Chapter 3: Basic Phenomenology: Decay Rates and Cross Section

3.1 Fermi's Golden Rule

The Fermi Golden Rule is a cornerstone in quantum mechanics, offering a formula to calculate the transition rate between different quantum states under the influence of a perturbation. Named after physicist Enrico Fermi, this principle finds applications across various domains such as quantum field theory, particle physics, and solid-state physics.

Consider a quantum system initially in a discrete, non-degenerate energy state $|i\rangle$, with energy E_i. When this system is subjected to a time-dependent perturbation, denoted as $V(t)$, which is turned on at $t = 0$, we aim to determine the rate of transition to a different state $|f\rangle$, possessing energy E_f. This scenario is addressed by employing time-dependent perturbation theory.

The system's Hamiltonian is represented as:

$$H = H_0 + V(t) \tag{3.1.1}$$

where H_0 signifies the time-independent unperturbed Hamiltonian, and $V(t)$ encapsulates the time-dependent perturbation.

The probability amplitude for the system to transition from the initial state $|i\rangle$ to the final state $|f\rangle$ over time t is given, to the first order in perturbation theory, by:

$$a_{fi}(t) = -\frac{i}{\hbar}\int_0^t \langle f|V(t')|i\rangle e^{i\omega_{fi}t'}\, dt' \tag{3.1.2}$$

where $\omega_{fi} = \frac{E_f - E_i}{\hbar}$ denotes the angular frequency associated with the energy difference between the final and initial states.

Accordingly, the probability P_{fi} for the system to transition from state $|i\rangle$ to state $|f\rangle$ is the modulus squared of the amplitude:

$$P_{fi}(t) = |a_{fi}(t)|^2 \tag{3.1.3}$$

When considering a continuous spectrum of final states, the transition rate W_{fi} per unit time to a state within an energy range from E_f to $E_f + dE_f$

is articulated by the Fermi Golden Rule as:

$$\Gamma_{fi} = \frac{2\pi}{\hbar}|\langle f|\hat{H}|i\rangle|^2\rho(E_f) \tag{3.1.4}$$

Here, $\rho(E_f)$ delineates the density of final states per unit energy interval at the energy E_f, and $|\langle f|\hat{H}|i\rangle|^2$ represents the squared matrix element of the perturbation between the initial and final states, which is averaged over initial states and summed over final states when necessary.

The Fermi's Golden Rule[1] thus serves as a powerful tool, linking the transition rate directly to both the density of final states and the magnitude of the perturbation, facilitating the computation of decay rates, scattering cross sections, and other pivotal transition processes in quantum systems.

3.2 Relativistic Gloden Rule

This section is devoted to the examination of the decay process of a particle at rest into multiple particles. We commence by considering the wave function's normalization, which is a fundamental aspect of quantum mechanics ensuring that the probability density integrates to unity. We proceed to discuss the implications of confining a particle within a finite volume and how this confinement leads to the quantization of momentum states. The density of states in momentum space is also analyzed, leading to insights into the density of final states at given energy levels. We conclude with an exploration of the energy-momentum relationship, which is pivotal in understanding particle behavior.

The wave function ψ, which encapsulates the quantum state of a system, is constrained by the normalization condition to ensure its probabilistic interpretation. This is mathematically articulated as:

$$\int \psi^*\psi dV = 1. \tag{3.2.1}$$

When hypothesizing the universe as a finite box, we infer that the particle's momentum components become discrete. This notion is an extension of the quantum mechanical treatment of particles in a 3D square potential well, leading to the following quantized momentum

[1]The rule is not given by Enrico Fermi, but Paul Dirac. However, Enrico Fermi gave it a good name, and people used it ever since. See a footnote in Griffiths's Quantum Mechanics book.

components:

$$p_x = \frac{2\pi n_x}{V^{1/3}}, \tag{3.2.2}$$

$$p_y = \frac{2\pi n_y}{V^{1/3}}, \tag{3.2.3}$$

$$p_z = \frac{2\pi n_z}{V^{1/3}}. \tag{3.2.4}$$

In the realm of momentum space, the separation between quantum states is uniformly defined by the box's dimensions, leading to a quantized spectrum of momentum states. This uniform spacing is given by the equation:

$$\left(\frac{2\pi}{V^{1/3}}\right)^3 = \frac{(2\pi)^3}{V}, \tag{3.2.5}$$

The concept of the density of states is crucial in statistical mechanics and quantum physics, representing the number of accessible quantum states per element of phase space. For an infinitesimal volume in momentum space, it is computed as follows:

$$dn = \frac{dN}{V} = \frac{d^3\mathbf{p}}{(2\pi)^3/V} \cdot \frac{1}{V} = \frac{d^3\mathbf{p}}{(2\pi)^3}. \tag{3.2.6}$$

The density of final states at a certain energy, denoted by E_f, quantifies the number of ways a system can be arranged compatibly with a given energy, which is fundamentally linked to the statistical nature of quantum systems. This is captured by:

$$\rho(E_f) = |\frac{dn}{dE}|_{E=E_f} = \left|\frac{dn}{d|\mathbf{p}|} \cdot \frac{d|\mathbf{p}|}{dE}\right|_{E=E_f}, \tag{3.2.7}$$

and detailed further as:

$$\rho(E_f) = \left|\frac{4\pi|\mathbf{p}|^2 d|\mathbf{p}|}{(2\pi)^3 d\mathbf{p}} \cdot \frac{d|\mathbf{p}|}{dE}\right|_{E_f}. \tag{3.2.8}$$

The energy-momentum relationship for a particle is a fundamental aspect of relativistic quantum mechanics, revealing the intrinsic link between a particle's energy and its momentum. It is eloquently described by the equation:

$$E = \sqrt{|\mathbf{p}|^2 + m^2}. \tag{3.2.9}$$

Moreover, the derivative of energy with respect to momentum magnitude provides insights into the particle's velocity in the context of special relativity, articulated as:

$$\frac{dE}{d|\mathbf{p}|} = \frac{|\mathbf{p}|}{\sqrt{|\mathbf{p}|^2 + m^2}} = \beta, \tag{3.2.10}$$

where β symbolizes the particle's velocity normalized by the speed of light.

In the context of quantum mechanics and particle physics, the density of states plays a crucial role in determining the statistical distribution of particles. For a given final state energy E_f, the density of states Ω can be expressed as a function of the particle's momentum P and its velocity β. This relationship is significant as it directly influences the probability of transition to a state at energy E_f:

$$\rho(E_f) = \left.\frac{4\pi|\mathbf{p}|^2}{(2\pi)^3\beta}\right|_{E=E_f} = \frac{4\pi E_f^2}{(2\pi)^3\beta}. \tag{3.2.11}$$

Here, $\beta \equiv v/c = |\mathbf{p}|/E$. This is useful for a one-particle state. However, what if I want to describe multiple particle states in the final state?

