P}^{\mu}\alpha_{\mu\nu}g^{\nu\alpha}-\hat{P}^{\nu}\alpha_{\mu\nu}g^{\mu\alpha})\\ [\hat{M}^{\mu\nu},\hat{P}^{\alpha}] &= -i(\hat{P}^{\mu}g^{\nu\alpha}-\hat{P}^{\nu}g^{\mu\alpha}).\end{aligned} \tag{2.3.36}$$

From the first line to the second line, we match the leading non-trivial term in a_{α} (it is a_{α} to the first order). From the second line to the third line, we match the leading non-trivial term in α (it is α to the first order). We use $g^{\nu\alpha}$ to lower the α index to match $\alpha_{\mu\nu}$ on both sides and get rid of it in the last line. We relabel the dummy indices and split the r.h.s. into two terms to make it an anti-symmetric form, which is merely an aesthetic choice.

The commutation relation between the generator of translation and Lorentz transformation, shown in Eq. (2.3.36), shows that these two transformations are "coupled", which are not simple ones that commute and can be exchanged. One sees again the semi-direct product nature of the Poincaré group from the Translation group and the Lorentz group.

Having understood the above, the first direct application of this space-time symmetry is how we define a one-particle state. One-particle state is the building block for particle phenomenology, and there are a lot to discuss in terms of interaction picture, canonical normalization, cluster decomposition, etc. For now, we simply want to have a clear definition of isolated pristine one-particle state that we can prepare and later

[8]Note that I can always relabel dummy indices that are contacted over, so I choose them "wisely"

experiment with. As already done in quantum mechanics, we want to find a maximally compatible set of operators such that we can define the wavefunction uniquely as a set of eigenstates of these operators that spans over the whole Hilbert space. A natural choice is to use the momentum operator, which in most cases is observable, and further, in high energy limit, the particle-wave duality does tell us the particle description provides a simpler calculation basis. Clearly, the Lorentz invariant operator $\hat{P}_\mu \hat{P}^\mu$ also commutes with $\hat{P}^\mu$, and this is the mass squared operator $\hat{M}^2$. How about Lorentz generators $M^{\mu\nu}$? Unfortunately, they do not commute with $\hat{P}^\mu$, as shown in Eq. (2.3.36). We will show in the next section that a subset of $\hat{M}^{\mu\nu}$ commutes with $\hat{P}^\mu$ and $\hat{M}^2$, and they are identified as the spin quantum numbers for a point-like one-particle state. So, the operator set is

$$\{\hat{P}^\mu, \hat{M}^2, \text{subsect of } \hat{M}^{\mu\nu} \text{ that communtes}\} \tag{2.3.37}$$

and we are able to specify the one-particle state by the eigenvalues,

$$|p, m, \text{some additional space-time quantum numbers}\rangle\,. \tag{2.3.38}$$

In the next section, we see how we can specify these additional space-time quantum numbers.

For now, the one-particle state have some degeneracy but we can still understand their behaviors, e.g.,

$$\begin{aligned}
\hat{P}^\mu |p, m\rangle &= p^\mu |p, m\rangle \\
\hat{U}(a) |p, m\rangle &= e^{i\hat{P}^\mu a_\mu} |p, m\rangle \\
\hat{M}^2 |p, m\rangle &= m^2 |p, m\rangle
\end{aligned} \tag{2.3.39}$$

What about $\hat{U}(\Lambda)|p, m\rangle$? From Eq. (2.3.35), write down the term linear in $\hat{P}$, and replace Λ^{-1} by Λ, we find

$$\begin{aligned}
\hat{U}(\Lambda^-1)\hat{P}^\mu \hat{U}(\Lambda) &= \Lambda^\mu{}_\nu \hat{P}^\nu \\
\hat{P}^\mu \hat{U}(\Lambda) &= \hat{U}(\Lambda)\Lambda^\mu{}_\nu \hat{P}^\nu \\
\hat{P}^\mu \left(\hat{U}(\Lambda)|p, m\rangle\right) &= \hat{U}(\Lambda)\left(\Lambda^\mu{}_\nu \hat{P}^\nu |p, m\rangle\right) \\
&= \Lambda^\mu{}_\nu p^\nu \left(\hat{U}(\Lambda)|p, m\rangle\right)
\end{aligned} \tag{2.3.40}$$

Hence $\left(\hat{U}(\Lambda)|p, m\rangle\right)$ is an eigenstate of $\hat{P}^\mu$ with eigenvalue of $\Lambda^\mu{}_\nu p^\nu$. Consequently, we conclude,

$$\hat{U}(\Lambda)|p, m\rangle = |\Lambda p, m\rangle\,. \tag{2.3.41}$$

However, as I declared that there are degeneracies (more non-trivial compatible operators with $\hat{P}^\mu$ exist from the Poincaré group, the resulting one-particle state not merely changed its momentum. We will get the fuller result in the next section.

