Topology

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CONTENTS

CHAPTER 1

SET THEORY AND LOGIC

We shall assume that what is meant by a *set* of objects is intuitively clear, and we shall proceed on that basis without analyzing the concept further.

1 Fundamental Concepts

1.1 Basic Notation

Commonly we shall use capital letters $A, B, \ldots$ to denote sets, an lowercase letters $a, b, \ldots$ to denote the ***objects*** or ***elements*** belonging to these sets.

We say that A is a ***subset*** of B is every element of A is also and element of B; and we express this fact by writing

$$A \subset B.$$

If $A \subset B$ and A is different from B, we say that A is a ***proper subset*** if B, and we write

$$A \subsetneq B.$$

The relations $\subset$ and $\subsetneq$ are called ***inclusion*** and ***proper inclusion***, respectively. If $A \subset B$, we also write $B \supset A$, which is read "B ***contains*** A."

1.2 The Union of Sets and the Meaning of "or"

Given two sets A and B, one can form a set from them that consists of all the elements of A together with all the elements of B. This set is called the ***union*** of A and B and is denoted by $A \cup B$.

1.3 The Intersection of Sets, the Empty Set, and the Meaning of "If ... Then"

Given sets A and B, another way one can form a set is to take the common part of A and B. This set is called the ***intersection*** of A and B and is denoted by $A \cap B$.

We introduce a special set that we call the ***empty set***, denoted by $\varnothing$, which we think of as "the set having no elements."

Using this convention, we express the statement that A and B have no elements in common by the equation

$$A \cap B = \varnothing.$$

We also express this fact by saying that A and B are ***disjoint***.

Mathematicians have agreed always to use "if ... then" in the first sense, so that a statement of the form "If P, then Q" means that if P is true, Q is true also, but if P is false, Q may be either true or false.

As an example, consider the following statement about real numbers:

$$If\ x > 0,\ then\ x^3 \neq 0.$$

It is a statement of the form, "If P, then Q," where P is the phrase "$x > 0$" (called the ***hypothesis*** of the statement) and Q is the phrase "$x^3 \neq 0$" (called the ***conclusion*** of the statement).

Another true statement about real numbers is the following:

$$If\ x^2 < 0,\ then\ x = 23;$$

in every case for which the hypothesis holds, the conclusion holds as well. Of course, it happens in this example that there are no cases for which the hypothesis holds. A statement of this sort is sometimes said to be ***vacuously true***.

1.4 Contrapositive and Converse

Give a statement of the form "If P, then Q," its ***contrapositive*** is defined to be the statement "If Q is not true, then P is not true."

There is another statement that can be formed from the statement $P \Rightarrow Q$. It is the statement

$$Q \Longrightarrow P,$$

which is called the ***converse*** of $P \Rightarrow Q$.

1.5 Negation

If one wishes to form the contrapositive of the statement $P \Rightarrow Q$, one has to know how to form the statement "not P", which is called the ***negation*** of P.

1.6 The Difference of Two Sets

There is one other operation on sets that is occasionally useful. It is the ***difference*** of two sets, denoted by $A - B$, and defined as the set conssiting of those elements of A that are not in B. It is sometimes called the ***complement*** of B relative to A, or the complement of B *in* A.

1.7 Collections of Sets

Given a set A, we can consider sets whose elements are subsets of A. In particular, we can consider the set of all subsets of A. This set is sometimes denoted by the symbol $\mathcal{P}(A)$ and is called the ***power set*** of A.

When we have a set whose elements are sets, we shall often refer to it as a ***collection*** of sets and denote it by a script letter.

1.8 Arbitrary Unions and Intersections

Given a collection $\mathcal{A}$ of sets, the ***union*** of the elements of $\mathcal{A}$ is defined by the equation

$$\bigcup_{A\in\mathcal{A}} A = \{x \mid x \in A \text{ for at least one } A \in \mathcal{A}\}.$$

The ***intersection*** of the element of $\mathcal{A}$ is defined by the equation

$$\bigcap_{A\in\mathcal{A}} A = \{x \mid x \in A \text{ for every } A \in \mathcal{A}\}.$$

2 Functions

First, we define the following:

Definition 2.1

A ***rule of assignment*** is a subset r of the cartesian product $C \times D$ of two sets, having the property that each element of C appears as the first coordinate of *at most one* ordered pair belonging to r.

Given a rule of assignment r, the ***domain*** of r is defined to be the subset of C consisting of all first coordinates of elements of r, and the ***image set*** of r is defined as the subset of D consisting of all second coordinates of elements of r.

