Title of Report

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Faculty Mentors: Mentor 1[6], Mentor 2[7]

Abstract

- Summarize the results presented in the report, and the contributions of your research. This is a test.
- Readers should not have to look at the rest of the paper in order to understand the abstract.
- Keep it short and to the point.

1 Introduction

Bathymetry is a measurement of submarine topography and can be used to understand shifts of the ocean floor and its depth. Knowledge of bathymetry is important for marine navigation, both civilian and military, as well as for monitoring the effects of storms on coastal environments. While direct measurement of bathymetry is possible, the process tends to be cost and time prohibitive. For example, amphibious vehicles (see Figure 2) are capable of spatially limited surveys of bathymetry in difficult surf-zone conditions but require significant resources to operate. Resulting surveys tend to be sparse in time as well due to these considerations.

More desirable would be a method to estimate bathymetry using surface measurements collected via remote sensing such as airborne and satellite platforms. While bathymetry data is currently sparse due to observational limitations, the physics of waves are reasonably well understood. In particular, a dispersion relationship can be used to relate water depth to surface properties such as wave length and wave period. It is therefore possible to estimate bathymetry given observations of the water surface. Light Detection And Ranging (LIDAR) has been used to determine wave heights and Argus land-mounted video has been analyzed photogrammetrically to determine wave frequency and wave number. Both of these sources therefore provide valuable inputs for estimating coastal bathymetry in a more efficient manner than is currently available.

Both wave and bathymetric data has been collected in Duck, NC by the U.S. Army Corps of Engineers Coastal and Hydraulics Laboratory, including in situ measurements of bathymetry and measurements of the water surface. This is a useful case for testing algorithms to invert for bathymetry because the true bathymetry is available for comparison to numerical estimates.

2 The Problem

Although there have been uncertainties in capturing the topography of the ocean nearshore, mathematical methods could prove to be possible solutions to this problem using the dispersion relationship between wavelength and the period. Stockdon and Holman used video imagery, which compared true wave signal and remotely sensed video signal to create a linear representation between wave amplitudes and phases. Holman used a 2-dimenisional method with Kalman filtering to estimate the depth, h.

Our research will compute the wave depth using wave length and wave number with a 1D model derived from using the energy flux method to create a correlation between the wave length and the wave depth from the surface. ¡¡¡¡¡¡¡ HEAD

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2.1 The Problem

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2.2 Forward Problem

¿¿¿¿¿¿¿ d803194192f48b5ea85a4e33663dee36530ee249 We consider the following models for the forward problem.

$$\begin{cases} \frac{d}{dx}\left(EC_g\right) = \delta, \\ \\ \sigma^2 = gk\tanh(kh), \end{cases} \tag{1}$$

¡¡¡¡¡¡¡ HEAD where the speed at which the energy is transmitted, C_g, called linear theory group speed, which is given as ======= where the speed at which the energy is transmitted, C_g, called linear theory group speed is given as ¿¿¿¿¿¿¿ d803194192f48b5ea85a4e33663dee36530ee249

$$C_g = \frac{c}{2}\left(1 + \frac{2kh}{\sinh(2kh)}\right), \tag{2}$$

with $c = \frac{\sigma}{k}$ and from the linear theory, the wave energy, E is given as

$$E = \frac{1}{8}\rho g H^2, \tag{3}$$

ρ is the density of water, g is the gravitational acceleration, σ is the angular frequency, c is the local wave phase speed, k is the wave number, h is the water depth and H is the wave height. Here, we assume T, the wave period, is constant.

Observe that equation 1 is coupled using the fact that $\sigma = \frac{2\pi}{T}$. Hence, using equation 2 and 3 in 1, we obtain

$$<<<<<<< HEAD =======>>>>>>> d803194192f48b5ea85a4e33663dee36530ee249 \frac{d}{dx}\left(\frac{\lambda}{k}\left(1 + \frac{2kh}{\sinh(2kh)}\right)H^2\right) = \tag{4}$$

and

$$f(k) = gk\tanh(kh) - \sigma^2, \tag{5}$$

where k is the zero of function f and $\lambda = \frac{\rho g \pi}{8T}$.
The wave breaking, δ, proposed by (Janssen and Battjes, 2007) given as

$$\delta = \frac{1}{4h}B\rho g f H_{rms}\left[(R^3 + \frac{3}{2}R)e^{-R^2} + \frac{3}{4}\sqrt{\pi}(1 - erf(R))\right], \tag{6}$$

