Comp. Bio. I. Continuous-time Markov processes

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March 16, 2017

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1. Superposition of independent Poisson processes

Let $\{N_{1,t}; t \geq 0\}$ and $\{N_{2,t}; t \geq 0\}$ be independent Poisson processes with rates λ_1 and λ_2. We are interested in the stochastic process $\{N_t; t \geq 0\}$ defined as

$N_t = N_{1,t} + N_{2,t}$.

From the fact that the sum of the number of arrivals in two non-overlapping intervals of a Poisson process follows a Poisson distribution, we have a reason to believe that $\{N_t; t \geq 0\}$ is a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$. This can be proven directly:

(1) $\Pr(N_t = k) = \sum_{l=0}^{k} \Pr(N_{1,t} = k - N_{2,t} \mid N_{2,t} = l) \Pr(N_{2,t} = l) = \frac{e^{-(\lambda_1+\lambda_2)t}[(\lambda_1+\lambda_2)t]^k}{k!} \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} (\frac{\lambda_1}{\lambda_1+\lambda_2})^{k-l} (\frac{\lambda_2}{\lambda_1+\lambda_2})^{l}$.

The sum in the last term equals 1, which gives us the desired result. (Implicit here is the fact that if a stochastic process $\{X_t; t \geq 0\}$ is such that X_t follows a Poisson distribution with mean λt for all $t \geq 0$, then $\{X_t; t \geq 0\}$ is a Poisson proces with rate λ.) By induction, the superposition of any finite number of independent Poisson processes is a Poisson process with rate equal to the sum of the rates of the individual processes.

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2. Thinning of a Poisson process.

Let us now consider dividing, or "thinning," $\{N_t; t \geq 0\}$ into two subprocesses. Every time an arrival occurs, it is classified as belonging to subprocess $N_{1,t}$ with probability p_1 and $N_{2,t}$ with probability $p_2 = 1 - p_1$. This classification occurs independently for each arrival. Then $\{N_{1,t}; t \geq 0\}$ and $\{N_{2,t}; t \geq 0\}$ are independent Poisson processes with rates λp_1 and λp_2:

(2) $\Pr(N_{1,t} = k) = \sum_{l=0}^{\infty} \Pr(N_t = k + l) \frac{(k+l)!}{k!l!} p_1^k p_2^l = e^{-\lambda t} \frac{(\lambda p_1 t)^k}{k!} \sum_{l=0}^{\infty} \frac{(\lambda p_2 t)^l}{l!} = e^{-\lambda p_1 t} \frac{(\lambda p_1 t)^k}{k!}$.

(3) $\Pr(N_{1,t} = k, N_{2,t} = l) = \Pr(N_t = k + l) \frac{(k+l)!}{k!l!} p_1^k p_2^l = e^{-\lambda(p_1+p_2)t} \frac{(\lambda p_1 t)^k (\lambda p_2 t)^l}{k!l!} = \Pr(N_{1,t} = k) \Pr(N_{2,t} = l)$.

The thinning of a Poisson process into more than two subprocesses can be described similarly.

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3. Non-stationary Poisson processes.

Let $\{N_t; t \geq 0\}$ be a stochastic process such that $N_0 = 0$, arrivals happen at most by one unit at a time, and the number of arrivals in any two non-overlapping intervals are independent. Unlike the stationary Poisson processes we have seen above, however, $\{N_t; t \geq 0\}$ is non-stationary: the rate of arrival $\lambda = \lambda(t)$ varies with time. From the independent increment property, $E(N_t)$ can be calculated:

(4) $E(N_t) = E(\sum_{i=0}^{n}[N_{(t/n)i} - N_{(t/n)(i-1)}]) = \sum_{i=0}^{n} E[N_{(t/n)i} - N_{(t/n)(i-1)}] \approx \sum_{i=0}^{n} \lambda[(t/n)i](t/n)$

In the limit that $n \to \infty$,

(5) $E(N_t) = \int_0^t \lambda(t')dt'$

Let $a(t) = E(N_t)$ and assume that $a(t)$ is continuous. We define the inverse of $a(t)$:

(6) $\tau(t) = inf\{s; a(s) > t\}$

We then define a new stochastic process $\{M_t; t \geq 0\}$ in the following way:

(7) $M_t = N_{\tau(t)}$.

It can be shown that $\{M_t; t \geq 0\}$ is a stationary Poisson process with rate 1. The arrival number and waiting time distributions for $\{N_t; t \geq 0\}$ can be calculated from Eq. 7. It turns out that

(8) $N_t - N_s$ follows a Poisson distribution with rate $a(t-s) = \int_s^t \lambda(t')dt'$.

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4. Non-stationary thinning of a Poisson process

Let $\{N_t; t \geq 0\}$ be a Poisson process with rate $\lambda(t)$, and suppose that thinning occurs with non-stationary probabilties $p_1(t)$ and $p_2(t) = 1 - p_1(t)$. The resulting subprocesses can be described as non-stationary Poisson processes with rates $\lambda(t)p_1(t)$ and $\lambda(t)p_2(t)$.