2.4 Little Group, Wigner Rotation, Spin, and Helicity

Instead of directly informing the reader of the representations and corresponding constructions of the Poincaré group for particle physics, here I show how to get general properties of the representation. In particular, we introduce the idea of Wigner's Little Group transformation and show how they can be utilized to specify the one-particle states. Another reason for me to introduce it here is that Little Group does buy us additional handles in particle physics, in particular the scattering processes, and in the modern helicity amplitude program.[9]

From the previous section, we know the key is to identify a subset of Lorentz generators $\hat{M}^{\mu\nu}$ that commutes with the translation generator $\hat{P}^\mu$ (and hence it automatically commutes with $\hat{M}^2$). We know Lorentz boost would change the momentum of a state, and hence, it would not commute with the momentum operator, which is the generator for translation. Would the answer be simply a rotation? The answer is no. Physically, the result of translating and then rotating would be different if one rotates and then translates (unless the translation direction is the same as the spatial rotation vector direction or the translation is in the time direction). Specifically,

$$[\hat{J}_k, \hat{P}^\mu] = -\frac{1}{2}\epsilon_{ijk}[\hat{M}_{ij}, \hat{P}^\mu] = i\frac{1}{2}\epsilon_{ijk}(\hat{P}^i g^{j\mu} - \hat{P}^j g^{i\mu}) = \begin{cases} 0 & \text{if } \mu = 0, k \\ i\epsilon_{ijk}\hat{P}^i g^{jj} & \text{if } \mu = j, j! = 0, k \end{cases} \tag{2.4.1}$$

We can see that two rotation generators commute with the translation generator. They will reappear later in a slightly subtle context.

We will need to construct some other generators of the Poincaré group, which depends on the original generators. We understand that the additional operators we can use to specify the one-particle state cannot be boost, which does not commute with $\hat{P}$, and it cannot be rotation, which does not commute with $\hat{P}$ in general, as shown in Eq. (2.4.1).

We introduce the concept of the little group [Wig39; Kim16], which is a subgroup of the Lorentz group that leaves the momentum four-vector invariant. (Isn't this what we are asking for in Eq. (2.3.37)?) An easy way to understand and construct the generators of this subgroup would be to see their effects on momentum in their so-called standard form, which is $\tilde{p} = (m, 0, 0, 0)$ for massive particles and $\tilde{p} = (p, 0, 0, p)$ for massless particles. The resulting Lie algebra for these two cases will be related but somewhat

[9] For reviews of the modern helicity amplitude program and the usage of helicity weights, see, e.g., [Che18; EH13].

different.[10]

2.4.1 Massive case

Let's denote the little group, a subset of Lorentz transformations Λ, as $\tilde{\Lambda}$. We then have,

$$\tilde{\Lambda} p^\mu = p^\mu. \tag{2.4.2}$$

In the rest frame, $\tilde{p} = (m, 0, 0, 0)$. We then have,

$$\tilde{\Lambda}^\mu{}_\nu \tilde{p}^\nu = (\delta^\mu{}_\nu + \tilde{\alpha}^\mu{}_\nu)\tilde{p}^\nu = \tilde{p}^\nu, \tag{2.4.3}$$

implying

$$\begin{aligned} \tilde{\alpha}^\mu{}_\nu \tilde{p}^\nu &= \tilde{\alpha}_{\mu\nu}\tilde{p}_\nu = 0 \\ \tilde{\alpha}^{i0} &= -\tilde{\alpha}^{0i} = 0. \end{aligned} \tag{2.4.4}$$

