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σ-Algebra

$\mathcal{F}$ is a σ-Algebra of A and B, so $A^c, B^c \in \mathcal{F}$ so $A^c \cup B^c \in \mathcal{F}$. Given that $(A \cap B)^c = A^c \cup B^c$, we have $(A \cap B)^c \in \mathcal{F}$. Thus, $A \cap B \in \mathcal{F}$. And since $B^c \in \mathcal{F}$, $A \setminus B = A \cap B^c \in \mathcal{F}$. And so $B \setminus A \in \mathcal{F}$, $A \oplus B = (A \setminus B) \cup (B \setminus A) \in \mathcal{F}$. Let $\Omega = 1, 2, 3$. Define $\mathcal{F} = \{\emptyset, \Omega, \{1\}, \{2, 3\}\}$. Then $\mathcal{F}$ is a σ-Algebra. Define $\mathcal{G} = \{\emptyset, \Omega, \{1, 3\}, \{2\}\}$. Then $\mathcal{G}$ is also a σ-Algebra. But $\mathcal{F} \cup \mathcal{G} = \{\emptyset, \Omega, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}\}$ is not a σ-Algebra, because $\{1\}, \{2\} \in \mathcal{F} \cup \mathcal{G}$ while $\{1\} \cup \{2\} = \{1, 2\} \notin \mathcal{F} \cup \mathcal{G}$.

Probability law

As $Pr(A \cup B) = Pr(A) + Pr(B)$, when $A \cap B = \emptyset$ Now we have $Pr(\Omega \cup \emptyset) = Pr(\Omega) + Pr(\emptyset) = 1$. So $Pr(\emptyset) = 0$.

Nonexistence of probability space

Prove non-existence on N

Suppose there exists a probability space $(N, 2^N, Pr)$ with $Pr(\{i\}) = Pr(\{j\}) = p$ for all $i, j \in N$. Then $Pr(N) = Pr(\cup_{i \in N}\{i\}) = \Sigma_{i \in N} Pr(\{i\}) = \Sigma_{i \in N} p$. If $p = 0$, then $Pr(N) = 0$, which contradicts the axiom that $Pr(\Omega) = 1$. If $p > 0$, then $\Sigma_{i \in N} p = +\infty$, which also contradicts the axiom that $Pr(\Omega) = 1$. So there does not exist such a probability space.

Reason of failure

N, the sample space, is countably infinite, while $[0, 1]$ is uncountably infinite. Intervals are uncountable and so there is no uniform probability law on the real interval $[0, 1]$.

Euler's totient function

Let $n = p_1^{c_1} p_2^{c_2} \cdots p_k^{c_k}$ be the prime factorization of n. Let A_i be the set of x that each satisfying $x \leq n$ and $p_i \mid x$. Then $\phi(n) = n - \mid A_1 \cup A_2 \cup \cdots \cup A_k \mid$. According to the principle of inclusion-exclusion, $\mid A_1 \cup A_2 \cup \cdots \cup A_k \mid = \Sigma_{i=1}^k \mid A_i \mid - \Sigma_{1 \leq i < j \leq k} \mid A_i \cap A_j \mid \cdots + (-1)^{k+1} \mid \cap_{i=1}^k A_i$ Then we calculate each term: $\mid A_i \mid = \frac{n}{p_i}$, because $p_i \mid n$ $\mid A_i \cap A_j \mid = \frac{n}{p_i p_j}$, because the one in $A_i \cap A_j$ should be divisible by both p_i and p_j ... $\mid \cap_{i=1}^k A_i \mid = \frac{n}{p_1 p_2 \cdots p_k}$ Now we have: $\phi(n) = n - n(\Sigma_{i=1}^k \frac{1}{p_i} - \Sigma_{1 \leq i < j \leq k} \frac{1}{p_i p_j} \cdots + (-1)^{k+1} \frac{1}{p_1 p_2 \cdots p_k})$ And by polynomial expansion we know this is equal to: $\phi(n) = n \prod_{i=1}^k (1 - \frac{1}{p_i})$

Ménages

Firstly we try to find the number of valid seating ways. We fix one person, assume him to be a man, and then there're n-1 men left to be seated, so $(n-1)!$ ways. Now we have n women to be seated, so $n!$ ways. And multiply them together, we have $T = (n-1)!n!$ ways to seat them without any restriction.

Then we wanna figure out number of seating ways when there is no adjacent couple. To begin with we fix one man and arrange the n-1 men left, so $(n-1)!$ ways. For women to not sit next to their partners, their seating must be a derangement relative to their husbands' seats. To arrange the n women, suppose every woman has one designated seat (her husband's seat). We want to count the number of permutations. Using the derangement formula, we have $n!\Sigma_{i=0}^{n}\frac{(-1)^i}{i!}$ ways. So the probability would be $\frac{!n}{n!} = \Sigma_{i=0}^{n}\frac{(-1)^i}{i!}$.

The problem of points

Now we say T for Telis, W for Wendy. Firstly we calculate the probability of Telis winning. $P_T = p^8 + C_8^1 p^7(1-p) + C_8^2 p^6(1-p)^2 + C_8^3 p^5(1-p)^3 + C_8^4 p^4(1-p)^4$ Then we calculate the probability of Wendy winning. $P_W = (1-p)^8 + C_8^1(1-p)^7 p + C_8^2(1-p)^6 p^2$ So Telis would get $10 \times \frac{P_T}{P_T+P_W}$ dollars

Certainly occur

According to Inclusion-Exclusion Principle, and since no more than 2 events occur simultaneously, we simplify the formula to: $Pr(\cup_{r=1}^{n} A_r) = \Sigma_{r=1}^{n} Pr(A_r) - \Sigma_{1\le r<s\le n} Pr(A_r \cap A_s)$ And since there must be one event to occur, it equals 1. Substitute the given values, we have: $np - C_n^2 q = 1$ So that $np = 1 + C_n^2 q \ge 1$, so $p \ge \frac{1}{n}$. And $q = \frac{np-1}{n(n-1)/2} = \frac{2np-2}{n(n-1)} \le \frac{2n-2}{n(n-1)} = \frac{2}{n}$.