9.13 The Riemann-Lebesgue Lemma and Its Applications in Quantum Field Theory

Introduction

The Riemann-Lebesgue lemma is a fundamental result in Fourier analysis and integral theory with profound implications in physics, particularly in quantum mechanics, signal processing, spectral theory, and quantum field theory.

The Riemann-Lebesgue lemma governs the decay of Fourier/inverse Fourier transforms for L^1-functions but does not impose pointwise decay on the original function. Lebesgue integrability is a broad class requiring finite total variation, while compact support provides stricter control over a function's asymptotic behavior. These concepts are crucial in QFT for understanding field behavior, ensuring causality, and managing divergences in calculations.

Key Concepts

Riemann-Lebesgue Lemma

The lemma states that for any Lebesgue integrable function $f \in L^1(\mathbb{R})$:

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0 \quad \text{and} \quad \lim_{|x|\to\infty} \check{f}(x) = 0,$$

where $\hat{f}$ is the Fourier transform and $\check{f}$ is the inverse Fourier transform.

Critical point: The vanishing behavior applies to the *transform*, not the original function. A function $f(x)$ need not vanish at infinity for its Fourier/inverse Fourier transform to decay.

Lebesgue Integrability

A function f is Lebesgue integrable ($f \in L^1$) if:

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

This is a weaker condition than pointwise decay: L^1-functions can have oscillations or sparse "spikes" at infinity.

Smoothness and Compact Support

Smoothness and compact support guarantee vanishing at infinity, but this is a stronger condition than Lebesgue integrability:

- Compact support directly enforces $f(x) = 0$ outside a finite interval.
- Smoothness ensures rapid decay of the Fourier transform, which is useful in spectral analysis and PDEs.

Physics Interpretations and Examples

Vanishing of Fourier/Inverse Fourier Transforms vs. Original Functions

In QFT, free field solutions (e.g., plane waves) have Fourier transforms concentrated at specific frequencies (on-shell momenta), but the fields themselves (e.g., $\phi(x)$) do *not* vanish at spatial infinity. Instead, their Fourier transforms (e.g., $\hat{\phi}(k)$) decay due to the lemma. We often argue that when Fourier transforms the fields at space-time infinity, the integrated oscillates rapidly and destructively interfere (inferences to Fourier spectrum), and it would vanish, and hence the boundary piece does not contribute to the spectrum.

Example: The electron field operator $\psi(x)$ in QFT is a distribution. Its Fourier transform $\hat{\psi}(k)$ decays at infinity, but $\psi(x)$ need not vanish spatially.

Smooth, Compactly Supported Functions in QFT

Test functions in QFT (e.g., $g(x) \in C_c^\infty(\mathbb{R}^n)$) are smooth and compactly supported, ensuring:

- Causality: Interactions are localized in spacetime.
- Energy finiteness: Avoids infinite energy from non-compact sources.
- Renormalization: UV/IR divergences are regulated via cutoff functions.

Example: In perturbative QFT, Feynman diagrams are regularized by inserting smooth cutoff functions like $e^{-\epsilon x^2}$ to suppress high-energy modes.

Connection to Quantum Mechanics and Hilbert Spaces

In quantum mechanics, $L^2(\mathbb{R}^3)$ (Lebesgue square-integrable wavefunctions) forms a Hilbert space, essential for the probabilistic interpretation. Riemann integrability fails here for functions like discontinuous potentials.

Key Takeaways for QFT

1. **Decoupling at infinity:** The Riemann-Lebesgue lemma underpins renormalization by ensuring high-momentum contributions vanish, allowing finite predictions despite divergent integrals.
2. **Tempered distributions:** Quantum fields are operator-valued tempered distributions. Their Fourier transforms decay, but the fields themselves may not vanish spatially.

3. **Test functions:** Smooth, compactly supported functions (e.g., C_c^∞) are critical for defining local observables and ensuring causality in QFT.