9.12 The $1 - i\epsilon$ Prescription in the Path Integral

Purpose of the $1 - i\epsilon$ Factor

The factor $1 - i\epsilon$ is introduced in the Hamiltonian to achieve two critical goals:

- Ensure convergence of oscillatory path integrals by suppressing high-energy contributions. The path integral is understood as an asymptotic theory and the $t \to \pm\infty$ proper quantities. Correspondingly, the transition amplitude between asymptotic states.
- Ensure the path integral project onto the vacuum (ground state) at asymptotic times $t \to \pm\infty$, which simplifies our treatment.

Ground State Projection at Asymptotic Times

Consider the transition amplitude between states $|i\rangle$ and $|f\rangle$:

$$\lim_{t\to\pm\infty} \langle f|e^{-iH(1-i\epsilon)t}|i\rangle.$$

As $t \to \pm\infty$, the damping factor $e^{-\epsilon Ht}$ suppresses all states except the ground state $|0\rangle$:

$$\lim_{t\to\pm\infty} e^{-\epsilon Ht}|n\rangle = \lim_{t\to\pm\infty} e^{-\epsilon E_n t}|n\rangle = \begin{cases} |0\rangle & \text{if } E_n = E_0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the amplitude reduces to the vacuum-to-vacuum transition.

Connection to Adiabatic Switching

The $1 - i\epsilon$ factor mimics adiabatically turning off interactions:

- In scattering theory, interactions are "switched off" adiabatically to define free in/out states.
- The prescription suppresses interaction terms in $\mathcal{H}$ at asymptotic times, projecting the interacting vacuum onto the free vacuum.

Physical Interpretation

- **Stability of the vacuum**: Ensures only the lowest-energy state survives at $t \to \pm\infty$.
- **Scattering theory**: Enforces free-field in/out states by "preparing" the vacuum.
- **LSZ reduction**: Necessary to extract connected Greens functions for S-matrix elements.