| **Lecturer:** | *October 2nd, 2023* | **Approved by:** | October 2nd, 2023 |
|---|---|---|---|
| *(Signature and Fullname)* | | *(Signature and Fullname)* | |

| UNIVERSITY OF TECHNOLOGY<br>FACULTY OF CSE | MIDTERM | Semester/Academic year | 1 | 2023-2024 |
|---|---|---|---|---|
| | | Date | | 18/10/2023 |
| | Course title | Mathematical Modeling | | |
| | Course ID | CO2011 | | |
| | Duration | 60 mins | Question sheet code | 1811 |

Notes: - Students do not use course materials except one A4 hand-written sheet.
- Submit the question sheet together with the answer sheet.
- Choose the best answer (only 1) for each question.

1. (L.O.1.2)
Which of the following statements is/are true?
(I) All finite languages are regular.
(II) If $L$ is regular so is the reverse language $L^R = \{w^R | w \in L\}$.
(III) $L = \{ww | w \in \{a, b\}^*\}$ is regular.
(IV) If $M = (Q, \Sigma, \delta, q_0, F)$ is a minimal DFA for a regular language $L$ then $\widehat{M} = (Q, \Sigma, \delta, q_0, Q - F)$ is a minimal DFA for $\overline{L}$.

A. (I), (II), (III). B. (I), (II), (IV). C. Only (I). D. (II), (III), (IV).

2. (L.O.1.3) Given a program finding max for 3 integers a,b,c:
```
max=0;
if(max<a) max=a;
if(max<b) max=b;
if(max<c) max=c;
```
Which of the following are the weakest pre-condition $\phi$ and post-condition $\psi$ for the program to be correct?

A. $\phi : a, b, c >= 0$, $\psi : a < b < c < max$
B. $\phi : a, b, c >= 0$, $\psi : a, b, c \leq max$
C. $\phi : a, b, c > 0$, $\psi : a, b, c \leq max$
D. $\phi : T$, $\psi : a, b, c \leq max$

3. Consider a program `Prod` $= Product(a, b)$ calculating the product $a \cdot (b-1)$ of two naturals $a, b$ where $b \geq 2$. Denote the Hoare triple of a suitable core program $P$ to correctly compute `Prod` as

$$\{\phi\}\ P\ \{\psi\}$$

where pre-conditions $\phi := (a \geq 1) \wedge (b \geq 2)$ and post-conditions $\psi$ should be suitably defined. A variant $K$ is an expression whose values are either strictly decreasing or strictly decreasing, and an invariant $E$ **does not** change its Boolean value in the loop. Both $K$ and $E$ depend on original inputs $a, b$ and also internal variables being newly declared in the program. Conditions and such variants are put in brackets $\{..\}$.

Let $E = \{\ u = (b-v) \cdot a\}$ be a "good" invariant of the program `Prod`, assuming the full format as follows.

$\{(a \geq 1) \wedge (b \geq 2)\}$
u = 0;
v = b;
$\{u = 0 \wedge v = b\}$
$E$
**while** ( v != Lb )
{
  $\vdots\ P$
}
$\{u = a \cdot (b-v)\}$

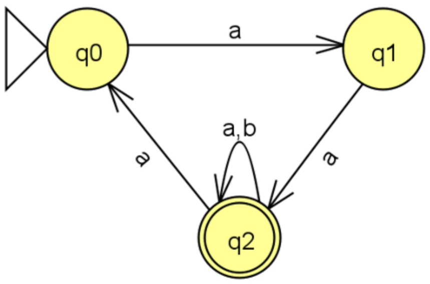
In the following options of core program $P$ and lower bound $Lb$:
(I) $P := [u = u + a;\ v = v - 1;]$ and $Lb := 0$
(II) $P := [u = u + a;\ v = v - 2;]$ and $Lb := 1$
(III) $P := [u = u + a;\ v = v - 1;]$ and $Lb := 1$
(IV) $P := [u = u + a;\ v = v + 1;]$ and $Lb := 1$.

Which one do you choose to obtain the totally correct statement?

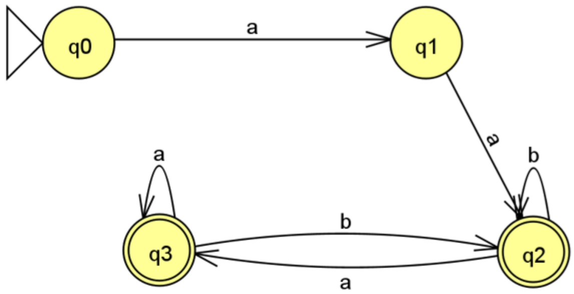
$$\vdash_{\text{tot}} \{\phi\}\ \texttt{Prod}\ \{u = a \cdot (b-1)\}?$$

A. (IV) B. (III) C. (II) D. (I)

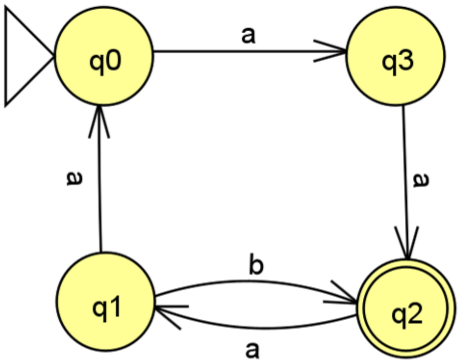
4. (L.O.2.3) Given 4 automata where $\Sigma = \{a, b\}$ and $\lambda$ is the empty string.




1




2

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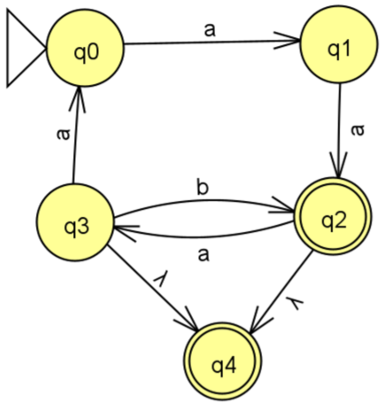



4

Which of the following couples are the same?

A. 1,2 B. 1,3 C. 3,4

D. All the other answers are incorrect.

5. (L.O.2.3) Let $L$ be the language consisting of string $w \in \{0, 1\}^+$ that represents a decimal number divisible by 3. Which choice is CORRECT?

A. All other choices are incorrect.

B. $L$ is not a regular language.

C. The number of states of a DFA recognizing only $L$ must be divisible by 3.

D. The number of bit 1s of any string of $L$ must be divisible by 3.

6. (L.O.1.3) Given a post-condition for a correct program:
$[|\forall q(i < q \leq n \implies a[i] \leq a[q]) \land a[i] \leq a[i-1]|]$
Which of the following post-conditions also correct for the same program?

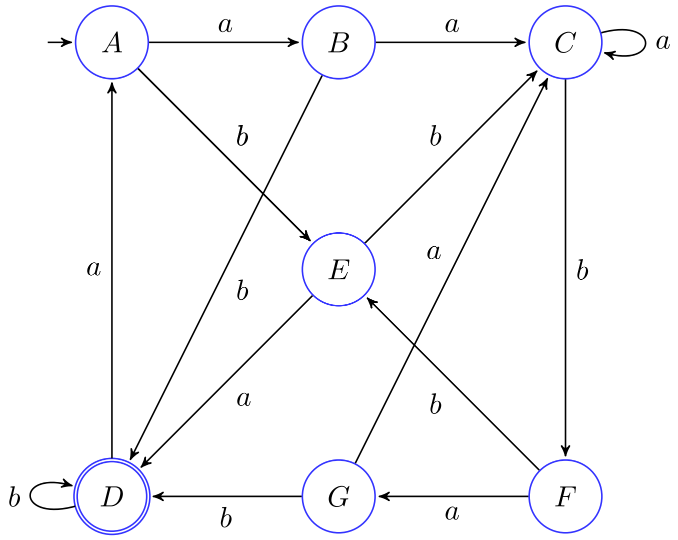
A. $[|\forall q(i - 1 < q \leq n \implies a[i-1] \leq a[q])|]$ B. $[|\forall q(i \leq q \leq n \implies a[i] \leq a[q])|]$

C. $[|\forall q(i < q \leq n \implies a[i-1] \leq a[q])|]$ D. All the other answers are correct.

7. (L.O.1.3)
Consider program `P` with precondition and postcondition as in Question 8. Then the invariant form to prove the partial correctness of `P` is

A. $z \in \mathbb{Z} \land yx^z = a^b$. B. $z \in \mathbb{N} \land yx^z = a^b$. C. $z \in \mathbb{N} \land y = x^z$. D. $z \in \mathbb{Z} \land y = x^z$.

8. (L.O.3.2)
Consider the following finite automaton:




What is the number of states in a minimal DFA (equivalent to the above FA)?

A. 4. B. 5. C. 6. D. 7.

9. (L.O.2.3) Let L be the language containing all the palindrome words where $\Sigma = \{a, b\}$, Which of the following regular expressions accepts the words in language L?

A. (a+b)*(b+a)* B. a*b*b*a* C. (abba)*
D. All the other answers are incorrect.

10. (L.O.2.3) Let $L$ be the language consisting of string $w \in \{0,1\}^*$ that the number of occurrences of sub-string 01 and the number of occurrences of sub-string 10 in $w$ are the same. Which choice is CORRECT?

A. A DFA recognizing only $L$ must have at least five states.
B. $L$ is not a regular language.
C. The number of bit 1s and the number of bit 0s of any string of $L$ must be the same.
D. All other choices are incorrect.

11. (L.O.1.2)
Consider the logical formula $\phi : \forall x \exists y \exists z (P(x,y) \land P(z,y) \land (P(x,z) \to P(z,x)))$ and the models $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ such that the universal set is the natural numbers set $\mathbb{N}$, $P^{\mathcal{M}_1} = \{(m,n) | n < m\}$, $P^{\mathcal{M}_2} = \{(m, 2m) | m \in \mathbb{N}\}$ and $P^{\mathcal{M}_3} = \{(m,n) | m < n+1\}$ Which of following statement is true?

A. $\mathcal{M}_1 \models \phi, \mathcal{M}_2 \not\models \phi, \mathcal{M}_3 \not\models \phi$.
B. $\mathcal{M}_1 \models \phi, \mathcal{M}_2 \models \phi, \mathcal{M}_3 \models \phi$.
C. $\mathcal{M}_1 \not\models \phi, \mathcal{M}_2 \models \phi, \mathcal{M}_3 \models \phi$.
D. $\mathcal{M}_1 \not\models \phi, \mathcal{M}_2 \not\models \phi, \mathcal{M}_3 \models \phi$.

12. (L.O.1.3)

Given the following program, with $\top$ as a precondition, determine the postcondition yourself. To prove the partial correctness of the corresponding Hoare triple, which of the following is an invariant form we should use?

```
i=0;
s=0;
while(i<=n){
    if(i%2==0)
        s+=i;
    i+=1;
}
```

A. $s = \sum_{k=0}^{(i-1)/2} 2k \land (i \leq n+1)$.
B. $s = \sum_{k=0}^{i/2} 2k \land (i \leq n)$.
C. $s = \sum_{k=0}^{(i-1)/2} 2k$.
D. $s = \sum_{k=0}^{i/2} 2k \land (i \leq n+1)$.

*Question 13– Question 14 use the same description as follows:* Denote an alphabet $\Sigma = \{b, c\}$, and let $u = bc$, $v = cb$ be strings of length 2 in $\Sigma^*$. Define $L(r)$ and $L(s)$ respectively be languages determined by regular expressions

$r = u^* = (bc)^*$ and $s = v^* = (cb)^*$.

13. The language $L_{bc}$ of all strings with $b$ and $c$ alternating, moreover starting with $b$ and ending with $c$, is

(I) $L(s)$; (II) $L(r) + L(s)$; (III) $L(r)$; (IV) $L(r) \cdot L(s)$.

A. (III) B. (IV) C. (II) D. (I)

14. The language $L$ of all strings with $b$ and $c$ alternating is determined by which of the following regular expressions?

(I) $r + s$; (II) $r + s + c\ r + b\ s$; (III) $r + s + b\ r + c\ s$; (IV) $s + r$.

A. IV B. III C. II D. I

15. Consider the following statement: "`A female athlete won every sports prize and her brother also won some mathematics prize`." We define some predicates below to symbolize the given statement

- $Girl(a)$ = "$a$ is a girl", $Boy(b)$ = "$b$ is a boy",
- $SportPrize(x)$ = "$x$ is a prize in Sport tournament",
- $MathPrize(y)$ = "$y$ is a prize in Mathematics competition",
- $Sibling(a, b)$ = "$a$ and $b$ are siblings in a family",
- $WinSport(w, s)$ = "$w$ won a sport prize $s$",
- $WinMath(u, m)$ = "$u$ won a mathematics prize $m$".

Which of the following options best represents the given statement?

(I) $\forall x\ [SportPrize(x) \longrightarrow \exists y\ (Girl(y) \wedge WinSport(y, x))]$

(II) $[\ (\exists f, Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ [\ (\exists b, Sibling(f, b) \wedge Boy(b) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(b, y)]$

(III) $\exists f\ [\ (Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ \exists m\ [\ (Sibling(f, m) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(m, y)\ ]$

(IV) $\exists f\ [\ (Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ \exists m\ [\ (Sibling(f, m) \wedge\ Boy(m) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(m, y)\ ]$

A. (IV) B. (III) C. (II) D. (I)

16. (L.O.1.3)

In the following program, % is the division with remainder and returns the remainder, and function abs() returns the absolute value of an input integer. Consider the precondition $x = x_0 \wedge y = y_0 \wedge (x_0 \neq 0 \vee y_0 \neq 0)$. *Which is the invariant form for proving the partial correctness of the program?* [In the choices, $GCD(a, b)$ denotes the greatest common divisor of two integers $a$ and $b$ that are not both 0, $a$ mod $b$ is the division with the remainder by positive $b$ and $|a|$ is the absolute value of $a$.]

```
while (x != 0) {
        z = x;
        x = y % x;
        y = z;
}
y = abs(y);
```

A. $GCD(x, y) = GCD(x_0, y_0)$
B. $y \neq 0 \rightarrow GCD(x, x \bmod |y|) = GCD(x_0, y_0)$
C. $x \neq 0 \rightarrow GCD(y, y \bmod |x|) = GCD(x_0, y_0)$
D. All other choices are correct.

17. (L.O.2.3) Which of the following is correct?

A. Given two languages $L_1, L_2$, if $L_1 \cup L_2$ is regular then both $L_1, L_2$ are also regular.

B. For two languages $L_1, L_2$, we have $L_1^* \circ L_2^* \subseteq (L_1 \cup L_2)^*$.

C. If $L_1, L_2$, are finite languages then $|L_1 \circ L_2| = |L_1| \cdot |L_2|$.

D. If $L_1$ is a regular language and the language $L_2$ is *not* regular then $L_1^{\circ} L_2$ is *not* regular.

18. (L.O.1.3)

Consider program P below. (Weakest) precondition $\phi$ and postcondition $\psi$ of P are

```
x := a;    y := 1;    z := b;
while true do
  if z = 0 then
    return y
  end if
  r := remainder(z, 2);    z := quotient(z, 2);
  if r = 1 then
    y := x * y;
  end if
  x := x * x;
end while
```

A. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{N}\},\ \psi = \{y = a^b\}$.

B. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{N}\},\ \psi = \{y = x^z\}$.

C. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{Z}^+\},\ \psi = \{y = a^b\}$.

D. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{Z}\},\ \psi = \{y = a^b\}$.

19. (L.O.1.1) Let $P$ and $Q$ be binary predicates and $S$ be nullary predicates. Which formulas are valid?

(I) $\forall x \forall y (P(x) \rightarrow P(y)) \wedge (P(y) \rightarrow P(x))$ (II) $\exists y ((\forall x P(x)) \rightarrow P(y))$

(III) $(\forall x P(x) \rightarrow S) \rightarrow \exists x (P(x) \rightarrow S)$ (IV) $\forall x (P(x) \vee Q(x)) \rightarrow (\forall x P(x)) \vee (\exists x Q(x))$

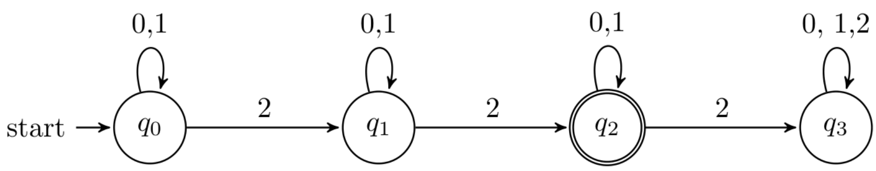
A. Only (II), (III), and (IV)

B. Only (II) and (III)

C. Only (I), (II), and (IV)

D. Only (III) and (IV)

20. (L.O.2.3) Which of the following languages is accepted by the DFA over $\Sigma = \{0, 1, 2\}$ given below?




A. $L = \{w \in \Sigma^* \mid w \text{ contains exactly two } 2's\}$.

B. $L = \{w \in \Sigma^* \mid w \text{ contains at least two } 2's\}$.

C. $L = \{w \in \Sigma^* \mid w \text{ contains at most two } 2's\}$.

D. $L = \{w \in \Sigma^* \mid w \text{ contains at least two 2's consecutively}\}$.

..............................................END OF EXAM..............................................

# Solution 1811

1. B.
2. D.
3. B.
4. A.
5. A.
6. B.
7. B.
8. B.
9. A.
10. A.
11. C.
12. B.
13. A.
14. C.
15. A.
16. A.
17. B.
18. A.
19. A.
20. A.

| UNIVERSITY OF TECHNOLOGY FACULTY OF CSE | MIDTERM | Semester/Academic year | 1 | 2023-2024 |
|---|---|---|---|---|
| | | Date | | 18/10/2023 |
| | Course title | Mathematical Modeling | | |
| | Course ID | CO2011 | | |
| | Duration | 60 mins | Question sheet code | 1812 |

Notes: - Students do not use course materials except one A4 hand-written sheet.
- Submit the question sheet together with the answer sheet.
- Choose the best answer (only 1) for each question.

