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2 Univariate Random Variables

2.1 Introduction to probability model

- **Probability model** is used to describe a random exprienment.
  It consists of three important components:
  i. **Sample space** S: a collection of all possible outcomes of one random experiment.
     > e.g. Toss a coin: $S = \{H, T\}$

     > e.g. Toss a coin twice: $S = \{(H, H), (H, T), (T, H), (T, T)\}$

  ii. **Event**: denoted by A, B, C, etc. It is a subset pf sample space.
     > e.g. Toss a coin twice:
     > Define A as 1st toss is tail, $A = \{(T, T), (T, H)\} \subseteq S$

  iii. **Probability function** P: It is a function of events.
     It satisfies properties (axioms):
     a. $0 \le P(A) \le 1$ for any event A.
     b. $P(S) = 1$
     c. Countable additivity: If $A_1, A_2, \ldots$ are assumed to be pairwise multually exclusive events (i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$), $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$.

     We can now prove the following properties:
     a. $P(\emptyset) = 0$.
        > Proof: Let $A_i = \emptyset$ for $i \ge 1$, $A_i \cap A_j = \emptyset$ for $i \neq j$, by axioms we have $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$, or in other words, $P(\emptyset) = \sum_{i=1}^{\infty} P(\emptyset)$. Additionally, $0 \le P(\emptyset) \le 1$, therefore, $P(\emptyset) = 0$.

     b. Let A denote an event. Let $\bar{A}$ denote the complementary event of A, which means $\bar{A}$ saitifies two conditions:
        a. $\bar{A} \cap A = \emptyset$, and
        b. $\bar{A} \cup A = S$.

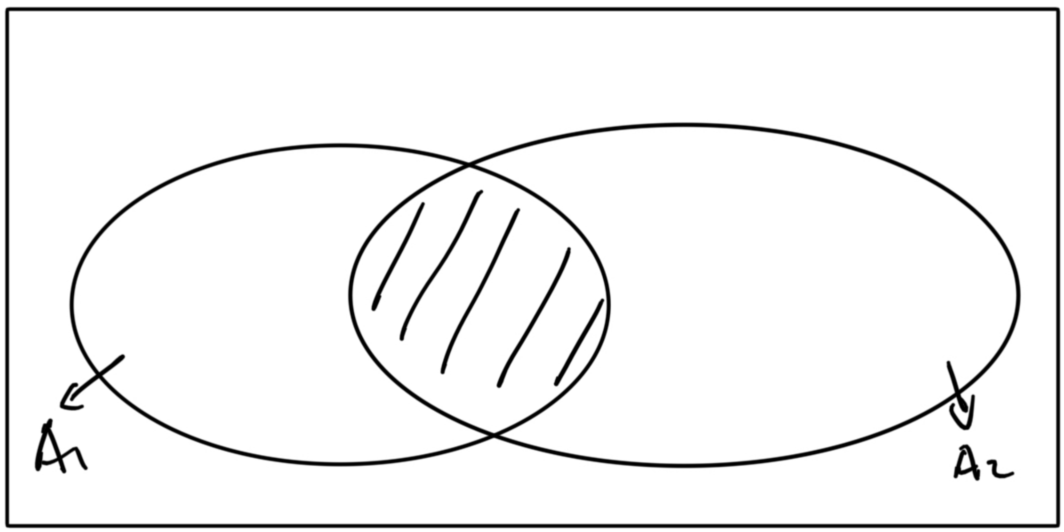
        Prove $P(A) + P(\bar{A}) = 1$:
        > Proof: Define $A_1 = A$, $A_2 = \bar{A}$, $A_i = \emptyset$ for $i \ge 3$, so $A_i \cap A_j = \emptyset$ for $i \neq j$, by axioms we have $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$, in other words, $P(S) = P(A) +$

> $P(\bar{A}) + \sum_{i=3}^{\infty} 0$, therefore, $P(A) + P(\bar{A}) = 1$.

c. If A_1 and A_2 are mutually exclusive, then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$.

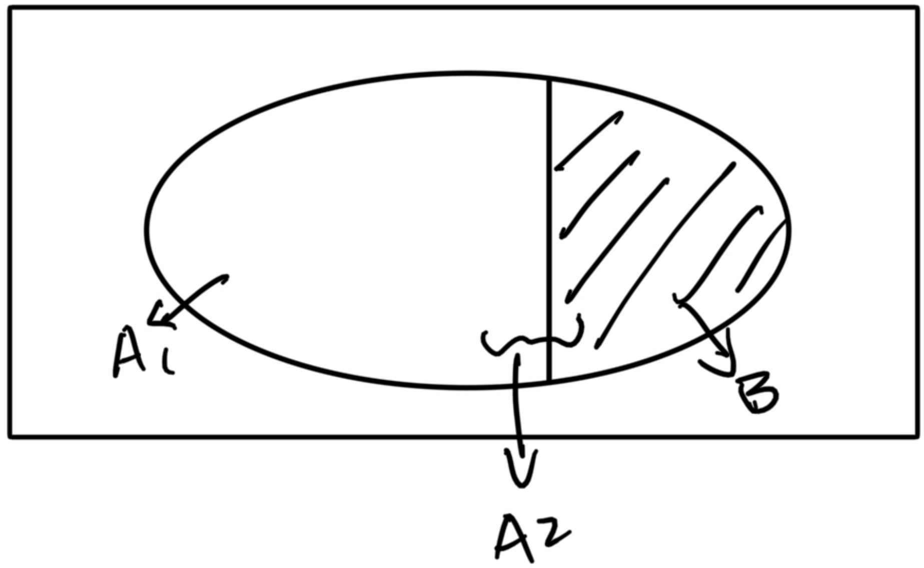
> Proof: Define $A_i = \emptyset$ for $i \geq 3$, so $S = A_i \cap A_j = \emptyset$, for $i \neq j$. Then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$, or in other words, $P(A_1 \cup A_2) = P(A_1) + P(A_2) + 0$.

d. In general, $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$.



> Proof: Define $B = \{\omega | \omega \in A_1, \omega \notin A_2\}$, since $A_1 = B \cup (A_1 \cap A_2)$, we can get $B \cap (A_1 \cap A_2) = \emptyset$, $B \cup (A_1 \cap A_2) = A_1$, $B \cap (A_1 \cap A_2) = \emptyset$, $B \cap A_2 = \emptyset$, and therefore $B \cup A_2 = A_1 \cup A_2$.
> Then $P(A_1 \cup A_2) = P(B \cup A_2) = P(B) + P(A_2)$. Note $P(A_1 \cup A_2) = P(A_2) + P(B)$ and $P(B) = P(A_1) - P(A_1 \cap A_2)$. Hence, $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$.

e. If $A_1 \subseteq A_2$, then $P(A_1) \leq P(A_2)$



> Proof: $A_2 \setminus A_1 := B = \{\omega | \omega \in A_2, \omega \notin A_1\}$, we have $B \cap A_1 = \emptyset$, $B \cup A_1 = A_2$. Then $P(A_2) = P(A_1 \cup B) = P(A_1) + P(B) \geq P(A_1)$.

> e.g. Toss a coin twice
> Then $S = \{(H,H),(H,T),(T,H),(T,T)\}$ for any event A,
>
> $$P(A) := \frac{\# \text{ of elements in } A}{4}$$
>
> Verify P is a probability function.

- **Conditional probability**
  Suppose A and B denote two events. Provided $P(B) > 0$, then the conditional probability of A given B is

  $$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

  - **Independence of two events**
    Suppose A and B denotes two events. We say A and B are independent if and only if

    $$P(A \cap B) = P(A)P(B)$$

    - Proposition: If A and B are independent, then $P(A|B) = P(A)$ (We assume $P(B) > 0$)
      > Proof: $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$

> e.g. Toss a coin twice
> $A := \text{1st toss is a head} = \{(H,T),(H,H)\}$

> $B := \text{2nd toss is a head} = \{(T,H),(H,H)\}$
> For any event C, $P(C) = \frac{\text{\# of elements in } C}{4}$
> Verify A and B are independent.
> $P(A \cap B) = P(A)P(B)$?
> By definition, $A \cap B = \{(H,H)\} \implies P(A \cap B) = \frac{1}{4}$
> $P(A) = \frac{2}{4}$, $P(B) = \frac{2}{4}$.
> Hence, $P(A \cap B) = P(A)P(B)$.

- **Random variable (r.v.)** X, Y, ζ, η

  Random variable is a function from sample space to real line.

$$X : S \to \mathbb{R}$$

  Specifically, given any $\omega \in S$, $X(\omega) \in \mathbb{R}$.

  This function satisfies that for any $x \in \mathbb{R}$, $\{X \leq x\} = \{\omega | X(\omega) \leq x\}$ is an event.

  > e.g. Toss a coin twice
  > $X : \text{\# of heads in two tosses.}$
  > $X : (H,H) \mapsto 2.$
  > We need to check for any x, $\{X \leq x\}$ is an event.
  > 1. $x \geq 2$, $\{X \leq x\} = \{\omega | X(\omega) \leq x\} = S$
  > 2. $x \in [1,2)$, what is $\{X \leq x\}$?
  > 3. $x \in [0,1)$, what is $\{X \leq x\}$?
  > 4. $x < 0$, what is $\{X \leq x\}$?

- **Cumulative distribution of X (c.d.f.)**

  For any $x \in \mathbb{R}$, the c.d.f. of X is defined as $F(x) = P(X \leq x)$.

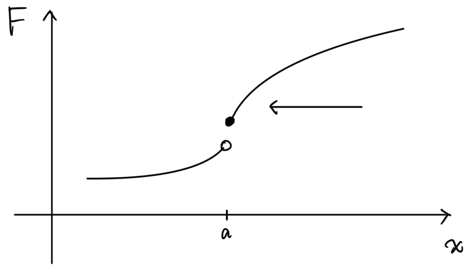
  It satisfies the following property:

  i. $F(x)$ is a non-decreasing fucntion, i.e., if $x_1 \leq x_2$, then $F(x_1) \leq F(x_2)$.

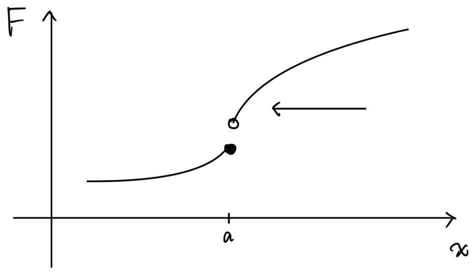
  > Proof: $\{X \leq x_1\}$ is an event. $\{X \leq x_1\} \subseteq \{X \leq x_2\}$ if $x_1 < x_2$, since $\{\omega | X(\omega) \leq x_1\} \leq \{\omega | X(\omega) \leq x_2\}$.

  ii. $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$.

  iii. $F(x)$ is a right-continuous function, i.e., for any $a \in \mathbb{R}$, $\lim_{x \to a^+} F(x) = F(a)$.



right-continuous



not right-continuous

1, 2 and 3 are three basic properties of a c.d.f.

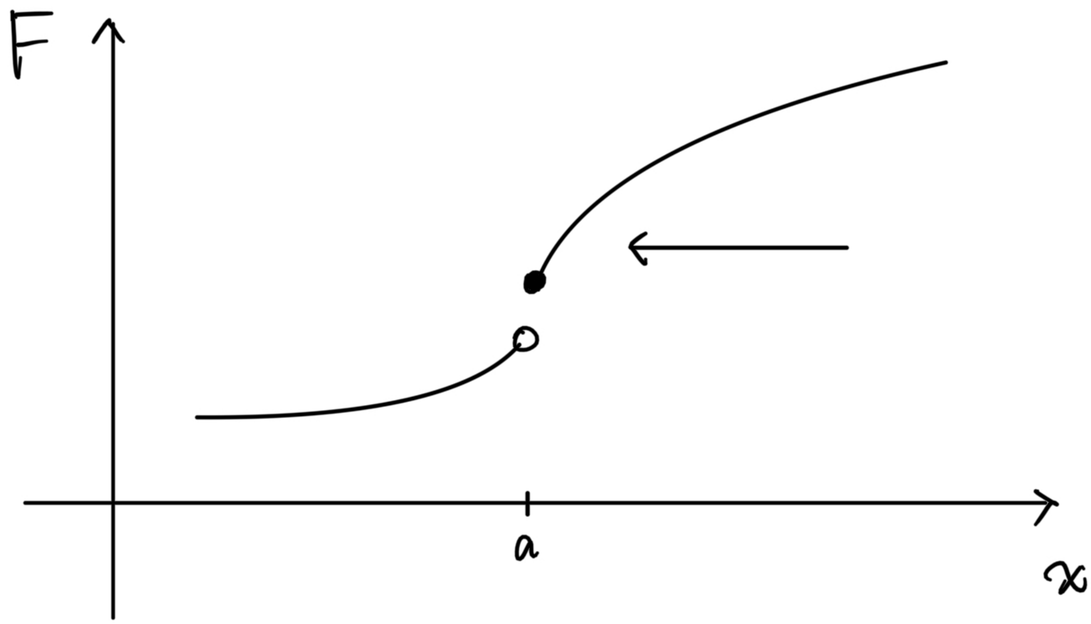
Some extra properties of a c.d.f.:

iv. $P(a < X \leq b) = F(b) - F(a)$.

> Proof: Define $A = \{X \leq b\}, B := \{X \leq a\}, C = \{a < x \leq b\}$, we want to prove: $P(a < X \leq b) = P(X \leq b) = P(X \leq a) \iff P(C) = P(A) - P(B)$. Note $B \cap C = \emptyset$, $B \cup C = A$. Then $P(A) = P(B \cup C) = P(B) + P(C)$.

v. $P(X = a) = P(X \leq a) - P(x < a) = F(a) - F(a^-)$.

> Proof: $P(X = a) = P(X \leq a) - P(X < a) = F(a) - \lim\limits_{x \to a^-} F(x) = \lim\limits_{x \to a^+} F(x) - \lim\limits_{x \to a^-} F(x)$.

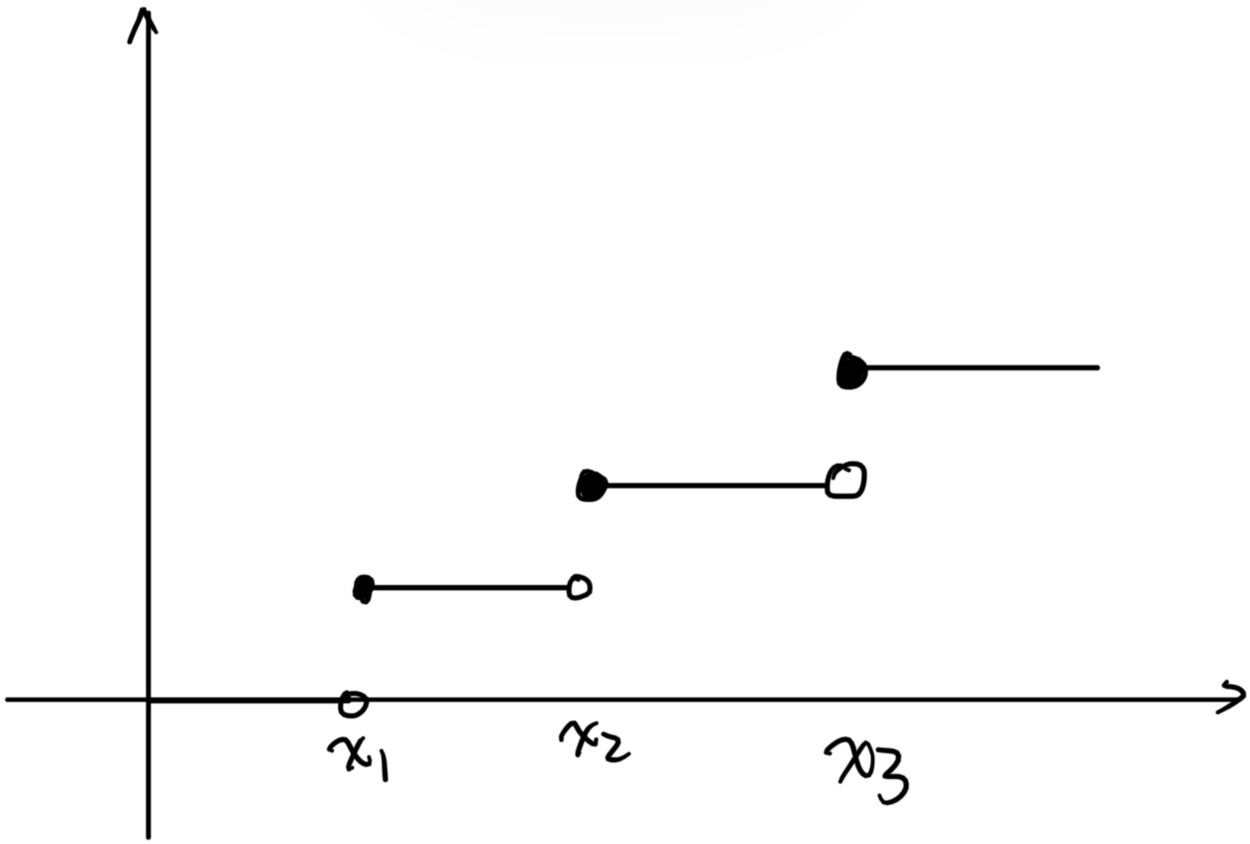


right-continuous

2.2 Discrete random variable

> Definition:
> If a random variable X can only take on a finite or countably infinite number of values, then X is called a discrete random variable.

- **cdf** of a discrete r.v. is a right continuous step funciton



- **Probability function (pf):** $f(x) = P(X = x)$.

  For a discrete r.v., $f(x) \begin{cases} > 0 & \text{if } X \text{ can take value } x \\ = 0 & \text{if } X \text{ cannot take value } x \end{cases}$
- **Support**: The set $A = \{x : f(x) > 0\}$ is called the support of X. These are all the possible values that X can take.
- Properties of a p.f. f for a discrete r.v. X.
  i. $f(x) \geq 0$ for any $x \in \mathbb{R}$.
  ii. $\sum_{x \in A} f(x) = 1$.

  > Proof: The support of X is a countable set, $A = \{x_1, \ldots, x_n\}$. Let $B_i = \{X = x_i\}$ is an event for $i = 1, ..., n$. B_i are pairwise mutually exclusive events, i.e. $B_i \cap B_j = \emptyset$ for $i \neq j$. Then, $\bigcup_{i=1}^{n} B_i = S$. Then, $1 = P(S) = P\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} P(B_i) = \sum_{i=1}^{n} P(X = x_i)$.

- Some commonly used discrete r.v.
  i. Bernoulli r.v. $X \sim \text{Bern}(p)$.
  X can only take two possible values, 0 and 1. $A = \{0, 1\}$.
  $f(1) = P(X = 1) = p$.

ii. Binomial distribution

Toss a coin n times.

a. different tosses are indepedent

b. probability of getting a head is fixed, which is denoted by p.

X: # of heads across n tosses, then $X \sim \mathrm{Bin}(n,p)$.

Hence the support of X, $A=\{0,1,2,\ldots,n\}$.

The p.f. of X is $f(x)=P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$, $x\in A$.

$$\sum_{x=0}^{n}\binom{n}{x}p^x(1-p)^{n-x}=[p+(1-p)]^n=1$$

iii. Geometric distribution

X: # of failures before the first success.

The support of X is $A=\{0,1,...\}$.

$f(x)=P(X=x)=(1-p)^x p$, $x\in A$.

$$\sum_{x=0}^{\infty}(1-p)^x p=\frac{p}{1-(1-p)}=1$$

iv. Negative binomial r.v. $X\sim \mathrm{NegBin}(r,p)$

X: # of failures before the rth success.

v. Poisson r.v. $X\sim \mathrm{Poisson}(\mu)$

The support of X, $A=\{0,1,\ldots\}$.

The probability function $f(x)=P(X=x)=\frac{\mu^x}{x!}e^{-\mu}$, $x\in A$.

$$\sum_{x\in A}f(x)=\sum_{x=0}^{\infty}\frac{\mu^x}{x!}e^{-\mu}=e^{-\mu}\sum_{x=0}^{\infty}\frac{\mu^x}{x!}=e^{-\mu}e^{\mu}=1.$$

Aside: $e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$.

2.3 Continuous random variable

> Definition: If the collection of all possible values X can take is an interval or the real line, then X is called a continuous r.v.
>
> - Remark: If X is continuous r.v., its cdf $F(x)$ is continuous everywhere. Moreover, F is differentialbe almost everywhere. It is not differentiable at atmost countable locations.

- Probability density function (pdf):

$$f(x)=\begin{cases}F'(x) & \text{if F is differentiable at } x\\ 0 & \text{otherwise}\end{cases}$$

- Support of X: $A = \{x | f(x) > 0\}$.
- Basic property of f:
  i. $f(x) \geq 0$ for any $x \in \mathbb{R}$.
  ii. $\int_{-\infty}^{\infty} f(x)dx = 1$.
- Extra properties of f:
  i. $F(x) = \int_{-\infty}^{x} f(t)dt = F(x) - F(-\infty)$ (find cdf from pdf).
  ii. $f(x) = \begin{cases} F'(x) & \text{if F is differentiable at } x \\ 0 & \text{otherwise} \end{cases}$ (find pdf from cdf).
  iii. $P(X = x) = 0$ and $f(x) \neq P(X = x)$ for any x.
  If F is differentiable at x, then $f(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$
  $\implies F(x+h) - F(x) \approx f(x) \cdot h$
  $\implies P(x < X \leq x + h) \approx f(x) \cdot h.$
  iv. $P(a < X \leq b) = F(b) - F(a) = P(a < X < b) = P(a \leq X \leq b)$

> Example (Uniform distribution):
>
> Suppose the cds if
>
> $$F(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \geq b \end{cases}$$
>
> Find pdf $f(x)$:
>
> The pdf is: $f(x) \begin{cases} 0 & x \leq a \\ \frac{1}{b-a} & a < x < b \\ 0 & x \geq b \end{cases}$

> Example:
>
> Define a function
>
> $$f(x) = \begin{cases} \frac{\theta}{x^{\theta+1}} & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$
>
> .
>
> i. Find for what values of θ, f is a pdf?
>
> > Solution: $f(x) \geq 0$ for any $x \in \mathbb{R}$, therefore $\theta \geq 0$. $\int_{-\infty}^{\infty} f(x)dx = \int_{1}^{\infty} \frac{\theta}{x^{\theta+1}} dx$.
> >
> > > Case 1: $\theta = 0$, $\int_{-\infty}^{\infty} f(x)dx = 0 \neq 1$.

> > Case 2: $\theta > 0$, $\int_{-\infty}^{\infty} f(x)dx = \int_{1}^{\infty} \frac{\theta}{x^{\theta+1}}dx = -\frac{1}{x^{\theta}}\Big|_{1}^{\infty} = 1.$

ii. Find $F(x)$ if f is a pdf.

> Solution: $F(x) = \int_{-\infty}^{x} f(t)dt$
>
> > Case 1: $x \leq 1$, $F(x) = \int_{-\infty}^{x} f(t)dt = 0.$
>
> > Case 2: $x > 1$, $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{1}^{x} \frac{\theta}{t^{\theta+1}}dt = -\frac{1}{t^{\theta}}\Big|_{1}^{x} = 1 - \frac{1}{x^{\theta}}.$

iii. Find $P(2 < X < 3)$ and $P(-2 < X < 3)$.

