Path Integral.

Sec. 6th

Consider a non-relativistic QM in one dimension.

$$\mathcal{H}(P, Q) = \frac{1}{2m} P^2 + V(Q).$$

P and Q are operators. $[Q, P] = i\hbar$.

We want to evaluate the particle transition probability.

starts at q' at t' and end at q'', t''

$$\langle q'' | e^{-i\mathcal{H}t''} e^{i\mathcal{H}t'} | q' \rangle.$$

where $|q''\rangle$ and $|q'\rangle$ are eigenstates of Q.

as the time evolution of $|q'\rangle$ is $e^{i\mathcal{H}t}$

formulating this question in the Heisenberg picture

$$Q(t) = e^{i\mathcal{H}t} Q(0) e^{-i\mathcal{H}t}$$

The instantaneous eigenstate of $Q(t)$ can be defined

via

$$Q(t) |q, t\rangle = q |q, t\rangle$$

and $\quad |q, t\rangle = e^{+i\mathcal{H}t} |q\rangle$

where $\quad \underset{t=0}{Q} |q\rangle = q |q\rangle$

Then the transition amplitude is as simple as

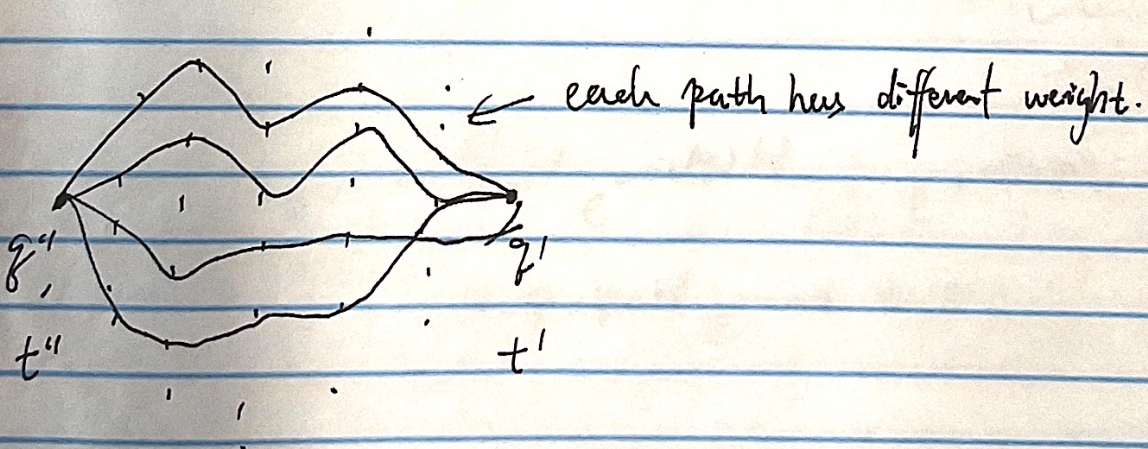
$\langle q'', t'' | q', t' \rangle$ in the Heisenberg picture.

To evaluate, we can do the following.

$$\langle q'', t'' | q', t' \rangle = \int \prod_{j=1}^{N} dq_j \, \langle q'' | e^{-i\mathcal{H}\delta t} | q_N \rangle \langle q_N | e^{-i\mathcal{H}\delta t} | q_{N-1} \rangle \langle q_{N-1} | e^{-i\mathcal{H}\delta t} | \cdots \cdots \langle q_1 | e^{-i\mathcal{H}\delta t} | q' \rangle$$

by inserting N complete states. $|q_i\rangle\langle q_i| \int_{-\infty}^{\infty} dq_i$

basically I am evaluating the transition as a product of small ~~tr~~ transitions $\delta t = \frac{t'' - t'}{N}$



For any small amplitude considered, using the Campbell - Baker - Hausdorf formula

$$\exp(A+B) = \exp A \, \exp B \, \exp(-\tfrac{1}{2}[A,B] + \cdots)$$

additional terms if $[A,[A,B]] \neq 0$ or $[B,[A,B]] \neq 0$

$$e^{-i\mathcal{H}\delta t} = e^{-i\frac{P^2}{2m}\delta t} \, e^{-iV(Q)\delta t} \, e^{[O(\delta t^2)]}$$