1. (L.O.1.2)
Which of the following statements is/are true?
(I) All finite languages are regular.
(II) If $L$ is regular so is the reverse language $L^R = \{w^R | w \in L\}$.
(III) $L = \{ww | w \in \{a,b\}^*\}$ is regular.
(IV) If $M = (Q, \Sigma, \delta, q_0, F)$ is a minimal DFA for a regular language $L$ then $\widehat{M} = (Q, \Sigma, \delta, q_0, Q - F)$ is a minimal DFA for $\overline{L}$.

A. (II), (III), (IV). B. (I), (II), (III). C. (I), (II), (IV). D. Only (I).

2. (L.O.1.3)
Given the following program, with $\top$ as a precondition, determine the postcondition yourself. To prove the partial correctness of the corresponding Hoare triple, which of the following is an invariant form we should use?

```
i=0;
s=0;
while(i<=n){
    if(i%2==0)
        s+=i;
    i+=1;
}
```

A. $s = \sum_{k=0}^{i/2} 2k \wedge (i \leq n+1)$.

B. $s = \sum_{k=0}^{(i-1)/2} 2k \wedge (i \leq n+1)$.

C. $s = \sum_{k=0}^{i/2} 2k \wedge (i \leq n)$.

D. $s = \sum_{k=0}^{(i-1)/2} 2k$.

3. Consider a program `Prod` $= Product(a, b)$ calculating the product $a \cdot (b-1)$ of two naturals $a, b$ where $b \geq 2$. Denote the Hoare triple of a suitable core program $P$ to correctly compute `Prod` as

$$\{\phi\}\ P\ \{\psi\}$$

where pre-conditions $\phi := (a \geq 1) \wedge (b \geq 2)$ and post-conditions $\psi$ should be suitably defined. A variant $K$ is an expression whose values are either strictly decreasing or strictly decreasing, and an invariant $E$ **does not** change its Boolean value in the loop. Both $K$ and $E$ depend on original inputs $a, b$ and also internal variables being newly declared in the program. Conditions and such variants are put in brackets $\{..\}$.

Let $E = \{\ u = (b - v) \cdot a\}$ be a "good" invariant of the program `Prod`, assuming the full format as follows.

$\{(a \geq 1) \wedge (b \geq 2)\}$
u = 0;
v = b;
$\{u = 0 \wedge v = b\}$
$E$
```
while ( v != Lb )
{
    ⋮ P
}
```
$\{u = a \cdot (b - v)\}$

In the following options of core program $P$ and lower bound $Lb$:
(I) $P := [u = u + a;\ v = v - 1;]$ and $Lb := 0$
(II) $P := [u = u + a;\ v = v - 2;]$ and $Lb := 1$
(III) $P := [u = u + a;\ v = v - 1;]$ and $Lb := 1$
(IV) $P := [u = u + a;\ v = v + 1;]$ and $Lb := 1$.

Which one do you choose to obtain the totally correct statement?

$$\vdash_{\text{tot}} \{\phi\}\ \texttt{Prod}\ \{u = a \cdot (b - 1)\}?$$

A. (I)  B. (IV)  C. (III)  D. (II)

4. (L.O.1.3) Given a post-condition for a correct program:
$[|\forall q(i < q \leq n \implies a[i] \leq a[q]) \wedge a[i] \leq a[i-1]|]$
Which of the following post-conditions also correct for the same program?

A. All the other answers are correct.
B. $[|\forall q(i - 1 < q \leq n \implies a[i-1] \leq a[q])|]$
C. $[|\forall q(i \leq q \leq n \implies a[i] \leq a[q])|]$
D. $[|\forall q(i < q \leq n \implies a[i-1] \leq a[q])|]$

5. (L.O.1.1) Let $P$ and $Q$ be binary predicates and $S$ be nullary predicates. Which formulas are valid?
(I) $\forall x \forall y(P(x) \to P(y)) \wedge (P(y) \to P(x))$
(II) $\exists y((\forall x P(x)) \to P(y))$
(III) $(\forall x P(x) \to S) \to \exists x(P(x) \to S)$
(IV) $\forall x(P(x) \vee Q(x)) \to (\forall x P(x)) \vee (\exists x Q(x))$

A. Only (III) and (IV)
B. Only (II), (III), and (IV)
C. Only (II) and (III)
D. Only (I), (II), and (IV)

6. (L.O.1.3) Given a program finding max for 3 integers a,b,c:
max=0;
if(max<a) max=a;
if(max<b) max=b;
if(max<c) max=c;
Which of the following are the weakest pre-condition $\phi$ and post-condition $\psi$ for the program to be correct?

A. $\phi : T,\ \psi : a, b, c \leq max$
B. $\phi : a, b, c >= 0,\ \psi : a < b < c < max$
C. $\phi : a, b, c >= 0,\ \psi : a, b, c \leq max$
D. $\phi : a, b, c > 0,\ \psi : a, b, c \leq max$

7. Consider the following statement: "`A female athlete won every sports prize and her brother also won some mathematics prize.`" We define some predicates below to symbolize the given statement

- $Girl(a)$ = "$a$ is a girl", $Boy(b)$ = "$b$ is a boy",
- $SportPrize(x)$ = "$x$ is a prize in Sport tournament",
- $MathPrize(y)$ = "$y$ is a prize in Mathematics competition",
- $Sibling(a, b)$ = "$a$ and $b$ are siblings in a family",
- $WinSport(w, s)$ = "$w$ won a sport prize $s$",
- $WinMath(u, m)$ = "$u$ won a mathematics prize $m$".

Which of the following options best represents the given statement?

(I) $\forall x\ [SportPrize(x) \longrightarrow \exists y\ (Girl(y) \wedge WinSport(y, x))]$

(II) $[\ (\exists f, Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ [\ (\exists b, Sibling(f, b) \wedge Boy(b) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(b, y)]$

(III) $\exists f\ [\ (Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ \exists m\ [\ (Sibling(f, m) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(m, y)\ ]$

(IV) $\exists f\ [\ (Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ \exists m\ [\ (Sibling(f, m) \wedge\ Boy(m) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(m, y)\ ]$

A. (I) B. (IV) C. (III) D. (II)

*Question 8– Question 9 use the same description as follows:* Denote an alphabet $\Sigma = \{b, c\}$, and let $u = bc$, $v = cb$ be strings of length 2 in $\Sigma^*$. Define $L(r)$ and $L(s)$ respectively be languages determined by regular expressions $r = u^* = (bc)^*$ and $s = v^* = (cb)^*$.

8. The language $L_{bc}$ of all strings with $b$ and $c$ alternating, moreover starting with $b$ and ending with $c$, is

(I) $L(s)$; (II) $L(r) + L(s)$; (III) $L(r)$; (IV) $L(r) \cdot L(s)$.

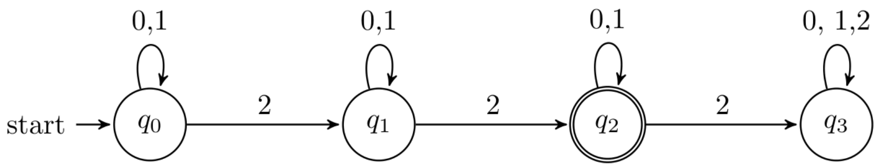
A. (I) B. (III) C. (IV) D. (II)

9. The language $L$ of all strings with $b$ and $c$ alternating is determined by which of the following regular expressions?

(I) $r + s$; (II) $r + s + c\ r + b\ s$; (III) $r + s + b\ r + c\ s$; (IV) $s + r$.

A. I B. IV C. III D. II

10. (L.O.2.3) Which of the following languages is accepted by the DFA over $\Sigma = \{0, 1, 2\}$ given below?




A. $L = \{w \in \Sigma^* \mid w$ contains at least two 2's consecutively$\}$.
B. $L = \{w \in \Sigma^* \mid w$ contains exactly two $2's\}$. C. $L = \{w \in \Sigma^* \mid w$ contains at least two $2's\}$.
D. $L = \{w \in \Sigma^* \mid w$ contains at most two $2's\}$.

11. (L.O.2.3) Let L be the language containing all the palindrome words where $\Sigma = \{a, b\}$, Which of the following regular expressions accepts the words in language L?

A. All the other answers are incorrect. B. (a+b)*(b+a)* C. a*b*b*a*
D. (abba)*

12. (L.O.1.3)

Consider program P below. (Weakest) precondition $\phi$ and postcondition $\psi$ of P are

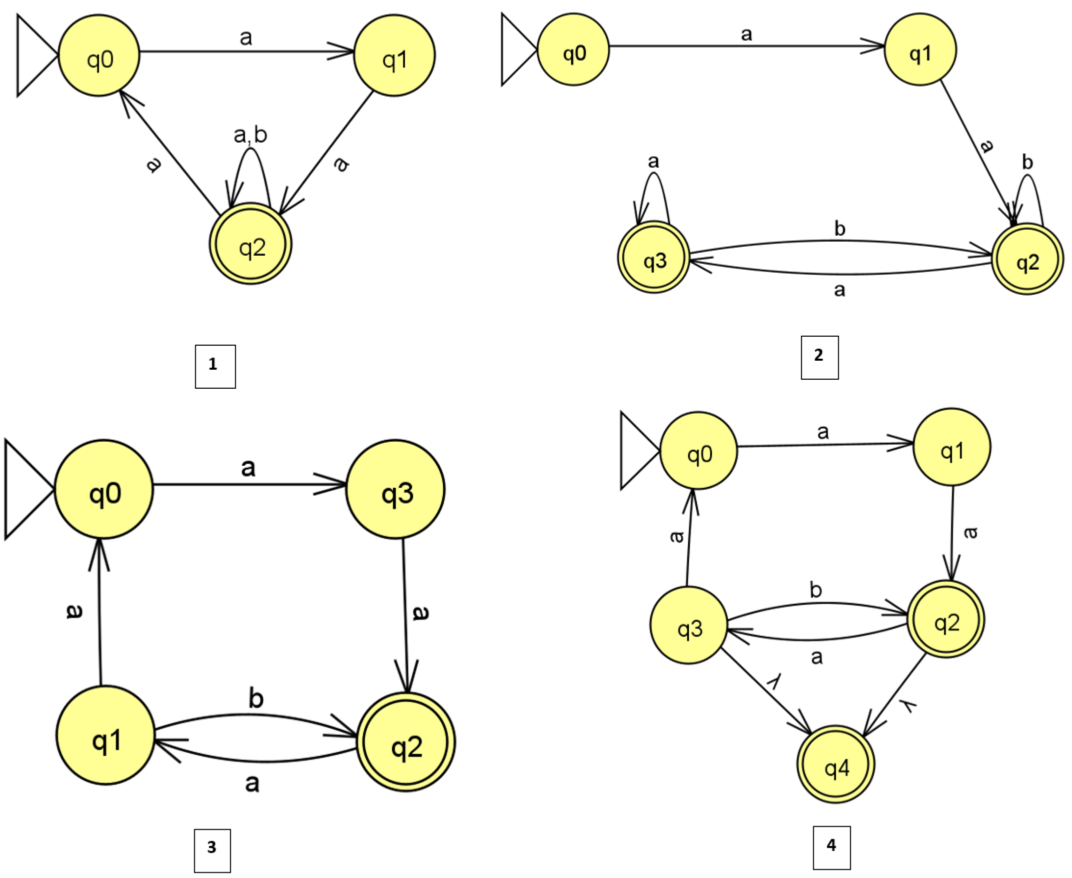
```
x := a;    y := 1;    z := b;
while true do
   if z = 0 then
      return y
   end if
   r := remainder(z, 2);    z := quotient(z, 2);
   if r = 1 then
      y := x * y;
   end if
   x := x * x;
end while
```

A. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{Z}\},\ \psi = \{y = a^b\}$.
B. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{N}\},\ \psi = \{y = a^b\}$.
C. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{N}\},\ \psi = \{y = x^z\}$.
D. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{Z}^+\},\ \psi = \{y = a^b\}$.

13. (L.O.1.3)
Consider program P with precondition and postcondition as in Question 8. Then the invariant form to prove the partial correctness of P is

A. $z \in \mathbb{Z} \land y = x^z$.
B. $z \in \mathbb{Z} \land yx^z = a^b$.
C. $z \in \mathbb{N} \land yx^z = a^b$.
D. $z \in \mathbb{N} \land y = x^z$.

14. (L.O.2.3) Given 4 automata where $\Sigma = \{a, b\}$ and $\lambda$ is the empty string.




Which of the following couples are the same?

A. All the other answers are incorrect.
B. 1,2
C. 1,3
D. 3,4

15. (L.O.2.3) Let $L$ be the language consisting of string $w \in \{0, 1\}^+$ that represents a decimal number divisible by 3. Which choice is CORRECT?

A. The number of bit 1s of any string of $L$ must be divisible by 3.
B. All other choices are incorrect.
C. $L$ is not a regular language.
D. The number of states of a DFA recognizing only $L$ must be divisible by 3.

16. (L.O.2.3) Let $L$ be the language consisting of string $w \in \{0,1\}^*$ that the number of occurrences of sub-string 01 and the number of occurrences of sub-string 10 in $w$ are the same. Which choice is CORRECT?

A. All other choices are incorrect.
B. A DFA recognizing only $L$ must have at least five states.
C. $L$ is not a regular language.
D. The number of bit 1s and the number of bit 0s of any string of $L$ must be the same.

17. (L.O.2.3) Which of the following is correct?

A. If $L_1$ is a regular language and the language $L_2$ is *not* regular then $L_1^{\circ}L_2$ is *not* regular.
B. Given two languages $L_1, L_2$, if $L_1 \cup L_2$ is regular then both $L_1, L_2$ are also regular.
C. For two languages $L_1, L_2$, we have $L_1^* \circ L_2^* \subseteq (L_1 \cup L_2)^*$.
D. If $L_1, L_2$, are finite languages then $|L_1 \circ L_2| = |L_1| \cdot |L_2|$.

18. (L.O.1.3)

In the following program, % is the division with remainder and returns the remainder, and function abs() returns the absolute value of an input integer. Consider the precondition $x = x_0 \wedge y = y_0 \wedge (x_0 \neq 0 \vee y_0 \neq 0)$. *Which is the invariant form for proving the partial correctness of the program?* [In the choices, $GCD(a, b)$ denotes the greatest common divisor of two integers $a$ and $b$ that are not both 0, $a$ mod $b$ is the division with the remainder by positive $b$ and $|a|$ is the absolute value of $a$.]

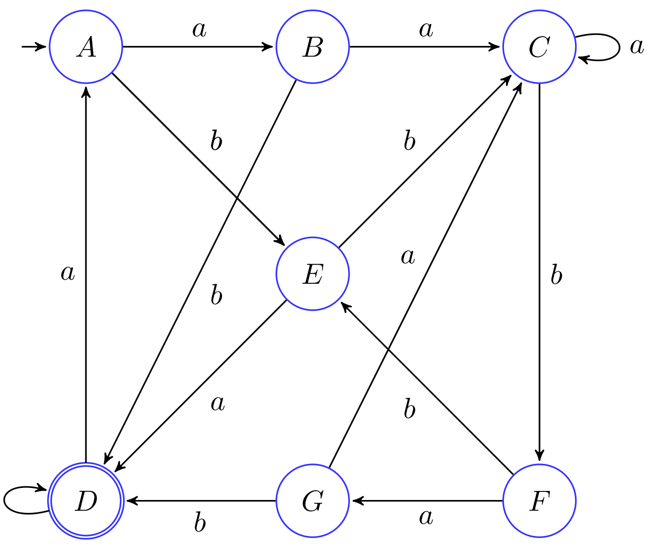
```
while (x != 0) {
        z = x;
        x = y % x;
        y = z;
}
y = abs(y);
```

A. All other choices are correct.
B. $GCD(x, y) = GCD(x_0, y_0)$
C. $y \neq 0 \rightarrow GCD(x, x \bmod |y|) = GCD(x_0, y_0)$
D. $x \neq 0 \rightarrow GCD(y, y \bmod |x|) = GCD(x_0, y_0)$

19. (L.O.1.2)
Consider the logical formula $\phi : \forall x \exists y \exists z (P(x,y) \wedge P(z,y) \wedge (P(x,z) \rightarrow P(z,x)))$ and the models $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ such that the universal set is the natural numbers set $\mathbb{N}$, $P^{\mathcal{M}_1} = \{(m,n) | n < m\}$, $P^{\mathcal{M}_2} = \{(m, 2m) | m \in \mathbb{N}\}$ and $P^{\mathcal{M}_3} = \{(m,n) | m < n+1\}$ Which of following statement is true?

A. $\mathcal{M}_1 \not\models \phi, \mathcal{M}_2 \not\models \phi, \mathcal{M}_3 \models \phi$.
B. $\mathcal{M}_1 \models \phi, \mathcal{M}_2 \not\models \phi, \mathcal{M}_3 \not\models \phi$.
C. $\mathcal{M}_1 \models \phi, \mathcal{M}_2 \models \phi, \mathcal{M}_3 \models \phi$.
D. $\mathcal{M}_1 \not\models \phi, \mathcal{M}_2 \models \phi, \mathcal{M}_3 \models \phi$.

20. (L.O.3.2)
Consider the following finite automaton:




What is the number of states in a minimal DFA (equivalent to the above FA)?

A. 7.
B. 4.
C. 5.
D. 6.

.........................................END OF EXAM..........................................