> Solution:
> $P(2 < X < 3) = F(3) - F(2) = \left(1 - \frac{1}{3^{\theta}}\right) - \left(1 - \frac{1}{2^{\theta}}\right) = \frac{1}{2^{\theta}} - \frac{1}{3^{\theta}}.$
> $P(-2 < X < 3) = F(3) - F(-2) = \left(1 - \frac{1}{3^{\theta}}\right) - 0 = 1 - \frac{1}{3^{\theta}}.$
> $P(-2 < X < 3) = \int_{-2}^{3} f(x)dx = \int_{-2}^{1} f(x)dx + \int_{1}^{3} f(x)dx = \int_{-2}^{1} 0dx + \int_{1}^{3} \frac{\theta}{x^{\theta+1}}dx = -\frac{1}{x^{\theta}}\Big|_{1}^{3} = 1 - \frac{1}{3^{\theta}}.$

- Gamma function, $\Gamma(\alpha), \alpha > 0$.

$$\Gamma(\alpha) = \int_{0}^{\infty} x^{\alpha-1}e^{-x}dx$$

a. $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$.
b. $\Gamma(n) = (n-1)!$ when n is a positive integer, $\Gamma(1) = 1$.
c. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

> Example (Gamma distribution):
>
> The pdf is
>
> $$f(x) = \begin{cases} \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$
>
> if $\alpha > 0, \beta > 0$ are constants.
>
> Verify f is a pdf.
>
> > Solution:
> >
> > a. $f(x) \geq 0$ for any $x \in \mathbb{R}$.
> > b. $\displaystyle\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = 0 + \int_{0}^{\infty} \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)}dx.$
> > Here, note $\displaystyle\int_{0}^{\infty} x^{\alpha-1}e^{-x}dx = \Gamma(\alpha).$
> > Let $y = \frac{x}{\beta} \implies x = \beta y, dx = \beta dy.$

Then, $\int_0^\infty \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)}dx = \int_0^\infty \frac{(\beta y)^{\alpha-1}e^{-y}}{\beta^\alpha\Gamma(\alpha)}\beta dy = \frac{1}{\Gamma(\alpha)}\int_0^\infty y^{\alpha-1}e^{-y}dy = \frac{1}{\Gamma(\alpha)}\Gamma(\alpha) = 1.$

Example (Weibull distribution):

The pdf is

$$f(x) = \begin{cases} \frac{\beta}{\theta^\beta}x^{\beta-1}\text{exp}\left\{-\left(\frac{x}{\theta}\right)^\beta\right\} & x > 0 \\ 0 & x < 0 \end{cases}$$

where $\alpha > 0, \beta > 0$ are constants, $X \sim \text{Weibull}(\theta, \beta)$.

Verify f is a pdf.

Solution:

a. $f(x) \geq 0$ for any $x \in \mathbb{R}$.

b. $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx = 0 + \int_0^\infty \frac{\beta}{\theta^\beta}x^{\beta-1}\text{exp}\left\{-\left(\frac{x}{\theta}\right)^\beta\right\}dx.$

Let $y = \ \ (\frac{x}{\theta}^\beta \implies x = \theta y^{\frac{1}{\beta}}, dx = \theta\frac{1}{\beta}y^{\frac{1}{\beta}-1}dy.$

Then, $\int_{-\infty}^\infty f(x)dx = \int_0^\infty \frac{\beta}{\theta^\beta}(\theta y^{\frac{1}{\beta}})^{\beta-1}\text{exp}\left\{-y\right\}\theta\frac{1}{\beta}y^{\frac{1}{\beta}-1}dy = \Gamma(1) = 1.$

Exmaple (Normal distribution/Gaussian distribution):

The pdf is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

for $x \in \mathbb{R}$,

where $\mu \in \mathbb{R}, \sigma > 0$ are constants, $X \sim \text{Normal}(\mu, \sigma)$.

Verify f is a pdf.

Solution:

a. $f(x) \geq 0$ for any $x \in \mathbb{R}$.

> > b. $\displaystyle\int_{-\infty}^{\infty} f(x)dx = 1.$
> > To verify 2, we start from a special case, where $\mu = 0$, $\sigma = 1$.
> > $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, i.e., $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1.$
> > $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$. Let $y = \frac{x^2}{2} \implies x = \sqrt{2y}, dx = \sqrt{2}dy.$
> > Then, $2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-y} y^{1-1/2} dy = \frac{1}{\sqrt{\pi}} \Gamma(1/2) = 1.$
>
> Prove $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is a pdf for any $\mu \in \mathbb{R}$, $\sigma > 0$.
>
> a. $f(x) \geq 0$ for any $x \in \mathbb{R}$.
> b. $\int_{-\infty}^{\infty} f(x)dx = 1$?
> $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$
> Let $z = \frac{x-\mu}{\sigma} \implies x = \mu + \sigma z, dx = \sigma dz$
> $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2}} dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2}} dx = 1.$

2.4 Expectation

- Definition of expectation for discrete r.v.
  Suppose that X is a discrete r.v. with support A and p.f. $f(x)$.
  Then, $E(X) = \sum_{x\in A} x f(x)$ provided $\sum_{x \in A} |x| f(x) < \infty$.
- Definition of expectation for continuous r.v.
  Suppose that X is a continuous r.v. with support A and pdf $f(x)$.
  Then $E(X) = \int_{-\infty}^{\infty} x f(x) dx$ provided $\int_{-\infty}^{\infty} |x| f(x) dx < \infty$.

> Example (Cauchy distribution):
> The pdf of X is $f(x) = \frac{1}{\pi(1+x^2)}$ for $x \in \mathbb{R}$.
> Find $E(X)$.
>
> > Solution:
> > $\int_{-\infty}^{\infty} |x| f(x) dx = \int_{-\infty}^{\infty} |x| \frac{1}{\pi(1+x^2)} dx = 2\int_0^{\infty} \frac{x}{\pi(1+x^2)} dx = \left. \frac{\ln(1+x^2)}{\pi} \right|_0^{\infty} = \infty.$
> > Therefore, $E(X)$ does not exist.

> Example:
> Suppose p.f. $f(x) = \frac{1}{x(x+1)}$ for $x = 1, 2, 3, \ldots$, the support of X is $A = \{1, 2, 3, \ldots\}$.
>
> i. Show f is a p.f.
> > Solution:
> >
> > i. $f(x) \geq 0$ for any $x \in \mathbb{R}$.

> > ii. $\sum_{x\in A} f(x) = \sum_{x\in A} \frac{1}{x(1+x)} = \sum_{x=1}^{\infty} \left(\frac{1}{x} - \frac{1}{x+1}\right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots = 1.$
>
> ii. Find $E(X)$.
> > Solution: $E(X) = \sum_{x\in A} xf(x) = \sum_{x\in A} x\frac{1}{x(x+1)} = \sum_{x\in A} \frac{1}{x+1} = \sum_{x=1}^{\infty} \frac{1}{x+1} = \infty$.
> > $E(X)$ does not exist.

> More examples of expectations:
>
> i. Binomial Distribution, $X \sim \text{Bin}(n, p)$.
> > Solution 1: $E(X) = \sum_{x\in A} xf(x) = \sum_{x=0}^{n} x\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x} = \sum_{x=1}^{n} x\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x} = \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!}p^x(1-p)^{n-x}$.
> > Let $y = x - 1$, then $\sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!}p^x(1-p)^{n-x} = np\sum_{y=0}^{n-1} \frac{(n-1)!}{y!(n-1-y)!}p^y(1-p)^{n-1-y} = np$, since $\sum_{y=0}^{n-1} \frac{(n-1)!}{y!(n-1-y)!}p^y(1-p)^{n-1-y}$ is a pf of $\text{Bin}(n-1, p)$.
>
> > Solution 2: For the ith trial, $X_i = \begin{cases} 1 & \text{if the } i\text{th outcome is a success} \\ 0 & \text{otherwise} \end{cases}$.
> > Then, $P(X_i = 1) = p$. Let $X = \sum_{i=1}^{n} X_i$, then $X \sim \text{Bin}(n, p)$.
> > $E(X) = E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} 1 \cdot P(X_i = 1) = np$.
>
> ii. Suppose X is a continuous r.v. with pdf $f(x) = \begin{cases} \frac{\theta}{x^{\theta+1}} & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$, where $\theta > 0$ is a constant. Find $E(X)$, and determine the values of θ for which $E(X)$ exists.
> > Solution: $\int_{-\infty}^{\infty} |x|f(x)dx = \int_{1}^{\infty} x\frac{\theta}{x^{\theta+1}}dx = \int_{1}^{\infty} \frac{\theta}{x^{\theta}}dx < \infty$ iff $\theta > 1$.
> > When $\theta > 1$, $E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{1}^{\infty} \frac{\theta x}{x^{\theta+1}}dx = \theta\int_{1}^{\infty} \frac{1}{x^{\theta}}dx = \left(\frac{\theta}{1-\theta}x^{1-\theta}\right)\Big|_{1}^{\infty} = \frac{\theta}{\theta-1}$.
> > When $\theta \leq 1$, $E(X)$ does not exist.

- Expectation of a function of X
  Suppose thar X is a r.v., what is $E(g(X))$, where g is a real function?
  For example, $g(x) = x^2$.
  Let $Y = g(X)$, find $E(Y)$.
    - Case 1: If X is a discrete r.v. with support A and p.f. $f(x)$, then $E(g(X)) = \sum_{x\in A} g(x)f(x)$ provided $\sum_{x\in A} |g(x)|f(x) < \infty$.
    - Case 2: If X is a continuous r.v. with support A and pdf $f(x)$, then $E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx$ provided $\int_{-\infty}^{\infty} |g(x)|f(x)dx < \infty$.
- Linearity Property: If a and b are two constants, then $E[ag(X) + bg(X)] = aE(g(X)) + bE(h(X))$.

- Variance: $Var(X) = E[(X-\mu)]^2 = E(X^2) - \mu^2 = E(X^2) - [E(X)]^2$ where $\mu = E(X)$.
- Moments:
  - kth moment about 0: $E(X^k)$.
  - kth moment about mean: $E[(X-\mu)^k]$, where $\mu = E(X)$.

> Example (Poission distribution):
> Suppose $X \sim \text{Poisson}(\mu)$, where $\mu > 0$ is a constant.
> Find $E(X)$ and $Var(X)$.
>
> > Solution: $E(X) = \sum_{x=0}^{\infty} x \frac{\mu^x}{x!} e^{-\mu} = \mu e^{-\mu} \sum_{x=1}^{\infty} \frac{\mu^{x-1}}{(x-1)!}$.
> > Let $y = x-1$, then $E(X) = \mu e^{-\mu} \sum_{y=0}^{\infty} \frac{\mu^y}{y!} = \mu e^{-\mu} e^{\mu} = \mu$.
> > $E(X^2) = \sum_{x=0}^{\infty} x^2 \frac{\mu^x}{x!} e^{-\mu} = \sum_{x=1}^{\infty} \frac{x\mu^x}{(x-1)!} e^{-\mu} = \sum_{x=1}^{\infty} \frac{(x-1+1)\mu^x}{(x-1)!} e^{-\mu} = \sum_{x=1}^{\infty} \frac{(x-1)^2\mu^x}{(x-1)!} e^{-\mu} + \sum_{x=1}^{\infty} \frac{\mu^x}{(x-1)!} e^{-\mu} = \sum_{x=2}^{\infty} \frac{(x-1)\mu^x}{(x-1)!} e^{-\mu} = \sum_{x=2}^{\infty} \frac{\mu^x}{(x-2)!} e^{-\mu}$.
> > Let $y = x-2$, then $\sum_{y=0}^{\infty} \frac{\mu^{y+2}}{y!} e^{-\mu} = \mu^2 e^{-\mu} \sum_{y=0}^{\infty} \frac{\mu^y}{y!} = \mu^2$.
> > That means $E(X^2) = \mu^2 + \mu$, and $Var(X) = E(X^2) - [E(X)]^2 = \mu^2 + \mu - \mu^2 = \mu$.

> Example (Gamma distribution):
> Suppose $X \sim \text{Gamma}(\alpha, \beta)$. Find $E(X^k)$, $k > 0$.
> pdf of X is $f(x) = \begin{cases} \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$.
>
> > Solution: $E(X^k) = \int_{-\infty}^{\infty} x^k f(x) dx = \int_0^{\infty} x^k \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} dx$. Let $y = \frac{x}{\beta} \implies x = \beta y$, $dx = \beta dy$.
> > Then, $E(X^k) = \int_0^{\infty} \frac{(\beta y)^k (\beta y)^{\alpha-1} e^{-y}}{\beta^\alpha \Gamma(\alpha)} \beta dy = \frac{\beta^k}{\Gamma(\alpha)} \int_0^{\infty} y^{k+\alpha-1} e^{-y} dy = \frac{\beta^k}{\Gamma(\alpha)} \Gamma(k+\alpha) = \frac{\beta^k \Gamma(k+\alpha)}{\Gamma(\alpha)}$.
> > In paticular, if $k=1$, $E(X) = \frac{\beta\Gamma(1+\alpha)}{\Gamma(\alpha)} = \frac{\beta\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha\beta$.
> > $k=2$, $E(X^2) = \frac{\beta^2\Gamma(2+\alpha)}{\Gamma(\alpha)} = \frac{\beta^2(\alpha+1)\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha(\alpha+1)\beta^2$.
> > $Var(X) = E(X^2) - [E(X)]^2 = \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2 = \alpha\beta^2$.
> > Alternatively:
> > $E(X^k) = \int_{-\infty}^{\infty} x^k f(x) dx = \int_0^{\infty} x^k \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} dx = \int_0^{\infty} \frac{x^{k+\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} dx$
> > Define $\alpha^* = k + \alpha$, then $E(X^k) = \int_0^{\infty} \frac{x^{\alpha^*-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} dx = \int_0^{\infty} \frac{x^{\alpha^*-1}e^{-x/\beta}}{\beta^{\alpha^*} \Gamma(\alpha^*)} \frac{\beta^{\alpha^*}\Gamma(\alpha^*)}{\beta^\alpha \Gamma(\alpha)} dx = \frac{\beta^{\alpha^*}\Gamma(\alpha^*)}{\beta^\alpha \Gamma(\alpha)} \int_0^{\infty} \frac{x^{\alpha^*-1}e^{-x/\beta}}{\beta^{\alpha^*} \Gamma(\alpha^*)} dx = \frac{\beta^{\alpha^*}\Gamma(\alpha^*)}{\beta^\alpha \Gamma(\alpha)} = \frac{\beta^{k+\alpha}\Gamma(k+\alpha)}{\beta^\alpha \Gamma(\alpha)} = \frac{\beta^k\Gamma(k+\alpha)}{\Gamma(\alpha)}$.

2.5 Moment generating function

- Definition: Suppose X is a random variable, then $M(t) = E(E^{tx})$ is called the moment generating function (mgf) of X if $M(t)$ exists for $t \in (-h, h)$ for some $h > 0$.

> Example (Gamma distribution):
> Suppose $X \sim \text{Gamma}(\alpha, \beta)$. Find the mgf of X.
>
> > Solution: $M(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx = \int_0^{\infty} e^{tx} \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} dx = \int_0^{\infty} \frac{x^{\alpha-1} e^{-(1/\beta - t)x}}{\beta^\alpha \Gamma(\alpha)} dx$. (Note: $1/\beta > t$, otherwise the integral diverges.)
> > Let $y = (1/\beta - t)x$, then $x = \frac{y}{1/\beta - t} = \frac{\beta y}{1 - t\beta}$, $dx = \frac{\beta}{1 - t\beta} dy$.
> > Then, $M(t) = \int_0^{\infty} \frac{(\beta y)^{\alpha-1} e^{-y}}{\beta^\alpha \Gamma(\alpha)} \frac{\beta}{1 - t\beta} dy = \frac{\beta^{\alpha-1}}{\Gamma(\alpha)(1 - t\beta)} \int_0^{\infty} y^{\alpha-1} e^{-y} dy = \frac{\beta^{\alpha-1}}{\Gamma(\alpha)(1 - t\beta)} \Gamma(\alpha) = \frac{\beta^{\alpha-1}\Gamma(\alpha)}{\Gamma(\alpha)(1 - t\beta)} = \frac{\beta^{\alpha-1}}{1 - t\beta}$.

> Example (Poisson distribution):
> Suppose $X \sim \text{Poisson}(\mu)$. Find the mgf of X.
>
> > Solution: $M(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \frac{\mu^x}{x!} e^{-\mu} = e^{-\mu} \sum_{x=0}^{\infty} \frac{(\mu e^t)^x}{x!} = e^{-\mu} e^{\mu e^t} \sum_{x=0}^{\infty} \frac{(\mu e^t)^x}{x!} e^{-e^t \mu} = e^{\mu(e^t - 1)}$.

> Example (Normal distribution):
> Suppose $X \sim N(0, 1)$. Find the mgf of X.
>
> > Solution: $M(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{tx - \frac{x^2}{2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2 - 2tx + t^2) + \frac{1}{2}t^2} dx = e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx = e^{\frac{1}{2}t^2}$.

> Question: How to find the mgf of $N(\mu, \sigma^2)$?

- Three important properties of mgf
  i. Suppose the mgf of X is $M(t)$. If $Y = aX + b$, where a and b are constants, then the mgf of Y is $M_Y(t) = e^{bt} M(at)$.
     If $Y \sim N(\mu, \sigma^2)$, then $X = \frac{Y - \mu}{\sigma} \sim N(0, 1)$.
     $\implies Y = \mu + \sigma X$, where $X \sim N(0, 1)$.
     $M_Y(t) = e^{\mu t} M_X(\sigma t) = e^{\mu t} e^{\frac{1}{2}\sigma^2 t^2}$.
  ii. Find the kth moment of X about 0 from $M(t)$:
     $E(X^k) = M^{(k)}(0) = \frac{d^k}{dt^k} M(t)\Big|_{t=0}$.
     $M(t) = E(e^{tX})$, $M'(t) = E(Xe^{tX})$.
     In particular, $E(X) = M'(0)$, $E(X^2) = M''(0)$. Then, $Var(X) = E(X^2) - [E(X)]^2 = M''(0) - [M'(0)]^2$.

> Example (Gamma distribution):
> If $X \sim \text{Gamma}(\alpha, \beta)$, $M(t) = \left(\frac{1}{1-t\beta}\right)^2$, where $t < \frac{1}{\beta}$.
> Find $E(X)$ and $Var(X)$.
>
> > Solution: $M'(t) = \alpha\beta(1-\beta t)^{-\alpha-1}$, $M''(t) = \alpha(\alpha+1)\beta^2(1-\beta t)^{-\alpha-2}$.
> > Then, $E(X) = M'(0) = \alpha\beta$, $E(X^2) = M''(0) = \alpha(\alpha+1)\beta^2$.

iii. Uniqueness of mgf.

Namely, X and Y have the same distribution iff X and Y have the same mgf.

> Example: X has mgf $M(t) = e^{t^2/2}$
>
> a. Find mgf of $Y = 2X - 1$.
> > Solution: $M_Y(t) = e^{-t}M_X(2t) = e^{-t}e^{2t^2}$.
>
> b. Find $E(Y)$ and $Var(Y)$.
> > Solution: $M'_Y(t) = (4t-1)e^{2t^2-t}$. $E(X) = M'Y(0) = -1$.
> > $M''_Y(t) = 4e^{2t^2-t} + (4t-1)^2e^{2t^2-t}$. $E(Y^2) = M''_Y(0) = 1 + 4 = 5$.
> > $Var(Y) = E(Y^2) - [E(Y)]^2 = 5 - (-1)^2 = 4$.
>
> c. What is the distribution of Y?
> > Solution: $Y \sim N(-1, 4)$, since $M_Y(t) = e^{-t}e^{2t^2}$.

3 Joint distribution

3.1 Joint and Marginal cdfs

- Definition of joint cdf

  Suppose that X and Y are two r.v.s. The joint cdf of X and Y is defined by $F(x, y) = P(X \le x, Y \le y)$ for $x, y \in \mathbb{R}$.

  > Remark: This definition can be extended to n r.v.s. $X_1, X_2, \dots, X_n$.
  > Joint cdf is $F(x_1, x_2, \dots, x_n) = P(X_1 \le x_1, X_2 \le x_2, \dots, X_n \le x_n)$.
  > However, we will focus on the case of $n = 2$.

- Properties of joint cdf

  i. Fix y, $F(x, y)$ is monotone increasing function of x, i.e., $F(x_1, y) \le F(x_2, y)$ if $x_1 < x_2$.
  > Proof: $F(x_1, y) = P(X \le x_1, Y \le y)$, since $\{X \le x_1, Y \le y\} \subset \{X \le x_2, Y \le y\}$, $F(x_1, y) \le F(x_2, y)$.

  ii. Fix x, $F(x, y)$ is monotone increasing function of y, i.e., $F(x, y_1) \le F(x, y_2)$ if $y_1 < y_2$.

  iii. $\lim_{x \to -\infty} F(x, y) = 0 = \lim_{y \to -\infty} F(x, y)$.

> Proof: $F(x,y) = P(X \le x, Y \le y) \le P(X \le x)$, and consider $\lim_{x\to-\infty} P(X \le x) = 0$, additionally, by property of joint cdf, $F(x,y) \ge 0$, then by squeeze theorem, $\lim_{x\to-\infty} F(x,y) = 0$.

iv. $\lim_{x\to\infty, y\to\infty} F(x,y) = 1$.

> Proof: Consider set $Axy = \{X \le x\} \cup \{Y \le y\}$, then as $x, y \to \infty$, $P(\overline{Axy}) \to 0$, then $F(x,y) = P(Axy) \to 1$.

v. How to find marginal cdf from the joint one?
$F_1(x) = P(X \le x) = \lim_{y\to\infty} F(x,y)$.
Define $Ax = \{X \le x\}, By = \{Y \le y\}$.
As $y \to \infty$, $Ax \cup By \to Ax$.
$F_2(y) = P(Y \le y) = \lim_{x\to\infty} F(x,y)$.

3.2 Joint Discrete r.v.s

- Definition: If both X and Y are discrete r.v.s, then as a pair, $X\&Y_{(X,Y)}$ are joint discrete r.v.s X and Y.
- Definition of joint p.f.:
  The joint p.f. of X and Y is given by $f(x,y) = P(X = x, Y = y)$ for any $x, y \in \mathbb{R}$.
- Definition of join support: The support of (X, Y) is the set $A = \{(x,y) \in \mathbb{R}^2 : f(x,y) > 0\}$.
- Basic properties of joint p.f.:
  i. $f(x,y) \ge 0$ for any $(x,y) \in \mathbb{R}^2$.
  ii. $\sum_{(x,y)\in A} f(x,y) = 1$.

  > Question: How to find probability over a region $C \subseteq \mathbb{R}^2$?

  iii. $P((X,Y) \in C) = \sum_{(x,y)\in C} f(x,y)$.

  > Question: How to find marginal p.f. from the joint one?

  iv. $f_1(x) = P(X = x) = P(X = xY < \infty) = \sum_{y\in\mathbb{R}} f(x,y)$.

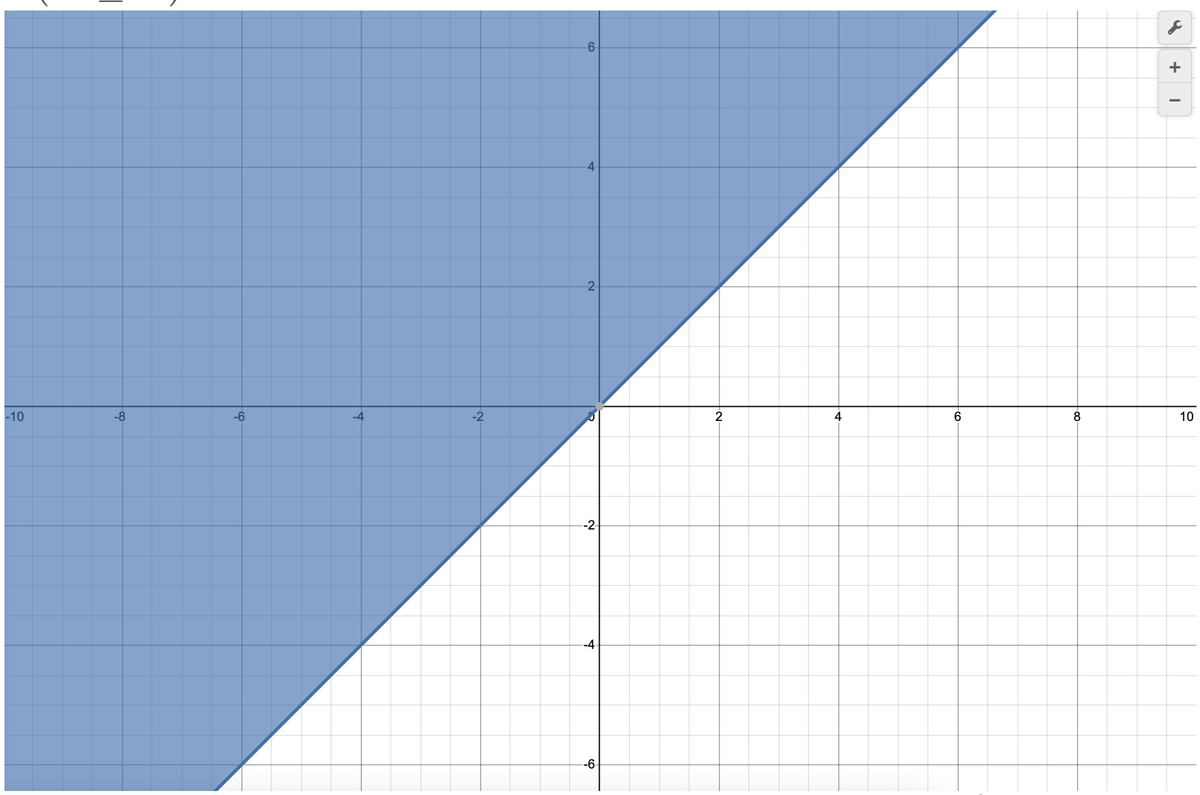
  > E.g. Suppose X and Y are independent discrete r.v.s with joint p.f. $f(x,y) = kq^2p^{x+y}$ for $x = 0, 1, ...$ and $y = 0, 1, ...$, and 0 elsewhere. Here $p \in (0,1)$ is a constant, $q = 1 - p$.
  >
  > a. Find k.
  >
  > > Solution: Since $f(x,y) \ge 0$ for any $(x,y) \in \mathbb{R}^2$, $k > 0$.Since $\sum_{x=0}^{\infty} f(x,y) = 1$, Then,
  > >
  > > $$k\left(\sum_{x=0}^{\infty} p^{x+y}q^2\right) = kq^2\left(\sum_{x=0}^{\infty} p^x\right)\left(\sum_{x=0}^{\infty} p^y\right) = kq^2\left(\frac{1}{1-p}\right)\left(\frac{1}{1-p}\right) = k$$

> Therefore, $k = 1$

b. Find the marginal p.f. of X and find marginal p.f. of Y.