¡¡¡¡¡¡¡ HEAD where

$$R = \frac{H_b}{H_{rms}}, \quad H_b = 2\sqrt{2}H_{m0}, \quad H_b = 0.78h, \quad f = \frac{1}{T}, \quad B = 1.$$

2.3 Numerical Solution of the Forward Model

Finite difference scheme is applied to obtain numerical solution with appropriate initial and boundary conditions. In the scheme of finite differences, the derivatives are replaced using their finite-difference approximations. The goal is to provide wave height using the finite difference method. In the process, MATLAB fsolve function is applied to obtain the wave number. Furthermore, Newton-Raphson method is used to verify solution obtained for the wave number.

2.3.1 Discretization of the Model

Forward difference is applied depending on the nature of the model given in equation (4). The ordinary differential equation of the model applied in the method is as follows

$$\frac{d}{dx}\left(c^{*}H^{2}\right) = -\delta \tag{7}$$

where

$$c^{*} = \frac{c_g\lambda}{k}, \quad c_g = nc = \frac{2\pi n}{T},$$

with

$$\lambda = \frac{1}{8}\rho g, \quad n = \frac{1}{2}\left(1 + \frac{2hk}{sinh(2hk)}\right).$$

For implementation purposes, we let

$$\tilde{H} = H^{2}.$$

Thus, the required expression becomes

$$\frac{d}{dx}\left(c^{*}\tilde{H}\right) = -\delta$$

Applying the finite difference method the above expression becomes

$$\frac{c_i^{*}\tilde{H}_i - c_{i-1}^{*}\tilde{H}_{i-1}}{\triangle x} = \delta$$

$$\tilde{H}_i = \frac{c_{i-1}^{*}\tilde{H}_{i-1} + \triangle x\delta}{c_i^{*}}$$

After determining $\tilde{H}$ we take the square root on that to determine H at each index point in the discretization.

$$H = \sqrt{\tilde{H}}$$

fsolve

Wave numbers are generated at each index points applying MATLAB function fsolve. fsolve is applied in the dispersion relationship

$$\sigma^{2} = gktanh(kh)$$

MATLAB function fsolve is used as non-linear solver. In this case $f(k) = gktanh(kh) - \sigma^{2}$ is provided as function handle in the input argument. Together to this, $\sigma/\sqrt{gh}$ is used as initial condition in fsolve.

Newton-Raphson Method

Newton-Raphson method is an widely used method for root finding. This method is used in the numerical experiment to verify the wave number extracted using fsolve. In Newton -Raphson method roots are calculated using the following expression

$$k_{i+1} = k_i - \frac{f(k_i)}{f'(k_i)}$$

This provides the wave number in each index.

2.3.2 Numerical Results

======= where

$$R = \frac{H_b}{H_{rms}}, \quad H_b = 2\sqrt{2}H_{m0}, \quad H_b = 0.78h, \quad f = \frac{1}{T}, \quad B = 1.$$

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3 Data

Data for this project was collected by the US. Army Corps of Engineers (USACE) Engineer Research and Development Center (ERDC) during October 2015 at the Field Research Facility (FRF) in Duck, NC on the Outerbanks shown in Figure 1.[8] The data was collected via the BathyDuck project conducted by the Coastal and Hydraulic Laboratory (CHL). Data of interest includes wave height (H), wave number (k), wave period (T), and bathymetry (h) measurements. These data combine information collected through a Nortek Acoustic Wave and Current (AWAC) Profiler, a Light Amphibious Resupply Cargo (LARC-5) vessel, a Coastal Research Amphibious Buggy (CRAB), and Argus Beach Monitoring systems. The LARC-5 and CRAB vessels are shown below in Figure 2.

