Interest Rate Models

2. Girsanov, Numeraires, and All That

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Outline

Self-financing portfolios and arbitrage

- This is a technical intermezzo in preparation for next few themes: valuation of options on interest rates, CMS based instruments, and term structure modeling.
- We start by reviewing briefly some basic concepts of arbitrage pricing theory, just enough to cover our upcoming needs. For a more complete account of this theory, I encourage you to consult e.g. Björk's book [1].
- We will skip over a lot of technicalities while discussing the probabilistic concepts underlying this framework, and, again, I recommend further study for a more in depth understanding of these concepts.
- Next, we will discuss the technique of *change of numeraire*, which will play a key role in the following lectures.

Frictionless markets

- We consider a model of a *frictionless* financial market in which which N (risky) assets are traded. By frictionless we mean that:
  (i) Each of the assets is *liquid*, i.e. at each time any bid or ask order can be immediately executed.
  (ii) There are no *transaction costs*, i.e. each bid and ask order for each security at time t is executed at the same unique price level.
- Very much like the formulation of Newtonian gravity in vacuum, this is obviously a gross oversimplification of reality, and much work has been done to relax these assumptions.
- From the conceptual point of view, however, it leads to a profound and workable framework of asset pricing theory.

Frictionless markets

- We model the price processes of these assets by the vector $S(t) = \big(S_1(t), \ldots, S_N(t)\big)$, where $S_i(t)$ denotes the price of i-th asset at time t. We note that these processes represent market observable prices, and not merely some convenient state variables.
- We assume that each price process is a diffusion process. In other words, there is an underlying probability space $(\Omega, (\mathscr{F}_t)_{t\geq 0}, \mathsf{P})$ generated by a multidimensional Wiener process $W_1(t), \ldots, W_d(t)$, and for each $j = 1, \ldots, N$,

$$dS_j(t) = \Delta_j(t, S(t))dt + \sum_{1\leq k\leq d} C_{jk}(t, S(t))dW_k(t)\,, \tag{1}$$

  with suitable drift and diffusion coefficients Δ_j and C_{jk}, respectively.
- In fact, one can weaken this assumption in many ways. It is, for example sufficient, to assume that the drift and diffusion coefficients are adapted with respect to $(\mathscr{F}_t)_{t\geq 0}$.

Black-Scholes model

- In order to develop an intuition for the concepts explained below we recall the Black-Schols from the world of equity derivatives.
- In this classic model of equity derivatives, $S_1(t) = B(t)$ is the riskless money market account, and $S_2(t) = S(t)$ is a risky stock, with the dynamics given by

$$\begin{aligned} dB(t) &= rB(t)\,dt, \\ dS(t) &= \mu S(t)\,dt + \sigma S(t)\,dW(t)\,. \end{aligned} \tag{2}$$

- The rate of return r on the money market account is called the *riskless rate*, while μ is the rate of return on the risky asset.
- As we already know, there is really no such thing as riskless rate (with the closest proxy being the overnight OIS rate), but its presence in the Black-Scholes model helps one understand the general framework of risk neutral valuation.
- The modeling framework for interest rate derivatives, which is the subject of these lectures, does not require invoking the risk free rate explicitly.

Self-financing portfolios and arbitrage

- A *portfolio* (or *trading strategy*) is specified by the vector of weights $w(t) = (w_1(t), \ldots, w_N(t))$, of the assets at time t. We assume, that the weights are adapted with respect to $(\mathscr{F}_t)_{t>0}$, and they add up to one. The value process of the portfolio is given by

$$V(t) = \sum_{1 \leq i \leq N} w_i(t) S_i(t). \tag{3}$$

- A portfolio is *self-financing*, if

$$dV(t) = \sum_{1 \leq i \leq N} w_i(t) dS_i(t). \tag{4}$$

- Equivalently,

$$V(t) = V(0) + \int_0^t \sum_{1 \leq i \leq N} w_i(s) dS_i(s). \tag{5}$$

- In other words, the price process of a self-financing portfolio does not allow for infusion or withdrawal of capital. It is entirely driven by price processes of the constituent instruments and their weights.