> Solution: The support of X is $Ax = \{0, 1, 2, ...\}$.
> Here, $f_1(x) = \sum_{y \in \mathbb{R}} f(x, y) = 0$ if $x \notin Ax$
> Given $X \in Ax$, then $f_1(x) = \sum_{y \in \mathbb{R}} f(x, y) = \sum_{y=0}^{\infty} f(x, y) = \sum_{y=0}^{\infty} p^{x+y} q^2 = q^2 p^x \sum_{y=0}^{\infty} p^y = q^2 p^x \frac{1}{1-p} = qp^x$.

c. $P(X \leq Y)$



> Solution: $P(X \leq Y) = \sum_{(x,y) \in C} f(x, y)$ where $C = \{(x, y) \in \mathbb{R}^2 = x \leq y\}$,
> therefore, $P(X \leq Y) = \sum_{y=0}^{\infty} \sum_{x=0}^{y} p^{x+y} q^2 = \sum_{x=0}^{\infty} p^x q^2 \sum_{y=x}^{\infty} p^y = \sum_{x=0}^{\infty} p^x q^2 \frac{p^x}{1-p} = q \sum_{x=0}^{\infty} p^{2x} = q \frac{1}{1-p^2} = \frac{1}{1+p}$.

3.3 Joint Continuous r.v.s

- Definition: If joint cdf of (X, Y) can be written as $F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) du dv$ then X and Y are joint continuous r.v.s with joint pdf $f(x, y)$.
  Namely, $f(x, y) = \begin{cases} \frac{\partial^2}{\partial x \partial y} F(x, y) & \text{if exists} \\ 0 & \text{o.w.} \end{cases}$.
- Definition of joint support: $A = \{(x, y) \in \mathbb{R}^2 : f(x, y) > 0\}$.
- Properties of joint pdf:
  i. $f(x, y) \geq 0$ for any $(x, y) \in \mathbb{R}^2$.

ii. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$.

> Question: How to find probability over a region $C \subseteq \mathbb{R}^2$?

iii. $P((X,Y) \in C) = \iint_{(x,y)\in C} f(x,y)dxdy$.

> Question: How to find marginal pdf from the joint one?

iv. $f_1(x) = \int_{-\infty}^{\infty} f(x,y)dy$ and $f_2(y) = \int_{-\infty}^{\infty} f(x,y)dx$.

> E.g. X and Y are joint continuous r.v.s with joint pdf $f(x,y) = \begin{cases} x+y & \text{if } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$.
>
> a. Show f is a joint pdf.
>
> > Solution: $f(x,y) \ge 0$ for any $(x,y) \in \mathbb{R}^2$.
> > $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \int_0^1\int_0^1 (x+y)dxdy = \int_0^1 \left(\frac{x^2}{2}+xy\right)\Big|_{x=0}^{x=1} dy = \int_0^1 \left(\frac{1}{2}+y\right)dy = \frac{1}{2}+\frac{1}{2} = 1$.
>
> b. Find
>
> > a. $P(X \le 1/3, Y \le 1/2)$
> >
> > > Solution: $P(X \le 1/3, Y \le 1/2) = \int_0^{1/3}\int_0^{1/2}(x+y)dydx = \int_0^{1/3}\left(xy+\frac{y^2}{2}\right)\Big|_{y=0}^{y=1/2} dx = \int_0^{1/3}\left(\frac{x}{2}+\frac{1}{8}\right)dx = \frac{1}{36}+\frac{1}{24} = \frac{5}{72}$.
> >
> > b. $P(X \le Y)$
> >
> > > Solution: $P(X \le Y) = \iint_C f(x,y)dxdy = \int_0^1 dx \int_x^1 (x+y)dy = \int_0^1 dy \int_0^y (x+y)dx = \int_0^1 \left(\frac{x^2}{2}+xy\right)\Big|_{x=0}^{x=y} dy = \int_0^1 \left(\frac{y^2}{2}+y^2\right)dy = \frac{1}{2}$.
> >
> > c. $P(X+Y \le 1/2)$
> >
> > > Solution: Let $C = \{(x,y) | x+y \le \frac{1}{2}, 0 \le x \le 1, 0 \le y \le 1\}$.
> > > Then, $P(X+Y \le 1/2) = \iint_C f(x,y)dxdy = \int_0^{1/2}\int_0^{1/2-x}(x+y)dydx = \int_0^{1/2}\left(xy+\frac{y^2}{2}\right)\Big|_{y=0}^{y=1/2-x} dx = \int_0^{1/2}\left(\frac{x}{2}-\frac{x^2}{+}\frac{1}{2}(x^2-x+\frac{1}{4}\right)dx = \int_0^{1/2}\left(-\frac{x^2}{2}+\frac{1}{8}\right)dx = \left(-\frac{x^3}{16}+\frac{x}{8}\right)\Big|_0^{1/2} = \frac{1}{24}$.
> >
> > d. $P(XY \le 1/2)$

Solution: Find $P(XY > 1/2)$ first.
$P(XY > 1/2) = \int_0^{1/2} \int_0^{1/2/x} (x+y) dy dx =$
$\int_0^{1/2} \left(xy + \frac{y^2}{2}\right)\Big|_{y=0}^{y=1/2x} dx = \int_0^{1/2} \left(x - \frac{1}{8x^2}\right) dx = \left(\frac{x^2}{2} + \frac{1}{8x}\right)\Big|_0^{1/2} = \frac{1}{4}.$
Therefore, $P(XY \leq 1/2) = 1 - P(XY > 1/2) = 1 - \frac{1}{4} = \frac{3}{4}$

c. Find marginal pdf of X and Y.

Solution: The support of X is $[0, 1]$.
Given $x \in [0, 1]$, $f_1(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_0^1 (x+y) dy =$
$\left(xy + \frac{y^2}{2}\right)\Big|_{y=0}^{y=1} = x + \frac{1}{2}.$

E.g. Suppose $f(x) = \begin{cases} ke^{-x-y} & 0 < x < y < \infty \\ 0 & \text{o.w.} \end{cases}$ is the joint pdf of (X, Y).

a. Find k.

Solution: $f(x, y) \geq 0$ for any $(x, y) \in \mathbb{R}^2$, therefore, $k \geq 0$.
Now, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = \int_0^{\infty} \int_x^{\infty} ke^{-x-y} dy dx =$
$\int_0^{\infty} ke^{-x} \left(-e^{-}y\right)\big|_x^{\infty} dx = \int_0^{\infty} ke^{-2x} dx = -\frac{k}{2} e^{-2x}\big|_0^{\infty} = \frac{k}{2} = 1$, therefore,
$k = 2$.

b. Find:

a. $P(X \leq \frac{1}{3}, Y \leq \frac{1}{2})$

Solution: Let $C = \{(x, y) | x \leq 1/3, y \leq 1/2, 0 < x < y\}$. Then, $P(X \leq \frac{1}{3}, Y \leq \frac{1}{2}) = \iint_C f(x, y) dx dy = \int_0^{1/3} \int_x^{1/2} 2e^{-x-y} dy dx =$
$\int_0^{1/3} 2e^{-x} \left(-e^{-y}\right)\Big|_x^{1/2} dx = \int_0^{1/3} 2e^{-x} \left(-e^{-1/2} + e^{-x}\right) dx =$
$\int_0^{1/3} 2e^{-x} \left(e^{-x} - e^{-1/2}\right) dx = \int_0^{1/3} 2e^{-2x} - 2e^{-1/2} e^{-x} dx =$
$-e^{-2x} + 2e^{-1/2} e^{-x}\big|_0^{1/3} = 1 - 2e^{-1/2} - e^{-2/3} - e^{-5/6}.$

b. $P(X \leq Y)$

Solution: $P(X \leq Y) = 1$

c. $P(X + Y \geq 1)$

Solution: Let $C = \{(x, y) | x + y \geq 1, 0 < x < y\}$
Let's find $P(X + Y < 1)$ first.
$P(X + Y < 1) = \iint_{x, y \in \mathbb{R}} 2e^{-x-y} dy dx = \int_0^{1/2} \int_x^{1-x} 2e^{-x-y} dy dx =$
$\int_0^{1/2} 2e^{-x} \left(-e^{-y}\right)\Big|_x^{1-x} dx = \int_0^{1/2} 2e^{-x} \left(-e^{x-1} + e^{-x}\right) dx =$

> > $\int_0^{1/2} 2e^{-2x} - 2e^{-1}dx = -e^{-2x} - 2e^{-1}x\Big|_0^{1/2} = 1 - 2e^{-1}$.
> > Hence, $P(X+Y \geq 1) = 1 - P(X+Y<1) = 2e^{-1}$.
>
> c. Find marginal pdf of X and Y.
>
> Joint support is $A = \{(x,y)|0<x<y<\infty\}$. The support of X is $A_X = \{0<x<\infty\}$.
> Given $x \in (0,\infty)$, $f_1(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_x^{\infty} 2e^{-x-y}dy = 2e^{-x}\left(-e^{-y}\right)\big|_x^{\infty} = 2e^{-2x}$.
> The support of Y is $A_Y = \{0<y<\infty\}$.
> Given $y \in (0,\infty)$, $f_2(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^{y} 2e^{-x-y}dx = 2e^{-y}\left(-e^{-x}\right)\big|_0^{y} = 2e^{-y} - 2e^{-2y}$.
>
> d. Find the distribution of $T = X+Y$.
>
> > Solution: The support of T is $A_T = \{0<t<\infty\}$.
> >
> > a. If $t \leq 0$, $P(T \leq t) = 0$.
> > b. If $t > 0$, $F_T(t) = P(T \leq t) = P(X+Y \leq t) = \iint_{(x,y)\in C} 2e^{-x-y}dxdy = \int_0^{t/2}\int_x^{t-x} 2e^{-x-y}dydx = \int_0^{t/2}\left(-2e^{-x}e^{-y}\right)\big|_x^{t-x} = -e^{-2x} - 2e^{-t}x\Big|_0^{t/2} = 1 - e^{-t} - te^{-t}$.
> > The pdf of T is $f_T(t) = \frac{d}{dt}F_T(t) = e^{-t} + te^{-t} = e^{-t} - e^{-t} + te^{-t} = te^{-t}$ for $t>0$ and 0 otherwise.

3.4 Independent of random variables

- Definition: For any two r.v.s X and Y, we say X and Y are independent if and only if $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ for any $A, B \subseteq \mathbb{R}$.
  Here, $X \in A$ is an event, meaning $\{\omega \in \Omega : X(\omega) \in A\}$.

  > e.g. Let $A = (-\infty, x), B = (-\infty, y)$, $x, y \in \mathbb{R}$.
  > Therefore, if X and Y are independent, $P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y) = F_1(x)F_2(y)$ for any $x, y \in \mathbb{R}$.
  > Conclusion: X and Y are independent if and only if $F(x,y) = F_1(x)F_2(y)$ for any $x, y \in \mathbb{R}$. (Above shows this is a necessary condition, proof of this is a sufficient condition is beyond the scope of this course.)

  Suppose X and Y has joint p.f. or joint p.d.f, which is denoted by $f(x,y)$, and marginal p.f. or marginal p.d.f, denoted by $f_1(x)$ and $f_2(y)$, then X and Y are independent iff $f(x,y) = f_1(x)f_2(y)$ for every $x, y \in \mathbb{R}$.
  Remark: If X and Y are independent, then $g(X)$ and $h(Y)$ must be independent for any real functions g and h.

> e.g. If X is independent of Y, then X^2 is independent of Y^2. But X^2 is independent of Y^2, we cannot conclude X is independent of Y.
> Suppose $P(X=1)=P(X=-1)=\frac{1}{2}$. Let $Y=X$. $P(X=1,Y=1)=P(X=1)=\frac{1}{2}$, but $P(X=1)P(Y=1)=\frac{1}{4}$.
> $P(Y^2=1)=P(X^2=1)=1$.

> Example: (Joint Discrete r.v.s)
> Consider the joint p.f. of X and Y is $f(x,y)=q^2p^{x+y}$ for $x=0,1,...$ and $y=0,1,...$, and 0 elsewhere. Here $p\in(0,1)$ is a constant, $q=1-p$.
> Marginal p.f. of X is $f_1(x)=qp^x$ for $x=0,1,...$ and 0 elsewhere.
> Marginal p.f. of Y is $f_2(y)=qp^y$ for $y=0,1,...$ and 0 elsewhere.
> Thus, $f(x,y)=f_1(x)f_2(y)$ for every $x,y\in\mathbb{R}$ therefore, X and Y are independent.

> Example (Joint Continuous r.v.s)
> Suppose the joint pdf of X and Y is $f(x,y)=\begin{cases}x+y & \text{if } 0\le x\le 1, 0\le y\le 1\\ 0 & \text{o.w.}\end{cases}$.
> The marginal pdf of X is $f_1(x)=x+\frac{1}{2}$ for $x\in[0,1]$ and 0 otherwise.
> The marginal pdf of Y is $f_2(y)=y+\frac{1}{2}$ for $y\in[0,1]$ and 0 otherwise.
> Hence, $f(x,y)\neq f_1(x)f_2(y)$ fpr $x\in(0,1)$ and $y\in(0,1)$, therefore, X and Y are not independent.

- Factorization theorem for independence
  Condition 1: $f(x,y)=g(x)h(y)$ for every $x,y\in\mathbb{R}$ for some function g and h where f(x,y) denotes the joint p.f. or joint p.d.f. of X and Y.
  Condition 2: Let A be the joint support of X and Y, and let A_1 be the marginal support of X and A_2 be the marginal support of Y. Then, $A=A_1\times A_2=\{(x,y)\in\mathbb{R}^2: x\in A_1, y\in A_2\}$. (Interpretation: A is a ractangle or the range of X and Y are independent.)
  Conditions 1 and 2 are satisfied if and only if X and Y are independent.

  > Example: If the joint p.f. of X and Y is $f(x,y)=\frac{\mu^{x+y}e^{-2\mu}}{x!y!}$ for $x=0,1,...$ and $y=0,1,...$ and 0 elsewhere.
  >
  > i. Is X independent of Y?
  >
  > > Solution: Condition 1: $f(x,y)=\frac{\mu^{x+y}e^{-2\mu}}{x!y!}=\frac{\mu^x e^{-\mu}}{x!}\frac{\mu^y e^{-\mu}}{y!}$. If we take $g(x)=\begin{cases}\frac{\mu^x e^{-\mu}}{x!} & \text{if } x=0,1,...\\ 0 & \text{o.w.}\end{cases}$ and $h(y)=\begin{cases}\frac{\mu^y e^{-\mu}}{y!} & \text{if } y=0,1,...\\ 0 & \text{o.w.}\end{cases}$, then $f(x,y)=g(x)h(y)$ for every $x,y\in\mathbb{R}$.
  > > Condition 2: $A=\{(x,y)\in\mathbb{R}^2: x\in A_1, y\in A_2\}$, where $A_1=\{0,1,...\}$ and $A_2=\{0,1,...\}$.
  > > Therefore, by factorization theorem, X and Y are independent.

ii. Find the marginal p.f. of X and Y.

Solution: A shortcut: $f_1(x) = C \cdot g(x)$ for some constant C.

Property 1: $f_1(x) \geq 0$ for any $x \in \mathbb{R}$. Here $g(x) = \begin{cases} \frac{\mu^x e^{-\mu}}{x!} & \text{if } x = 0, 1, \ldots \\ 0 & \text{o.w.} \end{cases}$,

therefore, $C \geq 0$.

Property 2: The support of X is $A_1 = \{0, 1, \ldots\}$. Therefore, $\sum_0^\infty f_1(x) = \sum_0^\infty C \frac{\mu^x e^{-\mu}}{x!} = C \sum_0^\infty \frac{\mu^x e^{-\mu}}{x!}$, then $C = 1$.

Therefore, $f_1(x) = \begin{cases} \frac{\mu^x e^{-\mu}}{x!} & \text{if } x = 0, 1, \ldots \\ 0 & \text{o.w.} \end{cases}$.

Similarly, $f_2(y) = \begin{cases} \frac{\mu^y e^{-\mu}}{y!} & \text{if } y = 0, 1, \ldots \\ 0 & \text{o.w.} \end{cases}$.

Example (Joint Continuous r.v.s)

Suppose the joint pdf of X and Y is $f(x, y) = \begin{cases} \frac{3}{2} y(1 - x^2) & -1 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$.

i. Is X independent of Y?

Solution: Condition 1: $f(x, y) = \left(\frac{3}{2} y\right)(1 - x^2)$, then $g(x) = \begin{cases} 1 - x^2 & \text{if } -1 \leq x \leq 1 \\ 0 & \text{o.w.} \end{cases}$ and $h(y) = \begin{cases} \frac{3}{2} y & \text{if } 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$.

Then $f(x, y) = g(x) h(y)$ for every $x, y \in \mathbb{R}$.

Condition 2: $A = \{(x, y) \in \mathbb{R}^2 : x \in A_1, y \in A_2\}$, where $A_1 = [-1, 1]$ and $A_2 = [0, 1]$.

Therefore, by factorization theorem, X and Y are independent.

ii. Find the marginal pdf of X and Y.

Solution: A shortcut: $f_1(x) = C \cdot g(x)$ for some constant C, the support of X is $A_1 = [-1, 1]$.

Property 1: $f_1(x) \geq 0$ for any $x \in \mathbb{R}$. Here $g(x) = \begin{cases} 1 - x^2 & \text{if } -1 \leq x \leq 1 \\ 0 & \text{o.w.} \end{cases}$,

therefore, $C \geq 0$.

Property 2: $\int_{-\infty}^{\infty} f_1(x) dx = \int_{-1}^{1} C(1 - x^2) dx = C\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1} = 2C\left(1 - \frac{1}{3}\right) = 1$, therefore, $C = \frac{3}{4}$.

Therefore, $f_1(x) = \begin{cases} \frac{3}{4}(1 - x^2) & \text{if } -1 \leq x \leq 1 \\ 0 & \text{o.w.} \end{cases}$.

Support of Y is $A_2 = [0, 1]$, given $y \in [0, 1]$, $f_2(y) = \frac{f(x,y)}{f_1(x)} = \frac{\frac{3}{2}y(1-x^2)}{\frac{3}{4}(1-x^2)} = 2y$.

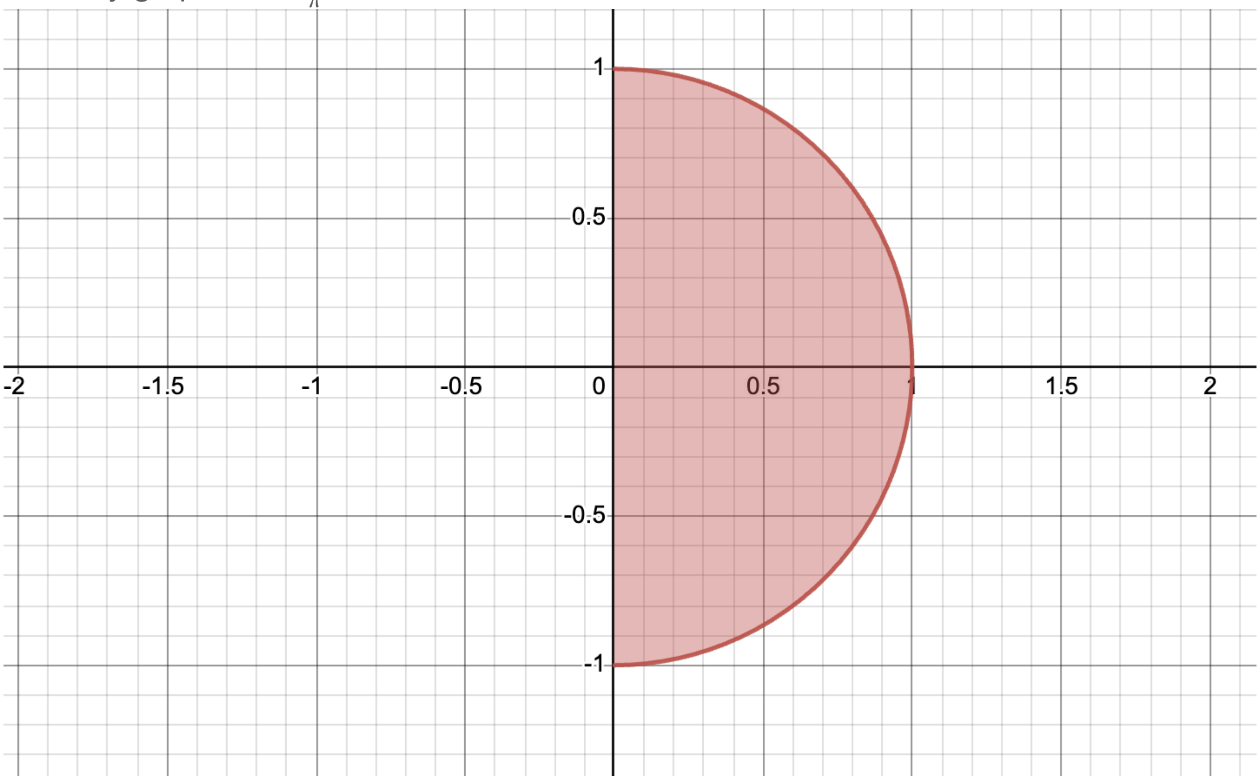
Therefore, $f_2(y) = \begin{cases} 2y & \text{if } 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$.

Example (Uniform distribution over a region)

Suppose (X, Y) follows a uniform distribution over $C = \{(x, y) | x \ge 0, x^2 + y^2 \le 1\}$.

Namely, $f(x, y) = \begin{cases} c & \text{if } (x, y) \in C \\ 0 & \text{o.w.} \end{cases}$.

Here, by graph, $c = \frac{2}{\pi}$.



i. Is X independent of Y?

Solution: Given $x \in [0, 1]$, Y can take value in $[-\sqrt{1-x^2}, \sqrt{1-x^2}]$, therefore, X and Y are not independent.

ii. Find the marginal pdf of X and Y.

Solution: The support of X is $A_1 = [0, 1]$, given $x \in [0, 1]$, $f_1(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{\pi} dy = \frac{4}{\pi}\sqrt{1-x^2}$.

The support of Y is $A_2 = [-1, 1]$, given $y \in [-1, 1]$, $f_2(y) = \int_0^{\sqrt{1-y^2}} \frac{2}{\pi} dx = \frac{2}{\pi}\sqrt{1-y^2}$.

3.5 Joint expectation

- Definition: Suppose $h(x,y)$ is a bivariate function, then $E[h(x,y)] = \begin{cases} \sum_x \sum_y h(x,y)f(x,y) & \text{if } X \text{ and } Y \text{ are joint discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)f(x,y)dxdy & \text{if } X \text{ and } Y \text{ are joint continuous} \end{cases}$, provided $E[|h(x,y)|] < \infty$.