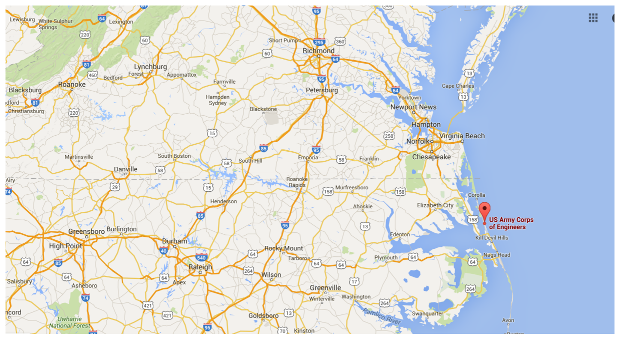


Figure 1: The location of the U.S. Army Corps of Engineers Field Research Facility in Duck, NC.

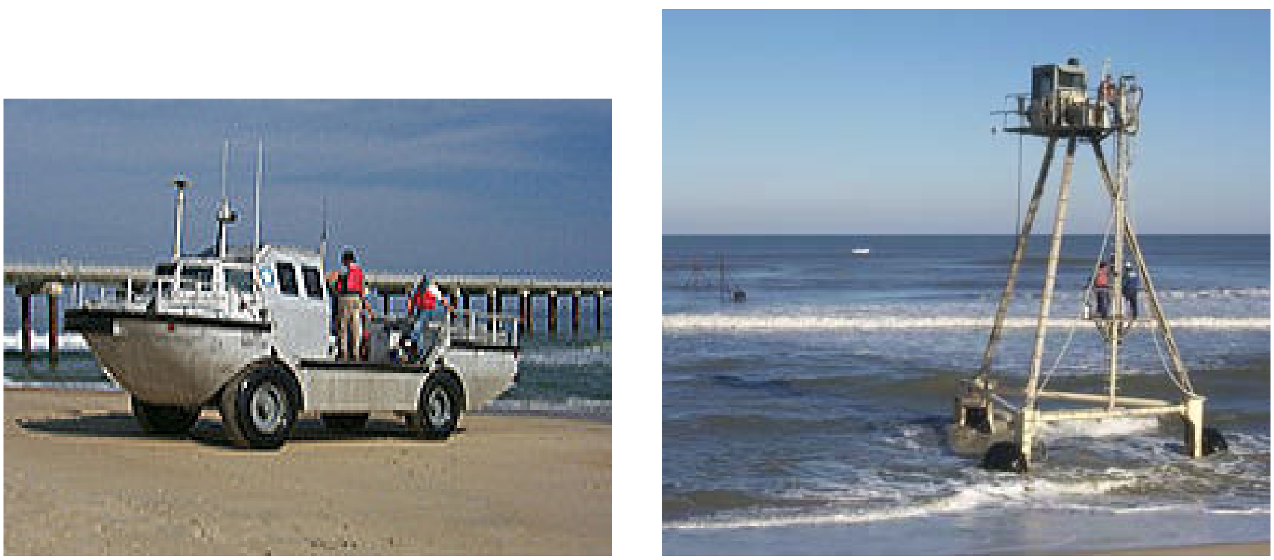


Figure 2: The LARC (left) and CRAB (right) instruments are used to measure near coastal bathymetry. Image source: http://www.frf.usace.army.mil/aboutUS/equipment.shtml

The following sections discuss in more detail the observations and how they were used. Please note that in physical space, the boundary point used for the 1-dimensional problem is located 1150 m offshore. For numerical simplicity, all observations are transformed such that $x = 0$ m corresponds to the offshore boundary point and $x = 1150$ m is the shoreline.

[8] The data can be accessed online at http://chlthredds.erdc.dren.mil/thredds/catalog/frf/projects/bathyduck/catalog.html

3.1 Boundary Condition

Boundary conditions for this project were collected through a bottom-mounted AWAC profiler located approximately 1150 meters offshore at a depth of 11 meters (Figure 3).

Figure 3: Acoustic Wave and Current Profiler. Image source: http://www.nortek-as.com/en/products/wave-systems/awac

Vast amounts of wave data has been collected by the AWAC system at the offshore boundary. For the 1-dimensional problem, the forcing condition at this boundary is the significant wave height and peak frequency.

3.2 Bathymetry

Survey data collected on 1 October 2015 was considered for this analysis. These data were measured via the CRAB and Trimble Real Time Kinematic (RTK) GPS system. Elevation data from six cross-sections[9], perpendicular to the shoreline, spaced over a 100-meter portion of the beach were combined to create the 2D surface shown in Figure 4.

