Title of Report

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Faculty Mentors: Mentor 1[6], Mentor 2[7]

Abstract

- Summarize the results presented in the report, and the contributions of your research. This is a test.
- Readers should not have to look at the rest of the paper in order to understand the abstract.
- Keep it short and to the point.

1 Introduction

Bathymetry is a measurement of submarine topography and can be used to understand shifts of the ocean floor and its depth. Knowledge of bathymetry is important for marine navigation, both civilian and military, as well as for monitoring the effects of storms on coastal environments. While direct measurement of bathymetry is possible, the process tends to be cost and time prohibitive. For example, amphibious vehicles (see Figure 8) are capable of spatially limited surveys of bathymetry in difficult surf-zone conditions but require significant resources to operate. As a result of these factors, surveys tend to be sparse in time.

~~More desirable would be a method to~~ estimate bathymetry using surface measurements collected via remote sensing platforms, i.e. airborne, satellite, or onshore platforms. While bathymetry data is currently sparse due to observational limitations, the physics of waves are reasonably well understood. In particular, a dispersion relationship can be used to relate water depth to surface properties such as wave length and wave period. It is therefore possible to estimate bathymetry given observations of these parameters. Light Detection And Ranging (LiDAR) has been used to determine wave heights and Argus land-mounted video has been analyzed photogrammetrically to determine wave frequency and wave number. Both of these sources therefore provide valuable inputs for estimating coastal bathymetry in a more efficient manner than is currently available.

Wave and bathymetric data has been collected in Duck, NC by the U.S. Army Corps of Engineers Coastal and Hydraulics Laboratory, including in situ measurements of bathymetry and measurements of the water surface. These measurements provide a method for testing algorithms to invert for bathymetry because the true bathymetry is available for comparison to the numerical estimates.

2 The Problem

Although there have been uncertainties in capturing the topography of the ocean near shore, mathematical methods can estimate bathymetry using the dispersion relationship between wavelength and the period. Stockdon et al. used video imagery, which compared true wave signal and remotely sensed video signal to create a linear representation between wave amplitudes and phases (**?**). Holman et al. used a 2-dimensional method with Kalman filtering to estimate the depth, h (**?**).

We invert for depth, h, using wave length and wave number with a 1D model derived using the energy flux method to create a correlation between wave length and depth from the water surface.

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2.1 Forward Problem

The forward problem is described by 1D linear wave theory:

$$\begin{cases} \frac{d}{dx}\left(EC_g\right) = -\delta, \\ \sigma^2 = gk\tanh(kh), \end{cases} \tag{1}$$

where δ is wave breaking parameter and the speed at which the energy is transmitted, C_g, called linear theory group speed, is given as

$$C_g = \frac{c}{2}\left(1 + \frac{2kh}{\sinh(2kh)}\right), \tag{2}$$

with $c = \frac{\sigma}{k}$ and from the linear theory, the wave energy, E is given as

$$E = \frac{1}{8}\rho g H^2, \tag{3}$$

ρ is the density of water, g is the gravitational acceleration, σ is the angular frequency, c is the local wave phase speed, k is the wave number, h is the water depth and H is the wave height. Here, we assume T, the wave period, is constant.

Observe that equation 1 is coupled using the fact that $\sigma = \frac{2\pi}{T}$. Hence, using equation 2 and 3 in 1, we obtain

$$\frac{d}{dx}\left(\frac{\lambda}{k}\left(1 + \frac{2kh}{\sinh(2kh)}\right)H^2\right) = -\delta, \tag{4}$$

and

$$f(k) = gk\tanh(kh) - \sigma^2, \tag{5}$$

where k is the zero of function f and $\lambda = \frac{\rho g \pi}{8T}$.

The wave breaking, δ, proposed by (Janssen and Battjes, 2007) is given as

$$\delta = \frac{1}{4h}B\rho g f H_{rms}\left[(R^3 + \frac{3}{2}R)e^{-R^2} + \frac{3}{4}\sqrt{\pi}(1 - erf(R))\right], \tag{6}$$

where

$$R = \frac{H_b}{H_{rms}}, \quad H_b = 2\sqrt{2}H_{m0}, \quad H_b = 0.78h, \quad f = \frac{1}{T}, \quad B = 1.$$

2.2 Numerical Solution of the Forward Model

A finite difference scheme is applied to obtain a numerical solution with appropriate initial and boundary conditions. In the scheme of finite differences, the derivatives are replaced using their finite-difference approximations. The goal is to provide wave height using the finite difference method. In the process, MATLAB's fsolve function is applied to obtain the wave number. Furthermore, the Newton-Raphson method is used to verify the solution of wave number k obtained by the fsolve function.

2.2.1 Discretization of the Model

A forward difference method is applied depending on the nature of the model given in equation (4).
Let

$$c^* = \frac{c_g\lambda}{k}, \quad c_g = nc = \frac{2\pi n}{T},$$

with

$$\lambda = \frac{1}{8}\rho g, \quad n = \frac{1}{2}\left(1 + \frac{2hk}{sinh(2hk)}\right).$$

Let $\widetilde{H} = H^2$. Then the ordinary differential equation of the model given in (4) becomes

$$\frac{d}{dx}\left(c^* \widetilde{H}\right) = -\delta \tag{7}$$

Applying the finite difference method, the above expression becomes

$$\frac{c_i^* \widetilde{H}_i - c_{i-1}^* \widetilde{H}_{i-1}}{\triangle x} = \delta \quad \Rightarrow \quad \widetilde{H}_i = \frac{c_{i-1}^* \widetilde{H}_{i-1} + \triangle x \delta}{c_i^*}$$

Hence, at each index point in the discretization, $H = \sqrt{\widetilde{H}}$.

