

Note: These lecture notes are still rough, and have only have been mildly proofread.

9.1 Continuous time processes (the exponential distribution)

The exponential distribution often appears as the distribution of waiting times until an event (i.e. arrival times).

$$f(x) = \lambda e^{-\lambda x}$$

Let T be a random variable representing the waiting time until an event happens. Then for $T \sim \text{exp}(\lambda)$ with expected value $E(T) = \frac{1}{\lambda}$. The cumulative distribution function is given by

$$\begin{aligned} F(X) &= P(T \leq X) = 1 - e^{-\lambda x} \\ P(T > X) &= e^{-\lambda x} \end{aligned}$$

An important property of the exponential distribution is that it is *memoryless*. Suppose that T is the waiting time until an event occurs. Then, for any given waiting times t and s ,

$$P(T > t + s | T > t) = P(T > s)$$

This is to say that if we have already waited t minutes for an event to occur, then the remaining time s for the event to occur is the same as if we hadn't waited the first t minutes to begin with. To show this, we use our definition of conditional probability:

$$P(T > t + s | T > t) = \frac{P(T > t + s, T > t)}{P(T > t)}$$

Since we have already waited t minutes, $P(T > t + s, T > t) = P(T > t + s)$. So,

$$\begin{aligned} P(T > t + s | T > t) &= \frac{P(T > t + s)}{P(T > t)} \\ &= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} \\ &= e^{-\lambda s} = P(T > s) \end{aligned}$$

9.2 Discrete time processes (the geometric distribution)

The geometric distribution can be thought of as the discrete analog of the exponential distribution. It is the probability distribution of X number of Bernoulli trials with success probability p before one success is obtained.

$$\text{Geom}(p) = P(X = k) = (1 - p)^{k-1}p$$

Because the trials are independent, the geometric distribution is also memoryless.

9.3 Poisson processes

9.3.1 Counting process and the poisson distribution

A poisson process can be defined as a counting process, where we are interested in the number of events (arrivals) that have occurred up until some time $N(t)$. Four properties of this counting process are the following

- 1) $N(0) = 0$
- 2) $N(t), t \geq 0$ has independent increments, or what happens in one time interval is independent of what happens in any other interval
- 3) $P(N(t_0 + t) - N(t_0) \geq 2) = o(t)$ as $t \rightarrow 0$

As the time between events gets smaller, the probability of observing two or more events in that time step goes to zero. In other words, events do not happen simultaneously.

- 4) $P(N(t_0 + t) - N(t_0) = 1) = \lambda t + o(t)$ as $t \rightarrow 0$

i.e. the number of events in an interval of length t is Poisson distributed with rate λt . We write this as

$$P(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

Here, $f(x) = o(g(x))$ means that $|f(x)| \leq k|g(x)|$ for all k . Intuitively this means that $g(x)$ grows faster than $f(x)$

9.3.2 Relationship to the binomial distribution

The poisson distribution can be derived as the limit of a binomial distribution as the number of trials n goes to infinity, and the probability of success p goes to zero, such that $np = \lambda$. To show this, we take an interval $(s, s+t]$ and divide it into n intervals of size t/n . The number of events in the i th interval is given by N_i . Since events do not happen simultaneously,

$$P(N_i \geq 2) = o(t/n), t \rightarrow 0$$

This is the equivalent of saying that $N_i \sim \text{Bernoulli}(p)$, where $p = \lambda(t/n) + o(t/n)$.

The total number of events in the interval is then Binomially distributed

$$N(t+s) - N(s) \sim \text{Binom}(n, p) = \text{Binom}(n, \lambda t/n)$$

So, as n approaches infinity, np approaches λt , and $N(t+s) - N(s) \sim \text{Pois}(\lambda t)$.