> e.g. $E[XY] = \begin{cases} \sum_x \sum_y (xy)f(x,y) & \text{if } X \text{ and } Y \text{ are joint discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (xy)f(x,y)dxdy & \text{if } X \text{ and } Y \text{ are joint continuous} \end{cases}$, provided $E[|XY|] < \infty$.

> e.g. E[X] (i.e. h(x,y) = x))
>
> - Method 1:
>
> $$E(X) = \begin{cases} \sum_x \sum_y xf(x,y) & \text{joint discrete} \\ \iint_{\mathbb{R}^2} xf(x,y)dxdy & \text{joint continuous} \end{cases}$$
>
> - Method 2: find the marginal distribution, i.e., the marginal p.f. or marginal p.d.f. of X first, denoted by $f_1(x)$, then
>
> $$E(X) = \begin{cases} \sum_x xf_1(x) & \text{joint discrete} \\ \int_{\mathbb{R}^2} xf_1(x)dx & \text{joint continuous} \end{cases}$$

- Properties of joint expectation:
  i. linearity: $E[ag(X,Y) + bh(X,Y)] = aE[g(X,Y)] + bE[h(X,Y)]$ where a, b are constants, g, h are bivariate functions.
  ii. Under independence assumption (X is independent of Y), $E(XY) = E(X)E(Y)$ and $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$. Further, if $X_1, ..., X_n$ are independent, then $E\left[\prod_{i=1}^n h_i(X_i)\right] = \prod_{i=1}^n E[h_i(X_i)]$.
- Covariance of X and Y
  Definition: Covariance of X and Y is defined as $Cov(X,Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y)$.
  If X and Y are independent, then $Cov(X,Y) = 0$.

> An example where X and Y are uncorrlated, but not independent.
> Let $X \sim N(0,1)$ and $Y = X^2$, then $E(X) = 0, E(XY) = E(X^3), Cov(X,Y) = 0$.
> Now, we find a pair of a and b such that $P(X \leq a, Y \leq b) \neq P(X \leq a)P(Y \leq b)$.
> Consider $a = -2, b = 1$, then $P(X \leq a) = P(X \leq -2) > 0, P(Y \leq b) = P(X^2 \leq 1) = P(-1 \leq X \leq 1) > 0$, but $P(X \leq a, Y \leq b) = P(X \leq -2, Y \leq 1) = 0$.

- Results for covariance

i. $Cov(X,X) = E[(X-\mu_X)(X-\mu_X)] = E[(X-\mu_X)^2] = Var(X)$.
ii. $Cov(X+Y,Z) = Cov(X,Z) + Cov(Y,Z)$.

- Variance formula

i. $Var(aX+bY) = Cov(aX+bY, aX+bY)$
$Cov(aX,aX) + Cov(aX,bY) + Cov(bY,aX) + Cov(bY,bY) = Var(aX) + 2abCov(X,Y) + Var(bY) = a^2Var(X) + 2abCov(X,Y) + b^2Var(Y)$

ii.
$$Var\left(\sum_{i=1}^{n}\right) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)$$

iii. If $X_1, ..., X_n$ are independent,
$$Var\left(\sum_{i=1}^{n}\right) = \sum_{i=1}^{n} Var(X_i)$$

> Example 1: Suppose the joint p.f. of X and Y is $f(x,y) = \begin{cases} \frac{\mu^{x+y}e^{-2\mu}}{x!y!} & \text{if } x = 0,1,... \text{ and } y = 0,1,... \\ 0 & \text{o.w.} \end{cases}$. Find $Var(2X+3Y) = 4Var(X) + 12Cov(X,Y) + 9Var(Y)$.
>
> > Solution: Since X and Y are independent, $Cov(X,Y) = 0$, therefore, $Var(2X + 3Y) = 4Var(X) + 9Var(Y)$.
> > Previously, we find $X \sim Poisson(\mu)$, $Y \sim Poisson(\mu)$, therefore $Var(X) = \mu$, $Var(Y) = \mu$.
> > Hence, $Var(2X+3Y) = 4\mu + 9\mu = 13\mu$.

> Example 2: Suppose the joint p.f. of X and Y is $f(x,y) = \begin{cases} x+y & \text{if } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$. Find $Var(X+Y)$.
>
> > Solution:
> >
> > $$\begin{aligned} Var(X+Y) &= Var(X) + 2Cov(X,Y) + Var(Y) \\ &= 2Var(X) + 2Cov(X,y) \end{aligned}$$
> >
> > the marginal pdf of X is $f_1(x) = \begin{cases} x + 1/2 & \text{if } 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$.
> > then, $E(X) = \int_0^1 x\left(x+\frac{1}{2}\right)dx = \int_0^1 \left(x^2 + \frac{x}{2}\right)dx = \left(\frac{x^3}{3} + \frac{x^2}{4}\right)\Big|_0^1 = \frac{7}{12}$.

> > $E(X^2)=\int_0^1 x^2\left(x+\frac{1}{2}\right)dx=\int_0^1\left(x^3+\frac{x^2}{2}\right)dx=\left(\frac{x^4}{4}+\frac{x^3}{6}\right)\Big|_0^1=\frac{5}{12}$.
> > $Var(X)=E(X^2)-(E(X))^2=\frac{5}{12}-\left(\frac{7}{12}\right)^2=\frac{11}{144}$.
> > $Cov(X,Y)=E(XY)-E(X)E(Y)$, where $E(X)E(Y)=\left(\frac{7}{12}\right)^2=\frac{49}{144}$.
> >
> > $$\begin{aligned} E(XY) &= \int_0^1\int_0^1 (xy)(x+y)dxdy \\ &= \int_0^1\int_0^1 (x^2y+xy^2)dxdy \\ &= \int_0^1 \left(\frac{x^3y}{3}+\frac{x^2y^2}{2}\right)\Big|_{x=0}^{x=1} dy \\ &= \int_0^1\left(\frac{y}{3}+\frac{y^2}{2}\right)dy \\ &= \left(\frac{y^2}{6}+\frac{y^3}{6}\right)\Big|_{y=0}^{y=1} \\ &= \frac{1}{3} \end{aligned}$$
> >
> > $Cov(X,Y)=1/3-49/144=-1/144$.
> > $Var(X+Y)=2Var(X)+2Cov(X,Y)=2\frac{11}{144}+2\frac{-1}{144}=\frac{20}{144}$.
> > Alternatively: Let $T=X+Y$, we can calculate the moment generating function: $E(e^{t(X+Y)})$.

- Corrlation coefficient

  Definition: Correlation coefficient of X and Y is defined as $\rho(X,Y)=\frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$.

  i. Used to describe linear association between X and Y.
  ii. Unit free
  iii. $-1\le\rho(X,Y)\le 1$.

  > (not required): Use the fact $|E(XY)|\le\sqrt{E(X^2)}\sqrt{E(Y^2)}$ to prove $-1\le\rho(X,Y)\le 1$.

- Properties of corrlation corfficient:

  i. $\rho(X,Y)=1 \implies Y=aX+b$ for some constants $a>0$ and b.
  ii. $\rho(X,Y)=-1 \implies Y=aX+b$ for some constants $a<0$ and b.

> Example: Suppose (X,Y) has joint pdf $f(x,y)=\begin{cases} x+y & 0\le x\le y, 0\le y\le 1 \\ 0 & \text{o.w.}\end{cases}$. Find $\rho(X,Y)$.

> > Solution: $Cov(X,Y) = -\frac{1}{144}$, $Var(X) = Var(Y) = \frac{11}{144}$, therefore, $\rho(X,Y) = \frac{-1/144}{\sqrt{11/144}\sqrt{11/144}} = -\frac{1}{11}$.

3.6 Conditional distribution

- Definition (Joint Discrete Case)
  Suppose X and Y are joint discrete random variable with joint p.f. denoted by $f(x,y)$. Then, conditional p.f. of X given $Y = y$ is $f_1(x|y) = \frac{f(x,y)}{f_2(y)}$, provided that $f_2(y) > 0$.
  > Idea: Let event $A = \{X = x\}, B = \{Y = y\}$, then $f_1(x|y) = P(X = x|Y = y) = \frac{P(A\cap B)}{P(B)} = \frac{f(x,y)}{f_2(y)}$.
  > Similarly, the conditional p.f. of Y given $X = x$ is $f_2(y|x) = \frac{f(x,y)}{f_1(x)}$, provided that $f_1(x) > 0$.

  - Property: Conditional p.f.s $f_1(x|y)$ and $f_2(x|y)$ are probability functions, i.e.:
    a. $f_1(x|y) \geq 0$ for any $x \in \mathbb{R}$, and y is fixed. Additionally, $\sum_{x\in\mathbb{R}} f_1(x|y) = 1$ for any y, where R is the conditional support of x and may depend on y.
    b. $f_2(y|x) \geq 0$ for any $y \in \mathbb{R}$, and x is fixed. Additionally, $\sum_{y\in\mathbb{R}} f_2(y|x) = 1$ for any x.
- Definition (Joint Continuous Case)
  Suppose X and Y are joint continuous random variable with joint p.d.f. denoted by $f(x,y)$. Then, conditional p.d.f. of X given $Y = y$ is $f_1(x|y) = \frac{f(x,y)}{f_2(y)}$, provided that $f_2(y) > 0$.
  Similarly, the conditional p.d.f. of Y given $X = x$ is $f_2(y|x) = \frac{f(x,y)}{f_1(x)}$, provided that $f_1(x) > 0$.
  - Property: Conditional p.d.f.s $f_1(x|y)$ and $f_2(x|y)$ are probability density functions, i.e.:
    a. $f_1(x|y) \geq 0$ for any $x \in \mathbb{R}$, and y is fixed. Additionally, $\int_{-\infty}^{\infty} f_1(x|y)dx = 1$ for any y.
    b. $f_2(y|x) \geq 0$ for any $y \in \mathbb{R}$, and x is fixed. Additionally, $\int_{-\infty}^{\infty} f_2(y|x)dy = 1$ for any x.

> Example 1: Let $f(x,y) = \begin{cases} 8xy & 0 < y < x < 1 \\ 0 & \text{o.w.} \end{cases}$.
>
> Find:
>
> 1. $f_1(x|y)$
>
> > Solution: $f_1(x|y) = \frac{f(x,y)}{f_2(y)}$.
> > The support of Y is $A_2 = (0,1)$, given $y \in (0,1)$, $f_2(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_y^1 8xydx = 4x^2y\Big|_y^1 = 4y - 4y^3$.
> > Therefore, $f_1(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{8xy}{4y-4y^3}$ for $0 < y < x < 1$ and 0 otherwise.

> 2. $f_2(y|x)$
>
> > Solution: $f_2(y|x) = \frac{f(x,y)}{f_1(x)}$.
> > The support of X is $A_1 = (0,1)$, given $x \in (0,1)$, $f_1(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^x 8xydy = 4xy^2\Big|_0^x = 4x^3$.
> > Therefore, $f_2(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{8xy}{4x^3}$ for $0 < y < x < 1$ and 0 otherwise.

> Example 2: The joint pdf is $f(x,y) = \begin{cases} x+y & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$.
>
> Find $f_1(x|y)$ and $f_2(y|x)$.
>
> > Solution: The marginal pdf of Y is $f_2(y) = \begin{cases} \frac{1}{2}+y & 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$.
> > Given $y \in [0,1]$ $f_1(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{x+y}{\frac{1}{2}+y}$ for $0 \le x \le 1$ and 0 otherwise.
> > The marginal pdf of X is $f_1(x) = \begin{cases} x+\frac{1}{2} & 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$.
> > Given $x \in [0,1]$ $f_2(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{x+y}{x+\frac{1}{2}}$ for $0 \le y \le 1$ and 0 otherwise.

> Example 3: The joint p.f. of X and Y is $f(x,y) = \begin{cases} q^2p^{x+y} & x = 0,1,... \text{ and } y = 0,1,... \\ 0 & \text{o.w.} \end{cases}$,
>
> where $p \in (0,1)$ is a constant, $q = 1-p$.
> Find $f_1(x|y)$ and $f_2(y|x)$.
>
> > Solution: The marginal p.f. of Y is $f_2(y) = \begin{cases} qp^y & y = 0,1,... \\ 0 & \text{o.w.} \end{cases}$.
> > Given $y \in \{0,1,...\}$, $f_1(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{q^2p^{x+y}}{qp^y} = qp^x$ for $x = 0,1,...$ and 0 otherwise.
> > The marginal p.f. of X is $f_1(x) = \begin{cases} qp^x & x = 0,1,... \\ 0 & \text{o.w.} \end{cases}$.
> > Given $x \in \{0,1,...\}$, $f_2(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{q^2p^{x+y}}{qp^x} = qp^y$ for $y = 0,1,...$ and 0 otherwise.

- Applications of conditional distribution:
  i. Check independence:
  X and Y are independent if and only if $f_1(x|y) = f_1(x)$ for any $x \in \mathbb{R}$, or $f_2(y|x) = f_2(y)$ for any $y \in \mathbb{R}$.
  > Proof sketch: X and Y are independent $\iff$ $f(x,y) = f_1(x)f_2(y)$ for any $x, y \in \mathbb{R}$. Then, $f_1(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{f_1(x)f_2(y)}{f_2(y)} = f_1(x)$ for any $x, y \in \mathbb{R}$.

ii. Use ocnditional distribution to find joint disteibution:
$f(x,y) = f_1(x|y)f_2(y) = f_2(y|x)f_1(x)$ as $f_1(x|y) = \frac{f(x,y)}{f_2(y)}$ and $f_2(y|x) = \frac{f(x,y)}{f_1(x)}$.

> Example 1: $Y \sim \text{Poisson}(\mu)$. $X|Y = y \sim \text{Binomial}(y,p)$, where $p \in (0,1)$ is a constant. Find the marginal p.f. of X.
>
>> Solution: The joint pf of (X,Y) is $f(x,y) = f_2(y)f_1(x|y) = \frac{\mu^y e^{-\mu}}{y!}\binom{y}{x}p^x(1-p)^{y-x}$ **for** $x = 0,1,...,y$ **and** $y = 0,1,...$.
>> The support of X is $A = \{0,1,...\}$, given $x \in \{0,1,...\}$, $f_1(x) = \sum_{y=x}^{\infty} f(x,y) = \sum_{y=x}^{\infty} \frac{\mu^y e^{-\mu}}{y!}\binom{y}{x}p^x(1-p)^{y-x} = \sum_{y=x}^{\infty} \frac{\mu^y e^{-\mu}}{y!}\frac{y!}{x!(y-x)!}p^x(1-p)^{y-x} = \frac{(\mu p)^x}{x!}e^{-\mu p}\sum_{y=x}^{\infty}\frac{(\mu(1-p))^{y-x}}{(y-x)!}$. Let $t = y - x$, then, $f_1(x) = \frac{(\mu p)^x}{x!}e^{-\mu p}\sum_{t=0}^{\infty}\frac{(\mu(1-p))^t}{t!} = \frac{(\mu p)^x}{x!}e^{-\mu p}e^{\mu(1-p)} = \frac{(\mu p)^x}{x!}e^{-\mu p}$. Then, $X \sim \text{Poisson}(\mu p)$.

> Example 2: Suppose Y has pdf $f_2(y) = \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}$ for $y > 0$, i.e. $Y \sim \text{Gamma}(\alpha, 1)$, and the conditional pdf of X given $Y = y$ is $f_1(x|y) = ye^{-xy}$ for $x > 0$, i.e. $X|Y = y \sim \text{Gamma}(1, 1/y)$. Find the marginal pdf of X.
>
>> Solution: $f(x,y) = f_2(y)f_1(x|y) = \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}ye^{-xy}$ for $x > 0$ and $y > 0$. The support of X is $(0,\infty)$
>> Given $x > 0$, $f_1(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^{\infty}\frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}ye^{-xy}dy = \int_0^{\infty}\frac{y^{(\alpha+1)-1}e^{-(x+1)y}}{\Gamma(\alpha)}$.
>> Aside: If $Y \sim \text{Gamma}(\alpha, \beta)$, then $f(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}}$ for $x > 0$.
>> Let $\bar{\alpha} = \alpha + 1$, $\beta = \frac{1}{x+1}$, then, $f_1(x) = \int_0^{\infty}\frac{y^{\bar{\alpha}-1}e^{-y/\beta}}{\Gamma(\bar{\alpha})\beta^{\bar{\alpha}}} = \frac{\beta^{\bar{\alpha}}}{\Gamma(\bar{\alpha})}\int_0^{\infty}\frac{y^{\bar{\alpha}-1}e^{-y/\beta}}{\beta^{\bar{\alpha}}} = \frac{(\frac{1}{x+1})^{\alpha+1}\Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)}\frac{1}{(x+1)^{\alpha+1}} = \frac{\alpha}{(x+1)^{\alpha+1}}$, $x > 0$.

3.7 Conditional expectation

Since $f_2(y|x)$ is a probability function (if X and Y are joint discrete) or probability density function (if X and Y are joint continuous). We can define expectation with respect to $f_2(y|x)$.

- Definition of conditional expectation (mean):
  The conditional expectation of $g(y)$ given $X = x$ is defined as $E[g(Y)|X = x] = \begin{cases} \sum_y g(y)f_2(y|x) & \text{if } X \text{ and } Y \text{ are joint discrete} \\ \int_{-\infty}^{\infty} g(y)f_2(y|x)dy & \text{if } X \text{ and } Y \text{ are joint continuous} \end{cases}$.
  In particular, we are particularly intrested in :
  i. $E[Y|X = x](g(y) = y)$
  ii. $Var(Y|X = x) = E[Y^2|X = x] - (E[Y|X = x])^2$.
  iii. $E(e^{tY}|X = x)(g(y) = e^{ty})$.

> Example: The joint pdf of X and Y is $f(x,y) = \begin{cases} 8xy & 0<y<x<1 \\ 0 & \text{o.w.} \end{cases}$. Find $E[X|Y=y]$ and $Var(X|Y=y)$.
>
>> Solution: The conditional pdf of X given $Y=y$ is $f_1(x|y) = \frac{2x}{1-y^2}, 0<y<x<1.$
>> Given $y \in (0,1)$, $E(X|Y=y) = \int_{-\infty}^{\infty} x \cdot f_1(x|y)dx = \int_y^1 x \cdot \frac{2x}{1-y^2}dx = \frac{2}{1-y^2}\int_y^1 x^2 dx = \frac{1}{1-y^2}\left(\frac{2x^3}{3}\right)\Bigg|_y^1 = \frac{2(1-y^3)}{3(1-y^2)}.$
>> Given $y \in (0,1)$, $E(X^2|Y=y) = \int_{-\infty}^{\infty} x^2 \cdot f_1(x|y)dx = \int_y^1 x^2 \cdot \frac{2x}{1-y^2}dx = \frac{2}{1-y^2}\int_y^1 x^3 dx = \frac{1}{1-y^2}\left(\frac{2x^4}{4}\right)\Bigg|_y^1 = \frac{2(1-y^4)}{4(1-y^2)} = \frac{1+y^2}{2}.$
>> $Var(X|Y=y) = E(X^2|Y=y) - (E(X|Y=y))^2 = \frac{1+y^2}{2} - \left(\frac{2(1-y^3)}{3(1-y^2)}\right)^2 = \frac{1+y^2}{2} - \left(\frac{2(1-y^3)}{3(1-y^2)}\right)^2$

- Some useful results regarding conditional expectation
  i. If X and Y are independent, then $E[g(Y)|X=x] = E[g(Y)]$ and $E[h(X)|Y=y] = E[h(X)]$.
  ii. Substitution rule: $E[h(X,Y)|X=x] = E[h(x,Y)|X=x] = h(x,Y)$.

  > e.g. $E[X+Y|X=x] = E[x+Y|X=x] = E[x|X=x] + E[Y|X=x] = x + E[Y|X=x]$.

  > e.g. $E(XY|X=x) = E(xY|X=x) = xE(Y|X=x)$.

  iii. Double Expectation Theorem: $E[E[g(Y)|X]] = E[g(Y)]$.
  Note: $E[g(Y)|X] \neq E[g(Y)|X=x]$.
  Two step method to find $E[g(Y)|X]$:
  Step 1: For any x taken from the support of X, calculate $E[g(Y)|X=x]$, denoted by $h(x)$.
  i.e. $h(x) = E[g(Y)|X=x] = \begin{cases} \sum_y g(y)f_2(y|x) & \text{if } X \text{ and } Y \text{ are joint discrete} \\ \int_{-\infty}^{\infty} g(y)f_2(y|x)dy & \text{if } X \text{ and } Y \text{ are joint continuous} \end{cases}$.
  Step 2: $E[g(Y)|X] = h(X)$.
  Hence, $E[g(y)|X]$ is a function of X, that is why it is a random variable.

  > Example 1: Suppose $Y \sim \text{Poisson}(\mu)$, $X|Y=y \sim \text{Binomial}(y,p)$, where $p \in (0,1)$ is a constant.
  >
  > a. Find $E[X]$.

Method 1: We've found $X \sim \text{Poisson}(\mu p)$, therefore, $E[X] = \mu p$. It is computationally intensive.
Method 2: $E[X] = E[E[X|Y]]$.
Apply the two step method:
Step 1: Given $y \in \{0, 1, ...\}$, $E[X|Y = y] = yp$.
Step 2: $E[X|Y] = Yp$.
Therefore, $E[X] = E[E[X|Y]] = E[Yp] = pE[Y] = p\mu$.
Method 3: $E(e^{tX}) = E[E(e^{tX}|Y)]$.
Apply the two step method:
Step 1: Given $y \in \{0, 1, ...\}$, $E(e^{tX}|Y = y) = [pe^t + (1-p)]^y$.
Step 2: $E(e^{tX}|Y) = [pe^t + (1-p)]^Y$.

b. Find $Var(X)$.

Method 1: We've found $X \sim \text{Poisson}(\mu p)$, therefore, $Var(X) = \mu p$.
Method 2: By double expectation theorem, $Var(X) = E[Var(X|Y)] + Var[E(X|Y)]$.
As $E(X|Y) = Yp$, $Var[E(X|Y)] = Var(Yp) = p^2 Var(Y) = p^2\mu$. ($Y \sim \text{Poisson}(\mu)$)
For $E(Var(X|Y))$, apply the two step method:
Step 1: Given $y \in \{0, 1, ...\}$, $Var(X|Y = y) = yp(1-p)$.
Step 2: $Var(X|Y) = Yp(1-p)$.
Therefore, $E[Var(X|Y)] = E[Yp(1-p)] = p(1-p)E[Y] = p(1-p)\mu$.
$Var(X) = E[Var(X|Y)] + Var[E(X|Y)] = p(1-p)\mu + p^2\mu = p\mu$.

Example 2 (Random variables of different types):
Suppose $X \sim \text{Unif}[0, 1], Y|X = x \sim \text{Binomial}(10, x)$, find E(Y) and Var(Y).

Solution: By double expectation theorem, $E(Y) = E[E(Y|X)]$.
Step 1: Given $x \in [0, 1]$, $E(Y|X = x) = 10x$.
Step 2: $E(Y|X) = 10X$.
Therefore, $E(Y) = E[E(Y|X)] = E(10X) = 10E(X) = 10 \cdot \frac{1}{2} = 5$.
Var(Y) = E[Var(Y|X)] + Var[E(Y|X)].
$Var[E(Y|X)] = Var(10X) = 100Var(X)$
For any $x \in [0, 1]$
Step 1: $Var(Y|X = x) = 10x(1-x)$.
Step 2: $Var(Y|X) = 10X(1-X)$.
Therefore, $E[Var(Y|X)] = E[10X(1-X)] = E(10X) - 10E(X^2) = 10E(X) - 10(Var(X) + (E(X))^2) = 10 \cdot \frac{1}{2} - 10\left(\frac{1}{12} + \frac{1}{4}\right) = 5 - 10 \cdot \frac{1}{3}$.
$Var(Y) = E[Var(Y|X)] + Var[E(Y|X)] = 5 - 10 \cdot \frac{1}{3} + 100 \cdot \frac{1}{12} = \frac{5}{3}$.

