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2 Univariate Random Variables

2.1 Introduction to probability model

- **Probability model** is used to describe a random exprienment.
  It consists of three important components:
  i. **Sample space** S: a collection of all possible outcomes of one random experiment.
    > e.g. Toss a coin: $S = \{H, T\}$

    > e.g. Toss a coin twice: $S = \{(H, H), (H, T), (T, H), (T, T)\}$

  ii. **Event**: denoted by A, B, C, etc. It is a subset pf sample space.
    > e.g. Toss a coin twice:
    > Define A as 1st toss is tail, $A = \{(T, T), (T, H)\} \subseteq S$

  iii. **Probability function** P: It is a function of events.
    It satisfies properties (axioms):
    a. $0 \leq P(A) \leq 1$ for any event A.
    b. $P(S) = 1$
    c. Countable additivity: If $A_1, A_2, \ldots$ are assumed to be pairwise multually exclusive events (i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$), $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$.

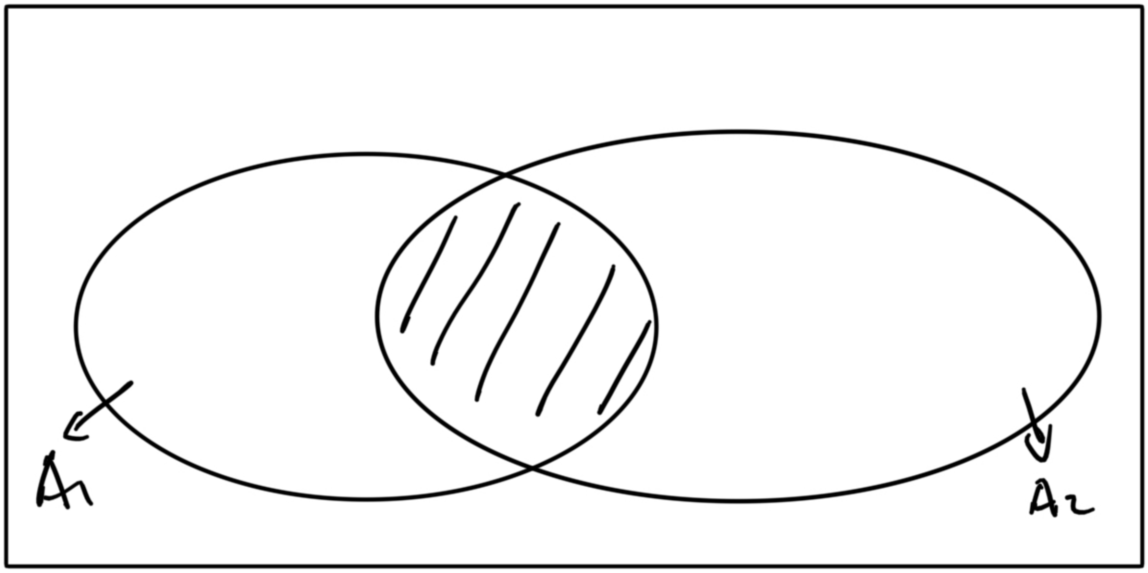
    We can now prove the following properties:
    a. $P(\emptyset) = 0$.
      > Proof: Let $A_i = \emptyset$ for $i \geq 1$, $A_i \cap A_j = \emptyset$ for $i \neq j$, by axioms we have $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$, or in other words, $P(\emptyset) = \sum_{i=1}^{\infty} P(\emptyset)$. Additionally, $0 \leq P(\emptyset) \leq 1$, therefore, $P(\emptyset) = 0$.

    b. Let A denote an event. Let $\bar{A}$ denote the complementary event of A, which means $\bar{A}$ saitifies two conditions:
      a. $\bar{A} \cap A = \emptyset$, and
      b. $\bar{A} \cup A = S$.

      Prove $P(A) + P(\bar{A}) = 1$:
      > Proof: Define $A_1 = A$, $A_2 = \bar{A}$, $A_i = \emptyset$ for $i \geq 3$, so $A_i \cap A_j = \emptyset$ for $i \neq j$, by axioms we have $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$, in other words, $P(S) = P(A) + P(\bar{A}) + \sum_{i=3}^{\infty} 0$, therefore, $P(A) + P(\bar{A}) = 1$.