[9] we could potentially add an overhead plot of the transects

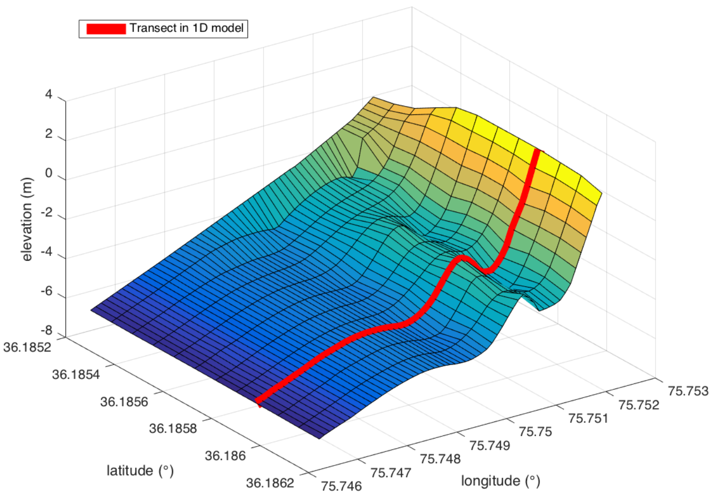


Figure 4: Measured and gridded 2D bathymetry in the survey area on 1 October 2015. The red line shows the transect considered in the 1D problem.

For the 1D problem, a single slice of the 2D bathymetry was used as model input, identified by the red line in Figure 4. In a cartesian coordinate system, this line (a.k.a. 'transect') is located at $y = 950$ meters.

