The Asymmetric Traveling Salesman Problem (ATSP)

1) ILP formulation with cut-set inequalities.

Let $G = (V, E)$ be a complete directed graph, with a cost $c_{ij} \in \mathbb{R}$ for each arc $(i, j) \in E$. Consider the following Integer Linear Programming formulation (Dantzig, Fulkerson, Johnson, 1959) for the ATSP:

$$
\begin{aligned}
(DFJ) \qquad & \min \sum_{(i,j)\in E} c_{ij}x_{ij} && && (1)\\
& s.t. \sum_{j\in V: j\neq i} x_{ij} = 1 && i \in V && (2)\\
& \sum_{i\in V: i\neq j} x_{ij} = 1 && j \in V && (3)\\
& \sum_{(i,j)\in \delta^+(S)} x_{ij} \geq 1 && S \subset V, 1 \in S, |S| > 1 && (4)\\
& x_{ij} \in \{0, 1\} && (i, j) \in E, && (5)
\end{aligned}
$$

where the constraints (4) are the *cut-set inequalities*, and its linear relaxation:

$$
\begin{aligned}
(DFJ^0) \qquad & \min \sum_{(i,j)\in E} c_{ij}x_{ij} && && (6)\\
& s.t. \sum_{j\in V: j\neq i} x_{ij} = 1 && i \in V && (7)\\
& \sum_{i\in V: i\neq j} x_{ij} = 1 && j \in V && (8)\\
& \sum_{(i,j)\in \delta^+(S)} x_{ij} \geq 1 && S \subset V, 1 \in S, |S| > 1 && (9)\\
& x_{ij} \geq 0 && (i, j) \in E. && (10)
\end{aligned}
$$

a) Write with AMPL and solve the original formulation (DFJ) and its linear relaxation (DFJ^0). A sketch of the file `.mod` is given on page 3 and following.

b) Compare the optimal solutions of (DFJ) and (DFJ^0). What do you observe?

2) **Compact extended ILP formulation.**

Consider the partial ILP formulation for the ATSP with only the assignment constraints:

$$\min \sum_{(i,j)\in E} c_{ij}x_{ij} \tag{11}$$

$$s.t. \sum_{j\in V:j\neq i} x_{ij} = 1 \forall i \in V \tag{12}$$

$$\sum_{i\in V:i\neq j} x_{ij} = 1 \forall j \in V \tag{13}$$

$$x_{ij} \in \{0,1\} \quad \forall (i,j) \in A. \tag{14}$$

c) Starting from the partial formulation (11)-(14), propose a compact extended formulation for ATSP. *Hint: add, for each $i \in V$, an integer variable t_i representing the "position" in which node i is visited in the tour, and an appropriate set of constraints used to avoid all possible subtours.*

d) Explain why such additional constraints exclude any subtour.

e) Is the integrality of variables t_i required?

f) The extended formulation of question c) is known in the literature as Miller, Tucker, Zemlin (MTZ) formulation. Write it in AMPL and find the solution of its linear relaxation for the given instance.

g) Compare the linear relaxation bound obtained with (MTZ^0) with that obtained with formulation (DFJ^0). What do you observe? Is the solution obtained with (MTZ^0) integer?

Solution sketch

The file `k6_dir.dat` contains a comlpete graph with 6 nodes (optimal value of the objective function: 137):

```
#k6_dir.dat

data;

set V := a b c d e f;

param c:  a    b    c    d   e    f   :=
     a    .    03   05   100  88   55
     b    02   .    04   69   75   55
     c    04   03   .    100  99   85
     d    99   68   99   .    03   04
     e    87   74   98   02   .    05
     f    54   54   84   03   04   .
     ;
```

The file `dfj-i.mod` contains the partial model for formulation (DFJ), to be completed:

```
#dfj-i.mod

set V ordered;
param n := card {V};

set POWERSET := 0 .. (2**n - 1);

set S {k in POWERSET} := {i in V: (k div 2**(ord(i)-1)) mod 2 = 1};

set E := {i in V, j in V: ord(i) <> ord(j)};

#add parameters declarations

#add variables declarations

#add model
```

Observation: a generic subset $S \subseteq V$ can be represented as a vector of $n = |V|$ bits with value in $\{0,1\}$, where position i has value 1 if the node i belongs to S, and 0 otherwise. Then the representation in base-2 of the integers between 0 and $2^n - 1$ gives all possible bit vectors of length n. Then, to build all possible subsets S of V, let

```
set POWERSET := 0 ..  (2**n - 1);
```

the set of indices from 0 to $2^n - 1$. We define the k-th subsets $S[k]$ with the binary representation of k: $S[k]$ contains the nodes whose corresponding bit is equal to 1. In base-2, the i-th figure of k_2 is equal to 1 if and only if

```
(k div 2**(ord(i)-1)) mod 2 = 1
```

then we can write all possible (exponentially many!) subsets as follows:

```
set S {k in POWERSET} := {i in V: (k div 2**(ord(i)-1)) mod 2 = 1};
```

The model can be solved running the file `dfj.run`:

```
model dfj.mod
data k6_dir.dat
option solver cplex;

solve;
display x;

printf "Linear relaxation\n\n";
option relax_integrality 1;

solve;
display x;
```

The file `MTZ-i.mod` contains the partial model of formulation (MTZ):

```
# SETS

set V ordered;

set V_reduced := {i in V : ord(i) > 1};

param n := card(V);

set E := {i in V, j in V: ord(i) <> ord(j)};

# PARAMETERS


# VARIABLES


# OBJECTIVE FUNCTION


# CONSTRAINTS
```

The model can be solved running the file `MTZ.run`:

```
model MTZ.mod
data k6_dir.dat
option solver cplex;

solve;
display x;
display t;

option relax_integrality 1;
solve;
display x;
display t;
```