# Solution 1812

1. C.
2. C.
3. C.
4. C.
5. B.
6. A.
7. B.
8. B.
9. D.
10. B.
11. B.
12. B.
13. C.
14. B.
15. B.
16. B.
17. C.
18. B.
19. D.
20. C.

| **Lecturer:** | *October 2nd, 2023* | **Approved by:** | October 2nd, 2023 |
|---|---|---|---|
| *(Signature and Fullname)* | | *(Signature and Fullname)* | |

| UNIVERSITY OF TECHNOLOGY FACULTY OF CSE | MIDTERM | Semester/Academic year | 1 | 2023-2024 |
|---|---|---|---|---|
| | | Date | | 18/10/2023 |
| | Course title | Mathematical Modeling | | |
| | Course ID | CO2011 | | |
| | Duration | 60 mins | Question sheet code | 1813 |

Notes: - Students do not use course materials except one A4 hand-written sheet.
- Submit the question sheet together with the answer sheet.
- Choose the best answer (only 1) for each question.

1. (L.O.1.3)

Consider program P below. (Weakest) precondition $\phi$ and postcondition $\psi$ of P are

```
x := a;   y := 1;   z := b;
while true do
  if z = 0 then
    return y
  end if
  r := remainder(z, 2);   z := quotient(z, 2);
  if r = 1 then
    y := x * y;
  end if
  x := x * x;
end while
```

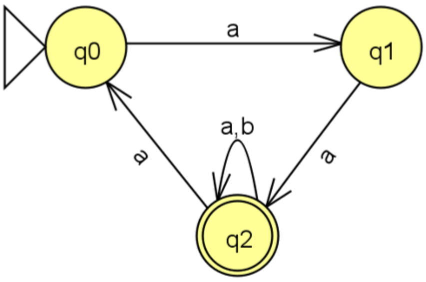
A. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{N}\},\ \psi = \{y = a^b\}$.

B. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{Z}\},\ \psi = \{y = a^b\}$.

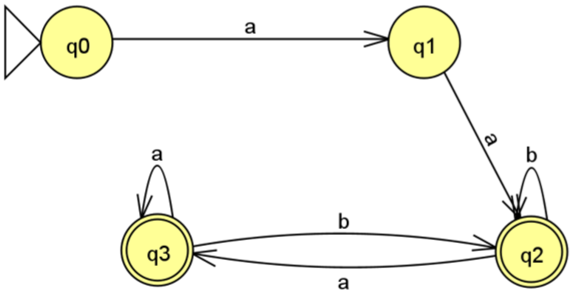
C. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{N}\},\ \psi = \{y = x^z\}$.

D. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{Z}^+\},\ \psi = \{y = a^b\}$.

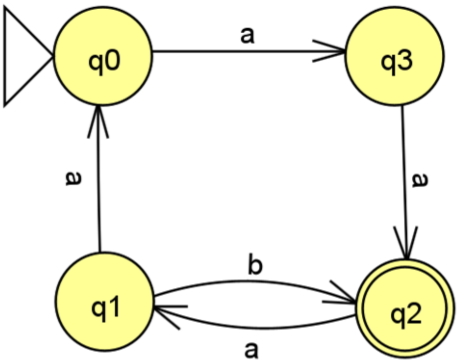
2. (L.O.2.3) Given 4 automata where $\Sigma = \{a, b\}$ and $\lambda$ is the empty string.



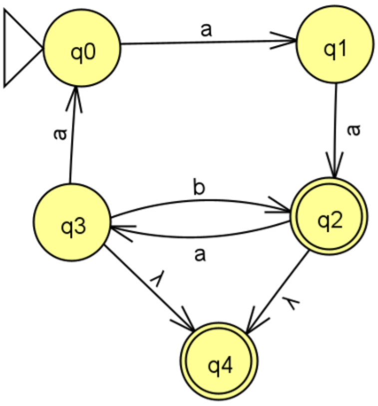

1




2




3




4

Which of the following couples are the same?

A. 1,2 B. All the other answers are incorrect. C. 1,3 D. 3,4

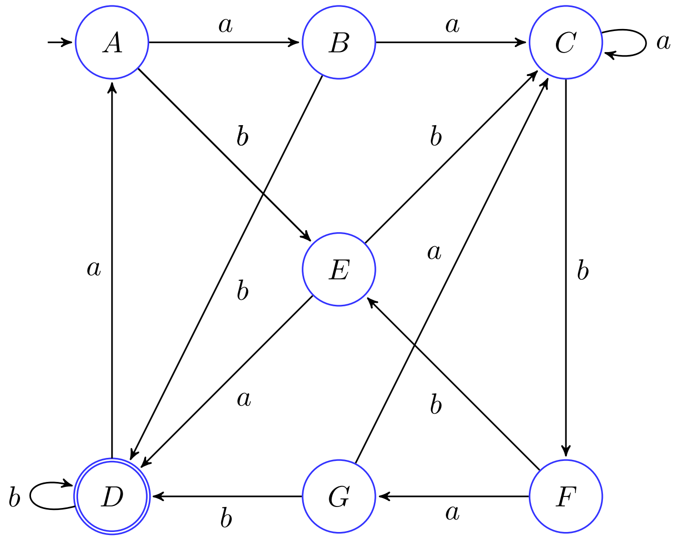
3. (L.O.1.3) Given a program finding max for 3 integers a,b,c:

max=0;
if(max<a) max=a;
if(max<b) max=b;
if(max<c) max=c;

Which of the following are the weakest pre-condition $\phi$ and post-condition $\psi$ for the program to be correct?

A. $\phi : a, b, c >= 0$, $\psi : a < b < c < max$ B. $\phi : T$, $\psi : a, b, c \leq max$
C. $\phi : a, b, c >= 0$, $\psi : a, b, c \leq max$ D. $\phi : a, b, c > 0$, $\psi : a, b, c \leq max$

4. (L.O.3.2)
Consider the following finite automaton:




What is the number of states in a minimal DFA (equivalent to the above FA)?

A. 4.  B. 7.  C. 5.  D. 6.

5. (L.O.1.2)
Consider the logical formula $\phi : \forall x \exists y \exists z (P(x, y) \wedge P(z, y) \wedge (P(x, z) \rightarrow P(z, x)))$ and the models $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ such that the universal set is the natural numbers set $\mathbb{N}$, $P^{\mathcal{M}_1} = \{(m, n) | n < m\}$, $P^{\mathcal{M}_2} = \{(m, 2m) | m \in \mathbb{N}\}$ and $P^{\mathcal{M}_3} = \{(m, n) | m < n + 1\}$ Which of following statement is true?

A. $\mathcal{M}_1 \models \phi, \mathcal{M}_2 \not\models \phi, \mathcal{M}_3 \not\models \phi$.
B. $\mathcal{M}_1 \not\models \phi, \mathcal{M}_2 \not\models \phi, \mathcal{M}_3 \models \phi$.
C. $\mathcal{M}_1 \models \phi, \mathcal{M}_2 \models \phi, \mathcal{M}_3 \models \phi$.
D. $\mathcal{M}_1 \not\models \phi, \mathcal{M}_2 \models \phi, \mathcal{M}_3 \models \phi$.

6. (L.O.1.2)
Which of the following statements is/are true?
(I) All finite languages are regular.
(II) If $L$ is regular so is the reverse language $L^R = \{w^R | w \in L\}$.
(III) $L = \{ww | w \in \{a, b\}^*\}$ is regular.
(IV) If $M = (Q, \Sigma, \delta, q_0, F)$ is a minimal DFA for a regular language $L$ then $\widehat{M} = (Q, \Sigma, \delta, q_0, Q - F)$ is a minimal DFA for $\overline{L}$.

A. (I), (II), (III).  B. (II), (III), (IV).  C. (I), (II), (IV).  D. Only (I).

7. (L.O.2.3) Which of the following is correct?
A. Given two languages $L_1, L_2$, if $L_1 \cup L_2$ is regular then both $L_1, L_2$ are also regular.
B. If $L_1$ is a regular language and the language $L_2$ is *not* regular then $L_1^{\circ} L_2$ is *not* regular.
C. For two languages $L_1, L_2$, we have $L_1^* \circ L_2^* \subseteq (L_1 \cup L_2)^*$.
D. If $L_1, L_2$, are finite languages then $|L_1 \circ L_2| = |L_1| \cdot |L_2|$.

8. Consider the following statement: "`A female athlete won every sports prize and her brother also won some mathematics prize`." We define some predicates below to symbolize the given statement

- $Girl(a)$ = "$a$ is a girl", $Boy(b)$ = "$b$ is a boy",
- $SportPrize(x)$ = "$x$ is a prize in Sport tournament",
- $MathPrize(y)$ = "$y$ is a prize in Mathematics competition",
- $Sibling(a, b)$ = "$a$ and $b$ are siblings in a family",
- $WinSport(w, s)$ = "$w$ won a sport prize $s$",
- $WinMath(u, m)$ = "$u$ won a mathematics prize $m$".

Which of the following options best represents the given statement?

(I) $\forall x\ [SportPrize(x) \longrightarrow \exists y\ (Girl(y) \wedge WinSport(y, x))]$

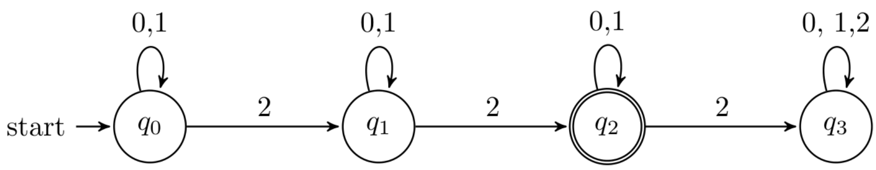
(II) $[\ (\exists f, Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ [\ (\exists b, Sibling(f, b) \wedge Boy(b) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(b, y)]$

(III) $\exists f\ [\ (Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ \exists m\ [\ (Sibling(f, m) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(m, y)\ ]$

(IV) $\exists f\ [\ (Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ \exists m\ [\ (Sibling(f, m) \wedge\ Boy(m) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(m, y)\ ]$

A. (IV) B. (I) C. (III) D. (II)

9. (L.O.2.3) Which of the following languages is accepted by the DFA over $\Sigma = \{0, 1, 2\}$ given below?




A. $L = \{w \in \Sigma^* \mid w \text{ contains exactly two } 2's\}$.
B. $L = \{w \in \Sigma^* \mid w \text{ contains at least two 2's consecutively}\}$.
C. $L = \{w \in \Sigma^* \mid w \text{ contains at least two } 2's\}$.
D. $L = \{w \in \Sigma^* \mid w \text{ contains at most two } 2's\}$.

10. (L.O.2.3) Let $L$ be the language consisting of string $w \in \{0, 1\}^+$ that represents a decimal number divisible by 3. Which choice is CORRECT?

A. All other choices are incorrect.
B. The number of bit 1s of any string of $L$ must be divisible by 3.
C. $L$ is not a regular language.
D. The number of states of a DFA recognizing only $L$ must be divisible by 3.

11. (L.O.1.3)
Consider program `P` with precondition and postcondition as in Question 8. Then the invariant form to prove the partial correctness of `P` is

A. $z \in \mathbb{Z} \wedge yx^z = a^b$. B. $z \in \mathbb{Z} \wedge y = x^z$. C. $z \in \mathbb{N} \wedge yx^z = a^b$. D. $z \in \mathbb{N} \wedge y = x^z$.

12. (L.O.1.3)

In the following program, % is the division with remainder and returns the remainder, and function abs() returns the absolute value of an input integer. Consider the precondition $x = x_0 \wedge y = y_0 \wedge (x_0 \neq 0 \vee y_0 \neq 0)$. *Which is the invariant form for proving the partial correctness of the program?* [In the choices, $GCD(a, b)$ denotes the greatest common divisor of two integers $a$ and $b$ that are not both 0, $a$ mod $b$ is the division with the remainder by positive $b$ and $|a|$ is the absolute value of $a$.]

```
while (x != 0) {
          z = x;
          x = y % x;
          y = z;
}
y = abs(y);
```

A. $GCD(x, y) = GCD(x_0, y_0)$
B. All other choices are correct.
C. $y \neq 0 \rightarrow GCD(x, x \bmod |y|) = GCD(x_0, y_0)$
D. $x \neq 0 \rightarrow GCD(y, y \bmod |x|) = GCD(x_0, y_0)$

13. (L.O.1.3)

Given the following program, with $\top$ as a precondition, determine the postcondition yourself. To prove the partial correctness of the corresponding Hoare triple, which of the following is an invariant form we should use?

```
i=0;
s=0;
while(i<=n){
    if(i%2==0)
        s+=i;
    i+=1;
}
```

A. $s = \sum_{k=0}^{(i-1)/2} 2k \wedge (i \leq n+1)$.
B. $s = \sum_{k=0}^{i/2} 2k \wedge (i \leq n+1)$.
C. $s = \sum_{k=0}^{i/2} 2k \wedge (i \leq n)$.
D. $s = \sum_{k=0}^{(i-1)/2} 2k$.

14. (L.O.1.3) Given a post-condition for a correct program:
$[|\forall q(i < q \leq n \implies a[i] \leq a[q]) \wedge a[i] \leq a[i-1]|]$
Which of the following post-conditions also correct for the same program?

A. $[|\forall q(i-1 < q \leq n \implies a[i-1] \leq a[q])|]$
B. All the other answers are correct.
C. $[|\forall q(i \leq q \leq n \implies a[i] \leq a[q])|]$
D. $[|\forall q(i < q \leq n \implies a[i-1] \leq a[q])|]$

15. (L.O.1.1) Let $P$ and $Q$ be binary predicates and $S$ be nullary predicates. Which formulas are valid?
(I) $\forall x \forall y(P(x) \rightarrow P(y)) \wedge (P(y) \rightarrow P(x))$
(II) $\exists y((\forall x P(x)) \rightarrow P(y))$
(III) $(\forall x P(x) \rightarrow S) \rightarrow \exists x(P(x) \rightarrow S)$
(IV) $\forall x(P(x) \vee Q(x)) \rightarrow (\forall x P(x)) \vee (\exists x Q(x))$

A. Only (II), (III), and (IV)
B. Only (III) and (IV)
C. Only (II) and (III)
D. Only (I), (II), and (IV)

*Question 16– Question 17 use the same description as follows:* Denote an alphabet $\Sigma = \{b, c\}$, and let $u = bc$, $v = cb$ be strings of length 2 in $\Sigma^*$. Define $L(r)$ and $L(s)$ respectively be languages determined by regular expressions $r = u^* = (bc)^*$ and $s = v^* = (cb)^*$.

16. The language $L_{bc}$ of all strings with $b$ and $c$ alternating, moreover starting with $b$ and ending with $c$, is

(I) $L(s)$;    (II) $L(r) + L(s)$;    (III) $L(r)$;    (IV) $L(r) \cdot L(s)$.

A. (III)
B. (I)
C. (IV)
D. (II)

17. The language $L$ of all strings with $b$ and $c$ alternating is determined by which of the following regular expressions?

(I) $r + s$;    (II) $r + s + c\ r + b\ s$;    (III) $r + s + b\ r + c\ s$;    (IV) $s + r$.

A. IV
B. I
C. III
D. II

18. Consider a program `Prod` $= Product(a, b)$ calculating the product $a \cdot (b-1)$ of two naturals $a, b$ where $b \geq 2$. Denote the Hoare triple of a suitable core program $P$ to correctly compute `Prod` as

$$\{\phi\}\ P\ \{\psi\}$$

where pre-conditions $\phi := (a \geq 1) \wedge (b \geq 2)$ and post-conditions $\psi$ should be suitably defined. A variant $K$ is an expression whose values are either strictly decreasing or strictly decreasing, and an invariant $E$ **does not** change its Boolean value in the loop. Both $K$ and $E$ depend on original inputs $a, b$ and also internal variables being newly declared in the program. Conditions and such variants are put in brackets $\{..\}$.

Let $E = \{\ u = (b - v) \cdot a\}$ be a "good" invariant of the program `Prod`, assuming the full format as follows.

$\{(a \geq 1) \wedge (b \geq 2)\}$
u = 0;
v = b;
$\{u = 0 \wedge v = b\}$
$E$
**while** ( v != Lb )
{
$\quad \vdots\ P$
}
$\{u = a \cdot (b - v)\}$

In the following options of core program $P$ and lower bound $Lb$:
(I) $P := [u = u + a;\ v = v - 1;]$ and $Lb := 0$
(II) $P := [u = u + a;\ v = v - 2;]$ and $Lb := 1$
(III) $P := [u = u + a;\ v = v - 1;]$ and $Lb := 1$
(IV) $P := [u = u + a;\ v = v + 1;]$ and $Lb := 1$.
Which one do you choose to obtain the totally correct statement?

$$\vdash_{\text{tot}} \{\phi\}\ \texttt{Prod}\ \{u = a \cdot (b-1)\}?$$

A. (IV) B. (I) C. (III) D. (II)

19. (L.O.2.3) Let $L$ be the language consisting of string $w \in \{0, 1\}^*$ that the number of occurrences of sub-string 01 and the number of occurrences of sub-string 10 in $w$ are the same. Which choice is CORRECT?
A. A DFA recognizing only $L$ must have at least five states.
B. All other choices are incorrect.
C. $L$ is not a regular language.
D. The number of bit 1s and the number of bit 0s of any string of $L$ must be the same.

20. (L.O.2.3) Let L be the language containing all the palindrome words where $\Sigma = \{a, b\}$, Which of the following regular expressions accepts the words in language L?
A. (a+b)*(b+a)* B. All the other answers are incorrect. C. a*b*b*a*
D. (abba)*

..........................................END OF EXAM..........................................

# Solution 1813

1. A.
2. A.
3. B.
4. C.
5. D.
6. C.
7. C.
8. A.
9. A.
10. A.
11. C.
12. A.
13. C.
14. C.
15. A.
16. A.
17. D.
18. C.
19. A.
20. A.

| **Lecturer:** | *October 2nd, 2023* | **Approved by:** | October 2nd, 2023 |
|---|---|---|---|
| *(Signature and Fullname)* | | *(Signature and Fullname)* | |

| UNIVERSITY OF TECHNOLOGY FACULTY OF CSE | MIDTERM | Semester/Academic year | 1 | 2023-2024 |
|---|---|---|---|---|
| | | Date | 18/10/2023 | |
| | Course title | Mathematical Modeling | | |
| | Course ID | CO2011 | | |
| | Duration | 60 mins | Question sheet code | 1814 |

Notes: - Students do not use course materials except one A4 hand-written sheet.
- Submit the question sheet together with the answer sheet.
- Choose the best answer (only 1) for each question.

