Lec 17

Dirac Equation & Spinor Fields

With all the invariant symbols we can construct a Lorentz invariant Lagrangian density for spinor fields.

as $$(i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi)^\dagger = i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi + \underbrace{-i\partial_\mu(\psi^\dagger\bar{\sigma}\psi)}_{\text{surface term.}}$$

and $(m\psi\psi)^\dagger = m^*\psi^\dagger\psi^\dagger$, we have

$$\mathcal{L} = i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi - \frac{1}{2}|m|(\psi\psi + \psi^\dagger\psi^\dagger)$$

complex phase absorbed in $\psi \to \psi e^{\frac{i\alpha}{2}}$

E.O.M.s is then

$$0 = -\frac{\delta S}{\delta\psi^\dagger} = -i\bar{\sigma}^\mu\partial_\mu\psi + m\psi^\dagger$$

$$0 = -\frac{\delta S}{\delta\psi} = -\left(\frac{\delta S}{\delta\psi^\dagger}\right)^\dagger$$

$$= +i(\bar{\sigma}^{\mu a\dot{c}})^*\partial_\mu\psi^\dagger_{\dot{c}} + m\psi^a \qquad \cancel{= i\bar{\sigma}^\mu\partial_\mu\psi^\dagger + m\psi}$$

$$= +i(\bar{\sigma}^{\mu\dot{c}a})\partial_\mu\psi^\dagger_{\dot{c}} + m\psi^a$$

$$\Rightarrow -0 = -i\sigma^\mu_{a\dot{c}}\partial_\mu\psi^{\dot{c}\dagger} + m\psi_a.$$

ϵ_{ba} $b \to a$ $\quad$ ${}_{\dot{c}} \to {}^{\dot{c}}$

We see the EOM of ψ and $\psi^\dagger$ are mixed.

$$\begin{pmatrix} m\delta_a{}^c & -i\sigma^\mu_{a\dot c}\partial_\mu \\ -i\bar\sigma^{\mu\dot a c}\partial_\mu & m\delta^{\dot a}{}_{\dot c} \end{pmatrix}\begin{pmatrix} \psi_c \\ \psi^{\dagger\dot c} \end{pmatrix} = 0$$

Traditionally, we define Gamma Matrices
(How people use to do calculations)

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{a\dot c} \\ \bar\sigma^{\mu\dot a c} & 0 \end{pmatrix}$$

It is ~~hermitian~~ ~~not~~ not hermitian, but Lorentz invariant as $\bar\sigma$ and σ are L.I. symbols that we constructed eariler.

Not that $\sigma^\mu = (I, \vec\sigma)$, $\bar\sigma^\mu = (I, -\vec\sigma)$

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}.$$

anticommutator as we already know from last lecture.

$$(\bar\sigma^\mu\sigma^\nu + \sigma^\nu\bar\sigma^\mu)^{\dot a}{}_{\dot c} = -2g^{\mu\nu}\delta^{\dot a}{}_{\dot c}$$

$$(\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu)_a{}^c = -2g^{\mu\nu}\delta_a{}^c$$

We can then introduce a four component field

$\Psi = \begin{pmatrix} \psi_c \\ \psi^{\dagger c} \end{pmatrix}$ that is a $\boxed{\text{majorana field.}}$

obeying the famous Dirac Equation. Which we haven't identified in Nature yet. And it doesn't conserve charge.

$$(-i\gamma^\mu \partial_\mu + m)\Psi = 0$$

Now, more typically, we talk about a Dirac Field.

