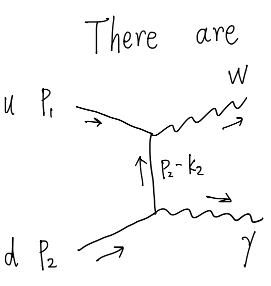
Thanks for the notes. It is nice and reasonable progress. Probably you can improve your notes by addressing the following:

1. Part 1, conclusion on fW<<1 is too quick and irrelevant for the process regions we are concerned about, as your calculation is IR divergent (logarithmically).
   1. To better connect with the W->3l\nu process understudy, you can put a simple pT cut of 1 GeV for the photons, can write down the results.
   2. I believe with a minimal photon cut, fW is already > 1 for the center of mass energy not too far away from the W pole.
   3. Again, when you say you cross-checked with Madgraph, write down what you did, otherwise you will get unphysical results, confuse yourself.
2. Part 2, conclusion on fW>(>)1 is also too quick,
   1. The decomposition of the calculation is cute.
   2. But the comparison between the |M_1+M_2+M_3|^2 from g^\mu\nu part where t-channel dominant by 1/m_\gamma^2 and the new term from k_2^2/m_\gamma^2 term in the s-channel is missing. You assumed M k_2 /m_\gamma dominant, which is not necessarily the case given, as you've shown, the 1/m_\gamma terms cancel.
   3. Again, verify your results with solid calculations and evidence. Integration matters often.
   4. (not that matters) M_2 k_2/m_\gamma expression has a typo.
3. In fact, I do not expect there would be much difference between the real photon calculation and the off-shell photon calculation (you almost prove that one can just do a 2->2 calculation with off-shell photons).
4. Your evaluations always miss an important step of Quantifying and Verifying your results. One should learn to be able to self-examine fully.

Note

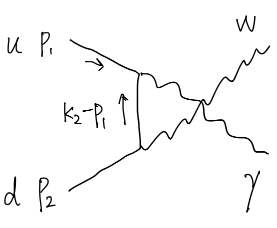
1. $u\bar{d} \to W^+\gamma$

There are 3 diagrams



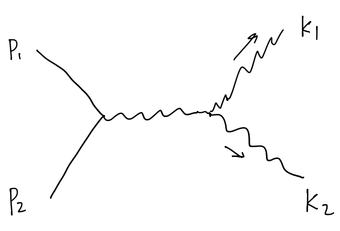
$$-\frac{e}{3(p_2-k_2)^2}\,\bar{v}(p_2)\,\not\epsilon_2\,(\not p_2-\not k_2)\,\not\epsilon_1\,u_L(p_1)$$

$$=\left(-\frac{e}{3}\right)\frac{1}{-2p_2\cdot k_2}\Big[2(p_2\cdot\epsilon_2)\,\bar{v}(p_2)\not\epsilon_1 u_L(p_1)-\bar{v}(p_2)\not\epsilon_2\not k_2\not\epsilon_1 u_L(p_1)\Big]$$



$$\frac{2e}{3(k_2-p_1)^2}\,\bar{v}(p_2)\,\not\epsilon_1\,(\not k_2-\not p_1)\,\not\epsilon_2\,u_L(p_1)$$

$$=\left(\frac{2e}{3}\right)\frac{1}{-2p_1\cdot k_2}\Big[\bar{v}(p_2)\not\epsilon_1\not k_2\not\epsilon_2 u_L(p_1)-2(p_1\cdot\epsilon_2)\,\bar{v}(p_2)\not\epsilon_1 u_L(p_1)\Big]$$



$$=(-e)\frac{1}{(k_1+k_2)^2}\Big[2(k_1\cdot\epsilon_2)\,\bar{v}(p_2)\not\epsilon_1 u_L(p_1)$$
$$-2(k_2\cdot\epsilon_1)\,\bar{v}(p_2)\not\epsilon_2 u_L(p_1)+2(\epsilon_1\cdot\epsilon_2)\,\bar{v}(p_2)\not k_2 u_L(p_1)\Big]$$

