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Problem 1

Proportional betting

- Setup: Let $F_0 > 0$ and independent multipliers Y_k with $\mathrm{P}(Y_k = 1.30) = \mathrm{P}(Y_k = 0.75) = 1/2$, so $F_n = F_0 \prod_{k=1}^n Y_k$.
- Expectation: $\mathrm{E}[Y_k] = (1.30 + 0.75)/2 = 1.025$. Independence gives $\mathrm{E}[F_n] = F_0\,(\mathrm{E}\,Y)^n = F_0 \cdot 1.025^n \to \infty$ as $n \to \infty$.
- Almost-sure decay: Put $S_n = \log F_n = \log F_0 + \sum_{k=1}^n \log Y_k$. By the Strong Law of Large Numbers,

  = \tfrac12\log(1.30)+\tfrac12\log(0.75) < 0.

Entropy

Let the partition of $[0, 1]$ have lengths $p_1, \dots, p_n$ and entropy $h = -\sum_{i=1}^n p_i \log p_i$. Draw $X_1, X_2, \dots$ i.i.d. $\mathrm{Unif}[0, 1]$, and let $Z_m^{(i)}$ be the count among $X_1, \dots, X_m$ falling in interval i.

- Multinomial structure: $(Z_m^{(1)}, \dots, Z_m^{(n)}) \sim \mathrm{Multinomial}(m; p_1, \dots, p_n)$.
- Empirical frequencies: $\hat{p}_i := Z_m^{(i)}/m \xrightarrow{a.s.} p_i$ by the Strong Law of Large Numbers.
- Define $R_m = \prod_{i=1}^n p_i^{Z_m^{(i)}}$. Then

  ;\xrightarrow[m\to\infty]{a.s.}; \sum_{i=1}^n p_i\log p_i = -h.

Mobilizing a Supermajority

Setup: For each of n baseline individuals, (i) attends with probability τ; (ii) if attending, votes Yes with probability p (No with $1-p$). A proposal is accepted if the fraction of Yes among attendees is at least $\theta \in (1/2, 1)$. A campaign can add m sure attendees who vote Yes.

Notation:

- Baseline attendees $A \sim \mathrm{Bin}(n, \tau)$.
- Baseline Yes votes among attendees, given $A = k$, are $Y \mid A = k \sim \mathrm{Bin}(k, p)$.
- After adding m supporters, acceptance is $(Y + m)/(A + m) \geq \theta$.

Exact acceptance probability for a given m:

$$P_{\mathrm{pass}}(m) = \sum_{k=0}^{n} \binom{n}{k} \tau^k (1-\tau)^{n-k} \ \Pr[\mathrm{Bin}(k, p) \geq \lceil \theta(k+m) - m \rceil].$$

The minimal m that achieves $P_{\mathrm{pass}}(m) \geq 1-\delta$ can be found exactly by increasing m until the sum exceeds $1-\delta$ (numerically tractable for moderate n).

Closed-form normal approximation (CLT): Define per-person contribution $T_i := \mathbf{1}\{\mathrm{attend{+}Yes}\} - \theta \mathbf{1}\{\mathrm{attend}\}$, so the baseline margin is $T := \sum_{i=1}^{n} T_i = Y - \theta A$. Passing with m supporters is

$$T + (1-\theta)m \geq 0.$$

Here

$$\mu = \mathrm{E}[T_i] = \tau(p-\theta), \qquad \sigma^2 = \mathrm{Var}(T_i) = \tau[p(1-\theta)^2 + (1-p)\theta^2] - \tau^2(p-\theta)^2.$$

Approximating $T \approx \mathrm{N}(n\mu,\ n\sigma^2)$, the δ-lower quantile is $n\mu + z_\delta\, \sigma\sqrt{n}$ (with $z_\delta = \Phi^{-1}(\delta)$). Ensuring $\Pr[T \geq -(1-\theta)m] \geq 1-\delta$ yields

$$m_{textCLT} \approx \lceil \max\{0,\ \frac{-n\mu - z_\delta\, \sigma\sqrt{n}}{1-\theta}\} \rceil.$$

Interpretation: If $p \geq \theta$, then $\mu \geq 0$ and often $m \approx 0$ suffices; if $p < \theta$, the formula gives the extra Yes votes needed to overcome the expected shortfall plus a δ-level safety margin.