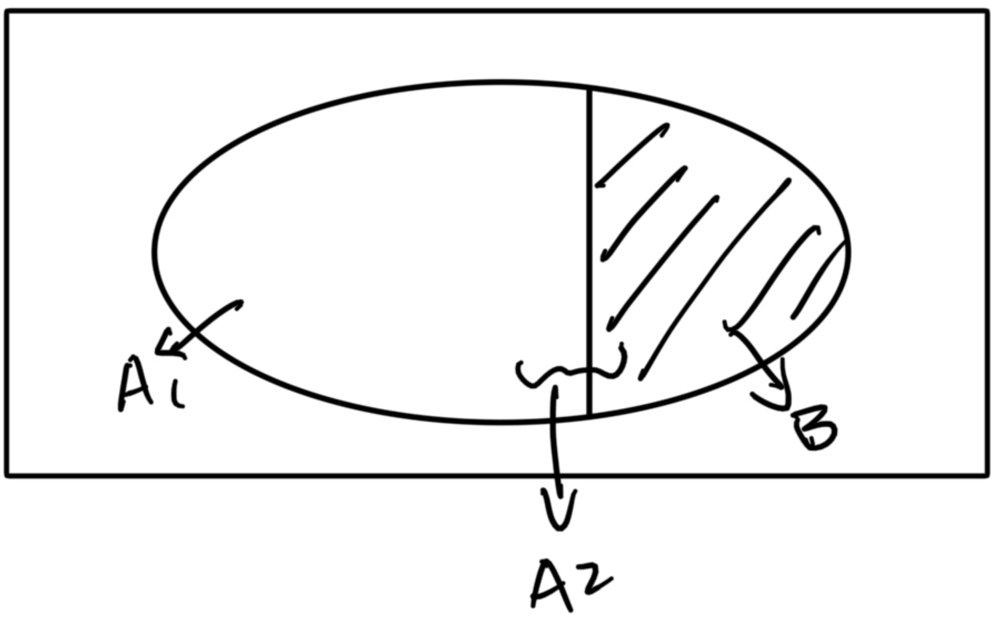
    c. If A_1 and A_2 are mutually exclusive, then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$.
      > Proof: Define $A_i = \emptyset$ for $i \geq 3$, so $S = A_i \cap A_j = \emptyset$, for $i \neq j$. Then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$, or in other words, $P(A_1 \cup A_2) = P(A_1) + P(A_2) + 0$.

    d. In general, $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$.



Proof: Define $B = \{\omega | \omega \in A_1, \omega \notin A_2\}$, since $A_1 = B \cup (A_1 \cap A_2)$, we can get $B \cap (A_1 \cap A_2) = \emptyset$, $B \cup (A_1 \cap A_2) = A_1$, $B \cap (A_1 \cap A_2) = \emptyset$, $B \cap A_2 = \emptyset$, and therefore $B \cup A_2 = A_1 \cup A_2$.
Then $P(A_1 \cup A_2) = P(B \cup A_2) = P(B) + P(A_2)$. Note $P(A_1 \cup A_2) = P(A_2) + P(B)$ and $P(B) = P(A_1) - P(A_1 \cap A_2)$. Hence, $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$.

e. If $A_1 \subseteq A_2$, then $P(A_1) \leq P(A_2)$



Proof: $A_2 \setminus A_1 := B = \{\omega | \omega \in A_2, \omega \notin A_1\}$, we have $B \cap A_1 = \emptyset$, $B \cup A_1 = A_2$. Then $P(A_2) = P(A_1 \cup B) = P(A_1) + P(B) \geq P(A_1)$.

> e.g. Toss a coin twice
> Then $S = \{(H,H), (H,T), (T,H), (T,T)\}$ for any event A,
>
> $$P(A) := \frac{\#\text{ of elements in } A}{4}$$
>
> Verify P is a probability function.

- **Conditional probability**
  Suppose A and B denote two events. Provided $P(B) > 0$, then the conditional probability of A given B is

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

- **Independence of two events**
  Suppose A and B denotes two events. We say A and B are independent if and only if

$$P(A\cap B) = P(A)P(B)$$

  - Proposition: If A and B are independent, then $P(A|B) = P(A)$ (We assume $P(B) > 0$)
    > Proof: $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$

  > e.g. Toss a coin twice
  > $A := \text{1st toss is a head} = \{(H,T),(H,H)\}$
  > $B := \text{2nd toss is a head} = \{(T,H),(H,H)\}$
  > For any event C, $P(C) = \frac{\#\text{ of elements in } C}{4}$
  > Verify A and B are independent.
  > $P(A\cap B) = P(A)P(B)$?
  > By definition, $A\cap B = \{(H,H)\} \implies P(A\cap B) = \frac{1}{4}$
  > $P(A) = \frac{2}{4}$, $P(B) = \frac{2}{4}$.
  > Hence, $P(A\cap B) = P(A)P(B)$.

- **Random variable (r.v.)** X, Y, ζ, η
  Random variable is a function from sample space to real line.

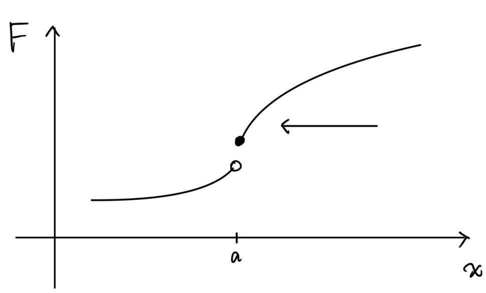
$$X : S \to \mathbb{R}$$

  Specifically, given any $\omega \in S$, $X(\omega) \in \mathbb{R}$.
  This function satisfies that for any $x \in \mathbb{R}$, $\{X \le x\} = \{\omega | X(\omega) \le x\}$ is an event.