3.8 Joint Moment Generating Function

- Definition: If X and Y are two r.v.s, then $M(t_1,t_2)=E(e^{t_1X+t_2Y})$ is called the joint moment generating function (mgf) of X and Y, if M(t_1,t_2) exists($M(t_1,t_2)<\infty$) for $|t_1|<h_1$, $|t_2|<h_2$, where $h_1,h_2>0$.
- Application of joint mgf
    i. Find marginal mgf from joint mgf.
    Given $M(t_1,t_2)<\infty$ for $|t_1|<h_1$ and $|t_2|<h_2$. Then, $M_X(t_1)=E(e^{t_1X})=M(t_1,0)$ for $|t_1|<h_1$ and $M_Y(t_2)=E(e^{t_2Y})=M(0,t_2)$ for $|t_2|<h_2$.
    ii. Independence of r.v.s
    X and Y are independent if and only if $M(t_1,t_2)=M_X(t_1)M_Y(t_2)$ for $|t_1|<h_1$ and $|t_2|<h_2$.

> Example 1 (Joint mgf):
>
> Suppose the joint pdf of X and Y is given by $f(x,y)=\begin{cases} e^{-y} & 0<x<y \\ 0 & \text{o.w.} \end{cases}$.
>
> i. Find the joint mgf of X and Y.
>
> > Solution: $M(t_1,t_2)=E(e^{t_1X+t_2Y})=\iint_{\mathbb{R}} e^{t_1x+t_2y}f(x,y)dxdy=$ $\int_0^\infty\int_x^\infty e^{t_1x+t_2y}e^{-y}dydx=\int_0^\infty e^{t_1x}\int_x^\infty e^{(t_2-1)y}dydx=$ $\int_0^\infty e^{t_1x}\left(\frac{e^{(t_2-1)y}}{t_2-1}\right)\Bigg|_x^\infty dx=\int_0^\infty e^{t_1x}\left(\frac{e^{(t_2-1)x}}{t_2-1}\right)dx=\frac{1}{t_2-1}\int_0^\infty e^{(t_1+t_2-1)x}dx=$ $\frac{1}{t_2-1}\left(\frac{e^{(t_1+t_2-1)x}}{t_1+t_2-1}\right)\Bigg|_0^\infty=\frac{1}{1-t_2}\left(\frac{1}{1-(t_1+t_2)}\right)$.
>
> ii. Are they independent?
>
> > Solution: $M_X(t_1)=M(t_1,0)=\frac{1}{1-t_1}$, $M_Y(t_2)=M(0,t_2)=\frac{1}{1-t_2}$. Therefore, $M_X(t_1)M_Y(t_2)=\frac{1}{(1-t_1)(1-t_2)}\neq M(t_1,t_2)$, therefore, X and Y are not independent.

> Example 2 (Additivity of Poisson r.v.s):
> Suppose $X\sim\text{Poisson}(\mu_1), Y\sim\text{Poisson}(\mu_2)$, X is independent of Y.
> Prove $X+Y\sim\text{Poisson}(\mu_1+\mu_2)$.
>
> > Solution: We first find the mgf of $X+Y$.
> > Let $Z=X+Y$, then the mgf of Z is $M_Z(t)=E(e^{tZ})=E(e^{t(X+Y)})=$ $E(e^{tX}e^{tY})=E(e^{tX})E(e^{tY})=e^{(\mu_1(e^t-1)+\mu_2(e^t-1+)}=e^{(\mu_1+\mu_2)(e^t-1)}$, which is the mgf of $\text{Poisson}(\mu_1+\mu_2)$.
> > By the uniqueness property of mgf, $X+Y\sim\text{Poisson}(\mu_1+\mu_2)$.

3.9 Multinomial Distribution

- Definition: $(X_1, ..., X_n)$ are joint discrete r.v.s with joint p.f. $f(x_1, .., x_k) = P(X_1 = x_1, ..., X_k = x_k) = \frac{n!}{x_1!...x_k!} p_1^{x_1} ... p_k^{x_k}$, where $x_i = 0, 1, ..., n$ for $i = 1, ..., k$. $\sum_i = 1^k x_i = n$, $0 < p_i < 1$ and $\sum_i = 1^k p_i = 1$. Then, $(X_1, ..., X_k)$ follows multinomial distribution, with notation $(X_1, ..., X_k) \sim \text{Mult}(n, p_1, ..., p_k)$.
- Properties of $\text{Mult}(n, p_1, ..., p_k)$:
  i. Joint mgf
     a. $M(t_1, ..., t_k) = E(e^{t_1 X_1 + ... + t_k X_k})$
     b. $M(t_1, ..., t_{k-1}) = E(e^{t_1 X_1 + ... + t_{k-1} X_{k-1}}) = (p_1 e^{t_1} + ... + p_{k-1} e^{t_{k-1}} + p_k)^n$

        > e.g. $k = 2$, $M(t_1) = E(e^{t_1 X_1}) = (p_1 e^{t_1} + p_2)^n$, where $p_1 + p_2 = 1$.

  ii. Marginal distribution
     $X_i \sim \text{Binomial}(n, p_i)$ for $i = 1, ..., k$.
  iii. Let $T = X_i + x_j, i \neq j$. Then, $T \sim \text{Binomial}(n, p_i + p_j)$.

     > e.g. Suppose $i = 1, j = 2$, set $t_1 = t_2 = t, t_3 = ... = t_k = 0$ in the joint mgf of $\text{Mult}(n, p_1, ..., p_k)$, then, $M_T(t) = [(p_1 + p_2)e^t + (1 - p_1 - p_2)]^n$.

  iv. Joint Moment
     $E(X_i) = np_i$ and $Var(X_i) = np_i(1 - p_i)$ for $i = 1, ..., k$.

     > Question: What is $Cov(X_i, X_j)$ for $i \neq j$?
     >
     > > $Var(X_i + X_j) = Var(X_i) + Var(X_j) + 2Cov(X_i, X_j)$.
     > > We know $Var(X_i) = np_i(1 - p_i)$, $Var(X_j) = np_j(1 - p_j)$, $Var(X_i + X_j) = n(p_i + p_j)[1 - (p_i + p_j)]$.
     > > Therefore, $Cov(X_i, X_j) = -np_i p_j$.

  v. Conditional distribution
     $X_i | X_i + X_j = t \sim \text{Binomial}(t, p_i/(p_i + p_j))$.
  vi. $X_i | X_j = t \sim \text{Binomial}(n - t, p_i/(1 - p_j))$.

3.10 Bivariate Normal Distribution

- Definition:
  Suppose that X_1 and X_2 are joint continuous r.v.s with joint pdf $f(x_1, x_2) = \frac{1}{2\pi|\Sigma|^{\frac{1}{2}}} \exp\{-\frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu)\}$, where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$, $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$, $\rho \in (-1, 1)$, and $|\Sigma|$ denotes the determinant of Σ, i,.e. $|\Sigma| =$

$\sigma_1^2\sigma_2^2(1-\rho^2)$.
Then, (X_1, X_2) follows bivariate normal distribution, with notation $X \sim \mathrm{BVN}(\mu, \Sigma)$.

- Properties:
  i. Joint mgf
  $M(t_1, t_2) = E(e^{t_1X_1+t_2X_2}) = E(e^{t^TX}) = e^{t^T\mu+\frac{1}{2}t^T\Sigma t}$, where $t = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$.
  ii. Marginally
  $M_{X_1}(t_1) = M(t_1, t_2 = 0) = e^{t_1\mu_1+\frac{1}{2}\sigma_1^2t_1^2}$, $M_{X_2}(t_2) = M(t_1 = 0, t_2) = e^{t_2\mu_2+\frac{1}{2}\sigma_2^2t_2^2}$.
  Then, $X_1 \sim \mathrm{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathrm{N}(\mu_2, \sigma_2^2)$, $E(X_1) = \mu_1$, $Var(X_1) = \sigma_1^2$, $E(X_2) = \mu_2$, $Var(X_2) = \sigma_2^2$.
  $Cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2)$.
  What is $E(X_1X_2)$?
  iii. We find the conditional distribution of X_1 given X_2, $X_1|X_2 = x_2$.
  Conclusion: $X_1|X_2 = x_2$ is normally distributed.
  Then, to find $E(X_1|X_2 = x_2)$ and $Var(X_1|X_2 = x_2)$.
  $E(X_1|X_2 = x_2) = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2 - \mu_2)$.
  $Var(X_1|X_2 = x_2) = \sigma_1^2(1-\rho^2)$.
  Finding $X_2|X_1 = x_1$ is normal.
  $E(X_2|X_1 = x_1) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1 - \mu_1)$.
  $Var(X_2|X_1 = x_1) = \sigma_2^2(1-\rho^2)$.
  iv. $Cov(X_1, X_2) = \rho\sigma_1\sigma_2$.

  > Proof: To find $E(X_1X_2)$, we apply double expectation theorem.
  > $E(X_1X_2) = E(E(X_1X_2|X_2))$
  > Step 1: $E(X_1X_2|X_1 = x_1) = x_1E(X_2|X_1 = x_1) = x_1(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1 - \mu_1))$
  > Step 2: $E(X_1X_2) = E(x_1(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1 - \mu_1))) = \mu_2E(X_1) + \rho\frac{\sigma_2}{\sigma_1}E(X_1^2) - \mu_1E(X_1) - \rho\frac{\sigma_2}{\sigma_1}\mu_1E(X_1) = \mu_2\mu_1 + \rho\frac{\sigma_2}{\sigma_1}(\sigma_1^2 + \mu_1^2) - \mu_1^2 - \rho\frac{\sigma_2}{\sigma_1}\mu_1^2 = \mu_1\mu_2 + \rho\sigma_1\sigma_2$.
  > Therefore, $Cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2) = \mu_1\mu_2 + \rho\sigma_1\sigma_2 - \mu_1\mu_2 = \rho\sigma_1\sigma_2$.
  > Furthermore, $\rho(X_1, X_1) = \rho = \frac{Cov(X_1,X_2)}{\sqrt{Var(X_1)}\sqrt{Var(X_2)}} = \frac{\rho\sigma_1\sigma_2}{\sigma_1\sigma_2}$.

  v. $\rho = 0$ if and only if X_1 and X_2 are independent.

  > Common Mistake: If Y_1 and Y_2 are normally distributed, and $Cov(Y_1, Y_2) = 0$, then Y_1 and Y_2 are independent.
  > Counter Example: $Y_1 \sim \mathrm{N}(0, 1)$, $Y_2 = RY_1$, where $P(R = 1) = P(R = -1) = 1/2$, R is independent of X.
  > Show that $Y_2 \sim \mathrm{N}(0, 1)$ and $Cov(Y_1, Y_2) = 0$.
  > If joint distribution (Y_1, Y_2) follows BVN, then $Y_1 + Y_2$ follows normal distribution, then

> $P(Y_1+Y_2=0)=0$, however, $P(Y_1+Y_2=0)=P(R=-1)=1/2$, then the joint distribution of (Y_1,Y_2) is not BVN.

vi. If $X\sim \mathrm{BVN}(\mu,\Sigma)$ and $C=\begin{pmatrix}c_1\\ c_2\end{pmatrix}$ is a constant vector, then $C^TX=c_1X_1+c_2X_2$ is normally distributed with mean $E(C^TX)=c_1\mu_1+c_2\mu_2=C^T\mu$ and variance $Var(C^TX)=C^T\Sigma C$.
Here we only consider a single linear combination of X_1 and X_2.
Furthermore, such a fact can be extend, and used to prove normal tests, i.e., if $X_1,\ldots,X_k$ are normally distributed with mean μ and variance σ^2, then $\bar{X}=\frac{1}{k}\sum_{i=1}^k X_i$ is normally distributed with mean μ and variance $\frac{\sigma^2}{k}$.

> Common Mistake: For normally distributed r.v.s Y_1 and Y_2, $c_1Y_1+c_2Y_2$ is normally distributed.

vii. If $A\in\mathbb{R}^{2\times 2}, b\in\mathbb{R}^{2\times 1}$, then $Y=AX+b\sim\mathrm{BVN}$, with mean vector $E(Y)=AE(X)+b=A\mu+b$, and variance $Var(Y)=Cov(AX+b,AX+b)=A\Sigma A^T$.

viii. $(X-\mu)^T\Sigma^{-1}(X-\mu)\sim\chi_2^2$
We define $\chi_1^2=Z^2$, where $Z\sim\mathrm{N}(0,1)$, and $\chi_k^2=\sum_{i=1}^k Z_i^2$, where $Z_1,\ldots,Z_k$ are independent and identically distributed as $\mathrm{N}(0,1)$.

> Proof: Since Σ is symmatric, then $\Sigma=Q\Lambda Q^T$, where Q is orthogonal (i.e. $QQ^T=Q^TQ=I$), and $\Lambda=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$, where λ_1,λ_2 are eigenvalues of Σ.
> Let $\Sigma^{1/2}=Q\Lambda^{1/2}Q^T$, where $\Lambda^{1/2}=\begin{pmatrix}\sqrt{\lambda_1} & 0\\ 0 & \sqrt{\lambda_2}\end{pmatrix}$, then $\Sigma^{1/2}\Sigma^{1/2}=\Sigma$, and $\Sigma^{-1/2}=Q\Lambda^{-1/2}Q^T$, where $\Lambda^{-1/2}=\begin{pmatrix}\frac{1}{\sqrt{\lambda_1}} & 0\\ 0 & \frac{1}{\sqrt{\lambda_2}}\end{pmatrix}$.
> Now, $(X-\mu)^T\Sigma^{-1}(X-\mu)=(X-\mu)^T\Sigma^{-1/2}\Sigma^{-1/2}(X-\mu)$. Let $Z=\Sigma^{-1/2}(X-\mu)$, then Z is normally distributed with mean $E(Z)=\Sigma^{-1/2}E(X-\mu)=\Sigma^{-1/2}(\mu-\mu)=0$, and variance $Var(Z)=\Sigma^{-1/2}Var(X-\mu)\Sigma^{-1/2}=\Sigma^{-1/2}\Sigma\Sigma^{-1/2}=I$, so Z_1,Z_2 are independent and identically distributed as $\mathrm{N}(0,1)$.
> Therefore, $(X-\mu)^T\Sigma^{-1}(X-\mu)=Z^TZ=Z_1^2+Z_2^2\sim\chi_2^2$.
> A simple fact: if $X\sim\mathrm{N}(\mu,\sigma^2)$, then $\left(\frac{X-\mu}{\sigma}\right)^2\sim\chi_1^2$.
> That also means if $X_1,\ldots,X_n$ are iid $\mathrm{N}(\mu,\sigma^2)$, then $\frac{\sum_{i=1}^n (X_i-\mu)^2}{\sigma^2}\sim\chi_n^2$.

4 Functions of Random Variables

Problems we want to answer:

- Given $X_1, ..., X_n$, which are continuous r.v., and their pdf is known, we are interested in finding the distribution of $Y = h(X_1, ..., X_n)$, where h is a function.
  Three main methods to be introduced:

1. cdf technique
2. one-to-one bivariate transformation
3. mgf technique

4.1 CDF Technique

Define $Y = h(X_1, ..., X_n)$, where h is a function.
Main idea:

- Step 1: Find the cdf of Y, $F_Y(y) = P(Y \leq y)$.
- Step 2: Find the pdf of Y, $f_Y(y) = \frac{d}{dy} F_Y(y)$.

Case 1: Y is a function of one single random variable ($n = 1$), i.e. $Y = h(X)$, where the distribution of X is known.

> Example (χ_1^2): If $X \sim \mathrm{N}(0,1)$, find the distribution of $Y = X^2$.
>
> > Solution: The support of Y is $A_Y = [0, \infty)$.
> >
> > 1. $y \leq 0$, $F_Y(y) = P(Y \leq y) = 0$.
> > 2. $y > 0$, $F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx$.
> >   The for $y \rightarrow 0$, the pdf of y us $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \frac{1}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \frac{1}{\sqrt{y}}$.
> >   Therefore, $f^Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \frac{1}{\sqrt{y}} & y > 0 \\ 0 & \text{o.w.} \end{cases}$, whcih is the pdf of $\mathrm{Gamma}(\alpha = \frac{1}{2}, \beta = \frac{1}{2})$.

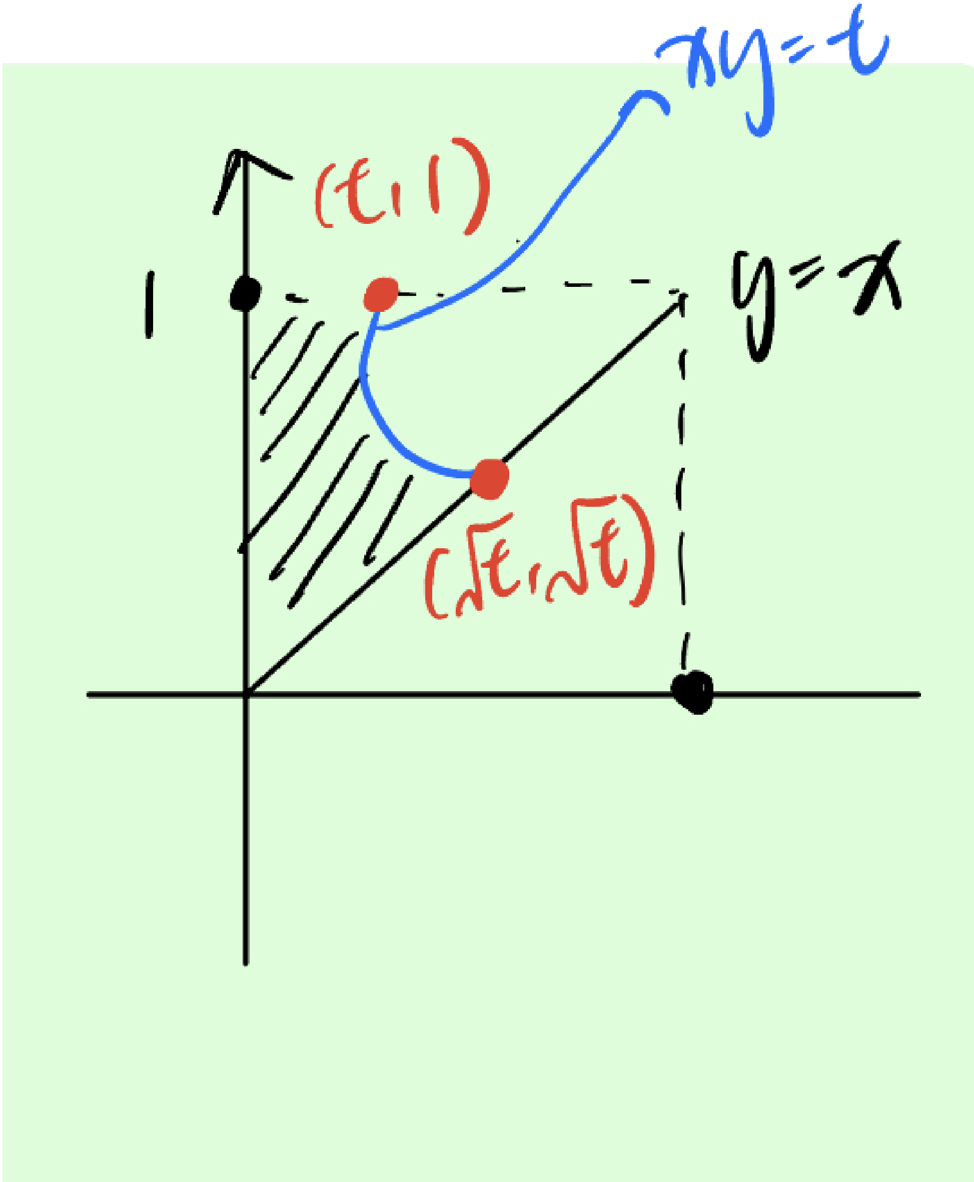
> Example 2: The pdf of X is $f(x) = \frac{\theta}{x^{\theta+1}}$ for $x \geq 1$, where $\theta > 0$ is a constant. Find the distribution of $Y = \log X (\ln X)$.
>
> > Solution: The support of Y is $A_Y = [0, \infty)$.
> >
> > 1. $y \leq 0$, $F_Y(y) = P(Y \leq y) = 0$.

> > 2. $y > 0$, $F_Y(y) = P(Y \le y) = P(\ln X \le y) = P(X \le e^y) = \int_1^{e^y} \frac{\theta}{x^{\theta+1}} dx = \left(-\frac{1}{x^\theta}\right)\Big|_1^{e^y} = 1 - e^{-\theta y}$.
> >
> > Therefore, $f_Y(y) = \begin{cases} \theta e^{-\theta y} & y \ge 0 \\ 0 & \text{o.w.} \end{cases}$, which is the pdf of $\text{Exponential}(\lambda = \theta)$.

Case 2: Y is a function of more than one random variable ($n > 1$), i.e. $Y = h(X_1, ..., X_n)$, where the distribution of $X_1, ..., X_n$ is known.

- Case 2.1: $n = 2, Y = h(X_1, X_2)$

  > Example: Joint pdf of X and Y is $f(x, y) = 3y$ if $0 \le x \le y \le 1$, and 0 otherwise. Find the distribution of $T = XY$ and $S = Y/X$.
  >
  > > Solution: The support of T is $A_T = [0, 1]$. Now we consider the cdf:
  > >
  > > i. $t \le 0$, $F_T(t) = P(T \le t) = 0$.
  > > ii. $t \ge 1$, $F_T(t) = P(T \le t) = 1$.
  > > iii. $0 < t < 1$, $F_T(t) = P(T \le t) = P(XY \le t)$.
  > > We calculate $P(T > t)$ instead.

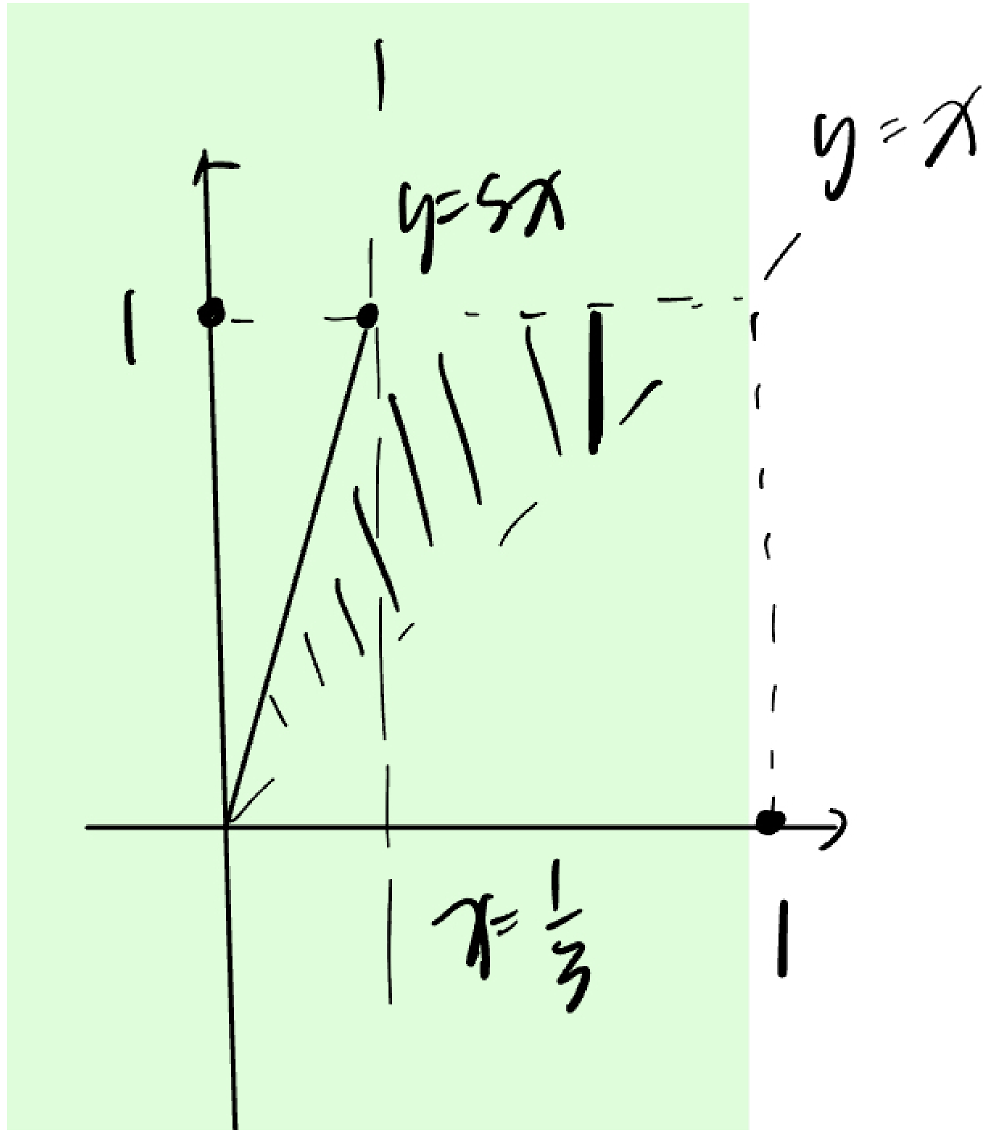


$P(T > t) = \int_{\sqrt{t}}^{1}\int_{t/y}^{y} 3y dx dy = \int_{\sqrt{t}}^{1} 3y(y - \frac{t}{y})dy = \int_{\sqrt{t}}^{1} 3y^2 - 3t dy = (1 - 3t) - (t^{3/2} - 3t^{1/2}) = 1 - 3t + 2t^{3/2}$.