MATLAB function fsolve

As part of the process of obtaining wave height, H, MATLAB function fsolve is used as non-linear solver to find the zeros of the function given in (5) obtained from the dispersion relationship

$$\sigma^2 = gktanh(kh).$$

So, at each index point, wave number, k, is generated with initial guess k_0:

$$k_0 = \frac{\sigma}{\sqrt{gh}}.$$

Newton-Raphson Method

The Newton-Raphson method is a widely used method for finding roots. This method is used in the numerical experiment to verify the wave number extracted using MATLAB function fsolve. Therefore, approximate solution using Newton-Raphson method is obtained as

$$k_{i+1} = k_i - \frac{gk_i \tanh(k_i h) - \sigma^2}{g \tanh(k_i h) - ghk_i sech^2(k_i h)}, \tag{8}$$

using the same initial guess as for fsolve. This provides the wave number, k, for each index.

2.2.2 Implementation

To apply the finite difference method we first discretize the space vector, x, depending on predefined mesh size, Δx. For test purposes, Δx is considered to be 10 m which means we estimate wave height, H, every 10 meters. The finite difference scheme provides a sparse bidiagonal matrix. In this experiment, a wave breaking condition is applied as a maximum value on estimates of wave height at each index point. The wave breaking condition applied in this experiment is

$$H = 0.78h$$

The implementation of the algorithm is as follows

2.2.3 Numerical Results

We present some of the numerical results in the following Figures.

3 Data

Data for this project were collected by the US. Army Corps of Engineers (USACE) Engineer Research and Development Center (ERDC) during October 2015 at the Field Research Facility (FRF) in Duck, NC on the Outerbanks shown in Figure 7.[8] The data was collected via the BathyDuck project conducted by the Coastal and Hydraulic Laboratory (CHL). Data of interest includes wave height (H), wave number (k), wave period (T), and bathymetry (h) measurements. These data combine information collected through a Nortek

Algorithm 1 Algorithm to estimate wave height H

procedure
Initialization:
  Initial depth: h
  Wave period: T
  Wave number estimated from fsolve: k
  Wave breaking: δ
Step 1:
  • *constant* $\lambda = \frac{\rho g \pi}{8T}$
Step 2:
  • *Find* c^* *depending on* k : $c^* = \frac{(1+(2kh)}{sinh(2kh)} \lambda \; k$
Step 3:Compute wave height H
  • *Compute:* $\widetilde{H}$: $\widetilde{H}_i = \frac{c^*_{i-1} \widetilde{H}_{i-1} + \triangle x \delta}{c^*_i}$
  • *Compute wave break condition* $Hmax : 0.78h$
  • *Take minimum between* $\widetilde{H}$ *and* $Hmax$ *at each index* i

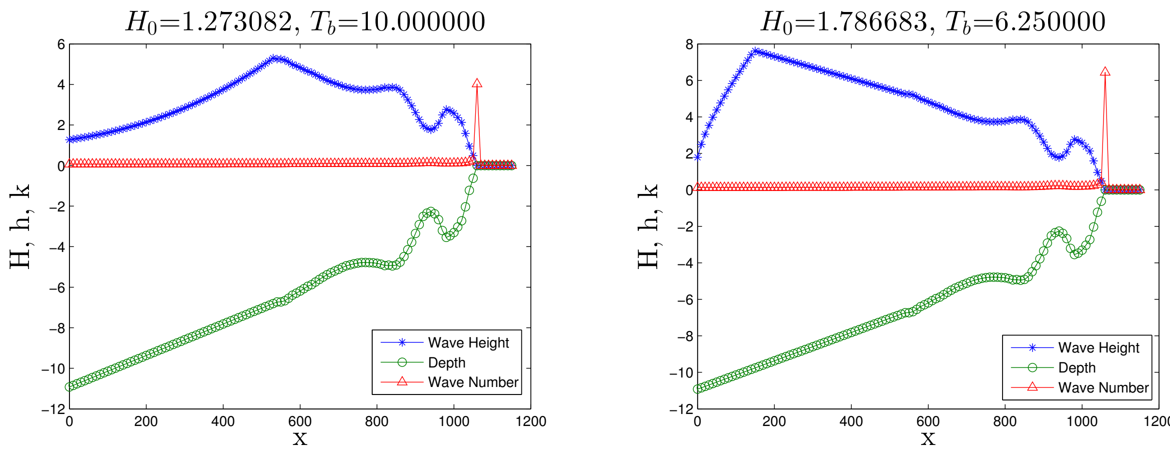


Figure 1: Water Depth(h), Wave Height(H) and Wave Number vary with x direction (Two sets of boundary conditions are applied for following figures)

Acoustic Wave and Current (AWAC) Profiler, a Light Amphibious Resupply Cargo (LARC-5) vessel, a Coastal Research Amphibious Buggy (CRAB), and Argus Beach Monitoring systems. The LARC-5 and CRAB vessels are shown below in Figure 8.

The following sections discuss in more detail the observations and how they were used. ~~Please~~ note ~~that~~ in physical space, the boundary point used for the 1-dimensional problem is located 1150 m offshore. For numerical simplicity, all observations are transformed such that $x = 0$ m corresponds to the offshore boundary point and $x = 1150$ m is the shoreline.

3.1 Boundary Condition

Boundary conditions for this project were collected through a bottom-mounted AWAC profiler located approximately 1150 meters offshore at a depth of 11 meters (Figure 9).

Vast amounts of wave data has been collected by the AWAC system at the offshore boundary. For the 1-dimensional problem, the forcing condition at this boundary is the significant wave height and peak frequency.

