**Problem 2.1.** Consider the 1d univariate function $f(x) = q(x) + \phi(x)$ where $q(x) = ax^2 + bx + c$ with $a > 0$ is convex quadratic and

$$\phi(x) = \begin{cases} \ell x & x < 0 \\ rx & x \geq 0 \end{cases}$$

(a) Under what conditions on $\ell, r \in \mathbb{R}$ is $\phi$ convex?

(b) Assuming the conditions from part (a), ie, assuming $\phi$ is convex, write down an explicit, closed form expression for $x^* = \operatorname{argmin} f(x)$.

(c) Explain in detail how the solution from part (b), together with coordinate descent, could be used to solve a LASSO regression problem very quickly. Augment your explanation with pseudocode or python which implements the procedure.

**Problem 2.2.** Consider a simple mean-variance optimization problem with trading cost given by paying half the bid-ask spread for every trade in either direction. The utility function of final wealth takes the form

$$u(h) = h'\alpha - \frac{\kappa}{2} h' \Sigma h - \sum_i \frac{1}{2} s_i \, |h_i - h_{0,i}| \tag{2.1}$$

for an initial portfolio $h_0 \in \mathbb{R}^n$ which gets traded into a final portfolio $h \in \mathbb{R}^n$ including all cost terms. Note that $h_0$ and $h$ have units of dollars.

Show that optimizing over $h_i$ while holding $h_j$ fixed for $i \neq j$ (in other words, coordinate descent over each asset) reduces to a sequence of 1d subproblems of the type considered in the previous problem. Write down the explicit solution for the $i$-th sub-problem in terms of the variables in (2.1).

**Problem 2.3** (Extra Credit - HARD)**.** Let $x \in \mathbb{R}^2$ and consider the problem $x^* = \operatorname{argmin} f(x)$ where $f(x) = q(x) + \phi(x)$ and $q(x) = \frac{1}{2} x^T A x + b^T x$ is convex quadratic ($A$ is positive definite). Also,

$$\phi(x) = c \, |x_1 - x_2| + b \mathbb{I}[x_2 < 0]$$

with $b, c > 0$. Can you find $x^*$ in closed form?

**Problem 2.4.** Suppose you are considering trading a single asset, and your current position is $h_0$ which is some constant determined by the past (not a decision variable). You trade into a target position $h_1$ in the morning, and then into a position $h_2$ just before the close, and finally liquidate the next morning. Suppose that the slippage of trading $\delta$ dollars is well approximated by $\frac{1}{2} s \, |\delta|$ where $s > 0$. Suppose also that you pay borrow cost $b h_2 \mathbb{I}[h_2 < 0]$ on the position you hold overnight where $\mathcal{I}$ denotes the indicator function and $b > 0$. Suppose your expected return on the asset is $\mu_1$ from open to close, and $\mu_2$ from close to the next open where $\mu_1 > 0$ and $\mu_2 < 0$ and the asset's volatility is $\sigma$.

(a) Let $w$ denote final wealth (in dollars) after all trading is complete the next morning. Write down the full mean-variance utility

$$u(h_1, h_2) = E[w] - (\kappa/2) V[w]$$

(where $w$ includes all costs), as a function of $h_1$ and $h_2$ where as always, $\kappa \geq 0$.

(b) Is $u(h_1, h_2)$ convex?

(c) If you solved the extra credit problem above, apply it to obtain the explicit solution to this one.