Suppose that I have two field related by $SO(2)$ transformation. (a, b) invariant under a^2+b^2

so $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \Rightarrow \underbrace{\boxed{\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}}}_{S} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

keeps a^2+b^2 invariant as. $S^T S = \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

such $SO(2)$ transformation can also rewritten as $U(1)$. (a phase).

$$\chi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$$

$$\xi = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2).$$

Now the two-~~field~~ left-handed spinor field Lagrangian density

$$\mathcal{L} = i\psi_1^\dagger \bar{\sigma}^\mu \partial_\mu \psi_1 + i\psi_2^\dagger \bar{\sigma}^\mu \partial_\mu \psi_2$$
$$- \frac{1}{2} m \left(\psi_1\psi_1 + \psi_1^\dagger\psi_1^\dagger + \psi_2\psi_2 + \psi_2^\dagger\psi_2^\dagger \right).$$

can be rewritten in such a new basis as

$$\mathcal{L} = i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \xi - m\chi\xi - m\xi^\dagger\chi^\dagger$$

And is invariant under

$$\chi \to e^{-i\alpha}\chi$$
$$\xi \to e^{+i\alpha}\xi$$

The equation of motion for this system is then

$$\begin{pmatrix} m\delta_a^{\ c} & -i\sigma^\mu_{a\dot{c}}\partial_\mu \\ -i\bar{\sigma}^{\mu\dot{a}c}\partial_\mu & m\delta^{\dot{a}}_{\ \dot{c}} \end{pmatrix} \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix} = 0$$

We can define the Dirac field

$$\Psi = \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix}, \quad \Psi^\dagger = (\chi^\dagger_{\dot{c}}, \xi^c)$$

Introducing a new matrix $\beta \equiv \cancel{\begin{pmatrix} 0 & \delta^{\dot{a}}{}_{\dot{c}} \\ \delta_a{}^c & 0 \end{pmatrix}} \begin{pmatrix} 0 & \delta^{\dot{c}}{}_{\dot{a}} \\ \delta_c{}^a & 0 \end{pmatrix}$

(this numerically equals γ^0 but meaning is different).

then

$$\bar{\Psi} \equiv \Psi^\dagger \beta = (\xi^a, \chi^\dagger_{\dot{a}})$$

that gives

$$\bar{\Psi}\Psi = \xi^a \chi_a + \chi^\dagger_{\dot{a}} \xi^{\dagger\dot{a}}$$, maintaining our convension

$$\bar{\Psi}\gamma^\mu \partial_\mu \Psi = (\xi^a, \chi^\dagger_{\dot{a}}) \begin{pmatrix} 0 & \sigma^\mu_{a\dot{c}} \partial_\mu \\ \bar{\sigma}^{\mu\dot{a}c} \partial_\mu & 0 \end{pmatrix} \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix}$$

$$= \xi^a \sigma^\mu_{a\dot{c}} \partial_\mu \xi^{\dagger\dot{c}} + \chi^\dagger_{\dot{a}} \bar{\sigma}^{\mu\dot{a}c} \partial_\mu \chi_c$$

Skip due to time

again IBP: $A\partial B = -(\partial A)B + \partial(AB)$.

$$-(\partial_\mu \xi^a) \sigma^\mu_{a\dot{c}} \xi^{\dagger\dot{c}} + \underbrace{\partial_\mu(\quad)}_{\text{surface term}}$$

$$= \xi^{\dagger\dot{c}} \sigma^\mu_{a\dot{c}} \partial_\mu \xi^a = \xi^\dagger_{\dot{c}} \bar{\sigma}^{\mu\dot{c}a} \partial_\mu \xi_a$$

number

"—"

Recall (35.19)

$$\bar{\sigma}^{\mu\dot{a}a} = \varepsilon^{ab} \varepsilon^{\dot{a}\dot{b}} \sigma^\mu_{b\dot{b}}$$

Hence

$$\bar{\Psi}\gamma^\mu\partial_\mu\Psi = \chi^\dagger\bar{\sigma}^\mu\gamma_\mu\chi + \xi^\dagger\bar{\sigma}^\mu\gamma_\mu\xi + \partial_\mu(\ )_{\text{surface}}.$$

We then transform the original Lagrangian to a very compact form

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi.$$

And such Lagrangian is invariant under

$$\Psi \to e^{-i\alpha}\Psi, \quad \bar{\Psi} \to e^{+i\alpha}\bar{\Psi}$$

which is the same as $\chi \to e^{-i\alpha}\chi$, $\xi^\dagger \to e^{-i\alpha}\xi^\dagger$

One can relate this to the charge, and the Noether current (of continuous symmetry)

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi = \chi^\dagger\bar{\sigma}\chi - \xi^\dagger\bar{\sigma}\xi.$$

Here we could have an operation called Charge Conjugation

$$C^{-1}\chi_a(x)C = \xi_a(x)$$

$$C^{-1}\xi_a(x)C = \chi_a(x)$$

And $C^{-1}\mathcal{L}(x)C = \mathcal{L}(x)$, the Langrangian conserves charge.

