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Problem 1

Triangle neighbors

Fix a vertex V. The expected number of triangles containing V is

$$\binom{n-1}{2} p^3 = O(\frac{1}{n})$$

For v to be in two triangles, either they share one other vertex or are disjoint aside from v. The expected number of such vertex-triangle pairs is $O(n \times^2 p^5 + n \times n^4 p^6) = O(n^{-2} + n^{-2}) \to 0.$

By Markov' s inequality, no vertex lies in more than one triangle.

Isolated vertices

Let I_v be indicator that vertex v is isolated.

$$E(I_v) = (1-p)^{n-1}$$

So

$$\mu = E(\sum_v I_v) = n(1-p)^{n-1}$$

For $p = \frac{c+\log n}{n}$,

$$\mu_n = n[1 - \frac{logn + c}{n}]^{n-1} \approx ne^{-(\log n + c)} = e^{-c}$$

Thus $E[X] \to e^{-c}$. For factorial moments, a set S of k vertices is isolated with probability

$$(1-p)^{k(n-k)+\binom{k}{2}} \approx n^{-k} e^{-ck}$$

Thus

$$E[\binom{X}{k}] = \binom{n}{k}(1-p)^{k(n-k)+\binom{k}{2}} \approx \frac{n^k}{k!} n^{-k} e^{-ck} = \frac{e^{-ck}}{k!}$$

By the method of moments, $X \xrightarrow{d} \text{Poisson}(e^{-c})$

Problem 2

Jointly continuous

(1) If U and V are jointly continuous, they have a joint probability density function $f_{U,V}(u,v)$. The event $U=V$ corresponds to the line $u=v$ in the plane, which has Lebesgue measure zero. Therefore:

$$\Pr(U=V)=\iint_{u=v} f_{U,V}(u,v)\,du\,dv=0$$

(2) Let $X\sim \mathrm{Uniform}(0,1)$ and $Y=X$. Then X and Y are continuous random variables, $but \Pr(X=Y)=1$. This does not contradict (1), because (X,Y) is not jointly continuous — its probability mass is concentrated on the line $y=x$, so there is no joint density. Part (1) only applies when a joint density exists.

Distribution function

Let $\mathrm{F}':\mathrm{R}\to[0,1]$ satisfy:

- non-decreasing
- $\lim_{x\to-\infty}F'(x)=0,\ \lim_{x\to\infty}F'(x)=1$
- continuous everywhere
- not different at the same point

Can F' be a cumulative distribution function (CDF) for some random variable? Yes.A CDF only needs to be non-decreasing, right-continuous, and have the correct limits.

In fact if F' is absolutely continuous, the random variable has a density; if not, it may correspond to a singular continuous distribution, but it is still a valid CDF.

Density Function

Given that:

$$f(x)=C\exp\left(-ax-e^{-x}\right),\quad x\in\mathrm{R},$$

find C such that f is a probability density function.

Normalization requires:

$$\int_{-\infty}^{\infty} f(x)\,dx=C\int_{-\infty}^{\infty} e^{-ax}e^{-e^{-x}}\,dx=1.$$

Substitute $t=e^{-x}$, so $x=-\ln t$, $dx=-dt/t$.

When $x: -\infty \to \infty$, $t: \infty \to 0$. Swapping limits:

$$\int_{-\infty}^{\infty} e^{-ax} e^{-e^{-x}}\, dx = \int_{0}^{\infty} t^{a-1} e^{-t}\, dt = \Gamma(a),$$

provided $a > 0$.

Thus:

$$C \cdot \Gamma(a) = 1 \quad \Rightarrow \quad C = \frac{1}{\Gamma(a)}, \quad a > 0.$$

So for $a > 0$, $C = \frac{1}{\Gamma(a)}$ makes f a PDF

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Let $\{X_r : r \geq 1\}$ be i.i.d. with distribution function F such that $F(y) < 1$ for all y. Define

$$Y(y) = \min\{k : X_k > y\}.$$

Then $Y(y)$ is geometric with success probability $p_y = 1 - F(y)$, and $\mathrm{E}[Y(y)] = 1/p_y$. We want to show:

$$\lim_{y \to \infty} \Pr\left(Y(y) \leq \beta \mathrm{E}[Y(y)]\right) = 1 - e^{-\beta}, \quad \beta > 0.$$

Proof outline:

$$\Pr(Y(y) > m) = [F(y)]^m.$$

Let $m = \beta/p_y = \beta/(1 - F(y))$. Then:

$$\Pr\left(Y(y) > \frac{\beta}{1 - F(y)}\right) = [F(y)]^{\beta/(1-F(y))}.$$

As $y \to \infty$, $F(y) \to 1$. Write $F(y) = 1 - \varepsilon$, then:

$$(1 - \varepsilon)^{\beta/\varepsilon} \to e^{-\beta}.$$

Hence:

$$Pr(Y(y) \leq \frac{\beta}{1 - F(y)}) = 1 - e^{-\beta}.$$

Tails and moments

For nonnegative X, we have $\mathrm{E}(X^r) = \int_0^\infty rx^{r-1}\Pr(X > x)\,dx$.