From the above, we can see that three degrees are possible in the little group. We are their generators, and how do we write them down with only Lorentz covariant quantities? Clearly, the index of $\mu = 0$ is special in this case, and the only covariant way to pick it would be multiplying the $\hat{P}^\mu$ operator. We can define,

$$\tilde{\alpha}_{\mu\nu}\hat{M}^{\mu\nu} = \hat{M}^{\mu\nu}\tilde{\alpha}_{\mu\nu} = -\frac{1}{2}\hat{M}^{\mu\nu}\epsilon_{\mu\nu\rho\sigma}\hat{P}^\rho n^\sigma, \tag{2.4.5}$$

so there is a one-to-one mapping between the possible non-zero little group expansion directions $\tilde{\alpha}_{\mu\nu}$ and the parameter n^σ for particle at rest. When Lorentz transformation act on a massive particle at rest (rest frame, r.f.),

$$-\frac{1}{2}\hat{M}^{\mu\nu}\epsilon_{\mu\nu\rho\sigma}\hat{P}^\rho n^\sigma \overset{\text{r.f.}}{=} -\frac{1}{2}\hat{M}^{\mu\nu}\epsilon_{\mu\nu 0\sigma}n^\sigma m = -\frac{1}{2}\hat{M}^{ij}\epsilon_{ijk}n^k m = m\hat{J}_k n^k. \tag{2.4.6}$$

Interestingly, in the rest frame of a massive particle, the little group transformation is equivalent to rotation (the rotation generator is defined as in Eq. (2.3.23)). From the above, one can define a Lorentz covariant generalized rotation using a new operator constructed from Eq. (2.4.5), called the Pauli-Lubanski operator, which is the generator of this generalized rotation,

$$\hat{W}_\sigma \equiv -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\hat{M}^{\mu\nu}\hat{P}^\rho, \tag{2.4.7}$$

$$\hat{U}(\tilde{\Lambda})\left|\tilde{p}, m\right\rangle \equiv \left(1 + i\, n^\sigma \hat{W}_\sigma + \ldots\right)\left|\tilde{p}, m\right\rangle. \tag{2.4.8}$$

[10] For a more systematic discussion, see Refs. [Kim16; Wei05].

We have

$$\hat{U}(\tilde{\Lambda})\,|\tilde{p},m\rangle = |\tilde{\Lambda}\tilde{p},m\rangle = |\tilde{p},m\rangle\,, \tag{2.4.9}$$

$$\hat{W}_\sigma\,|\tilde{p},m\rangle = m\hat{J}_i\,|\tilde{p},m\rangle = ms_i\,|\tilde{p},m\rangle\,, \tag{2.4.10}$$

$$\hat{W}_\sigma\hat{W}^\sigma\,|\tilde{p},m\rangle = -m^2\hat{J}^2\,|\tilde{p},m\rangle = m^2 s(s+1)\,|\tilde{p},m\rangle\,, \tag{2.4.11}$$

where s and s_i are the eigenvalues of the $\hat{J}^2$ and $\hat{J}_i$ operators, from Eq. (2.4.6) and the fact $[\hat{J}_i\hat{J}_j] = i\epsilon_{ijk}\hat{J}_k$ (thus all the nice relations with raising and lowering operators and relations to $\hat{J}^2$).

Q & As

Q: (Chimeln) What is $\hat{W}_\sigma\hat{P}^\sigma$?
Q: (Chimeln) What is $[\hat{W}_\sigma,\hat{P}^\mu]$?

We can work out its commutation relations,

$$\hat{W}_\sigma\hat{P}^\sigma = 0, \tag{2.4.12}$$

$$[\hat{W}_\sigma,\hat{P}^\mu] = 0, \tag{2.4.13}$$

$$[\hat{M}_{\mu\nu},\hat{W}_\sigma] = -i\left(\hat{W}_\mu g_{\nu\sigma} - \hat{W}_\nu g_{\mu\sigma}\right). \tag{2.4.14}$$

The last line above is the same commutation relation as momentum operators $\hat{P}$, which is a Lorentz four-vector. It implies that $\hat{W}_\sigma$ transforms as a Lorentz four-vector. Now, we feel more confident to say $\hat{W}^\sigma$ is a generalization of the three rotation generators in a Lorentz covariant four-vector form. Its commutation relation is,

$$[\hat{W}_\mu,\hat{W}_\nu] = i\epsilon_{\mu\nu\rho\sigma}\hat{W}^\rho\hat{P}^\sigma. \tag{2.4.15}$$

Knowing $\hat{W}_\sigma$ is a Lorentz four-vector with three effective angles n^σ (one degree is mapped out by p^ρ), we can now declare and complete Eq. (2.3.37),

$$\{\hat{P}^\mu,\hat{M}^2,\frac{\hat{W}_\sigma\hat{W}^\sigma}{m^2},\frac{\hat{W}_\sigma}{m}\}. \tag{2.4.16}$$

