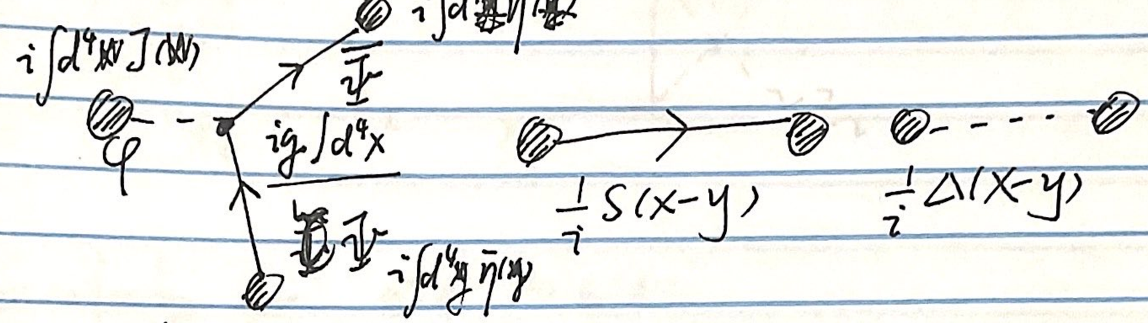
lec 18

Now we have three fields φ, Ψ, $\bar{\Psi}$

the vertex is then



For each vertex

one away

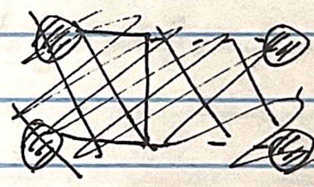
one toward.

$$\downarrow$$

$$i^3\left(\frac{1}{i}\right)^3(ig)\int d^4x\, d^4y\, d^4z\, \overset{d^4w}{\downarrow}\left[\bar{\eta}(x)S(x-y)S(y-z)\eta(z)\right]\Delta(y-w)J(w)$$

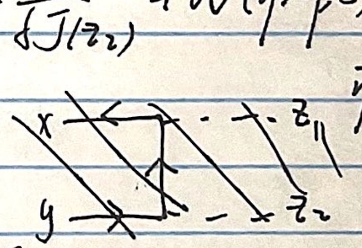
Let's say Ψ is e^- and $\bar{\Psi}$ is e^+

and we can calculate $e^+e^- \to \varphi\varphi$.



$$\langle 0|T\,\Psi\bar{\Psi}\varphi\varphi|0\rangle_c$$

$$= \frac{1}{i}\frac{\delta}{\delta\bar{\eta}_\alpha(x)}\, i\frac{\delta}{\delta\eta_\beta(y)}\, \frac{1}{i}\frac{\delta}{\delta J(z_1)}\, \frac{1}{i}\frac{\delta}{\delta J(z_2)}\, iW(\bar{\eta},\eta,J)\Big|_{\bar{\eta}=\eta=J=0}$$

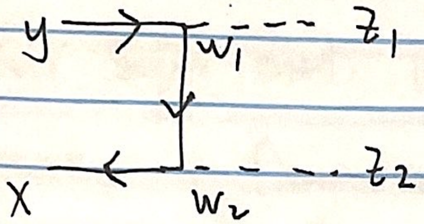


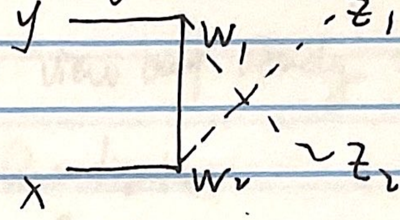
$$= \left(\frac{1}{i}\right)^5(ig)^2\int d^4w_1\, d^4w_2$$

$$[S(x-w_2)S(w_2-w_1)S(w_1-y)]_{\alpha\beta}$$

$$\Delta(z_1-w_1)\Delta(z_2-w_2) + (z_1\leftrightarrow z_2) + O(g^4)$$

Which correspond to diagrams





Similarly for $e^-e^- \to e^-e^-$

$e^-e^+ \to e^-e^+$

~~wh~~ we need $\langle 0|T\psi\bar\psi\psi\bar\psi|0\rangle_c$

$$= \frac{1}{i}\frac{\delta}{\delta\bar\eta_{\alpha_1}(x_1)}\, i\frac{\delta}{\delta\eta_{\beta_1}(y_1)}\, \frac{1}{i}\frac{\delta}{\delta\bar\eta_{\alpha_2}(x_2)}\, i\frac{\delta}{\delta\eta_{\beta_2}(y_2)}$$