1. (L.O.1.2)
Consider the logical formula $\phi : \forall x \exists y \exists z (P(x,y) \wedge P(z,y) \wedge (P(x,z) \to P(z,x)))$ and the models $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ such that the universal set is the natural numbers set $\mathbb{N}$, $P^{\mathcal{M}_1} = \{(m,n) | n < m\}$, $P^{\mathcal{M}_2} = \{(m, 2m) | m \in \mathbb{N}\}$ and $P^{\mathcal{M}_3} = \{(m,n) | m < n+1\}$ Which of following statement is true?

A. $\mathcal{M}_1 \models \phi, \mathcal{M}_2 \not\models \phi, \mathcal{M}_3 \not\models \phi$.
B. $\mathcal{M}_1 \not\models \phi, \mathcal{M}_2 \models \phi, \mathcal{M}_3 \models \phi$.
C. $\mathcal{M}_1 \models \phi, \mathcal{M}_2 \models \phi, \mathcal{M}_3 \models \phi$.
D. $\mathcal{M}_1 \not\models \phi, \mathcal{M}_2 \not\models \phi, \mathcal{M}_3 \models \phi$.

2. (L.O.2.3) Let L be the language containing all the palindrome words where $\Sigma = \{a, b\}$, Which of the following regular expressions accepts the words in language L?

A. (a+b)*(b+a)*
B. (abba)*
C. a*b*b*a*
D. All the other answers are incorrect.

3. (L.O.1.3) Given a program finding max for 3 integers a,b,c:
max=0;
if(max<a) max=a;
if(max<b) max=b;
if(max<c) max=c;
Which of the following are the weakest pre-condition $\phi$ and post-condition $\psi$ for the program to be correct?

A. $\phi : a,b,c >= 0,\ \psi : a < b < c < max$
B. $\phi : a,b,c > 0,\ \psi : a,b,c \leq max$
C. $\phi : a,b,c >= 0,\ \psi : a,b,c \leq max$
D. $\phi : T,\ \psi : a,b,c \leq max$

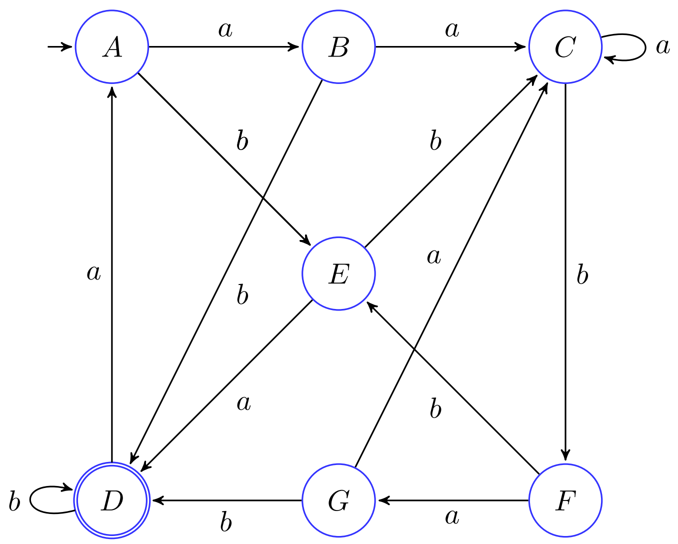
4. (L.O.1.2)
Which of the following statements is/are true?
(I) All finite languages are regular.
(II) If $L$ is regular so is the reverse language $L^R = \{w^R | w \in L\}$.
(III) $L = \{ww | w \in \{a,b\}^*\}$ is regular.
(IV) If $M = (Q, \Sigma, \delta, q_0, F)$ is a minimal DFA for a regular language $L$ then $\widehat{M} = (Q, \Sigma, \delta, q_0, Q - F)$ is a minimal DFA for $\overline{L}$.

A. (I), (II), (III).
B. Only (I).
C. (I), (II), (IV).
D. (II), (III), (IV).

5. (L.O.2.3) Let $L$ be the language consisting of string $w \in \{0,1\}^*$ that the number of occurrences of sub-string 01 and the number of occurrences of sub-string 10 in $w$ are the same. Which choice is CORRECT?

A. A DFA recognizing only $L$ must have at least five states.
B. The number of bit 1s and the number of bit 0s of any string of $L$ must be the same.
C. $L$ is not a regular language.
D. All other choices are incorrect.

6. (L.O.3.2)
Consider the following finite automaton:




What is the number of states in a minimal DFA (equivalent to the above FA)?

A. 4.   B. 6.   C. 5.   D. 7.

*Question 7– Question 8 use the same description as follows:* Denote an alphabet $\Sigma = \{b, c\}$, and let $u = bc$, $v = cb$ be strings of length 2 in $\Sigma^*$. Define $L(r)$ and $L(s)$ respectively be languages determined by regular expressions $r = u^* = (bc)^*$ and $s = v^* = (cb)^*$.

7. The language $L_{bc}$ of all strings with $b$ and $c$ alternating, moreover starting with $b$ and ending with $c$, is

$$\text{(I) } L(s); \qquad \text{(II) } L(r) + L(s); \qquad \text{(III) } L(r); \qquad \text{(IV) } L(r) \cdot L(s).$$

A. (III)   B. (II)   C. (IV)   D. (I)

8. The language $L$ of all strings with $b$ and $c$ alternating is determined by which of the following regular expressions?

$$\text{(I) } r + s; \qquad \text{(II) } r + s + c\,r + b\,s; \qquad \text{(III) } r + s + b\,r + c\,s; \qquad \text{(IV) } s + r.$$

A. IV   B. II   C. III   D. I

9. (L.O.2.3) Let $L$ be the language consisting of string $w \in \{0,1\}^+$ that represents a decimal number divisible by 3. Which choice is CORRECT?

A. All other choices are incorrect.
B. The number of states of a DFA recognizing only $L$ must be divisible by 3.
C. $L$ is not a regular language.
D. The number of bit 1s of any string of $L$ must be divisible by 3.

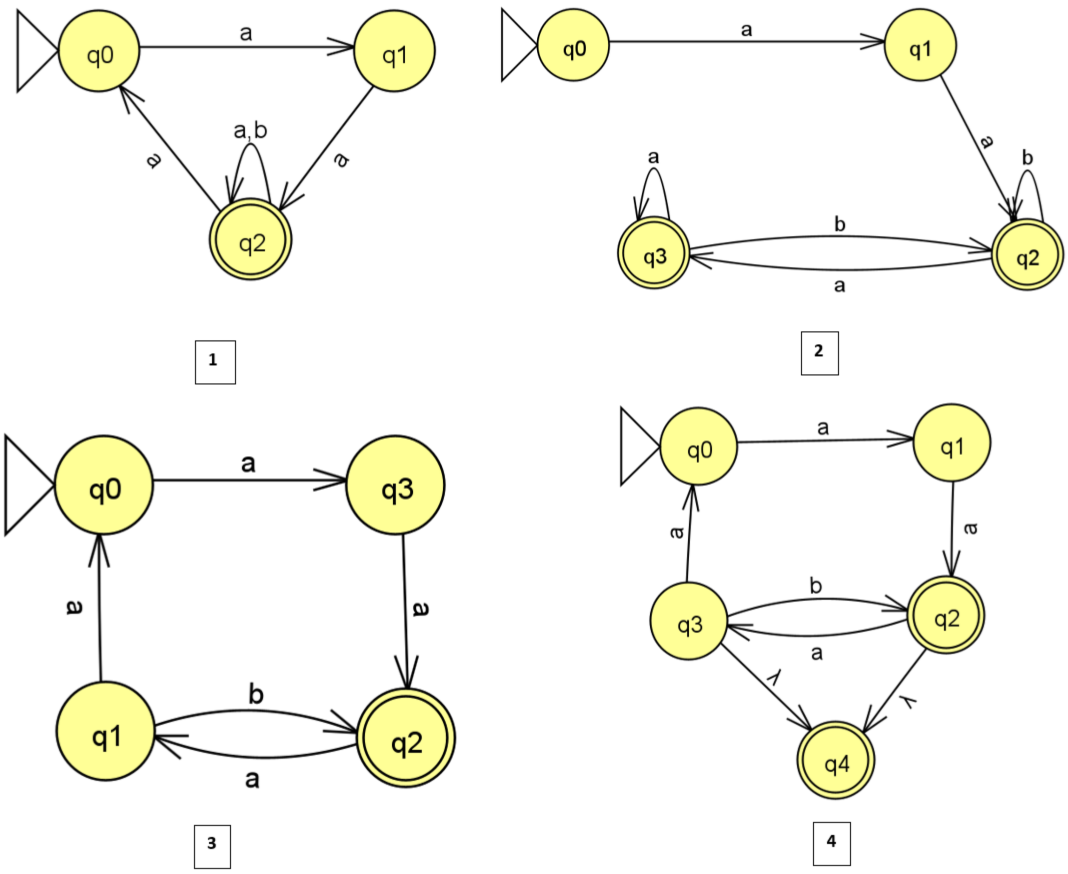
10. (L.O.1.3) Given a post-condition for a correct program:
$[|\forall q(i < q \le n \implies a[i] \le a[q]) \wedge a[i] \le a[i-1]|]$
Which of the following post-conditions also correct for the same program?

A. $[|\forall q(i-1 < q \le n \implies a[i-1] \le a[q])|]$
B. $[|\forall q(i < q \le n \implies a[i-1] \le a[q])|]$
C. $[|\forall q(i \le q \le n \implies a[i] \le a[q])|]$
D. All the other answers are correct.

11. (L.O.1.3)
Consider program P with precondition and postcondition as in Question 8. Then the invariant form to prove the partial correctness of P is

A. $z \in \mathbb{Z} \wedge yx^z = a^b$.   B. $z \in \mathbb{N} \wedge y = x^z$.   C. $z \in \mathbb{N} \wedge yx^z = a^b$.   D. $z \in \mathbb{Z} \wedge y = x^z$.

12. (L.O.2.3) Given 4 automata where $\Sigma = \{a, b\}$ and $\lambda$ is the empty string.




Which of the following couples are the same?

A. 1,2 B. 3,4 C. 1,3

D. All the other answers are incorrect.

13. Consider a program `Prod` $= Product(a, b)$ calculating the product $a \cdot (b-1)$ of two naturals $a, b$ where $b \geq 2$. Denote the Hoare triple of a suitable core program $P$ to correctly compute `Prod` as

$$\{\phi\}\ P\ \{\psi\}$$

where pre-conditions $\phi := (a \geq 1) \wedge (b \geq 2)$ and post-conditions $\psi$ should be suitably defined. A variant $K$ is an expression whose values are either strictly decreasing or strictly decreasing, and an invariant $E$ **does not** change its Boolean value in the loop. Both $K$ and $E$ depend on original inputs $a, b$ and also internal variables being newly declared in the program. Conditions and such variants are put in brackets $\{..\}$.

Let $E = \{\ u = (b - v) \cdot a\}$ be a "good" invariant of the program `Prod`, assuming the full format as follows.

$\{(a \geq 1) \wedge (b \geq 2)\}$
u = 0;
v = b;
$\{u = 0 \wedge v = b\}$
$E$
**while** ( v != Lb )
{
  $\vdots\ P$
}
$\{u = a \cdot (b - v)\}$

In the following options of core program $P$ and lower bound $Lb$:
(I) $P := [u = u + a;\ v = v - 1;]$ and $Lb := 0$
(II) $P := [u = u + a;\ v = v - 2;]$ and $Lb := 1$
(III) $P := [u = u + a;\ v = v - 1;]$ and $Lb := 1$
(IV) $P := [u = u + a;\ v = v + 1;]$ and $Lb := 1$.

Which one do you choose to obtain the totally correct statement?

$$\vdash_{\text{tot}} \{\phi\}\ \texttt{Prod}\ \{u = a \cdot (b - 1)\}?$$

A. (IV) B. (II) C. (III) D. (I)

14. (L.O.1.1) Let $P$ and $Q$ be binary predicates and $S$ be nullary predicates. Which formulas are valid?
(I) $\forall x \forall y (P(x) \to P(y)) \wedge (P(y) \to P(x))$ (II) $\exists y ((\forall x P(x)) \to P(y))$
(III) $(\forall x P(x) \to S) \to \exists x (P(x) \to S)$ (IV) $\forall x (P(x) \vee Q(x)) \to (\forall x P(x)) \vee (\exists x Q(x))$

A. Only (II), (III), and (IV)
B. Only (I), (II), and (IV)
C. Only (II) and (III)
D. Only (III) and (IV)

15. (L.O.1.3)

Consider program P below. (Weakest) precondition $\phi$ and postcondition $\psi$ of P are

```
x := a;   y := 1;   z := b;
while true do
  if z = 0 then
    return y
  end if
  r := remainder(z, 2);   z := quotient(z, 2);
  if r = 1 then
    y := x * y;
  end if
  x := x * x;
end while
```

A. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{N}\},\ \psi = \{y = a^b\}$.
B. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{Z}^+\},\ \psi = \{y = a^b\}$.
C. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{N}\},\ \psi = \{y = x^z\}$.
D. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{Z}\},\ \psi = \{y = a^b\}$.

16. (L.O.1.3)

Given the following program, with $\top$ as a precondition, determine the postcondition yourself. To prove the partial correctness of the corresponding Hoare triple, which of the following is an invariant form we should use?

```
i=0;
s=0;
while(i<=n){
    if(i%2==0)
        s+=i;
    i+=1;
}
```

A. $s = \sum_{k=0}^{(i-1)/2} 2k \land (i \leq n+1)$.

B. $s = \sum_{k=0}^{(i-1)/2} 2k$.

C. $s = \sum_{k=0}^{i/2} 2k \land (i \leq n)$.

D. $s = \sum_{k=0}^{i/2} 2k \land (i \leq n+1)$.

17. (L.O.1.3)

In the following program, % is the division with remainder and returns the remainder, and function abs() returns the absolute value of an input integer. Consider the precondition $x = x_0 \land y = y_0 \land (x_0 \neq 0 \lor y_0 \neq 0)$. *Which is the invariant form for proving the partial correctness of the program?* [In the choices, $GCD(a, b)$ denotes the greatest common divisor of two integers $a$ and $b$ that are not both 0, $a$ mod $b$ is the division with the remainder by positive $b$ and $|a|$ is the absolute value of $a$.]

```
while (x != 0) {
        z = x;
        x = y % x;
        y = z;
}
y = abs(y);
```

A. $GCD(x, y) = GCD(x_0, y_0)$

B. $x \neq 0 \to GCD(y, y \bmod |x|) = GCD(x_0, y_0)$

C. $y \neq 0 \to GCD(x, x \bmod |y|) = GCD(x_0, y_0)$

D. All other choices are correct.

18. Consider the following statement: "`A female athlete won every sports prize and her brother also won some mathematics prize`." We define some predicates below to symbolize the given statement

- $Girl(a)$ = "$a$ is a girl", $Boy(b)$ = "$b$ is a boy",
- $SportPrize(x)$ = "$x$ is a prize in Sport tournament",
- $MathPrize(y)$ = "$y$ is a prize in Mathematics competition",
- $Sibling(a, b)$ = "$a$ and $b$ are siblings in a family",
- $WinSport(w, s)$ = "$w$ won a sport prize $s$",
- $WinMath(u, m)$ = "$u$ won a mathematics prize $m$".

Which of the following options best represents the given statement?

(I) $\forall x\ [SportPrize(x) \longrightarrow \exists y\ (Girl(y) \wedge WinSport(y, x))]$

(II) $[\ (\exists f, Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ [\ (\exists b, Sibling(f, b) \wedge Boy(b) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(b, y)]$

(III) $\exists f\ [\ (Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ \exists m\ [\ (Sibling(f, m) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(m, y)\ ]$

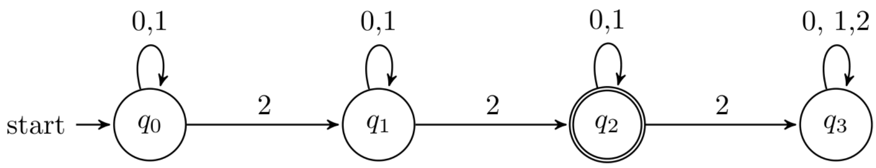
(IV) $\exists f\ [\ (Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ \exists m\ [\ (Sibling(f, m) \wedge\ Boy(m) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(m, y)\ ]$

A. (IV) B. (II) C. (III) D. (I)

19. (L.O.2.3) Which of the following is correct?

A. Given two languages $L_1, L_2$, if $L_1 \cup L_2$ is regular then both $L_1, L_2$ are also regular.
B. If $L_1, L_2$, are finite languages then $|L_1 \circ L_2| = |L_1| \cdot |L_2|$.
C. For two languages $L_1, L_2$, we have $L_1^* \circ L_2^* \subseteq (L_1 \cup L_2)^*$.
D. If $L_1$ is a regular language and the language $L_2$ is *not* regular then $L_1^{\circ} L_2$ is *not* regular.

20. (L.O.2.3) Which of the following languages is accepted by the DFA over $\Sigma = \{0, 1, 2\}$ given below?




A. $L = \{w \in \Sigma^* \mid w$ contains exactly two $2's\}$.
B. $L = \{w \in \Sigma^* \mid w$ contains at most two $2's\}$.
C. $L = \{w \in \Sigma^* \mid w$ contains at least two $2's\}$.
D. $L = \{w \in \Sigma^* \mid w$ contains at least two 2's consecutively$\}$.