Equivalently, we can choose to the set to be

$$\{\hat{P}^\mu,\hat{M}^2,\hat{J}^2,\hat{J}_i\}. \tag{2.4.17}$$

Note that $\hat{W}^\sigma\hat{W}_\sigma$ is a Lorentz invariant quantity, m^2 is a Lorentz invariant quantity. One might worry about the compatibility between $\hat{J}_i$ with $\hat{P}^\mu$, we can declare that we want to specify the quantum state in the rest frame of the particle, or we declare that we want to specify the eigenvalue of $\hat{W}_\sigma/m$. In fact, an even better definition, motivated by our observation in

Eq. (2.4.1), is to define the J_k along the momentum direction, and we have proven that they commute. Nevertheless, for massive particles, we can fully specify the state (according to its space-time symmetries) via

$$|p, m, s, s_i\rangle\,. \tag{2.4.18}$$

Now that we have fully specified the state (that there is no degeneracy in space-time symmetries), we can understand how such a state transforms in a general Lorentz transformation beyond those specified in Eq. (2.3.39). Before that, let's define a pure boost operator,

$$\begin{aligned}\hat{L}(p)^{\mu}{}_{\nu}\tilde{p}^{\nu} &= p^{\mu},\\ \hat{U}(L(p))\,|\tilde{p}, m\rangle &= |p, m\rangle\,,\end{aligned} \tag{2.4.19}$$

which transforms a massive state (momentum as a Lorentz four-vector) from its rest frame to a frame with momentum of p^{μ}.

We want to complete the results in Eq. (2.3.39),

$$\begin{aligned}\hat{U}(\Lambda)\,|p, m, s, s_i\rangle &= \text{(unknown coefficient)}\,|\Lambda p, m, s, s_i'\rangle\\ &= \hat{U}(\Lambda)\hat{U}(L(p))\,|\tilde{p}, m, s, s_i\rangle \qquad (2.4.20)\\ &= \hat{U}(L(\Lambda p))\hat{U}^{-1}(L(\Lambda p))\hat{U}(\Lambda)\hat{U}(L(p))\,|\tilde{p}, m, s, s_i\rangle\\ &= \hat{U}(L(\Lambda p))\hat{U}(L^{-1}(\Lambda p))\hat{U}(\Lambda)\hat{U}(L(p))\,|\tilde{p}, m, s, s_i\rangle \qquad (2.4.21)\end{aligned}$$

We understand the first (from right to left) three operators do not change the four-momentum at rest[11],

$$\begin{aligned}L^{-1}(\Lambda p)\Lambda L(p)\tilde{p} &= L^{-1}(\Lambda p)\Lambda p\\ &= L^{-1}(\Lambda p)(\Lambda p)\\ &= \tilde{p}.\end{aligned} \tag{2.4.22}$$

Hence, $L^{-1}(\Lambda p)\Lambda L(p)$ is a member of the little group, and we can define, in short, a **Winger Rotation**,

$$R(\Lambda p, p) \equiv L^{-1}(\Lambda p)\Lambda L(p). \tag{2.4.23}$$

We can work its impact on a state by inserting a complete set of states,

$$\begin{aligned}\hat{U}(R(\Lambda p, p))\,|\tilde{p}, m, s, s_i\rangle &= \sum_{s_i'} |\tilde{p}, m, s, s_i'\rangle\,\langle\tilde{p}, m, s, s_i'|\,\hat{U}(R(\Lambda p, p))\,|\tilde{p}, m, s, s_i\rangle\\ &= \sum_{s_i'} D^{(s)}_{s_i', s_i}(R(\Lambda p, p))\,|\tilde{p}, m, s, s_i'\rangle\,.\end{aligned} \tag{2.4.24}$$

[11] Here, we can work with the parameters of the operators directly in the defining representation.

Here $D^{(s)}_{s'_i,s_i}(R(\Lambda p, p))$ are Wigner rotation matrices in the spin s representation, with[12]

$$D^{(s)}_{s'_i,s_i}(R(\Lambda p, p)) \equiv (e^{-i\theta\hat{n}\cdot\vec{J}^{(s)}})_{s'_i,s_i}. \tag{2.4.25}$$

The matrix elements are called Wigner d-functions.