Let me make a detour here. The Dirac delta function, denoted by δ, is a fundamental concept used in physics, especially in the fields of quantum mechanics and electrodynamics. It is defined as:

$$\delta(x - x_0) = \begin{cases} 0, & x \neq x_0, \\ \infty, & x = x_0, \end{cases} \tag{3.2.12}$$

with the integral property:

$$\int_{-\infty}^{\infty} \delta(x - x_0)dx = 1. \tag{3.2.13}$$

This peculiar function, despite being infinitely narrow and infinitely high at x_0, integrates to 1 over the entire real line. It is used in physics to model quantities that are concentrated at a single point. For any regular, well-behaved function $f(x)$, the Dirac delta function has the sifting property:

$$\int_{-\infty}^{\infty} \delta(x - x_0)f(x)dx = f(x_0). \tag{3.2.14}$$

The delta function can be represented as the limit of a series of functions. For example, it can be approximated by a normalized Gaussian function as the standard deviation approaches zero:

$$\delta(x) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \quad \text{(Infinitely sharp Unit Gaussian)}. \tag{3.2.15}$$

Moving on to the properties of the delta function when applied to a function argument, we consider the dimensionality of $\delta(E - E_0)$ where E is energy. The dimension of the delta function is the inverse of the dimension of its argument, thus in this case, it is the inverse of energy.

When the delta function is applied to a function $f(x)$, its integral over an interval containing the root x_0 (where $f(x_0) = 0$) is given by:

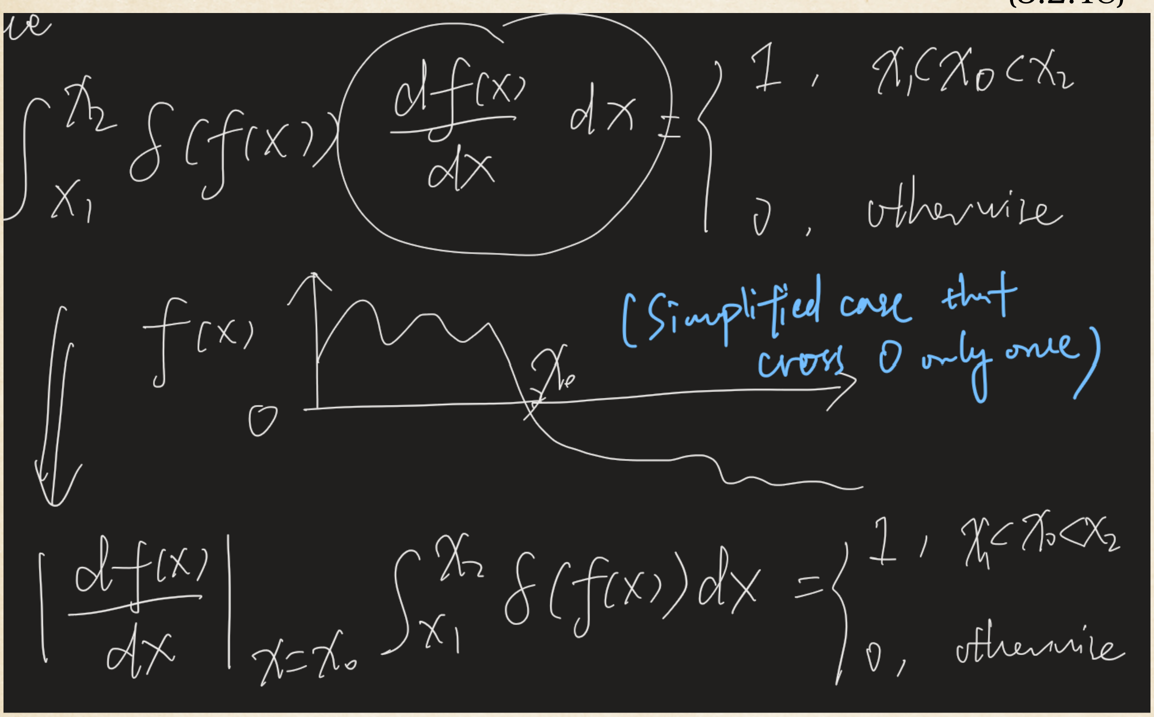
$$\int_{x_1}^{x_2} \delta(f(x)) \left| \frac{df(x)}{dx} \right| dx = \begin{cases} 1, & x_1 < x_0 < x_2, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2.16}$$

This property is particularly useful for transforming integrals over one variable into integrals over another variable where the function is zero, effectively 'picking out' the value of the new variable that corresponds to the zero of the function. The delta function applied to a function $f(x)$ is then:

$$\delta(f(x)) = \left| \frac{d(f(x))}{dx} \right|_{x=x_0}^{-1} \delta(x - x_0) \quad \text{(assuming non-singularity at } x = x_0\text{)}. \tag{3.2.17}$$

More generally, for a function with multiple poles x_i,

$$\delta(f(x)) = \sum_{x_i} \left| \frac{d(f(x))}{dx} \right|_{x=x_i}^{-1} \delta(x - x_i) \quad \text{(assuming non-singularity at } x = x_i\text{)}. \tag{3.2.18}$$



This approach simplifies many problems in physics where the evaluation of an integral is needed at a specific point, particularly in the analysis of resonant frequencies or energy levels.

The density of states at a specific final state energy E_f can be calculated using the formula:

$$\rho(E_f) = \frac{dn}{dE}\bigg|_{E=E_f} = \int \delta(E - E_f)\frac{dn}{dE}dE = \int \delta(E - E_f)dn. \tag{3.2.19}$$

The transition probability per unit time from an initial state i to a final state f, within the framework of Fermi's golden rule, is then given by:

$$T_{fi} = 2\pi|T_{fi}|^2\rho(E_f) = \int 2\pi|T_{fi}|^2\delta(E - E_f)dn, \tag{3.2.20}$$

where T_{fi} is the transition matrix element and $\rho(E_f)$ is the density of final states at the energy E_f.

One shall realize that a typical QM transition involves energy exchange from potentials, so the particle state there does not necessarily conserve energy alone. In fact, typically, E_f is the allowed energy state of the transition. On the other hand, the study of QFT in particle physics, most of the time, is about isolated systems (which obey the cluster decomposition principle)[2]. We will need to incorporate energy and momentum conservation into the transition probability, for an $i \to 1 + 2 + ... + n'$ process, and we get,

$$T_{fi} = \int (2\pi)^4|T_{fi}|^2\delta^3(\mathbf{p}_i - \sum_{i=1..n'}\mathbf{p}_{f,i})\delta(E_i - \sum_{i=1...n'}E_{f,i})\prod_{i=1...n'}\frac{d^3\mathbf{p}_{f,i}}{(2\pi)^3}, \tag{3.2.21}$$

which ensures that energy and momentum are conserved in the transition process.