Now we can say what a function is:

Definition 2.2

A ***function*** f is a rule of assignment r, together with a set B that contains the image set of r. The domain A of the rule r is also called the ***domain*** of the function f; the image set of r is also called the ***image set*** of f; and the set B is called the ***range*** of f.

If $f : A \to B$ and if a is an element of A, we denote by $f(a)$ the unique element of B that the rule determining f assigns to a; it is called the ***value*** of f at a, or sometimes the ***image*** of a under f.

Definition 2.3

If $f : A \to B$ and if A_0 is a subset of A, we define the ***restriction*** of f to A_0 to be the function mapping A_0 into B whose rule is

$$\{(a, f(a)) \mid a \in A_0\}.$$

Definition 2.4

Given functions $f : A \to B$ and $g : B \to C$, we define the ***composite*** $g \circ f$ of f and g as the function $g \circ f : A \to C$ defined by the equation $(g \circ f)(a) = g(f(a))$.

Definition 2.5

A function $f : A \to B$ is said to be ***injective*** (or ***one-to-one***) if for each pair of distinct points of A, their images under f are distinct. It is said to be ***surjective*** (or f is said to map A ***onto*** B) if every element of B is the image of some element of A under the function f. If f is both injective and surjective, it is said to be ***bijective*** (or is called a ***one-to-one correspondence***).

If f is bijective, there exists a function from B to A called the ***inverse*** of f.
A useful criterion for showing that a given function f is bijective is the following,

Lemma 2.6

Let $f : A \to B$. If there are functions $g : B \to A$ and $h : B \to A$ such that $g(f(a)) = a$ for every a in A and $f(h(b)) = b$ for every b in B, then f is bijective and $g = h = f^{-1}$.

Definition 2.7

Let $f : A \to B$. If A_0 is a subset of A, we denote by $f(A_0)$ the set of all images of points of A_0 under the function f; this set is called the ***image*** of A_0 under f. Formally,

$$f(A_0) = \{b \mid b = f(a) \text{ for at least one } a \in A_0\}.$$

On the other hand, if B_0 is a subset of B, we denote by $f^{-1}(B_0)$ the set of all elements of A whose images under f lie in B_0; it is called the ***preimage*** of B_0 under f (or the "counterimage," or the "inverse image," of B_0).

3 Relations

Definition 3.1

A ***relation*** on a set A is a subset C of the cartesian product $A \times A$.

3.1 Equivalence Relations and Partitions

An ***equivalence relation*** on a set A is a relation C on A having the following three properties:

1. (Reflexivity) xCx for every x in A.
2. (Symmetry) If xCy, then yCx.
3. (Transitivity) If xCy and yCz, then xCz.

Given an equivalence relation $\sim$ on a set A and an element x of A, we define a certain subset E of A, called the ***equivalence class*** determined by x, by the equation

$$E = \{y \mid y \sim x\}.$$

Equivalence classes have the following property:

Lemma 3.2

Two equivalence classes E and E' are either disjoint or equal.

Given an equivalence relation on a set A, let us denote $\mathcal{E}$ the collection of all the equivalence classes determined by this relation. The collection $\mathcal{E}$ is a particular example of what is called a partition of A:

Definition 3.3

A ***partition*** of a set A is a collection of disjoint nonempty subsets of A whose union is all of A.

3.2 Order Relations

A relation C on a set A is called an ***order relation*** (or a ***simple order****, or a* ***linear order***) if it has the following properties:

1. (Comparability) For every x and y in A for which $x \neq y$, either xCy or yCx.
2. (Nonreflexivity) For no x in A does the relation xCx hold.
3. (Transitivity) If xCy and yCz, then xCz.

Definition 3.4

If X is a set and $<$ is an order relation on X, and if $a < b$, we use the notation (a, b) to denote the set

$$\{x \mid a < x < b\};$$

it is called an ***open interval*** in X. If this set is empty, we call a the ***immediate predecessor*** of b, and we call b the ***immediate successor*** of a.

Definition 3.5

Suppose that A and B are two sets with order relations $<_A$ and $<_B$ respectively. We say that A and B have the same ***order type*** if there is a bijective correspondence between them that preserves order; that is, if there exists a bijective function $f : A \to B$ such that

$$a_1 <_A a_2 \Longrightarrow f(a_1) <_B f(a_2)$$

One interesting way of defining an order relation, which will be useful to us later in dealing with some examples, is the following:

Definition 3.6

Suppose that A and B are two sets with order relations $<_A$ and $<_B$ respectively. Define an order relation $<$ on $A \times B$ by defining

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$, or if $a_1 = a_2$ and $b_1 <_B b_2$. It is called the ***dictionary order relation*** on $A \times B$.