Canonical Quantization of Scalar Fields

(Also known as Second Quantization) [comment on this v.s Path-Integral].

Back to free particles.

$$\mathcal{H} = \int d^3x \, a^\dagger(\vec{x}) \left(-\frac{1}{2m}\vec{\nabla}^2\right) a(\vec{x}).$$

$$= \int d^3p \, \hat{a}^\dagger(\vec{p}) \left(\frac{1}{2m}\vec{p}^{\,2}\right) \hat{a}(\vec{p})$$

Will this easy transform to a Lorentzian (Respect special relativity easily?)

here $\hat{a}(\vec{p}) = \int d^3x / (2\pi)^{3/2} \, e^{-i\vec{p}\cdot\vec{x}} \, a(\vec{x})$

You should all have seen such "Dual" Representation in QM.

Here I dropped $\hbar$ and c (we're in Natural Units)

$$\hbar = c = 1.$$

And we again have

$$[\hat{a}(\vec{p}), \hat{a}(\vec{p}')]_\mp = 0$$

$$[\hat{a}^\dagger(\vec{p}), \hat{a}^\dagger(\vec{p}')]_\mp = 0$$

$$[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{p}')]_\mp = \delta^3(\vec{p}-\vec{p}')$$

$$[A, B]_\mp = AB \mp BA.$$

$-$ for Bosons

$+$ for Fermions.

$\hat{a}^{+}(\vec{p})$ is the creation operator of a particle with momentum $\vec{p}$. and annihilated by $\hat{a}(\vec{p})$

H is a theory of free particles.

Vacuum $|0\rangle$ has

$$\hat{a}(\vec{p})|0\rangle = 0$$

The other eigenstates of H are all of the form

$$\hat{a}^{+}(\vec{p}_1)\hat{a}^{+}(\vec{p}_2)\cdots\hat{a}^{+}(\vec{p}_n)|0\rangle$$

corresponds to n-particles ~~of~~ with energy

$$E = E_1(\vec{p}_1) + E_2(\vec{p}_2) + \cdots + E_n(\vec{p}_n)$$

with $E_i(\vec{p}_i) = \frac{1}{2m}\vec{p}_i^{\,2}$

Now we can generalize~~d~~ it to relativistic one.

$$E(\vec{p}) = \sqrt{\vec{p}^{\,2} + m^2}$$

we again can guess

$$H = \int d^3p \, (\vec{p}^{\,2} + m^2)^{1/2} \, \hat{a}^{+}(\vec{p})\hat{a}(\vec{p})$$

now this is a relativistic theory of free (spin zero) particles.

Is it ~~Lorentz~~ the theory Invariant? Not in an Obvious way.

Let's see from a different view (emphasizing Lorentz Invariance)

Begin with a classical "Scalar field" $\varphi(x)$.

Scalar field means it is invariant under coordinate transformation.

For a same field, two people at different inertial frames find identical values of them.

$$\boxed{\varphi(x)} = \boxed{\bar{\varphi}(\bar{x})}, \quad \bar{x}^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}.$$

This is in contrast to other covariant fields, such as vector fields

~~to avoid confusion, let me remove the ‾ on $\bar{\varphi}$ [in general] $\varphi(x) \xrightarrow{\Lambda} \varphi'(x) = \bar{\varphi}(\bar{x})$. Given the scalar feature $\bar{\varphi}$ is the same function as φ.~~

We already know that

$$(-\partial^2 + m^2)\varphi(x) = 0.$$

recall: $(-\bar{\partial}^2 + m^2)\bar{\varphi}(\bar{x}) = 0$

The equation of motion (EoM) can be derived from variation of an action. $S = \int dt\, L$ here L is the Lagrangian

Action is local and can be written as

$$L = \int d^3x\, \underbrace{\mathcal{L}}_{\text{Lagrangian Density}}.$$

$$\rightarrow \quad S = \int dt\, L = \underbrace{\int d^4x}_{\text{Lorentz invariant}}\, \mathcal{L}.$$

$$d^4\bar{x} = |\det \Lambda|\, d^4x = d^4x$$

For the action S to be Lorentz Invariant, we need $\mathcal{L}$ to be Lorentz Invariant:

$$\tilde{\mathcal{L}}(\bar{x}) = \underline{\mathcal{L}(x)}$$

Let's construct it:

any simple function of $\underbrace{\varphi(x)}_{\text{L. Scalar}}$ would be L. Scalar.

we can take

$$\mathcal{L}(x) = \underbrace{-\frac{1}{2}\partial^\mu\varphi(x)\,\partial_\mu\varphi(x)}_{[2]+[2]} \cancel{+} \underbrace{-\frac{1}{2}m^2}_{+2}\underbrace{\varphi^2(x)}_{+2} + \underbrace{\Omega_0}_{+4}$$

Dimensional Analysis $\qquad \varphi(x)\ [1]$

Teach students quickly. (Here declare Action S is dimensionless, actually $\hbar$).