..............................................END OF EXAM..............................................

# Solution 1814

1. B.
2. A.
3. D.
4. C.
5. A.
6. C.
7. A.
8. B.
9. A.
10. C.
11. C.
12. A.
13. C.
14. A.
15. A.
16. C.
17. A.
18. A.
19. C.
20. A.

| **Lecturer:** | *October 2nd, 2023* | **Approved by:** | October 2nd, 2023 |
|---|---|---|---|
| *(Signature and Fullname)* | | *(Signature and Fullname)* | |

| UNIVERSITY OF TECHNOLOGY<br>FACULTY OF CSE | MIDTERM | Semester/Academic year | 1 | 2023-2024 |
|---|---|---|---|---|
| | | Date | | 18/10/2023 |
| | Course title | Mathematical Modeling | | |
| | Course ID | CO2011 | | |
| | Duration | 60 mins | Question sheet code | 1815 |

Notes: - Students do not use course materials except one A4 hand-written sheet.
- Submit the question sheet together with the answer sheet.
- Choose the best answer (only 1) for each question.

*Question 1– Question 2 use the same description as follows:* Denote an alphabet $\Sigma = \{b, c\}$, and let $u = bc$, $v = cb$ be strings of length 2 in $\Sigma^*$. Define $L(r)$ and $L(s)$ respectively be languages determined by regular expressions $r = u^* = (bc)^*$ and $s = v^* = (cb)^*$.

1. The language $L_{bc}$ of all strings with $b$ and $c$ alternating, moreover starting with $b$ and ending with $c$, is

(I) $L(s)$; (II) $L(r) + L(s)$; (III) $L(r)$; (IV) $L(r) \cdot L(s)$.

A. (I) B. (III) C. (II) D. (IV)

2. The language $L$ of all strings with $b$ and $c$ alternating is determined by which of the following regular expressions?

(I) $r + s$; (II) $r + s + c\ r + b\ s$; (III) $r + s + b\ r + c\ s$; (IV) $s + r$.

A. I B. IV C. II D. III

3. Consider the following statement: "`A female athlete won every sports prize and her brother also won some mathematics prize.`" We define some predicates below to symbolize the given statement

- $Girl(a)$ = "$a$ is a girl", $Boy(b)$ = "$b$ is a boy",
- $SportPrize(x)$ = "$x$ is a prize in Sport tournament",
- $MathPrize(y)$ = "$y$ is a prize in Mathematics competition",
- $Sibling(a, b)$ = "$a$ and $b$ are siblings in a family",
- $WinSport(w, s)$ = "$w$ won a sport prize $s$",
- $WinMath(u, m)$ = "$u$ won a mathematics prize $m$".

Which of the following options best represents the given statement?

(I) $\forall x\ [SportPrize(x) \longrightarrow \exists y\ (Girl(y) \wedge WinSport(y, x))]$

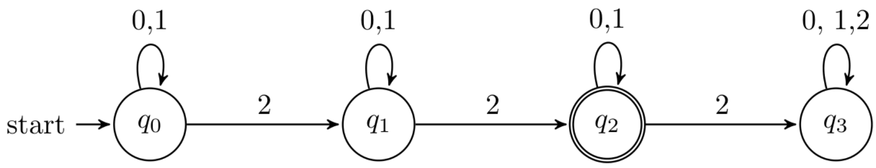
(II) $[\ (\exists f, Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ [\ (\exists b, Sibling(f, b) \wedge Boy(b) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(b, y)]$

(III) $\exists f\ [\ (Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ \exists m\ [\ (Sibling(f, m) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(m, y)\ ]$

(IV) $\exists f\ [\ (Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ \exists m\ [\ (Sibling(f, m) \wedge\ Boy(m) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(m, y)\ ]$

A. (I) B. (IV) C. (II) D. (III)

4. (L.O.2.3) Which of the following languages is accepted by the DFA over $\Sigma = \{0, 1, 2\}$ given below?




A. $L = \{w \in \Sigma^* \mid w$ contains at least two 2's consecutively$\}$.
B. $L = \{w \in \Sigma^* \mid w$ contains exactly two $2's\}$.
C. $L = \{w \in \Sigma^* \mid w$ contains at most two $2's\}$.
D. $L = \{w \in \Sigma^* \mid w$ contains at least two $2's\}$.

5. (L.O.1.3) Given a post-condition for a correct program:
$[|\forall q(i < q \leq n \implies a[i] \leq a[q]) \wedge a[i] \leq a[i-1]|]$
Which of the following post-conditions also correct for the same program?

A. All the other answers are correct.
B. $[|\forall q(i-1 < q \leq n \implies a[i-1] \leq a[q])|]$
C. $[|\forall q(i < q \leq n \implies a[i-1] \leq a[q])|]$
D. $[|\forall q(i \leq q \leq n \implies a[i] \leq a[q])|]$

6. (L.O.1.3)

Given the following program, with $\top$ as a precondition, determine the postcondition yourself. To prove the partial correctness of the corresponding Hoare triple, which of the following is an invariant form we should use?

```
i=0;
s=0;
while(i<=n){
    if(i%2==0)
        s+=i;
    i+=1;
}
```

A. $s = \sum_{k=0}^{i/2} 2k \wedge (i \leq n+1)$.
B. $s = \sum_{k=0}^{(i-1)/2} 2k \wedge (i \leq n+1)$.
C. $s = \sum_{k=0}^{(i-1)/2} 2k$.
D. $s = \sum_{k=0}^{i/2} 2k \wedge (i \leq n)$.

7. (L.O.2.3) Let $L$ be the language consisting of string $w \in \{0,1\}^*$ that the number of occurrences of sub-string 01 and the number of occurrences of sub-string 10 in $w$ are the same. Which choice is CORRECT?

A. All other choices are incorrect.
B. A DFA recognizing only $L$ must have at least five states.
C. The number of bit 1s and the number of bit 0s of any string of $L$ must be the same.
D. $L$ is not a regular language.

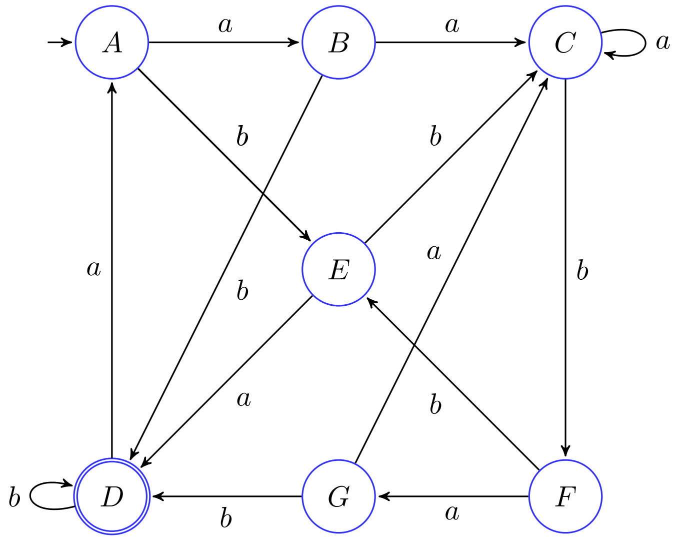
8. (L.O.1.3) Given a program finding max for 3 integers a,b,c:
max=0;
if(max<a) max=a;
if(max<b) max=b;
if(max<c) max=c;
Which of the following are the weakest pre-condition $\phi$ and post-condition $\psi$ for the program to be correct?

A. $\phi : T$, $\psi : a, b, c \leq max$
B. $\phi : a, b, c >= 0$, $\psi : a < b < c < max$
C. $\phi : a, b, c > 0$, $\psi : a, b, c \leq max$
D. $\phi : a, b, c >= 0$, $\psi : a, b, c \leq max$

9. (L.O.1.3)
Consider program `P` with precondition and postcondition as in Question 8. Then the invariant form to prove the partial correctness of `P` is

A. $z \in \mathbb{Z} \wedge y = x^z$.
B. $z \in \mathbb{Z} \wedge yx^z = a^b$.
C. $z \in \mathbb{N} \wedge y = x^z$.
D. $z \in \mathbb{N} \wedge yx^z = a^b$.

10. (L.O.3.2)
Consider the following finite automaton:




What is the number of states in a minimal DFA (equivalent to the above FA)?

A. 7. B. 4. C. 6. D. 5.

11. (L.O.1.2)
Which of the following statements is/are true?
(I) All finite languages are regular.
(II) If $L$ is regular so is the reverse language $L^R = \{w^R | w \in L\}$.
(III) $L = \{ww | w \in \{a, b\}^*\}$ is regular.
(IV) If $M = (Q, \Sigma, \delta, q_0, F)$ is a minimal DFA for a regular language $L$ then $\widehat{M} = (Q, \Sigma, \delta, q_0, Q - F)$ is a minimal DFA for $\overline{L}$.

A. (II), (III), (IV). B. (I), (II), (III). C. Only (I). D. (I), (II), (IV).

12. (L.O.1.2)
Consider the logical formula $\phi : \forall x \exists y \exists z (P(x, y) \wedge P(z, y) \wedge (P(x, z) \rightarrow P(z, x)))$ and the models $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ such that the universal set is the natural numbers set $\mathbb{N}$, $P^{\mathcal{M}_1} = \{(m, n) | n < m\}$, $P^{\mathcal{M}_2} = \{(m, 2m) | m \in \mathbb{N}\}$ and $P^{\mathcal{M}_3} = \{(m, n) | m < n + 1\}$ Which of following statement is true?

A. $\mathcal{M}_1 \not\models \phi, \mathcal{M}_2 \not\models \phi, \mathcal{M}_3 \models \phi$.
B. $\mathcal{M}_1 \models \phi, \mathcal{M}_2 \not\models \phi, \mathcal{M}_3 \not\models \phi$.
C. $\mathcal{M}_1 \not\models \phi, \mathcal{M}_2 \models \phi, \mathcal{M}_3 \models \phi$.
D. $\mathcal{M}_1 \models \phi, \mathcal{M}_2 \models \phi, \mathcal{M}_3 \models \phi$.

13. (L.O.2.3) Let L be the language containing all the palindrome words where $\Sigma = \{a, b\}$, Which of the following regular expressions accepts the words in language L?

A. All the other answers are incorrect. B. (a+b)*(b+a)* C. (abba)*
D. a*b*b*a*

14. (L.O.2.3) Let $L$ be the language consisting of string $w \in \{0, 1\}^+$ that represents a decimal number divisible by 3. Which choice is CORRECT?

A. The number of bit 1s of any string of $L$ must be divisible by 3.
B. All other choices are incorrect.
C. The number of states of a DFA recognizing only $L$ must be divisible by 3.
D. $L$ is not a regular language.

15. (L.O.1.3)

In the following program, % is the division with remainder and returns the remainder, and function abs() returns the absolute value of an input integer. Consider the precondition $x = x_0 \land y = y_0 \land (x_0 \neq 0 \lor y_0 \neq 0)$. *Which is the invariant form for proving the partial correctness of the program?* [In the choices, $GCD(a, b)$ denotes the greatest common divisor of two integers $a$ and $b$ that are not both 0, $a$ mod $b$ is the division with the remainder by positive $b$ and $|a|$ is the absolute value of $a$.]

```
while (x != 0) {
        z = x;
        x = y % x;
        y = z;
}
y = abs(y);
```

A. All other choices are correct.

B. $GCD(x, y) = GCD(x_0, y_0)$

C. $x \neq 0 \to GCD(y, y \bmod |x|) = GCD(x_0, y_0)$

D. $y \neq 0 \to GCD(x, x \bmod |y|) = GCD(x_0, y_0)$

16. Consider a program `Prod` $= Product(a, b)$ calculating the product $a \cdot (b-1)$ of two naturals $a, b$ where $b \geq 2$. Denote the Hoare triple of a suitable core program $P$ to correctly compute `Prod` as

$$\{\phi\}\ P\ \{\psi\}$$

where pre-conditions $\phi := (a \geq 1) \land (b \geq 2)$ and post-conditions $\psi$ should be suitably defined. A variant $K$ is an expression whose values are either strictly decreasing or strictly decreasing, and an invariant $E$ **does not** change its Boolean value in the loop. Both $K$ and $E$ depend on original inputs $a, b$ and also internal variables being newly declared in the program. Conditions and such variants are put in brackets {..}.

Let $E = \{\ u = (b - v) \cdot a\}$ be a "good" invariant of the program `Prod`, assuming the full format as follows.

$\{(a \geq 1) \land (b \geq 2)\}$
u = 0;
v = b;
$\{u = 0 \land v = b\}$
$E$
**while** ( v != Lb )
{
  $\vdots\ P$
}
$\{u = a \cdot (b - v)\}$

In the following options of core program $P$ and lower bound $Lb$:

(I) $P := [u = u + a;\ v = v - 1;]$ and $Lb := 0$
(II) $P := [u = u + a;\ v = v - 2;]$ and $Lb := 1$
(III) $P := [u = u + a;\ v = v - 1;]$ and $Lb := 1$
(IV) $P := [u = u + a;\ v = v + 1;]$ and $Lb := 1$.

Which one do you choose to obtain the totally correct statement?

$$\vdash_{\text{tot}} \{\phi\}\ \texttt{Prod}\ \{u = a \cdot (b-1)\}?$$

A. (I)

B. (IV)

C. (II)

D. (III)

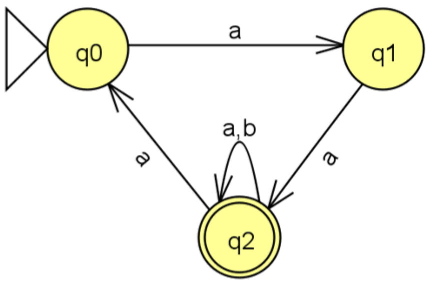
17. (L.O.1.3)

Consider program P below. (Weakest) precondition $\phi$ and postcondition $\psi$ of P are

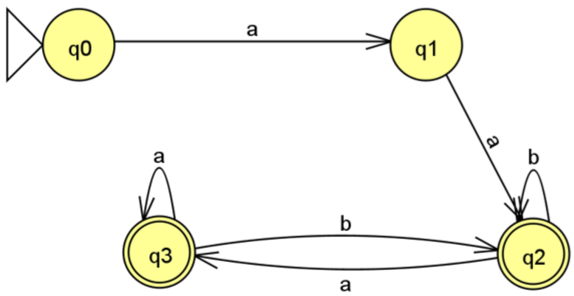
```
x := a;    y := 1;    z := b;
while true do
   if z = 0 then
      return y
   end if
   r := remainder(z, 2);    z := quotient(z, 2);
   if r = 1 then
      y := x * y;
   end if
   x := x * x;
end while
```

A. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{Z}\},\ \psi = \{y = a^b\}$.
B. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{N}\},\ \psi = \{y = a^b\}$.
C. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{Z}^+\},\ \psi = \{y = a^b\}$.
D. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{N}\},\ \psi = \{y = x^z\}$.

18. (L.O.2.3) Given 4 automata where $\Sigma = \{a, b\}$ and $\lambda$ is the empty string.




1

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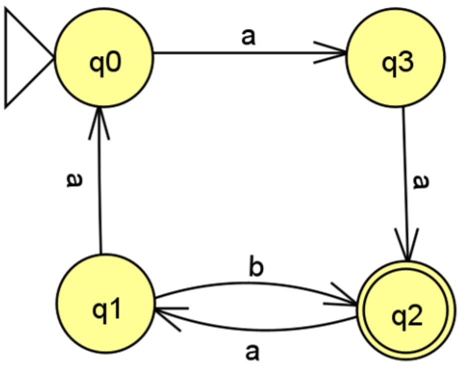


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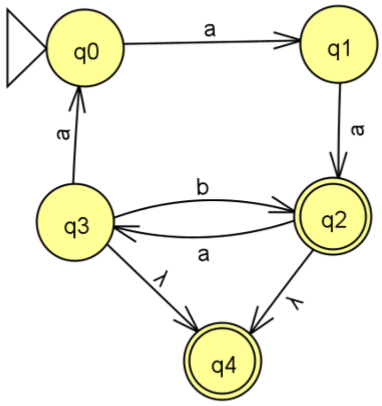


4

Which of the following couples are the same?

A. All the other answers are incorrect.
B. 1,2
C. 3,4
D. 1,3

19. (L.O.2.3) Which of the following is correct?

A. If $L_1$ is a regular language and the language $L_2$ is *not* regular then $L_1^\circ L_2$ is *not* regular.
B. Given two languages $L_1, L_2$, if $L_1 \cup L_2$ is regular then both $L_1, L_2$ are also regular.
C. If $L_1, L_2$, are finite languages then $|L_1 \circ L_2| = |L_1| \cdot |L_2|$.
D. For two languages $L_1, L_2$, we have $L_1^* \circ L_2^* \subseteq (L_1 \cup L_2)^*$.

20. (L.O.1.1) Let $P$ and $Q$ be binary predicates and $S$ be nullary predicates. Which formulas are valid?

(I) $\forall x \forall y (P(x) \to P(y)) \land (P(y) \to P(x))$
(II) $\exists y((\forall x P(x)) \to P(y))$
(III) $(\forall x P(x) \to S) \to \exists x(P(x) \to S)$
(IV) $\forall x(P(x) \lor Q(x)) \to (\forall x P(x)) \lor (\exists x Q(x))$

A. Only (III) and (IV)
B. Only (II), (III), and (IV)
C. Only (I), (II), and (IV)
D. Only (II) and (III)

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . END OF EXAM . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

# Solution 1815

1. B.
2. C.
3. B.
4. B.
5. D.
6. D.
7. B.
8. A.
9. D.
10. D.
11. D.
12. C.
13. B.
14. B.
15. B.
16. D.
17. B.
18. B.
19. D.
20. B.