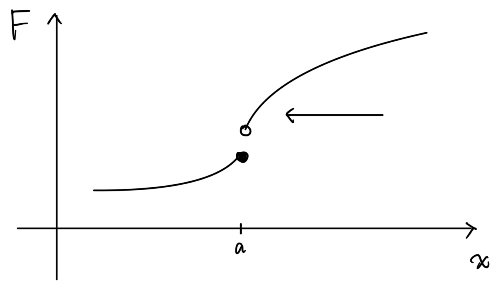
  > e.g. Toss a coin twice
  > $X : \#\text{ of heads in two tosses.}$
  > $X : (H,H) \mapsto 2.$
  > We need to check for any x, $\{X \le x\}$ is an event.
  > 1. $x \ge 2$, $\{X \le x\} = \{\omega | X(\omega) \le x\} = S$
  > 2. $x \in [1,2)$, what is $\{X \le x\}$?
  > 3. $x \in [0,1)$, what is $\{X \le x\}$?
  > 4. $x < 0$, what is $\{X \le x\}$?

- **Cumulative distribution of X (c.d.f.)**
  For any $x \in \mathbb{R}$, the c.d.f. of X is defined as $F(x) = P(X \le x)$.
  It satisfies the following property:
  i. $F(x)$ is a non-decreasing fucntion, i.e., if $x_1 \le x_2$, then $F(x_1) \le F(x_2)$.
    > Proof: $\{X \le x_1\}$ is an event. $\{X \le x_1\} \subseteq \{X \le x_2\}$ if $x_1 < x_2$, since $\{\omega | X(\omega) \le x_1\} \le \{\omega | X(\omega) \le x_2\}$.

  ii. $\lim\limits_{x\to -\infty} F(x) = 0$, $\lim\limits_{x\to\infty} F(x) = 1$.
  iii. $F(x)$ is a right-continuous function, i.e., for any $a \in \mathbb{R}$, $\lim\limits_{x\to a^+} F(x) = F(a)$.



right-continuous



not right-continuous

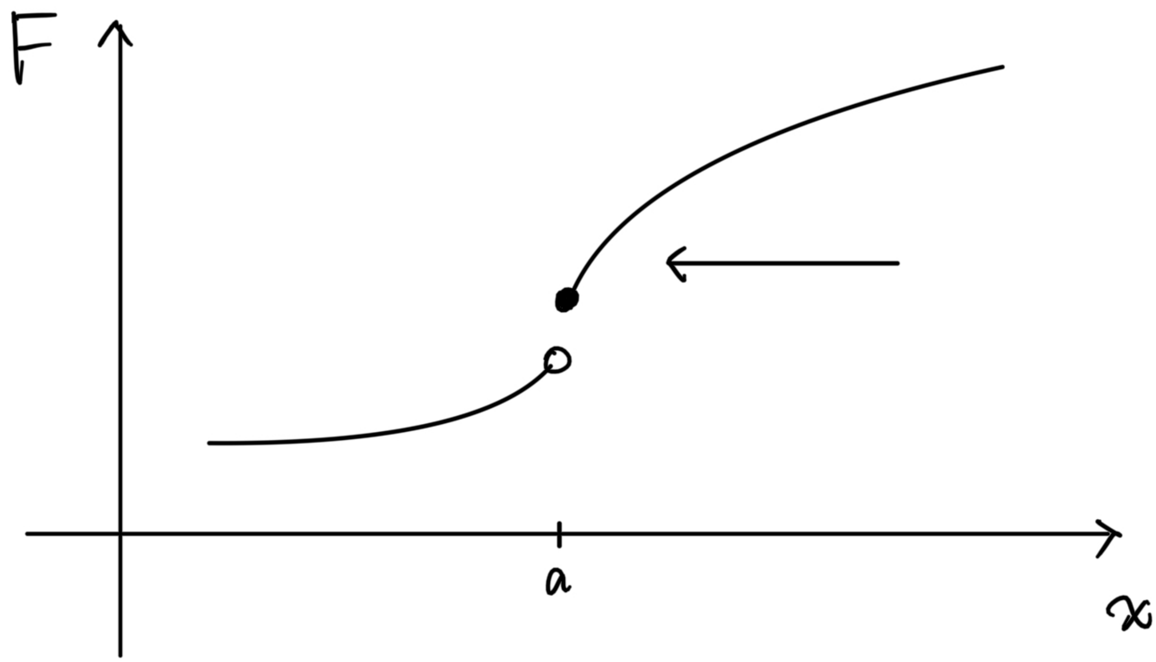
1, 2 and 3 are three basic properties of a c.d.f.

Some extra properties of a c.d.f.:

iv. $P(a < X \leq b) = F(b) - F(a)$.

> Proof: Define $A = \{X \leq b\}, B := \{X \leq a\}, C = \{a < x \leq b\}$, we want to prove: $P(a < X \leq b) = P(X \leq b) = P(X \leq a) \iff P(C) = P(A) - P(B)$. Note $B \cap C = \emptyset, B \cup C = A$. Then $P(A) = P(B \cup C) = P(B) + P(C)$.