On the other hand, we understand that the wavefunctions are normalized in quantum mechanics, as in Eq. (3.2.1). However, the integration variable, dV, is a Lorentz varying quantity, e.g., the volume in the observer's frame is enlarged by γ for a boost in an arbitrary direction. The wavefunctions, hence, are normalized in a Lorentz varying way in other Lorentz quantities (since we integrated over space already, we only have energy-momentum) by the state's Energy. The relativistic QM particle states are normalized hence by[3]

$$\int \psi^*\psi dV = 2E, \tag{3.2.22}$$

modifies when volume contracts in relativistic theory. Specifically, in the direction of motion, Lorentz contraction is accounted for, leading to the introduction of the Lorentz invariant matrix element M_{fi}, which is defined as:

$$M_{fi} = \langle\psi_{f1}\psi_{f2}\psi_{f3}...|\mathcal{O}|\psi_{i1}\psi_{i2}\psi_{i3}...\rangle = (2E_{i1}2E_{i2}2E_{i3}...2E_{f1}2E_{f2}2E_{f3}...)^{1/2}T_{fi}. \tag{3.2.23}$$

[2]See discussion in Weinberg Vol1 [Wei05].

[3]Again, the QFT approach will yield a same result, including the factor of two.

Here, $\mathcal{O}$ represents the operator associated with the physical process, and the E terms correspond to the energies of the particles involved. Here T_{fi} is the non-relativistic QM transition amplitude. We can relate it to the Lorentz Invariant matrix element by understanding the difference in the wavefunction normalization.

For a general n initial-state to n' final-state particle transition, given that each particle takes a volumn density of $d^3p/(2\pi)^3$, we have a general rate formula,[4]

$$
\begin{aligned}
T_{fi} &= \int (2\pi)^4 |T_{fi}|^2 \delta^3(\sum_{i=1...n} \mathbf{p}_{i,i} - \sum_{i=1...n'} \mathbf{p}_{f,i}) \delta(\sum_{i=1...n} E_{i,i} - \sum_{i=1...n'} E_{f,i}) \prod_{i=1...n'} \frac{d^3\mathbf{p}_{f,i}}{(2\pi)^3} \\
&= \int \frac{|M_{fi}|^2}{(2E_{i,1}...2E_{i,n}\ 2E_{f,1}...\ 2E_{f,n'})} (2\pi)^4 \delta^{(4)}(\sum_{i=1...n} p_{i,i} - \sum_{i=1...n'} p_{f,i}) \prod_{i=1...n'} \frac{d^3\mathbf{p}_{f,i}}{(2\pi)^3} \\
, &= \frac{1}{2E_{i,1}...2E_{i,n}} \int |M_{fi}|^2 (2\pi)^4 \delta^{(4)}(\sum_{i=1...n} p_{i,i} - \sum_{i=1...n'} p_{f,i}) \prod_{i=1...n'} \frac{d^3\mathbf{p}_{f,i}}{2E_{f,i}(2\pi)^3}
\end{aligned}
\tag{3.2.24}
$$

We updated the Golden Rule to be a product of several Lorentz Invariant (L.I.) quantities. These L.I. quantities are $|M_{fi}|^2$ for the dynamics of the underlying theory, $(2\pi)^4$, $\delta^{(4)}(p_i - p_f)$ for energy-momentum conservation, and $d^3p_{f,i}/(2\pi)^2/(2E_{f,i})$ for partial phase space volumn and the product of them are the phase-space volumn of all the final state particles. On the other hand, the prefactor is Lorentz varying quantity and we will explore their physics meaning next.

Note that $d^3p_{f,i}/(2\pi)^2/(2E_{f,i})$ is L.I. as it is equivalent to an on-shell particle phase-space, upon integration,

$$
\frac{d^4p}{(2\pi)^3} \delta(p^2 - m^2) \Theta(p^0) = \left. \frac{d^3\mathbf{p}}{(2\pi)^3 2E} \right|_{E=+\sqrt{|\mathbf{p}|^2+m^2}} . \tag{3.2.25}
$$

Here $\Theta(x)$ is the Heaviside step function, 0 for $x < 0$ and 1 for $x \geq 1$.

Before we further into the decay rate and cross sections. It is meaningful to understand certain basic properties of the dLIPS as they will be useful in our analysis various universal features of the physics involving increasing number of particles.

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[4] Note that in this class, I introduce this formula from a bottom-up construction. In QFT, one can arrive at exactly the same equation within itself, where the Lorentz invariant matrix elements squared the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula [Sre07; Wei05] is accompanied by 4D Dirac delta function for energy-momentum conservation with $(2\pi)^4$, and there are some state normalization cancellations in V. For decay rates, V is cancelled, and for cross sections, V is cancelled by another concept called flux [Sre07].

3.3 Differential Lorentz Invariant Phase Space (dLIPS)

This is an important and useful topic in particle physics to enable your quick thinking and analysis of different processes. For a quick discussion and summary, see the kinematics section in Reviews in Particle Physics [Wor+22] and the appendix in Barger&Phillips [BP87].

The differential Lorentz invariant phase space of n particles are defined as[5],

$$dLIPS_n(P; p_1, p_2, ...p_n) \equiv \delta^{(4)}(P - \sum_{i=1...n} P_i) \prod_{i=1}^{n} \frac{d^3\mathbf{p}_i}{(2\pi)^3 2E_i}, \tag{3.3.1}$$

where P is the total initial momentum that flows into the n-particle final state. For m identical particles, there will be an additional symmetry factor $1/m!$ multiplying the phase space.

Q & As

Q: (ChimeIn) What's the mass dimension of $\delta(E - E_0)$ where E is energy?
Q: (ChimeIn) What's the mass dimension of dLIPS2?
Q: (ChimeIn) What's the mass dimension of $|T_{fi}|^2$ for a $1 \to 2$ process?
Q: What's the mass dimension of dLIPS2n?
Q: Can you guestimate the phase-space volume of dLIPSn? There is a closed form for all particles being massless.

There are several basic properties we can explore. First, the mass dimension of the dLIPS$_n$ fixed,

$$[dLIPS_n]_{\text{mass}} = 2n - 4. \tag{3.3.2}$$

Second, every time one increases the number of final state particles by one, there is a generic "counting" of the phase space volumn to say that,

$$dLIPS_{n+1}(P; p_1...p_{n+1}) \simeq \frac{P^2}{16\pi^2} dLIPS_n. \tag{3.3.3}$$

Note that this is just to say the more particles in the final state, the "smaller" the phase space volumne it would be. This lead to a generic anticipation that, if there are no additional suppressions in the dyanmics part, two-body process, such as decays will be higher in rate than three-body process by about two orders of magnitude.

[5] Note that some people defines the n-body phase space with $(2\pi)^4$ in front, with which one has to modify the transition rate formular consistently. I liked the version with $(2\pi)^4$ more, in particular when it comes to recursion relations, but to avoid confusion, I decided here to follow the convention from Review of Particle Physics.