Note that $\hat{W}_\sigma$ is the generator for such class of rotations with three d.o.f.s, and they are the rotation generator in the defining representation in the rest frame, multiplying mass m. Hence, for a general representation, the Wigner rotation in the rest frame would rotate the spin components without changing the total spin quantum number as an eigenstate of the $\hat{J}^2$ operator with eigenvalue $s(s+1)$. Consequently, the Wigner rotation matrices correspond to a general rotation with a compound direction specified by $\hat{n}$ (a normalized three vector) and an angle θ in the spin-s representation. We somewhat anticipate this result for a few reasons. The boost and rotation generators do not commute with each other; hence, also the Wigner rotation does not change momentum; it is a nest operation of boost, general Lorentz transformation (including boost and rotation), and then boost the new result momentum back to the rest four-vector. It cannot correspond to another boost as it will change the momentum, so it can only change the spin quantum number via rotation. Furthermore, we've also seen that the generator of Wigner rotation, the Pauli-Lubanski operator, is a three-rotation, and hence we expect it to appear here. Last but not least, since this Winger rotation is acting on a particle at rest, it is identical to the rotation operator, and consequently, we get this result.

With the Wigner rotation matrices (elements) in Eq. (2.4.24), we can continue the evaluation in Eq. (2.4.21),

$$\begin{aligned}\hat{U}(\Lambda)\left|p, m, s, s_i\right\rangle &= \hat{U}(L(\Lambda p))\hat{U}(L^{-1}(\Lambda p))\hat{U}(\Lambda)\hat{U}(L(p))\left|\tilde{p}, m, s, s_i\right\rangle \\ &= \hat{U}(L(\Lambda p))\hat{U}(R(\Lambda p, p))\left|\tilde{p}, m, s, s_i\right\rangle \\ &= \sum_{s'_i} D^{(s)}_{s'_i,s_i}(R(\Lambda p, p))\hat{U}(L(\Lambda p))\left|\tilde{p}, m, s, s'_i\right\rangle \\ &= \sum_{s'_i} D^{(s)}_{s'_i,s_i}(R(\Lambda p, p))\left|\Lambda p, m, s, s'_i\right\rangle . \end{aligned} \tag{2.4.26}$$

We can see that general Lorentz transformations change the momentum of a particle and rotate its spin. Now, let's move to the massless case and try to unify the results in a more commonly used version.

2.4.2 Massless case

For the massless case, we can go through a similar procedure, but we will find the commuting rotation corresponds to two degrees of rotation,

[12] The minus sign in the exponent is simply a result of the Lorentz inner product on the space components.

motivating us to define a new quantity called helicity. For simplicity of the discussion, we work with the standard form of the momentum[13]

$$\tilde{p}^\mu \equiv (p, 0, 0, p)\,. \tag{2.4.27}$$

We first check how Pauli-Lubanski operator acts on a massless state with momentum $\tilde{p}$. From $\hat{W}_\sigma \hat{P}^\sigma = 0$ (see Eq. (2.4.13)),[14]

$$\hat{W}_\sigma \hat{P}^\sigma \left|\tilde{p}, 0\right\rangle = (\hat{W}_0 + \hat{W}_3) p \left|\tilde{p}, 0\right\rangle = 0. \tag{2.4.28}$$

On the other hand, $\hat{W}^\sigma \hat{W}_\sigma$ is a Lorentz Invariant (L.I.), mass dimension two quantity. The only scale in $|\tilde{p}, 0\rangle$ is p, which is not L.I. but covariant. One can Lorentz-boost along or against the z-direction to get arbitrary momentum (including approaching zero not zero). Hence, the only possible outcome of this operator on the state is zero,[15]

$$\begin{aligned} \hat{W}_\sigma \hat{W}^\sigma \left|\tilde{p}, 0\right\rangle &= 0, && (2.4.29)\\ (\hat{W}_0)^2 - (\hat{W}_1)^2 - (\hat{W}_2)^2 - (\hat{W}_3)^2 \left|\tilde{p}, 0\right\rangle &= 0, \\ (\hat{W}_1)^2 + (\hat{W}_2)^2 \left|\tilde{p}, 0\right\rangle &= 0. && (2.4.30) \end{aligned}$$

To obtain the last line, we use that (with Eq. (2.4.28)),

$$(\hat{W}_0)^2 - (\hat{W}_3)^2 \left|\tilde{p}, 0\right\rangle = (\hat{W}_0 - \hat{W}_3)(\hat{W}_0 + \hat{W}_3) \left|\tilde{p}, 0\right\rangle = 0. \tag{2.4.31}$$

and (from Eq. (2.4.15))

$$[\hat{W}_0, \hat{W}_3] \left|\tilde{p}, 0\right\rangle = i\epsilon_{03ij} \hat{W}^i \hat{P}^j \left|\tilde{p}, 0\right\rangle = 0. \tag{2.4.32}$$