$P(T \le t) = 1 - P(T > t) = 3t - 2t^{3/2}$. Therefore, the p.d.f. of T is $f_T(t) = 3 - 3t^{1/2}$ when $0 < t < 1$, and 0 otherwise.

For S, the support of S is $A_S = [1, \infty)$. Now we consider the cdf:

iv. $s \le 1$, $F_S(s) = P(S \le s) = 0$.



v. $s > 1$, $F_S(s) = P(S \leq s) = P(Y/X \leq s) = P(Y \leq sX) = \int_0^1 \int_{y/s}^{y} 3y dx dy = \int_0^1 3y(y - y/s)dy = \int_0^1 (3y^2 - 3y^2/s)dy = (y^3 - 3y^3/2s)\Big|_0^1 = 1 - 1/s.$

Hence, the pdf of S is $f_S(s) = \frac{1}{s^2}$ when $s > 1$, and 0 otherwise.

- Case 2.2: $n > 2, Y = h(X_1, ..., X_n)$

  In particular, we are interested in the distribution of order statistics. More specifically, assume $X_1, ..., X_n$ are iid r.v.s with pdf $f(x)$. Define the order statistics $Y_1 = \min\{X_1, ..., X_n\}$, denoted as $X(1)$, and $Y_n = \max\{X_1, ..., X_n\}$, denoted as $X(n)$.

  > Example (Order Statistics): Suppose $X_1, ..., X_n \overset{iid}{\sim} \text{Unif}[0, \theta]$. Find the distribution of $X(1)$ and $X(n)$.
  >
  > > Solution: For $X(n), thesupportof$X(n)isA_{X(n)} = [0,\theta]$. Now we consider the cdf:
  > >
  > > i. $x \leq 0$, $F_{X(n)}(x) = P(X(n) \leq x) = 0.$
  > > ii. $x \geq \theta$, $F_{X(n)}(x) = P(X(n) \leq x) = 1.$

> iii. $0 < x < \theta$, $F_{X(n)}(x) = P(X(n) \le x) = P(\max\{X_1, ..., X_n\} \le x) = P\left(\bigcap_{i=1}^n \{X_i \le x\}\right) = \prod_{i=1}^n P(X_i \le x) = \prod_{i=1}^n \frac{x}{\theta} = \left(\frac{x}{\theta}\right)^n$.
> Then the pdf of $X(n)$ is $f_{X(n)}(x) = \frac{nx^{n-1}}{\theta^n}$ when $0 < x < \theta$, and 0 otherwise.
> For $X(1)$, the support of $X(1)$ is $A_{X(1)} = [0, \theta]$. Now we consider the cdf:
> iv. $x \le 0$, $F_{X(1)}(x) = P(X(1) \le x) = 0$.
> v. $x \ge \theta$, $F_{X(1)}(x) = P(X(1) \le x) = 1$.
> vi. $0 < x < \theta$, $F_{X(1)}(x) = P(X(1) \le x) = P(\min\{X_1, ..., X_n\} \le x) = 1 - P(\min\{X_1, ..., X_n\} > x) = 1 - P\left(\bigcap_{i=1}^n \{X_i > x\}\right) = 1 - \prod_{i=1}^n P(X_i > x) = 1 - \prod_{i=1}^n \left(\int_x^\theta \frac{1}{\theta} dx_i\right) = 1 - \left(\frac{\theta - x}{\theta}\right)^n$.
> Then the pdf of $X(1)$ is $f_{X(1)}(x) = \frac{n(\theta - x)^{n-1}}{\theta^n}$ when $0 < x < \theta$, and 0 otherwise.

4.2 One-to-One Bivariate Transformation

Problem we are going to solve:
Given the joint pdf of (X, Y) denoted by $f(x, y)$, we want to find $U = h_1(X, Y)$ and $V = h_2(X, Y)$.

- Definition of one-to-one function: These two transfomations (h_1 and h_2) is one-to-one bivariate transformation if there exist other two functions (ω_1 and ω_2) such that $x = \omega_1(U, V)$ and $y = \omega_2(U, V)$. Note: $U = h_1(x, y)$ and $V = h_2(x, y)$.
- Notation: Jacobian of $U = h_1(x, y)$ and $V = h_2(x, y)$:

$$\frac{\partial(U, V)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

  , determinet of 2×2 matrix.
- Theorem: The p.d.f. of U and V is $f_{U,V}(u, v) = f_{X,Y}(\omega_1(u, v), \omega_2(u, v)) \left|\frac{\partial(U,V)}{\partial(x,y)}\right|$.

> Example 1: $X \sim N(0, 1)$ and $Y \sim N(0, 1)$, assume X and Y are independent. Find the joint pdf of $U = X + Y$ and $V = X - Y$.
>
> > Solution: Since $U = X + Y$ and $V = X - Y$, then support of U and V is $A_U = (-\infty, \infty)$ and $A_V = (-\infty, \infty)$.
> > then, $x = \frac{U+V}{2}$ and $y = \frac{U-V}{2}$.
> > $\frac{\partial(U,V)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$.
> > Then, the joint pdf of U and V is $g(u, v) = f(x, y) \cdot |J| = f_1(x) \cdot f_2(y) \cdot 1/2 = \frac{1}{2\pi} e^{-\frac{x^2}{2}} \cdot \frac{1}{2\pi} e^{-\frac{y^2}{2}} \cdot \frac{1}{2} = \frac{1}{4\pi} e^{-\frac{u^2+v^2}{4}}$.

> Example 2: Suppose the joint pdf of X and Y is $f(x,y) = e^{-x-y}$ for $0 < X < \infty$ and $0 < Y < \infty$, and 0 elsewhere. Find the pdf of $U = X + Y$.
>
> > Solution: Define $V = X$, then $U = X + Y$ and $V = X$, therefoer, $x = v$ and $y = u - v$, i.e., $v > 0, u - v > 0$. Therefore, $0 < v < u$ is the joint support of U and V.
> > The Jacbian of x and y with respect to u and v is $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$.
> > Therefore, the joint pdf of U and V is $g(u,v) = f(x,y) \cdot |J| = e^{-x-y} = e^{-u}$ for $0 < v < u < \infty$. The support of U is $A_U = (0, \infty)$.
> > Given $u \in (0,\infty)$, $f_U(u) = \int_{-\infty}^{\infty} g(u,v)dv = \int_0^u e^{-u}dv = ue^{-u}$.

- How to find the support of transformations?
  $\begin{cases} U = h_1(x,y) \\ V = h_2(x,y) \end{cases}$, what is the support of U and V?

> Example 1: Suppose the support of X and Y is $0 < x < y < 1$. Let $U = X$ and $V = XY$.
> Question: Find the support of U and V.
>
> > Solution: $\begin{cases} u = x \\ x = xy \end{cases} \implies \begin{cases} x = u \\ y = v/u \end{cases}$
> > Therefore, $0 < u < v/u < 1 \implies 0 < u^2 < v < u$ is the support of U and V.
> > Example 2: Suppose the support of X and Y is $0 < x < 1$ and $0 < y < 1$. Find the support of $U = X/Y$ and $V = XY$.
> > Solution: $\begin{cases} u = x/y \\ v = xy \end{cases} \implies \begin{cases} x = \sqrt{uv} \\ y = \sqrt{v/u} \end{cases}$
> > Therefore, $0 < \sqrt{uv} < 1$ and $0 < \sqrt{v/u} < 1$, which tells us $uv < 1, v/u < 1$.
> > Thus, $0 < v < u < 1/v$ is the joint support.

4.3 MGF Technique

Main idea:

1. Find the mgf of the random variable of interest.
2. By the uniqueness property of mgf, we can identify the distribution of the random variable of interest.

- Highlight one special case where the mgf technique is useful:
  Suppose $X_1, ..., X_n$ are independent and $T = \sum_{i=1}^n X_i$. Then, the mgf of T is $M_T(t) = E(e^{tT}) = E(e^{t\sum_{i=1}^n X_i}) = E(\prod_{i=1}^n e^{tX_i}) = \prod_{i=1}^n E(e^{tX_i}) = \prod_{i=1}^n M_{X_i}(t)$.

In particular, if $X_1, ..., X_n$ are iid, i.e., they have a common distribution, then having a common mgf, denoted by $M(t)$, then $M_T(t) = [M(t)]^n$.

Next, we introduce properties of some important distributions (normal, χ^2, t, F).

1. Normal Distribution
   - If $X \sim N(\mu, \sigma^2)$, then $aX + b \sim N(a\mu + b, a^2\sigma^2)$.

     > Proof: Let $Y = aX + b$, then $M_Y(t) = M_X(at)e^{bt} = e^{bt}e^{a\mu t + \frac{1}{2}a^2\sigma^2t^2} = e^{(a\mu+b)t+\frac{1}{2}a^2\sigma^2t^2}$.
     > Hence, by the uniqueness property of mgf, $Y \sim N(a\mu + b, a^2\sigma^2)$.
     > An immediate result (z-score): If $X \sim N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(0, 1)$.

   - If $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, ..., n$ and $X_1, ..., X_n$ are independent, then $\sum_{i=1}^n a_iX_i \sim N\left(\sum_{i=1}^n a_i\mu_i, \sum_{i=1}^n a_i^2\sigma_i^2\right)$.

     > Proof: Let $T = \sum_{i=1}^n a_iX_i$, then the mgf of T is $M_T(t) = \prod_{i=1}^n M_{a_iX_i}(t) = \prod_{i=1}^n M_{X_i}(a_it) = \prod_{i=1}^n e^{a_i\mu_it+\frac{1}{2}a_i^2\sigma_i^2t^2} = e^{\sum_{i=1}^n(a_i\mu_i)t+\frac{1}{2}\sum_{i=1}^n a_i^2\sigma_i^2t^2}$.
     > By the uniqueness property of mgf, $T \sim N\left(\sum_{i=1}^n a_i\mu_i, \sum_{i=1}^n a_i^2\sigma_i^2\right)$.
     > In particular, if $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, then $\bar(X) = 1/n\sum_{i=1}^n X_i \sim N(\mu, \sigma^2/n)$.
     > Hence $\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim N(0, 1)$.

2. χ^2 distribution
   - $\chi_1^2 = Z^2$, where $Z \sim N(0, 1)$.
     $\chi_k^2 = \sum_{i=1}^k Z_i^2$, where $Z_i \overset{iid}{\sim} N(0, 1)$.
     An immediate result:
     If $X \sim \mathrm{N}(\mu, \sigma^2)$, then $\left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi_1^2$.
   - If $Y_i \sim \chi_{k_i}^2$, where $k_i \in \mathbb{N}^+$ and $Y_1, ..., Y_n$ are independent.
     Then $T = \sum_{i=1}^n Y_i \sim \chi_d^2$, where $d = \sum_{i=1}^n k_i$.

     > Proof:
     >
     > - Step 1: χ_1^2 is the same as $\mathrm{Gamma}(\alpha = \frac{1}{2}, \beta = \frac{1}{2})$ (See example of $Y = Z^2$ in 4.1).
     > - Step 2: for $\chi_n^2 = \sum_{i=1}^n Z_i$, where $Z_i \overset{iid}{\sim} N(0, 1)$. Let $S = \sum_{i=1}^n Z_i^2$, then $S \sim \chi_n^2$. So the mgf of S is $M_S(t) = E(e^{tS}) = E(e^{t\sum_{i=1}^n Z_i^2}) = \prod_{i=1}^n M_{Z_i^2}(t) = [M_{Z_i^2}(t)]^n$. Since $Z_i^2 \sim \mathrm{Gamma}(\alpha = \frac{1}{2}, \beta = \frac{1}{2})$, then $M_{Z_i^2}(t) = \left(\frac{1}{1-\beta t}\right)^\alpha = \left(\frac{1}{1-2t}\right)^{\frac{1}{2}}$. Then, $M_S(t) = \left(\frac{1}{1-2t}\right)^{\frac{n}{2}}$.
     > - Step 3: Since $Y_i \sim \chi_{k_i}^2$, then mgf of Y_i is $M_{Y_i}(t) = \left(\frac{1}{1-2t}\right)^{\frac{k_i}{2}}$. Now $T = \sum_{i=1}^n Y_i$ and Y_is are independent, then $M_T(t) = \prod_{i=1}^n M_{Y_i}(t) = \prod_{i=1}^n \left(\frac{1}{1-2t}\right)^{\frac{k_i}{2}} =$

> $\left(\frac{1}{1-2t}\right)^{\frac{d}{2}}$. Therefore, by uniquesness property of mgf, $T \sim \chi^2_d$.

3. t distribution

   Definition: If $X \sim N(0,1)$ and $Y \sim \chi^2_n$, $n \in \mathbb{N}^+$, and X and Y are independent, then $\frac{X}{\sqrt{Y/n}} \sim t_n$.

   Note the support of t_n is $A_{t_n} = (-\infty, \infty)$.

   - Conclusion:

     If $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$ for $i = 1, ..., n$, let $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$, then

     a. $\bar{X}$ is independent of S^2.

     > Proof: To show this, we only need to prove $\bar{X}$ is independent of $(X_1 - \bar{X}, ..., X_n - \bar{X})^T$.
     >
     > Consider $\begin{pmatrix} \bar{X} \\ X_1 - \bar{X} \\ ... \\ X_n - \bar{X} \end{pmatrix} = A \begin{pmatrix} X_1 \\ ... \\ X_n \end{pmatrix}$, where $A \in \mathbb{R}^{(n+1)\times n}$ and first row of A is $(1/n, ..., 1/n)$.
     >
     > Here $\begin{pmatrix} X_1 \\ ... \\ X_n \end{pmatrix}$ follows MVN, the joint distribution of $\begin{pmatrix} \bar{X} \\ X_1 - \bar{X} \\ ... \\ X_n - \bar{X} \end{pmatrix}$ is also MVN.
     >
     > Hence, it suffices to prove $\bar{X}$ and $(X_1 - \bar{X}, ..., X_n - \bar{X})^T$ are uncorrelated, i.e., we need to show $Cov(\bar{X}, X_i - \bar{X}) = 0$ for $i = 1, ..., n$.
     >
     > for $i = 1, ..., n, Cov(\bar{X}, X_i - \bar{X}) = Cov(\bar{X}, X_i) - Cov(\bar{X}, \bar{X}) = \frac{1}{n}\sum_{j=1}^n Cov(X_j, X_i) - Var(X) = \sigma^2/n - \sigma^2/n = 0$. Hence, $\bar{X}$ is independent of $(X_1 - \bar{X}, ..., X_n - \bar{X})^T$, which implies $\bar{X}$ is independent of S^2.

     b. $\frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^n (X_1 - \bar{X})^2}{\sigma^2} \sim \chi^2_{n-1}$.

     > Proof: Firstly, note $\frac{\sum_{i=1}^n (X_i - \mu)^2}{\sigma^2} \sim \chi^2_n$. (let $Z_i = \frac{X-i-\bar{X}}{\sigma}$, then $\Sigma Z_i = 0$, thus $\sum_{i=1}^n (X_i - \bar{X}) = \sum_{i=1}^n X_i - \sum_{i=1}^n \bar{X} = 0$). $\frac{\sum_{i=1}^n (X_i - \mu)^2}{\sigma^2} = \frac{\sum_{i=1}^n (X_i - \bar{X} + \bar{X} - \mu)^2}{\sigma^2} = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} + 2\frac{\sum_{i=1}^n (X_i - \bar{X})(\bar{X} - \mu)}{\sigma^2} + \frac{\sum_{i=1}^n (\bar{X} - \mu)^2}{\sigma^2}$. Now, $\frac{\sum_{i=1}^n (X_i - \mu)^2}{\sigma^2}(A) = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2}(B) + \frac{n(\bar{X} - \mu)}{sigma^2}(C)$.
     >
     > Facts: $A \sim \chi^2_n = \text{Gamma}(\alpha = n/2, \beta = 2), C \sim \chi^2_1 = \text{Gamma}(\alpha = 1/2, \beta = 2)$, and B and C are independent.
     >
     > Question: $B \sim \chi^2_{n-1}$?
     >
     > We use the mgf technique. The mgf of A is $M_A(t) = \left(\frac{1}{1-2t}\right)^{n/2}$, and the mgf of C is $M_C(t) = \left(\frac{1}{1-2t}\right)^{1/2}$. In addition, $M_A(t) = M_B(t)M_C(t)$, then $M_B(t) =$

> $\left(\frac{1}{1-2t}\right)^{(n-1)/2}$, which is the mgf of χ^2_{n-1}. Thus, by the uniqueness property of mgf, $B \sim \text{Gamma}(\alpha = n, \beta = 2) = \chi^2_{n-1}$.

c. $\frac{(\bar{X}-\mu)}{S/\sqrt{n}} \sim t_{n-1}$

> Proof: Rewrite $\frac{(\bar{X}-\mu)}{S/\sqrt{n}}$ as $\frac{\sqrt{n}(\bar{X}-mu)/\sigma}{\sqrt{S^2/\sigma^2}} = \frac{\sqrt{n}(\bar{X}-\mu)/\sigma}{\sqrt{\frac{(n-1)S^2}{\sigma^2(n-1)}}} \sim t_{n-1}$.

iv. F distribution

Definition: If $X \sim \chi^2_n$ and $Y \sim \chi^2_m$, where $n, m \in \mathbb{N}^+$ and X and Y are independent, then $\frac{X/n}{Y/m} \sim F_{n,m}$.

> Question: If $X \sim \chi^2_n, Y \sim \chi^2_m$ and X and Y are independent, then $X + Y \sim \chi^2_{n+m}$. Does $\frac{X/n}{(X+Y)/(n+m)} \sim F_{n,n+m}$?
>
> > Solution: No. Consider $Cov(X, X+Y) = Cov(X, X) + Cov(X, Y) > 0$. Hence, X and $X+Y$ are not independent. Therefore, $\frac{X/n}{(X+Y)/(n+m)} \sim F_{n,n+m}$ is not true.
> >
> > If $X_1, ..., X_n \overset{iid}{\sim} N(\mu_1, \sigma_1^2), Y_1, ..., Y_m \overset{iid}{\sim} N(\mu_2, \sigma_2^2)$ are independent, let $S_1^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$ and $S_2^2 = \frac{1}{m-1}\sum_{i=1}^m (Y_i - \bar{Y})^2$, then $\frac{(n-1)S_1^2}{\sigma_1^2} \sim \chi^2_{n-1}$, $\frac{(m-1)S_2^2}{\sigma_2^2} \sim \chi^2_{n-1}$, and $\frac{\frac{(n-1)S_1^2}{\sigma_1^2}/n-1}{\frac{(m-1)S_2^2}{\sigma_2^2}/m-1} = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{n-1,m-1}$.

5 Limiting (Asymptotic) Distribution

Problem: We are interested in the distribution of $\sqrt{n}(\bar{X} - \mu)$, where $X_1, ..., X_n \overset{iid}{\sim} f(X)$ with $E(X_i) = \mu$, $Var(X_i) = \sigma^2$, $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$.

- Note f is unknown. Therefore, it's impossible to find the exact distribution of T.
  Solution: Find a approximate distribution of T.
  Roughly speaking, we find a cdf F such that when n is sufficiently large, $F(x) \approx P(\sqrt{n}(\bar{X} - \mu) \leq x)$.

5.1 Convergence in Distribution

- Definition: Let $X_1, X_2, ...$ be a sequence of r.v.s with cdf $F_1(x), F_2(x), ...$ Let X be a r.v. with cdf $F(x)$. If $\lim_{n\to\infty} F_n(x) = F(x)$ for all x at which F is continuous, then we say X_n converges in distribution to X, denoted by $X_n \xrightarrow{d} X$.
- Remark:
  i. $F(x)$ is called the limiting distribution of X_n as $n \to \infty$.

ii. Note it's the convergence of cdf, rather than X_n.

> Assume $X_1 = ... = X_N = Z \sim N(0,1)$. Take $X = -Z \sim N(0,1)$. Then $X_n \xrightarrow{d} X$.

iii. We only need to require $\lim_{n\to\infty} F_n(x) = F(x)$ holds for continuous points of F.

> e.g. If $F(x) = \begin{cases} 1 & x \geq a \\ 0 & x < a \end{cases}$ denotes the cdf of X, then $P(X = a) = 1$. Obviously the cdf F is not continuous at $x = a$. Hence, if we want to prove $X_n \xrightarrow{d} X$, where $P(X = a) = 1$, we only need to prove $\lim_{n\to\infty} F_n(x) = \begin{cases} 1 & x \geq a \\ 0 & x < a \end{cases}$, we are not interested in $\lim_{n\to\infty} F_n(a) = F(a)$.

iv. This definition applies to both discrete and continuous r.v.s

> E.g. Suppose $X_1, ..., X_n \overset{iid}{\sim} \text{Unif}[0,1]$. Let $X_{(1)} = \min_{1\leq i\leq n} X_i$ and $X_{(n)} = \max_{1\leq i\leq n} X_i$. Find the limiting distribution of
>
> i. $nX_{(1)}$ and $n(1 - X_{(n)})$.
> ii. $X_{(1)}$ and $X_{(n)}$.
>
> > Solution:
> >
> > i. The support of $nX_{(1)}$ is $[0, n]$. Now we consider the cdf:
> > For $x \leq 0$, $F_n(x) = P(nX_{(1)} \leq x) = 0$.
> > For $x \geq n$, $F_n(x) = P(nX_{(1)} \leq x) = 1$.
> > For $0 < x < n$, $F_n(x) = P(nX_{(1)} \leq x) = P(X_{(1)} \leq x/n) = 1 - P(X_{(1)} > x/n) = 1 - \prod_{i=1}^n P(X_i > x/n) = 1 - \prod_{i=1}^n (1 - x/n) = 1 - (1 - x/n)^n$.
> > Thus, $F_n(x) = \begin{cases} 0 & x \leq 0 \\ 1 - (1 - x/n)^n & 0 < x < n \\ 1 & x \geq n \end{cases}$.
> > Therefore, $\lim_{n\to\infty} F_n(x) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-x} & x > 0 \end{cases}$.
> > Thus the limiting distribution of $nX_{(1)}$ is $F(x) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-x} & x > 0 \end{cases}$.
> > The support of $n(1 - X_{(n)})$ is $[0, n]$. Now we consider the cdf:
> > For $x \leq 0$, $F_n(x) = P(n(1 - X_{(n)}) \leq x) = 0$.
> > For $x \geq n$, $F_n(x) = P(n(1 - X_{(n)}) \leq x) = 1$.
> > For $0 < x < n$, $F_n(x) = P(n(1 - X_{(n)}) \leq x) = P(1 - X_{(n)} \leq x/n) = P(X_{(n)} \geq 1 - x/n) = 1 - P(X_{(n)} < 1 - x/n) = 1 - \prod_{i=1}^n P(X_i < 1 -

$x/n) = 1 - (1 - x/n)^n$.

Thus, $F_(x) = \begin{cases} 0 & x \le 0 \\ 1 - (1 - x/n)^n & 0 < x < n \\ 1 & x \ge n \end{cases}$. (i.e., same as $nX_{(1)}$)

Thus the limiting distribution of $n(1 - X_{(n)})$ is $F(x) = \begin{cases} 0 & x \le 0 \\ 1 - e^{-x} & x > 0 \end{cases}$.

Note: This result is because if $X \sim \mathrm{Unif}[0, 1]$, then $1 - X \sim \mathrm{Unif}[0, 1]$. Then, $X_{(i)} = \min_{1 \le i \le n} X_i \overset{d}{=} \min_{1 \le i \le n}(1 - X_i) = 1 - \max_{1 \le i \le n} X_i = 1 - X_{(n)}$.

ii. The support of $X_{(1)}$ is $[0, 1]$. Now we consider the cdf:

For $x \le 0$, $F_n(x) = P(X_{(1)} \le x) = 0$.