In fact, in the massless limit, the n-body phase-space volumn for n identical particles can be evaluated in a closed form,

$$\int dLIPS_n(P) = \frac{(P^2)^{n-2}}{8\pi(16\pi^2)^{n-2}n!\,(n-1)!}. \tag{3.3.4}$$

More commonly, we express the Lornetz invariant quanity P^2 as the Mandelstam variable s. I will introduce these variables soon. One can understand the above results as the volumn of hyper-spheres, as P sets the “length” of the sphere in energy through the delta function, and the momentum for massless particles has to add up in quadrature to be the energy of individual particles. We can clearly see where the common wisdom on phase-space volumn comes from. It ignored the additional $1/(n(n+1))$ suppressions.

For general phase space, there is a useful recursion relation that relates the n-particle phase space as a product of sub-phase space,

$$dLIPS_n(P;p_1...p_n) = dLIPS_j(q;p_1...p_j)dLIPS_{n-j+1}(P;q,p_{j+1}...p_n)(2\pi)^3dq^2. \tag{3.3.5}$$

The physical interpretation is clear: that n-body phase space can be viewed as a process where the initial momentum first split into $n-j$ particles and another particle with momentum q, and then q split into j particles. Note that this formula is general and it does not require the intermediate phase space q to be on-shell (i.e., a real particle); hence, we need to integrate over dq^2, which is the “mass” of the immedidate momentum q. The above formular is a general recursive formula and arbitrary division of the n-particles are allowed.

We can prove it in general using the properties of delta functions.

$$\begin{aligned}
dLIPS_n(P;p_1,p_2,...p_n) &= \delta^{(4)}(P-\sum_{i=1...n}p_i)\prod_i^n \frac{d^3\mathbf{p}_i}{(2\pi)^3 2E_i} \\
&= \delta^{(4)}(q-\sum_{i=1...j}p_i)\prod_{i=1}^j \frac{d^3\mathbf{p}_i}{(2\pi)^3 2E_i}\delta^{(4)}(P-q-\sum_{i=j+1...n}p_i)\prod_{i=j+1}^n \frac{d^3\mathbf{p}_i}{(2\pi)^3 2E_i}d^4q \\
&= dLIPS_j(q;p_1...p_j)\delta^{(4)}(P-q-\sum_{i=j+1...n}p_i)\prod_{i=j+1}^n \frac{d^3\mathbf{p}_i}{(2\pi)^3 2E_i}\frac{d^3\mathbf{q}}{(2\pi)^3 2E_q}\times\frac{(2\pi)^3 2E_q d^3\mathbf{q} dE_q}{d^3\mathbf{q}} \\
&= dLIPS_j(q;p_1...p_j)dLIPS_{n-j+1}(P;q,p_{j+1}...p_n)(2\pi)^3 dE_q^2 \\
&= dLIPS_j(q;p_1...p_j)dLIPS_{n-j+1}(P;q,p_{j+1}...p_n)(2\pi)^3 dq^2
\end{aligned} \tag{3.3.6}$$

In the last line, we made use of the fact that the measure does not change between dE_q^2 and dq^2 upon integration, as the Jacobian is one. One might ask the question that in the recursion relation it looks like the intermediate momentum as if it were an on-shell particle with an unfixed mass q^2, what happens to the positive energy requirement, $\Theta(E_q)$, as defined in Eq. (3.2.25)? Does the energy need to be positive here? The answer is yes, it is identical if we put in this step function. From the delta functions of energy conservation in the subsystems, all energies are positive semi-definite.

With enough general understanding, let's explore some of the most useful formulaes for two-body phase space. As argued before, if not dynamically suppressed, two-body process will dominate the rate. There are also one-body process, which I will discuss separately when we encourage it, those processes are called resonance production, and many additional "tricks" can exploited for those processes.

The two body phase space only have 2 degree of freedom upon integration over the delta functions,

$$
\begin{aligned}
(2\pi)^4 dLIPS_2(P;p_1,p_2) &= (2\pi)^4\delta^4(P-p_1-p_2)\frac{d^3\mathbf{p}_1}{(2\pi)^3 2E_1}\frac{d^3\mathbf{p}_2}{(2\pi)^3 2E_2}\\
&= (2\pi)^4\delta(P^0-E_1-E_2)\frac{d^3\mathbf{p}_1}{(2\pi)^3 2E_1}\frac{1}{(2\pi)^3 2E_2}\\
&= \frac{1}{16\pi^2 E_1 E_2}\delta(P^0-E_1-E_2)d^3\mathbf{p}_1\\
&= \frac{1}{16\pi^2 E_1 E_2}\delta\left(P^0-\sqrt{m_1^2+|\mathbf{p}_1|^2}-\sqrt{m_2^2+|\mathbf{p}_2|^2|\mathbf{p}_1|^2}\right)d\Omega_1 d|\mathbf{p}_1|.
\end{aligned}
\tag{3.3.7}
$$

In the second line, we integrated over the three-momentum delta function using $d^3\mathbf{q}_2$. In the lat line we change to the spheretical coordinate system for $\mathbf{q}_1$ Note that the $\mathbf{p}_2$ is a function of $\mathbf{p}_1$ due to the momentum conservation. To evaluate and perform in the integration in $d|\mathbf{q}_1|$, we use the property of the Dirac delta-funciton $\delta(f(x))$ and that dLIPS are Lorentz invariant. Let us work in the physicists' favorite reference frame, the center of mass frame (CM frame), where we have,

$$
\begin{aligned}
&P^* = (P^{*0},0,0,0) = p_1^* + p_2^* = (\sqrt{s},0,0,0) = (E_1^*+E_2^*,0,0,0)\\
&p_1^* = (E_1^*,\mathbf{p}_1^*)\\
&p_2^* = (E_2^*,\mathbf{p}_2^*) = (E_2,-\mathbf{p}_1^*).
\end{aligned}
\tag{3.3.8}
$$

We typically use to explicitly denote the quantities in the CM frame.

Then, from the Dirac delta function property Eq. (3.2.17), we have, in the CM frame

$$
\begin{aligned}
\delta(P^0 - &\sqrt{m_1^2+|\mathbf{p}_1|^2} - \sqrt{m_2^2+|\mathbf{p}_2|^2}) d|\mathbf{p}_1| \\
&= \delta(P^0 - \sqrt{m_1^2+|\mathbf{p}_1^*|^2} - \sqrt{m_2^2+|\mathbf{p}_1^*|^2}) d|\mathbf{p}_1^*| \\
&= |\frac{P^0 - \sqrt{m_1^2+|\mathbf{p}_1^*|^2} - \sqrt{m_2^2+|\mathbf{p}_1^*|^2}}{d|\mathbf{p}_1^*|}|^{-1}_{|\mathbf{p}_1^*|=|\mathbf{p}_1^*|_{\text{solution}}} \\
&= \left(\frac{|\mathbf{p}_1^*|}{E_1^*} + \frac{|\mathbf{p}_1^*|}{E_2^*}\right)^{-1} \\
&= \frac{E_1^* E_2^*}{(E_1^* + E_2^*)|\mathbf{p}_1^*|}.
\end{aligned}
\tag{3.3.9}
$$

Here the function needs to be evaluated only at values of $|\mathbf{p}_1^*|$ solves the energy conservation Dirac delta function $|\mathbf{p}_1^*| = |\mathbf{p}_1^*|_{\text{solution}}$.