Since $\mathrm{E}(|X|^r)$ exists, applying this to $|X|$ gives

$$\int_0^\infty rx^{r-1}\Pr(|X| > x)\,dx = \mathrm{E}(|X|^r) < \infty,$$

so $\int_0^\infty x^{r-1}\Pr(|X| > x)\,dx < \infty$.

Also, since the integrand $x^{r-1}\Pr(|X| > x)$ is integrable, we must have $x^{r-1}\Pr(|X| > x) \to 0$ as $x \to \infty$, hence $x^r\Pr(|X| > x) \to 0$.

Conditional expectation

By definition, $\psi(X) = \mathrm{E}(Y \mid X)$ is a function of X such that for any measurable g,

$$\mathrm{E}[\psi(X)g(X)] = \mathrm{E}[\mathrm{E}(Y \mid X)g(X)] = \mathrm{E}[g(X)\mathrm{E}(Y \mid X)].$$

But by the law of total expectation, $\mathrm{E}[g(X)\mathrm{E}(Y \mid X)] = \mathrm{E}[\mathrm{E}(g(X)Y \mid X)] = \mathrm{E}[g(X)Y]$.

Thus $\mathrm{E}(\psi(X)g(X)) = \mathrm{E}(Y g(X))$.

Correlated? Independent?

Let $X \sim \mathrm{Uniform}[-1, 1]$.

$$\mathrm{E}[Z_n] = \frac{1}{2}\int_{-1}^{1}\cos(n\pi x)\,dx = \frac{\sin(n\pi)}{n\pi} = 0.$$

For $m \neq n$,

$$\mathrm{E}[Z_m Z_n] = \frac{1}{2}\int_{-1}^{1}\cos(m\pi x)\cos(n\pi x)\,dx.$$

Using $\cos A\cos B = \frac{1}{2}[\cos((m+n)\pi x) + \cos((m-n)\pi x)]$, integral over symmetric interval of cosine with nonzero frequency is zero. So covariance is $0 \to$ uncorrelated.

But not independent: Z_1 and Z_2 are deterministic functions of X, so knowing Z_1 may determine X up to sign and thus determine Z_2.

Aliasing method

Any probability vector $p \in \mathbb{R}^n$ can be written as

$$p = \frac{1}{n} \sum_{i=1}^{n} v_i$$

where each v_i is a probability vector with at most 2 nonzero entries.

Construction: Think of p as a histogram. The method:

Pick i with $p_i \leq 1/n$, j with $p_j \geq 1/n$, form v_1 by putting mass np_i at i and $1 - np_i$ at j, update p by removing v_1/n, repeat.

Sampling method: Choose I uniformly in $\{1, \ldots, n\}$, then sample from the 2-point distribution v_I.

Stochastic domination

X stochastically dominates Y means $F_X(t) \leq F_Y(t)$ for all t, equivalently $\Pr(X > t) \geq \Pr(Y > t)$ for all t.

If $E[f(X)] \geq E[f(Y)]$ for all non-decreasing f, take $f(x) = \mathbf{1}_{x>t}$ to get $\Pr(X > t) \geq \Pr(Y > t)$, so stochastic domination holds.

Conversely, if X stochastically dominates Y, then by coupling there exists $X' \overset{d}{=} X$, $Y' \overset{d}{=} Y$ with $X' \geq Y'$ a.s. Then for non-decreasing f, $f(X') \geq f(Y')$ a.s., so $E[f(X)] \geq E[f(Y)]$.

Problem 3

Uniform Distribution (i)

Let $U \sim \mathrm{Uniform}[0, 1]$, $0 < q < 1$.

Define $X = 1 + \lfloor \frac{\ln U}{\ln q} \rfloor$.

Since $\ln U \le 0$ and $\ln q < 0$, $\frac{\ln U}{\ln q} \ge 0$.