Self-financing portfolios and arbitrage

- A fundamental assumption of arbitrage pricing theory is that financial markets (or at least, their models) are free of arbitrage opportunities[1].
- An *arbitrage opportunity* arises if one can construct a self-financing portfolio such that:
  (a) The initial value of the portfolio is zero, $V(0) = 0$.
  (b) With probability one, the portfolio has a non-negative value at maturity, $\mathrm{P}(V(T) \geq 0) = 1$.
  (c) With a positive probability, the value of the portfolio at maturity is positive, $\mathrm{P}(V(T) > 0) > 0$.
- We say a market is *arbitrage free* if it does not allow arbitrage opportunities. Requiring arbitrage freeness has important consequences for price dynamics.

[1] This assumption is, mercifully, violated frequently enough so that much of the financial industry can sustain itself exploiting the market's lack of respect for arbitrage freeness.

Complete markets

- Consider a time horizon $T > 0$. A *contingent claim* expiring at T is a random variable $X \in \mathscr{F}_T$. It is usually assumed that X is square integrable, i.e. $\mathrm{E}[X^2] < \infty$.
- For our needs, we can focus attention on simple contingent claims which are of the form $X = g(S(T))$, where the *payoff* $g(x)$ is a function on $\mathbb{R}^N$.
- For example, the payoff of a call option is $(S(T) - K)^+$, where K is the strike on the option.
- A financial market is called *complete* if each contingent claim can be obtained as the terminal value of a self-financing trading strategy, i.e. if there exists a self-financing portfolio $w(t)$ such that $X = V(T)$.
- This somewhat technical sounding condition has a natural interpretation. Completeness means that it can be replicated by way of a self-financing trading strategy.
- Indeed, in order to avoid arbitrage, the time $t < T$ value $V(t)$ of the self-financing portfolio has to equal to the time t value of the claim (since they have the same value at T and rebalancing of the portfolio $w(t)$ does not cost anything).

Complete markets

- To illustrate this concept, we consider the Black-Scholes model. Let $g(S(T))$ be a time T payoff.
- Let φ be the solution to the terminal value problem:

$$\begin{aligned} \frac{\partial\varphi}{\partial t} + rx\,\frac{\partial\varphi}{\partial x} + \frac{1}{2}\,\sigma^2 x^2\,\frac{\partial^2\varphi}{\partial x^2} - r\varphi = 0, \\ \varphi(T,x) = g(x)\,. \end{aligned} \tag{6}$$

- Then, the portfolio

$$\begin{aligned} w_1(t) &= \frac{\varphi(t,S(t)) - S(t)\,\frac{\partial\varphi(t,S(t)}{\partial x}}{B(t)}\,, \\ w_1(t) &= \frac{\partial\varphi(t,S(t))}{\partial x} \end{aligned} \tag{7}$$

is the replicating portfolio for the payoff g, and its value is equal to

$$V(t) = \varphi(t,S(t)). \tag{8}$$

Complete markets

- The proof of this fact follows from an application of Ito's lemma.
- Multi-asset markets of the form (1) are, in general, not complete.
- As a rule of thumb, a multi-asset market (1) is complete, if and only if
    (i) $N = d$, i.e., the number of assets equals the number of stochastic factors,
    (ii) The matrix $C(t, S(t))$ is invertible for all $t \leq T$ with probability 1.
    (iii) Additional technical integrability condition is satisfied.

Numeraires and EMMs

- A key concept in modern asset pricing theory is that of a *numeraire*. A numeraire is any tradeable asset with price process $\mathcal{N}(t)$ such that $\mathcal{N}(t) > 0$, for all times t.
- The *relative price* process of asset I_i is defined by

$$S_i^{\mathcal{N}}(t) = \frac{S_i(t)}{\mathcal{N}(t)} . \tag{9}$$

- In other words, the relative price of an asset is its price expressed in the units of the numeraire.

