Poisson Processes

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1 Counting Processes

1.1 Background and Criteria

A counting process $N(t)$ satisfies the following criteria:

1. $N(0) = 0$
2. If $s < t$, $N(s) \leq N(t)$
3. $N(t) > 0, t \geq 0$
4. $N(t) - N(s) = x$ where x is the number of arrivals in the interval (s, t)

DEFINITION:

THE WAITING TIME/ARRIVAL TIME has the MEMORYLESS PROPERTY in a special case. The MEMORYLESS PROPERTY is the following:

$$\mathbb{P}(X > s + t | X > t) = \mathbb{P}(X > s)$$

REMARK:

The EXPONENTIAL DISTRIBUTION has this MEMORYLESS PROPERTY.

$$\text{Exponential Density: } f_\lambda(t) = \lambda e^{-\lambda t}$$
$$\text{Exponential CDF:} \mathbb{P}(X > t) = e^{-\lambda t}$$

PROOF:

$$\begin{aligned}
\mathbb{P}(X > s + t | X > t) &= \frac{\mathbb{P}(X > s + t, X > t)}{\mathbb{P}(X > t)} \\
&= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} \\
&= e^{-\lambda s} \\
&= \mathbb{P}(X > s)
\end{aligned}$$

DEFINITION:

The analogous distribution for discrete distributions (the counterpart to the continuous exponential) is the GEOMETRIC DISTRIBUTION:

$$\mathbb{P}(T = t) = (1 - p)^{t-t} p$$

Examples of Counting Processes in Biological Settings:

1. # of mutations on a branch of a tree
2. # recombinations along a chromosome
3. Counting processes also show up often in Queuing Theory

1.2 Properties of Counting Processes

There are two important properties that counting processes have:

1. Stationary increments: $\mathbb{P}(N(t+s) - N(t))$ does not depend on t
2. Independent Increments: Any pair of increments along the process are independent from one another. That is $N(t_2) - N(t_1), N(t_3) - N(t_2), ..., N(t_n) - N(t_{n-1})$ are all independent of one another.

2 Poisson Processes

2.1 Definition of the Poisson Process

Definition:
A Counting Process is a Poisson Process if:

1. $N(0) = 0$
2. Stationary Increments is satisfied
3. Independent Increments is satisfied

2.2 Aside: Little-O notation

Definition:
A function $f(x)$ is $o(g(x))$ if:

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 0$$

Example:
The function $f(x) = x^2$ is $o(x)$ as:

$$\begin{aligned}\lim_{x \to 0} \frac{f(x)}{g(x)} &= \lim_{x \to 0} \frac{x^2}{x} \\ &= \lim_{x \to 0} x \\ &= 0\end{aligned}$$

2.3 Poisson Process Properties

Properties:

1. $N(0) = 0$
2. $N(t), t \geq 0$ has independent increments
3. $\mathbb{P}(N(t+h) - N(t) \geq 2) = o(h)$
4. $\mathbb{P}(N(t+h) - N(t) = 1) = \lambda h + o(h)$

Note that λ is the *instantaneous rate.*

2.4 Relationship to the Bernoulli Process

Suppose we have a one dimensional line segment starting at point s and going to point $s+t$. Let this line be divided into n segments where $\delta = \frac{t}{n}$ is the width of one of the sub intervals. Let $N_i = \#$ events in interval i.

$$\mathbb{P}(N_i \geq 2) = o(\frac{t}{n})$$
$$\mathbb{P}(N_i = 1) = \lambda(\frac{t}{n}) + o(\frac{t}{n})$$

Therefore we have that $N_i \sim Bernoulli(\frac{\lambda t}{n})$.

$$N(s+t) - N(s) = \sum N_i$$
$$N(s+t) - N(s) \sim Bin(n, \frac{\lambda t}{n})$$

Therefore by property of binomial we have $\mathbb{E}[N(s+t) - N(s)] = \lambda t$.

2.5 Aside: Limit of the Binomial Distribution to Poisson

Let $X \sim Bin(p,n)$. As $n \to \infty$, $X \sim Bin(p,n)$ if we have $np \to c$ we have $X \sim Pois(c)$ as $n \to \infty$.

2.6 Returning to the Bernoulli Process Limit

Therefore, using the previous aside:
$\implies N(s+t) - N(s)$ with n large $\implies \lim_{n\to\infty} N(s+t) - N(s) \sim Pois(\lambda t)$ as desired.

2.7 Exponentially Distributed Interarrival Times

The interarrival times are exponentially distributed with parameter λ.

PROOF:
Let X_n be the interrarrival times.

$$\begin{aligned}
\mathbb{P}(X_n > s | X_{n-1} = t) &= \mathbb{P}(N(t+s) - N(t) = 0 | X_{n-1} = t) \\
&= \mathbb{P}(N(s) = 0) \\
&= e^{-\lambda s}
\end{aligned}$$

Therefore the interarrival times X_n are distributed exponentially with parameter λ.

2.8 Total Waiting Time

Let S_n be the total waiting time until the n^{th} event.

$$\begin{aligned}
S_n &= \sum_{j=1}^{n} X_j \\
&\sim Gamma(\lambda, n)
\end{aligned}$$

This follows as the sum of n exponential random variables with identical parameters λ is distributed Gamma.

2.9 Distribution of Arrivals within an Interval

The distribution of arrivals in the interval are $Uniform$.

PROOF:

$$\begin{aligned}
\mathbb{P}(T_1 < s | N(t) = 1) &= \frac{\mathbb{P}(\text{1 point in } [0, s],\ \text{0 events in } [s, t])}{\mathbb{P}(\text{1 event in [0,t]})} \\
&= \frac{(\lambda s)e^{-\lambda s} e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} \\
&= \frac{s}{t}
\end{aligned}$$

Therefore, this is a property of the uniform distribution so the distribution of arrivals in the interval is $Uniform$.

2.10 Combining Different Poisson Processes

For example, let there be two Poisson Processes. Let there be Poisson Process 1 (PP_1) $N_1(t)$, Poisson Process 2 (PP_2) $N_2(t)$ and let the combined Poisson Process be $N(t)$.
$N_1(t)$ is $PP(\lambda_1)$
$N_2(t)$ is $PP(\lambda_2)$
$N(t)$ is $PP(\lambda_1 + \lambda_2)$

Therefore, the superposition of Poisson Processes is a Poisson Process.

If we have a Poisson Process that is the combination of several Poisson Processes we have the following:

At any time points, the probability of the next event being type i (from Poisson Process i), where $i \in \{1, 2, ..., k\}$, in the combined Poisson Process:

$$\mathbb{P}(\text{next event being type i}) = \frac{\lambda_i}{\sum_{j=1}^{k} \lambda_j}$$

THINNING PROPERTY -

PP with rate λ and arrivals are assigned to type i in this PP with probability p_i. Then $N_i(t) \sim PP(\lambda_{p_i})$.