Now we can continue evlauating the 2-body phase space in Eq. (3.3.7), in CM frame,

$$
\begin{aligned}
(2\pi)^4 dLIPS_2(P; p_1, p_2) &= \frac{1}{16\pi^2 E_1 E_2} |\mathbf{p_1}|^2 \frac{E_1^* E_2^*}{(E_1^* + E_2^*)|\mathbf{p}_1^*|} d\Omega_1^* \\
&= \frac{1}{16\pi^2} \frac{|\mathbf{p}_1^*|}{E_1^* + E_2^*} d\Omega_1^* \\
&= \frac{1}{16\pi^2} \frac{|\mathbf{p}_1^*|}{\sqrt{s}} d\Omega_1^*.
\end{aligned}
\tag{3.3.10}
$$

For many typical systems, the two-body phase space is often also symmetric along the spin-quanitization axis (or momentum direction of the initial momentum P as the helicity quantization axis) and hence we are often able to integrate over the azimulthal angle trivially. We then have,

$$
(2\pi)^4 dLIPS_2(P; p_1, p_2) = \frac{1}{8\pi} \frac{|\mathbf{p}_1^*|}{\sqrt{s}} dcos\theta^*. \tag{3.3.11}
$$

This is my favorite one to remember.

There are additional simplifications, for special case. Often, the final state involves particles with identical masses $m_1 = m_2 = m$ (the non-identical mass case is left as a homework problem). In such a case, we have $E_1^* = E^* 2_2 = \sqrt{s}/2$, and use

$$
\beta^* = \beta_1^* = \beta_2^* = |\mathbf{p}_1^*|/E_1^* = \sqrt{1 - \frac{4m^2}{s}}, \tag{3.3.12}
$$

we get

$$
(2\pi)^4 dLIPS_2(P; p_1, p_2)_{m_1=m_2=m} = \frac{1}{16\pi} \beta^* d\cos\theta^*. \tag{3.3.13}
$$

We can see a few limits: for $m << \sqrt{s}$, we have a non-suppressed phase space volumn as $\beta^* = 1$. For $m \simeq \sqrt{s}/2$, we have suppressed phase-space volumn by a linear order in β^*. For $m \geq \sqrt{s}/2$, the process is kinematically forbidden.

3.4 Decay Rate

Considering a particle with a lifetime τ in its rest frame, the decay probability is given by:

$$P(t) = \frac{1}{\tau} e^{-t/\tau} \tag{3.4.1}$$

However, in a laboratory frame, this particle might have been boosted by a Lorentz factor γ, modifying the decay probability as follows:

$$P(t_{\text{lab}}) = \frac{1}{\gamma\tau} e^{-t_{\text{lab}}/(\gamma\tau)} \tag{3.4.2}$$

This leads to the relation between the transition probabilities in different frames, which is exactly the relation between Energy in different frames, for decay rate at $t = 0$

$$P(0)_{\text{lab}} = \frac{1}{\tau_{\text{lab}}} = \frac{1}{\gamma_{\text{lab}}\tau} = \frac{m}{E\tau}. \tag{3.4.3}$$

We find it is exactly what we have in the relativistic Golden rule formulae for the one-particle initial state, from the relativistic wave-function normalization that is not absorbed in the dLIPS. The master formula for decay is then, updated from Eq. (3.2.24),

$$T_{i\to n} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 dLIPS_n. \tag{3.4.4}$$

In particle physics, the total decay width Γ_{tot} and branching fractions are vital for understanding particle lifetimes and decay processes. The decay width is inversely related to the particle's lifetime, and the branching fraction represents the probability of a particular decay channel among all possible decays.

The total decay width is the sum of the decay widths for all possible final states:

$$\Gamma_{\text{tot}} = \sum_f \Gamma_{i\to f} \tag{3.4.5}$$

where a larger Γ_{tot} implies a shorter lifetime dominated by the large decay width.

The branching fraction for a decay from initial state i to final state f is defined as:

$$\text{Br}(i \to f) = \frac{\Gamma(i \to f)}{\Gamma_{\text{tot}}} \tag{3.4.6}$$

and the sum of the branching fractions for all possible final states equals 100%.

The sum of all possible decay partial widths determines the lifetime of a particle.

3.5 Scattering and Mandelstam Variables

Apart from decay, scattering processes are a major probe of particle physics, and one can already understand the rule of scattering through the famous Rutherford experiment that reveals the structure of atoms. For high-energy particle physics, we typically do two kinds of experiments when smashing particles: we either smash two particles towards each other or smash one energetic particle to another at rest. The former is called colliders, and the latter is called beamdumps. The most common scattering is a 2 to n process, in which we smash two particles and study the outcome of the experiment.

There are natural frames to use for these two kinds of experiments; hence, the two prevailing frame choices in particle physics, the CM frame (which was introduced in the previous section) and the fixed-target frame (FT frame).[6]

The FT frame kinematics are defined as,

$$\begin{aligned} p_1 &= (E_1, \mathbf{p}_1) \\ p_2 &= (m_2, 0, 0, 0) \\ (p_1 + p_2)^2 &= 2E_1 m_2 + m_2^2 + m_1^2. \end{aligned} \tag{3.5.1}$$

For high-energy collisions with $E_1 >> m_1, m_2$, we have, in the FT frame,

$$(p_1 + p_2)^2 \simeq 2E_1 m_2. \tag{3.5.2}$$

Of course, we'd love to easily relate our results in different frames, and we'd like to express quantities in a Lorentz invariant way as much as possible. As discussed in the previous section on dLIPS, two-body phase space is particularly important as it often dominates the physical process when the process is not suppressed. 2-to-2 scattering is one of the most basic scattering topologies, and we use a series of Lorentz invariant quantities to describe them. Apart from the one-particle L.I. quantities of masses m_1, m_2, m_3 and m_4. For a 2-to-2 process $1 + 2 \;\to\; 3 + 4$ where particles are abstractly labeled as numbers, with momentum conservation ($p_1 + p_2 \;=\; p_3 + p_4$), there are only a finite number of multi-particle L.I. quantities. We call them Mandelstam variables, s, t, u, defined as,

$$s \equiv (p_1 + p_2)^2 = (p_3 + p_4)^2 t \equiv (p_1 - p_3)^2 = (p_4 - p_2)^2 u \equiv (p_1 - p_4)^2 = (p_3 - p_2)^2. \tag{3.5.3}$$

