9.10 Lorentz Transformation of Scalar Fields in QFT

Key Concepts and Definitions

- **Active Lorentz Transformation**: Physically alters the field configuration while keeping the spacetime coordinate system fixed. For example, boosting or rotating the field itself.
- **Passive Lorentz Transformation**: Changes the observer's coordinate system while leaving the physical field unchanged. Coordinates relabel $x \to \bar{x} = \Lambda x$, where Λ is a Lorentz transformation matrix.
- **Scalar Field**: A field $\phi(x)$ that transforms trivially under Lorentz transformations. Its *value* at a spacetime point P is observer-independent. If Alice labels P as x and Bob (using Λ) labels it as $\bar{x} = \Lambda x$, then:

$$\phi(x) = \bar{\phi}(\bar{x}) \quad \text{(Scalar invariance)}.$$

Passive Transformation of the Scalar Field

Consider two inertial observers, Alice and Bob, related by a Lorentz transformation Λ:

- Alice's coordinates: x^μ
- Bob's coordinates: $\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu$

The scalar field's value at spacetime point P must agree for both observers:

$$\phi(x) = \bar{\phi}(\bar{x}) = \bar{\phi}(\Lambda x).$$

Rewriting this in terms of Bob's coordinates ($\bar{x} = \Lambda x \implies x = \Lambda^{-1}\bar{x}$):

$$\bar{\phi}(\bar{x}) = \phi(\Lambda^{-1}\bar{x}) \quad \text{or} \quad \bar{\phi}(x) = \phi(\Lambda^{-1}x).$$

This defines how Bob's field $\bar{\phi}$ relates to Alice's field ϕ under coordinate relabeling.

Operator Transformation in Quantum Field Theory

In QFT, Lorentz transformations are implemented as unitary operators $U(\Lambda)$ acting on the Hilbert space. For a scalar field operator $\phi(x)$, the transformation law is:

$$U(\Lambda)^{-1}\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x). \quad \text{(Operator transformation)}$$

This ensures consistency with the passive interpretation:

- *Left-hand side*: $U(\Lambda)^{-1}\phi(x)U(\Lambda)$ represents the transformed operator at x in the new state.
- *Right-hand side*: $\phi(\Lambda^{-1}x)$ is the original operator evaluated at $\Lambda^{-1}x$.

Connecting Active and Passive Views

Passive Perspective

Bob measures the field at $\bar{x} = \Lambda x$ using his coordinates:

$$\bar{\phi}(\bar{x}) = \phi(\Lambda^{-1}\bar{x}).$$

Substituting $\bar{x} = \Lambda x$, we recover:

$$\bar{\phi}(\Lambda x) = \phi(x),$$

which matches the scalar invariance principle.

Active Perspective

Actively transforming the field with $U(\Lambda)$ gives:

$$\phi'(x) \equiv U(\Lambda)^{-1}\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x).$$

If we compare $\phi'(x)$ (actively transformed field) with $\bar{\phi}(x)$ (passively transformed field), they satisfy the same relation:

$$\phi'(x) = \bar{\phi}(x).$$

This equivalence shows that active and passive transformations are two sides of the same coin.

9.10.1 Summary

- **Scalar invariance**: $\phi(x) = \bar{\phi}(\Lambda x)$ ensures observers agree on field values at spacetime points.
- **Operator transformation**: $U(\Lambda)^{-1}\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)$ encodes how quantum fields transform under Lorentz symmetry.
- **Equivalence**: Active transformations (operator acting on Hilbert space) and passive transformations (coordinate relabeling) yield identical physical predictions for scalar fields.