$$iW(\bar\eta,\eta,J)\big|_{\bar\eta=\eta=J=0}$$

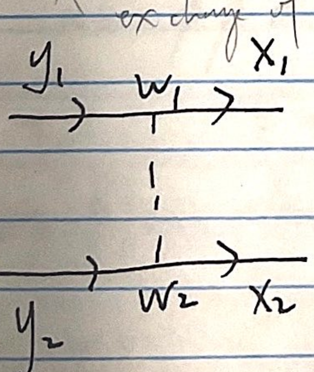
$$= \left(\frac{1}{i}\right)^5 (ig)^2 \int d^4w_1\, d^4w_2\, [S(x_1-w_1)S(w_1-y_1)]_{\alpha_1\beta_1}$$

$$\Delta(w_1-w_2)\,[S(x_2-w_2)S(w_2-y_2)]_{\alpha_2\beta_2}$$

$$-\,((y_1\beta_1)\leftrightarrow(y_2\beta_2)) + O(g^4)$$

exchange of fermions

for



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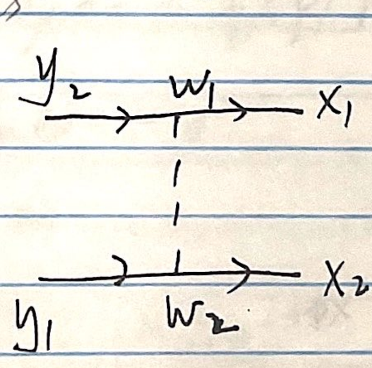

Next, we will "plug in" the LSZ reduction.
But before that, let's view and study $\Psi(x)$ a bit more. (Canonical Quantization a/co the wave-function) & Spinor Technology.

For the Lagrangian

$$\mathcal{L} = i\bar{\Psi}\not{\partial}\Psi - m\bar{\Psi}\Psi$$

The E.O.M. contains

$$(-i\not{\partial} + m)\Psi = 0$$

Left multiply by $(i\not{\partial} + m)$

$$(i\not{\partial} + m)(-i\not{\partial} + m)\Psi = (\not{\partial}\not{\partial} + m^2)\Psi = 0$$

$$= (-\partial^2 + m^2)\Psi = 0$$

as $\not{\partial}\not{\partial} = \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}\{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\}\partial_\mu\partial_\nu$

$$= -g^{\mu\nu}\partial_\mu\partial_\nu = -\partial^2.$$

This is Klein-Gordon, so

$$\Psi(x) = u(\vec{p})\,\underline{e^{ipx}} + v(\vec{p})\,\underline{e^{-ipx}}$$

$\downarrow$ $\begin{pmatrix} \chi_a \\ \xi^{\dagger\dot{a}} \end{pmatrix}$ 4-component spinors.

put back to the equations of motion, we get

$$(\not{p}+m)u(\vec{p})e^{ipx} + (-\not{p}+m)\cancel{e} \; v(\vec{p})e^{-ipx} = 0$$

To be valid for arbitrary x and $\vec{p}$, we will need

$$(\not{p}+m)u(\vec{p})=0 \quad \text{(we can see next)}.$$
$$(-\not{p}+m)v(\vec{p})=0.$$

$u(\vec{p})$ and $V(\vec{p})$ each have two linearly independent solutions. $u_\pm(\vec{p})$, $V_\pm(\vec{p})$. then the general solution to the Dirac Equation is

$$\Psi(x) = \sum_{s=\pm} \int \underbrace{\widetilde{dp}}_{\frac{d^3p}{(2\pi)^3 2w}} \left[b_s(\vec{p})u_s(\vec{p})e^{ipx} + d_s^\dagger(\vec{p})V_s(\vec{p})e^{-ipx} \right]$$