[8] The data is available in netcdf format at http://chlthredds.erdc.dren.mil/thredds/catalog/frf/projects/bathyduck/catalog.html

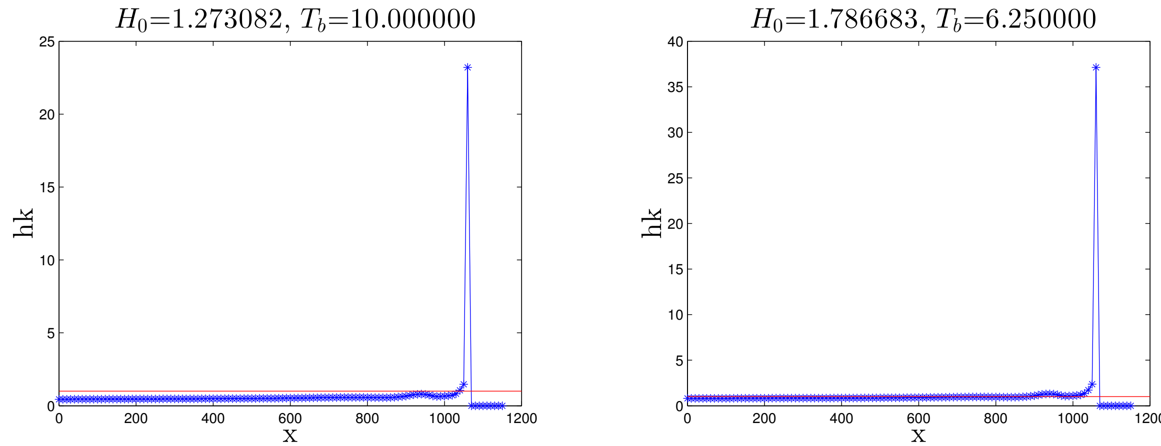


Figure 2: Relative Depth varies with x direction

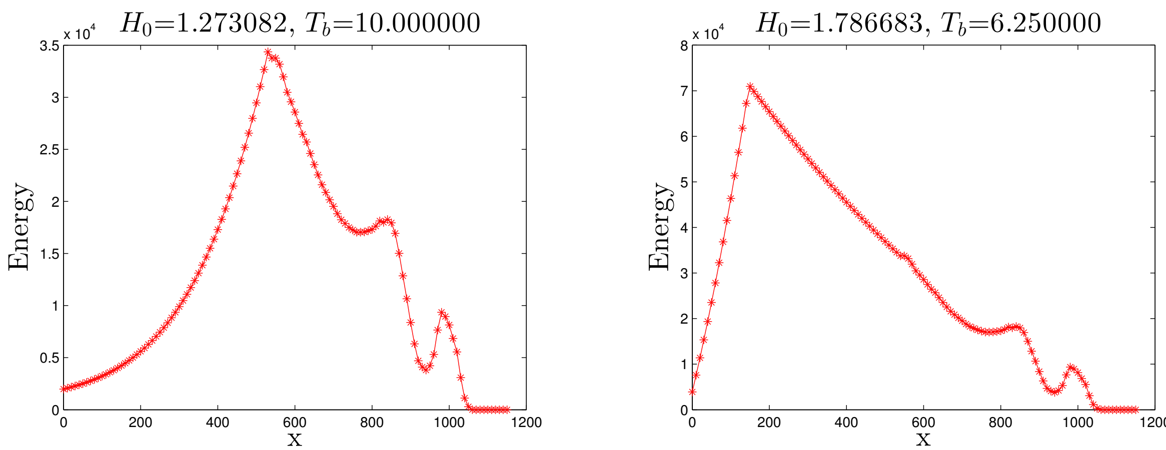


Figure 3: Wave Energy varies with x direction

3.2 Bathymetry

Survey data collected on 1 October 2015 was considered for this analysis. These data were measured via the CRAB and Trimble Real Time Kinematic (RTK) GPS system. Elevation data from six cross-sections[9], perpendicular to the shoreline, spaced over a 100-meter portion of the beach were combined to create the 2D surface shown in Figure 10.

