Consider estimating the coefficients from data which was actually generated from the model $y = 3 + x_1 + x_2$, and compute the OLS coefficients in R using the following code snippet. Be sure to keep the set.seed command intact so that you get the same random data that the grader will use.

```
set.seed(10)
N <- 20;
x1 <- rnorm(N);
x2 <- rnorm(N, mean = x1, sd = .01);
y <- rnorm(N, mean = 3 + x1 + x2)
lm(y ~ x1 + x2)$coef
```

Note that this model has the form $y = X\beta + \epsilon$ discussed in class, where $x_1$ and $x_2$ are the columns of $X$ and the coefficients output by the last command are the elements of $\beta$.

**Problem 1.1.** Interpret the results. Do they seem reasonable? Would you be willing to use the estimated coefficients to make predictions out of sample if the design matrix $X$ had changed? What can you infer about the matrix $X'X$ from seeing coefficients like these?

**Problem 1.2.** Also compute the full summary of the regression via

```
summary(lm(y ~ x1 + x2))
```

Based on the output of this, which of the coefficients are significantly different from zero? Does this make sense, given how you know the data was generated?

**Problem 1.3.** Re-estimate the coefficients using ridge regression and experiment with various values of the ridge parameter.

**Problem 1.4.** One definition of the *Moore-Penrose pseudoinverse* of a matrix $A$ is

$$A^+ = \lim_{\delta \searrow 0} (A'A + \delta I)^{-1} A' = \lim_{\delta \searrow 0} A'(AA' + \delta I)^{-1}$$

Prove that these limits exist for any matrix $A$ with real entries, and equal each other, even if $(AA')^{-1}$ or $(A'A)^{-1}$ do not exist.

**Problem 1.5.** Compute the Moore-Penrose inverse of the design matrix $X$ in the toy problem above and verify that $X^+y$ equals the limit of the ridge regression coefficients as the ridge parameter tends to zero.

**Problem 1.6.** Consider the minimum-norm solution of a set of linear equations:

$$\text{minimize } \|x\|_Q \text{ subject to } \; Ax = b$$

where the norm is given by

$$\|x\|_Q^2 := \langle x, Qx \rangle$$

and $Q$ is symmetric and positive-definite (hence $Q^{1/2}$ exists.)

Compute the Lagrangian $L(x, \nu)$ and the Lagrange dual function $g(\nu)$ explicitly. Show that strong duality holds. Solve the dual problem and find $d^*$. Express the solution to the primal problem in terms of the Moore-Penrose pseudoinverse.

**Problem 1.7.** Verify directly from the definition of convexity that the Legendre-Fenchel conjugate $f^*(y)$ is always a convex function of $y$, even if $f$ isn't convex.