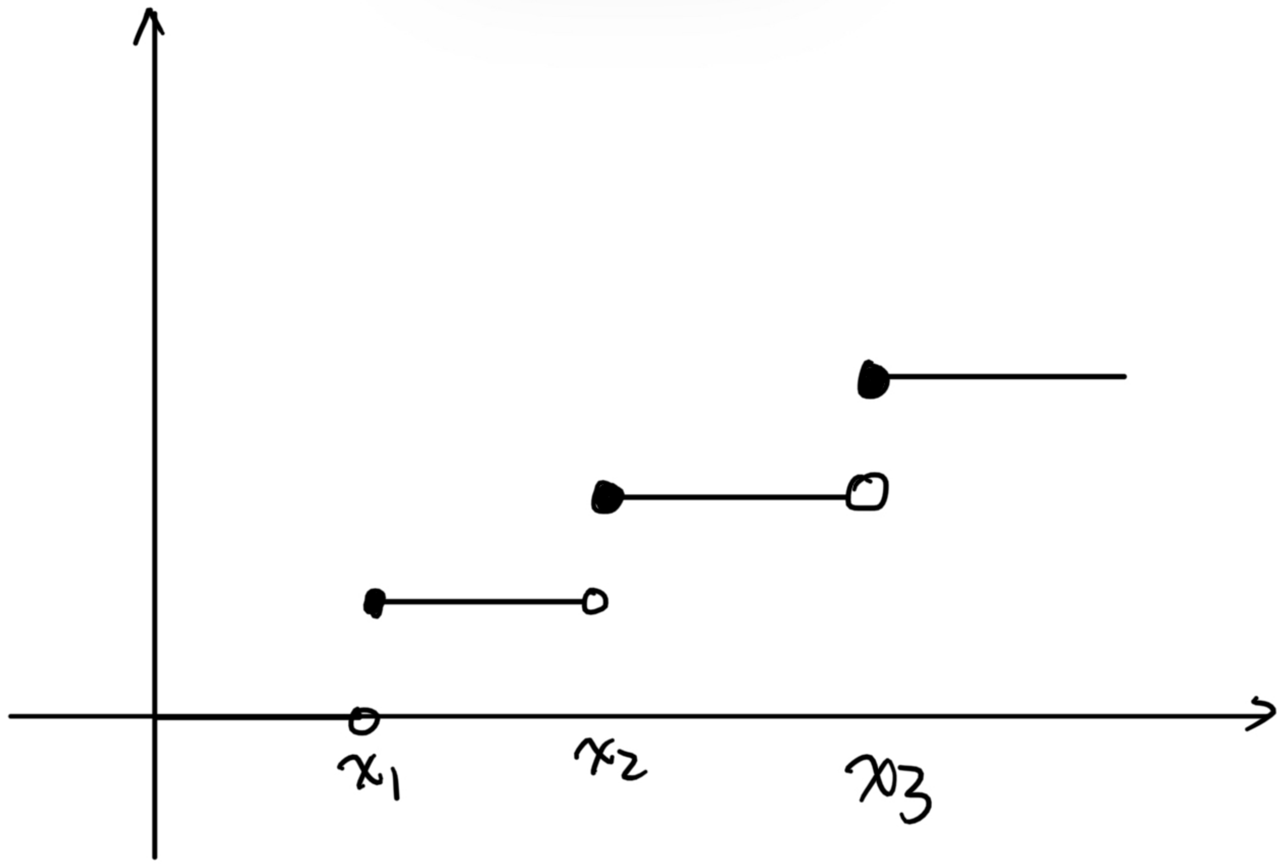
v. $P(X = a) = P(X \leq a) - P(x < a) = F(a) - F(a^-)$.

> Proof: $P(X = a) = P(X \leq a) - P(X < a) = F(a) - \lim\limits_{x \to a^-} F(x) = \lim\limits_{x \to a^+} F(x) - \lim\limits_{x \to a^-} F(x)$.
>
> 
>
>
> right-continuous

2.2 Discrete random variable

> Definition:
> If a random variable X can only take on a finite or countably infinite number of values, then X is called a discrete random variable.

- **cdf** of a discrete r.v. is a right continuous step funciton



- **Probability function (pf):** $f(x) = P(X = x)$.

  For a discrete r.v., $f(x) \begin{cases} > 0 & \text{if } X \text{ can take value } x \\ = 0 & \text{if } X \text{ cannot take value } x \end{cases}$
- **Support**: The set $A = \{x : f(x) > 0\}$ is called the support of X. These are all the possible values that X can take.
- Properties of a p.f. f for a discrete r.v. X.
  i. $f(x) \geq 0$ for any $x \in \mathbb{R}$.
  ii. $\sum_{x \in A} f(x) = 1$.

  > Proof: The support of X is a countable set, $A = \{x_1, \ldots, x_n\}$. Let $B_i = \{X = x_i\}$ is an event for $i = 1, ..., n$. B_i are pairwise mutually exclusive events, i.e. $B_i \cap B_j = \emptyset$ for $i \neq j$. Then, $\bigcup_{i=1}^{n} B_i = S$. Then, $1 = P(S) = P\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} P(B_i) = \sum_{i=1}^{n} P(X = x_i)$.

- Some commonly used discrete r.v.
  i. Bernoulli r.v. $X \sim \text{Bern}(p)$.

     X can only take two possible values, 0 and 1. $A = \{0, 1\}$.