We can introduce the charge conjugation matrix

$$C = \begin{pmatrix} \varepsilon_{ac} & \\ & \varepsilon^{\dot{a}\dot{c}} \end{pmatrix}$$

To get $C^{-1} \Psi C = \Psi^c = \begin{pmatrix} \xi_a \\ \chi^{+\dot{a}} \end{pmatrix}$

notice that $\bar{\Psi}^T = \begin{pmatrix} \xi^a \\ \chi^+{}_{\dot{a}} \end{pmatrix}$

$$\Psi^c \equiv C \bar{\Psi}^T = \begin{pmatrix} \xi_a \\ \chi^{+\dot{a}} \end{pmatrix}.$$

Hence we have $C^{-1} \Psi C = \Psi^c$

Numerically $C^T = C^\dagger = C^{-1} = -C$

$$C^{-1} \gamma^\mu C = \begin{pmatrix} \varepsilon^{ab} & \\ & \varepsilon_{\dot{a}\dot{b}} \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu_{b\dot{c}} \\ \bar{\sigma}^{\mu \dot{b} c} & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{ce} & \\ & \varepsilon^{\dot{c}\dot{e}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \varepsilon^{ab} \sigma^\mu_{b\dot{c}} \varepsilon^{\dot{c}\dot{e}} \\ \varepsilon_{\dot{a}\dot{b}} \bar{\sigma}^{\dot{b}c} \varepsilon_{ce} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\bar{\sigma}^{\mu a \dot{e}} \\ -\sigma^\mu_{\dot{a}e} & 0 \end{pmatrix}$$

but $\gamma^\mu = \cancel{\begin{pmatrix} & \sigma^\mu_{\dot{a}e} \\ \bar{\sigma}^{\mu a \dot{e}} & \end{pmatrix}} = \begin{pmatrix} & \sigma^\mu_{e\dot{a}} \\ \bar{\sigma}^{\mu \dot{e} a} & \end{pmatrix}$

Hence $C^{-1}\gamma^\mu C = -(\gamma^\mu)^T$.

For Majorana field $\mathcal{L} = \frac{i}{2}\bar{\Psi}\gamma^\mu\partial_\mu\Psi - \frac{1}{2}m\bar{\Psi}\Psi$

but here we yet impose the majorana condition.

$$\Psi^C = \Psi.$$

Skip due to time.

we can use the above relations to write

$$\bar{\Psi} = \Psi^T C$$

$\mathcal{L}_{\text{majorana}}$

$$= \frac{i}{2}\Psi^T C\gamma^\mu\partial_\mu\Psi - \frac{1}{2}m\Psi^T C\Psi.$$

$$\Psi^C = C\bar{\Psi}^T$$

$$C^T\Psi^C = C^{-1}C\bar{\Psi}^T$$

$$\bar{\Psi} = (C^T\Psi^C)^T = \Psi^{CT}C$$

$$= \Psi^T C.$$

Here we can also find another important γ matrix, which do not have Lorentz vector index

$$\gamma_5 \equiv \begin{pmatrix} -\delta_a{}^c & \\ & +\delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix} \text{ such that}$$

Projection operator

$$P_L \equiv \frac{1}{2}(\mathbb{1} - \gamma_5) = \begin{pmatrix} \delta_a{}^c & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_R = \frac{1}{2}(\mathbb{1} + \gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix}.$$