For $x \ge 1$, $F_n(x) = P(X_{(1)} \le x) = 1$.

For $0 < x < 1$, $F_n(x) = P(X_{(1)} \le x) = 1 - P(X_{(1)} > x) = 1 - \prod_{i=1}^n P(X_i > x) = 1 - \prod_{i=1}^n (1 - x) = 1 - (1 - x)^n$.

Thus, $F_(x) = \begin{cases} 0 & x \le 0 \\ 1 - (1 - x)^n & 0 < x < 1 \\ 1 & x \ge 1 \end{cases}$.

Therefore, $\lim_{n \to \infty} F_n(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$.

Thus the limiting distribution of $X_{(1)}$ is $F(x) = \begin{cases} 0 & x < 0 \\ 1 & x \ge 0 \end{cases}$, or we can say $X_{(1)} \xrightarrow{d} 0$ (equivalently, $X_{(1)} \xrightarrow{d} X$ for $P(X = 0) = 1$).

(Since $\lim_{n \to \infty} F_n(x) = F(x)$ for any $x \ne 0$ and $F(x)$ is not continuous at $x = 0$, we do not require $\lim_{n \to \infty} F_n(0) = F(0)$.)

The support of $X_{(n)}$ is $[0, 1]$. Now we consider the cdf:

For $x \le 0$, $F_n(x) = P(X_{(n)} \le x) = 0$.

For $x \ge 1$, $F_n(x) = P(X_{(n)} \le x) = 1$.

For $0 < x < 1$, $F_n(x) = P(X_{(n)} \le x) = P(X_{(n)} < x) = \prod_{i=1}^n P(X_i < x) = \prod_{i=1}^n x = x^n$.

Thus, $F_(x) = \begin{cases} 0 & x \le 0 \\ x^n & 0 < x < 1 \\ 1 & x \ge 1 \end{cases}$.

Therefore, $\lim_{n \to \infty} F_n(x) = \begin{cases} 0 & x < 1 \\ 1 & x \ge 1 \end{cases}$.

> Thus the limiting distribution of $X_{(n)}$ is $F(x) = \begin{cases} 0 & x < 1 \\ 1 & x \geq 1 \end{cases}$, or we can say $X_{(n)} \xrightarrow{d} 1$ (equivalently, $X_{(n)} \xrightarrow{d} X$ for $P(X = 1) = 1$).

5.2 Convergence in Probability

- Definition: Let $X_1, X_2, ...$ be a sequence of r.v.s with cdf $F_1(x), F_2(x), ...$. Let X be a r.v. with cdf $F(x)$. If for any (given) $\epsilon > 0$, $\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$ or equivalently $\lim_{n\to\infty} P(|X_n - X| \leq \epsilon) = 1$, then we say X_n converges in probability to X, denoted by $X_n \xrightarrow{p} X$.
- Remark:
  i. It is the limit for a probability. That is why we call it convergence in probability.
  ii. Interpretation of $X_n \xrightarrow{p} X$: as $n \to \infty$, X_n cannot be ϵ away from X, that is, X_n is close to X as $n \to \infty$. Because of this, we expect that $F_n(x)$ becomes close to $F(x)$ if $X_n \xrightarrow{p} X$.

  > Theorem: If $X_n \xrightarrow{p} X$, then $X_n \xrightarrow{d} X$, that is to say, convergence in probability implies convergence in distribution.
  > However, the converse is not true:
  > Example: if we take $X_1 = ... = X_n = Z \sim N(0,1)$, let $X = -Z \sim N(0,1)$ then $X_n \xrightarrow{p} X$.
  >
  > Next we show $X_n \not\xrightarrow{d} X$.
  > For $\epsilon = 1$, $P(|X_n - X| > \epsilon) = P(|2Z| > \epsilon) = P(|Z| > 1/2) = 2P(Z > 1/2) > 0$ for all n.

- Convergence in probability to a constant: Let $X_1, X_2, ...$ be a sequence of r.v.s and a be a constant. If $\lim_{n\to\infty} P(|X_n - a| > \epsilon) = 0$ for any (given) $\epsilon > 0$, then we say X_n converges in probability to a, denoted by $X_n \xrightarrow{p} a$.

  > Theorem: $X_n \xrightarrow{p} a \iff X_n \xrightarrow{d} a$. We say $X_n \xrightarrow{d} a$ if $\lim_{n\to\infty} P(X_n \leq x) = \begin{cases} 0 & x < a \\ 1 & x \geq a \end{cases}$, or $X_n \xrightarrow{d} X$, where $P(X = a) = 1$.
  > Proof: Since convergence in probability implies convergence in distribution, we only need to show $F_n(x) \to F(x)$ for all x at which F is continuous.
  > We only need to show $\lim_{n\to\infty} P(|X_n - a| > \epsilon) = 0$ for any $\epsilon > 0$, if $X_n \xrightarrow{d} a$.
  >
  > i. $P(|X_n - a| > \epsilon) \geq 0$
  > ii. $P(|X_n - a| > \epsilon) = P(X_n > a + \epsilon) + P(X_n < a - \epsilon) \leq 1 - P(X_n \leq a + \epsilon) + P(X_n \leq a - \epsilon)$. Since $X_n \xrightarrow{d} a$, then $\lim_{n\to\infty} P(X_n \leq a + \epsilon) = 1$ and

> $\lim_{n\to\infty} 1 - P(X_n \le a+\epsilon) + P(X_n \le a-\epsilon) = 0$. Hence, by sqeezing theorem, $\lim_{n\to\infty} P(|X_n - a| > \epsilon) = 0$, and therefore $X_n \xrightarrow{d} a \Longrightarrow X_n \xrightarrow{p} a$.

> Example 1: $X_1, ..., X_n \overset{iid}{\sim} Unif(0,1)$. Let $X_{(1)} = \min_{1\le i\le n} X_i$ and $X_{(n)} = \max_{1\le i\le n} X_i$. Find the limiting distribution of $X_{(1)}$ and $X_{(n)}$.
>
>> Solution: We have shown that $X_{(1)} \xrightarrow{d} 0$ and $X_{(n)} \xrightarrow{d} 1$. Hence, by the theorem above, $X_{(1)} \xrightarrow{p} 0$ and $X_{(n)} \xrightarrow{p} 1$.

> Example 2: We assume $X_1, ..., X_n \overset{iid}{\sim} f(x,\theta) = e^{-(x-\theta)}$ for $x \ge \theta$ and 0 elsewhere. Define $X_{(1)} = \min_{1\le i\le n} X_i$ and $X_{(n)} = \max_{1\le i\le n} X_i$. Prove $X_{(1)} \xrightarrow{p} \theta$.
>
>> Method 1: By definition, we only need to show for any $\epsilon > 0$, $\lim_{n\to\infty} P(|X_{(1)} - \theta| > \epsilon) = 0$.
>>
>> i. $P(|X_{(1)} - \theta| > \epsilon) \ge 0$
>> ii. $P(|X_{(1)} - \theta| > \epsilon) = P(\{X_{(1)} > \epsilon + \theta\} \cup \{X_{(1)} > \theta - \epsilon\}) = P(X_{(1)} > \epsilon + \theta) + P(X_{(1)} > \theta - \epsilon) = P(X_{(1)} > \epsilon + \theta) = P(\bigcup_{i=1}^n X_i > \epsilon + \theta) = P(X_1 > \epsilon + \theta)^n = (1 - P(X_1 \le \epsilon + \theta))^n = (1 - (1 - e^{-(\epsilon+\theta)}))^n = (e^{-(\epsilon+\theta)})^n = e^{-n(\epsilon+\theta)} \to 0$ as $n \to \infty$.
>> Hence, $X_{(1)} \xrightarrow{p} \theta$.
>> Method 2: To show $X_{(1)} \xrightarrow{p} \theta$, we only need to show $X_{(1)} \xrightarrow{d} \theta$. In other words, we need to prove $\lim_{n\to 0} P(X_{(1)} \le x) = \begin{cases} 0 & x < \theta \\ 1 & x \ge \theta \end{cases}$.
>> For $x < \theta$, $P(X_{(1)} \le x) = 0$.
>> For $x \ge \theta$, $P(X_{(1)} \le x) = 1 - P(X_{(1)} > x) = 1 - P(\bigcap_{i=1}^n X_i > x) = 1 - \prod_{i=1}^n P(X_i > x) = 1 - \prod_{i=1}^n e^{-(x-\theta)} = 1 - e^{-n(x-\theta)} \to 1$ as $n \to \infty$.
>> Hence, $F_n(x) = \begin{cases} 0 & x < \theta \\ 1 - e^{-n(x-\theta)} & x \ge \theta \end{cases}$, thus $\lim_{n\to 0} P(X_{(1)} \le x) = \begin{cases} 0 & x \ge \theta \\ 1 & x > \theta \end{cases}$. Therefore, the limiting cdf is: $F(x) = \begin{cases} 0 & x < \theta \\ 1 & x \ge \theta \end{cases}$ since F is not continuous at $x = \theta$.

A brief summary:

So far we have two convergence modes:

1. Convergence in distribution: $X_n \xrightarrow{d} X$.

2. Convergence in probability: $X_n \overset{p}{\to} X$.

   Generally speaking: $X_n \overset{d}{\to} X \implies X_n \overset{p}{\to} X$, but the converse is not true (consider previous example).

   But there is one special case in which two modes are equivalent, i.e., $X_n \overset{d}{\to} X \iff X_n \overset{p}{\to} X$, for this setting, we focus on $X_{(1)} = \min_{1 \leq i \leq n} X_i$ and $X_{(n)} = \max_{1 \leq i \leq n} X_i$.

- Next, we focus on convergence in distribution in probability for $\bar{X_n} = \frac{1}{n}\sum_{i=1}^n X_i$, where we assume $X_1, ..., X_n \overset{iid}{\sim} f(x, \theta)$, where θ is unknown.
  i. Convergence in Probability: Does $\bar{X_n} \overset{p}{\to} \mu$? (Weak Law of Large Numbers)
  ii. Convergence in distribution: Does $\sqrt{n}(\bar{X_n} - \mu) \overset{d}{\to} N$? (Central Limit Theorem)
  - To prove the WLLN, Markov inequality is useful.
    Suppose X is an r.v.. For any $k > 0$ and C is a positive constant. Then we have $P(|X| \geq C) \leq \frac{E(|X|^k)}{C^k}$. (bound probability by the kth moment)
    In particular, we consider $k = 2$ and replace X with $X - \mu$ where $\mu = E(X)$, then we have $P(|X - \mu| \geq C) \leq \frac{E(|X-\mu|^2)}{C^2} = \frac{Var(X)}{C^2}$.
  - The Weak Law of Large Numbers (WLLN)
    Suppose $X_1, ..., X_n$ are independent with a common mean $\mu < \infty$ and a common variance $\sigma^2 < \infty$.
    Then $\bar{X_n} \overset{p}{\to} \mu$ where $\bar{X_n} = \frac{1}{n}\sum_{i=1}^n X_i$.

    > Proof: We only neeed to show for any $\epsilon > -$, $\lim_{n\to\infty} P(|\bar{X_n} - \mu| > \epsilon) = 0$.
    >
    > a. P(|\bar{X_n} - \mu| > \epsilon) \geq 0
    > b. By the Markov inequality we have $P(|\bar{X_n} - \mu| > \epsilon) \leq \frac{E[(\bar{X_n}-\mu)^2]}{\epsilon^2} = \frac{Var(\bar{X_n})}{\epsilon^2} = \frac{Var(\frac{1}{n}\sum_{i=1}^n X_i)}{\epsilon^2} = \frac{1}{n^2\epsilon^2}Var(\sum_{i=1}^n X_i) = \frac{1}{n^2\epsilon^2}\sum_{i=1}^n Var(X_i) = \frac{1}{n\epsilon^2}\sigma^2 \to 0$ as $n \to \infty$. By squeeze theorem, $\lim_{n\to\infty} P(|\bar{X_n} - \mu| > \epsilon) = 0$, therefore $\bar{X_n} \overset{p}{\to} \mu$.

    > Example 3: (Application of WLLN) Suppose $X_1, ..., X_n \overset{iid}{\sim} \chi_1^2$, then $\bar{X_n} \overset{p}{\to} 1$.
    >
    > > Proof:
    > >
    > > a. $X_1, ..., X_n$ are iid
    > > b. $E(X_1) = E(\chi_1^2) = E(Z^2) = Var(Z) + (E(Z))^2 \leq \infty$, where $Z \sim N(0, 1)$.
    > > c. $Var(X_1) = Var(\chi_1^2) = Var(Z^2) < \infty$.
    > >   Then, 1st way, $Var(Z^2) = E(Z^4) - (E(Z^2))^2$.
    > >   2nd way, $\chi_1^2 \overset{d}{=} Gamma(\alpha = 1/2, \beta = 2) Then,$Var(\chi_1^2) = \alpha \beta^2 = 2 < \infty.$Lastly, by the WLLN,$\bar{X_n} \overset{p}{\to} 1$.

> Example 4: Suppose $Y_n \sim \chi_n^2$. Then, $\frac{Y_n}{n} \overset{p}{\to} 1$.
>
>> Proof: Since $Y_n \sim \chi_n^2, Y_n = \sum_{i=1}^n Z_i^2, where Z_1, ..., Z_n \overset{iid}{\sim} N(0,1)$, then $Y_n/n = \frac{1}{n}\sum_{i=1}^n Z_i^2$.
>>
>> a. $Z_1^2, ..., Z_n^2$ are iid.
>> b. E(Z_1^2) = 1 < \infty$.
>> c. Var(Z_1^2) = 2 < \infty$.Then, by the WLLN,$\frac{Y_n}{n} \overset{p}{\to} 1$.

> Example 5: Suppose $X_1, ..., X_n \overset{iid}{\sim} \text{Poisson}\mu$, then $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i \overset{p}{\to} E(X_i) = \mu$.
>
>> Solution:
>>
>> a. $X_1, ..., X_n$ are iid.
>> b. $E(X_1) = \mu < \infty$.
>> c. $Var(X_1) = \mu < \infty$.
>> Then, by the WLLN, $\bar{X}_n \overset{p}{\to} \mu$.

> Practice 6: If $Y_n \sim \text{Poisson}(n), does \frac{Y_n}{n} \overset{p}{\to} \mu$?

5.3 Some Useful Limiting Theorems

In this section, we will discuss some theorems regarding the convergecne in distribution of $\bar{X}_n$ and $g(\bar{X}_n)$, wherre g is a known function.

- The Central Limit Theorem (CLT)
  Let $X_1, X_2, ...$ be iid with $E(X_i) = \mu < \infty$ and $Var(X_i) = \sigma^2 < \infty$. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Then, the limiting distribution of $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}$ is the cdf of $N(0,1)$, i.e., $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \overset{d}{\to} N(0,1)$.
  Proof of CLT relies on the following theorem:
  - Theorem: Let $X_1, X_2, ...$ be a sequence of r.v.s such that X_n has mgf $M_n(t)$ and X be a r.v. with mgf $M(t)$. If there exist some $h > 0$ such that $\lim_{h\to\infty} M_n(t) \to M(t)$ for any $|t| < h$, then $X_n \overset{d}{\to} X$.
    In other words, convergence in mgf implies convergence in distribution.

  > Proof: By the theorem above, to show $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \overset{d}{\to} N(0,1)$, we only need to show the mgf of $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}$ converges to the mgf of $N(0,1)$, which is $M(t) = e^{t^2/2}$.
  > Step 1: Find the mgf of $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}$, denoted by $M_n(t)$. Note: $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \frac{\sqrt{n}(\frac{1}{n}\sum_{i=1}^n X_i - n\mu)}{\sigma} = \frac{(\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i - n\mu)}{\sigma}$. Let $Y_i = \frac{X_i - \mu}{\sigma}$, then $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \frac{1}{\sqrt{n}}\sum_{i=1}^n Y_i$.

> Obviously $Y_1, ..., Y_n$ are iid, and $E(Y_i) = 0, Var(Y_i) = 1$. Suppose the mgf of Y_i exists and is $M_Y(t)$, then $M_n(t) = E(e^{t\frac{1}{\sqrt{n}}\sum_{i=1}^n Y_i}) = E(\prod_{i=1}^n e^{\frac{t}{\sqrt{n}}Y_i}) = \prod_{i=1}^n E(e^{\frac{t}{\sqrt{n}}Y_i}) = \prod_{i=1}^n M_Y(\frac{t}{\sqrt{n}}) = (M_Y(\frac{t}{\sqrt{n}}))^n = [M_Y(0) + M_Y'(0) \cdot t/\sqrt{n} + \frac{M_Y''(0)}{2} \cdot (t/\sqrt{n})^2 + o(1/2)]^n$. Here, we have definition of small-o notation.
> Aside: $\lim_{n\to\infty}(1 + \frac{x}{n} + o(1/n))^n = e^x$. Then, $\lim_{n\to\infty} M_n(t) = e^{t^2/2}$, which is the mgf of $N(0, 1)$.
> Step 2: Since $\lim_{n\to\infty} M_n(t) = M(t)$ for any $|t| < h$, by the theorem above, $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} N(0, 1)$.

Examples of CLT:

> Example 1: Suppose $X_1, X_2, ... \overset{iid}{\sim} \chi_1^2$. Let $Y_n = \sum_{i=1}^n X_i$. Show that $\frac{Y_n - n}{\sqrt{2n}} \xrightarrow{d} N(0, 1)$.
>
> > Solution: Let $\bar{X}_n = Y_n/n = \frac{1}{n}\sum_{i=1}^n X_i$.
> >
> > i. $X_1, X_2, ...$ are iid.
> > ii. $E(X_1) = E(\chi_1^2) = E(Z^2) = Var(Z) + (E(Z))^2 \leq \infty$, where $Z \sim N(0, 1)$
> > .
> > iii. $Var(X_1) = Var(\chi_1^2) = Var(Z^2) < \infty$.
> > By the CLT, $\frac{\sqrt{n}(\bar{X}_n - 1)}{\sqrt{2}} \xrightarrow{d} N(0, 1)$.
> > Then, $\frac{Y_n - n}{\sqrt{2n}} = \frac{\sum_{i=1}^n X_i - n}{\sqrt{2n}} = \frac{\sqrt{n}(\bar{X}_n - 1)}{\sqrt{2}} \xrightarrow{d} N(0, 1)$.

> Practice 2: Suppose $Y_n \sim \chi_n^2$. Show that $\frac{Y_n - n}{\sqrt{2n}} \xrightarrow{d} N(0, 1)$.

> Example 3: Suppose $X_1, ..., X_n \overset{iid}{\sim} \text{Poisson}(\mu)$. Let $Y_n = \sum_{i=1}^n X_i$. Find the limiting distribution of $\frac{Y_n - n\mu}{\sqrt{n\mu}}$.
>
> > Solution: Since $X_1, ..., X_n \overset{iid}{\sim} \text{Poisson}(\mu)$.
> >
> > i. $X_1, ..., X_n$ are iid.
> > ii. $E(X_i) = \mu < \infty$.
> > iii. $Var(X_i) = \mu < \infty$.
> > Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$, then by the CLT, $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sqrt{\mu}} \xrightarrow{d} N(0, 1)$.
> > Note that $\frac{Y_n - n\mu}{\sqrt{n\mu}} = \frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n\mu}} = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sqrt{\mu}} \xrightarrow{d} N(0, 1)$.

> Practice 4: Suppose $Y_n \sim \text{Poisson}(n\mu)$. Find the limiting distribution of $\frac{Y_n - n\mu}{\sqrt{n\mu}}$.

- Continuous mapping theorem:
  Suppose that g is a continuous function,

1. If $X_n \overset{d}{\to} X$, then $g(X_n) \overset{d}{\to} g(X)$.
2. If $X_n \overset{p}{\to} X$, then $g(X_n) \overset{p}{\to} g(X)$.

> Example 1: $X_n \overset{p}{\to} a \implies X_n^2 \overset{p}{\to} a^2$, if $X_n \geq 0$, and $a \geq 0$, then $X_n \overset{p}{\to} a \implies \sqrt{X_n} \overset{p}{\to} \sqrt{a}$.

> Example 2: If $X_n \overset{d}{\to} Z \sim N(0,1)$, then 2X_n \overset{d}{\to} 2Z \sim N(0,4), $X_n^2 \overset{d}{\to} Z^2 \sim \chi_1^2$.

- Slutsky's theorem:

  Suppose that $X_n \overset{d}{\to} X$ and $Y_n \overset{p}{\to} a$, where a is a constant. Then,

  1. $X_n + Y_n \overset{d}{\to} X + a$.
  2. $X_n Y_n \overset{d}{\to} aX$.
  3. $\frac{X_n}{Y_n} \overset{d}{\to} \frac{X}{a}$, if $a \neq 0$.

  > Comment: If $X_n \overset{d}{\to} X$ and $Y_n \overset{p}{\to} Y$, then $X_n + Y_n \overset{d}{\to} X + Y$ does not hold in general.
  > Example 1: Take $X_1 = X_2 = ... = Z \sim N(0,1), Y_1 = Y_2 = ... = Z \sim N(0,1)$, let $X = -Z \sim N(0,1), Y = Z \sim N(0,1)$, then $X_n \overset{d}{\to} X$ and $Y_n \overset{d}{\to} Y$, but $X_n + Y_n \overset{d}{\not\to} X + Y$, as $X + Y = 0$.

  > Example 2: If $X_n \overset{d}{\to} X \sim N(0,1)$ and $Y_n \overset{p}{\to} b \neq 0$, then $X_n + Y_n \overset{d}{\to} X + b \sim N(b,1)$, $X_n Y_n \overset{d}{\to} bX \sim N(0,b^2)$, $\frac{X_n}{Y_n} \overset{d}{\to} \frac{X}{b} \sim N(0, \frac{1}{b^2})$.

  > Example 3: Assume $X_1, X_2, ... \overset{iid}{\sim} \text{Poisson}(\mu)$. Find the limiting distribution of $U_n = \frac{\sqrt{n}(X_n - \mu)}{\sqrt{\bar{X}_n}}$ and $V_n = \sqrt{n}(X_n - \mu)$.
  >
  > > Solution: $U_n = \frac{\sqrt{n}(X_n-\mu)}{\sqrt{\bar{X}_n}} = \frac{\sqrt{n}(X_n-\mu)}{\sqrt{\mu}} \cdot \frac{\sqrt{\mu}}{\sqrt{\bar{X}_n}}$. By the CLT, $\frac{\sqrt{n}(X_n-\mu)}{\sqrt{\mu}} \overset{d}{\to} N(0,1)$. By the WLLN, $\bar{X}_n \overset{p}{\to} \mu$. Now if we take $g(x) = \frac{\sqrt{\mu}}{\sqrt{x}}$, then by the continuous mapping theorem $g(\bar{X}_n) \overset{p}{\to} g(\mu) = 1$. Lastly, by Slutsky's Theorem, $U_n = \frac{\sqrt{n}(X_n-\mu)}{\sqrt{\bar{X}_n}} = \frac{\sqrt{n}(X_n-\mu)}{\sqrt{\mu}} \cdot \frac{\sqrt{\mu}}{\sqrt{\bar{X}_n}} \overset{d}{\to} N(0,1) \cdot 1 = N(0,1)$.
  > > $V_n = \sqrt{n}(X_n - \mu) = \frac{\sqrt{n}(X_n-\mu)}{\sqrt{\mu}} \cdot \sqrt{\mu}$. By the CLT, $\frac{\sqrt{n}(X_n-\mu)}{\sqrt{\mu}} \overset{d}{\to} N(0,1)$. If we take $g(x) = \sqrt{\mu} x$, then by the continuous mapping theorem $g(\frac{\sqrt{n}(X_n-\mu)}{\sqrt{\mu}}) \overset{d}{\to} g(N(0,1)) = \sqrt{mu} Z \sim N(0, \mu^2)$.
  >
  > > Note: This proof is identical to the proof of $P(-1.96 < \frac{\sqrt{n}(\bar{X}_n-\theta)}{\sqrt{\bar{X}_n(1-\bar{X}_n)}} < 1.96) \approx 0.95$ as $n \to \infty$. (Confidence interval for θ)

> Example 4: Assume $X_1, X_2, ... \overset{iid}{\sim} \mathrm{Unif}[0,1]$. Let $Y_n = \max_{1\le i\le n} X_i$ for $n \ge 1$. Find the limiting distribution of 1) e^{Y_n}, 2) $\sin(1-Y_n)$, 3) $e^{-n(1-Y_n)}$, 4) $(Y_n-1)^2[n(1-Y_n)]$.
>
>> Solution: $Y_n \xrightarrow{d} 1$, then by continuoud mapping theorem, 1) $e^{Y_n} \xrightarrow{d} e^1 = e$, 2) $\sin(1-Y_n) \xrightarrow{d} \sin(1-1) = 0$.
>> 3) $n(1-Y_n) \xrightarrow{d} X \sim \exp(1)$, then by the continuous mapping theorem, $e^{-n(1-Y_n)} \xrightarrow{d} e^{-X}$. We let $Y = e^{-X}$, then the support of Y is $(0,1)$ for $y \le 0$, $F_Y(y) = P(Y \le y) = 0$; for $y \ge 1$, $F_Y(y) = P(Y \le y) = 1$; for $0 < y < 1$, $F_Y(y) = P(Y \le y) = P(e^{-X} \le y) = P(-X \le \ln y) = P(X \ge -\ln y) = 1 - P(X \le -\ln y) = 1 - F_X(-\ln y) = 1 - (1 - e^{-\ln y}) = y$. Therefore, $F_Y(y) = \begin{cases} 0 & y \le 0 \\ y & 0 < y < 1 \\ 1 & y \ge 1 \end{cases}$, thus $Y \sim \mathrm{Unif}[0,1]$.
>> 4) Take $g(x) = (1+x)^2$, by continuous mapping theorem, $(Y_n+1)^2 \xrightarrow{p} 4$. Since $n(1-Y_n) \xrightarrow{d} X \sim \exp(1)$, By Slutsky's theorem, $(Y_n-1)^2[n(1-Y_n)] \xrightarrow{d} 4X$, where $X \sim \exp(1)$. Let $Y = 4X$, the support of Y is $(0,\infty)$, for $y \le 0$, $F_Y(y) = P(Y \le y) = 0$; for $y > 0$, $F_Y(y) = P(Y \le y) = P(4X \le y) = P(X \le \frac{y}{4}) = F_X(\frac{y}{4}) = 1 - e^{-\frac{y}{4}}$. Therefore, $F_Y(y) = \begin{cases} 0 & y \le 0 \\ 1 - e^{-\frac{y}{4}} & 0 < y < 1 \\ 1 & y \ge 1 \end{cases}$, thus $Y \sim \exp(4)$.