First we can sum up the black terms in the bracket

Convec Current

$$2(p_2\cdot\epsilon_2)\,\bar{v}(p_2)\not\epsilon_1 u_L(p_1)+2(p_1\cdot\epsilon_2)\,\bar{v}(p_2)\not\epsilon_1 u_L(p_1)-2(k_1\cdot\epsilon_2)\,\bar{v}(p_2)\not\epsilon_1 u_L(p_1)=0$$

Summing up the purple terms

$$\bar{v}(p_2)\Big[\not\epsilon_2\not k_2\not\epsilon_1+\not\epsilon_1\not k_2\not\epsilon_2-(\not k_2\not\epsilon_1+\not\epsilon_1\not k_2)\not\epsilon_2+\not k_2(\not\epsilon_1\not\epsilon_2+\not\epsilon_2\not\epsilon_1)\Big]u_L(p_1)$$

$$=\bar{v}(p_2)\Big[\not\epsilon_2\not k_2\not\epsilon_1+\not k_2\not\epsilon_2\not\epsilon_1\Big]u_L(p_1)=\bar{v}(p_2)\not\epsilon_1 u_L(p_1)\,(k_2\cdot\epsilon_2)=0$$

So if

$$\frac{e}{3}\frac{1}{p_2\cdot k_2}=\frac{2e}{3}\frac{1}{p_1\cdot k_2}=\frac{e}{k_1\cdot k_2}\qquad \sum J_i=0\qquad \mathcal{M}=0$$

In the CMS frame, we set

$$p_1=(E_0,0,0,E_0)\qquad p_2=(E_0,0,0,-E_0)$$
$$k_1=\left(E_0+\frac{m_W^2}{4E_0},\ P_W\sin\theta,\ 0,\ P_W\cos\theta\right)$$
$$k_2=(P_W,\ -P_W\sin\theta,\ 0,\ -P_W\cos\theta)$$

$$p_2\cdot k_2=E_0P_W(1-\cos\theta)$$
$$p_1\cdot k_2=E_0P_W(1+\cos\theta)$$
$$k_1\cdot k_2=E_0P_W+\frac{m_W^2}{4E_0}P_W+P_W^2=2E_0P_W$$

$$P_W=E_0-\frac{m_W^2}{4E_0}$$

if $\frac{1}{3}\frac{1}{1-\cos\theta}=\frac{2}{3}\frac{1}{1+\cos\theta}=\frac{1}{2}$ $\quad\begin{cases}1+\cos\theta=\frac{4}{3}\\ 1-\cos\theta=\frac{2}{3}\end{cases}$ $\quad\cos\theta=\frac{1}{3}$

RAZ

Cross checking with MadGraph, we find that

$$\frac{\int d\Pi\ |\ \ |^2}{\int d\Pi\ |\mathcal{M}_{tot}|^2}\ll 1$$

This is due to the t-channel dominance.

$|t|$ can approach 0

2. $u\bar{d}\to W^+l^+l^-$

Comparing to $u\bar{d}\to W^+\gamma$, the l^+l^- pair just attaches the photon. primary contribution (other diagrams can be ignored)

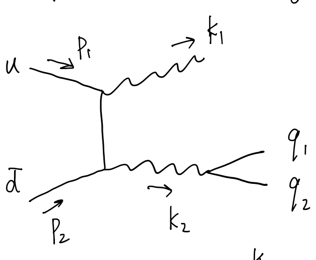
The two cases are quite different.

$u\bar{d}\to W^+\gamma$ on shell photon

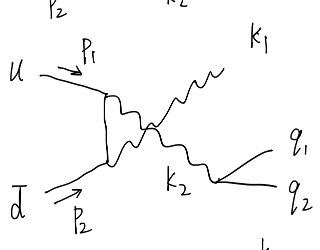
$u\bar{d}\to W^+\gamma^*\to W^+l^+l^-$ off-shell photon