For a Dirac field, we then have

$$P_L \Psi = \begin{pmatrix} \chi_a \\ 0 \end{pmatrix}, \quad P_R \Psi = \begin{pmatrix} 0 \\ \xi^{\dagger \dot{a}} \end{pmatrix}$$

$\gamma 5$ is an invariant symbol as (and we do not write it as γ^5 or γ_5)

$$\gamma 5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\frac{i}{24} \varepsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$$

Now, finally, let's come back to Lorentz transformation under our current convention.

$$\mathcal{U}(\Lambda)^{-1} \psi_a(x) \mathcal{U}(\Lambda) = L(\Lambda)_a{}^c \psi_c(\Lambda^{-1} x)$$

$$\mathcal{U}(\Lambda)^{-1} \psi^\dagger_{\dot{a}}(x) \mathcal{U}(\Lambda) = R(\Lambda)_{\dot{a}}{}^{\dot{c}} \psi_{\dot{c}}(\Lambda^{-1} x).$$

$$L = (1+\delta\omega)_a{}^c = \delta_a{}^c + \frac{i}{2} \delta\omega_{\mu\nu} (S_L{}^{\mu\nu})_a{}^c$$

$$R = (1+\delta\omega)_{\dot{a}}{}^{\dot{c}} = \delta_{\dot{a}}{}^{\dot{c}} + \frac{i}{2} \delta\omega_{\mu\nu} (S_R{}^{\mu\nu})_{\dot{a}}{}^{\dot{c}}$$

and

$$(S_L{}^{\mu\nu})_a{}^c = +\frac{i}{4} (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)_a{}^c$$

$$(S_R{}^{\mu\nu})_{\dot{a}}{}^{\dot{c}} = -\frac{i}{4} (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)_{\dot{a}}{}^{\dot{c}}$$

we can then define

$$\frac{i}{4} [\gamma^\mu, \gamma^\nu] = \begin{pmatrix} +(S_L{}^{\mu\nu})_a{}^c & \\ & -(S_R{}^{\mu\nu})^{\dot{a}}{}_{\dot{c}} \end{pmatrix} \equiv S^{\mu\nu}$$

Then we have

$$\mathcal{U}(\Lambda)^{-1}\Psi(x)\mathcal{U}(\Lambda) = D(\Lambda)\Psi(\Lambda^{-1}x)$$

and $D(1+\delta\omega) = 1 + \frac{i}{2}\delta\omega_{\mu\nu}S^{\mu\nu}$

and here the "$-$" for S_R is taken

care by the $\frac{i}{i}$ to $\frac{i}{i}$.

$S^{\mu\nu}$ is the generator for L.T. for Dirac field

$\Psi(x)$.

Gamma Matrices Gymnastics.

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$$

$$\gamma_5^2 = 1$$

$$\boxed{\{\gamma^\mu, \gamma_5\} = 0}.$$

$$\mathrm{Tr}\,1 = 4.$$

Now Tr[odd no. of γ_s] $= 0$

as $\mathrm{Tr}[\gamma^{\mu_1}\dots\gamma^{\mu_n}] = \mathrm{Tr}[\gamma_5^2\gamma^{\mu_1}\dots\gamma_5^2\gamma^{\mu_n}]$

$$= \mathrm{Tr}[(\gamma_5\gamma^{\mu_1}\gamma_5)\dots(\gamma_5\gamma^{\mu_n}\gamma_5)]$$

$$= (-1)^n\,\mathrm{Tr}[\gamma^{\mu_1}\dots\gamma^{\mu_n}].$$

$$\gamma^\mu \partial_\mu = \not\partial$$
$$\gamma^\mu p_\mu = \not p$$

Similarly

$$\mathrm{Tr}[\gamma_5 \ldots \underbrace{\gamma^{\mu_1} \ldots \gamma^{\mu_n}}_{\text{odd } \#}] = 0.$$