For $m\neq 0$, (any massless fermions in Nature?) ← neutrino would be

we can go to the rest frame and have $\vec{p}=0$, $E=m$.
Then

$$\not{p}+m = \begin{pmatrix} -m & m \\ m & -m \end{pmatrix}, \quad -\not{p}+m = \begin{pmatrix} m & m \\ m & m \end{pmatrix}$$

and the two solutions can be separated by the spin matrix (along z)

$$S_z = \frac{i}{4}[\gamma^1,\gamma^2] = \frac{i}{2}\gamma^1\gamma^2 = \begin{pmatrix} \frac{1}{2}\sigma_3 & \\ & \frac{1}{2}\sigma_3 \end{pmatrix}$$

Real Spin

We can require

$$S_z\, u_\pm(\vec{0}) = \pm\tfrac{1}{2}\, u_\pm(\vec{0})$$

$$S_z\, v_\pm(\vec{0}) = \underline{\mp}\tfrac{1}{2}\, v_\pm(\vec{0})$$

Such choice conveniently have

$$[J_z, b_\pm^\dagger(\vec{0})] = \pm\tfrac{1}{2}\, b_\pm^\dagger(\vec{0})$$

$$[J_z, d_\pm^\dagger(\vec{0})] = \pm\tfrac{1}{2}\, d_\pm^\dagger(\vec{0})$$

(recall N_i, $N_i^\dagger$).

This means $b_+^\dagger(\vec{0})$ and $d_+^\dagger(\vec{0})$ each creates a particle with spin up along the z-axis.

Now, $u_+(\vec{0}) = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$, $u_-(\vec{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$

$v_+(\vec{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$, $v_-(\vec{0}) = \sqrt{m}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}$

For the barred spinors $\beta = \begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$ (again $\beta^T = \beta^\dagger = \beta^{-1} = \beta$).

$\bar{u}_+(\vec{0}) = \sqrt{m}\,(1, 0, 1, 0)$, $\bar{v}_+(\vec{0}) = \sqrt{m}\,(0, 1, 0, 1)$

$\bar{u}_-(\vec{0}) = \sqrt{m}\,(0, 1, 0, 1)$, $\bar{v}_-(\vec{0}) = \sqrt{m}\,(1, 0, -1, 0)$

We can apply boost operation

$$D(\Lambda) = \exp(-i\eta\,\hat{p}\cdot K).$$

unit vector in p direction

$$K^j = \frac{i}{4}[\gamma^j, \gamma^0] = \frac{i}{2}\gamma^j\gamma^0$$

$$\eta = \sinh^{-1}(|\vec{p}|/m)$$

$$u_s(\vec{p}) = e^{-i\eta\hat{\vec{p}}\cdot\vec{K}}u_s(\vec{0})$$

$$v_s(\vec{p}) = e^{-i\eta\hat{\vec{p}}\cdot\vec{K}}v_s(\vec{0})$$

Also. $$\overline{u}_s(\vec{p}) = \overline{u}_s(\vec{0})\,e^{-i\eta\hat{\vec{p}}\cdot\vec{K}}$$

$$\overline{v}_s(\vec{p}) = \overline{v}_s(\vec{0})\,e^{-i\eta\hat{\vec{p}}\cdot\vec{K}}$$

(this follows that $\overline{K^j} = \beta K^{j\dagger}\beta = K^j$

$$= -\frac{i}{2}\gamma^0\gamma^{0\dagger}\gamma^{j\dagger}\gamma^0$$

$$= -\frac{i}{2}\gamma^0\gamma^0\gamma^0\gamma^j\gamma^0\gamma^0 = \frac{i}{2}\gamma^j\gamma^0 = K^j$$

as for general matrix $\overline{A} \equiv \beta A^\dagger\beta$.