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|---|---|---|---|---|---|
| | | | Date | 18/10/2023 | |
| | Course title | Mathematical Modeling | | | |
| | Course ID | CO2011 | | | |
| | Duration | 60 mins | Question sheet code | | 1816 |

Notes: - Students do not use course materials except one A4 hand-written sheet.
- Submit the question sheet together with the answer sheet.
- Choose the best answer (only 1) for each question.

1. (L.O.2.3) Let L be the language containing all the palindrome words where $\Sigma = \{a, b\}$, Which of the following regular expressions accepts the words in language L?

A. (a+b)*(b+a)* B. All the other answers are incorrect. C. (abba)*
D. a*b*b*a*

2. Consider a program `Prod` $= Product(a, b)$ calculating the product $a \cdot (b-1)$ of two naturals $a, b$ where $b \geq 2$. Denote the Hoare triple of a suitable core program $P$ to correctly compute `Prod` as

$$\{\phi\}\ P\ \{\psi\}$$

where pre-conditions $\phi := (a \geq 1) \wedge (b \geq 2)$ and post-conditions $\psi$ should be suitably defined. A variant $K$ is an expression whose values are either strictly decreasing or strictly decreasing, and an invariant $E$ **does not** change its Boolean value in the loop. Both $K$ and $E$ depend on original inputs $a, b$ and also internal variables being newly declared in the program. Conditions and such variants are put in brackets {..}.

Let $E = \{\ u = (b - v) \cdot a\}$ be a "good" invariant of the program `Prod`, assuming the full format as follows.

$\{(a \geq 1) \wedge (b \geq 2)\}$
u = 0;
v = b;
$\{u = 0 \wedge v = b\}$
$E$
**while** ( v != Lb )
{
$\vdots\ P$
}
$\{u = a \cdot (b - v)\}$

In the following options of core program $P$ and lower bound $Lb$:
(I) $P := [u = u + a;\ v = v - 1;]$ and $Lb := 0$
(II) $P := [u = u + a;\ v = v - 2;]$ and $Lb := 1$
(III) $P := [u = u + a;\ v = v - 1;]$ and $Lb := 1$
(IV) $P := [u = u + a;\ v = v + 1;]$ and $Lb := 1$.
Which one do you choose to obtain the totally correct statement?

$$\vdash_{\text{tot}} \{\phi\}\ \texttt{Prod}\ \{u = a \cdot (b - 1)\}?$$

A. (IV) B. (I) C. (II) D. (III)

3. (L.O.1.3)
Consider program `P` with precondition and postcondition as in Question 8. Then the invariant form to prove the partial correctness of `P` is

A. $z \in \mathbb{Z} \wedge yx^z = a^b$. B. $z \in \mathbb{Z} \wedge y = x^z$. C. $z \in \mathbb{N} \wedge y = x^z$. D. $z \in \mathbb{N} \wedge yx^z = a^b$.

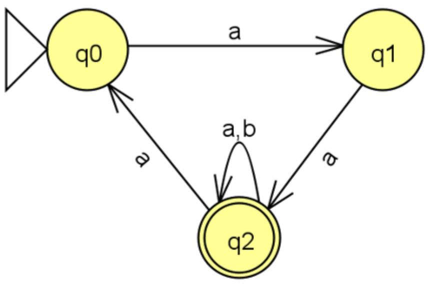
4. (L.O.1.2)
Consider the logical formula $\phi : \forall x \exists y \exists z (P(x, y) \wedge P(z, y) \wedge (P(x, z) \rightarrow P(z, x)))$ and the models $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ such that the universal set is the natural numbers set $\mathbb{N}$, $P^{\mathcal{M}_1} = \{(m, n) | n < m\}$, $P^{\mathcal{M}_2} = \{(m, 2m) | m \in \mathbb{N}\}$ and $P^{\mathcal{M}_3} = \{(m, n) | m < n + 1\}$ Which of following statement is true?

A. $\mathcal{M}_1 \models \phi, \mathcal{M}_2 \not\models \phi, \mathcal{M}_3 \not\models \phi$.
B. $\mathcal{M}_1 \not\models \phi, \mathcal{M}_2 \not\models \phi, \mathcal{M}_3 \models \phi$.
C. $\mathcal{M}_1 \not\models \phi, \mathcal{M}_2 \models \phi, \mathcal{M}_3 \models \phi$.
D. $\mathcal{M}_1 \models \phi, \mathcal{M}_2 \models \phi, \mathcal{M}_3 \models \phi$.

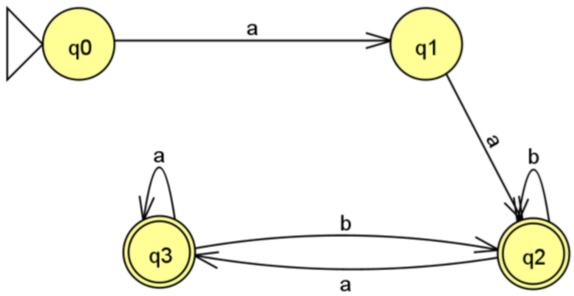
5. (L.O.2.3) Which of the following is correct?

A. Given two languages $L_1, L_2$, if $L_1 \cup L_2$ is regular then both $L_1, L_2$ are also regular.
B. If $L_1$ is a regular language and the language $L_2$ is *not* regular then $L_1^{\circ} L_2$ is *not* regular.
C. If $L_1, L_2$, are finite languages then $|L_1 \circ L_2| = |L_1| \cdot |L_2|$.
D. For two languages $L_1, L_2$, we have $L_1^* \circ L_2^* \subseteq (L_1 \cup L_2)^*$.

6. (L.O.2.3) Given 4 automata where $\Sigma = \{a, b\}$ and $\lambda$ is the empty string.




1

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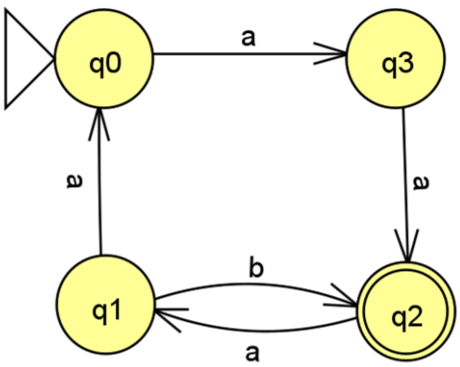


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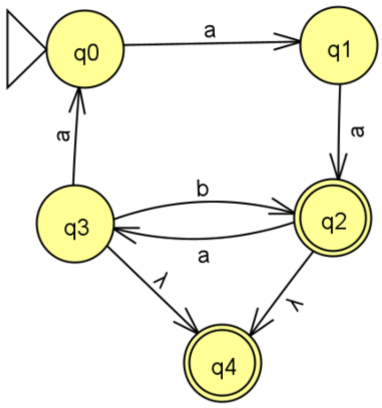



4

Which of the following couples are the same?

A. 1,2
B. All the other answers are incorrect.
C. 3,4
D. 1,3

7. (L.O.1.3)

In the following program, % is the division with remainder and returns the remainder, and function abs() returns the absolute value of an input integer. Consider the precondition $x = x_0 \wedge y = y_0 \wedge (x_0 \neq 0 \vee y_0 \neq 0)$. *Which is the invariant form for proving the partial correctness of the program?* [In the choices, $GCD(a, b)$ denotes the greatest common divisor of two integers $a$ and $b$ that are not both 0, $a$ mod $b$ is the division with the remainder by positive $b$ and $|a|$ is the absolute value of $a$.]

```
while (x != 0) {
        z = x;
        x = y % x;
        y = z;
}
y = abs(y);
```

A. $GCD(x, y) = GCD(x_0, y_0)$
B. All other choices are correct.
C. $x \neq 0 \rightarrow GCD(y, y \bmod |x|) = GCD(x_0, y_0)$
D. $y \neq 0 \rightarrow GCD(x, x \bmod |y|) = GCD(x_0, y_0)$

8. (L.O.1.2)
Which of the following statements is/are true?
(I) All finite languages are regular.
(II) If $L$ is regular so is the reverse language $L^R = \{w^R | w \in L\}$.
(III) $L = \{ww | w \in \{a, b\}^*\}$ is regular.
(IV) If $M = (Q, \Sigma, \delta, q_0, F)$ is a minimal DFA for a regular language $L$ then $\widehat{M} = (Q, \Sigma, \delta, q_0, Q - F)$ is a minimal DFA for $\overline{L}$.

A. (I), (II), (III).
B. (II), (III), (IV).
C. Only (I).
D. (I), (II), (IV).

9. (L.O.1.3) Given a post-condition for a correct program:
$[|\forall q(i < q \leq n \implies a[i] \leq a[q]) \wedge a[i] \leq a[i-1]|]$
Which of the following post-conditions also correct for the same program?

A. $[|\forall q(i - 1 < q \leq n \implies a[i-1] \leq a[q])|]$
B. All the other answers are correct.
C. $[|\forall q(i < q \leq n \implies a[i-1] \leq a[q])|]$
D. $[|\forall q(i \leq q \leq n \implies a[i] \leq a[q])|]$

10. (L.O.2.3) Let $L$ be the language consisting of string $w \in \{0, 1\}^+$ that represents a decimal number divisible by 3. Which choice is CORRECT?

A. All other choices are incorrect.
B. The number of bit 1s of any string of $L$ must be divisible by 3.
C. The number of states of a DFA recognizing only $L$ must be divisible by 3.
D. $L$ is not a regular language.

11. (L.O.1.3)

Given the following program, with $\top$ as a precondition, determine the postcondition yourself. To prove the partial correctness of the corresponding Hoare triple, which of the following is an invariant form we should use?

```
i=0;
s=0;
while(i<=n){
    if(i%2==0)
        s+=i;
    i+=1;
}
```

A. $s = \sum_{k=0}^{(i-1)/2} 2k \wedge (i \leq n+1)$.
B. $s = \sum_{k=0}^{i/2} 2k \wedge (i \leq n+1)$.
C. $s = \sum_{k=0}^{(i-1)/2} 2k$.
D. $s = \sum_{k=0}^{i/2} 2k \wedge (i \leq n)$.

12. Consider the following statement: "`A female athlete won every sports prize and her brother also won some mathematics prize`." We define some predicates below to symbolize the given statement

- $Girl(a)$ = "$a$ is a girl",  $Boy(b)$ = "$b$ is a boy",
- $SportPrize(x)$ = "$x$ is a prize in Sport tournament",
- $MathPrize(y)$ = "$y$ is a prize in Mathematics competition",
- $Sibling(a, b)$ = "$a$ and $b$ are siblings in a family",
- $WinSport(w, s)$ = "$w$ won a sport prize $s$",
- $WinMath(u, m)$ = "$u$ won a mathematics prize $m$".

Which of the following options best represents the given statement?

(I) $\forall x\ [SportPrize(x) \longrightarrow \exists y\ (Girl(y) \wedge WinSport(y, x))]$

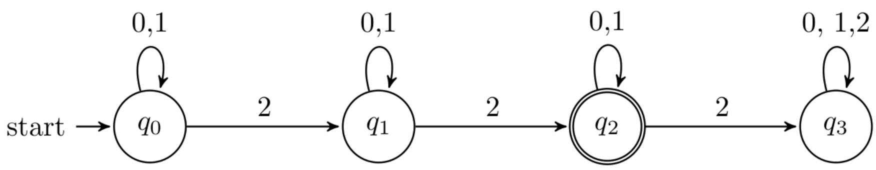
(II) $[\ (\exists f, Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ [\ (\exists b, Sibling(f, b) \wedge Boy(b) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(b, y)]$

(III) $\exists f\ [\ (Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ \exists m\ [\ (Sibling(f, m) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(m, y)\ ]$

(IV) $\exists f\ [\ (Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ \exists m\ [\ (Sibling(f, m) \wedge\ Boy(m) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(m, y)\ ]$

A. (IV)  B. (I)  C. (II)  D. (III)

13. (L.O.2.3) Which of the following languages is accepted by the DFA over $\Sigma = \{0, 1, 2\}$ given below?




A. $L = \{w \in \Sigma^* \mid w$ contains exactly two $2's\}$.
B. $L = \{w \in \Sigma^* \mid w$ contains at least two 2's consecutively$\}$.
C. $L = \{w \in \Sigma^* \mid w$ contains at most two $2's\}$.
D. $L = \{w \in \Sigma^* \mid w$ contains at least two $2's\}$.

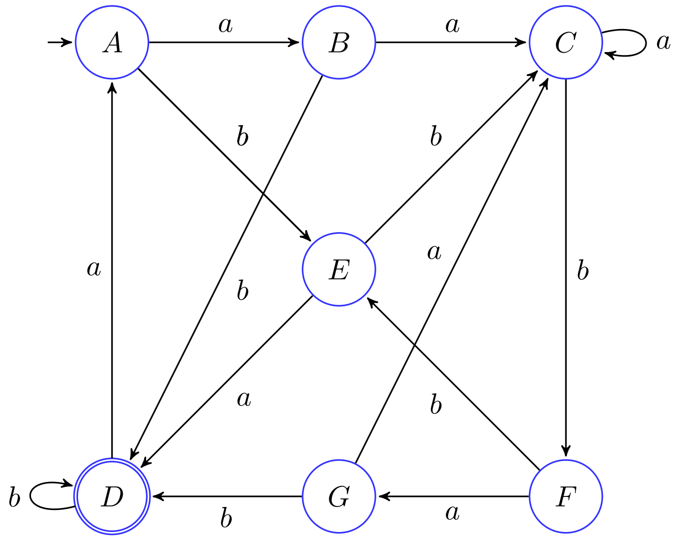
14. (L.O.1.3)

Consider program `P` below. (Weakest) precondition $\phi$ and postcondition $\psi$ of `P` are

```
x := a;   y := 1;   z := b;
while true do
  if z = 0 then
    return y
  end if
  r := remainder(z, 2);   z := quotient(z, 2);
  if r = 1 then
    y := x * y;
  end if
  x := x * x;
end while
```

A. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{N}\},\ \psi = \{y = a^b\}$.
B. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{Z}\},\ \psi = \{y = a^b\}$.
C. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{Z}^+\},\ \psi = \{y = a^b\}$.
D. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{N}\},\ \psi = \{y = x^z\}$.

15. (L.O.3.2)
Consider the following finite automaton:




What is the number of states in a minimal DFA (equivalent to the above FA)?

A. 4. B. 7. C. 6. D. 5.

16. (L.O.2.3) Let $L$ be the language consisting of string $w \in \{0,1\}^*$ that the number of occurrences of sub-string 01 and the number of occurrences of sub-string 10 in $w$ are the same. Which choice is CORRECT?

A. A DFA recognizing only $L$ must have at least five states.
B. All other choices are incorrect.
C. The number of bit 1s and the number of bit 0s of any string of $L$ must be the same.
D. $L$ is not a regular language.

17. (L.O.1.1) Let $P$ and $Q$ be binary predicates and $S$ be nullary predicates. Which formulas are valid?
(I) $\forall x \forall y (P(x) \rightarrow P(y)) \wedge (P(y) \rightarrow P(x))$ (II) $\exists y((\forall x P(x)) \rightarrow P(y))$
(III) $(\forall x P(x) \rightarrow S) \rightarrow \exists x(P(x) \rightarrow S)$ (IV) $\forall x(P(x) \vee Q(x)) \rightarrow (\forall x P(x)) \vee (\exists x Q(x))$

A. Only (II), (III), and (IV) B. Only (III) and (IV)
C. Only (I), (II), and (IV) D. Only (II) and (III)

18. (L.O.1.3) Given a program finding max for 3 integers a,b,c:
max=0;
if(max<a) max=a;
if(max<b) max=b;
if(max<c) max=c;
Which of the following are the weakest pre-condition $\phi$ and post-condition $\psi$ for the program to be correct?

A. $\phi : a, b, c >= 0$, $\psi : a < b < c < max$ B. $\phi : T$, $\psi : a, b, c \leq max$
C. $\phi : a, b, c > 0$, $\psi : a, b, c \leq max$ D. $\phi : a, b, c >= 0$, $\psi : a, b, c \leq max$

*Question 19– Question 20 use the same description as follows:* Denote an alphabet $\Sigma = \{b, c\}$, and let $u = bc$, $v = cb$ be strings of length 2 in $\Sigma^*$. Define $L(r)$ and $L(s)$ respectively be languages determined by regular expressions $r = u^* = (bc)^*$ and $s = v^* = (cb)^*$.

19. The language $L_{bc}$ of all strings with $b$ and $c$ alternating, moreover starting with $b$ and ending with $c$, is

(I) $L(s)$; (II) $L(r) + L(s)$; (III) $L(r)$; (IV) $L(r) \cdot L(s)$.

A. (III) B. (I) C. (II) D. (IV)

20. The language $L$ of all strings with $b$ and $c$ alternating is determined by which of the following regular expressions?

(I) $r + s$; (II) $r + s + c\ r + b\ s$; (III) $r + s + b\ r + c\ s$; (IV) $s + r$.

A. IV B. I C. II D. III

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . END OF EXAM . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

# Solution 1816

1. A.
2. D.
3. D.
4. C.
5. D.
6. A.
7. A.
8. D.
9. D.
10. A.
11. D.
12. A.
13. A.
14. A.
15. D.
16. A.
17. A.
18. B.
19. A.
20. C.

| **Lecturer:** | *October 2nd, 2023* | **Approved by:** | October 2nd, 2023 |
|---|---|---|---|
| *(Signature and Fullname)* | | *(Signature and Fullname)* | |

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|---|---|---|---|---|---|
| | | | Date | | 18/10/2023 |
| | Course title | Mathematical Modeling | | | |
| | Course ID | CO2011 | | | |
| | Duration | 60 mins | Question sheet code | | 1817 |

Notes: - Students do not use course materials except one A4 hand-written sheet.
- Submit the question sheet together with the answer sheet.
- Choose the best answer (only 1) for each question.