Also

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu] = \frac{1}{2} \mathrm{Tr}[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu]$$
$$= -4g^{\mu\nu}.$$

$$\mathrm{Tr}[\not a \not b] = -4(ab).$$

$$\mathrm{Tr}[\not a \not b \not c \not d] = 4[(ad)(bc) - (ac)(bd) + (ab)(cd)]$$

this is ~~prove~~ proven by

$$\not a \not b = -\not b \not a - 2(a\cdot b)$$

$$\mathrm{Tr}[\not a \not b \not c \not d] = -\mathrm{Tr}[\not b \not a \not c \not d] - 2(ab)\,\mathrm{Tr}[\not c \not d]$$

$$\mathrm{Tr}[\not c \not d] \to -4(cd)$$

$$-\mathrm{Tr}[\not b \not a \not c \not d] \to -\mathrm{Tr}[\not b \not c \not a \not d] - 2(ac)\,\mathrm{Tr}[\not b \not d]$$

$$\downarrow$$

$$\ldots$$

$$\downarrow$$

$$\ldots$$

$$= -\mathrm{Tr}[\not d \not b \not c \not d] + \cdots$$

One can ~~they~~ then perform arbitrary even traces.

Now, γ_5. $\{\gamma_5, \gamma^\mu\} = 0$ and $\gamma_5^2 = 1$

Then and traxe with γ_5s is only one γ_5.

Also $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ ~~and~~ $\mathrm{Tr}[\gamma_5] = 0$

$$\mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu] = 0$$

$$\cancel{= \mathrm{Tr}[\gamma\ \ = -\mathrm{Tr}[\gamma_5\gamma^\nu\gamma^\mu] + 2\mathrm{Tr}[\gamma_5]g^{\mu\nu}}$$

$\mu = \nu$, $\gamma^2 = 1$, $\mathrm{Tr}[\gamma_5] = 0$

$$\mu \neq \nu \begin{cases} 0, i & \mathrm{Tr}\left[\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} & \vec{\sigma} \\ -\vec{\sigma} & \end{pmatrix}\right] = 0 \\ i, j & \mathrm{Tr}\left[\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}\begin{pmatrix} & \vec{\sigma} \\ -\vec{\sigma} & \end{pmatrix}\begin{pmatrix} & \vec{\sigma} \\ -\vec{\sigma} & \end{pmatrix}\right] = \mathrm{Tr}\begin{bmatrix} +\vec{\sigma}\cdot\vec{\sigma} & \\ & -\vec{\sigma}\cdot\vec{\sigma} \end{bmatrix} = 0 \end{cases}$$

And $\mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = -4i\varepsilon^{\mu\nu\rho\sigma}$

Another set of useful trick/identity is

$$\gamma^\mu\gamma_\mu = g_{\mu\nu}\,\cancel{\gamma^\mu\gamma^\nu}\,\gamma^\mu\gamma^\nu = \frac{1}{2}g_{\mu\nu}\{\gamma^\mu, \gamma^\nu\} = -d.$$

need for dim-reg.

$$\gamma^\mu \not{a}\gamma_\mu = \gamma^\mu(-\gamma_\mu\not{a} - 2a_\mu)$$
$$= (d-2)\not{a}$$

$$\gamma^\mu\not{a}\not{b}\gamma_\mu = 4(ab) - (d-4)\not{a}\not{b}$$

- - - -.

Let's do path integral, Recall for real scalar

$$\mathcal{L}_0 = -\frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}m\varphi^2$$

$$= -\frac{1}{2}\varphi(-\partial^2+m^2)\varphi - \frac{1}{2}\partial_\mu(\varphi\partial^\mu\varphi)$$

We had

$$\langle 0|T\varphi(x_1)\ldots \cancel{\varphi(x_m)}|0\rangle = \frac{1}{i}\frac{\delta}{\delta J_\varepsilon(x_1)}\cdots Z_0(J)\Big|_{J=0}$$

$$Z_0(J) = \int \mathcal{D}\varphi\, \exp[i\int d^4x(\mathcal{L}+J\varphi)]$$

we used the $i\varepsilon$ to make sure the initial and final states are vacuum state (as others it exponentially suppresses and state with non-zero energy)

And then (perform the Gaussian integration...)