Furthermore, we can get, from Eq. (2.4.15),

$$\begin{aligned} [\hat{W}_1, \hat{W}_2] \left|\tilde{p}, 0\right\rangle &= i(\epsilon_{1203} \hat{W}^0 \hat{P}^3 + \epsilon_{1230} \hat{W}^3 \hat{P}^0) \left|\tilde{p}, 0\right\rangle \\ &= ip\epsilon_{1203} (\hat{W}^0 - \hat{W}^3) \left|\tilde{p}, 0\right\rangle \\ &= ip{\epsilon_{12}}^{03} (\hat{W}_0 + \hat{W}_3) \left|\tilde{p}, 0\right\rangle \\ &= 0. \end{aligned} \tag{2.4.33}$$

Similarly, one can get that,[16]

$$[\hat{W}_0, \hat{W}_1] \left|\tilde{p}, 0\right\rangle = i\epsilon_{0123} \hat{W}^2 \hat{P}^3 \left|\tilde{p}, 0\right\rangle = -ip\hat{W}_2 \left|\tilde{p}, 0\right\rangle \tag{2.4.34}$$

$$[\hat{W}_3, \hat{W}_1] \left|\tilde{p}, 0\right\rangle = i\epsilon_{3120} \hat{W}^2 \hat{P}^0 \left|\tilde{p}, 0\right\rangle = ip\hat{W}_2 \left|\tilde{p}, 0\right\rangle \tag{2.4.35}$$

$$[\hat{W}_0, \hat{W}_2] \left|\tilde{p}, 0\right\rangle = i\epsilon_{0213} \hat{W}^1 \hat{P}^3 \left|\tilde{p}, 0\right\rangle = ip\hat{W}_1 \left|\tilde{p}, 0\right\rangle \tag{2.4.36}$$

$$[\hat{W}_3, \hat{W}_2] \left|\tilde{p}, 0\right\rangle = i\epsilon_{3210} \hat{W}^1 \hat{P}^0 \left|\tilde{p}, 0\right\rangle = -ip\hat{W}_1 \left|\tilde{p}, 0\right\rangle\,. \tag{2.4.37}$$

[13] There is no magic here; you can choose general momentum, and we need to adjust the statements from z-direction to a linear combination proportional to the momentum direction.

[14] Note here the Lorentz indices are not contracted yet so no need to implement the metric.

[15] For a more systematic discussion, see Weinberg Vol.I [Wei05].

[16] Note that I flipped the sign when I lower the indices in the spatial part, consistent with the metric choice.

Hence we have,

$$[\hat{W}_3, \hat{W}_1 \pm i\hat{W}_2] |\tilde{p}, 0\rangle = \pm p(\hat{W}_1 \pm i\hat{W}_2) |\tilde{p}, 0\rangle \tag{2.4.38}$$

This is our familiar raising and lowering operator. Let's label a state with its eigenvalue on $\hat{W}_3/p$, $|\tilde{p}, 0, s_3\rangle$. Then, one have

$$\hat{W}_3/p(\hat{W}_1 \pm i\hat{W}_2) |\tilde{p}, 0, s_3\rangle = (s_3 \pm 1)(\hat{W}_1 \pm i\hat{W}_2) |\tilde{p}, 0, s_3\rangle \tag{2.4.39}$$

and expect $(\hat{W}_1 \pm i\hat{W}_2) |\tilde{p}, 0, s_3\rangle \propto |\tilde{p}, 0, s_3 \pm 1\rangle$, i.e., the raising and lowering operators change the eigenstate of s_3 to $s_3 \pm 1$. However, there is a different possibility for Eq. (2.4.39) to hold, that is,

$$(\hat{W}_1 \pm i\hat{W}_2) |\tilde{p}, 0, s_3\rangle = 0. \tag{2.4.40}$$

It has to be the case that if I can raise or lower my state without getting a null state, at least one of the following be nonzero,

$$(\hat{W}_1 \pm i\hat{W}_2)(\hat{W}_1 \mp i\hat{W}_2) |\tilde{p}, 0, s_3\rangle \neq 0, \tag{2.4.41}$$

which contradicts Eq. (2.4.30). Consequently, we have

$$\hat{W}_1 |\tilde{p}, 0\rangle = \hat{W}_2 |\tilde{p}, 0\rangle = 0. \tag{2.4.42}$$