Problem 2

Tossing coins

Let X be the number of throws needed to obtain n heads when tossing a fair coin. Then X is a sum of n i.i.d. geometric($p = 1/2$) variables, i.e., a negative-binomial $\mathrm{NB}(n, 1/2)$ random variable with moment generating function

$$M_X(t) = \left(\frac{\frac{1}{2}e^t}{1 - \frac{1}{2}e^t}\right)^n, \quad 0 < t < \ln 2.$$

By Chernoff's method, for any $t \in (0, \ln 2)$,

$$\Pr[X \geq x] \leq e^{-tx} M_X(t).$$

Set $x = 2n + \Delta$ with $\Delta > 0$. Minimizing the RHS over t is equivalent to minimizing over $y = e^t \in (1,2)$ the function

$$f(y) = -x \ln y + n[\ln(\tfrac{y}{2}) - \ln(1 - \tfrac{y}{2})].$$

Writing $u = y/2$ and optimizing gives $u^* = (n + \Delta)/(2n + \Delta)$ and

$$\ln \Pr[X \geq 2n + \Delta] \leq -(3n + \Delta) \ln 2 + (2n + \Delta) \ln(2n + \Delta) - (n + \Delta) \ln(n + \Delta) - n \ln n.$$

Using the inequality $\ln(1 + x) \leq x - x^2/2 + x^3/3$ for $x \in [0, 1]$ and the constraint $0 < \Delta \leq 2n$ (equivalently $0 < \delta < \sqrt{4n/\ln n}$ when $\Delta = \delta\sqrt{n \ln n}$), one checks that

$$\ln \Pr[X \geq 2n + \Delta] \leq -\frac{\Delta^2}{6n}.$$

Therefore for any $0 < \delta < \sqrt{\frac{4n}{\ln n}}$,

$$\Pr\left[X > 2n + \delta\sqrt{n \ln n}\right] \leq \exp\left(-\frac{\delta^2}{6} \ln n\right) = n^{-\delta^2/6}.$$

k-th moment bound

Assume $\mathrm{E}[e^{t|X|}] < \infty$ for some $t > 0$ and $\mathrm{E}[X] = 0$.

- Chernoff bound: $\Pr(|X| \geq \delta) \leq \inf_{s>0} e^{-s\delta}\,\mathrm{E}[e^{s|X|}]$.
- k-th moment bound: $\Pr(|X| \geq \delta) \leq \mathrm{E}[|X|^k]/\delta^k$.

1. Existence of a k matching Chernoff (probabilistic method). Let $K \sim \mathrm{Poisson}(s\delta)$ be independent of X. Then

$$\mathrm{E}\left[\frac{|X|^K}{\delta^K}\right] = \mathrm{E}[e^{s(|X|-\delta)}] = e^{-s\delta}\,\mathrm{E}[e^{s|X|}].$$

Since $\Pr(|X| \geq \delta) \leq \mathrm{E}[|X|^k]/\delta^k$ for each fixed k, averaging this bound over K yields exactly the Chernoff bound for the same s. Hence there exists a realization k of K such that

$$\Pr(|X| \geq \delta) \leq \frac{\mathrm{E}[|X|^k]}{\delta^k} \leq e^{-s\delta}\,\mathrm{E}[e^{s|X|}] \quad \text{for that } k.$$

Minimizing over $s > 0$ shows that for every δ, one can choose an integer k so that the k-th moment bound is no weaker than the optimal Chernoff bound.

2. Why still prefer Chernoff? The Chernoff method

- provides smoothly tunable bounds via a continuous parameter s rather than discrete k;
- avoids factorial/large-k constants and is usually sharper for moderate deviations;
- requires only finiteness of the mgf near 0, whereas high-order moments may not exist or be hard to evaluate;
- yields clean, distribution-agnostic exponential tails with standard templates (Hoeffding/Bernstein/KL forms).