$$Z_0(J) = \cancel{\int} \exp[\frac{i}{2}\int d^4x\,d^4y\, J(x)\Delta(x-y)J(y)]$$

$$\Delta(x-y) = \int\frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2+m^2-i\varepsilon}$$

with is the inverse of the Klein-Gordon wave operator

$$(-\partial_x^2+m^2)\Delta(x-y) = \delta^4(x-y).$$

And for complex scalar field (through your HWs)

$$\mathcal{L}_0 = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi$$

$$= -\varphi^\dagger(-\partial^2 + m^2)\varphi - \partial_\mu(\varphi^\dagger \partial^\mu \varphi).$$

such that the path integral

$$Z_0(J^\dagger, J) = \int D\varphi^\dagger D\varphi \, \exp\left[i\int d^4x\,(\mathcal{L}_0 + J^\dagger\varphi + J\varphi^\dagger)\right]$$

$$= \exp\left[i\int d^4x\, d^4y\, J^\dagger(x)\Delta(x-y)J(y)\right].$$

and

$$\langle 0|T[\varphi(x_1)\ldots\varphi^\dagger(y_1)\ldots|0\rangle = \frac{1}{i}\frac{\delta}{\delta J^\dagger(x_1)}\cdots\frac{\delta}{i\delta J(y_1)}\cdots Z_0(J^\dagger,J)\Big|_{J=J^\dagger=0}$$

So now for Dirac Fermions, we do things similarly, but now pay special attention that $\Psi(x)$ and $\bar{\Psi}(x)$ are spinors ~~and~~ (not scalars that can be moved around), and the source terms are also spinors $\bar{\eta}(x)$ and $\eta(x)$ such that $\mathcal{L}$ remains a scalar quantity.

also, there are anti-commutation relations. so

$$\frac{\delta}{\delta\eta(x)}\int d^4y\,[\underbrace{\bar\eta(y)}_{\text{row vector}}\underbrace{\Psi(y)}_{\text{column vector}}+\bar\Psi(y)\eta(y)] = -\bar\Psi(x)$$

from $\eta(x)$ passing through $\bar\Psi(y)$
Anti-commutation.

$$\frac{\delta}{\delta\bar\eta(x)}\int d^4y\,[\bar\eta(y)\Psi(y)+\bar\Psi(y)\eta(y)] = \Psi(x)$$

Now the free Dirac field

$$\mathcal{L}_0 = i\bar\Psi\partial\!\!\!/\Psi - m\bar\Psi\Psi$$

$$= \cancel{-\bar\Psi(\partial\!\!\!/ \pm m)}\ \cancel{i}\ -\bar\Psi(-i\partial\!\!\!/+m)\Psi$$

We shall have

$$\langle 0|T\,\Psi_{\alpha_1}(x_1)\cdots\bar\Psi_{\beta_1}(y_1)\cdots|0\rangle$$

$$=\frac{1}{i}\frac{\delta}{\delta\bar\eta_{\alpha_1}(x_1)}\cdots\underset{(-1)/i}{i}\frac{\delta}{\delta\eta_{\beta_1}(y_1)}\cdots Z_0(\bar\eta,\eta)\Big|_{\bar\eta=\eta=0}$$

$$Z_0(\bar\eta,\eta)=\int D\bar\Psi\, D\Psi\,\exp\left[i\int d^4x(\mathcal{L}_0+\bar\eta\Psi+\bar\Psi\eta)\right]$$

$$=\exp\left[i\int d^4x\,d^4y\,\bar\eta(x)\,S(x-y)\,\eta(y)\right]$$

Here $S(x-y)$ is the Feynman propagator.

$$S(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{(-\not p + m)}{p^2+m^2-i\epsilon} e^{ip(x-y)}$$

which is the inverse of the Dirac wave operator:

$$(-i\not\partial_x + m)\, S(x-y) = \delta^4(x-y).$$

Skip the following due to time

The above can be done more rigorously, see Sec. 44.
Here I sketch the key components.