We do the following parametrization

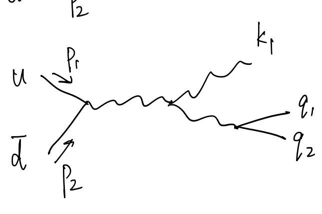
$$k_2=q_1+q_2$$



$$=\mathcal{M}_1^\mu\frac{1}{(q_1+q_2)^2}\,\bar{u}(q_1)\gamma_\mu v(q_2)$$



$$=\mathcal{M}_2^\mu\frac{1}{(q_1+q_2)^2}\,\bar{u}(q_1)\gamma_\mu v(q_2)$$



$$=\mathcal{M}_3^\mu\frac{1}{(q_1+q_2)^2}\,\bar{u}(q_1)\gamma_\mu v(q_2)$$

We can think now the off-shell photon has an effective mass

$$m_\gamma^2=(q_1+q_2)^2=2q_1\cdot q_2$$

$$\mathcal{M}_{2\to3}=(\mathcal{M}_1^\mu+\mathcal{M}_2^\mu+\mathcal{M}_3^\mu)\frac{1}{m_\gamma^2}\,\bar{u}(q_1)\gamma_\mu v(q_2)$$

We can first sum over the spins of l^+l^-

$$\sum_{s_{q_1},s_{q_2}}|\mathcal{M}_{2\to3}|^2=(\mathcal{M}_1^\mu+\mathcal{M}_2^\mu+\mathcal{M}_3^\mu)(\mathcal{M}_1^\nu+\mathcal{M}_2^\nu+\mathcal{M}_3^\nu)\frac{1}{m_\gamma^4}\mathrm{Tr}(\not q_1\gamma_\mu\not q_2\gamma_\nu)$$

$$=(\mathcal{M}_1^\mu+\mathcal{M}_2^\mu+\mathcal{M}_3^\mu)(\mathcal{M}_1^\nu+\mathcal{M}_2^\nu+\mathcal{M}_3^\nu)\underbrace{\frac{1}{m_\gamma^4}4(q_1^\mu q_2^\nu+q_2^\mu q_1^\nu-g^{\mu\nu}q_1\cdot q_2)}_{\text{similar to the massive photon polarization sum}}$$

Now we try to convert the 3 body phase to 2 body phase space using the formula

$$\int\frac{d^3q_1}{|\vec{q}_1|}\frac{d^3q_2}{|\vec{q}_2|}\,q_1^\mu q_2^\nu\,\delta^{(4)}(k_2-q_1-q_2)=\frac{\pi}{6}(g^{\mu\nu}k_2^2+2k_2^\mu k_2^\nu)$$

$$\int\frac{d^3q_1}{|\vec{q}_1|}\frac{d^3q_2}{|\vec{q}_2|}\,\delta^{(4)}(k_2-q_1-q_2)=2\pi$$

Hence we can have

$$\int\frac{d^3q_1}{(2\pi)^3 2|\vec{q}_1|}\frac{d^3q_2}{(2\pi)^3 2|\vec{q}_2|}\Big[q_1^\mu q_2^\nu+q_2^\mu q_1^\nu-g^{\mu\nu}(q_1\cdot q_2)\Big](2\pi)^4\delta^{(4)}(k_2-q_1-q_2)$$

$$=\frac{1}{4(2\pi)^2}\Big[\frac{\pi}{3}(g^{\mu\nu}k_2^2+2k_2^\mu k_2^\nu)-2\pi(q_1\cdot q_2)g^{\mu\nu}\Big]$$

$$=\frac{1}{16\pi^2}\left(-\frac{2\pi}{3}k_2^2g^{\mu\nu}+\frac{2\pi}{3}k_2^\mu k_2^\nu\right)$$

$$=\frac{k_2^2}{24\pi}\left(-g^{\mu\nu}+\frac{k_2^\mu k_2^\nu}{m_\gamma^2}\right)$$ polarization sum for massive photon