For $k \ge 1$,

$$P(X = k) = P\left(1 + \lfloor \frac{\ln U}{\ln q} \rfloor = k\right) = P\left(k - 1 \le \frac{\ln U}{\ln q} < k\right).$$

Because $\ln q < 0$, inequalities reverse when multiplying:

$$= P(q^k < U \le q^{k-1}) = q^{k-1} - q^k = q^{k-1}(1 - q).$$

This is the geometric distribution with success probability $1 - q$ on $\{1, 2, \dots\}$.

Uniform Distribution (ii)

Suppose $U = X + Y$ with $U \sim \text{Uniform}[0, 1]$, X, Y i.i.d.

Then $E[U] = 1/2$, so ☑ = 1/4.

Also $\text{Var}(U) = 1/12$, but $\text{Var}(X + Y) = 2\text{Var}(X)$, so $\text{Var}(X) = 1/24$.

But $X \in [0, 1]$ a.s. (since $U \in [0, 1]$ and $X, Y \ge 0$ by $U = X + Y$), so ☑ = 1/4, no contradiction yet.

Better: Consider $P(U \le 1/2) = 1/2$.

But $P(X + Y \le 1/2) \ge P(X \le 1/4, Y \le 1/4) = [P(X \le 1/4)]^2$.

If X has mean $1/4$, by Markov $P(X > 1/4) \le 1$, but sharper: By uniformity of U, $F_U(t) = t$.

But $F_{X+Y}(t)$ is continuous and $F_{X+Y}(1) = 1$, $F_{X+Y}(0) = 0$, but for i.i.d. nonnegative X, Y, $F_{X+Y}(t)$ for small t is $O(t^2)$ if X has density at 0, but here $F_{X+Y}(t) = t$ for all $t \in [0, 1]$ implies $F_{X+Y}(t)$ linear, which is impossible for i.i.d. sum unless degenerate, contradiction.

Uniform Distribution (iii)

Suppose uniform distribution on (a, ∞) exists. Then PDF $f(x) = c$ for $x \ge a$.

But $\int_a^\infty c\,dx$ diverges unless $c = 0$, then total probability 0. Contradiction.

So no uniform distribution on infinite intervals.

Exponential distribution (i)

Let X be continuous, memoryless: $P(X > s + t \mid X > s) = P(X > t)$ for $s, t \geq 0$.

Let $\bar{F}(t) = P(X > t)$. Then $\bar{F}(s + t) = \bar{F}(s)\bar{F}(t)$.

The only continuous solution with $\bar{F}(0) = 1$, $\bar{F}(\infty) = 0$ is $\bar{F}(t) = e^{-\lambda t}$ for $\lambda > 0$.

Thus $X \sim \mathrm{Exponential}(\lambda)$.

Exponential distribution (ii)

Let $X \sim \mathrm{Exp}(\lambda)$, $N = \lfloor X \rfloor$, $M = X - N \in [0, 1)$.

For $n \in Z_{\geq 0}$, $m \in [0, 1)$,

$$P(N = n, M \leq m) = P(n \leq X \leq n + m) = e^{-\lambda n} - e^{-\lambda(n+m)}.$$

Also $P(N = n) = P(n \leq X < n + 1) = e^{-\lambda n} - e^{-\lambda(n+1)} = e^{-\lambda n}(1 - e^{-\lambda})$.

Thus

$$P(M \leq m \mid N = n) = \frac{e^{-\lambda n} - e^{-\lambda(n+m)}}{e^{-\lambda n}(1 - e^{-\lambda})} = \frac{1 - e^{-\lambda m}}{1 - e^{-\lambda}},$$

independent of n. So M and N are independent.

M has CDF $F_M(m) = \frac{1-e^{-\lambda m}}{1-e^{-\lambda}}$ for $m \in [0, 1)$, density $f_M(m) = \frac{\lambda e^{-\lambda m}}{1-e^{-\lambda}}$.

N is geometric: $P(N = n) = (1 - e^{-\lambda})e^{-\lambda n},\ n = 0, 1, 2, \ldots$.

Waiting for offers

Offers $X_i \sim F$ i.i.d. Accept first offer $> K$.

Number of offers N before sale is geometric with success probability $p = 1 - F(K)$
.

So $\mathrm{E}[N] = \frac{1}{p} - 1 = \frac{F(K)}{1-F(K)}$ if counting only rejected offers before acceptance.

If including the accepted offer: $\mathrm{E}[N] = \frac{1}{1-F(K)}$.