[9]we could potentially add an overhead plot of the transects

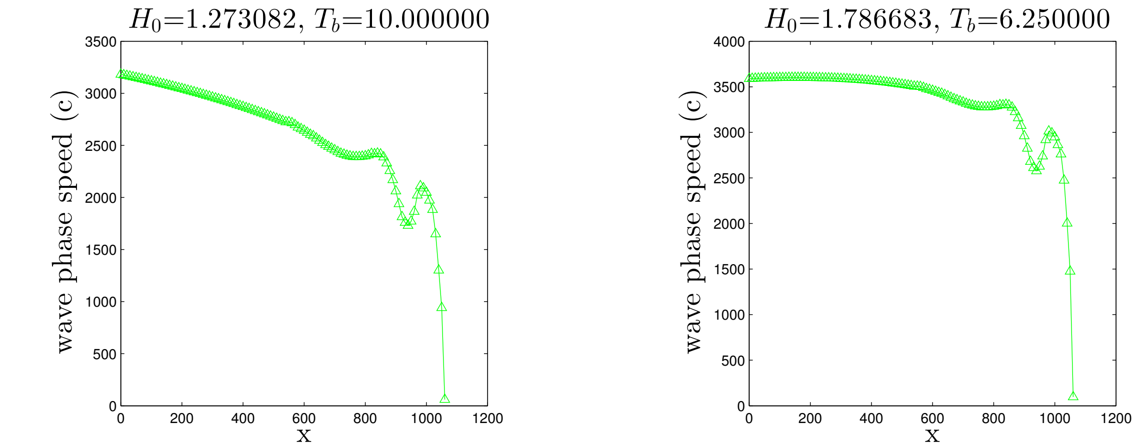


Figure 4: Wave Phase Speed (c) varies with x direction

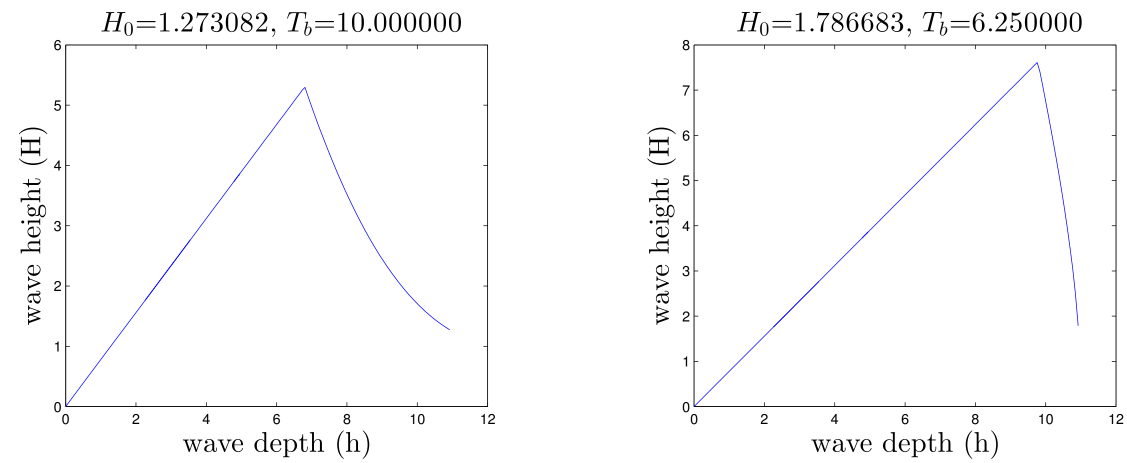


Figure 5: Wave Height(H) varies with Water Depth(h)

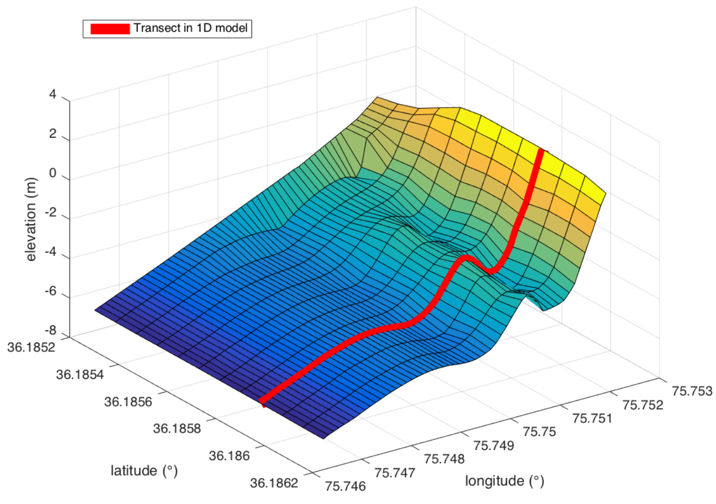


Figure 10: Measured and gridded 2D bathymetry in the survey area on 1 October 2015. The red line shows the transect considered in the 1D problem.

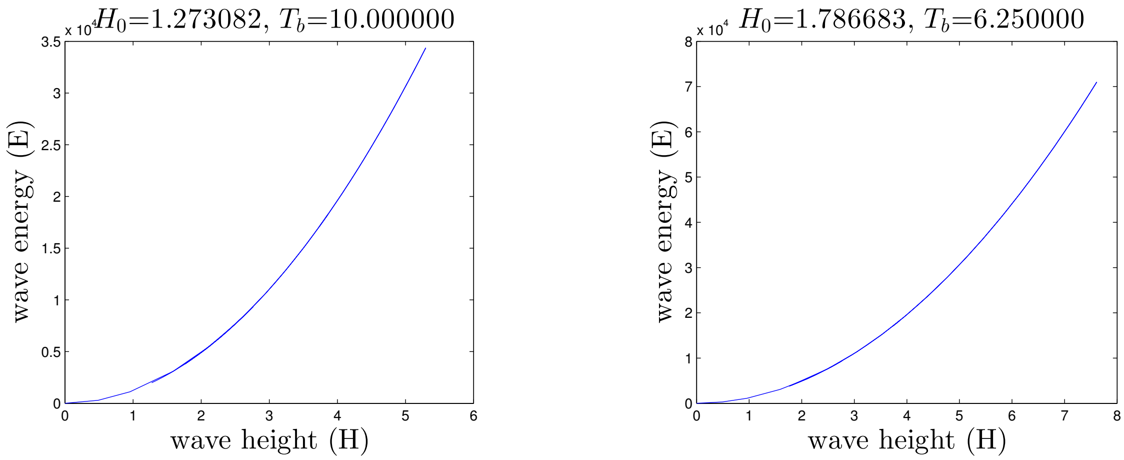


Figure 6: Wave Energy varies with Wave Height(H)

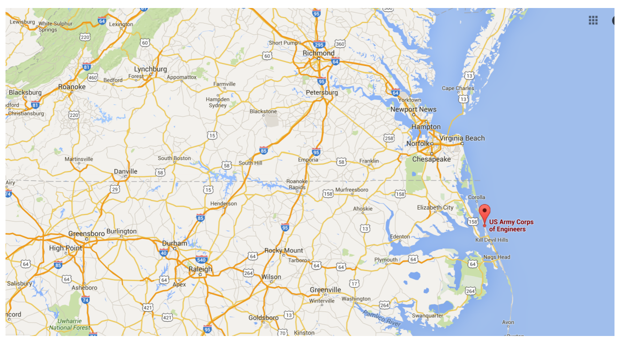


Figure 7: The location of the U.S. Army Corps of Engineers Field Research Facility in Duck, NC.

For the 1D problem, a single slice of the 2D bathymetry was used as model input, identified by the red line in Figure 10. In a cartesian coordinate system, this line (a.k.a. 'transect') is located at $y = 950$ meters.