So $$\overline{\gamma^\mu} = \gamma^\mu, \quad \overline{S^{\mu\nu}} = S^{\mu\nu}, \quad \overline{i\gamma_5} = i\gamma_5$$

$$\overline{\gamma^\mu\gamma_5} = \gamma^\mu\gamma_5, \quad \overline{i\gamma_5 S^{\mu\nu}} = i\gamma_5 S^{\mu\nu}$$

then we also have

$$\overline{u}_s(\vec{p})(\not{p}+m) = 0$$

$$\overline{v}_s(\vec{p})(-\not{p}+m) = 0.$$

Then we have some useful relations.

$$\bar{u}_{s'}(\vec{p})\,u_s(\vec{p}) = \bar{u}_{s'}(\vec{0})\,u_s(\vec{0}) = +2m\,\delta_{s's}$$

$$\bar{v}_{s'}(\vec{p})\,v_s(\vec{p}) = \bar{v}_{s'}(\vec{0})\,v_s(\vec{0}) = -2m\,\delta_{s's}$$

$$\bar{u}_{s'}(\vec{p})\,v_s(\vec{p}) = 0$$

$$\bar{v}_{s'}(\vec{p})\,u_s(\vec{p}) = 0.$$

Now study the spin sums.

$$\sum_{s=\pm} u_s(\vec{p})\,\bar{u}_s(\vec{p}) = e^{i\eta\hat{p}\cdot\vec{K}}(-m\gamma^0+m)\,e^{-i\eta\hat{p}\cdot\vec{K}}$$

$$= -\not{p}+m$$

$$\sum_{s=\pm} v_s(\vec{p})\,\bar{v}_s(\vec{p}) = -\not{p}-m$$

The spin quantization axis is arbitrary,

the above sum get rid of it and is useful for a broad class of observables.

Helicity:

We can also get non-summed relations by projection

$$\frac{1}{2}(1+2sS_z)\,u_{s'}(\vec{0}) = \delta_{ss'}\,u_{s'}(\vec{0})$$

$$\frac{1}{2}(1-2sS_z)\,v_{s'}(\vec{0}) = \delta_{ss'}\,v_{s'}(\vec{0}).$$

Skip

Avoid confusing students, just mention Helicity

Recall $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$

then $S_z = \frac{i}{2}\gamma^1\gamma^2 = -\frac{1}{2}\gamma_5\gamma^3\gamma^0$

In the rest frame, we have γ^0 as $-\not{p}/m$, γ^3 as $\not{z}$

$(z)^\mu = (0, \vec{z}))$

Then $S_z = \frac{1}{2m}\gamma_5 \not{z}\not{p}$

this motivates the helicity eigenstate such that

$$u_s(\vec{p})\bar{u}_s(\vec{p}) \to \frac{1}{2}(1+s\gamma_5)(-\not{p}). + O(e^{-\eta})$$

$$v_s(\vec{p})\bar{v}_s(\vec{p}) \to \frac{1}{2}(1-s\gamma_5)(-\not{p}) + O(e^{-\eta})$$

Back to the solutions.

$$\Psi = \sum_{s=\pm} \int \widetilde{dp}\, [b_s(\vec{p}) u_s(\vec{p}) e^{ipx} + d_s^\dagger(\vec{p}) v_s(\vec{p}) e^{-ipx}]$$

using the completeness relation

$$\int d^3x\, e^{-ipx} \Psi(x) = \sum_{s'=\pm} [\frac{1}{2\omega} b_{s'}(\vec{p}) u_{s'}(\vec{p}) + \frac{1}{2\omega} e^{2i\omega t} d_{s'}^\dagger(-\vec{p}) v_{s'}(-\vec{p})]$$

Given the relations above on spinors

$$b_s(\vec{p}) = \int d^3x\, e^{-ipx} \bar{u}_s(\vec{p}) \gamma^0 \Psi(x).$$

$$b_s^\dagger(\vec{p}) = [b_s(\vec{p})]^\dagger = \int d^3x\, e^{ipx} \bar{\Psi}(x) \gamma^0 u_s(\vec{p}).$$

In class. I wrongfully wrote $\int d^3y$ It should be $e^{-ip'x}$

Also, we have

$$\bar{u}_{s'}(\vec{p})\gamma^\mu u_s(\vec{p}) = 2p^\mu \delta_{s's}$$

$$\bar{v}_{s'}(\vec{p})\gamma^\mu v_s(\vec{p}) = 2p^\mu \delta_{s's}$$

and $\bar{u}_{s'}(\vec{p})\gamma^0 v_s(-\vec{p}) = 0$

$$\bar{v}_{s'}(\vec{p})\gamma^0 u_s(-\vec{p}) = 0.$$

E.q. 38.18, 38.22. (see note ⑪). used above.

$$d_s^\dagger(\vec{p}) = \int d^3x\, e^{-ipx}\, \bar{v}_s(\vec{p})\gamma^0 \Psi(x)$$

$$d_s(\vec{p}) = \int d^3x\, e^{-ipx}\, \bar{\Psi}(x)\gamma^0 v_s(\vec{p})$$

They have all the commutation relations of creation/annihilation operators that we want.