9.3.3 Inter-arrival times

Waiting times between events (inter-arrival times) in a poisson process are exponentially distributed. To show this, we consider the waiting times between the first and second arrival T_1 and T_2 respectively.

$$\begin{aligned} P(T_1 > t) &= P(N(t) = 0) \\ &= \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t} \end{aligned}$$

So the waiting time until the first arrival is exponentially distributed. For the second arrival event, given the first arrival occurred at time s , the waiting time is

$$\begin{aligned} P(T_2 > t) &= \int_s^\infty P(T_2 > t | T_1 = s) P(T_1 = s) \\ &= \int_s^\infty P(0 \text{ events in } (s, s+t] | T_1 = s) P(T_1 = s) \end{aligned}$$

By independence,

$$\begin{aligned} &= \int_s^\infty P(0 \text{ events in } (s, s+t]) P(T_1 = s) \\ &= e^{-\lambda t} \int_s^\infty P(T_1 = s) \\ &= e^{-\lambda t} \end{aligned}$$

By the same logic, the inter-arrival time (waiting time in between events) for any i th interval is also exponentially distributed $T_i \sim \exp(\lambda)$. Then, the waiting time until the n th arrival is the sum of all i events $S_n = \sum_{i=1}^n T_i$. Since the inter-arrival times are exponentially distributed, the n th arrival time is gamma distributed $S_n \sim \text{Gamma}(\lambda, n)$

If we know the number of events in some given interval $(0, t]$, then the distribution of inter-arrival times conditioned on the number of events is uniformly distributed on $(0, t]$. We show this for the simple case with one event.

$$\begin{aligned} P(T_1 > s | N(t) = 1) &= \frac{P(T_1 > s, N(t) = 1)}{P(N(t) = 1)} \\ &= P(0 \text{ events in } (0, s], 1 \text{ event in } (s, t]) \\ &= \frac{e^{-\lambda s} \lambda(t-s) e^{-\lambda(t-s)}}{\lambda t e^{-t}} \\ &= \frac{t-s}{t} \end{aligned}$$

To get to the uniform distribution,

$$\begin{aligned} P(T_1 \leq s | N(t) = 1) &= 1 - P(T_1 > s | N(t) = 1) \\ &= \frac{s}{t} \end{aligned}$$

9.3.4 Splitting and superposition

In order to develop point processes suitable for certain models, one can employ mathematical operations that remove points from Poisson process (splitting i.e. thinning), or combining points from multiple Poisson processes (superposition).

Superposition

If we combine the points from two Poisson processes with rates r_1 and r_2 , the result is a Poisson process with rate $r_1 + r_2$. More formally, suppose we have two Poisson processes of rate λ_1 and λ_2 , given by $\{N_1(t); t > 0\}$ and $\{N_2(t); t > 0\}$. The two are independent if for all t_1, \dots, t_n , the random variables $N_1(t_1), \dots, N_1(t_n)$ are independent of $N_2(t_1), \dots, N_2(t_n)$. Suppose $N(t) = N_1(t) + N_2(t)$ for all $t > 0$. Then $\{N_t; t > 0\}$ then a Poisson process with rate $\lambda_1 + \lambda_2$.

Splitting

Now suppose we have a Poisson process $\{N_t; t > 0\}$ with rate λ . Suppose each arrival is switched to $\{N_1(t); t > 0\}$ with probability p and switched to $\{N_2(t); t > 0\}$ with probability $(1-p)$. Then $\{N_1(t); t > 0\}$ is a Poisson process with rate λp and $\{N_2(t); t > 0\}$ is a Poisson process with rate $\lambda(1-p)$.

9.3.5 Compound Poisson processes

Compound Poisson processes assign a random value or weight to each point in a Poisson process.

9.3.6 Non-homogenous Poisson processes

So far, we have considered Poisson processes where the rate parameter λ is constant for all time $t > 0$. A nonhomogenous Poisson process is one where the ‘rate,’ referred to as an intensity function $\lambda(t)$ can vary over time. $\{N_t; t > 0\}$ is a nonhomogenous or inhomogenous Poisson process if

1. $N(0) = 0$
2. $\{N(t), t \geq 0\}$ has independent increments
3. $P(N(t+h) - N(t) \geq 2) = o(h)$
4. $P(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$

Note that a nonhomogenous Poisson process with $\lambda(t) = \lambda$ for all $t > 0$ is a homogenous Poisson process.

Also note that for the nonhomogenous Poisson process, the interarrival times are no longer exponentially distributed, nor are they independent.

9.3.7 Compound or mixed ‘Poisson’ processes