9.18 The $i\epsilon$ Prescription and Logarithms in Loop Integrals

Introduction: The Problem of the Imaginary Part

When calculating loop integrals in Quantum Field Theory, one frequently encounters logarithmic terms that depend on Mandelstam variables, such as $\log(-s/m^2)$ or $\log(t/m^2)$. A critical question arises when these kinematic invariants are in their physical regions (e.g., $s > 0$ and $t < 0$), as the argument of the logarithm becomes negative. This introduces an imaginary part of the form $\pm i\pi$. Note that such integrals come from the integrand of $D(x_1, x_2 ...; k_1, k_2 ...)$ where we can restore the $-i\epsilon$ easily since it always contains a $+m^2$ term (following our metric choice).[2]

The sign is not arbitrary; it is unambiguously fixed by the fundamental principles of causality. This physical requirement is mathematically encoded in the Feynman $i\epsilon$ prescription, see earlier notes Section 9.12. This section explains how this one fundamental rule determines the correct imaginary parts for all resulting functions.

The Fundamental Rule: Causality and the Feynman Propagator

The origin of all $i\epsilon$ rules in a scattering amplitude can be traced back to a single source: the Feynman propagator. For a scalar particle of mass m, the propagator in momentum space is:

$$\frac{i}{p^2 + m^2 - i\epsilon} \quad \text{where } \epsilon \to 0^+ \tag{9.18.1}$$

This $+i\epsilon$ (which can be thought of as giving the mass a small negative imaginary part, $m^2 \to m^2 - i\epsilon$) is the precise mathematical statement of causality. It ensures that positive-energy states propagate forward in time. Every other prescription for handling singularities in the final, integrated amplitude is a **consequence** of applying this fundamental rule to all propagators in the loop *before* integration.

[2] The m^2 in the denominator is nothing but a dimensionful quantity to make the argument dimensionless, so it doesn't come with a $-i\epsilon$.

The Consequence: A "Simple Rule" for Kinematic Invariants

When we perform the loop integration (e.g., using Feynman parameters), the $i\epsilon$ terms from all internal propagators combine. They work together to define the analytic properties of the final integral, which we can call $I(s, t, u, \dots)$. The result of this process is that the final amplitude I is an analytic function of its kinematic variables.
This leads to a simple, practical shortcut for evaluating the final expressions, which we can call the "Simple Rule":

> To find the correct imaginary part of a final amplitude, replace all independent kinematic invariants x (like s, t, and u) with $\mathrm{x} + \mathrm{i}\epsilon$.

This rule is the *result* of the fundamental $m^2 \to m^2 - i\epsilon$ prescription, combined with the knowledge that all products of x_is are positive, not a new or independent assumption.

Applying the Simple Rule to Logarithms

We use the standard "principal branch" convention for the complex logarithm, where the argument $\mathrm{Arg}(z)$ is defined to be in the range $(-\pi, \pi]$. The imaginary part of the log is determined by the quadrant of its argument.

Case 1: $\log(t/m^2)$

In this region, t is the (spacelike) momentum transfer, so $t < 0$. Let $t = -|t|$.

- **Apply Rule:** We replace $t \to t + i\epsilon = -|t| + i\epsilon$.
- **Evaluate Log:** $\log\left(\frac{t+i\epsilon}{m^2}\right) = \log\left(\frac{-|t|+i\epsilon}{m^2}\right)$.
- **Analysis:** The argument $z = (-|t| + i\epsilon)/m^2$ has a negative real part and a small positive imaginary part. It is in the **second quadrant**.
- **Result:** $\mathrm{Arg}(z) = +\pi$. Therefore:

$$\log(t/m^2) \to \log\left(\frac{|t|}{m^2}\right) + i\pi$$

Case 2: $\log(-s/m^2)$

In this region, s is the (timelike) center-of-mass energy squared, so $s > 0$.

- **Apply Rule:** We replace $s \to s + i\epsilon$.
- **Evaluate Log:** $\log\left(\frac{-(s+i\epsilon)}{m^2}\right) = \log\left(\frac{-s-i\epsilon}{m^2}\right)$.
- **Analysis:** The argument $z = (-s - i\epsilon)/m^2$ has a negative real part and a small negative imaginary part. It is in the **third quadrant**.

- **Result:** $\text{Arg}(z) = -\pi$. Therefore:

$$\log(-s/m^2) \to \log\left(\frac{s}{m^2}\right) - i\pi$$

This imaginary part is physically crucial. It is related via the Optical Theorem to the discontinuity of the amplitude, which corresponds to the probability of the intermediate particles going on-shell.