Euler Lagrange Equation. (E.o.M.)

$$0 = \delta S = \int d^4x\, \delta\mathcal{L}$$

(Same as textbook)

$$= \int d^4x \left[-\frac{1}{2}\partial^\mu\delta\varphi\,\partial_\mu\varphi - \frac{1}{2}m^2\cdot 2\cancel{\delta\varphi}\varphi\delta\varphi - \frac{1}{2}\partial^\mu\varphi\,\partial_\mu\delta\varphi\right]$$

In class, say we restrict ourselves to $\delta\varphi = 0$ at boundary

see discussion in additional notes

$$\cancel{= \int d^4x\, [-\partial^\mu\varphi\partial_\mu\varphi - m^2\varphi^2}$$

$$= \int d^4x\, [\partial^\mu\partial_\mu\varphi - m^2\varphi]\,\delta\varphi$$

$$- \text{surface terms} \left\{ - \boxed{\delta\varphi(x)}\, \partial\varphi(x)\Big|_{x^i\to\pm\infty}\right.$$

← sloppy way of writting

An alternative $\boxed{\text{Skipping Pg ⑤-⑥}}$

(quick solution to $\varphi(x)$).

Recall we need to solve

$$(\partial^2 - m^2)\varphi(x) = 0 \Rightarrow (\vec{\partial}^2 - \partial_0^2 - m^2) \cancel{\varphi_0}\, \varphi(x) = 0$$

$$\cancel{\varphi(x) = \int d^3k / f(k)\, e^{ikx}}$$

the solution should be as simple as

$$\varphi(x) = \int \frac{d^3k}{f(k)}\, a(\vec{k})\, e^{ikx} + b(\vec{k})\, e^{-ikx}$$

where $kx = \vec{k}\cdot\vec{x} - \omega t$, $\omega = \pm\sqrt{\vec{k}^2 + m^2}$

$$\varphi^*(x) = \boxed{\int \frac{d^3k}{f(k)}}\, a^*(\vec{k})\, e^{-ikx} + b^*(\vec{k})\, e^{+ikx}$$

So $\varphi(x) = \varphi^*(x)$ by reality condition

we find

$$a^*(\vec{k}) = b(\vec{k}), \quad b^*(\vec{k}) = a(\vec{k})$$

Hence

$$\varphi(x) = \int \frac{d^3k}{f(k)}\, a(\vec{k})\, e^{ikx} + a^*(\vec{k})\, e^{-ikx}$$

Here we didn't spend much time discussing the energy term. We ~~might~~ encounter it again later.

$$kx = \vec{k}\cdot\vec{x} - \omega t = k^\mu x_\mu = k_\mu x^\mu = g^{\mu\nu} k_\mu x_\nu = g_{\mu\nu} k^\nu x^\mu$$

$$k^2 = k^\mu k_\mu = \vec{k}^2 - \omega^2 = -m^2.$$

A four momentum k^μ that obeys $k^2 = -m^2$

is "on-shell" (on the mass ~~sell~~ $\underbrace{\text{shell}}$).

in k^μ space configuration).

Now let's see what makes

$$\int \frac{d^3k}{f(k)} \quad \text{L.I.}$$ [and select the positive energy state]

$SO(1,3)^\uparrow$

Obviously d^4k is L.I.

$$d^4k\, \delta(\underline{k^2} + \underline{m^2})\, \theta(k^0)$$ is L.I. for on-shell positive energy

L.I. L.I. Selecting the positive energy "orthochronous"

↑ Dirac Delta function.

↑ Unit step function

$$\int_{-\infty}^{\infty} dk^0\, \delta(k^2+m^2)\,\theta(k^0) = \frac{1}{2\omega}.$$

as $$\int_{-\infty}^{\infty} dx\, \delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|}$$

$g(x)$ being any smooth function with simple zeros at $x = x_i$

thus for $f(k) \propto \omega$,

$d^3k / f(k)$ would be Lorentz Invariant
(for on-shell states)

$$\widetilde{dk} \equiv \frac{d^3k}{(2\pi)^3 2\omega}.$$

Hence

$$\varphi(x) = \int \widetilde{dk}\,[a(\vec{k})e^{ikx} + a^*(\vec{k})e^{-ikx}].$$