Figure 8: The LARC (left) and CRAB (right) instruments are used to measure near coastal bathymetry. Image source: http://www.frf.usace.army.mil/aboutUS/equipment.shtml

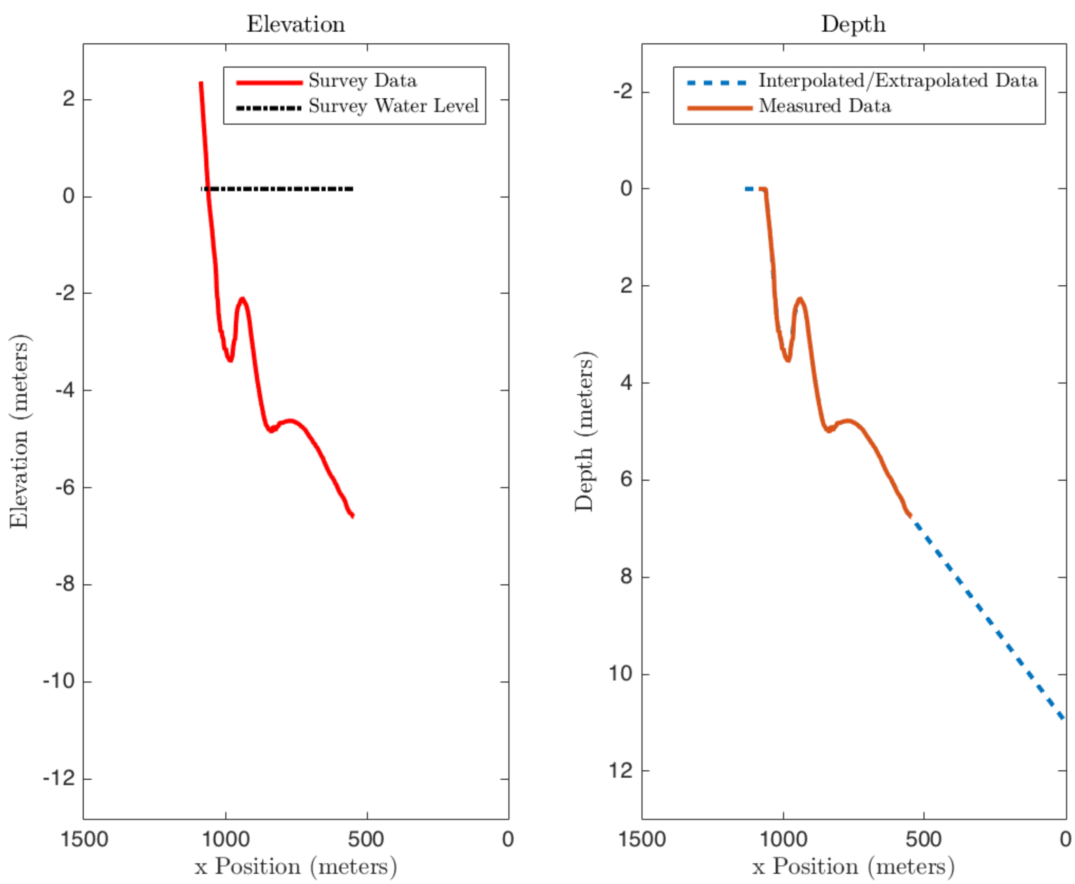


Figure 11: 1D Bathymetry - elevation data (left) and depth data (right).

Figure 9: Acoustic Wave and Current Profiler. Image source: http://www.nortek-as.com/en/products/wave-systems/awac

Survey data provided sea floor elevation referenced to the North American Vertical Datum of 1988 (NAVD88) (Figure 11, *left*). To provide proper input data for the 1D model, elevation was transformed to depth data (Figure 11, *right*). Once transformed, depth was discretized by interpolating between measured data points via Matlab's built-in pchip method. Pchip was chosen for the interpolation due to its shape-preserving nature so as to not introduce non-physical oscillations. Between the boundary condition and the nearest measured depth point, linear interpolation was used to fill in missing data.

3.3 Wave number

Wave number is a measure of the number of waves per unit distance and is inversely proportional to wave speed. Hourly observations using an Argus video monitoring system mounted on shore are available during October 2015. Photogrammetry is performed on the video to derive the dominant wave frequencies and wave numbers in the survey area (**?**). Data is available for a 2D area at the FRF survey site. A 1D profile is extracted from the 2D data along a transect corresponding to the position of the model boundary point (y = 950 m). Figure 12 shows statistics for wave number, k, along the 1D transect. Wave number is shown to be slightly more variable over time close to the coast. Mean wave number increases toward the shoreline, as expected.

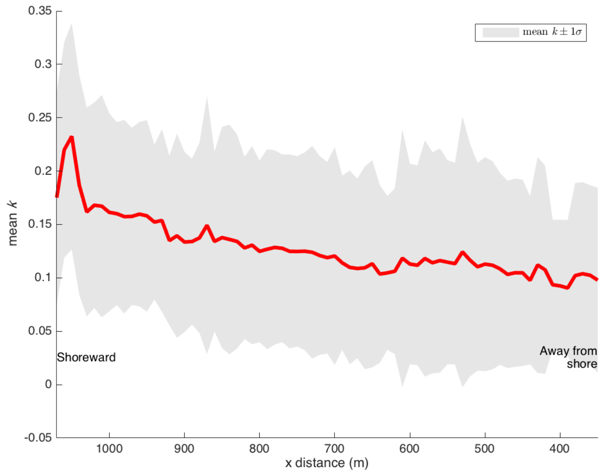


Figure 12: Wave number along the transect where the model boundary condition is located. Mean wave number, k, during October 2015 is shown in red. Gray envelopes show $\pm$ 1σ standard deviation in k. Wave number is observed to be relatively larger and more variable closer to shore.