The error I am making is

$$e^{-i\mathcal{H}\delta t} - e^{-iA\delta t} e^{-iB\delta t} = -\frac{[A,B]}{2}\delta t^2 + \cdots$$ upon integration, $-\frac{[A,B]}{2}\delta t \, t \xrightarrow{} 0$ for $\delta t \to 0$ limit.

for small δt ~~$\ll \mathcal{H}^{-1}$ ... & other scales of this EFT~~

we approximatelly have

$$\langle q_2 | e^{-i\mathcal{H}\delta t} | q_1 \rangle = \int dP_1 \langle q_2 | e^{-i\frac{P^2}{2m}\delta t} | P_1 \rangle \langle P_1 | e^{-iV(Q)\delta t} | q_1 \rangle$$

$$= \int dP_1 \, e^{-i\frac{P_1^2}{2m}\delta t} e^{-iV(q_1)\delta t} \underbrace{\langle q_2 | P_1 \rangle \langle P_1 | q_1 \rangle}_{\frac{1}{(2\pi)^{1/2}} e^{-ipq}}$$

$$= \int dP_1 \, e^{-i\frac{P_1^2}{2m}\delta t} e^{-iV(q_1)\delta t} \frac{1}{(2\pi)} e^{-i P_1(q_1 - q_2)}$$

$$= \int \frac{dP_1}{(2\pi)} e^{-i\mathcal{H}(P_1, q_1)\delta t} e^{-iP_1(q_1 - q_2)}$$

Trotter Product Formula:

$$\lim_{N\to\infty} (e^{-iA\epsilon} e^{-iB\epsilon})^N = e^{-i(A+B)t}$$

$t \equiv N\epsilon$.

Here $\mathcal{H}(P,Q)$ is simple and we could do separation of variables, but for more complex ones, e.g. $V(Q) \to V(P,Q)$. We can adopt

$\boxed{\text{Wyle Ordering}}$ ← from classical hamiltonian $\mathcal{H}(p,q)$

$$\mathcal{H}(P,Q) \equiv \int \frac{dx}{2\pi}\frac{dk}{2\pi}\, \underbrace{e^{ixP+ikQ}}_{\uparrow} \int dp\,dq\, \underbrace{e^{-ixp-ikq}}_{\uparrow}\, \mathcal{H}(p,q)$$

$\delta(P-p)$

Then the corrected transitions amplitude would be

$$\langle q_2 | e^{-i\mathcal{H}\delta t} | q_1 \rangle = \int \frac{dP_1}{2\pi}\, e^{-i\mathcal{H}(p_1,\, \frac{1}{2}(q_1+q_2))\delta t}\, e^{ip_1(q_2-q_1)}$$

Adopting Weyl ordering for the general case:

$$\langle q'', t'' | q', t' \rangle = \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dP_j}{2\pi}\, e^{iP_j(q_{j+1}-q_j)}\, e^{-i\mathcal{H}(P_j,\bar{q}_j)\delta t}.$$

Isn't averaging over initial & final more satisfying?

$$\bar{q}_j = \boxed{\frac{1}{2}(q_j + q_{j+1})}, \quad q_0 = q', \quad q_{N+1} = q''$$

If we now define $\dot{q}_j = \frac{q_{j+1}-q_j}{\delta t}$ and take the limit $\delta t \to 0$, we get

$$\langle q'', t'' | q', t' \rangle = \int \mathcal{D}q\,\mathcal{D}p\, \exp\left[i\int_{t'}^{t''} dt\, \left(p(t)\dot{q}(t) - \mathcal{H}(p(t), q(t))\right)\right] \quad \ldots (6.8)$$

(5)

One can further simplify by noticing that

for simple enough $\mathcal{H}(p, q)$

(p less or equal to than p^2, and p^2 has ~~not~~ q dependence)

The E.O.M is the replacement equation for p to finish the Gaussian integral.

