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Problem 1 Moments and (co)variances

Run

1

Denote number of runs by R_n For n tosses, we can think of $n−1$ opportunities for a "**switch**" to occur between heads and tails. The total number of runs will be 1 (the first toss) plus the number of switches between heads and tails. The probability of a switch occurring between the ith and $i+1$th tosses is $p(1-p)+(1-p)p=2p(1-p)$, so the expectation: $$ E(R_n) = 1+(n-1)\times 2p(1-p) $$

As for the variance, the number of runs can be thought of as the sum of indicator variables for whether a switch occurs between each pair of consecutive tosses. Each switch is a **Bernoulli trial,** where the probability of success (a switch) is $2p(1−p)$. The variance is therefore derived from the **binomial distribution**, where the total number of trials is $n−1$, and the probability of success (a switch) is $2p(1−p)$ $$ Var(R_n)=(n-1)\times 2p(1-p)\times(1-2p(1-p)) $$

2

The expected number of runs of heads is given by the number of times we expect a transition from tails to heads in the random arrangement. This is roughly the number of times a tail precedes a head, and can be approximated by: $$ E(\text{runs of heads})=\frac{h}{n} $$ On average, each head will start a run in a random sequence.

And the variance can be computed as: $$ Var(\text{runs of heads})=\frac{h(1-p)}{n} $$

Gambler Lester Savage

1 Possible values of G and corresponding probabilities

1. If head comes on the first flip Then $G=x$ and he stops, probability is $\frac{1}{2}$
2. If tails comes on the first flip, and heads comes on the second flip: $G=-x+y$, and he stops, with the probability of $\frac{1}{4}$
3. If tails comes on the first flip, tails on the second flip, and heads on the third flip: $G=-x-y+z$, with the probability of $\frac{1}{8}$
4. If tails comes on all three flips: $G=-(x+y+z)$, with the probability of $\frac{1}{8}$

2 Expectation and Variance of G

$$ E(G)=x\times$\frac{1}{2}$+(-x+y)\times$\frac{1}{4}$+(-x-y+z)\times$\frac{1}{8}$-(x+y+z)\times$\frac{1}{8}$=0 $$ $$ Var(G)=E(G^2)-(E(G))^2=E(G^2)=x^2+\frac{2y^2+z^2}{4} $$

$$ Pr(G<0)=\frac{1}{4}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2} $$

3 Vary the Order

We should put the relative smaller ones before the bigger ones, so z for first bet, y for second, and x for the third. In this way, with probability $\frac{1}{4}$ we get G=y-z which is positive, and with some probability $frac{1}{8}$ G=x-y-z might be positive, so in this way we minimize $Pr(G<0)$ and $Var(G)$

Moments

$$ E(X^{\alpha})=\sum_{x=1}^{\alpha}x^{\alpha}f(x) $$ Substituting $f(x)=\frac{1}{x(x+1)}$ $$ E(X^{\alpha})=\sum_{x=1}^{\alpha}x^{\alpha}f(x)=\sum_{x=1}^{\alpha}\frac{x^{\alpha}}{x(x+1)} $$

For large x, the series is dominated by $x^{\alpha-2}$ So the series $\sum_{x=1}^{\alpha}x^{\alpha-2}$ coverages if $\alpha-2<-1$ Therefore $\alpha < 1$

Covariance and correlation (i)

Firstly, prove $|ρ|=1 \Rightarrow X=aY+b$ Now that X and Y are perfectly linear related, $Cov(X, Y)=± \sigma_X\sigma_Y$ Thus $X=aY+b$ for some constants a and b.

Then, prove $ X=aY+b \Rightarrow |ρ|=1$ Then $$ Cov(X, Y)=Cov(aY+b, Y)=a\times \sigma_Y^2 $$ Therefore $$ ρ=\frac{Cov(X, Y)}{\sigma_X\sigma_Y}=\frac{a\sigma_Y}{\sigma_X} $$ Since $X=aY+b$, |ρ|=1

Covariance and correlation (ii)

Let X, Y, Z be the random variables that: $X=-Y$, $Z=Y$, X and Z are independent. In this case $Cov(X, Y)<0$, $Cov(Y, Z)<0$ but $Cov(X, Z)=0$, which implies there is no relation between X and Z.