Numeraires and EMMs

- A probability measure Q is called an *equivalent martingale measure* (EMM) for the above market, with numeraire $\mathcal{N}(t)$, if it has the following properties:
  (i) Q is equivalent to P, i.e.
  $$dP(\omega) = D_{PQ}(\omega)\, dQ(\omega),$$
  and
  $$dQ(\omega) = D_{QP}(\omega)\, dP(\omega),$$
  with some $D_{PQ}(\omega) > 0$ and $D_{QP}(\omega) > 0$.
  (ii) The relative price processes $S_i^{\mathcal{N}}(t)$ are martingales under Q,
  $$S_i^{\mathcal{N}}(s) = E^{Q}\left[S_i^{\mathcal{N}}(t) \,\middle|\, \mathscr{F}_s\right]. \tag{10}$$
- Before we proceed, let us briefly review Girsanov's theorem.

Mathematical interlude: Girsanov's theorem

- Girsanov's theorem is a culmination of efforts by a number of researchers studying the effect of "change of variables" in the measure P on the properties of martingales under that measure. Girsanov's theorem plays a key conceptual role in arbitrage free pricing theory, a fact that will be explained below.
- We consider a Brownian motion $W(t)$, and the associated probability space $(\Omega, \mathscr{F}, \mathrm{P})$, where Ω is the sample space, $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$, is the filtered information set, and P is the probability measure. By E (or E^P, when we want to be precise) we denote the expected value with respect to the measure P.
- We say that a measure Q on Ω is *absolutely continuous* with respect to P if there exists a positive function D (called the *Radon-Nikodym derivative*) such that

$$\mathrm{Q}(A) = \int_A D(\omega)\, d\mathrm{P}(\omega), \tag{11}$$

  for $A \subset \Omega$.
- Equivalently,

$$\frac{d\mathrm{Q}}{d\mathrm{P}}(\omega) = D(\omega). \tag{12}$$

Mathematical interlude: Girsanov's theorem

- In other words, the "volume element" $d\mathrm{Q}$ is always proportional to the "volume element" $d\mathrm{P}$, with the proportionality factor being a positive function throughout the probability space.
- In the context of a Brownian motion, we also require that the Radon-Nikodym derivative respect the filtration by time, i.e. the identity above holds if we condition on the information up to time t:

$$\frac{d\mathrm{Q}}{d\mathrm{P}}(\omega)\Big|_t = D(\omega, t)\,. \tag{13}$$

- Two probability measures Q and P are called *equivalent*, if Q is absolutely continuous with respect to P and P is absolutely continuous with respect to Q.

Mathematical interlude: Girsanov's theorem

- Consider now a diffusion process:

$$dX(t) = \Delta(t, X(t))dt + C(t, X(t))dW(t) . \tag{14}$$

- A natural question arises: can we transform a diffusion process into a diffusion process with a different drift,

$$dX(t) = \widetilde{\Delta}(t, X(t))dt + C(t, X(t))d\widetilde{W}(t) . \tag{15}$$

  by a change to an equivalent probability measure Q?

- In particular, can we make the new process a martingale? Recall that if the process $X(t)$ is a *martingale*, the diffusion above is driftless, i.e. $\widetilde{\Delta}(t, X(t)) = 0$. Recall that a process $X(t)$ is a martingale if $\mathrm{E}^{\mathrm{Q}}[|X(t)|] < \infty$, for all t, and

$$X(s) = \mathrm{E}^{\mathrm{Q}}[X(t) \,|\mathscr{F}_s] , \tag{16}$$

  where $\mathrm{E}^{\mathrm{Q}}[\,\cdot\,|\mathscr{F}_s]$ denotes the conditional expected value.

Mathematical interlude: Girsanov's theorem

- In other words, given all information up to time s, the expected value of future values of a martingale is $X(s)$. An affirmative answer to this question is provided by Girsanov's theorem.
- One might heuristically proceed like this. Write

$$\begin{aligned} dX(t) &= \widetilde{\Delta}(t)\,dt + C(t)\left(\frac{\Delta(t)-\widetilde{\Delta}(t)}{C(t)}\,dt + dW(t)\right) \\ &= \widetilde{\Delta}(t)\,dt + C(t)\,d\widetilde{W}(t)\,, \end{aligned} \tag{17}$$

where

$$\begin{aligned} \widetilde{W}(t) &= W(t) + \int_0^t \frac{\Delta(s)-\widetilde{\Delta}(s)}{C(s)}\,ds \\ &\equiv W(t) - \int_0^t \theta(s)\,ds. \end{aligned} \tag{18}$$

- This looks like a new Brownian motion! Girsanov's theorem asserts that, under some technical assumptions on the drift and diffusion coefficients, $\widetilde{W}(t)$ is indeed a Brownian motion provided that the probability measure is modified appropriately.