Let me introduce Grassmann numbers ψ_i
(anti commuting numbers)

$$\{\psi_i, \psi_j\} = 0, \quad i,j = 1,\dots,n.$$

For $n=1$, $\psi_1^2 + \psi_1^2 = 0 \to \psi_1^2 = 0$

So any function $f(\psi) = a + \psi b$ as high orders vanish.

Let's demand $f(\psi)$ is a regular commuting number/function
then we need to have $\{\psi, b\} = \{b, b\} = 0$
and hence

$$f(\psi) = a + \psi b = a - b\psi.$$

then we can define derivatives

$$\partial_\psi f(\psi) = b$$

$$f(\psi)\overleftarrow{\partial}_\psi = -b$$

Similarly, integrals demanding

$$\int_{-\infty}^{+\infty} dx\, c f(x) = c \int_{-\infty}^{\infty} dx\, f(x)$$

$$\int_{-\infty}^{+\infty} dx\, f(x+a) = \int_{-\infty}^{\infty} dx\, f(x)$$

$\int dx\, c f(x+d) = \int dx\, c f(x)$

$\Rightarrow \int dx\, ca + cdb = \int dx\, ca \Rightarrow \underbrace{\int dx\, \text{const}}_{=0}$

$\cancel{\int (a + xb) \leftrightarrow (a + (x+d)b)}$ tells us that consistant definition is

$$\int_{-\infty}^{+\infty} dx\, f(x) = b\,, \quad \int_{-\infty}^{\infty} dx = 0$$

Similarly for $i \to n$ for ψ_i

$$\frac{\partial}{\partial \psi_j} f(\psi) = b_j + \psi_i c_{ji} + \cdots + \frac{1}{(n-1)!} \psi_{i_2} \cdots \psi_{i_n} \underline{d_{j i_2 \cdots i_n}}$$

totally antisymmetric symbol

$$d_{i_1 \ldots i_n} = d\, \varepsilon_{i_1 \ldots i_n}$$

Levi-Civita

So $\int d^n\psi\, f(\psi) = d.$

$$\int d\psi_n\, d\psi_{n-1} \ldots d\psi_1$$

Also, consistently

$\int d\psi_j = 0, \quad \int d\psi_j\, \psi_i = \delta_{ji}$

Now, if we change coordinates linearly

$\psi_i = J_{ij}\psi_j$, J is a normal matrix

$f(\psi) = a + \cdots + \frac{1}{n!}(J_{i_1 j_1}\psi'_{j_1})\cdots(J_{i_n j_n}\psi'_{j_n})\underline{\varepsilon_{i_1\ldots i_n} d}.$

$$\prod_j \psi'_j \det J\ \varepsilon_{i_1\ldots i_n}$$

$f(\psi) = a + \cdots + \frac{1}{n!}\psi'_{j_1}\cdots\psi'_{j_n}(\det J)\varepsilon_{j_1\ldots j_n} d.$

$$\rightarrow \int d^n\psi\, f(\psi) = (\det J)^{-1}\int d^n\psi'\, f(\psi').$$

Recall for normal $\widehat{\text{commuting}}$ numbers, we have

$$\boxed{\int d^n x\, f(x) = (\det J)^{+1}\int d^n x'\, f(x').}$$

Grassmannian numbers are different.

Now, back to fermions, we can define the following

$\chi \equiv \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$

$\bar{\chi} = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2)$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \bar{x} \\ x \end{pmatrix}.$$

so $d^2\psi = d\psi_2 \, d\psi_1 = (-i)^{-1} dx \, d\bar{x}$

also $\psi_1 \psi_2 = -i \bar{x} x$, hence

$$\int dx \, d\bar{x} \; \bar{x} x = (-i)(-i)^{-1} \int d\psi_2 \, d\psi_1 \; \psi_1 \psi_2 = 1.$$

thus, for a general fuction

$$f(x, \bar{x}) = a + x b + \bar{x} c + \bar{x} x d.$$

$$\int dx \, d\bar{x} \; f(x, \bar{x}) = d$$

$$\int dx \, d\bar{x} \; \underbrace{\exp(\bar{x} m x)}_{\text{only linear order matters}} = m$$