Usually "number of offers received before I sell" means before acceptance: $\frac{F(k)}{1-F(k)}$

Geometric distribution

Let $X \sim \exp(\lambda)$, $N = \lfloor X \rfloor$.

For $k \geq 0$,

$$\mathrm{P}(N = k) = \mathrm{P}(k \leq X < k+1) = e^{-\lambda k} - e^{-\lambda(k+1)} = e^{-\lambda k}\left(1 - e^{-\lambda}\right).$$

This is geometric distribution with success probability $p = 1 - e^{-\lambda}$ on $\{0, 1, 2, \ldots\}$.

Poisson clocks

Let $N_1(t), \ldots, N_k(t)$ be independent Poisson processes rate λ.

Their superposition $N(t) = \Sigma_{i=1}^{k} N_i(t)$ is a Poisson process with rate $k\lambda$ by additive property of Poisson processes: independent increments, rate sums.

Poissonian bears

Brown bears $B \sim \mathrm{PP}(\beta)$, grizzly bears $G \sim \mathrm{PP}(\gamma)$, independent.

(1) First bear arrival time: superposition is Poisson rate $\beta + \gamma$.

Probability first is brown: $\beta/(\beta + \gamma)$ by thinning property.

(2) Between two consecutive brown bears: time between browns $\sim \exp(\beta)$.

Given inter-brown time T, number of grizzlies in that interval $\sim \mathrm{Poisson}(\gamma T)$.

So

$$\mathrm{P}(r\,\text{grizzlies}) = \int_0^\infty e^{-\gamma T}\frac{(\gamma T)^r}{r!} \cdot \beta e^{-\beta T}\, dT = \frac{\beta\gamma^r}{r!}\int_0^\infty T^r e^{-(\beta+\gamma)T}\, dT.$$

Integral $= r!/(\beta + \gamma)^{r+1}$, so

$$P(r \text{ grizzlies}) = \frac{\beta \gamma^r}{(\beta + \gamma)^{r+1}}.$$

Bivariate normal distributions (i)

From given density $f_{XY}(x, y)$, means: ☑ = \mu_1, $E[Y] = \mu_2$.

Variances: $\mathrm{Var}(X) = \sigma_1^2$, $\mathrm{Var}(Y) = \sigma_2^2$.

Covariance: $\mathrm{Cov}(X, Y) = \rho \sigma_1 \sigma_2$ (since $Q(x, y)$ is standard bivariate normal form).

Bivariate normal distributions (ii)

Let $X \sim N(0, 1)$, $a > 0$,

$$Y = \{ \begin{matrix} X & |X| < a \\ -X & |X| \geq a \end{matrix}.$$

Check $Y \sim N(0, 1)$:

For $y \geq 0$,

$$F_Y(y) = P(Y \leq y) = P(|X| < a, X \leq y) + P(|X| \geq a, -X \leq y).$$

For $y \geq a$, $P(|X| < a, X \leq y) = P(|X| < a)$ since $y \geq a$, and $P(|X| \geq a, -X \leq y) = P(|X| \geq a)$ since $y \geq a > 0$, so sum = 1? Wait, need careful symmetry:

Better: Y is symmetric and $|Y| = |X|$, so Y has same distribution as X by symmetry of $N(0, 1)$ and the transformation preserves measure.

Covariance:

$$\rho(a) = E[XY] = E\left[X^2 \mathbf{1}_{|X| < a}\right] + E\left[-X^2 \mathbf{1}_{|X| \geq a}\right].$$

$$= E\left[X^2\right] - 2E\left[X^2 \mathbf{1}_{|X| \geq a}\right] = 1 - 2 \int_{|x| \geq a} x^2 \phi(x) dx.$$

Let $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$,

$$\rho(a) = 1 - 4 \int_a^\infty x^2 \phi(x) dx.$$

Integrate by parts: $\int_a^\infty x^2\phi(x)dx = \int_a^\infty x \cdot x\phi(x)dx$, $\int x\phi(x)dx = -\phi(x)$, so

$$\int_a^\infty x^2\phi(x)dx = a\phi(a) + \int_a^\infty \phi(x)dx = a\phi(a) + \bar{\Phi}(a),$$

where $\bar{\Phi}(a) = 1 - \Phi(a)$.

Thus

$$\rho(a) = 1 - 4\left[a\phi(a) + \bar{\Phi}(a)\right].$$

Pair (X, Y) is not bivariate normal: e.g., $X + Y$ is not normal (it's $2X$ for $|X| < a$, 0 for $|X| \geq a$, not normal).