4 The Approach

4.1 The Additive Gaussian Noise Model

By assuming the measurements are corrupted by the additive Gaussian noise, the linear estimation model can be written as

$$\mathbf{d} \sim \text{Gaussian}(\mathbf{A}\mathbf{h}_t), \tag{9}$$

where

$$\begin{aligned}
\mathbf{d} &= \text{a vector of measurements,} \\
\mathbf{A} &= \text{a linear forward operator,} \\
\mathbf{h}_t &= \text{the true bathymetry (depth).}
\end{aligned}$$

Therefore, the Gaussian noise $\boldsymbol{\epsilon}$ corrupted measurements $\mathbf{d}$ with variance ν is given by

$$\mathbf{d} = \mathbf{A}\mathbf{h}_t + \boldsymbol{\epsilon}.$$

4.2 Bathymetric Inversion Methods

4.2.1 Least-Squares Inversion

As a beginning to approximate the topographic heights of the near-shore sea-floor, we consider the following least-squares minimization problem,

$$\hat{\mathbf{h}} = \underset{\mathbf{h}\in\mathbb{R}^n}{\arg\min} \quad f(\mathbf{h}) = \|\mathbf{A}\mathbf{h} - \mathbf{d}\|_2^2, \tag{10}$$

where we minimize the data misfit between the forward predictions and the measurements, in the least-squares sense. In order to test possible Matlab inbuilt solvers to solve the minimization problem in (10), we have generated dummy measurements with $\nu = 0.1$. In particular, in this test, the forward operator `A = rand(50)`, the true bathymetry `ht = -linspace(-11,0,N)' - 11`, and the Gaussian noise corrupted measurements `b = A * h_t + 0.1 * randn(N,1)`. Recovered bathymetries from different Matlab solvers are given below. Note that the initial guess for all methods is zero.

(1) *Nonnegative least-squares method:* `lsqnonneg(A,b, options)`. This Matlab function uses the algorithm so called *active-set* and note that it requires the matrix $\mathbf{A}$ explicitly. Residual norm error for this nonnegativity reconstruction is 8.88×10^{-26} (see Fig. 13).

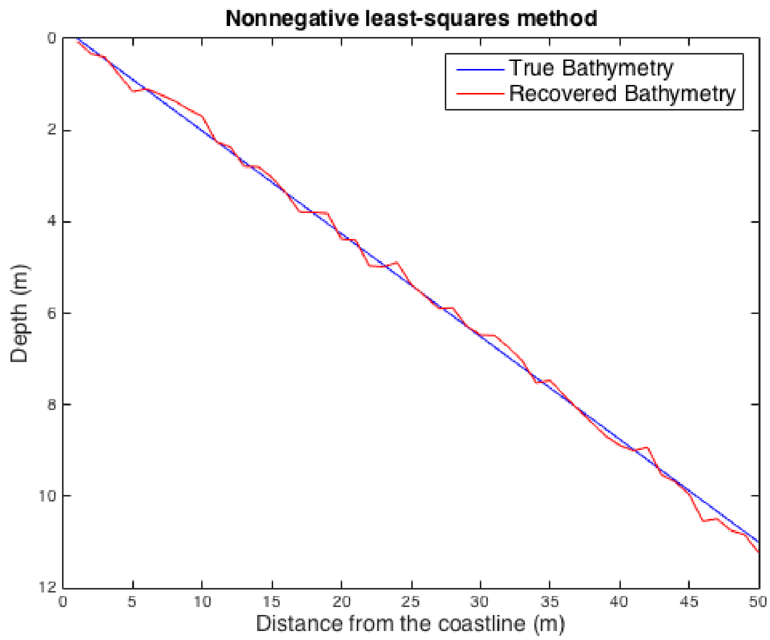


Figure 13: Nonnegative least-squares method reconstruction of depth $\mathbf{h}$ using the dummy dataset.

(2) *Trust-Region-Reflective method:* `lsqnonlin(@(h) A * h - b, zeros(N,1), zeros(N,1),inf(N,1), options)`. Note that the reconstruction of $\mathbf{h}$ is restricted to the positive x-axis using the function arguments. Residual norm error for the reconstruction is 4.32×10^{-10}.

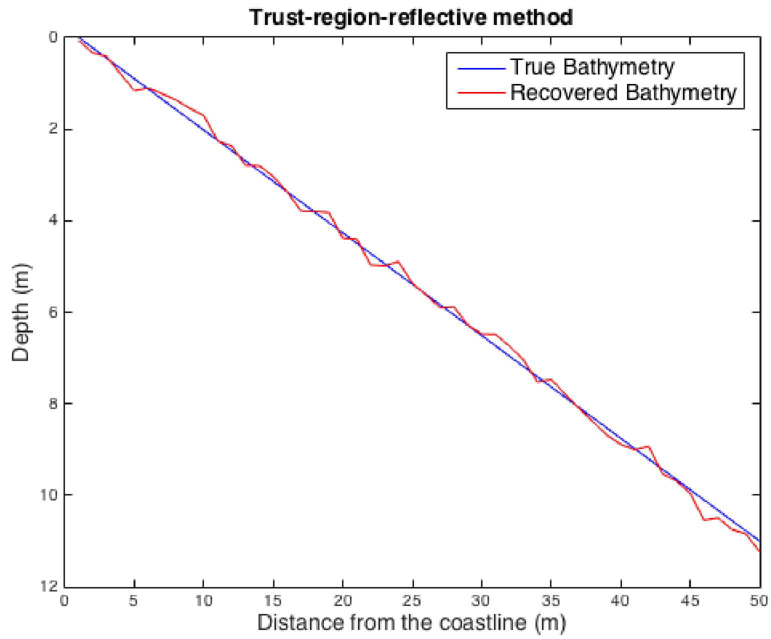


Figure 14: Trust-Region-Reflective method reconstruction of depth $\mathbf{h}$ using the dummy data.

(3) *Levenberg-Marquardt (LM) method:* `lsqnonlin(@(h) A * h - b, 'Algorithm', 'levenberg-marquardt')` Residual norm error for the reconstruction is 6.39×10^{-13}. The Levenberg-Marquardt algorithm does not handle bound constraints.