Mathematical interlude: Girsanov's theorem

- More precisely, define the stochastic process:

$$D(t) = \exp\left(\int_0^t \theta(s)\, dW(s) - \frac{1}{2}\int_0^t \theta(s)^2\, ds\right). \tag{19}$$

- Note that we have changed our notation: as always when dealing with stochastic processes, we have suppressed the argument ω in D, and made the dependence on t explicit. We now define the equivalent measure Q with

$$\left.\frac{dQ}{dP}\right|_t = D(t). \tag{20}$$

Mathematical interlude: Girsanov's theorem

- *Girsanov's theorem*. Assume that the following technical condition (*Novikov's condition*) holds:

$$\mathsf{E}^{\mathsf{P}}\left[\exp\left(\frac{1}{2}\int_0^t \theta(s)^2\, ds\right)\right] < \infty. \tag{21}$$

Then

(a) The process $D(t)$ is a martingale under P. Furthermore, it satisfies the following stochastic differential equation:

$$dD(t) = \theta(t)\, D(t)\, dW(t)\,. \tag{22}$$

(b) $\widetilde{W}(t)$ is a Wiener process under Q.

Mathematical interlude: Girsanov's theorem

- We have stated Girsanov's theorem for a one-dimensional Brownian motion. This assumption is not essential and, using a bit of linear algebra, one can easily formulate a version of Girsanov's theorem for an arbitrary multidimensional Brownian motion.
- Consider a standard d-dimensional Brownian motion $W(t) = \big(W_1(t), \ldots, W_d(t)\big)$ (i.e. the components of $W(t)$ are uncorrelated), and let $X(t) = \big(X_1(t), \ldots, X_d(t)\big)$ be a d-dimensional diffusion process:

$$dX(t) = \Delta(t, X(t))dt + C(t, X(t))dW(t)\,. \tag{23}$$

- Define the stochastic process:

$$D(t) = \exp\Big(\int_0^t \theta(s)^{\mathrm{T}}\, dW(s) - \frac{1}{2}\int_0^t \theta(s)^{\mathrm{T}}\,\theta(s)\, ds\Big). \tag{24}$$

  where

$$\theta(t) = C(t, X(t))^{-1}\big(\widetilde{\Delta}(t, X(t)) - \Delta(t, X(t))\big). \tag{25}$$

Mathematical interlude: Girsanov's theorem

- *Girsanov's theorem (multidimensional version).* Assume that the following technical condition (*Novikov's condition*) holds:

$$\mathrm{E}^{\mathrm{P}}\left[\exp\left(\frac{1}{2}\int_0^t \theta(s)^{\mathrm{T}}\,\theta(s)\,ds\right)\right] < \infty. \tag{26}$$

Then

(a) The process $D(t)$ is a martingale under P. Furthermore, it satisfies the following stochastic differential equation:

$$dD(t) = \theta(t)^{\mathrm{T}}\,D(t)\,dW(t)\,. \tag{27}$$

(b) $\widetilde{W}(t)$ is a Wiener process under Q.

The First Fundamental Theorem

- We now formulate two central results of the arbitrage pricing theory, namely the *two fundamental theorems.*
- *First Fundamental Theorem of arbitrage free pricing*. A market is arbitrage free if and only if for each numeraire there exists an equivalent martingale measure Q.
- This theorem is formulated in a somewhat cavalier way, as we have suppressed some important technical assumptions.
- What the theorem says is that arbitrage freeness means the existence of a numeraire $\mathcal{N}(t)$, and an equivalent measure such that the relative price process is a martingale under this measure. In other words, in an arbitrage free market, we can express the prices of all assets in the units of a single asset so that the prices are martingales.