(this is Gaussian and the normalization / prefactor)

$$\langle q'', t'' | q', t' \rangle = \int \mathcal{D}q \exp . \left[i \int_{t'}^{t''} dt \, L(\dot{q}(t), q(t)) \right]$$

$L(\dot{q}, q)$ is computed by first finding the stationary point of the p integral by solving

$$0 = \frac{\partial}{\partial p} (p\dot{q} - \mathcal{H}(p,q)) = \dot{q} - \frac{\partial \mathcal{H}(p,q)}{\partial p}$$

for p in terms of $\dot{q}$ and q, and then plugging back into $p\dot{q} - \mathcal{H}$. to get L.

Gaussian integral

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}. \quad \longrightarrow \quad \int_{-\infty}^{+\infty} e^{-(x+a)^2} d(x+a) = \sqrt{\pi}$$

$$\int_{-\infty}^{+\infty} e^{-(x/a)^2} dx = a\sqrt{\pi}.$$

Quick derivation

$$I \equiv \int_{-\infty}^{+\infty} e^{-x^2} dx$$

$$I^2 = \int_{-\infty}^{+\infty} dx\,dy \; e^{-(x^2+y^2)}$$

$$= \int_0^{2\pi} d\theta \, dr \cdot r \; e^{-r^2}$$

$$= 2\pi \times \frac{1}{2} \int_0^{\infty} dr^2 e^{-r^2} = -\pi \left(e^{-r^2} \right) \Big|_0^{\infty} = \pi .$$

Now let's try to generalize:

$$\langle q'', t'' | Q(t_1) | q', t' \rangle = \langle q'' | e^{-iH(t''-t_1)} Q e^{-iH(t_1-t')} | q' \rangle$$

In the path-integral formula.
this $Q(t_1)$ will generate an extra $q(t_1)$

[e.g. from Eq 6.] $\langle q'', t'' | q', t' \rangle = \int \prod dq_k \prod \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-iH(p_j, \bar{q}_j)\delta t}$)

$$\langle q'', t'' | Q(t_1) | q', t' \rangle = \int \mathcal{D}p \mathcal{D}q \, q(t_1) e^{iS}$$

where $S = \int_{t'}^{t''} dt (p\dot{q} - H)$

Now consider the opposite side $\int \mathcal{D}p \mathcal{D}q \, q(t_1) q(t_2) e^{iS}$

this clearly requires time ordering $Q(t_1)$ and $Q(t_2)$

If $t_1 < t_2$ $\quad Q(t_2) Q(t_1)$

If $t_1 > t_2$ $\quad Q(t_1) Q(t_2)$

so $\int \mathcal{D}p \mathcal{D}q \, q(t_1) q(t_2) e^{iS}$

$$= \langle q'', t'' | \underbrace{T}_{\uparrow} Q(t_1) Q(t_2) | q', t' \rangle.$$

this is what appears in LSZ.

We can quickly realize that we are dealing with functionals.

We can definite the functional ~~derivative~~ variation

$$\frac{\delta f(t_2)}{\delta f(t_1)} = \delta(t_2 - t_1)$$

(it satisfies the product rule, chain rule &.

Now, consider modifications to the Lagrangian.

$$H(p,q) \to H(p,q) - f(t)q(t) - h(t)p(t).$$

$f(t)$ and $h(t)$ are specified functions

$$\langle q'', t'' | q', t' \rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\, e^{i\int_{t'}^{t''} dt(p\dot{q} - H + fq + hp)}$$

then

$$\frac{1}{i}\frac{\delta}{\delta f(t_1)} \langle q'', t'' | q', t' \rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\; q(t_1)\, e^{iS_{f,h}}.$$

$$\frac{1}{i}\frac{\delta}{\delta f(t_1)}\frac{\delta}{\delta f(t_2)} \langle \quad \rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\, q(t_1) q(t_2)\, e^{iS_{f,h}}$$

$$\frac{1}{i}\frac{\delta}{\delta h(t_1)} \langle \quad \rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\; p(t_1) e^{iS_{f,h}}.$$

So we can get any # of p, q and then set $h(t) = f(t) = 0$. that get us back to the original Lagrangian.

$$\langle q'', t'' | T(Q(t_1) \cdots P(t_n) \cdots | q', t' \rangle$$

$$= \frac{1}{i}\frac{\delta}{\delta f(t_1)} \cdots \frac{\delta}{\delta h(t_n)} \cdots \langle q'', t'' | q', t' \rangle_{f,h} \Big|_{f=h=0}$$