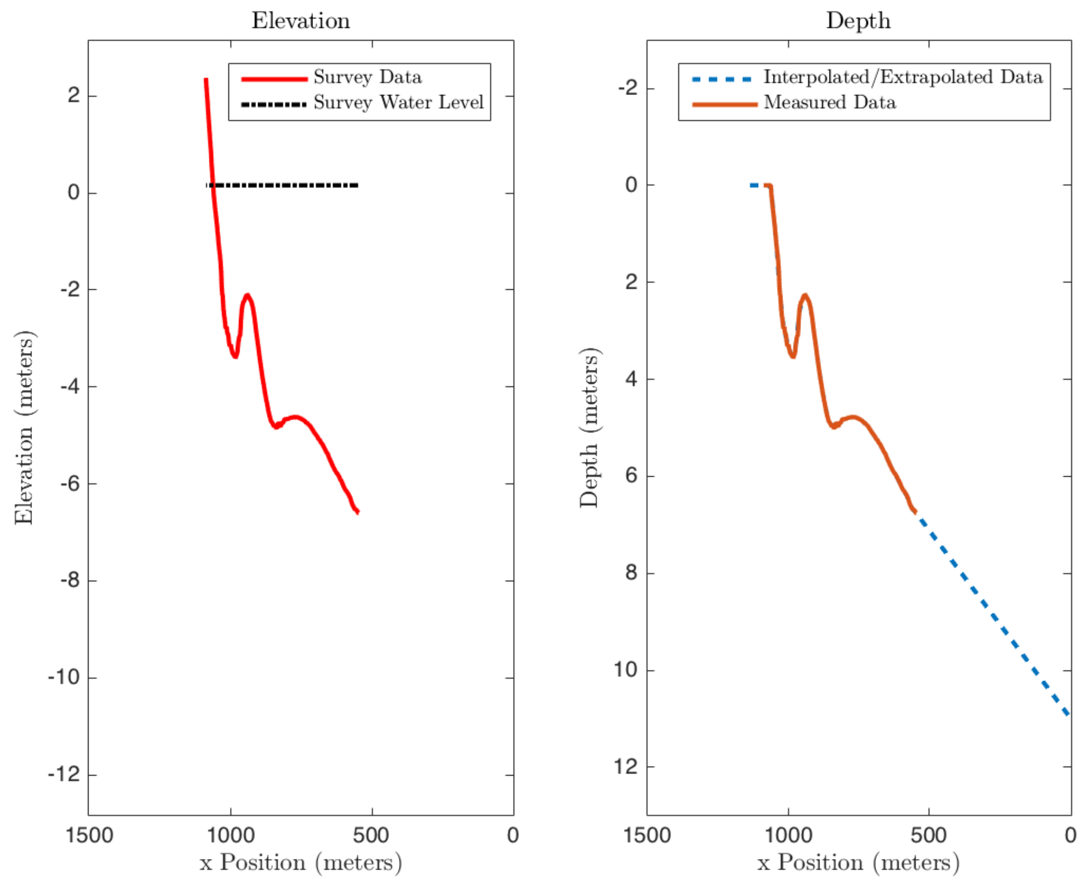


Figure 5: 1D Bathymetry - elevation data (left) and depth data (right).

Survey data provided sea floor elevation referenced to the North American Vertical Datum of 1988 (NAVD88) (Figure 5, *left*). To provide proper input data for the 1D model, elevation was transformed to depth data (Figure 5, *right*). Once transformed, depth was discretized by interpolating between measured data points via Matlab's built-in pchip method. Pchip was chosen for the interpolation due to its shape-preserving nature so as to not introduce non-physical oscillations. Between the boundary condition and the nearest measured depth point, linear interpolation was used to fill in missing data.

3.3 Wave number

Wave number is a measure of the number of waves per unit distance and is inversely proportional to wave speed. Hourly observations using an Argus video monitoring system mounted on shore are available during October 2015. Photogrammetry is performed on the video to derive the dominant wave frequencies and wave numbers in the survey area (**?**). Data is available for a 2D area at the FRF survey site. A 1D profile is extracted from the 2D data along a transect corresponding to the position of the model boundary point ($y =$ 950 m). Figure 6 shows statistics for wave number, k, along the 1D transect. Wave number is shown to be more variable over time close to the coastline. Mean wave number increases toward the shoreline, as expected.

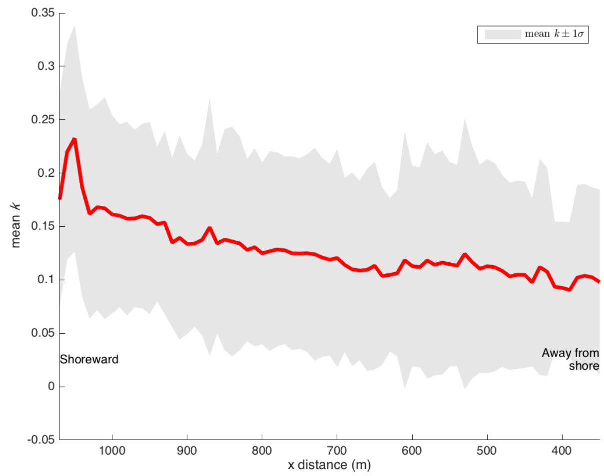


Figure 6: Wave number along the transect where the model boundary condition is located. Mean wave number, k, during October 2015 is shown in red. Gray envelopes show $\pm$ 1σ standard deviation in k. Wave number is observed to be relatively larger and more variable closer to shore.

4 The Approach

4.1 The Additive Gaussian Noise Model

By assuming the measurements are corrupted by the additive Gaussian noise, the linear estimation model can be written as

$$\mathbf{d} \sim \text{Gaussian}(\mathbf{A}\mathbf{h}_t), \tag{8}$$

where

$\mathbf{d}$ = a vector of measurements,
$\mathbf{A}$ = a linear forward operator,
$\mathbf{h}_t$ = the true bathymetry (depth).

Therefore, the Gaussian noise $\boldsymbol{\epsilon}$ corrupted measurements $\mathbf{d}$ with variance ν is given by

$$\mathbf{d} = \mathbf{A}\mathbf{h}_t + \boldsymbol{\epsilon}.$$

4.2 Bathymetric Inversion Method

As a beginning to approximate the topographic heights of the sea-floor, we consider the following least-squares minimization problem,

$$\hat{\mathbf{h}} = \underset{\mathbf{h} \in \mathbb{R}^n}{\arg\min} \quad f(\mathbf{h}) = \|\mathbf{A}\mathbf{h} - \mathbf{d}\|_2^2, \tag{9}$$

where we minimize the data misfit between the forward predictions and the measurements in least-squares sense. This least-squares minimization problem (9) can be solved using the following MATLAB functions:

(1) Nonnegative least-squares method: `lsqnonneg(A,b)`. This Matlab function uses the algorithm so called *active-set* and note that it requires matrix $\mathbf{A}$ explicitly. Residual norm error for the sample dummy data set with $\nu = 0.1$ is 8.88×10^{-26} (see Fig. 7).