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5. M/G/∞ queue

Consider a thinning scheme where arrivals are classified not at the moment they occur but later at a certain given time. Such a stochastic process is called the M/G/∞ queue. Suppose that a stationary Poisson process $\{N_t; t \geq 0\}$ is being thinned into n states. Starting at time t_0, arrival times are recorded and arrivals classified at time t_1. If $u \in (t_0, t_1)$ is an arrival time, then this arrival has probability $p_i(t_1 - u)$ of being classified as state i. Thus, according to Sections 3 and 4, the number of state i observed at t_1 follows a Poisson distribution with mean $\lambda \int_{t_0}^{t_1} p_i(t_1 - u)du$.

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6. A differential equation framework

Let us take a different approach to characterizing continuous-time stochastic processes. Let $\{N_t; t \geq 0\}$ be a stochastic process and define $P_i(t) = \Pr(N_t = i)$. From known structural features of $\{N_t; t \geq 0\}$, a set of differential equations for $P_i(t)$ can be derived and solved. Let us take a stationary Poisson process with rate λ as an example. For visibility, we write $N(t)$ instead of N_t. The following equation can be derived from the independent increment property.

(9) $\Pr[N(t+h) = j] = \sum_{i \leq j} \Pr[N(t) = i] \Pr[N(t+h) - N(t) = j - i]$

Only $i \leq j$ is considered because $N(t)$ is non-decreasing. Notice that

(10) $\Pr[N(t+h) - N(t) \geq 2] = \Pr(T_2 \leq h)$,

where T_2 is the sum of two interarrival times. From the independent increment and stationarity properties, it can be shown that the interarrival times of a Poisson process follows an exponential distribution with rate λ and hence that T_2 follows a $\Gamma(2, \lambda)$ distribution. Then,

(11) $\lim_{h\to 0} \frac{\Pr[N(t+h)-N(t)\geq 2]}{h} = \lim_{h\to 0} \frac{\Pr(T_2 \leq h)}{h} = \frac{d}{dt}\Pr(T_2 \leq t)\mid_{t=0} = \lambda^2 t e^{-\lambda t}\mid_{t=0} = 0.$

Notice also that since $\Pr[N(t+h) - N(t) = 1] = \Pr(T_1 \leq h)$ where T_1 follows an exponential distribution with rate λ,

(12) $\lim_{h\to 0} \frac{\Pr[N(t+h)-N(t)=1]}{h} = \lim_{h\to 0} \frac{\Pr(T_1 \leq h)}{h} = \frac{d}{dt}\Pr(T_1 \leq t)\mid_{t=0} = \lambda e^{-\lambda t}\mid_{t=0} = \lambda$

Let $o(h)$ denote functions for which $\lim_{h\to 0} o(h)/h = 0$. Expanding Eq. 9,

(13) $\Pr[N(t+h) = j] = \Pr[N(t) = j][1 - \lambda h + o(h)] + \Pr[N(t) = j-1][\lambda h + o(h)] + o(h)$

Rearranging Eq. 13, the following differential equation can be obtained.

(14) $\frac{dP_j(t)}{dt} = -\lambda P_j(t) + \lambda P_{j-1}(t).$

(15) $\frac{dP_0(t)}{dt} = -\lambda P_0(t)$

By introducing the initial condition $P_0(0) = 1$, Eqs. 14 and 15 can be solved to give the Poisson distribution probabilities.

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7. Birth-death processes

The differential equation framework outlined above allows a precise description of stochastic processes more complex than Poisson processes. For example, by letting λ to vary as follows,

(16) $\lambda_j = \lambda j$,

Eqs. 14 and 15 can be modified as

(17) $\frac{dP_j(t)}{dt} = -\lambda j P_j(t) + \lambda(j-1)P_{j-1}(t)$

(18) $\frac{dP_1(t)}{dt} = -\lambda P_1(t)$

The stochastic process described by these equations is called the pure-birth process or the Yule process. Notice that unlike Poisson processes, the state space of a pure-birth process consists of $E = (1, 2, \dots)$.

By introducing death parameter μ_j, the following general birth-death process can also be defined.

(19) $\frac{dP_j(t)}{dt} = -(\lambda_j + \mu_j)P_j(t) + \lambda_{j-1}P_{j-1}(t) + \mu_{j+1}P_{j+1}(t)$

(20) $\frac{dP_0(t)}{dt} = -\lambda_0 P_j(t) + \mu_1 P_1(t)$

These differential equations are complex enough that only stationary distribution has an analytic form.

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8. Continuous-time Markov processes

Finally, we consider a continuous-time Markov process on bounded, finite state space $E = \{1, 2, \ldots, s\}$. Let $P_{ij}(t)$ denote the probability of ending at state j at time t when the system began at state i. The following differential equation, called the Kolmogorov forward equation, describes the forward evolution of the system.

(21) $\frac{dP_{ij}(t)}{dt} = -vP_{ij}(t) + \sum_{k \neq j} P_{ik}(t)q_{kj}$

where q_{kj} is the instantaneous rate of the transition $k \to j$, and $v_j = \sum_{k \neq j} q_{jk}$. By constructing the rate matrix Q,

(22) $Q_{ij} = q_{ij}$ for $i \neq j$ and $Q_{ii} = -v_i$,

the following matrix-form differential equation can be derived:

(23) $\frac{dP}{dt} = PQ$.