Universite Paris 1 Pantheon - Sorbonne 2025
Master MMMEF

Project B (C. CHORRO)

Your report (written using Word or Latex or Jupyter notebook) has to be in my mailbox at the latest March 24 2025 (send me the electronic pdf version by email : christophe.chorro@gmail.com, please give an appropriate name to your file projectB-name1-name2.pdf)

The notations are taken from the course. You may use (with justifications) any proposition seen in class. For the numerical part you may use R, Mathlab, Scilab, VBA... The code has to be joined to the document.

Useful documents :

- Your lecture notes
- https ://sites.google.com/view/chorro-christophe/enseignement ?authuser=0

Part A : An integration by part formula

We consider $X \in \mathbb{D}$, $\Phi \in C^1 \cap Lip(\mathbb{R}, \mathbb{R})$ (C^1 and Lipschitz), $Y \in L^2(\Omega)$ and $h \in L^2([0,T] \times \Omega)$ such that

$$\int_0^T h(t) D_t X dt > 0 \quad a.s \tag{1}$$

and

$$u(t) = \frac{h(t)Y}{\int_0^T h(t) D_t X dt} \in dom(\delta). \tag{2}$$

a) Prove that

$$E[< DZ, u >_{L^2([0,T])}] = E[\Phi'(X)Y]. \tag{3}$$

where $Z = \Phi(X)$

b) Deduce that

$$E[\Phi(X)\delta(u)] = E[\Phi'(X)Y]. \tag{4}$$

Part B : Delta for asian options

We consider a risky asset whose dynamic on $[0, T]$ is given by the following equation

$$dX_t^x = rX_t^x dt + \sigma X_t^x dB_t;\ X_0^x = x$$

where $r > 0$ is the risk free rate, $x > 0$ the initial condition, $\sigma >$ and $(B_t)_{t\in[0,T]}$ the standard Brownian motion defined in the course.

Let $F = \Phi\left(\int_0^T X_s^x ds\right)$ where $\Phi \in C^1 \cap Lip(\mathbb{R}, \mathbb{R})$. We consider

$$P(x) = e^{-rT} E\left[\Phi\left(\int_0^T X_s^x ds\right)\right]. \tag{5}$$

a) Prove that $P \in C^1$ and that

$$\Delta(x) = \frac{dP(x)}{dx} = \frac{e^{-rT}}{x} E\left[\Phi'\left(\int_0^T X_s^x ds\right)\int_0^T X_s^x ds\right];\ \forall \Phi \in C^1 \cap Lip(\mathbb{R}, \mathbb{R}). \tag{6}$$

b) Taking $h = 1$ and supposing that we may apply the results of part A, show that

$$\Delta(x) = \frac{dP(x)}{dx} = \frac{e^{-rT}}{x} E\left[\Phi\left(\int_0^T X_s^x ds\right)\Pi_1\right];\ \forall \Phi \in C^1 \cap Lip(\mathbb{R}, \mathbb{R}) \tag{7}$$

where

$$\Pi_1 = \frac{\int_0^T X_s^x ds}{\int_0^T sX_s^x ds}\left[\frac{B_T}{\sigma} + \frac{\int_0^T s^2 X_s^x ds}{x\int_0^T sX_s^x ds}\right]. \tag{8}$$

c) Taking $h(t) = S_t$ and supposing that we may apply the results of part A, show that

$$\Delta(x) = \frac{dP(x)}{dx} = \frac{e^{-rT}}{x} E\left[\Phi\left(\int_0^T X_s^x ds\right)\Pi_2\right];\ \forall \Phi \in C^1 \cap Lip(\mathbb{R}, \mathbb{R}) \tag{9}$$

where

$$\Pi_2 = \frac{2}{\sigma^2}\left[\frac{X_T^x - x}{\int_0^T X_s^x ds} - r\right] + 1. \tag{10}$$

d) Let $\mathcal{W}$ be the set of all the random variables Π in $L^2(\Omega)$ such that

$$\Delta(x) = e^{-rT} E\left[\Phi\left(\int_0^T X_s^x ds\right)\Pi\right];\ \forall \Phi \in C^1 \cap Lip(\mathbb{R}, \mathbb{R}). \tag{11}$$

Prove that the weight $\Pi_0 \in \mathcal{W}$ minimizing the variance of $\Phi\left(\int_0^T X_s^x ds\right)\Pi$ is given by

$$\Pi_0 = E\left[\Pi \mid \sigma\left(\int_0^T X_s^x ds\right)\right]$$

where Π is an arbitrary element of $\mathcal{W}$.

e) Are Π_1 and Π_2 optimal in the sense of the preceding question ?

f) For $\Pi \in L^2(\Omega)$ prove that

$$g(x) = e^{-rT} E\left[\Phi(\int_0^T X_s^x ds)\Pi\right] \; ; \; \forall \Phi \in C^1 \cap Lip(\mathbb{R}, \mathbb{R})$$

$$\Updownarrow$$

$$g(x) = e^{-rT} E\left[\Phi(\int_0^T X_s^x ds)\Pi\right] \; ; \; \forall \Phi \in C(\mathbb{R}, \mathbb{R}) \;\; \text{with linear growth.}$$

Part C : Numerical part

For each question I want explanations concerning the methods you use and the choice you do for the parameters. You have to present in the clearest way your results (using graphics, tables, etc...) in the spirit of what I have done during the first 2 lectures.

The aim is to compute financial quantities associated to asian options by Monte Carlo Methods in a Black-Scholes model where $x = 100$, $r = 3\%$, $\sigma = 20\%$, $T = 1$.

In order to use monte Carlo Methods we may use the following approximation

$$\int_0^T X_s^x ds \approx \frac{T}{M} \sum_{k=0}^{M} X_{\frac{Tk}{M}}^x . \tag{12}$$

(In the following you will take $M = 50, M = 150$ and $M = 250$ and will compare the associated results.)

a) Using Monte Carlo Methods compute the prices of asian options with payoff $(\int_0^T X_s^x ds - K_1)_+$ and $1_{K_1 < \int_0^T X_s^x ds < K_2}$ where $K_1 = 100$ and $K_2 = 110$. Precise the empirical variance of each estimator and the corresponding confidence intervals. Study empirically the convergence of each estimators when the number of simulations increase (for example : from $N = 1000$ to $N = 51000$ by 2000).

b) Compute $\Delta(x)$ for the two options using the finite difference method. Precise the empirical variance of each estimator and the corresponding confidence intervals. What happens when you make ε varies ?

c) Compute $\Delta(x)$ for the two options using the two methods that may be deduced from part B. Precise the empirical variance of each estimator and the corresponding confidence intervals.

d) Compare (graphically and in terms of empirical variance) the 3 methods.

e) Conclusion