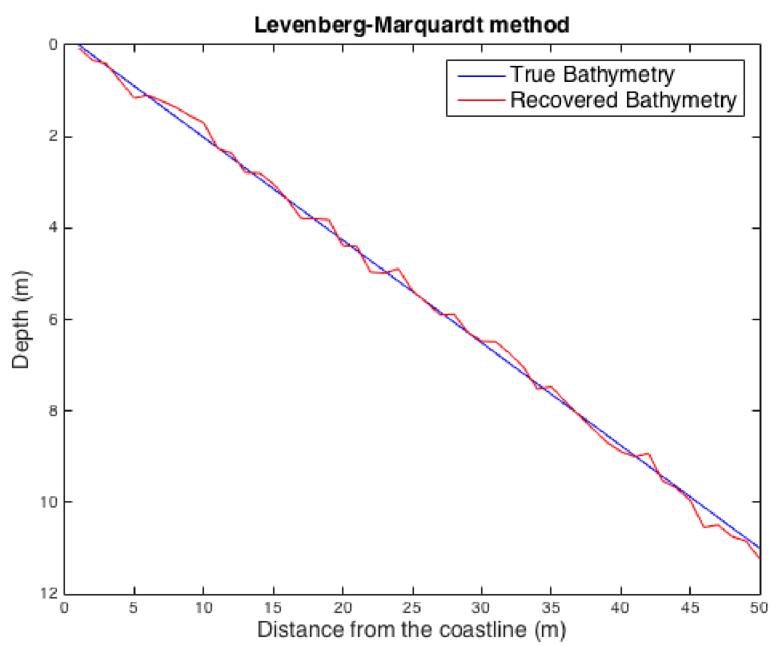


Figure 15: Levenberg-Marquardt (LM) method reconstruction of depth $\mathbf{h}$ using the dummy data.

(4) *Interior-point method:* `fmincon(f, zeros(N,1), [],[],[],[], zeros(N,1), inf(N,1))`. Residual norm error for the sample dummy data set with $\nu = 0.1$ is 1.29.

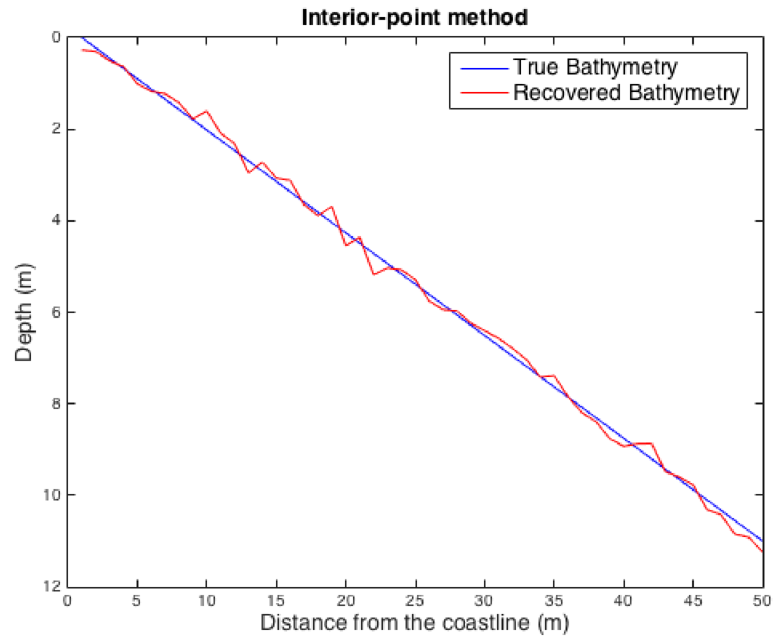
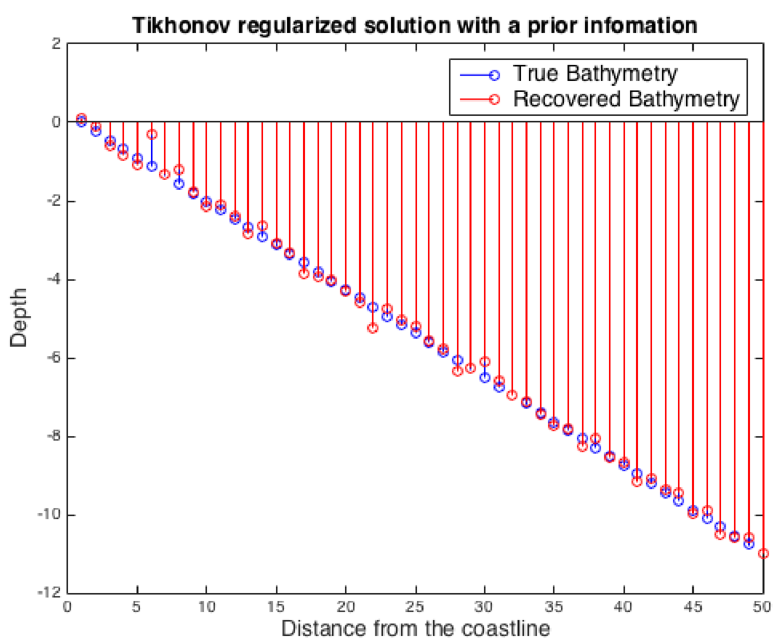


Figure 16: fmincon method reconstruction of depth $\mathbf{h}$ using the dummy data.

If the inverse problem in (10) is ill-posed we have to consider a regularized version of it. If we have some prior estimate for the $\mathbf{h}$, i.e., $\mathbf{h}_p$, then the Tikhonov regularized solution with a prior information can be written as

$$\hat{\mathbf{h}} = \arg\min_{\mathbf{h}\in\mathbb{R}^n} \|\mathbf{A}\mathbf{h} - \mathbf{d}\|_2^2 + \alpha\|\mathbf{h} - \mathbf{h}_p\|_2^2,$$

where α is a regularization parameter (> 0). To test this method, Tikhonov method is implemented and test with sample data set with $\nu = 0.2$. This reconstruction of depth has 0.038 residual norm error. In order to run this `tikhonov` Matlab function, matrix $\mathbf{A}$ should be explicitly known.