Similarly, for

$$\cancel{d^n x \, d^m \bar{x}} \; d^n x \, d^n \bar{x} = (\det J)^{-1} (\det K)^{-1} d^n x' \, d^n \bar{x}'$$

for $x_i \to J_{ij} x'_j$

$\bar{x}_i \to K_{ij} \bar{x}'_j$

Now

if we evaluate a general matrix M.

$$\int d^n x \, d^n \bar{x} \ \exp(\underbrace{\bar{x} M x}_{\text{not diagonal}})$$

but M can be diagonalized via

$$M = VMU = M_{diag}, \qquad VV^\dagger = V^\dagger V = 1, \quad U^\dagger U = UU^\dagger = 1$$

$$\int d^n x \, d^n \bar{x} \ \exp(\bar{x} M x)$$

$$= \int d^n x \, d^n \bar{x} \ \exp(\bar{x} V^\dagger V M U U^\dagger x)$$

$$= \int d^n x' \, d^n \bar{x}' \ \exp(\bar{x}' M_{diag} x') \ \det(V^\dagger) \det(U^\dagger)$$

$$= \det(V)^{-1} \det(U)^{-1} \ \cancel{M_{diag}} \ \det M_{diag}$$

$$\det M = \det(V^{-1} M_{diag} U^{-1})$$
$$= \det V^{-1} \det M_{diag} \ dU^{-1}$$

$$\rightarrow \int d^n x \, d^n \bar{x} \ \exp(\bar{x} M x) = \det M.$$

and again, for commuting numbers

$$\int d^n z \, d^n \bar{z} \ \exp(-z^\dagger M z) = (2\pi)^n (\det M)^{-1}$$

can you prove it?

So now

$$\int d^n x \, d^n \bar{x} \, \exp(\bar{x} M x + \bar{\eta} x + \bar{x} \eta)$$

$$= (\det M) \exp(-\bar{\eta} M^{-1} \eta)$$

as the numerator is a complex square via

$$x \to x - M^{-1}\eta, \quad \bar{x} \to \bar{x} - \bar{\eta} M^{-1}$$

this justifies our path-integral.

Skip above

So now let's talk about interacting theory & Feynman Rules.

adding $\mathcal{L}_1 = g\varphi \bar{\Psi}\Psi$.

real scalar.

tree-level only.

We see at $d=4$, g is dimensionless.

This interacting still preserves charge. as

$$\Psi \to e^{-i\alpha}\Psi \text{ makes } \mathcal{L}_1 \text{ invariant.}$$

This theory is called "Scalar QED"

$$\text{Now} \quad Z(\underset{\bar\Psi}{\underset{\uparrow}{\bar\eta}}, \underset{\Psi}{\underset{\uparrow}{\eta}}, \underset{\varphi}{\underset{\uparrow}{J}}) \propto \exp\left[ig\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\, \frac{1}{i}\frac{\delta}{\delta \bar\eta_\alpha(x)}\, \frac{(-1)}{i}\frac{\delta}{\delta \eta(x)}\right)\right] Z_0(\bar\eta, \eta, J).$$

where

$$Z_0(\bar\eta, \eta, J) = \exp\left[i\int d^4x\, d^4y\; \bar\eta(x) S(x-y) \eta(y)\right]$$
$$\times \exp\left[i\int d^4x\, d^4y\; J(x) \Delta(x-y) J(y)\right]$$

$$\text{with} \quad S(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{(-\not{p}+m)}{p^2+m^2-i\epsilon} e^{ip(x-y)}$$

$$\Delta(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2+m^2-i\epsilon} e^{ip(x-y)}.$$

Similarly to the scalar field theory case, we have

$$Z(\bar\eta, \eta, J) = \underbrace{\exp\left[iW(\bar\eta, \eta, J)\right]}_{\substack{\text{connected Feynman diagrams} \\ \text{(with sources).}}}$$