The First Fundamental Theorem

- The proof of this theorem is quite technical, and is outside of the scope of this course. However, we shall indicate, how the existence of an EMM implies lack of arbitrage.
- Let $\mathcal{N}$ be a numeraire, and let Q be a measure under which all $S_i^{\mathcal{N}}$'s are martingales.
- From Homework Assignment #1, Problem 3, we know that a self-financing portfolio, when expressed in terms of the relative prices $S_i^{\mathcal{N}}$, is self-financing, i.e.

$$dV^{\mathcal{N}}(t) = \sum_{1 \leq i \leq N} w_i(t)\, dS_i^{\mathcal{N}}(t)\,.$$

- Since, all $S_i^{\mathcal{N}}$'s are martingales,

$$dS_i^{\mathcal{N}}(t) = \sum_{1 \leq j \leq N} C_{ij}(t) dW_j(t)$$

(no drift terms!).

The First Fundamental Theorem

- As a consequence, there is no drift term in $dV^{\mathcal{N}}(t)$, and so $V^{\mathcal{N}}(t)$ is a martingale.
- Since the measures P and Q are equivalent,

$$
\begin{aligned}
\mathrm{Q}(V^{\mathcal{N}}(T) > 0) &> 0, \\
\mathrm{Q}(V^{\mathcal{N}}(T) \geq 0) &= 1.
\end{aligned}
$$

- Therefore,

$$
\begin{aligned}
V^{\mathcal{N}}(0) &= \mathrm{E}^{\mathrm{Q}}[V^{\mathcal{N}}(T) \mid \mathscr{F}_T] \\
&> 0
\end{aligned}
$$

and so $V^{\mathcal{N}}(0) > 0$, contradicting the assumption.

The First Fundamental Theorem

- Consider, for example, the Black-Scholes model introduced above. First, we choose the money market account as the numeraire,

$$\begin{aligned}\mathcal{N}(t) &= B(t)\\ &= e^{rt}.\end{aligned}$$

- With this choice of numeraire,

$$dS^B(t) = (\mu - r)S^B(t)\,dt + \sigma S^B(t)\,dW(t)\,,$$

where $S^B(t)$ denotes the relative price process,

$$\begin{aligned}S^B(t) &= \frac{S(t)}{B(t)}\\ &= e^{-rt}S(t)\,.\end{aligned}$$

The First Fundamental Theorem

- Next, we use Girsanov's theorem to change the probability measure so that the relative price process is driftless,

$$dS^B(t) = \sigma S^B(t)\, dW(t)\,.$$

- Explicitly, this amounts to the following change of measure

$$\frac{d\mathsf{Q}}{d\mathsf{P}}(t) = \exp\Big(- \lambda W(t) - \frac{1}{2}\lambda^2 t\Big),$$

where

$$\lambda = \frac{\mu - r}{\sigma}$$

is known as the *market price of risk*.

The Second Fundamental Theorem

- *Second Fundamental Theorem of arbitrage free pricing*. An arbitrage free market is complete if and only if, for each numeraire $\mathcal{N}(t)$, the equivalent martingale measure Q is unique.
- For example, in the Black-Scholes model, the EMM constructed above, is unique.

Change of numeraire

- An important consequence of the Second Fundamental Theorem is the arbitrage pricing law:
$$\frac{V(s)}{\mathcal{N}(s)} = \mathsf{E}^{\mathsf{Q}}\Big[\frac{V(t)}{\mathcal{N}(t)}\,\Big|\mathscr{F}_s\Big], \tag{28}$$
for all $s < t$.
- One is, of course, free to use a different numeraire $\mathcal{N}(t) \to \mathcal{N}'(t)$. Girsanov's theorem implies that there exists a martingale measure Q' such that
$$\frac{V(s)}{\mathcal{N}'(s)} = \mathsf{E}^{\mathsf{Q}'}\Big[\frac{V(t)}{\mathcal{N}'(t)}\,\Big|\mathscr{F}_s\Big]. \tag{29}$$