You can see why I like to use s a lot. It is the L.I. total energy of the collision. I declare generically it labels scale the experiment probes. Since

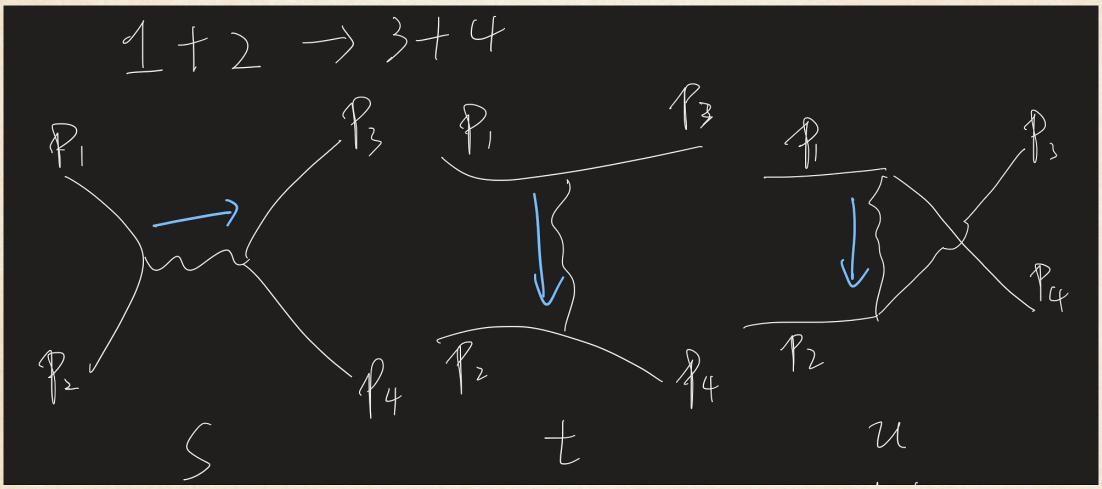
[6]Somebooks call it the lab frame, which I am opposing as the lab frame really differs in different experimental setups.

these are L.I. quantities, one can evaluate them in any frame. One can find that t and u are of the order s, with some angular dependence and finite mass correction.

If one stares at these variables long enough, one realizes that these are not linearly independent; we have,

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2. \tag{3.5.4}$$

These kinematical variables are also impornt in the sense that they are exactly the momentum-flow of the intermediate particles of different 2-to-2 processes, shown in the following diagrams.



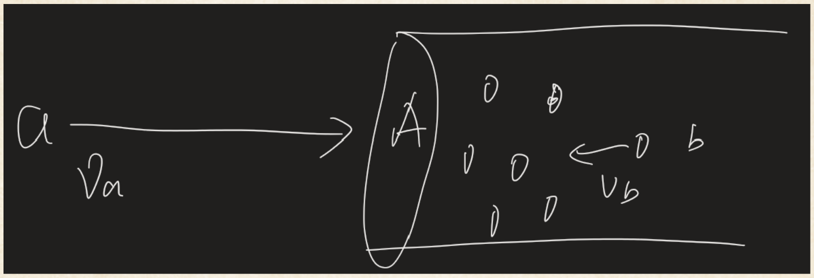
Hence, we call the first topology s-channel and the other two t-channel and u-channel, respectively.

These variables are also extremely helpful when we study the symmetries in the scattering process that we call crossing symmetry. In the s-matrix program, which leads to the String theory, and in the modern amplitude progress, they appear everywhere and make symmetries of the dynamics transparent. For interested readers, I recommend a light reading on the Veneziano amplitude [Ven68].

3.6 Cross Sections

The cross section is an intermediate useful quanity of dimension area that helps describe how easily a process can happen when two particles "meet." This quanity causes many frowned upon when first introduced but I hope to go through some details in this section to show you the rationals.

Let's think about a generic process: particle "a" traveling towards particle "b", and particles a and b move towards each other and scatter.



In this picture, for the duration of Δt, particle "a" sees a large number of particle "b"s,[7][8]

$$N_b = n_b \times (v_a - v_b) \times A \times \Delta t, \tag{3.6.1}$$

How likely do they interact? We introduce the concept of cross section, typically denoted by σ, which is an area perpendicular to the direction of relative motion. It is an intuitive definition as if, for a particular interaction, the particle has a size, so the probability of interaction would be

$$\frac{N_b \sigma}{A}. \tag{3.6.2}$$

Clearly, here I only talked about one particle a, entering an area A filled with b. The interaction rate will also be proportional to the number of particle a that enters the area. So overall, the rate of interaction would be,

$$\frac{N_a \times N_b \times \sigma}{\Delta t \times A} = \frac{N_a \times n_b \times (v_a - v_b) \times A \times \Delta t \times \sigma}{\Delta t \times A} = N_a \times n_b \times (v_a - v_b) \times \sigma. \tag{3.6.3}$$

In principle, one can find a consistent definition of the relativistic Golden rule rate to directly correspond to the above. However, clearly, many parameters depend on how the beam is prepared. We'd like to divide up the information to derive the rate depending only on one particle state; after all, this is the simplest initial state one can study using QM and QFT. We define the quantity σ as the effective area of $N_a = 1$ and $n_b = 1/\text{unit volumn}$ quantity, the per particle effective area that characterizes the interaction rate would be (for "head-on" collision; see the footnote earlier),

$$\sigma = \frac{\mathcal{T}_{a+b\to n}}{v_a - v_b} = \frac{(2\pi)^4}{2E_a 2E_b (v_a - v_b)} \int |M_{a+b\to n}|^2 dLIPS_n. \tag{3.6.4}$$

[7]Note that for typically very high energy collisions, our control of the positions of the target particles is far below the scale of the interaction, which forces us to use an "averaged", statistical treatment. Furthermore, we should also interpret this as very thin layers for a short amount of time Δt, so we avoid counting on multiple interactions at this per-collision level definition.

[8]Here we are in a frame where a and b are moving towards each other. Sometimes people write this term $v_a - v_b$ as $|\vec{v}_a - \vec{v}_b|$. These are equivalent in the frame I stated here, but I also want emphasis on their "head-on" collision/scattering nature.