One can already get a hint of the difference between the massive case and the massless case in the sense that,

$$-\hat{W}_i\hat{W}^i/m^2 |\tilde{p}, m, s, s_i\rangle = \hat{J}^2 |\tilde{p}, m, s, s_i\rangle = s(s+1) |\tilde{p}, m, s, s_i\rangle , \tag{2.4.43}$$

$$-\hat{W}_i\hat{W}^i/p^2 |\tilde{p}, 0, s_3\rangle = \hat{W}_3^2/p^2 |\tilde{p}, 0, s_3\rangle = s_3^2 |\tilde{p}, 0, s_3\rangle . \tag{2.4.44}$$

The total spin is just the spin in the z-direction for massless particles with standard momentum; the action of raising and lowering operators gets zero. So, I do not need another total spin quantum number for the massless case.

Furthermore, from the definition,

$$\begin{aligned}\hat{W}_1 |\tilde{p}, 0\rangle &= -\frac{1}{2}\epsilon_{\mu\nu 01}\hat{M}^{\mu\nu}\hat{P}^0 - \frac{1}{2}\epsilon_{\mu\nu 31}\hat{M}^{\mu\nu}\hat{P}^3 |\tilde{p}, 0\rangle \\ &= p(\hat{J}_1 + \hat{K}_2) |\tilde{p}, 0\rangle \end{aligned}\tag{2.4.45}$$

$$\begin{aligned}\hat{W}_2 |\tilde{p}, 0\rangle &= -\frac{1}{2}\epsilon_{\mu\nu 02}\hat{M}^{\mu\nu}\hat{P}^0 - \frac{1}{2}\epsilon_{\mu\nu 32}\hat{M}^{\mu\nu}\hat{P}^3 |\tilde{p}, 0\rangle \\ &= -p(\hat{J}_2 + \hat{K}_1) |\tilde{p}, 0\rangle \end{aligned}\tag{2.4.46}$$

$$\begin{aligned}\hat{W}_0 |\tilde{p}, 0\rangle &= -\frac{1}{2}\epsilon_{\mu\nu 30}\hat{M}^{\mu\nu}\hat{P}^3 |\tilde{p}, 0\rangle \\ &= p\hat{J}_3 |\tilde{p}, 0\rangle \end{aligned}\tag{2.4.47}$$

$$\begin{aligned}\hat{W}_3 |\tilde{p}, 0\rangle &= -\frac{1}{2}\epsilon_{\mu\nu 03}\hat{M}^{\mu\nu}\hat{P}^0 |\tilde{p}, 0\rangle \\ &= -p\hat{J}_3 |\tilde{p}, 0\rangle , \end{aligned}\tag{2.4.48}$$

Combining with the commutation relations between rotation and boost generators, We can see clearly many equations in this section, e.g., $\hat{W}_0 + \hat{W}_3 \left|\tilde{p}, 0\right\rangle = 0$, confirming Eq. (2.4.28), and $[\hat{W}_1, \hat{W}_2] \left|\tilde{p}, 0\right\rangle = 0$, confirming Eq. (2.4.33).

The above motivates us to define a quantum number as an eigenstate of the Pauli-Lubanski operator in the non-trivial direction of $\hat{W}_0$, $\hat{W}_3$ or a linear combination that is not $\hat{W}_0 + \hat{W}_3$. Furthermore, we know $\hat{W}_\sigma$ transforms as a Lorentz 4-vector, from the commutation relations with Lorentz generator in Eq. (2.4.14). There are no other Lorentz covariant symbols in momentum space that are intrinsic to the state (e.g., not a reference four-vector); we do foresee that $\hat{W}_\sigma \propto \hat{P}_\sigma$. We can specify a new (mass-)dimensionless quantity, **helicity**,

$$\lambda \equiv -\frac{\hat{W}_\mu n^\mu}{\hat{P}_\mu n^\mu}. \tag{2.4.49}$$

Here n^μ is a reference vector, and we typically choose it as $n^\mu = (1, 0, 0, 0)$. This corresponds to defining the one-particle eigenstate according to its eigenvalue with respect to the rotation generator $\hat{J}_3$. Compared to the case of massive particles, we lose the freedom to define the one-particle state with its eigenvalue on an arbitrary spatial axis. On the other hand, this quantum number choice is still consistent with what we find in Eq. (2.4.1), that rotation along the momentum direction is compatible with the momentum operator. In other words, we already foresee that we want to project the spin quantum number along the momentum direction from the very beginning of this section! This choice is independent of whether the underlying state is massive or massless. A more *commonly used* **helicity** definition, which is equivalent to Eq. (2.4.49) with the n^μ choice, is

$$\lambda \equiv -\frac{\hat{W}_0 n^0}{\hat{P}_0 n^0} = -\frac{\hat{W}_0 P^0}{\hat{P}_0 P^0} = \frac{\hat{W}_i \cdot \hat{P}^i}{|\vec{P}|^2} = \frac{\vec{J} \cdot \vec{p}}{|\vec{p}|}, \tag{2.4.50}$$

where we have used Eq. (2.4.12) Both definitions are consistent with each other, and they are *invariant* under rotation. one can see that this rightmost definition can be extendable to massive particles, which we will explore later.