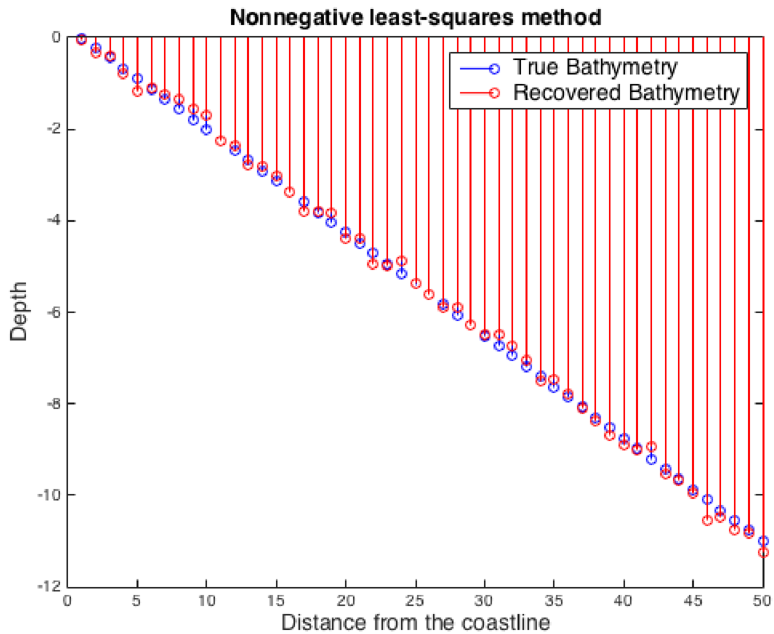


Figure 7: Nonnegative least-squares method reconstruction of depth $\mathbf{h}$ using sample data.

(2) Trust-Region-Reflective method: `lsqnonlin(f)`. Residual norm error for the sample dummy data set with $\nu = 0.1$ is 4.32×10^{-10}.

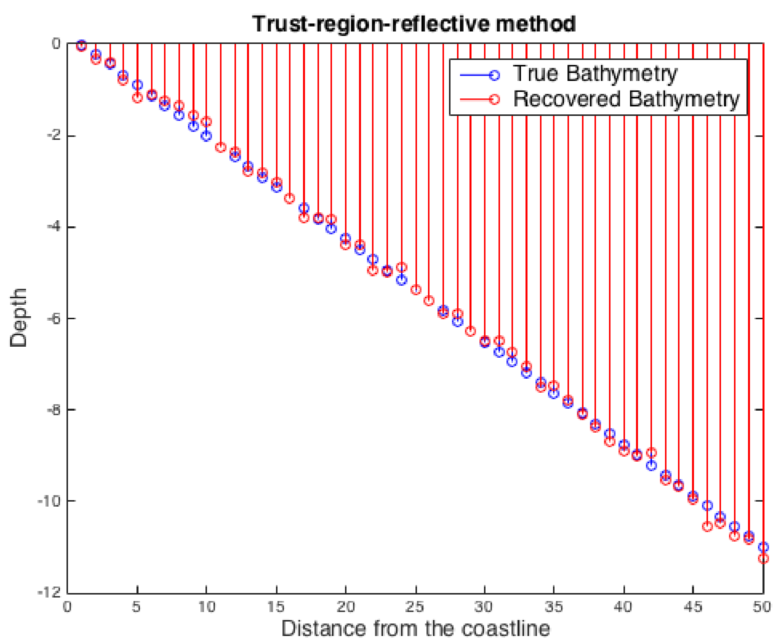


Figure 8: Trust-Region-Reflective method reconstruction of depth $\mathbf{h}$ using sample data.