     $f(1) = P(X = 1) = p$.
  ii. Binomial distribution

     Toss a coin n times.

     a. different tosses are indepedent

     b. probability of getting a head is fixed, which is denoted by p.

     X: # of heads across n tosses, then $X \sim \text{Bin}(n, p)$.

     Hence the support of X, $A = \{0, 1, 2, \ldots, n\}$.

     The p.f. of X is $f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$, $x \in A$.

     $\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = [p + (1-p)]^n = 1$
  iii. Geometric distribution

     X: # of failures before the first success.

     The support of X is $A = \{0, 1, ...\}$.

$f(x) = P(X = x) = (1-p)^x p,\ x \in A.$

$\sum_{x=0}^{\infty}(1-p)^x p = \frac{p}{1-(1-p)} = 1$

iv. Negative binomial r.v. $X \sim \mathrm{NegBin}(r, p)$

X: # of failures before the rth success.

v. Poisson r.v. $X \sim \mathrm{Poisson}(\mu)$

The support of X, $A = \{0, 1, \dots\}$.

The probability function $f(x) = P(X = x) = \frac{\mu^x}{x!}e^{-\mu},\ x \in A.$

$\sum_{x \in A} f(x) = \sum_{x=0}^{\infty} \frac{\mu^x}{x!}e^{-\mu} = e^{-\mu}\sum_{x=0}^{\infty}\frac{\mu^x}{x!} = e^{-\mu}e^{\mu} = 1.$

Aside: $e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}.$

2.3 Continuous random variable

> Definition: If the collection of all possible values X can take is an interval or the real line, then X is called a continuous r.v.
>
> - Remark: If X is continuous r.v., its cdf $F(x)$ is continuous everywhere. Moreover, F is differentialbe almost everywhere. It is not differentiable at atmost countable locations.

- Probability density function (pdf):

$$f(x) = \begin{cases} F'(x) & \text{if F is differentiable at } x \\ 0 & \text{otherwise} \end{cases}$$

- Support of X: $A = \{x | f(x) > 0\}$.
- Basic property of f:
  i. $f(x) \geq 0$ for any $x \in \mathbb{R}$.
  ii. $\int_{-\infty}^{\infty} f(x)dx = 1.$
- Extra properties of f:
  i. $F(x) = \int_{-\infty}^{x} f(t)dt = F(x) - F(-\infty)$ (find cdf from pdf).
  ii. $f(x) = \begin{cases} F'(x) & \text{if F is differentiable at } x \\ 0 & \text{otherwise} \end{cases}$ (find pdf from cdf).
  iii. $P(X = x) = 0$ and $f(x) \neq P(X = x)$ for any x.
  If F is differentiable at x, then $f(x) = \lim_{h \to 0}\frac{F(x+h) - F(x)}{h}$
  $\implies F(x+h) - F(x) \approx f(x) \cdot h$
  $\implies P(x < X \leq x + h) \approx f(x) \cdot h.$
  iv. $P(a < X \leq b) = F(b) - F(a) = P(a < X < b) = P(a \leq X \leq b)$

> Example (Uniform distribution):
> Suppose the cds if
>
> $$F(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \geq b \end{cases}$$
>
> Find pdf $f(x)$:
> The pdf is: $f(x) \begin{cases} 0 & x \leq a \\ \frac{1}{b-a} & a < x < b \\ 0 & x \geq b \end{cases}$

> Example:
> Define a function
>
> $$f(x) = \begin{cases} \frac{\theta}{x^{\theta+1}} & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

.

> i. Find for what values of θ, f is a pdf?
> Solution: $f(x) \geq 0$ for any $x \in \mathbb{R}$, therefore $\theta \geq 0$. $\int_{-\infty}^{\infty} f(x)dx = \int_1^{\infty} \frac{\theta}{x^{\theta+1}}dx$.
>
> > Case 1: $\theta = 0$, $\int_{-\infty}^{\infty} f(x)dx = 0 \neq 1$.
>
> > Case 2: $\theta > 0$, $\int_{-\infty}^{\infty} f(x)dx = \int_1^{\infty} \frac{\theta}{x^{\theta+1}}dx = -\frac{1}{x^\theta}\Big|_1^\infty = 1$.
>
> ii. Find $F(x)$ if f is a pdf.
> Solution: $F(x) = \int_{-\infty}^{x} f(t)dt$
>
> > Case 1: $x \leq 1$, $F(x) = \int_{-\infty}^{x} f(t)dt = 0$.
>
> > Case 2: $x > 1$, $F(x) = \int_{-\infty}^{x} f(t)dt = \int_1^x \frac{\theta}{t^{\theta+1}}dt = -\frac{1}{t^\theta}\Big|_1^x = 1 - \frac{1}{x^\theta}$.
>
> iii. Find $P(2 < X < 3)$ and $P(-2 < X < 3)$.
> Solution:
> $P(2 < X < 3) = F(3) - F(2) = \left(1 - \frac{1}{3^\theta}\right) - \left(1 - \frac{1}{2^\theta}\right) = \frac{1}{2^\theta} - \frac{1}{3^\theta}$.
> $P(-2 < X < 3) = F(3) - F(-2) = \left(1 - \frac{1}{3^\theta}\right) - 0 = 1 - \frac{1}{3^\theta}$.
> $P(-2 < X < 3) = \int_{-2}^{3} f(x)dx = \int_{-2}^{1} f(x)dx + \int_1^3 f(x)dx = \int_{-2}^{1} 0dx + \int_1^3 \frac{\theta}{x^{\theta+1}}dx = -\frac{1}{x^\theta}\Big|_1^3 = 1 - \frac{1}{3^\theta}$.