Unlike the case for massive particles, one cannot define raising and lowering operators for helicity. Hence, a particle does not need to fill the full $2s + 1$ spin d.o.f.s. A one-particle state with specific helicity lives on its own. Why do we often see opposite helicities for massless particles? It is because we require some more symmetries of the theory.

Helicity is rotation invariant but boosts variant quantity, but the spin projection is a rotation variant quantity. It is even convenient many times to define the last quantum number for massive particles' helicity instead of spin projection.

For massless particles, the total spin quantum number is its spin projection along the momentum direction. The algebra is closed, so there are no $2s+1$ states. Note that, under parity transformation,

$$\begin{aligned} \vec{p} &\xrightarrow{\mathcal{P}} -\vec{p} \\ s_i &\xrightarrow{\mathcal{P}} s_i \\ \lambda &\xrightarrow{\mathcal{P}} -\lambda. \end{aligned} \tag{2.4.51}$$

Spin projection being invariant as its generator $\hat{M}^{ij}$ gets flipped twice. Helicity, as the spin projection along the momentum direction, gets flipped under parity. If we require a theory (with its corresponding one-particle state) to be an eigenstate of the parity transformation, we would need the existence of the opposite helicity state. Hence, we often see massless states come in helicity pairs.

One can unify the notation for one-particle state for massive and massless particles, both expressed as

$$|p, m, s, \lambda\rangle\,, \tag{2.4.52}$$

and for massless state $m=0$, s is redundant with $s=|\lambda|$ (again, note that $-\hat{W}_i\hat{W}^i = W_3^2$, differ from the massive case), but it doesn't hurt to keep it.

Finally, we can also unify the discussion regarding a general Lorentz transformation's effect on a one-particle state. With helicity, we are motivated to exploit rotation, as both S and λ are invariant under rotation. We can define a general standard form momentum,

$$\tilde{p}' \equiv (E, 0, 0, p), \tag{2.4.53}$$

as we put all the momentum in the positive z-direction. For massless case, $E = p$ and massive case has $E = \sqrt{m^2+p^2}$. For the massless case, the helicity is already defined to be projected in this direction. For the massive case, with a general momentum $p \equiv \Lambda\tilde{p}$, we can write

$$\begin{aligned} \hat{U}(\Lambda)\,|\tilde{p}, m, s, \lambda\rangle &= \hat{U}(R(p,\tilde{p}'))\hat{U}(L(\tilde{p}'))\,|\tilde{p}, m, s, \lambda\rangle \\ &= \hat{U}(R(p,\tilde{p}'))\,|\tilde{p}', m, s, \lambda\rangle\,. \end{aligned} \tag{2.4.54}$$

Basically, for a general momentum direction, one can define the helicity for a massive particle in its rest frame as the spin projection along the positive z-direction. The general momentum p is rewritten as a rotation of the general standard form momentum $\tilde{p}'$. The first step of boost along the positive z-direction will not change the spin projection along the positive z-direction; it transforms the state momentum from rest $\tilde{p}$ to a new one that has the same three-momentum magnitude as that of the final p. Then one just performs a rotation, which again does not change λ. If we do this in another order, it often involves other helicity values (for massive cases).For a three momentum with Euler angle (ϕ, θ), the rotation $R(p, \tilde{p}')$ can be expressed as a series of operation,

$$\hat{R}(p, \tilde{p}') = \hat{R}_3(-\phi) R_2(-\theta) R_3(\phi) = e^{-i\phi \hat{J}_3} e^{-i\theta \hat{J}_2} e^{+i\phi \hat{J}_3}. \tag{2.4.55}$$

For massless case, one again can boost the standard form momentum further along the z-direction to match the magnitude of the resulting three-momentum $p = \Lambda \tilde{p} = R(\Lambda \tilde{p}, \tilde{p}') L(\tilde{p}', p) \tilde{p}$. The resulting effect is analogous to the above.

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