We can get $a(\vec{k})$ via ~~fourier~~ A general transformation (motivated by Fourier transformation between $\vec{x} \leftrightarrow \vec{p}$)

$$\int d^3x\, e^{-ikx}\varphi(x) = \frac{1}{2\omega}a(\vec{k}) + \frac{1}{2\omega}e^{2i\omega t}a^*(-\vec{k})$$

$$\int d^3x\, e^{-ikx}\partial_0\varphi(x) = -\frac{i}{2}a(\vec{k}) + \frac{i}{2}e^{2i\omega t}a^*(-\vec{k})$$

Note:
(3d and 4d)

$$\int_{-\infty}^{\infty} e^{ik(x-x')}\,\boxed{d^dk} = (2\pi)^{(d)}\delta^{(d)}(x-x')$$

← for consistant notation.

$$\int_{-\infty}^{\infty} e^{i(k-k')x}\,\boxed{d^dx} = (2\pi)^{(d)}\delta^{(d)}(k-k')$$

$$\Rightarrow a(\vec{k}) = \int d^3x\, e^{-ikx}[i\partial_0\varphi(x) + \omega\varphi(x)]$$

$$= i\int d^3x\, e^{-ikx}\overleftrightarrow{\partial_0}\varphi(x)$$

here $f\overleftrightarrow{\partial_\mu}g = f(\partial_\mu g) - (\partial_\mu f)g$, $\partial_0\varphi = \frac{\partial\varphi}{\partial t} = \dot{\varphi}$

$a(\vec{k})$ is time-independent.

With Lagrangian, we can construct Hamiltonian.

Classical mechanics:

given $L(q_i, \dot{q}_i)$ of coordinate q_i and their time derivative, the conjugate momenta are given

$p_i = \frac{\partial L}{\partial \dot{q}_i}$, and the Hamiltonian will be

$$H = \sum_i p_i \dot{q}_i - L.$$

Now q_i are replaced by $\varphi(x)$, x is an index (continuous)

The corresponding generalization would be

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)}$$

$$\mathcal{H} = \Pi \dot{\varphi} - \mathcal{L}$$

Hamiltonian density $\quad H = \int d^3x\, \mathcal{H}$

In our scalar Lagrangian density, we have

$$\Pi(x) = \dot{\varphi}(x)$$

$$\Rightarrow \mathcal{H} = \frac{1}{2}\Pi^2(x) + \frac{1}{2}(\vec{\nabla}\varphi(x))^2 + \frac{1}{2}m^2\varphi^2(x) - \Omega_0$$

Don't this look familiar?

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 \quad (\text{in QM}) \quad a_+, a_- \ldots$$

Now we can get $\mathcal{H}$. in terms of $a(\vec{k})$. $a^*(\vec{k})$ $\quad + a^*(\vec{k}) e^{-ikx}]$ (10)

$$\mathcal{H} = \int d^3x\, \mathcal{H}'$$

$\leftarrow \int d^3x = V$

$$= -\Omega_0 V + \frac{1}{2} \int \widehat{dk}\, \widehat{dk'}\, d^3x \Big\{$$

$$(-i\omega a(\vec{k}) e^{-ikx} + i\omega a^*(\vec{k}) e^{-ikx})(-i\omega' a(\vec{k'}) e^{ik'x} + i\omega' a^*(\vec{k'}) e^{-ik'x})$$

$$+ (+i\vec{k} a(\vec{k}) e^{ikx} - i\vec{k} a^*(\vec{k}) e^{-ikx})(+i\vec{k'} a(\vec{k'}) e^{ik'x} - i\vec{k'} a^*(\vec{k'}) e^{-ik'x})$$

$$+ m^2 (a(\vec{k}) e^{ikx} + a^*(\vec{k}) e^{-ikx})(a(\vec{k'}) e^{ik'x} + a^*(\vec{k'}) e^{-ik'x}) \Big\}$$

$$= -\Omega_0 V + \frac{1}{2} (2\pi)^3 \int \widehat{dk}\, \widehat{dk'} \Big\{$$

$$\delta(\vec{k} + \vec{k'}) (-\omega\omega' - \vec{k}\cdot\vec{k'} + m^2)(a(\vec{k}) a(\vec{k'}) e^{-i(\omega+\omega')t} + a^*(\vec{k}) a^*(\vec{k'}) e^{i(\omega+\omega')t})$$

$$+ \delta(\vec{k} - \vec{k'}) (\omega\omega' + \vec{k}\cdot\vec{k'} + m^2)(a(\vec{k}) a^*(\vec{k'}) e^{-i(\omega-\omega')t} + a^*(\vec{k}) a(\vec{k'}) e^{i(\omega-\omega')t}) \Big\}$$

$$= -\Omega_0 V + \frac{1}{2} \cdot \overset{\frac{1}{2\omega}}{\frac{1}{2\omega'}} \int \widehat{dk} \Big\{$$

$$(-\omega^2 + \vec{k}^2 + m^2)(a(\vec{k}) a(-\vec{k}) e^{-2i\omega t} + a^*(\vec{k}) a^*(-\vec{k}) e^{2i\omega t})$$

$$(\omega^2 + \vec{k}^2 + m^2)(a(\vec{k}) a^*(\vec{k}) + a^*(\vec{k}) a(\vec{k})) \Big\}$$

$$= -\Omega_0 V + \frac{1}{2} \int \widehat{dk}\, \underline{\omega}\, (a(\vec{k}) a^*(\vec{k}) + a^*(\vec{k}) a(\vec{k})).$$