- Gamma function, $\Gamma(\alpha), \alpha > 0$.

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}e^{-x}dx$$

  a. $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$.
  b. $\Gamma(n) = (n-1)!$ when n is a positive integer, $\Gamma(1) = 1$.
  c. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

> Example (Gamma distribution):
> The pdf is
>
> $$f(x) = \begin{cases} \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$
>
> if $\alpha > 0, \beta > 0$ are constants.
> Verify f is a pdf.
>
> > Solution:
> >
> > a. $f(x) \geq 0$ for any $x \in \mathbb{R}$.
> > b. $\displaystyle\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{\infty} f(x)dx = 0 + \int_0^{\infty} \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}dx.$
> > Here, note $\displaystyle\int_0^{\infty} x^{\alpha-1}e^{-x}dx = \Gamma(\alpha)$.
> > Let $y = \frac{x}{\beta} \implies x = \beta y, dx = \beta dy$.
> > Then, $\displaystyle\int_0^{\infty} \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}dx = \int_0^{\infty} \frac{(\beta y)^{\alpha-1}e^{-y}}{\beta^\alpha \Gamma(\alpha)}\beta dy = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} y^{\alpha-1}e^{-y}dy = \frac{1}{\Gamma(\alpha)}\Gamma(\alpha) = 1.$

> Example (Weibull distribution):
> The pdf is
>
> $$f(x) = \begin{cases} \frac{\beta}{\theta^\beta}x^{\beta-1}\exp\left\{-\left(\frac{x}{\theta}\right)^\beta\right\} & x > 0 \\ 0 & x < 0 \end{cases}$$
>
> where $\alpha > 0, \beta > 0$ are constants, $X \sim \text{Weibull}(\theta, \beta)$.
> Verify f is a pdf.
>
> > Solution:
> >
> > a. $f(x) \geq 0$ for any $x \in \mathbb{R}$.

b. $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = 0 + \int_{0}^{\infty} \frac{\beta}{\theta^\beta} x^{\beta-1} \exp\left\{-\left(\frac{x}{\theta}\right)^\beta\right\} dx.$
Let $y =$ $(\frac{x}{\theta}^\beta \implies x = \theta y^{\frac{1}{\beta}}, dx = \theta \frac{1}{\beta} y^{\frac{1}{\beta}-1} dy.$
Then, $\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} \frac{\beta}{\theta^\beta} (\theta y^{\frac{1}{\beta}})^{\beta-1} \exp\{-y\}\, \theta \frac{1}{\beta} y^{\frac{1}{\beta}-1} dy = \Gamma(1) = 1.$

Exmaple (Normal distribution/Gaussian distribution):
The pdf is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

for $x \in \mathbb{R}$,
where $\mu \in \mathbb{R}, \sigma > 0$ are constants, $X \sim \text{Normal}(\mu, \sigma)$.
Verify f is a pdf.

Solution:

a. $f(x) \geq 0$ for any $x \in \mathbb{R}$.
b. $\int_{-\infty}^{\infty} f(x)dx = 1.$
To verify 2, we start from a special case, where $\mu = 0, \sigma = 1$.
$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, i.e., $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1.$
$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$. Let $y = \frac{x^2}{2} \implies x = \sqrt{2y}, dx = \sqrt{2}dy.$
Then, $2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-y} y^{1-1/2} dy = \frac{1}{\sqrt{\pi}} \Gamma(1/2) = 1.$

Prove $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is a pdf for any $\mu \in \mathbb{R}, \sigma > 0$.

a. $f(x) \geq 0$ for any $x \in \mathbb{R}$.
b. $\int_{-\infty}^{\infty} f(x)dx = 1$?
$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$
Let $z = \frac{x-\mu}{\sigma} \implies x = \mu + \sigma z, dx = \sigma dz$
$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2}} dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2}} dx = 1.$

2.4 Expectation

- Definition of expectation for discrete r.v.
  Suppose that X is a discrete r.v. with support A and p.f. $f(x)$.
  Then, $E(X) = \sum_{x \in A} x f(x)$ provided $\sum_{x \in A} |x| f(x) < \infty$.
- Definition of expectation for continuous r.v.
  Suppose that X is a continuous r.v. with support A and pdf $f(x)$.
  Then $E(X) = \int_{-\infty}^{\infty} x f(x)dx$ provided $\int_{-\infty}^{\infty} |x| f(x)dx < \infty$.

  Example (Cauchy distribution):
  The pdf of X is $f(x) = \frac{1}{\pi(1+x^2)}$ for $x \in \mathbb{R}$.
  Find $E(X)$.