Densest induced subgraph in random graph

Let $G \sim G(n, 1/2)$. For $S \subseteq [n]$, denote by $e(S)$ the number of edges with both endpoints in S and define

$$\mathrm{dens}(S) := \frac{2e(S)}{|S|} \quad \text{(average degree in the induced subgraph).}$$

Fix $s = |S| \geq 2$. Then $e(S) \sim \mathrm{Bin}(\binom{s}{2}, \frac{1}{2})$ with mean $\mu_s = \frac{1}{2}\binom{s}{2}$ and variance $\sigma_s^2 = \frac{1}{4}\binom{s}{2}$. By Bernstein (or Hoeffding), for any $t > 0$,

$$\Pr[\mathrm{dens}(S) \geq \frac{s}{2} + t] = \Pr[e(S) \geq \mu_s + \frac{ts}{2}] \leq \exp(-\frac{t^2 s^2}{2\binom{s}{2}}) \leq \exp(-\frac{t^2}{s}).$$

Apply a union bound over all $\binom{n}{s}$ subsets of size s:

$$\Pr[\exists S : \mathrm{dens}(S) \geq \frac{s}{2} + t] \leq \binom{n}{s} \exp(-\frac{t^2}{s}) \leq \exp(s \ln \frac{en}{s} - \frac{t^2}{s}).$$

Choose $t = C\sqrt{n}$ with an absolute constant $C > 0$ large enough; then the exponent is $\leq -\ln 3$ for all $s \in [2, n]$, so summing over s gives

$$\Pr[\max_S \mathrm{dens}(S) \leq n/2 + C\sqrt{n}] \geq 2/3,$$

that is, with probability at least $2/3$, the densest induced subgraph in $G(n, 1/2)$ has average degree at most $n/2 + Cn^{1/2}$.

Problem 3

High-dimensional random walk (martingale)

Let $(X_t)_{t \geq 0}$ be a random walk in $\mathbb{R}^n$ with increments $\Delta_t := X_{t+1} - X_t$ such that $\mathbb{E}[\Delta_t \mid \mathcal{F}_t] = 0$ for all t (unbiased step in every dimension, possibly with dependent coordinates and arbitrary distributions), where $\mathcal{F}_t$ is the natural filtration. Then

$$\mathbb{E}[X_{t+1} \mid \mathcal{F}_t] = X_t + \mathbb{E}[\Delta_t \mid \mathcal{F}_t] = X_t,$$

so (X_t) is a martingale in any dimension. Coordinate-wise, each $X_t^{(i)}$ is a martingale by the same argument.

Pólya's urn martingale

Initially $r > 0$ red and $b > 0$ blue balls. At time n, draw one ball uniformly, return it and add another of the same color. Let R_n be the number of red balls after n draws and define

$$Y_n := \frac{R_n}{r + b + n}.$$

Conditional on $\mathcal{F}_{n-1}$, $R_n = R_{n-1} + I_n$ where $I_n \in \{0, 1\}$ is the indicator of drawing red, with

$$\mathbb{P}(I_n = 1 \mid \mathcal{F}_{n-1}) = \frac{R_{n-1}}{r + b + n - 1}.$$

Hence

$$\mathrm{E}[Y_n \mid \mathcal{F}_{n-1}] = \frac{\mathrm{E}[R_{n-1} + I_n \mid \mathcal{F}_{n-1}]}{r+b+n} = \frac{R_{n-1} + R_{n-1}/(r+b+n-1)}{r+b+n} = \frac{R_{n-1}}{r+b+n-1} = Y_{n-1}.$$

Thus (Y_n) is a martingale.