Here we

$$b_s(\vec{p})|0\rangle = d_s(\vec{p})|0\rangle = 0.$$

Recall

$$Q = \int d^3x\, \bar{\Psi}\gamma^0\Psi$$

$$= \sum_{s=\pm}\int \widetilde{dp}\, [b_s^\dagger(\vec{p})b_s(\vec{p}) + d_s(\vec{p})d_s^\dagger(\vec{p})]$$

$$= \sum_{s=\pm}\int \widetilde{dp}\, [\underbrace{b_s^\dagger(\vec{p})b_s(\vec{p})}_{\#\text{ of } b\text{-type}} - \underbrace{d_s^\dagger(\vec{p})d_s(\vec{p})}_{\#\text{ of } d\text{-type particles.}}] + \text{const}$$

Note that what we have done is corresponding to Canonical quantization, which means

$$\{\Psi_\alpha(\vec{x},t), \Psi_\beta(\vec{y},t)\} = 0$$

$$\{\Psi_\alpha(\vec{x},t), \bar{\Psi}_\beta(\vec{y},t)\} = (\gamma^0)_{\alpha\beta}\,\delta^3(\vec{x}-\vec{y})$$

α, β are just which element $\alpha, \beta = 1,2,3,4$.

Recall γ^0 is a 4x4 matrix

Alternatively, one can also find this is consistent with per Wyle spinor quantization

e.g. For general one wyle field Lagrangian density

$$\mathcal{L} = i\psi^\dagger \bar{\sigma}^\mu \psi - \frac{1}{2}m(\psi\psi + \psi^\dagger\psi^\dagger)$$

conjugate momentum is then

$$\Pi^a(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_a(x))}$$

$$\mathcal{H} = \cancel{-i\psi^\dagger\bar{\sigma}^\mu\psi} - i\psi^\dagger\bar{\sigma}^i\partial_i\psi + \frac{1}{2}(\psi\psi + \psi^\dagger\psi^\dagger)$$

Fermions are anti-commutting, so we have

$$\{\psi_a(\vec{x},t), \psi_b(\vec{y},t)\} = 0$$

$$\{\psi_a(\vec{x},t), \Pi^c(\vec{y},t)\} = i\delta_a{}^c\,\delta^3(\vec{x}-\vec{y})$$

with substitution, we get

$$\{\psi_a(\vec{x},t),\ \psi_{\dot{c}}^\dagger(\vec{y},t)\}\ \bar{\sigma}^{0\dot{c}a} = \delta_a{}^c\,\delta^3(\vec{x}-\vec{y})$$

Noting that $\bar{\sigma}^0 = \mathbb{1}_{2\times 2}$, $\sigma = \mathbb{1}_{2\times 2}$

$$\{\psi_a(\vec{x},t),\ \psi_{\dot{c}}^\dagger(\vec{y},t)\} = \sigma^0_{a\dot{c}}\,\delta^3(\vec{x}-\vec{y})$$

$$\{\psi^a(\vec{x},t),\ \psi^{\dot{c}\dagger}(\vec{y},t)\} = \bar{\sigma}^{\dot{c}a}\,\delta^3(\vec{x}-\vec{y}).$$

Eq. 38.13, Gordon Identity

$$\gamma^\mu \not{p} = \frac{1}{2}\{\gamma^\mu,\not{p}\} + \frac{1}{2}[\gamma^\mu,\not{p}]$$

$$= -p^\mu - 2iS^{\mu\nu}p_\nu$$

and

$$\not{p}'\gamma^\mu = -p'^\mu + 2iS^{\mu\nu}p_\nu.$$

$$2m\,\bar{u}_{s'}(\vec{p}')\gamma^\mu u_s(\vec{p}) = -\bar{u}_{s'}(\vec{p}')(\not{p}'\gamma^\mu + \gamma^\mu\not{p})u_s(\vec{p}).$$