Change of numeraire

- The Radon-Nikodym derivative is thus given by the ratio of the numeraires:

$$\begin{aligned} \frac{dQ'}{dQ}\Big|_t &= \frac{\dfrac{\mathcal{N}(0)}{\mathcal{N}(t)}}{\dfrac{\mathcal{N}'(0)}{\mathcal{N}'(t)}} \\ &= \frac{\mathcal{N}(0)}{\mathcal{N}(t)}\,\frac{\mathcal{N}'(t)}{\mathcal{N}'(0)}\,. \end{aligned} \tag{30}$$

- The choice of numeraire and the corresponding martingale measure is very much a matter of convenience, and is motivated by the problem at hand. We will see in the following lectures how this important technique works in practice. In the meantime, let us review some of the most important numeraires encountered in interest rates modeling.

Examples of EMMs: spot measure

- The *banking account numeraire* is simply a $1 deposited in a bank and accruing the instantaneous rate. Its price process $\mathcal{N}(t)$ is given by

$$\mathcal{N}(t) = \exp\left(\int_0^t r(s)ds\right). \tag{31}$$

- Here,

$$r(t) = f(t, t), \tag{32}$$

  where $f(t, s)$ is the instantaneous forward rate introduced in Lecture 1.
- The corresponding EMM is called the *spot measure*.
- The special case of a constant riskless rate $r(t) = r$ plays a key role in the Black-Scholes model. Valuation under this measure is referred to as *risk neutral* valuation.

Examples of EMMs: forward measure

- Another choice of a numeraire is the zero coupon bond for maturity T. Its price at $t < T$ is given by
$$\mathcal{N}_T(t) = P(t, t, T). \tag{33}$$
- The corresponding EMM is called the *T-forward measure*.
- We will see in Lecture 3 that the T-forward numeraire arises naturally in pricing instruments based of forwards maturing at T. Forward rates for maturity at T are martingales under the measure associated with this numeraire.

Examples of EMMs: annuity measure

- Consider a forward starting swap which settles in T_0 and matures T years from now, respectively. Consider the annuity (or level function) of this swap as a numeraire.
- Recall from Lecture 1 that the latter pays \$1 (per annum) on each coupon day of the swap, accrued according to the swap's day count day conventions. Its present value for the valuation date T_{val} as observed at time $t \leq T_{\text{val}}$ is given by:

$$A(t, T_{\text{val}}, T_0, T) = \sum_{j=1}^{n_c} \alpha_j P(t, T_{\text{val}}, T_j^c), \tag{34}$$

  where the summation runs over the coupon dates of the swap.
- We define the annuity numeraire (associated with the given swap) as

$$\mathcal{N}_{T_0,T}(t) = A(t, t, T_0, T). \tag{35}$$

- The corresponding EMM is called the *swap measure*.
- This numeraire arises as naturally when valuing swaptions. In Lecture 3 we will see that the swap rate $S(t, T_0, T)$ is a martingale under the corresponding swap measure.

Mechanics of change of numeraire

- Choice of a numeraire is a matter of convenience and is dictated by the valuation problem at hand.
- Asset valuation leads frequently to complicated stochastic processes, and one way of making the problem easier is to eliminate the drift term from the stochastic differential equation defining the process.
- The change of numeraire technique allows us to achieve precisely this: modify the probability law (the measure) of the process so that, under this new measure, the process is driftless, i.e. it is a martingale.

Mechanics of change of numeraire

- Consider a financial asset whose dynamics is given in terms of the state variable $X(t)$. Under the measure P this dynamics reads:

$$dX(t) = \Delta^{\mathrm{P}}(t)\,dt + C(t)\,dW^{\mathrm{P}}(t)\,. \tag{36}$$

- Our goal is to relate this dynamics to the dynamics of the same asset under an equivalent measure Q:

$$dX(t) = \Delta^{\mathrm{Q}}(t)\,dt + C(t)\,dW^{\mathrm{Q}}(t)\,. \tag{37}$$

- Remember that the diffusion coefficients in these equations are unaffected by the change of measure! We assume that P is associated with the numeraire $\mathcal{N}(t)$ whose dynamics is given by:

$$d\mathcal{N}(t) = A_{\mathcal{N}}(t)\,dt + B_{\mathcal{N}}(t)\,dW^{\mathrm{P}}(t)\,, \tag{38}$$

  while Q is associated with the numeraire $\mathcal{M}(t)$ whose dynamics is given by:

$$d\mathcal{M}(t) = A_{\mathcal{M}}(t)\,dt + B_{\mathcal{M}}(t)\,dW^{\mathrm{P}}(t)\,. \tag{39}$$

Mechanics of change of numeraire

- According to Girsanov's theorem, the Radon-Nikodym derivative

$$D(t) = \frac{d\mathsf{Q}}{d\mathsf{P}}\Big|_t \tag{40}$$

is a martingale under the measure P, which satisfies the stochastic differential equation:

$$dD(t) = \theta(t)\, D(t)\, dW^{\mathsf{P}}(t), \tag{41}$$

with

$$\theta(t) = \frac{\Delta^{\mathsf{Q}}(t) - \Delta^{\mathsf{P}}(t)}{C(t)}. \tag{42}$$

- Explicitly, the likelihood process $D(t)$ is given by the stochastic exponential of the martingale $\int_0^t \theta(s)\, dW^{\mathsf{P}}(s)$:

$$D(t) = \exp\Big(\int_0^t \theta(s)\, dW^{\mathsf{P}}(s) - \frac{1}{2}\int_0^t \theta(s)^2\, ds\Big). \tag{43}$$

Mechanics of change of numeraire

- On the other hand, from the fundamental theorem of asset pricing we infer that

$$D(t) = \frac{\mathcal{N}(0)}{\mathcal{M}(0)} \frac{\mathcal{M}(t)}{\mathcal{N}(t)}. \tag{44}$$

- Since $D(t)$ is a martingale under P, we conclude that the process $\mathcal{M}(t)/\mathcal{N}(t)$ is driftless under P. As a consequence,

$$d\left(\frac{\mathcal{M}(t)}{\mathcal{N}(t)}\right) = \frac{\mathcal{M}(t)}{\mathcal{N}(t)} \left(\frac{B_{\mathcal{M}}(t)}{\mathcal{M}(t)} - \frac{B_{\mathcal{N}}(t)}{\mathcal{N}(t)}\right) dW^{\mathsf{P}}(t).$$

- Comparing this with (40) we infer that

$$\theta(t) \frac{\mathcal{M}(t)}{\mathcal{N}(t)} = \frac{\mathcal{M}(t)}{\mathcal{N}(t)} \left(\frac{B_{\mathcal{M}}(t)}{\mathcal{M}(t)} - \frac{B_{\mathcal{N}}(t)}{\mathcal{N}(t)}\right). \tag{45}$$

Mechanics of change of numeraire

- This leads to the following drift transformation law:

$$\begin{aligned}\Delta^{\mathrm{Q}}(t)-\Delta^{\mathrm{P}}(t)&=C(t)\left(\frac{B_{\mathcal{M}}(t)}{\mathcal{M}(t)}-\frac{B_{\mathcal{N}}(t)}{\mathcal{N}(t)}\right)\\&=\frac{d}{dt}\int_0^t dX(s)\,d\Big(\log\frac{\mathcal{M}(s)}{\mathcal{N}(s)}\Big).\end{aligned}\tag{46}$$

- The formula above expresses the change in the drift in the dynamics of the state variable, which accompanies a change of numeraire, in terms of the processes themselves.
- We can rewrite (46) in a more intrinsic form. Note that the integral in the equation above defines the quadratic covariation between X and $\log(\mathcal{M}/\mathcal{N})$. Consequently, the change of numeraire formula can be stated in the elegant, easy to remember form:

$$\Delta^{\mathrm{Q}}(t)=\Delta^{\mathrm{P}}(t)+\frac{d}{dt}\left[X,\log\frac{\mathcal{M}}{\mathcal{N}}\right](t)\,.\tag{47}$$

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