In fact, the denominator is so important that we give it a name, Flux,

$$F \equiv 2E_a 2E_b (v_a - v_b). \tag{3.6.5}$$

There are several useful expressions for flux (for "head-on" collisions)

$$\begin{aligned} F &= 2E_a 2E_b (v_a - v_b) = 4E_a E_b \left(\frac{|\mathbf{p}_a|}{E_a} + \frac{|\mathbf{p}_b|}{E_b} \right) \\ &= 4\left(|\mathbf{p}_a| E_b + |\mathbf{p}_b| E_a \right) \\ &= 4\left((p_a \cdot p_b)^2 - m_a^2 m_b^2 \right)^{1/2} \end{aligned} \tag{3.6.6}$$

In the last line above, we use the fact that in a head-on collision,

$$p_a \cdot p_b = E_a E_b + |\mathbf{p}_a||\mathbf{p}_b| \tag{3.6.7}$$

and one can show that, in this case,

$$\left(\frac{F}{4} \right)^2 - (p_a \cdot p_b)^2 = -m_a^2 m_b^2. \tag{3.6.8}$$

Another way to express the flux factor F, using the Mandelstam variable s, is

$$\begin{aligned} F &= 4\left((p_a \cdot p_b)^2 - m_a^2 m_b^2 \right)^{1/2} \\ &= 4\left(\left(\frac{s - m_a^2 - m_b^2}{2} \right)^2 - m_a^2 m_b^2 \right)^{1/2} \\ &= 2\left(s^2 - 2s(m_a^2 + m_b^2) + (m_a^2 - m_b^2)^2 \right)^{1/2}. \end{aligned} \tag{3.6.9}$$

While the above expression is hard to remember, the mostly used version in high energy collisions, where $\sqrt{s} >> m_a, m_b$ has,

$$F \simeq 2s. \tag{3.6.10}$$

Without making approximations, as one often needs to consider a class of experiments and compare them, here are some other forms of F that are useful.

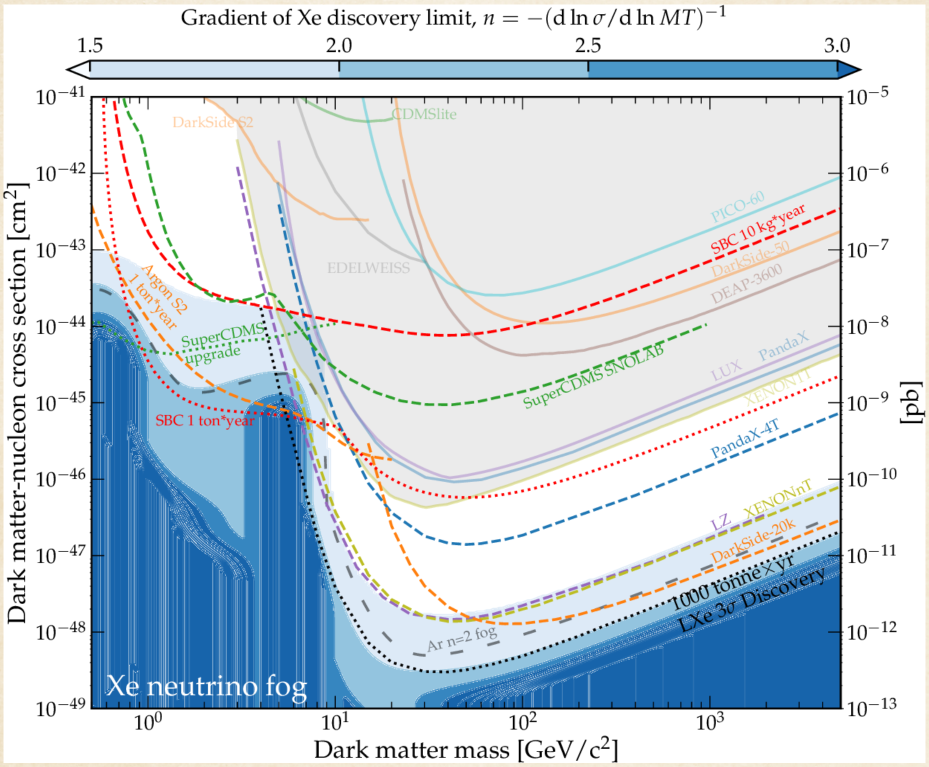
$$\begin{aligned} \text{CM frame:} F &= 4\left(|\mathbf{p}_a| E_b + |\mathbf{p}_b| E_a \right) \to 4|\mathbf{p}_a^*|(E_a^* + E_b^*) = 4|\mathbf{p}_a^*|\sqrt{s} \\ \text{FT frame:} F &= 4\left(|\mathbf{p}_a| E_b + |\mathbf{p}_b| E_a \right) \to 4|\mathbf{p}_a| m_b. \end{aligned} \tag{3.6.11}$$

Combing with the phase-space formula in Eq. (3.3.10), one can obtain a useful formula in the CM frame for 2-to-2 scattering,

$$\sigma(2 \to 2) = \frac{1}{64\pi^2 s} \frac{|\mathbf{p}_f^*|}{|\mathbf{p}_i^*|} \int |M|^2 d\Omega_1^*. \tag{3.6.12}$$

Units for cross section

Clearly, from the definition, the unit for the cross section is length-squared. Depending on the subfield of high energy physics, one used different units. One natural choice is still (almost) the metric system, so one can see people express the cross section in cm^2 (somehow, we use centimeters instead of meters). For example, one sees the typical constraints from the dark matter direct detection cross section in cm^2 in a typical plot. On the other hand, even for that kind of plot, we still often have, on the right-axis panel, a unit system of "barn".



The prevailing unit for cross section is "barn", which is a result of the famous Manhattan Project. It comes from the American idiom "hitting the broader side of the barn" to describe something that is very easy to happen. Back then, scientists wanted to have a coded language to describe the interaction rates, in particular, the neutron-to-nucleus (guess which nucleus) interactions. They decide to use the coded unit "barn",

$$1b = 100fm^{-2} = 10^{-28}m^2 = 10^{-24}cm^2. \tag{3.6.13}$$

A typical nucleon-nucleon (e.g. proton-on-proton) [9] is

$$\sigma_{pp\to X} \sim 10\ mb. \tag{3.6.14}$$

[9] This is an energy scale dependent question, we will see later roughly why and how.

Now you can see, if I smash a neutron at a nucleus that contains $O(100)$ neucleons, the cross section is around 1 barn.

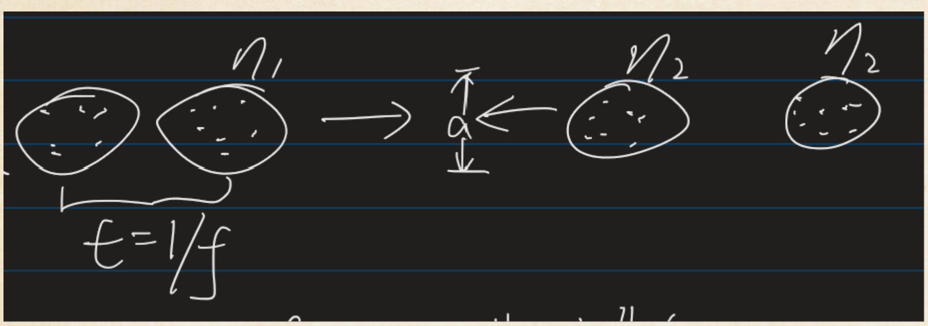
Now is a good time to comment on the meaning of the cross section, which, in some sense, is the size of a particle. This “size” is not the geometric spatial size of a particle, as so far, we find the fundamental particles are still consistent with point-like assumption. This size is a process-dependent, effective size that characterizes interaction probability, which is to be combined with the beam property to derive and compare the rates at different experiments.