(3) Levenberg-Marquardt (LM) method: `lsqnonlin(f, 'Algorithm', 'levenberg-marquardt')`. Residual norm error for the sample dummy data set with $\nu = 0.1$ is 6.39×10^{-13}. The Levenberg-Marquardt algorithm does not handle bound constraints.

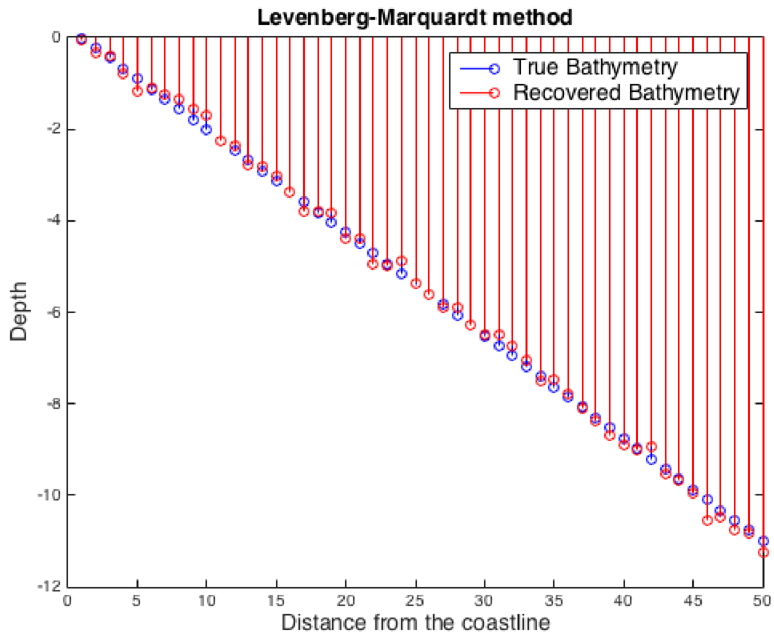


Figure 9: Levenberg-Marquardt (LM) method reconstruction of depth $\mathbf{h}$ using sample data.

(4) fmincon method: `fmincon(f)`. Residual norm error for the sample dummy data set with $\nu = 0.1$ is 1.29.

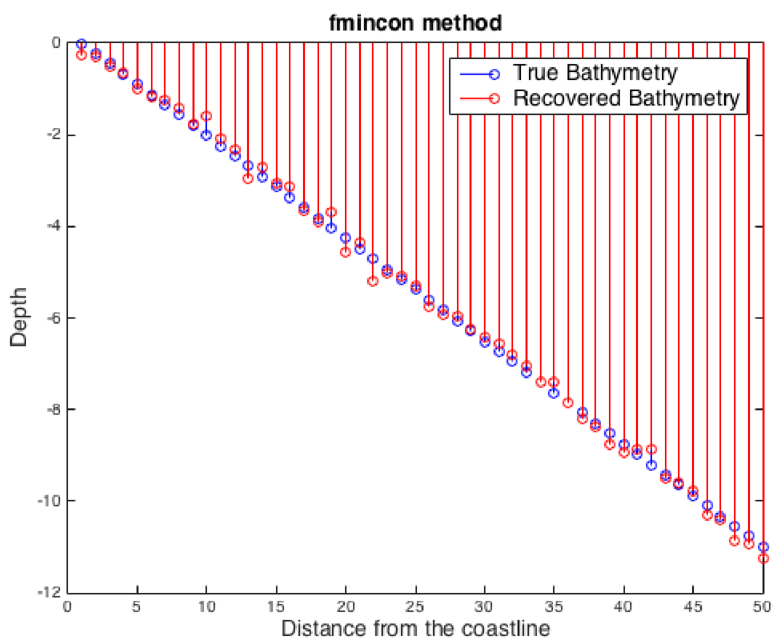
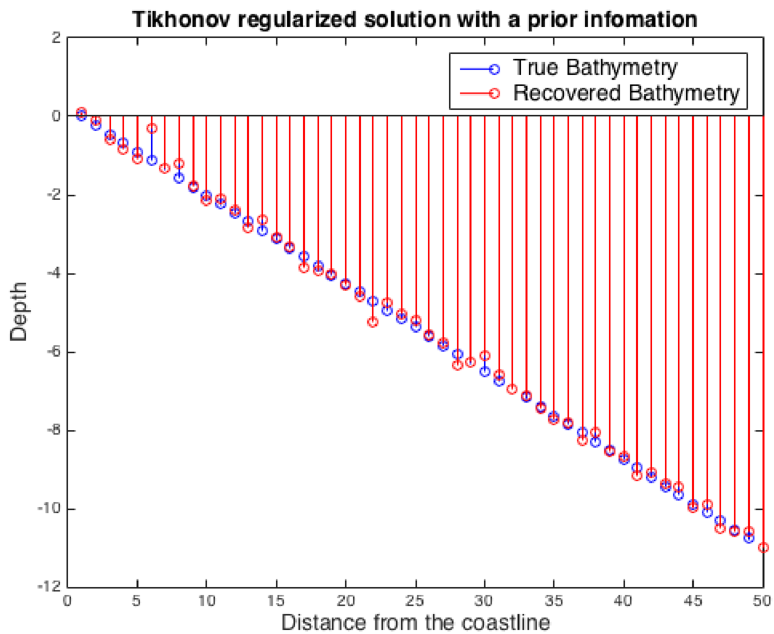