1. (L.O.1.3)

Given the following program, with $\top$ as a precondition, determine the postcondition yourself. To prove the partial correctness of the corresponding Hoare triple, which of the following is an invariant form we should use?

```
i=0;
s=0;
while(i<=n){
    if(i%2==0)
        s+=i;
    i+=1;
}
```

A. $s = \sum_{k=0}^{(i-1)/2} 2k \wedge (i \leq n+1)$.
B. $s = \sum_{k=0}^{(i-1)/2} 2k$.
C. $s = \sum_{k=0}^{i/2} 2k \wedge (i \leq n+1)$.
D. $s = \sum_{k=0}^{i/2} 2k \wedge (i \leq n)$.

2. Consider the following statement: "`A female athlete won every sports prize and her brother also won some mathematics prize`." We define some predicates below to symbolize the given statement

- $Girl(a)$ = "$a$ is a girl", $Boy(b)$ = "$b$ is a boy",
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- $Sibling(a, b)$ = "$a$ and $b$ are siblings in a family",
- $WinSport(w, s)$ = "$w$ won a sport prize $s$",
- $WinMath(u, m)$ = "$u$ won a mathematics prize $m$".

Which of the following options best represents the given statement?

(I) $\forall x\ [SportPrize(x) \longrightarrow \exists y\ (Girl(y) \wedge WinSport(y, x))]$

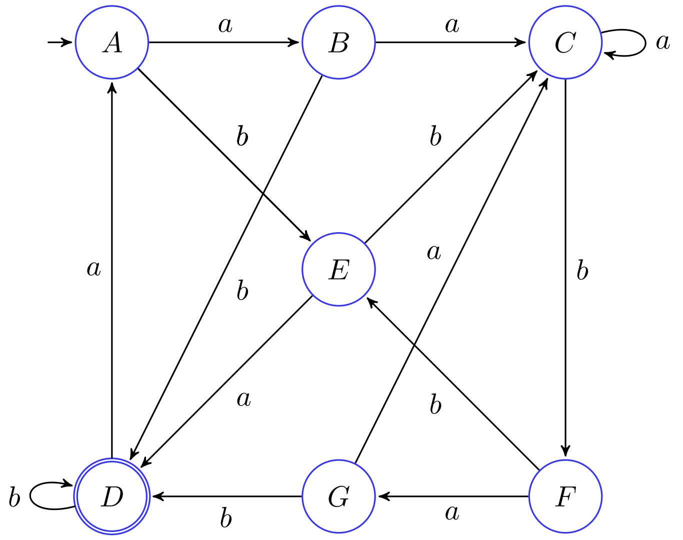
(II) $[\ (\exists f, Girl(f) \wedge \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ [\ (\exists b, Sibling(f, b) \wedge Boy(b) \wedge \exists y, MathPrize(y)) \longrightarrow WinMath(b, y)]$

(III) $\exists f\ [\ (Girl(f) \wedge \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ \exists m\ [\ (Sibling(f, m) \wedge \exists y, MathPrize(y)) \longrightarrow WinMath(m, y)\ ]$

(IV) $\exists f\ [\ (Girl(f) \wedge \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ \exists m\ [\ (Sibling(f, m) \wedge Boy(m) \wedge \exists y, MathPrize(y)) \longrightarrow WinMath(m, y)\ ]$

A. (IV) B. (II) C. (I) D. (III)

3. (L.O.3.2)
Consider the following finite automaton:




What is the number of states in a minimal DFA (equivalent to the above FA)?

A. 4. B. 6. C. 7. D. 5.

4. (L.O.2.3) Let L be the language containing all the palindrome words where $\Sigma = \{a, b\}$, Which of the following regular expressions accepts the words in language L?

A. (a+b)*(b+a)* B. (abba)* C. All the other answers are incorrect.
D. a*b*b*a*

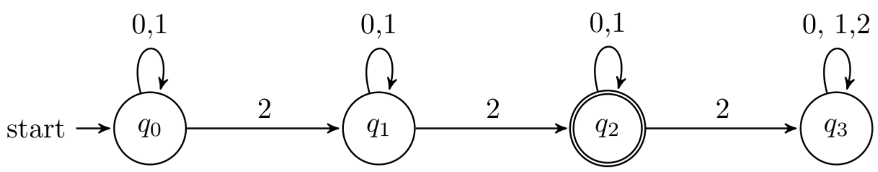
5. (L.O.1.3)

In the following program, % is the division with remainder and returns the remainder, and function abs() returns the absolute value of an input integer. Consider the precondition $x = x_0 \wedge y = y_0 \wedge (x_0 \neq 0 \vee y_0 \neq 0)$. *Which is the invariant form for proving the partial correctness of the program?* [In the choices, $GCD(a, b)$ denotes the greatest common divisor of two integers $a$ and $b$ that are not both 0, $a$ mod $b$ is the division with the remainder by positive $b$ and $|a|$ is the absolute value of $a$.]

```
while (x != 0) {
        z = x;
        x = y % x;
        y = z;
}
y = abs(y);
```

A. $GCD(x, y) = GCD(x_0, y_0)$ B. $x \neq 0 \rightarrow GCD(y, y \bmod |x|) = GCD(x_0, y_0)$
C. All other choices are correct. D. $y \neq 0 \rightarrow GCD(x, x \bmod |y|) = GCD(x_0, y_0)$

6. (L.O.2.3) Which of the following languages is accepted by the DFA over $\Sigma = \{0, 1, 2\}$ given below?




A. $L = \{w \in \Sigma^* \mid w$ contains exactly two $2's\}$. B. $L = \{w \in \Sigma^* \mid w$ contains at most two $2's\}$.
C. $L = \{w \in \Sigma^* \mid w$ contains at least two 2's consecutively$\}$.
D. $L = \{w \in \Sigma^* \mid w$ contains at least two $2's\}$.

7. (L.O.1.3)
Consider program `P` with precondition and postcondition as in Question 8. Then the invariant form to prove the partial correctness of `P` is

A. $z \in \mathbb{Z} \wedge yx^z = a^b$. B. $z \in \mathbb{N} \wedge y = x^z$. C. $z \in \mathbb{Z} \wedge y = x^z$. D. $z \in \mathbb{N} \wedge yx^z = a^b$.

8. (L.O.1.1) Let $P$ and $Q$ be binary predicates and $S$ be nullary predicates. Which formulas are valid?

(I) $\forall x \forall y (P(x) \rightarrow P(y)) \wedge (P(y) \rightarrow P(x))$ (II) $\exists y ((\forall x P(x)) \rightarrow P(y))$

(III) $(\forall x P(x) \rightarrow S) \rightarrow \exists x (P(x) \rightarrow S)$ (IV) $\forall x (P(x) \vee Q(x)) \rightarrow (\forall x P(x)) \vee (\exists x Q(x))$

A. Only (II), (III), and (IV)
B. Only (I), (II), and (IV)
C. Only (III) and (IV)
D. Only (II) and (III)

9. (L.O.2.3) Let $L$ be the language consisting of string $w \in \{0,1\}^*$ that the number of occurrences of sub-string 01 and the number of occurrences of sub-string 10 in $w$ are the same. Which choice is CORRECT?

A. A DFA recognizing only $L$ must have at least five states.
B. The number of bit 1s and the number of bit 0s of any string of $L$ must be the same.
C. All other choices are incorrect.
D. $L$ is not a regular language.

10. (L.O.1.3)

Consider program P below. (Weakest) precondition $\phi$ and postcondition $\psi$ of P are

```
x := a;    y := 1;    z := b;
while true do
  if z = 0 then
    return y
  end if
  r := remainder(z, 2);    z := quotient(z, 2);
  if r = 1 then
    y := x * y;
  end if
  x := x * x;
end while
```

A. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{N}\},\ \psi = \{y = a^b\}$.
B. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{Z}^+\},\ \psi = \{y = a^b\}$.
C. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{Z}\},\ \psi = \{y = a^b\}$.
D. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{N}\},\ \psi = \{y = x^z\}$.

11. (L.O.1.3) Given a post-condition for a correct program:
$[|\forall q (i < q \leq n \implies a[i] \leq a[q]) \wedge a[i] \leq a[i-1]|]$
Which of the following post-conditions also correct for the same program?

A. $[|\forall q (i-1 < q \leq n \implies a[i-1] \leq a[q])|]$
B. $[|\forall q (i < q \leq n \implies a[i-1] \leq a[q])|]$
C. All the other answers are correct.
D. $[|\forall q (i \leq q \leq n \implies a[i] \leq a[q])|]$

12. (L.O.2.3) Let $L$ be the language consisting of string $w \in \{0,1\}^+$ that represents a decimal number divisible by 3. Which choice is CORRECT?

A. All other choices are incorrect.
B. The number of states of a DFA recognizing only $L$ must be divisible by 3.
C. The number of bit 1s of any string of $L$ must be divisible by 3.
D. $L$ is not a regular language.

13. (L.O.1.3) Given a program finding max for 3 integers a,b,c:
max=0;
if(max<a) max=a;
if(max<b) max=b;
if(max<c) max=c;
Which of the following are the weakest pre-condition $\phi$ and post-condition $\psi$ for the program to be correct?

A. $\phi : a, b, c >= 0,\ \psi : a < b < c < max$
B. $\phi : a, b, c > 0,\ \psi : a, b, c \leq max$
C. $\phi : T,\ \psi : a, b, c \leq max$
D. $\phi : a, b, c >= 0,\ \psi : a, b, c \leq max$

*Question 14– Question 15 use the same description as follows:* Denote an alphabet $\Sigma = \{b, c\}$, and let $u = bc$, $v = cb$ be strings of length 2 in $\Sigma^*$. Define $L(r)$ and $L(s)$ respectively be languages determined by regular expressions $r = u^* = (bc)^*$ and $s = v^* = (cb)^*$.

14. The language $L_{bc}$ of all strings with $b$ and $c$ alternating, moreover starting with $b$ and ending with $c$, is

$$\text{(I) } L(s); \qquad \text{(II) } L(r) + L(s); \qquad \text{(III) } L(r); \qquad \text{(IV) } L(r) \cdot L(s).$$

A. (III) B. (II) C. (I) D. (IV)

15. The language $L$ of all strings with $b$ and $c$ alternating is determined by which of the following regular expressions?

$$\text{(I) } r + s; \qquad \text{(II) } r + s + c\ r + b\ s; \qquad \text{(III) } r + s + b\ r + c\ s; \qquad \text{(IV) } s + r.$$

A. IV B. II C. I D. III

16. (L.O.2.3) Which of the following is correct?

A. Given two languages $L_1, L_2$, if $L_1 \cup L_2$ is regular then both $L_1, L_2$ are also regular.

B. If $L_1, L_2$, are finite languages then $|L_1 \circ L_2| = |L_1| \cdot |L_2|$.

C. If $L_1$ is a regular language and the language $L_2$ is *not* regular then $L_1^{\circ} L_2$ is *not* regular.

D. For two languages $L_1, L_2$, we have $L_1^* \circ L_2^* \subseteq (L_1 \cup L_2)^*$.

17. Consider a program `Prod` $= Product(a, b)$ calculating the product $a \cdot (b - 1)$ of two naturals $a, b$ where $b \geq 2$. Denote the Hoare triple of a suitable core program $P$ to correctly compute `Prod` as

$$\{\phi\}\ P\ \{\psi\}$$

where pre-conditions $\phi := (a \geq 1) \wedge (b \geq 2)$ and post-conditions $\psi$ should be suitably defined. A variant $K$ is an expression whose values are either strictly decreasing or strictly decreasing, and an invariant $E$ **does not** change its Boolean value in the loop. Both $K$ and $E$ depend on original inputs $a, b$ and also internal variables being newly declared in the program. Conditions and such variants are put in brackets {..}.

Let $E = \{\ u = (b - v) \cdot a\}$ be a "good" invariant of the program `Prod`, assuming the full format as follows.

```
{(a ≥ 1) ∧ (b ≥ 2)}
u = 0;
v = b;
{u = 0 ∧ v = b}
E
while ( v != Lb )
{
   ⋮ P
}
{u = a · (b − v)}
```

In the following options of core program $P$ and lower bound $Lb$:

(I) $P := [u = u + a;\ v = v - 1;]$ and $Lb := 0$

(II) $P := [u = u + a;\ v = v - 2;]$ and $Lb := 1$

(III) $P := [u = u + a;\ v = v - 1;]$ and $Lb := 1$

(IV) $P := [u = u + a;\ v = v + 1;]$ and $Lb := 1$.

Which one do you choose to obtain the totally correct statement?

$$\vdash_{\text{tot}} \{\phi\}\ \texttt{Prod}\ \{u = a \cdot (b - 1)\}?$$

A. (IV) B. (II) C. (I) D. (III)

18. (L.O.1.2)
Consider the logical formula $\phi : \forall x \exists y \exists z (P(x, y) \wedge P(z, y) \wedge (P(x, z) \rightarrow P(z, x)))$ and the models $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ such that the universal set is the natural numbers set $\mathbb{N}$, $P^{\mathcal{M}_1} = \{(m, n) | n < m\}$, $P^{\mathcal{M}_2} = \{(m, 2m) | m \in \mathbb{N}\}$ and $P^{\mathcal{M}_3} = \{(m, n) | m < n + 1\}$ Which of following statement is true?

A. $\mathcal{M}_1 \models \phi, \mathcal{M}_2 \not\models \phi, \mathcal{M}_3 \not\models \phi$.

B. $\mathcal{M}_1 \not\models \phi, \mathcal{M}_2 \models \phi, \mathcal{M}_3 \models \phi$.

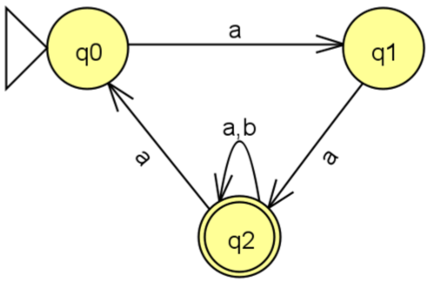
C. $\mathcal{M}_1 \not\models \phi, \mathcal{M}_2 \not\models \phi, \mathcal{M}_3 \models \phi$.

D. $\mathcal{M}_1 \models \phi, \mathcal{M}_2 \models \phi, \mathcal{M}_3 \models \phi$.

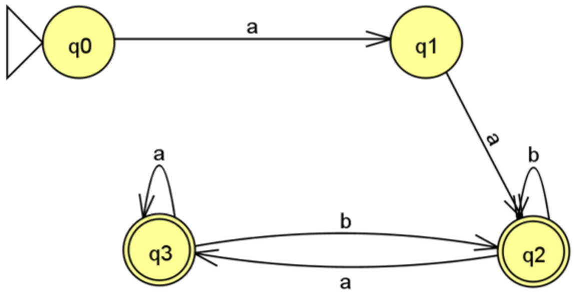
19. (L.O.1.2)
Which of the following statements is/are true?
(I) All finite languages are regular.
(II) If $L$ is regular so is the reverse language $L^R = \{w^R | w \in L\}$.
(III) $L = \{ww | w \in \{a, b\}^*\}$ is regular.
(IV) If $M = (Q, \Sigma, \delta, q_0, F)$ is a minimal DFA for a regular language $L$ then $\widehat{M} = (Q, \Sigma, \delta, q_0, Q - F)$ is a minimal DFA for $\overline{L}$.

A. (I), (II), (III). B. Only (I). C. (II), (III), (IV). D. (I), (II), (IV).

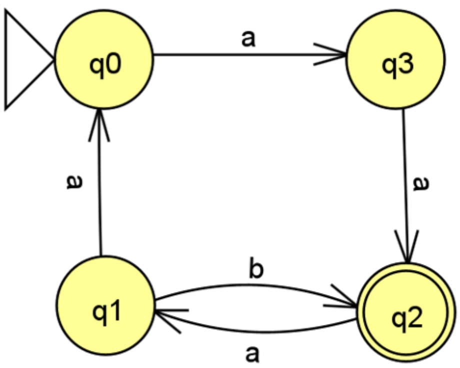
20. (L.O.2.3) Given 4 automata where $\Sigma = \{a, b\}$ and $\lambda$ is the empty string.



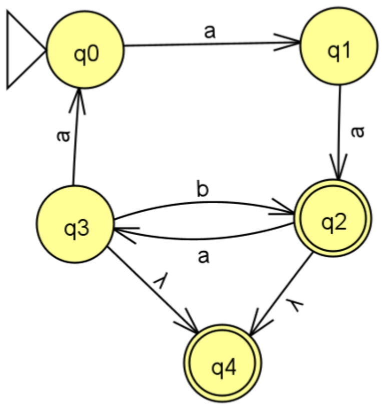

1




2




3




4

Which of the following couples are the same?

A. 1,2 B. 3,4 C. All the other answers are incorrect. D. 1,3

............................................END OF EXAM............................................

# Solution 1817

1. D.
2. A.
3. D.
4. A.
5. A.
6. A.
7. D.
8. A.
9. A.
10. A.
11. D.
12. A.
13. C.
14. A.
15. B.
16. D.
17. D.
18. B.
19. D.
20. A.

| **Lecturer:** | *October 2nd, 2023* | **Approved by:** | October 2nd, 2023 |
|---|---|---|---|
| *(Signature and Fullname)* | | *(Signature and Fullname)* | |

| UNIVERSITY OF TECHNOLOGY FACULTY OF CSE | MIDTERM | | Semester/Academic year | 1 | 2023-2024 |
|---|---|---|---|---|---|
| | | | Date | | 18/10/2023 |
| | Course title | Mathematical Modeling | | | |
| | Course ID | CO2011 | | | |
| | Duration | 60 mins | Question sheet code | | 1818 |

Notes: - Students do not use course materials except one A4 hand-written sheet.
- Submit the question sheet together with the answer sheet.
- Choose the best answer (only 1) for each question.

1. (L.O.2.3) Let L be the language containing all the palindrome words where $\Sigma = \{a, b\}$, Which of the following regular expressions accepts the words in language L?