In particle physics, we have established many large-rate (which is a large cross section) processes, and we are studying the rare, low cross section events that reveal either weakly coupled underlying theory or very high-scale physics imprints in low energies. In typical modern-era collider searches, the target search cross sections varies between millibarn (mb, 10^{-3}b), microbarn (μb, 10^{-6}b), nanobarn (nb, 10^{-9}b), picobarn (μb, 10^{-12}b), femtobarn (fb, 10^{-15}b), attobarn (ab, 10^{-18}b). Should our technology improve in the future, we might have access to even rarer cross sections.

Luminosity

Defining the cross section is one size; on the other hand, we still need to define the rest of the quantities so that we can derive the rate of interaction and make connections with experimentally observable data.

In a typical particle collider experiment, we smash particles batch by batch. We basically collect particles in batches (called “spills"), accelerate them, (focus them more, and) smash them. Near the interaction point, where particles collide, the particles look like the following,



We basically smashed two spills of particles with a relative motion to each other at the speed of light over a small area with a typical radius a. The (averaged between spills) instantaneous luminosity, $\mathcal{L}$, is defined as,

$$\mathcal{L} \equiv \frac{N_a N_b}{A \Delta t} = \frac{N_a N_b}{a^2} f. \tag{3.6.15}$$

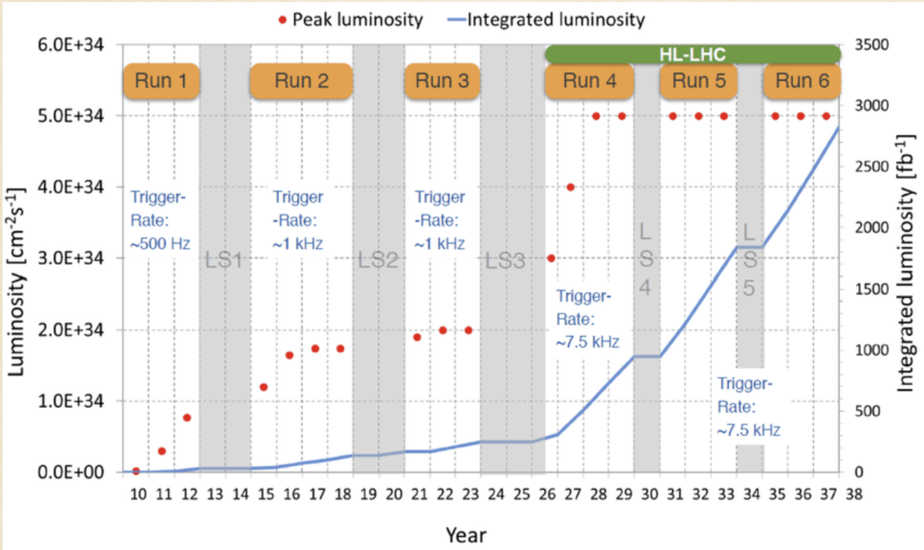
This instantaneous luminosity is exactly what we need to complement the cross section and give us a rate prediction. You can see from dimensional analysis, $\mathcal{L}$ is inverse area inverse time. Following the left-most expression in Eq. (3.6.3), the event rate is,

$$\mathcal{L}\sigma. \tag{3.6.16}$$

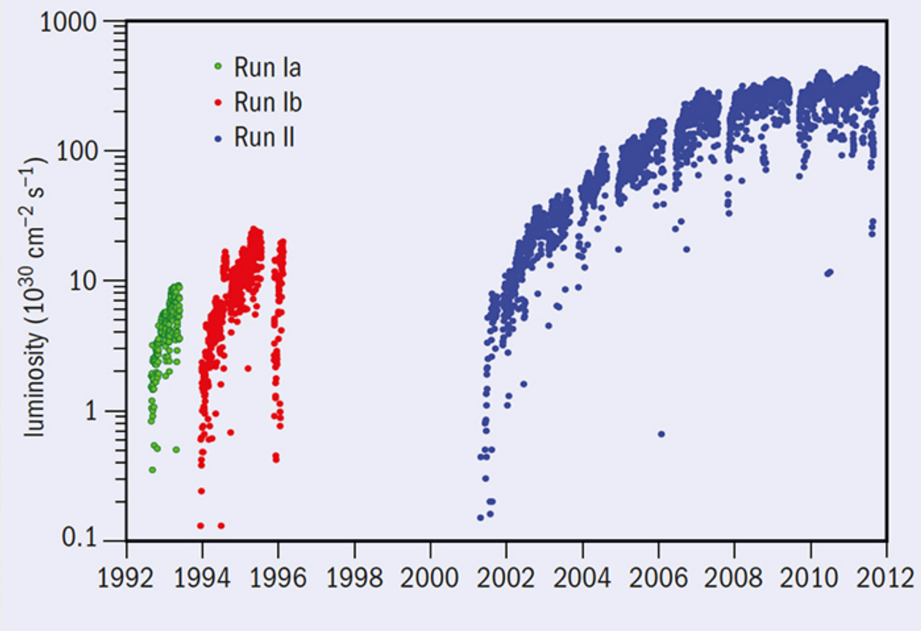
As a benchmark case, we can actually understand the instantaneous luminosity of the LHC. In the high-luminosity run of the LHC, we collide spills of protons every 25 ns. Each spill contains around 10^{11} protons, and the root-mean-square transverse size of the spills is 64 microns. Hence, for the HL-LHC, in instantaneous luminosity is,

$$\mathcal{L}_{\mathrm{HL-LHC}} = \frac{10^{11} \times 10^{11}}{64\ \mu m^2 \times 25ns} \simeq 10^{34}(cm^2 \cdot s)^{-1} = 10^{10}(b \cdot s)^{-1} = 10(nb \cdot s)^{-1} \tag{3.6.17}$$

One can already see the concepts introduced here allows us to understand the LHC run plans and as well as the Tevatron legacy[10],



[10] https://cerncourier.com/a/the-tevatron-legacy-a-luminosity-story/.



One can further introduce another often used concept, called **integrated luminosity, L**, which is to integrate the instantaneous luminosity for some time, e.g., a month, a year, 10 years, and the lifetime of the experiment, to quantify the total luminosity delivered. Then, if we multiply it with the cross section, we obtain how many collision events take place. When we count time, one year is approximately $\pi \times 10^7$ seconds, and a "collider year" is 10^7 seconds where we take into account typical downtime and maintenance and others for a running collider. Then we know the integrated luminosity of one collider year running of the HL-LHC would collect,

$$L = \mathcal{L} \times t = 10(nb \cdot s)^{-1} \times 10^7 s = 100 fb^{-1}. \tag{3.6.18}$$

Again, if one multiplies the integrated luminosity with the cross section, one can obtain the number of expected events for a given process.

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