Similarly,

$$2m\,\bar{v}_{s'}(\vec{p}')\gamma^\mu v_s(\vec{p}) = \bar{v}_s(\vec{p}')(\not{p}'\gamma^\mu + \gamma^\mu\not{p})v_s(\vec{p})$$

when $p = p'$

$$\bar{u}_s(\vec{p})\gamma^\mu u_s(\vec{p}) = 2p^\mu\delta_{s's}$$

$$\bar{v}_{s'}(\vec{p})\gamma^\mu v_s(\vec{p}) = 2p^\mu\delta_{s's}$$

~~whi~~ With $\vec{p}' = -\vec{p}$, $p'^0 = p^0$

$$\bar{u}_{s'}(\vec{p})\gamma^0 u_s(\vec{p}) = 0$$

$$\bar{v}_{s'}(\vec{p})\gamma^0 v_s(-\vec{p}) = 0$$

The full list of anti-commutation relations on the solutions we got are (momentum space)

$$\{b_s(\vec{p}), b_{s'}(\vec{p})\} = 0$$

$$\{d_s(\vec{p}), d_{s'}(\vec{p})\} = 0 \quad \Leftarrow \quad \{\Psi_\alpha(\vec{x},t), \Psi_\beta(\vec{y},t)\} = 0$$

$$\{b_s(\vec{p}), d^\dagger_{s'}(\vec{p}')\} = 0$$

and also the Hermitian conjugated relations are true.

Now we also have non-trivial ones.

$$\{b_s(\vec{p}), b_s^\dagger(\vec{p}')\} = \int d^3x\, d^3y\, e^{-ipx+ip'y}\bar{u}_s(\vec{p})\gamma^0\{\Psi(x), \bar{\Psi}(y)\}\gamma^0 u_s(\vec{p}')$$

$$= \int d^3x d^3y\, e^{-ipx+ip'y}\bar{u}_s(\vec{p})\gamma^0\gamma^0\gamma^0\delta^3(\vec{x}-\vec{y})u_s(\vec{p}')$$

$$= (2\pi)^3\delta^3(\vec{p}-\vec{p}')\bar{u}_s(\vec{p})\gamma^0 u_s(\vec{p}')$$

$$= (2\pi)^3\delta^3(\vec{p}-\vec{p}')\,2\omega\,\delta_{ss'}$$

Similarly

$$\{ d_s^\dagger(\vec{p}), d_{s'}(\vec{p}') \} = (2\pi)^3 \delta^3(\vec{p}-\vec{p}') 2\omega \delta_{ss'}$$

And in momentum space

$$H = \int d^3x \, \bar{\Psi} (-i\gamma^i \partial_i + m) \Psi$$

$$= \sum_{s=\pm} \int \widetilde{dp} \; \omega \underbrace{[b_s^\dagger(\vec{p}) b_s(\vec{p}) + d_s^\dagger(\vec{p}) d_s(\vec{p})]}_{\text{Recall SHO?}} - 4\mathcal{E}_0 V$$

Now LSZ reduction.

let's denote

$$|p, s, +\rangle = b_s^\dagger(\vec{p})|0\rangle$$

$$|p, s, -\rangle = d_s^\dagger(\vec{p})|0\rangle$$

"$\pm$" to denote particle type $b_s^\dagger$ and $d_s^\dagger$ (which have $U(1)$ charge oppositely)

and as

$$b_s^\dagger(\vec{p}) = \int d^3x \, e^{ipx} \, \bar{\Psi}(x) \gamma^0 u_s(\vec{p})$$

$$d_s^\dagger(\vec{p}) = \int d^3x \, e^{ipx} \, \bar{v}_s(\vec{p}) \gamma^0 \Psi(x)$$

$\leftarrow$ for free theory.

We have

$$\boxed{\langle p, s, q | p', s', q' \rangle = (2\pi)^3 2\omega \, \delta^3(\vec{p}-\vec{p}') \delta_{ss'} \delta_{qq'}}$$