Black-Litterman Optimization

Black and Litterman (1991) suggest that a portfolio manager's "views" can be expressed in the form

$$\mathbb{E}[p'R] = q$$

where $p \in \mathbb{R}^n$ is a portfolio, and R is the random vector of asset returns over some subsequent interval. If there are multiple such views, say

$$\mathbb{E}[p_i'R] = q_i, \quad i = 1 \ldots k$$

then the portfolios p_i are more conveniently arranged as rows of a matrix P, and the statement of views becomes

$$\mathbb{E}[PR] = Q \text{ for } Q \in \mathbb{R}^k. \tag{1}$$

The model of black1991asset is essentially to treat the portfolio manager's views as noisy observations which are useful for performing statistical inference concerning the parameters in the model for R. For example, if

$$R \sim N(\theta, \Sigma)$$

with Σ known, then the views (1) can be seen as observations relevant for inference on the parameter θ. A subjective view about what will happen to a certain portfolio in the future is conceptually distinct from a noisy observation, unless of course one models a portfolio manager's forecasts as an observation of the future in which the measuring device is a rather murky and unreliable crystal ball.

Specifically, the "uncertain observations" take the form $P\theta = Q$. It is a desirable feature of the model for the practitioner to be able to specify an uncertainty for each view, which amounts to the statement that each row of $P\theta = Q$ is a *noisy* observation. This is expressed mathematically as

$$P\theta = Q + \epsilon^{(v)}, \quad \epsilon^{(v)} \sim N(0, \Omega), \quad \Omega = \mathrm{diag}(\omega_1, \ldots, \omega_k) \tag{2}$$

In fact, to perform Bayesian inference, observations alone are not sufficient; one needs to fully specify the statistical model, which includes a likelihood and a prior. In fact (2) specifies the likelihood as

$$p(Q \,|\, \theta) \propto \exp\big[-\frac{1}{2}(P\theta - Q)'\Omega^{-1}(P\theta - Q)\big] \tag{3}$$

which is the standard normal likelihood for a multiple linear regression problem. One caveat to note is that, if the number of views is less than the number of assets, then this is a so-called "$p > n$ regression" meaning that the number of parameters is larger than the number of observations. Such regressions are encountered quite frequently in biostatistics, such as gene microarray analysis, and they are perfectly well-behaved as long as there is a prior.

We have not yet specified the prior, but Black and Litterman were motivated by the guiding principle that in the absence of any sort of information (or "views") which could constitute alpha over the benchmark, the optimization procedure should simply return the global CAPM equilibrium portfolio, with weights denoted w_{eq}. The behavior in the absence of views should be modeled by the prior. Hence in the absence of any views, and with prior mean equal to Π, the investor's model of the world is that

$$R \sim N(\theta, \Sigma), \quad \text{and} \quad \theta \sim N(\Pi, C) \tag{4}$$

for some covariance C representing the amount of precision in the prior. For any portfolio p, then, according to (4) we have

$$\mathbb{E}[p'R] = p'\Pi \quad \mathbb{V}[p'R] = p'(\Sigma + C)p\,.$$

In fact one must make a choice whether to use the conditional or unconditional variance in optimization: $\mathbb{V}(R \,|\, \theta) = \Sigma$ but $\mathbb{V}(R) = \Sigma + C$. Since investors are presumably concerned with unconditional variance of wealth, the unconditional variance form is preferable. Hence mean-variance optimization with risk-aversion parameter δ gives

$$w_{eq} = \delta^{-1}(\Sigma + C)^{-1}\Pi.$$

Any combination of Π, C satisfying this will lead to a model with the desired property – that the optimal portfolio with only the information given in the prior is the prescribed portfolio w_{eq}. In particular, taking $C = \tau\Sigma$ with $\tau > 0$ as did the original authors leads to

$$\Pi = \delta(1 + \tau)\Sigma w_{eq}$$

We thus have the normal likelihood (3) and the normal prior (4) which we know is a conjugate prior for that likelihood. Conjugacy is the key to being able to write down the posterior in closed form, but the prior considered here isn't the only conjugate prior. Moreover, if one is willing to give up the requirement of working only with closed-form expressions, then many other distributional assumptions both for the prior and likelihood are possible.

The negative log posterior is thus proportional to (neglecting terms that don't contain θ):

(5) $(P\theta - Q)'\Omega^{-1}(P\theta - Q) + (\theta - \Pi)'C^{-1}(\theta - \Pi)]$

(6) $= \theta'P'\Omega^{-1}P\theta - \theta'P'\Omega^{-1}Q - Q'\Omega^{-1}P\theta + \theta'C^{-1}\theta - \theta'C^{-1}\Pi - \Pi'C^{-1}\theta$

(7) $= \theta'[P'\Omega^{-1}P + C^{-1}]\theta - 2(Q'\Omega^{-1}P + \Pi'C^{-1})\theta$

We now need to complete the square. Noting that for any symmetric matrix H and vector v, one has

$$x'Hx - 2v'Hx = (x - v)'H(x - v) - v'Hv$$

For the quadratic term to match (7) one must have $H = P'\Omega^{-1}P + C^{-1}$ and to find v we must solve

$$Hv = P'\Omega^{-1}Q + C^{-1}\Pi$$

Fortunately, H is invertible so this is straightforward,

$$v = [P'\Omega^{-1}P + C^{-1}]^{-1}[P'\Omega^{-1}Q + C^{-1}\Pi]$$

Hence completing the square is finished: the posterior has mean v and covariance $H^{-1} = [P'\Omega^{-1}P + C^{-1}]^{-1}$.

Investors with CARA utility of final wealth will want to solve

$$w^* = \underset{h}{\operatorname{argmax}} \left\{ \mathbb{E}[h'R] - \frac{\delta}{2}\mathbb{V}[h'R] \right\}$$

where $\mathbb{E}[R]$ and $\mathbb{V}[R]$ denote, respectively, the unconditional mean and covariance of R under the posterior. The unconditional covariance is a sum of variance due to parameter uncertainty, and variance due to the randomness in R. In other words,

$$\mathbb{V}[h'R] = h'[P'\Omega^{-1}P + C^{-1}]^{-1}h + h'\Sigma h$$

The optimal portfolio accounting for both types of variance is then

$$w^* = \delta^{-1}[H^{-1} + \Sigma]^{-1}H^{-1}[P'\Omega^{-1}Q + C^{-1}\Pi]$$

The topic of Black-Litterman optimization seems to have generated more than its share of confusion over the years, as evidenced by the appearance of articles with titles such as *"A demystification of the Black–Litterman model"* (Satchell and Scowcroft, 2000), etc. The source of all this confusion is perhaps related to the fact that the original authors didn't use standard statistical terminology. A particularly clear reference is He and Litterman (1999), but even there, aspects of Bayesian inference are glossed over: there are many references to a "prior" but no mention of a "likelihood" anywhere at all. This is surprising for a paper ostensibly dedicated to a Bayesian method.

References

Black, Fischer and Robert B Litterman (1991). "Asset allocation: combining investor views with market equilibrium". In: *The Journal of Fixed Income* 1.2, pp. 7–18.

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