And the theory is L.I. (from Lagrangian).

exactly what we want!
Relativistic QM (that reduce back to S.E.) with correct energy

$$\int dx\, e^{iax} = (2\pi)\,\delta(a)$$

$$\int d^3x\, e^{i\vec{p}\cdot\vec{x}} = (2\pi)^3\,\delta^3(\vec{p})$$

The above are all classical field theory. Below we start 2nd quantization.

We can take up the quantum theory now (from the classical equation of motion) through

Canonical Quantization.

(promoting q_i and p_i to operators, with

$$[q_i, q_j] = 0, \quad [p_i, p_j] = 0, \quad [q_i, p_j] = i\hbar\,\delta_{ij}$$

In Heisenberg Picture all operators are taken at Equal times.

$$[\varphi(\vec{x},t), \varphi(\vec{x}',t)] = 0$$

$$[\Pi(\vec{x},t), \Pi(\vec{x}',t)] = 0$$

$$[\varphi(\vec{x},t), \Pi(\vec{x}',t)] = i\cancel{\hbar}\,\delta^3(\vec{x}-\vec{x}')$$

Then we ~~need to~~ equivalently can have

$$[a(\vec{k}), a(\vec{k}')] = 0$$

$$[a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0$$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3\, 2\omega\, \delta^3(\vec{k}-\vec{k}')$$

Here $a^\dagger(\vec{k})$ replaced $a^*(\vec{k})$ for more general states.

Comment on $\varepsilon^0 V$ part.

Now

$$H = \int \widetilde{dk}\ \omega\, a^{+}(\vec{k})\, a(\vec{k}) + (\mathcal{E}_0 - \Omega_0) V$$

$$[a, a^{+}] = i\delta^3(\vec{k} - \vec{k}')\,(2\pi)^3 \cdot 2\omega .$$

$$\mathcal{E}_0 V = \frac{1}{2}\int \widetilde{dk}\ \omega\ \delta^3(\vec{k} - \vec{k})\,(2\pi)^3 2\omega$$

infinite zero point energy.

$$\mathcal{E}_0 \cancel{V} = \frac{1}{2}\frac{1}{(2\pi)^3}\int d^3k\, \omega$$

as $\delta^3(\vec{0})(2\pi)^3 = \int d^3x\, e^{i\vec{0}\cdot\vec{x}} = \int d^3x = V.$

$e^{i\vec{0}\cdot\vec{x}}$

$\cancel{\int_0 \widetilde{d^3k}\ dk}$

$$\int d^3k = \int_0^{\Lambda} d|k|\, d\Omega.$$

if we integrate up to Λ.

which means we only work within a scale Λ.

$$\mathcal{E}_0 = \frac{1}{2} \times \frac{1}{2\pi^2} \cdot \Lambda^4.$$

Now, a first encounter of ultraviolate divergence.

$$\hat{H} = \int d\tilde{k}\, \omega\, a^{+}(\vec{k})\, a(\vec{k}) + (\varepsilon_0 - \Omega_0) V$$

$$\varepsilon_0 = \frac{1}{2} \times \frac{1}{(2\pi)^3} \boxed{\int d^3k\, \omega} \Leftarrow \int_0^{\Lambda} d|k|\, |\vec{k}|^2 \cdot d\Omega \cdot |\vec{k}| = \frac{4\pi}{4} |\vec{k}|^4 \Big|_0^{\Lambda} = \pi \Lambda^4.$$

we have interpreted $(2\pi)^3 \delta^{(3)}(\vec{0})$ as the volume of V

Recall $\int d^3x\, e^{-i\vec{q}\vec{x}} = (2\pi)^3 \delta^{(3)}(\vec{q})$.

$$\int d^3x\, e^{-i0} = \int d^3x = V$$

~~then~~ ε_0 if we view there is a maximum of momentum we can integrate over to Λ

then ~~$\int d^3k\, \omega$~~ $\varepsilon_0 = \frac{\Lambda^4}{16\pi^2}$

Λ is ~~view~~ call the ultraviolate cutoff.

we choose $\Omega_0 = \varepsilon_0$ so the ground state has zero energy.

About fermions, we should invoke anticommutation for both the field part and "spinor part".