4.3 Bayesian MCMC Inverse Method

4.3.1 Bayesian Formulation

We use a Bayesian Markov Chain Monte Carlo (MCMC) method to estimate depth, h, given wave number, k. This method computes the posterior probability distribution of h by combining prior information about h with a likelihood function according to Bayes Rule in Equation 11:

$$P(h|k) \propto \Pi(h)L(h|k), \tag{11}$$

where $P(h|k)$ represents the posterior probability function, $\Pi(h)$ is the prior distribution function and $L(h|k)$ is the likelihood function.

We use a random walk Metropolis algorithm (?) which samples proposed depth profiles and updates the posterior probability distribution of h according to the likelihood that the proposed depth profiles describe the true bathymetry.

To estimate depth, we created a prior distribution function of h given 500 realistic approximations of bathymetry [which come from somewhere we should probably mention]. We assume a normal prior with mean and standard deviation determined by the distribution of simulated bathymetry as shown in equation 12.

$$\Pi(h) \sim N(\bar{h}_{sim}, \sigma^2_{sim}) \tag{12}$$

where $\bar{h}_{sim}$ is the mean and σ^2_{sim} is the variance of the simulated h profiles.

Given the sample bathymetry, we evaluated the forward model to compute wave numbers along the 1D profile. A likelihood function (Equation 13) compares the computed k profile to k values observed along the 1D profile.

$$L(h|k) = exp(-\frac{\sum_{i=1}^{n}(k_{m,i} - k_{d,i})^2}{2\sigma_d^2}) \tag{13}$$

Modeled and observed k are $k_{m,i}$ and $k_{d,i}$, respectively, where i corresponds to the points along the 1D profile for which we have inferred k measurements and σ_d^2 is the variance of k values observed along the 1D profile during October 2015. (See section 3.3 for more information about the observations.)

The likelihood uses the sum of square errors between simulated and observed k to quantify the probability that the modeled k represents the true k profile as observed. Inclusion of the σ_d^2 acknowledges that a perfect match between modeled and observed k cannot be expected due to uncertainty in the measurements of k.

4.3.2 Metropolis Algorithm

The Metropolis algorithm begins with an initial h profile sampled from the simulated distribution of bathymetry. This initial estimate of h is used to compute an initial prior probability distribution. An initial likelihood probability distribution is computed using observed values of k, modeled values of k determined by using the sampled h as input to the forward model (see Section 2.1), and σ_d^2. The prior and likelihood are combined to compute an initial posterior probability distribution of h, shown in equation 14:

$$P(h|k) = log(\Pi(h)) + log(L(h|k)) \tag{14}$$

The algorithm then uses a markov chain random walk to propose new h profiles. Each proposed h is evaluated in the forward model to get a proposed k profile. The proposed k is then compared to observed k using the likelihood function and a proposed posterior distribution is computed.

A unique characteristic of the Metropolis algorithm is the way in which proposed posteriors are accepted or rejected by comparison to the previous step's posterior probability. The probabilty of accepting a proposed posterior is

$$\rho = exp(P(h|k)_{prop} - P(h|k)_{current}) \tag{15}$$

The formulation in equation 15 implies that if the proposed posterior probability is higher than the current posterior probability than the proposed posterior probability replaces the current posterior probability and the proposed h is also accepted as a solution. The algorithm is then incremented and a new proposal is selected and evaluated. We perform 10,000 iterations to arrive at a posterior distribution of h profiles which are expected to describe the true bathymetry.

4.4 Estimating Bathymetry using Simulated Data

4.4.1 Simulated Data

[How we obtained the simulated data]

4.4.2 Using the Matlab's lsqnonlin function

4.4.3 Using the Matlab's fmincon function

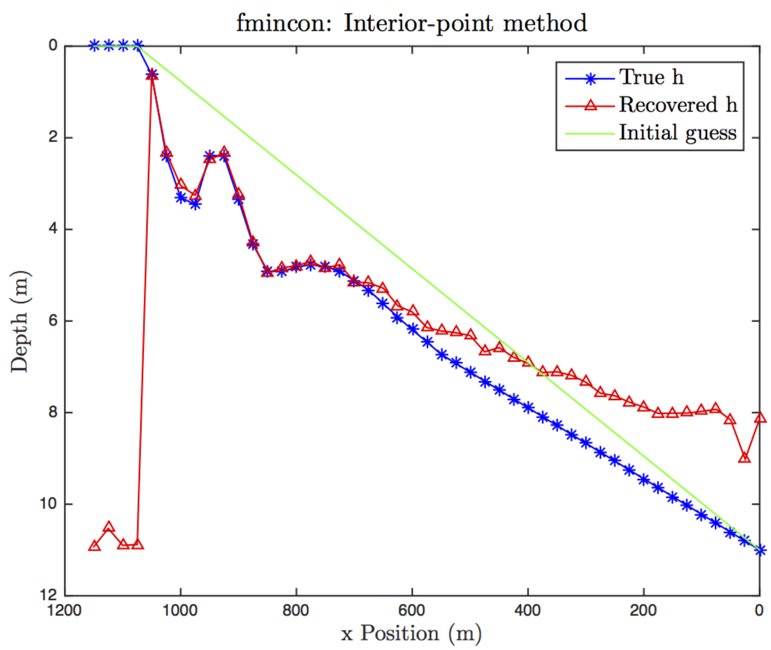


Figure 17: fmincon method reconstruction of depth **h** using the simulated data.

4.4.4 Using the MCMC method

4.5 Estimating Bathymetry using Real Data

5 Computational Experiments

Give enough details so that readers can duplicate your experiments.

- Describe the precise purpose of the experiments, and what they are supposed to show.
- Describe and justify your test data, and any assumptions you made to simplify the problem.
- Describe the software you used, and the parameter values you selected.
- For every figure, describe the meaning and units of the coordinate axes, and what is being plotted.
- Describe the conclusions you can draw from your experiments

6 Summary and Future Work

- Briefly summarize your contributions, and their possible impact on the field (but don't just repeat the abstract or introduction).
- Identify the limitations of your approach.
- Suggest improvements for future work.
- Outline open problems.