A. a*b*b*a* B. (a+b)*(b+a)* C. (abba)*
D. All the other answers are incorrect.

2. Consider the following statement: "`A female athlete won every sports prize and her brother also won some mathematics prize`." We define some predicates below to symbolize the given statement

- $Girl(a)$ = "$a$ is a girl", $Boy(b)$ = "$b$ is a boy",
- $SportPrize(x)$ = "$x$ is a prize in Sport tournament",
- $MathPrize(y)$ = "$y$ is a prize in Mathematics competition",
- $Sibling(a, b)$ = "$a$ and $b$ are siblings in a family",
- $WinSport(w, s)$ = "$w$ won a sport prize $s$",
- $WinMath(u, m)$ = "$u$ won a mathematics prize $m$".

Which of the following options best represents the given statement?

(I) $\forall x\ [SportPrize(x) \longrightarrow \exists y\ (Girl(y) \wedge WinSport(y, x))]$

(II) $[\ (\exists f, Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ [\ (\exists b, Sibling(f, b) \wedge Boy(b) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(b, y)]$

(III) $\exists f\ [\ (Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ \exists m\ [\ (Sibling(f, m) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(m, y)\ ]$

(IV) $\exists f\ [\ (Girl(f) \wedge\ \forall x, SportPrize(x)) \longrightarrow WinSport(f, x)\ ]$
$\wedge\ \exists m\ [\ (Sibling(f, m) \wedge\ Boy(m) \wedge\ \exists y, MathPrize(y)) \longrightarrow WinMath(m, y)\ ]$

A. (III) B. (IV) C. (II) D. (I)

3. (L.O.1.3)

Given the following program, with $\top$ as a precondition, determine the postcondition yourself. To prove the partial correctness of the corresponding Hoare triple, which of the following is an invariant form we should use?

```
i=0;
s=0;
while(i<=n){
    if(i%2==0)
        s+=i;
    i+=1;
}
```

A. $s = \sum_{k=0}^{i/2} 2k \wedge (i \leq n)$.

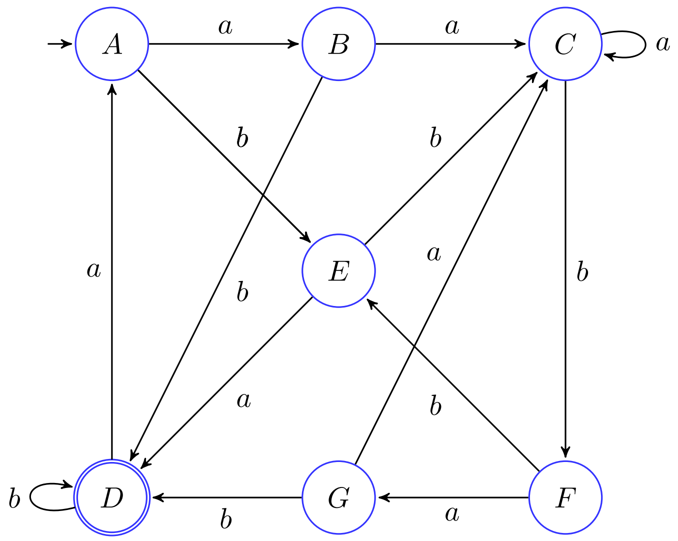
B. $s = \sum_{k=0}^{(i-1)/2} 2k \wedge (i \leq n+1)$.

C. $s = \sum_{k=0}^{(i-1)/2} 2k$.

D. $s = \sum_{k=0}^{i/2} 2k \wedge (i \leq n+1)$.

4. (L.O.3.2)

Consider the following finite automaton:




What is the number of states in a minimal DFA (equivalent to the above FA)?

A. 5.

B. 4.

C. 6.

D. 7.

5. Consider a program `Prod` $= Product(a, b)$ calculating the product $a \cdot (b-1)$ of two naturals $a, b$ where $b \geq 2$. Denote the Hoare triple of a suitable core program $P$ to correctly compute `Prod` as

$$\{\phi\}\ P\ \{\psi\}$$

where pre-conditions $\phi := (a \geq 1) \wedge (b \geq 2)$ and post-conditions $\psi$ should be suitably defined. A variant $K$ is an expression whose values are either strictly decreasing or strictly decreasing, and an invariant $E$ **does not** change its Boolean value in the loop. Both $K$ and $E$ depend on original inputs $a, b$ and also internal variables being newly declared in the program. Conditions and such variants are put in brackets $\{..\}$.

Let $E = \{\ u = (b-v) \cdot a\}$ be a "good" invariant of the program `Prod`, assuming the full format as follows.

```
{(a ≥ 1) ∧ (b ≥ 2)}
u = 0;
v = b;
{u = 0 ∧ v = b}
E
while ( v != Lb )
{
    ⋮ P
}
{u = a · (b − v)}
```

In the following options of core program $P$ and lower bound $Lb$:

(I) $P := [u = u + a;\ v = v - 1;]$ and $Lb := 0$

(II) $P := [u = u + a;\ v = v - 2;]$ and $Lb := 1$

(III) $P := [u = u + a;\ v = v - 1;]$ and $Lb := 1$

(IV) $P := [u = u + a;\ v = v + 1;]$ and $Lb := 1$.

Which one do you choose to obtain the totally correct statement?

$$\vdash_{\text{tot}} \{\phi\}\ \texttt{Prod}\ \{u = a \cdot (b-1)\}?$$

A. (III) B. (IV) C. (II) D. (I)

6. (L.O.1.3)

Consider program `P` below. (Weakest) precondition $\phi$ and postcondition $\psi$ of `P` are

```
x := a;   y := 1;   z := b;
while true do
  if z = 0 then
    return y
  end if
  r := remainder(z, 2);   z := quotient(z, 2);
  if r = 1 then
    y := x * y;
  end if
  x := x * x;
end while
```

A. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{N}\},\ \psi = \{y = x^z\}$.

B. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{N}\},\ \psi = \{y = a^b\}$.

C. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{Z}^+\},\ \psi = \{y = a^b\}$.

D. $\phi = \{a \in \mathbb{R},\ b \in \mathbb{Z}\},\ \psi = \{y = a^b\}$.

7. (L.O.1.2)
Which of the following statements is/are true?
(I) All finite languages are regular.
(II) If $L$ is regular so is the reverse language $L^R = \{w^R | w \in L\}$.
(III) $L = \{ww | w \in \{a, b\}^*\}$ is regular.
(IV) If $M = (Q, \Sigma, \delta, q_0, F)$ is a minimal DFA for a regular language $L$ then $\widehat{M} = (Q, \Sigma, \delta, q_0, Q - F)$ is a minimal DFA for $\overline{L}$.

A. (I), (II), (IV). B. (I), (II), (III). C. Only (I). D. (II), (III), (IV).

8. (L.O.1.3)
Consider program `P` with precondition and postcondition as in Question 8. Then the invariant form to prove the partial correctness of `P` is

A. $z \in \mathbb{N} \wedge yx^z = a^b$. B. $z \in \mathbb{Z} \wedge yx^z = a^b$. C. $z \in \mathbb{N} \wedge y = x^z$. D. $z \in \mathbb{Z} \wedge y = x^z$.

9. (L.O.1.1) Let $P$ and $Q$ be binary predicates and $S$ be nullary predicates. Which formulas are valid?
(I) $\forall x \forall y(P(x) \to P(y)) \wedge (P(y) \to P(x))$ (II) $\exists y((\forall x P(x)) \to P(y))$
(III) $(\forall x P(x) \to S) \to \exists x(P(x) \to S)$ (IV) $\forall x(P(x) \vee Q(x)) \to (\forall x P(x)) \vee (\exists x Q(x))$

A. Only (II) and (III) B. Only (II), (III), and (IV)
C. Only (I), (II), and (IV) D. Only (III) and (IV)

10. (L.O.1.3) Given a program finding max for 3 integers a,b,c:
max=0;
if(max<a) max=a;
if(max<b) max=b;
if(max<c) max=c;
Which of the following are the weakest pre-condition $\phi$ and post-condition $\psi$ for the program to be correct?

A. $\phi : a, b, c >= 0$, $\psi : a, b, c \leq max$ B. $\phi : a, b, c >= 0$, $\psi : a < b < c < max$
C. $\phi : a, b, c > 0$, $\psi : a, b, c \leq max$ D. $\phi : T$, $\psi : a, b, c \leq max$

11. (L.O.2.3) Let $L$ be the language consisting of string $w \in \{0, 1\}^*$ that the number of occurrences of sub-string 01 and the number of occurrences of sub-string 10 in $w$ are the same. Which choice is CORRECT?

A. $L$ is not a regular language.
B. A DFA recognizing only $L$ must have at least five states.
C. The number of bit 1s and the number of bit 0s of any string of $L$ must be the same.
D. All other choices are incorrect.

*Question 12– Question 13 use the same description as follows:* Denote an alphabet $\Sigma = \{b, c\}$, and let $u = bc$, $v = cb$ be strings of length 2 in $\Sigma^*$. Define $L(r)$ and $L(s)$ respectively be languages determined by regular expressions $r = u^* = (bc)^*$ and $s = v^* = (cb)^*$.

12. The language $L_{bc}$ of all strings with $b$ and $c$ alternating, moreover starting with $b$ and ending with $c$, is

(I) $L(s)$; (II) $L(r) + L(s)$; (III) $L(r)$; (IV) $L(r) \cdot L(s)$.

A. (IV) B. (III) C. (II) D. (I)

13. The language $L$ of all strings with $b$ and $c$ alternating is determined by which of the following regular expressions?

(I) $r + s$; (II) $r + s + c\ r + b\ s$; (III) $r + s + b\ r + c\ s$; (IV) $s + r$.

A. III B. IV C. II D. I

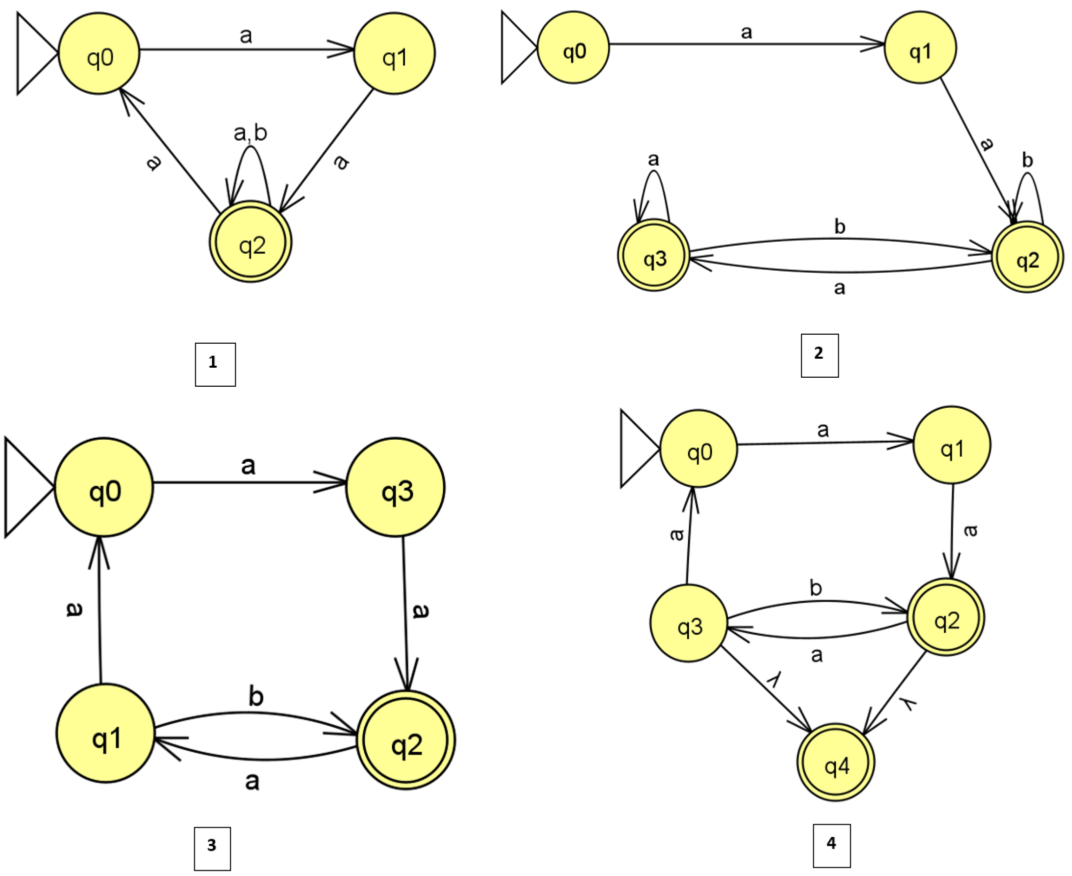
14. (L.O.2.3) Which of the following is correct?

A. For two languages $L_1, L_2$, we have $L_1^* \circ L_2^* \subseteq (L_1 \cup L_2)^*$.
B. Given two languages $L_1, L_2$, if $L_1 \cup L_2$ is regular then both $L_1, L_2$ are also regular.
C. If $L_1, L_2$, are finite languages then $|L_1 \circ L_2| = |L_1| \cdot |L_2|$.
D. If $L_1$ is a regular language and the language $L_2$ is *not* regular then $L_1^\circ L_2$ is *not* regular.

15. (L.O.1.2)
Consider the logical formula $\phi : \forall x \exists y \exists z (P(x, y) \wedge P(z, y) \wedge (P(x, z) \rightarrow P(z, x)))$ and the models $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ such that the universal set is the natural numbers set $\mathbb{N}$, $P^{\mathcal{M}_1} = \{(m, n) | n < m\}$, $P^{\mathcal{M}_2} = \{(m, 2m) | m \in \mathbb{N}\}$ and $P^{\mathcal{M}_3} = \{(m, n) | m < n + 1\}$ Which of following statement is true?

A. $\mathcal{M}_1 \models \phi, \mathcal{M}_2 \models \phi, \mathcal{M}_3 \models \phi$.
B. $\mathcal{M}_1 \models \phi, \mathcal{M}_2 \not\models \phi, \mathcal{M}_3 \not\models \phi$.
C. $\mathcal{M}_1 \not\models \phi, \mathcal{M}_2 \models \phi, \mathcal{M}_3 \models \phi$.
D. $\mathcal{M}_1 \not\models \phi, \mathcal{M}_2 \not\models \phi, \mathcal{M}_3 \models \phi$.

16. (L.O.2.3) Given 4 automata where $\Sigma = \{a, b\}$ and $\lambda$ is the empty string.



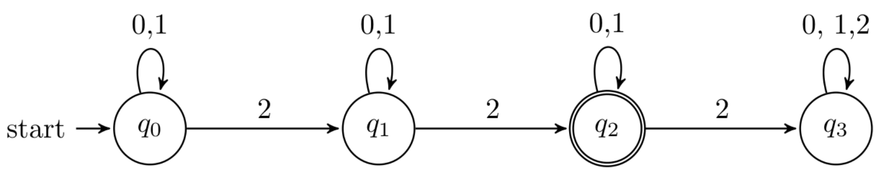

Which of the following couples are the same?

A. 1,3
B. 1,2
C. 3,4
D. All the other answers are incorrect.

17. (L.O.1.3) Given a post-condition for a correct program:
$[|\forall q(i < q \leq n \implies a[i] \leq a[q]) \wedge a[i] \leq a[i-1]|]$
Which of the following post-conditions also correct for the same program?

A. $[|\forall q(i \leq q \leq n \implies a[i] \leq a[q])|]$
B. $[|\forall q(i - 1 < q \leq n \implies a[i-1] \leq a[q])|]$
C. $[|\forall q(i < q \leq n \implies a[i-1] \leq a[q])|]$
D. All the other answers are correct.

18. (L.O.2.3) Which of the following languages is accepted by the DFA over $\Sigma = \{0, 1, 2\}$ given below?




A. $L = \{w \in \Sigma^* \mid w \text{ contains at least two } 2's\}$.
B. $L = \{w \in \Sigma^* \mid w \text{ contains exactly two } 2's\}$.
C. $L = \{w \in \Sigma^* \mid w \text{ contains at most two } 2's\}$.
D. $L = \{w \in \Sigma^* \mid w \text{ contains at least two 2's consecutively}\}$.

19. (L.O.1.3)

In the following program, % is the division with remainder and returns the remainder, and function abs() returns the absolute value of an input integer. Consider the precondition $x = x_0 \land y = y_0 \land (x_0 \neq 0 \lor y_0 \neq 0)$. *Which is the invariant form for proving the partial correctness of the program?* [In the choices, $GCD(a, b)$ denotes the greatest common divisor of two integers $a$ and $b$ that are not both 0, $a$ mod $b$ is the division with the remainder by positive $b$ and $|a|$ is the absolute value of $a$.]

```
while (x != 0) {
        z = x;
        x = y % x;
        y = z;
}
y = abs(y);
```

A. $y \neq 0 \rightarrow GCD(x, x \bmod |y|) = GCD(x_0, y_0)$
B. $GCD(x, y) = GCD(x_0, y_0)$
C. $x \neq 0 \rightarrow GCD(y, y \bmod |x|) = GCD(x_0, y_0)$
D. All other choices are correct.

20. (L.O.2.3) Let $L$ be the language consisting of string $w \in \{0, 1\}^+$ that represents a decimal number divisible by 3. Which choice is CORRECT?

A. $L$ is not a regular language.
B. All other choices are incorrect.
C. The number of states of a DFA recognizing only $L$ must be divisible by 3.
D. The number of bit 1s of any string of $L$ must be divisible by 3.

..............................................END OF EXAM..............................................

# Solution 1818

1. B.
2. B.
3. A.
4. A.
5. A.
6. B.
7. A.
8. A.
9. B.
10. D.
11. B.
12. B.
13. C.
14. A.
15. C.
16. B.
17. A.
18. B.
19. B.
20. B.