Figure 10: fmincon method reconstruction of depth $\mathbf{h}$ using sample data.

If the inverse problem in (9) is ill-posed, we have to consider a regularized version of it. If we have some prior estimate for the $\mathbf{h}$, i.e., $\mathbf{h}_p$, then the Tikhonov regularized solution with a prior information can be written as

$$\hat{\mathbf{h}} = \underset{\mathbf{h}\in\mathbb{R}^n}{\arg\min} \|\mathbf{A}\mathbf{h} - \mathbf{d}\|_2^2 + \alpha\|\mathbf{h} - \mathbf{h}_p\|_2^2,$$

where α is a regularization parameter (> 0). To test this method, Tikhonov method is implemented and test with sample data set with $\nu = 0.2$. This reconstruction of depth has 0.038 residual norm error. In order to run this `tikhonov` Matlab function, matrix $\mathbf{A}$ should be explicitly known.



Reference: I found a MATLAB package for analysis and solution of discrete ill-posed problems, which is available in http://www2.imm.dtu.dk/ pcha/Regutools/

4.3 Help

LSQR method: $\hat{\Psi}$ = lsqr(A,d,tol,maxit) it attempts to solve the least squares solution x that minimizes norm($\mathbf{d} - \mathbf{A\Psi}$) Note that $\mathbf{A}$ need not be square.

Conjugate gradients: $\hat{\Psi}$ = cgs(A,b) attempts to solve the system of linear equations $\mathbf{A\Psi} - \mathbf{d}$ for Ψ.

4.4 Bayesian MCMC Inverse Method

The Bayesian MCMC Method can be used with the Forward Model to gather an estimate for depth, h, given wave number, k, and wave height, H. To estimate depth, we had to create a prior distribution function of bathymetry with given samples of depths from (add location). Given the sample bathymetry, we inputed these values in the Forward Model to produce wavelengths and wave numbers. This model is then compared with a function of parameters k and H given depth, formally known as a likelihood function to compare the simulated data with the survey data from (location). This makes a good estimate of the depths to create a distribution of depths profile, known as the posterior probability function. The modeled relationship is given by equation below

$$P(h|H, k) \propto P(h)L(h|H, k),$$

$$P(h) =$$

$$L(h|H, k) = \sum_{i=1}^{n} \frac{(\bar{k} - k_i)^2}{\sigma_i^2}$$

and $P(h|H,k)$ represents the posterior probability function, $P(h)$ is the prior distribution function and $L(h|H,k)$ is the likelihood function.

5 Computational Experiments

Give enough details so that readers can duplicate your experiments.

- Describe the precise purpose of the experiments, and what they are supposed to show.
- Describe and justify your test data, and any assumptions you made to simplify the problem.
- Describe the software you used, and the parameter values you selected.
- For every figure, describe the meaning and units of the coordinate axes, and what is being plotted.
- Describe the conclusions you can draw from your experiments

6 Summary and Future Work

- Briefly summarize your contributions, and their possible impact on the field (but don't just repeat the abstract or introduction).
- Identify the limitations of your approach.
- Suggest improvements for future work.
- Outline open problems.