Now we can write down

$$d\sigma=\frac{1}{2E_1 2E_2|v_1-v_2|}\int\frac{d^3k_1}{(2\pi)^3 2E_W}\frac{1}{24\pi}\frac{1}{m_\gamma^2}(\mathcal{M}_1^\mu+\mathcal{M}_2^\mu+\mathcal{M}_3^\mu)(\mathcal{M}_1^\nu+\mathcal{M}_2^\nu+\mathcal{M}_3^\nu)\left(-g_{\mu\nu}+\frac{k_{2\mu}k_{2\nu}}{m_\gamma^2}\right)$$

the contraction with $g_{\mu\nu}$ part should be similar to $u\bar{d}\to W^+\gamma$

the contraction with $\frac{k_{2\mu}k_{2\nu}}{m_\gamma^2}$ is related to WT identity

replace ϵ_2 with k_2/m_γ

$$\frac{\mathcal{M}_1^\mu k_{2\mu}}{m_\gamma}=\left(-\frac{e}{3}\right)\frac{1}{-2p_2\cdot k_2}\left[\frac{2p_2\cdot k_2}{m_\gamma}\bar{v}(p_2)\not\epsilon_1 u_L(p_1)-m_\gamma\bar{v}(p_2)\not\epsilon_1 u_L(p_1)\right]$$
$$=\left(-\frac{e}{3}\right)\bar{v}(p_2)\not\epsilon_1 u_L(p_1)\left(-\frac{1}{m_\gamma}+\frac{m_\gamma}{2p_2\cdot k_2}\right)$$

$$\frac{\mathcal{M}_2^\mu k_{2\mu}}{m_\gamma}=\left(\frac{2e}{3}\right)\frac{1}{-2p_1\cdot k_2}\bar{v}(p_2)\not\epsilon_1 u_L(p_1)\left(m_\gamma^2-\frac{2p_1\cdot k_2}{m_\gamma}\right)$$
$$=\left(\frac{2e}{3}\right)\bar{v}(p_2)\not\epsilon_1 u_L(p_1)\left(\frac{1}{m_\gamma}-\frac{m_\gamma^2}{2p_1\cdot k_2}\right)$$

$$\frac{\mathcal{M}_3^\mu k_{2\mu}}{m_\gamma}=(-e)\frac{1}{(k_1+k_2)^2}\frac{1}{m_\gamma}\Big[2(k_1\cdot k_2)\bar{v}(p_2)\not\epsilon_1 u_L(p_1)-2(k_2\cdot\epsilon_1)\bar{v}(p_2)\not k_2 u_L(p_1)$$
$$+2(\epsilon_1\cdot k_2)\bar{v}(p_2)\not k_2 u_L(p_1)\Big]$$
$$=(-e)\frac{1}{m_\gamma}\bar{v}(p_2)\not\epsilon_1 u_L(p_1)$$

Hence we have

$$(\mathcal{M}_1^\mu+\mathcal{M}_2^\mu+\mathcal{M}_3^\mu)\frac{k_{2\mu}}{m_\gamma}=\left(-\frac{e}{3}\frac{m_\gamma}{2p_2\cdot k_2}-\frac{2e}{3}\frac{m_\gamma}{2p_1\cdot k_2}\right)\bar{v}(p_2)\not\epsilon_1 u_L(p_1)$$

One can see that

$$\mathcal{M}_3^\mu\frac{k_{2\mu}}{m_\gamma}=\frac{(-e)}{m_\gamma}\bar{v}(p_2)\not\epsilon_1 u_L(p_1)$$

$\left|\frac{\mathcal{M}_3^\mu k_{2\mu}}{m_\gamma}\right|^2$ should be dominat over $|\mathcal{M}_3^\mu\mathcal{M}_{3\mu}|$ for low m_γ

the typical t-channel propagator is smaller

$$\frac{1}{m_\gamma}\gg\frac{m_\gamma}{p_1\cdot k_2}\ \text{or}\ \frac{m_\gamma}{p_2\cdot k_2}\qquad |t|>\frac{m_W^2}{s-m_W^2}m_\gamma^2\gg m_\gamma^2$$

Hence we can get

$$\left|\mathcal{M}_3^\mu\frac{k_{2\mu}}{m_\gamma}\right|^2\gg\left|(\mathcal{M}_1^\mu+\mathcal{M}_2^\mu+\mathcal{M}_3^\mu)\frac{k_{2\mu}}{m_\gamma}\right|^2$$

This can explain

$$\frac{\int d\Pi\ |\ \ |^2}{\int d\Pi\ |\mathcal{M}_{tot}|^2}>1$$