Optional stopping for 1-D symmetric walk

Let $S_n = a + \sum_{r=1}^{n} X_r$ where $X_r \in \{\pm 1\}$ are i.i.d. symmetric. Compute

$$\mathrm{E}[S_{n+1}^3 \mid \mathcal{F}_n] = S_n^3 + 3S_n,$$

so $S_{n+1}^3 - S_n^3 - 3S_n$ has zero conditional mean and

$$M_n := \sum_{r=0}^{n} S_r - \tfrac{1}{3} S_n^3$$

is a martingale. Let T be the hitting time of $\{0, K\}$ with $0 < a < K$. Under standard integrability conditions, optional stopping yields

$$\mathrm{E}[\sum_{r=0}^{T} S_r - \tfrac{1}{3} S_T^3] = \mathrm{E}[M_T] = \mathrm{E}[M_0] = a - \tfrac{1}{3} a^3.$$

For the simple symmetric walk, $\mathrm{P}(S_T = K) = a/K$, therefore $\mathrm{E}[S_T^3] = K^3 \cdot (a/K) = aK^2$. Rearranging gives

$$\mathrm{E}[\sum_{r=0}^{T} S_r] = \tfrac{1}{3} aK^2 + a - \tfrac{1}{3} a^3 = \tfrac{1}{3}(K^2 - a^2)\, a + a.$$

Random walk on a graph: stationary distribution

Let G be a finite connected simple undirected graph with η edges and degrees d_v. The simple random walk moves from v to a uniformly chosen neighbor; hence $P_{v,u} = 1/d_v$ if $u \sim v$. Define $\pi(v) = d_v/(2\eta)$. Then for any edge $u \sim v$,

$$\pi(v) P_{v,u} = \frac{d_v}{2\eta} \cdot \frac{1}{d_v} = \frac{1}{2\eta} = \frac{d_u}{2\eta} \cdot \frac{1}{d_u} = \pi(u) P_{u,v}.$$

Thus detailed balance holds and π is stationary.

Reversibility versus periodicity

Yes. A reversible chain can be periodic. Example: the simple random walk on any bipartite graph (e.g., the path $\{0, 1, \dots, n\}$ or an even cycle) is reversible with respect to degree measure, yet has period 2 because it alternates between the two partitions.

Metropolis–Hastings algorithm

1. With neighborhood structure $N(x)$ and $M \geq \max_x |N(x)|$, define

$$P_{x,y} = \begin{cases} frac 1 M \min\{1, \pi_y/\pi_x\}, & y \in N(x),\ y \neq x, \\ 0, & y \notin N(x),\ y \neq x, \\ 1 - \sum_{z \neq x} P_{x,z}, & y = x. \end{cases}$$

Assuming $y \in N(x) \iff x \in N(y)$ (undirected proposals),

$$\pi_x P_{x,y} = \frac{\pi_x}{M} \min\{1, \pi_y/\pi_x\} = \frac{1}{M} \min\{\pi_x, \pi_y\} = \pi_y P_{y,x},$$

so detailed balance holds and π is stationary. Irreducibility and aperiodicity are assumed.

2. Target on positive integers: $\pi_i = 1/(Si^2)$ where $S = \sum_{i \geq 1} i^{-2} = \pi^2/6$. Use linear neighbors: for $i > 1$, $N(i) = \{i-1, i+1\}$ and $N(1) = \{2\}$. Take $M = 2$. Then for $i \geq 2$,

$$P_{i,i-1} = \tfrac{1}{2} \min\{1, \frac{\pi_{i-1}}{\pi_i}\} = \tfrac{1}{2} \min\{1, (\frac{i}{i-1})^2\} = \tfrac{1}{2}, \quad P_{i,i+1} = \tfrac{1}{2} \min\{1, \frac{\pi_{i+1}}{\pi_i}\} = \tfrac{1}{2} \min\{1, (\frac{i}{i+1})^2\} = \tfrac{1}{2}(\frac{i}{i+1})^2$$

Set $P_{i,i} = 1 - P_{i,i-1} - P_{i,i+1}$. For $i = 1$,

$$P_{1,2} = \tfrac{1}{2} \min\{1, \frac{\pi_2}{\pi_1}\} = \tfrac{1}{2}(\frac{1}{2})^2 = \tfrac{1}{8}, \quad P_{1,1} = 1 - P_{1,2} = \tfrac{7}{8}.$$

This MH chain is irreducible and reversible w.r.t. $\pi_i \propto i^{-2}$, giving the desired stationary distribution.