Uncorrelation versus Independence

(a) To make X and Y uncorrelated, we need to make $Cov(X, Y)=0$ $E(X)=c+d$, $E(Y)=b+d$, $E(XY)=d$, so $$ Cov(X, Y)=E(XY)-E(X)E(Y)=d-cb-cd-db-d^2 $$ SO when $d-cb-cd-db-d^2=0$, X and Y are uncorrelated.

(b) For the two to be independent, we need $$ f(x, y)=f_X(x)f_Y(y) $$ And by calculation we get the conditions:

- $a=(a+b)(a+c)$
- $b=(a+b)(b+d)$
- $c=(c+d)(a+c)$
- $d=(c+d)(b+d)$

Covariance Matrix

The determinant of the covariance matrix $V(X)$ is zero if and only if the rows (or columns) of the matrix are linearly dependent. If the random variables are linearly dependent, then there exists a nontrivial linear combination of the variables that equals a constant with probability 1. This implies that the covariance matrix does not have full rank, leading to a determinant of zero.

So the determinant of $V(X)$ is 0 if and only if the random variables $X_1, X_2,\dots, X_n$ are linearly dependent with probability 1

Conditional variance formula

The conditional variance of Y given by X can be defined as: $$ Var(Y|X)=E[(Y-E[Y|X])^2|X] $$ $$ Var(Y)=E(Y^2)-(E[Y])^2 $$ And: $$ E[Y^2]E[E[Y^2|X]]=E[Var(Y|X)]+(E[Y|X])^2 $$ So in conclusion: $$

Var(Y)=E[Var(Y|X)]+Var(E[Y|X]) $$ $$ E[S|N=n]=E[X_1+X_2+\dots+X_n]=n\mu $$ So $$ E[S|N]=\mu N $$ Therefore $$ E[S]=E[E[S|N]]=E[\mu N]=\mu E[N] $$

$$ E[Var(S|N)]=E[\sigma^2N]=\sigma^2E[N] $$

So $$ Var(S)=\sigma^2E[N]+\mu^2Var(N) $$

Problem 2 Markov and Chebyshev

Markov's inequality

For any non-negative random variable Y and a > 0: $$ Pr(Y\geq a)\leq\frac{E(Y)}{a} $$ Now take $Y=e^{\beta X}$, where $\beta > 0$, we have: $$ Pr[e^{\beta X\geq\beta x}]\leq \frac{E[e^{\beta X}]}{e^{\beta x}} $$ Thus we have $$ Pr[X\geq x]\leq E[e^{\beta X}]e^{-\beta x} $$

Chebyshev's inequality (i) (Cantelli's inequality)

$$ Pr[X-E[X]>t]\leq Pr[|X-E[X]|\geq t] $$ From Chebyshev's inequality $$ Pr[|X-E[X]|\geq t]\leq\frac{Var(X)}{t^2} $$ Therefore $$ Pr[|X-E[X]|> t]\leq\frac{Var(X)}{t^2+Var(X)} $$

Chebyshev's inequality (ii) (Paley–Zygmund inequality)

1

$$ E[X]=E[XI_{X\leq \theta E[X]}]+E[XI_{X> \theta E[X]}] $$ So $$ P(X>\theta E[X])\geq\frac{(1-\theta)^2E[X]^2}{E[X^2]} $$ Let $\theta=a$, we have $$ P(X>aE[X])\geq\frac{(1-a)^2E[X]^2}{E[X^2]} $$

2

$$ 1-P(X=0)\geq\frac{E[X]^2}{E[X^2]} $$ So $$ P(X=0)\leq 1-\frac{E[X]^2}{E[X^2]}=\frac{Var(X)}{E[X^2]} $$ So $$ P(X=0)\leq\frac{Var(X)}{E[X]^2} $$