  Solution:
  $\int_{-\infty}^{\infty} |x| f(x)dx = \int_{-\infty}^{\infty} |x| \frac{1}{\pi(1+x^2)} dx = 2\int_{0}^{\infty} \frac{x}{\pi(1+x^2)} dx = \left.\frac{\ln(1+x^2)}{\pi}\right|_0^\infty = \infty.$
  Therefore, $E(X)$ does not exist.

  Example:
  Suppose p.f. $f(x) = \frac{1}{x(x+1)}$ for $x = 1, 2, 3, \ldots$, the support of X is $A = \{1, 2, 3, \ldots\}$.

  i. Show f is a p.f.
  Solution:

  i. $f(x) \geq 0$ for any $x \in \mathbb{R}$.
  ii. $\sum_{x \in A} f(x) = \sum_{x \in A} \frac{1}{x(1+x)} = \sum_{x=1}^{\infty} \left(\frac{1}{x} - \frac{1}{x+1}\right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots = 1.$

  ii. Find $E(X)$.

> Solution: $E(X) = \sum_{x\in A} xf(x) = \sum_{x\in A} x\frac{1}{x(x+1)} = \sum_{x\in A}\frac{1}{x+1} = \sum_{x=1}^{\infty}\frac{1}{x+1} = \infty$.
> $E(X)$ does not exist.

> More examples of expectations:
>
> i. Binomial Distribution, $X \sim \mathrm{Bin}(n,p)$.
>
> > Solution 1: $E(X) = \sum_{x\in A} xf(x) = \sum_{x=0}^{n} x\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x} = \sum_{x=1}^{n} x\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x} = \sum_{x=1}^{n}\frac{n!}{(x-1)!(n-x)!}p^x(1-p)^{n-x}$.
> > Let $y = x-1$, then $\sum_{x=1}^{n}\frac{n!}{(x-1)!(n-x)!}p^x(1-p)^{n-x} = np\sum_{y=0}^{n-1}\frac{(n-1)!}{y!(n-1-y)!}p^y(1-p)^{n-1-y} = np$, since $\sum_{y=0}^{n-1}\frac{(n-1)!}{y!(n-1-y)!}p^y(1-p)^{n-1-y}$ is a pf of $\mathrm{Bin}(n-1,p)$.
>
> > Solution 2: For the ith trial, $X_i = \begin{cases} 1 & \text{if the } i\text{th outcome is a success} \\ 0 & \text{otherwise} \end{cases}$.
> > Then, $P(X_i = 1) = p$. Let $X = \sum_{i=1}^{n} X_i$, then $X \sim \mathrm{Bin}(n,p)$.
> > $E(X) = E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} 1\cdot P(X_i = 1) = np$.
>
> ii. Suppose X is a continuous r.v. with pdf $f(x) = \begin{cases} \frac{\theta}{x^{\theta+1}} & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$, where $\theta > 0$ is a constant. Find $E(X)$, and determine the values of θ for which $E(X)$ exists.
>
> > Solution: $\int_{-\infty}^{\infty} |x| f(x)dx = \int_{1}^{\infty} x\frac{\theta}{x^{\theta+1}}dx = \int_{1}^{\infty}\frac{\theta}{x^{\theta}}dx < \infty$ iff $\theta > 1$.
> > When $\theta > 1$, $E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{1}^{\infty}\frac{\theta x}{x^{\theta+1}}dx = \theta\int_{1}^{\infty}\frac{1}{x^{\theta}}dx = \left(\frac{\theta}{1-\theta}x^{1-\theta}\right)\Big|_{1}^{\infty} = \frac{\theta}{\theta-1}$.
> > When $\theta \leq 1$, $E(X)$ does not exist.

- Expectation of a function of X
  Suppose thar X is a r.v., what is $E(g(X))$, where g is a real function?
  For example, $g(x) = x^2$.
  Let $Y = g(X)$, find $E(Y)$.
  - Case 1: If X is a discrete r.v. with support A and p.f. $f(x)$, then $E(g(X)) = \sum_{x\in A} g(x)f(x)$ provided $\sum_{x\in A}|g(x)|f(x) < \infty$.
  - Case 2: If X is a continuous r.v. with support A and pdf $f(x)$, then $E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx$ provided $\int_{-\infty}^{\infty}|g(x)|f(x)dx < \infty$.
- Linearity Property: If a and b are two constants, then $E[ag(X) + bg(X)] = aE(g(X)) + bE(h(X))$.
- Variance: $Var(X) = E[(X-\mu)]^2 = E(X^2) - \mu^2 = E(X^2) - [E(X)]^2$ where $\mu = E(X)$.
- Moments:
  - kth moment about 0: $E(X^k)$.
  - kth moment about mean: $E[(X-\mu)^k]$, where $\mu = E(X)$.