If we are interested in initial and final states other than position Eigenstates, we will need to integrate them over. We are also taking $t' \to -\infty$, $t'' \to +\infty$

$$\langle 0|0\rangle_{f,h} = \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} \int dq'' dq' \, \psi_0^*(q'') \langle q'', t'' | q', t' \rangle \psi_0(q')$$

$$\psi_0(q) = \underbrace{\langle q|0\rangle}_{\text{ground state wave function in } q\text{-space}}, \quad \psi_0^* = \langle 0|q''\rangle$$

let $|n\rangle$ denote an eigenstate of H with eigenergy E_n
suppose that $E_0 = 0$. If this is not true, we will shift H by such a constant.

$$|q', t'\rangle = e^{iHt'}|q'\rangle = \sum_{n=0}^{\infty} e^{iHt'}|n\rangle\langle n|q'\rangle$$

$$= \sum_{n=0}^{\infty} \psi_n^*(q') e^{iE_n t}|n\rangle$$

$\psi_n(q) = \langle q|n\rangle$ as the wave function of the nth eigenstate.

Now replace H with $(1-i\epsilon)H$, ϵ infinitesimal (positive)
Then take the limit $t' \to -\infty$
Every state except the ground state is multiplied by a ~~fixed constant~~ exponentially vanishing factor of $\lim_{t \to -\infty} e^{+\epsilon E_n t}$.

Similarly, this also picks out the ground state for $t'' \to +\infty$

Futher, we can multiply arbitrary functions $\chi(q')$ and integrate over, which will yield a constant shift that can be absorbed into $\mathcal{D}p\mathcal{D}q$ [We then only need to care $\langle 0|0\rangle_{f,h}$] (Damping & project onto vacuum)

This implies that if we use

$$\mathcal{H} \to (1-\epsilon)\mathcal{H}, \quad \text{we can be}$$

cavalier about boundary conditions ~~as~~.

Any reasonable boundary conditions will result in good ground state as both the initial & final state

$$\langle 0|0\rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\, e^{i\int_{-\infty}^{+\infty} dt\,(p\dot{q} - (1-\epsilon)\mathcal{H} + fq + hp)}$$

Suppose $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$

(note the connection with interaction picture, in contrast to Schrödinger Picture & Heisenberg Picture)

$\mathcal{H}_0$ is solvable (e.g. free theory)

$\mathcal{H}_1$ is a perturbation.

$$\langle 0|0\rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\, e^{i\int_{-\infty}^{\infty} dt\,(p\dot{q} - \underset{\uparrow\,(1-\epsilon)}{\mathcal{H}_0(p,q)} - \underset{\uparrow\,(1-\epsilon)}{\mathcal{H}_1(p,q)} + fq + hp)}$$

but we just for shorthand not write them out explicitly.

$$= e^{[-i\int_{-\infty}^{\infty} dt\, \mathcal{H}_1(\frac{\delta}{\delta h(t)}, \frac{\delta}{\delta f(t)})]} \times$$

$$\underbrace{\int \mathcal{D}p\,\mathcal{D}q\, e^{i\int_{-\infty}^{\infty} dt\,(p\dot{q} - \mathcal{H}_0(p,q) + fq + hp)}}$$

we assume we know how to calculate this. (We shall see next a calculable example)

Now, if H_1 only on q (not on p)

and if we (surely) are only interested in time-ordered

products of Q_s (not P_s), and if H is no

more than quadratic on P and the term quadratic in P

not dependent on Q, then we can simplify

the above equations to

$$\langle 0|0\rangle_f = e^{i\int_{-\infty}^{+\infty} dt\, L_1\left(\frac{1}{i}\frac{\delta}{\delta f(t)}\right)} \times$$

$$\int \mathcal{D}q\, \cancel{\mathcal{D}p}\, \exp\left[i\int_{-\infty}^{+\infty} dt\left(L_0(\dot{q}, q) + fq\right)\right].$$

Here $L_1(q) = -H_1(q)$.