- Delta method:
  Question: We want to find the limiting distribution of $\sqrt{n}[g(\bar{X}_n) - g(\mu)]$, where g is a differentiable function. $X_1, ..., X_N \overset{iid}{\sim} \exp(\lambda)$, $f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & x < 0 \end{cases}$. The MLE for λ is $\hat{\lambda} = \frac{1}{\bar{X}}$. How to establish $\sqrt{n}(\hat{\lambda} - \lambda) \xrightarrow{d} N(0,1)$?
  Delta method: Suppose that $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0,\sigma^2)$, and g is a differentiable at $x = \mu$, $g'(\mu) \ne 0$, then $\sqrt{n}[g(\bar{X}_n) - g(\mu)] \xrightarrow{d} N(0, \sigma^2[g'(\mu)]^2)$.
  How to prove this result?
  We consider the first-order Taylor expansion of $g(x)$ around $x = \mu$, i.e., $g(\bar{X}_n) = g(\mu) + g'(\mu)(\bar{X}_n - \mu) + R_n$, where R_n is the remainder term and is ignigible, then $\sqrt{n}[g(\bar{X}_n) - g(\mu)] \approx \sqrt{n} g'(\mu)(\bar{X}_n - \mu)$. By continuous mapping theorem, let $h(x) = g'(\mu)x$, then $\sqrt{n} g'(\mu)(\bar{X}_n - \mu) \xrightarrow{d} g'(\mu)Z$, where $Z \sim N(0,\sigma^2)$. Therefore, $\sqrt{n}[g(\bar{X}_n) - g(\mu)] \xrightarrow{d} N(0, \sigma^2[g'(\mu)]^2)$.

> Example 1: Suppose $X_1, ..., X_n \overset{iid}{\sim} f(x) = \begin{cases} 0 & x \le 0 \\ \frac{1}{\theta} e^{-\frac{x}{\theta}} & x > 0 \end{cases}$. Find the limiting distribution of 1) X_n, 2) $Z_n = \frac{\sqrt{n}(X_n - \theta)}{\bar{X}_n}$, 3) $U_n = \sqrt{n}(\bar{X}_n - \theta)$, 4) $V_n = \sqrt{n}(\ln \bar{X}_n - \ln \theta)$.
>
>> Solution:
>>
>> i. $X_1, ..., X_n$ are iid; $E(X_1) = \theta < \infty$; $Var(X_1) = \theta^2 < \infty$; By the CLT, $\bar{X}_n \xrightarrow{p} \theta$.
>> ii. $Z_n = \frac{\sqrt{n}(X_n - \theta)}{\theta} \cdot \frac{\theta}{\bar{X}_n}$. By the CLT, \frac{\sqrt{n} (\bar{X_n}-\theta)}{\theta} \overset{d}{\to} N(0,1). By the WLLN, $\bar{X}_n \xrightarrow{p} \theta$. By the continuous mapping theorem, $\frac{\theta}{\bar{X}_n} \xrightarrow{p} \frac{\theta}{\theta} = 1$. By Slutsky's theorem, $Z_n = \frac{\sqrt{n}(X_n - \theta)}{\theta} \cdot \frac{\theta}{\bar{X}_n} \xrightarrow{d} N(0,1) \cdot 1 = N(0,1)$.
>> iii. $U_n = \sqrt{n}(\bar{X}_n - \theta)$. By the CLT, $\sqrt{n}(\bar{X}_n - \theta) \xrightarrow{d} N(0, \theta^2)$. By the continuous mapping theorem, $U_n = \sqrt{n}(\bar{X}_n - \theta) \xrightarrow{d} N(0, \theta^2)$.
>> iv. $V_n = \sqrt{n}(\ln \bar{X}_n - \ln \theta) = \sqrt{n}(\ln \frac{\bar{X}_n}{\theta}) = \sqrt{n}(\ln(1 + \frac{\bar{X}_n - \theta}{\theta}))$. By the Taylor expansion, $\ln(1 + \frac{\bar{X}_n - \theta}{\theta}) \approx \frac{\bar{X}_n - \theta}{\theta}$. By the CLT, $\sqrt{n}(\bar{X}_n - \theta) \xrightarrow{d} N(0, \theta^2)$. By the continuous mapping theorem, $\frac{\bar{X}_n - \theta}{\theta} \xrightarrow{d} \frac{N(0, \theta^2)}{\theta} = N(0, \frac{\theta^2}{\theta^2}) = N(0,1)$. By Slutsky's theorem, $V_n = \sqrt{n}(\ln \bar{X}_n - \ln \theta) = \sqrt{n}(\ln \frac{\bar{X}_n}{\theta}) = \sqrt{n}(\ln(1 + \frac{\bar{X}_n - \theta}{\theta})) \xrightarrow{d} N(0,1) \cdot 1 = N(0,1)$.

> Example 2: Suppose $X_1, ..., X_n \overset{iid}{\sim} \text{Poisson}(\mu)$. Find the limiting distribution of \$Z_n = \frac{\sqrt{n}(\sqrt{X_n} - \sqrt{\mu})}.
>
>> Solution: We have shown that $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \mu)$. If we take $g(x) = \sqrt{x}$, then $g'(\mu) = \frac{1}{2\sqrt{\mu}} \neq 0$. By the Delta method, $\sqrt{n}(\sqrt{\bar{X}_n} - \sqrt{\mu}) = \sqrt{n}(g(\bar{X}_n - g(\mu))) \xrightarrow{d} N(0, \mu(g'(\mu))^2) = N(0, \frac{1}{4})$.

6 Point Estimation

6.1 Backgroud and Notation

Suppose $X_1, ..., X_n$ are iid (is random sample) from $f(x; \theta)$. Here $f(x; \theta)$ is a p.f. for discrete r.v. or p.d.f. for continuous r.v. and θ is unknown and consist of finite number of parameters, i.e. $\theta = (\theta_1, ..., \theta_k)^T$, θ can be a scalar ($k = 1$) or a vector ($k > 1$).
\> For example:

> 1. $X_1, ..., X_n \overset{iid}{\sim} \text{Poisson}(\mu)$, then $\theta = \mu$ is a scalar ($k = 1$).
> 2. $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, then $\theta = (\mu, \sigma^2)^T$ is a vector ($k > 1$).

Some useful notations:
- Θ: parameter space. It consists of all possible values θ can take.
1. $X_1, ..., X_n \overset{iid}{\sim} \text{Poisson}(\mu)$, then $\Theta = \{\mu | \mu > 0\}$.
2. $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, then $\Theta = \{(\mu, \sigma^2) | \mu \in \mathbb{R}, \sigma^2 > 0\}$.
- Data: $X_1, ..., X_n \overset{iid}{\sim} f(x; \theta)$, are random variables
- Observation: $x_1, ..., x_n$ are observed values of $X_1, ..., X_n$, they are not random.
- Statistic: a function of data, and cannot depend on θ.
> e.g. $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ is a statistic, but $\sqrt{n}(\bar{X}_n - \mu)$ is not a statistic if μ is unknown.
- Estimator: If a statistic $T = T(X_1, ..., X_n)$ is used to estimate θ, then T is an estimator of θ, it is a random variable.
- Estimate: An observed value of $T = T(X_1, ..., X_n)$, also known as realization of T, denoted by $t = T(x_1, ..., x_n)$, it is not random.
> e.g. $\bar{X}_n$ is an estimator of μ if $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. If we observe $x_1, ..., x_n$, then $\bar{x_n} = \frac{1}{n}\sum_{i=1}^n x_i$ is an estimate of μ.
> Remark: In statistics, we use $\hat{\theta} = \hat{\theta}(X_1, ..., X_n)$ to denote an estimator of θ. If $\hat{\theta}$ is an observed value, not a r.v., then $\hat{\theta} = \hat{\theta}(x_1, ..., x_n)$ to denote an estimate of θ.

6.2 Method of Moments

Problem Setup: Suppose $X_1, ..., X_n \overset{iid}{\sim} f(x; \theta)$, we want to estimate $\theta = (\theta_1, ..., \theta_k)^T$.

1. Let $\mu_j = E(X_i^j), j = 1, ..., k$ denote the jth moment of X_i. (population moment)
   i. μ_j is called the jth population moment.
   ii. $\mu_j = \mu_j(\theta)$, i.e. μ_j is a function of θ.
   > e.g. $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Here, $\theta = (\mu, \sigma^2)^T$.
   > Then, $\mu_1 = E(X_1) = \mu = \mu_1(\theta)$, $\mu_2 = E(X_1^2) = \sigma^2 + \mu^2 = \mu_2(\theta)$.

2. Let $\hat{\mu_j} = \frac{1}{n}\sum_{i=1}^n X_i^j, j = 1, ..., k$ denote the jth sample moment of X_i. (sample moment)
3. Idea of method of moment estimator (MM estimator)
   Find estimator of θ, denoted by $\hat{\theta}$, such that $\hat{\mu_j} = \mu_j(\hat{\theta})$. Recall $\theta \overset{\mu_j}{\to} \mu_j(\theta) = \mu_j$. Intuitively speaking,
   > i. $X_1 \sim \text{Poisson}(\theta)$, $\mu_1 = E(X_1) = \theta = \mu_1(\theta)$. Then the MM estimator satisfies $\mu_1(\hat{\theta}) = \hat{\theta} = \frac{1}{n}\sum_{i=1}^n X_i$.
   > ii. $X_i \sim \text{Unif}[0, \theta]$. Then, $\mu_1 = E(X_1) = \frac{\theta}{2} = g_1(\theta)$. So the MM estimator satisfies $g_1(\hat{\theta}) = \hat{\theta}/2 = \frac{1}{n}\sum_{i=1}^n X_i$, then the MM estimator of θ is $\hat{\theta} = 2\bar{X}_n$, where $\bar{X}_n =$

> $\frac{1}{n}\sum_{i=1}^n X_i$.
>
> iii. $X_i \sim f(x;\theta) = \begin{cases} \theta x^{\theta-1} & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$. ...... $\hat{\theta} = \frac{\bar{X}}{1-\bar{X}}, where \bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$.
>
> iv. Two parameter case: $X_i \sim N(\mu,\sigma^2)$. Hence $\theta = (\mu,\sigma^2)^T$. Then, $\mu_1 = E(X_1) = \mu = g_1(\theta)$, $\mu_2 = E(X_1^2) = \sigma^2+\mu^2 = g_2(\theta)$. Then, the MM estimator of θ satisfies $g_1(\hat{\theta}) = \hat{\mu} = \bar{X}_n$, $g_2(\hat{\theta}) = (\hat{\mu})^2 + \hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n X_i^2$. Then, $\hat{\mu} = \bar{X}_n$, $\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n X_i^2 - (\bar{X}_n)^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)^2$. Note $E(\hat{\sigma^2}) \neq \sigma^2$, so $\hat{\sigma^2}$ is a biased estimator of σ^2.

6.3 Maximum Likelihood Method

In this section we will introduce the most commonly used method for estimating unknown parameters.

- Likelihood function
  i. Suppose that $X_1, \ldots, X_n$ are iid from $f(x;\theta)$, a p.f. if X_i is discrete, or a p.d.f. if X_i is continuous.
  ii. Given $x_1, \ldots, x_n$, which denote the observed values of $X_1, \ldots, X_n$, we calculate the joint p.f. or p.d.f. of $(X_1, \ldots, X_n)$ wrt observed values $(x_1, \ldots, x_n)$
     a. Discrete case: joint p.f. of $X_1, \ldots, X_n$ wrt $x_1, \ldots, x_n$ is $P(X_1 = x_1, \ldots, X_n = x_n) = \prod_{i=1}^n P(X_i = x_i) = \prod_{i=1}^n f(x_i;\theta)$.
     b. Continuous case: joint p.d.f. of $X_1, \ldots, X_n$ wrt $x_1, \ldots, x_n$ is $f(x_1, \ldots, x_n;\theta) = \prod_{i=1}^n f(x_i;\theta)$.
  iii. We use $L(\theta; x_1, \ldots, x_n)$ or $L(\theta)$ to denote the joint p.f. or pdf evaluated at $x_1, \ldots, x_n$. That is to say, $L(\theta; x_1, \ldots, x_n) = \begin{cases} P(X_1 = x_1, \ldots, X_n = x_n) & \text{discrete case} \\ f_{X_1,\ldots,X_n}(x_1, \ldots, x_n;\theta) & \text{continuous case} \end{cases} = \prod_{i=1}^n f(x_i;\theta)$. Here, $L(\theta; x_1, \ldots, x_n)$ is called the likelihood of θ.

  > Comments:
  >
  > i. Likelihood function measures how likely we get the observed data $x_1, \ldots, x_n$ for a given θ.
  > ii. Smaller $L(\theta)$ indicates it is less likely for such θ to generate the observed data $x_1, \ldots, x_n$.
  > iii. Larger $L(\theta)$ indicates it is more likely for such θ to generate the observed data $x_1, \ldots, x_n$.

  - Idea of Maximum Likelihood Method:
    Choose θ to maximize $L(\theta)$. In other words, we choose θ such that it is most lilely to generate the observed data $x_1, \ldots, x_n$.

- Maximum Likelihood Estimator (MLE)
  i. MLE maximizes $L(\theta)$, i.e., if we use $\hat{\theta} = \hat{\theta}(x_1, ..., x_n)$ to denote the ML estimate, then $\hat{\theta} = \arg\max_{\theta \in \Theta} L(\theta; x_1, ..., x_n)$.
  ii. ML estimator: $\hat{\theta} = \hat{\theta}(X_1, ..., X_n)$.
  iii. log-likelihood function: $l(\theta) = \ln L(\theta)$. Then the ML estimator satisfies $\hat{\theta} = \arg\max_{\theta \in \Theta} L(\theta) = \arg\max_{\theta \in \Theta} l(\theta)$.
  iv. Invariance property of ML estimator: Let $\eta = g(\theta)$, i.e., η is a function of θ. Then, the MLE of η is given by $\hat{\eta} = g(\hat{\theta})$, where θ denotes the MLE of θ.

> Examples of MLE
>
> Suppose $X_1, ..., X_n \overset{iid}{\sim}$
>
> i. $\text{Poisson}(\theta)$
> ii. $\text{Unif}[0, \theta]$
> iii. $f(x; \theta) = \begin{cases} \theta x^{\theta - 1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$
> iv. $N(\mu, \sigma^2)$
>
> > Solution:
> >
> > i. Likelihood function for $x_1, ..., x_n$ is $L(\theta; x_1, ..., x_n) = \prod_{i=1}^n f(x_i; \theta) = \prod_{i=1}^n \frac{\theta^{x_i}}{x_i!} e^{-\theta}$, $x_i = 0, 1, 2, ...$ Then, the log-likelihood is, $l(\theta) = \ln L(\theta) = \sum_{i=1}^n \ln f(x_i; \theta) = \sum_{i=1}^n \ln \frac{\theta^{x_i}}{x_i!} e^{-\theta} = \sum_{i=1}^n \ln \theta^{x_i} - \sum_{i=1}^n \ln x_i! - n\theta$. Then the ML estimate of θ satisfies $\frac{dl(\theta)}{d\theta} = \sum_{i=1}^n \frac{x_i}{\theta} - n = 0$, then $\hat{\theta} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x_n}$. Therefore, the MLE of θ is $\hat{\theta} = \bar{X}_n$.
> > ii. Likelihood function for $x_1, ..., x_n$ is $L(\theta; x_1, ..., x_n) = 1/\theta^n \prod_{i=1}^n 1(0 \leq x_i \leq \theta) = 1/\theta^n 1(x_{(1)} \geq 0) 1(x_{(n)} \leq \theta)$, where $x_{(1)} = \min\{x_1, ..., x_n\}$, $x_{(n)} = \max\{x_1, ..., x_n\}$. Then, when $\theta < x_{(n)}$, $L(\theta) = 0$, when $\theta \geq x_{(n)}$, $L(\theta)$ is a monotone decreasing function of θ. Therefore, the ML estimate of θ is $\hat{\theta} = x_{(n)}$. Therefore, the MLE of θ is $\hat{\theta} = x_{(n)} = \max\{X_1, ..., X_n\}$.
> > iii. Likelihood function for $x_1, ..., x_n$ is $L(\theta; x_1, ..., x_n) = \prod_{i=1}^n f(x_i; \theta) = \prod_{i=1}^n \theta x_i^{\theta - 1} = \theta^n (\prod_{i=1}^n x_i)^{\theta - 1}$. Then, the log-likelihood is, $l(\theta) = \ln L(\theta) = \ln \theta^n + (\theta - 1) \sum_{i=1}^n \ln x_i$. Then the ML estimate of θ satisfies $\frac{dl(\theta)}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^n \ln x_i = 0$, then $\hat{\theta} = -\frac{n}{\sum_{i=1}^n \ln x_i}$. Therefore, the MLE of θ is $\hat{\theta} = -\frac{n}{\sum_{i=1}^n \ln X_i}$.
> > iv. Likelihood function for $x_1, ..., x_n$ is $L(\mu, \sigma^2; x_1, ..., x_n) = \prod_{i=1}^n f(x_i; \mu, \sigma^2) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i - \mu)^2}{2\sigma^2}}$. Then, the log-likelihood is, $l(\mu, \sigma^2) = \ln L(\mu, \sigma^2) = \sum_{i=1}^n \ln f(x_i; \mu, \sigma^2) = \sum_{i=1}^n \ln \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i - \mu)^2}{2\sigma^2}} = -\frac{n}{2} \ln 2\pi - \frac{n}{2} \ln \sigma^2 -$

> > $\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2$. Then the ML estimate of μ satisfies $\frac{\partial l(\mu,\sigma^2)}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^n (x_i-\mu) = 0$, then $\hat{\mu} = \frac{1}{n}\sum_{i=1}^n x_i = \bar{x_n}$ and the ML estimate of σ^2 satisfies $\frac{\partial l}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^n (x_i-\mu)^2 = 0$, then $\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n (x_i-\mu)^2 = \frac{1}{n}\sum_{i=1}^n (x_i-\bar{x_n})^2$. Therefore, the MLE of μ is $\hat{\mu} = \bar{X}_n$ and the MLE of σ^2 is $\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n (X_i-\bar{X}_n)^2$.

6.4 Properties of ML Estimator

In this section:

1. We consider θ is a scalar, i.e., $k=1$.
2. We consider the ML estimator (a r.v.)
3. The support of $X_1, ..., X_n$ does not depend on θ.

   For example, if $X_1, ..., X_n \overset{iid}{\sim} \text{Unif}[0,\theta]$, then the support of $X_1, ..., X_n$ depends on θ. Then the theories developed in this section do not apply.

We define some notation first:

1. Score function, dentoed as $S(\theta)$: $S(\theta) = \frac{dl(\theta)}{d\theta} = \frac{d\ln L(\theta;x_1,...,x_n)}{d\theta}$. Typically, the MLE $\hat{\theta}$ satisfies $S(\hat{\theta})=0$, when the support of $X_1, ..., X_n$ does not depend on θ.
2. Information function, denoted as $I(\theta)$: $I(\theta) = -\frac{d^2 l(\theta)}{d\theta^2} = -\frac{d^2\ln L(\theta;x_1,...,x_n)}{d\theta^2}$.
3. Fisher information, denoted as $J(\theta)$: $J(\theta) = E(I(\theta)) = E(-\frac{d^2 l(\theta)}{d\theta^2}) = E(-\frac{d^2\ln L(\theta;x_1,...,x_n)}{d\theta^2})$
   .

> Example: If $X_1, ..., X_n \overset{iid}{\sim} f(x;\theta)$. Then $L(\theta;x_1,...,x_n) = \prod_{i=1}^n f(x_i;\theta)$, $l(\theta) = \ln L(\theta) = \sum_{i=1}^n \ln f(x_i;\theta)$. Then, $S(\theta) = \frac{dl(\theta)}{d\theta} = \sum_{i=1}^n \frac{d\ln f(x_i;\theta)}{d\theta}$, $I(\theta) = -\frac{d^2 l(\theta)}{d\theta^2} = -\sum_{i=1}^n \frac{d^2\ln f(x_i;\theta)}{d\theta^2}$, $J(\theta) = -E(\frac{dl(\theta)}{d\theta})^2 = -E(\sum_{i=1}^n \frac{d\ln f(x_i;\theta)}{d\theta})^2 = \sum_{i=1}^n -E(\frac{d\ln f(x_i;\theta)}{d\theta})^2$. Let $J_1(\theta) = -E(\frac{dl(\theta)}{d\theta})^2 = -E(\frac{d\ln f(x_1;\theta)}{d\theta})^2$, then $J(\theta) = nJ_1(\theta)$.

> Example: If $X_1, ..., X_n \overset{iid}{\sim} \text{Poisson}(\theta)$. Likelihood function: $L(\theta;x_1,...,x_n) = \prod_{i=1}^n f(x_i;\theta) = \prod_{i=1}^n \frac{\theta^{x_i}}{x_i!}e^{-\theta}$, log-likelihoodfunction: $l(\theta) = \ln L(\theta) = \sum_{i=1}^n \ln f(x_i;\theta) = \sum_{i=1}^n \ln\frac{\theta^{x_i}}{x_i!}e^{-\theta} = \sum_{i=1}^n \ln\theta^{x_i} - \sum_{i=1}^n \ln x_i! - n\theta$. Then, Score function $S(\theta) = \frac{dl(\theta)}{d\theta} = \sum_{i=1}^n \frac{x_i}{\theta} - n$, Information function $I(\theta) = -\frac{dS}{d\theta} = -\sum_{i=1}^n \frac{x_i}{\theta^2}$, Fisher information $J(\theta) = E(I(\theta)) = E(-\frac{dS}{d\theta}) = E(\sum_{i=1}^n \frac{x_i}{\theta^2}) = \sum_{i=1}^n E(\frac{x_i}{\theta^2}) = \sum_{i=1}^n \frac{1}{\theta^2}E(X_i) = \frac{n}{\theta^2}E(X_1) = \frac{n}{\theta^2}\theta = \frac{n}{\theta}$.