> Example (Poission distribution):
> Suppose $X \sim \mathrm{Poisson}(\mu)$, where $\mu > 0$ is a constant.
> Find $E(X)$ and $Var(X)$.
>
> > Solution: $E(X) = \sum_{x=0}^{\infty} x\frac{\mu^x}{x!}e^{-\mu} = \mu e^{-\mu}\sum_{x=1}^{\infty}\frac{\mu^{x-1}}{(x-1)!}$.
> > Let $y = x-1$, then $E(X) = \mu e^{-\mu}\sum_{y=0}^{\infty}\frac{\mu^y}{y!} = \mu e^{-\mu}e^{\mu} = \mu$.
> > $E(X^2) = \sum_{x=0}^{\infty} x^2\frac{\mu^x}{x!}e^{-\mu} = \sum_{x=1}^{\infty}\frac{x\mu^x}{(x-1)!}e^{-\mu} = \sum_{x=1}^{\infty}\frac{(x-1+1)\mu^x}{(x-1)!}e^{-\mu} = \sum_{x=1}^{\infty}\frac{(x-1)^2\mu^x}{(x-1)!}e^{-\mu} + \sum_{x=1}^{\infty}\frac{\mu^x}{(x-1)!}e^{-\mu} = \sum_{x=2}^{\infty}\frac{(x-1)\mu^x}{(x-1)!}e^{-\mu} = \sum_{x=2}^{\infty}\frac{\mu^x}{(x-2)!}e^{-\mu}$.
> > Let $y = x-2$, then $\sum_{y=0}^{\infty}\frac{\mu^{y+2}}{y!}e^{-\mu} = \mu^2 e^{-\mu}\sum_{y=0}^{\infty}\frac{\mu^y}{y!} = \mu^2$.
> > That means $E(X^2) = \mu^2 + \mu$, and $Var(X) = E(X^2) - [E(X)]^2 = \mu^2 + \mu - \mu^2 = \mu$.

> Example (Gamma distribution):
> Suppose $X \sim \mathrm{Gamma}(\alpha,\beta)$. Find $E(X^k)$, $k > 0$.
> pdf of X is $f(x) = \begin{cases} \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$.
>
> > Solution: $E(X^k) = \int_{-\infty}^{\infty} x^k f(x)dx = \int_{0}^{\infty} x^k\frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)}dx$. Let $y = \frac{x}{\beta} \implies x = \beta y, dx = \beta dy$.
> > Then, $E(X^k) = \int_{0}^{\infty}\frac{(\beta y)^k(\beta y)^{\alpha-1}e^{-y}}{\beta^\alpha\Gamma(\alpha)}\beta dy = \frac{\beta^k}{\Gamma(\alpha)}\int_{0}^{\infty} y^{k+\alpha-1}e^{-y}dy = \frac{\beta^k}{\Gamma(\alpha)}\Gamma(k+\alpha) = \frac{\beta^k\Gamma(k+\alpha)}{\Gamma(\alpha)}$.
> > In paticular, if $k = 1$, $E(X) = \frac{\beta\Gamma(1+\alpha)}{\Gamma(\alpha)} = \frac{\beta\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha\beta$.
> > $k = 2$, $E(X^2) = \frac{\beta^2\Gamma(2+\alpha)}{\Gamma(\alpha)} = \frac{\beta^2(\alpha+1)\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha(\alpha+1)\beta^2$.
> > $Var(X) = E(X^2) - [E(X)]^2 = \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2 = \alpha\beta^2$.
> > Alternatively:
> > $E(X^k) = \int_{-\infty}^{\infty} x^k f(x)dx = \int_{0}^{\infty} x^k\frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)}dx = \int_{0}^{\infty}\frac{x^{k+\alpha-1}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)}dx$

> > Define $\alpha^*=k+\alpha$, then $E(X^k)=\int_0^\infty \frac{x^{\alpha^*-1}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)}dx=\int_0^\infty \frac{x^{\alpha^*-1}e^{-x/\beta}}{\beta^{\alpha^*}\Gamma(\alpha^*)}\frac{\beta^{\alpha^*}\Gamma(\alpha^*)}{\beta^\alpha\Gamma(\alpha)}dx=\frac{\beta^{\alpha^*}\Gamma(\alpha^*)}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty \frac{x^{\alpha^*-1}e^{-x/\beta}}{\beta^{\alpha^*}\Gamma(\alpha^*)}dx=\frac{\beta^{\alpha^*}\Gamma(\alpha^*)}{\beta^\alpha\Gamma(\alpha)}=\frac{\beta^{k+\alpha}\Gamma(k+\alpha)}{\beta^\alpha\Gamma(\alpha)}=\frac{\beta^k\Gamma(k+\alpha)}{\Gamma(\alpha)}$.

2.5 Moment generating function

- Definition: Suppose X is a random variable, then $M(t)=E(E^{tx})$ is called the moment generating function (mgf) of X if $M(t)$ exists for $t\in(-h,h)$ for some $h>0$.

> Example (Gamma distribution):
> Suppose $X\sim\mathrm{Gamma}(\alpha,\beta)$. Find the mgf of X.
>
> > Solution: $M(t)=E(e^{tX})=\int_{-\infty}^\infty e^{tx}f(x)dx=\int_0^\infty e^{tx}\frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)}dx=\int_0^\infty \frac{x^{\alpha-1}e^{-(1/\beta-t)x}}{\beta^\alpha\Gamma(\alpha)}dx$. (Note: $1/\beta>t$, otherwise the integral diverges.)
> > Let $y=(1/\beta-t)x$, then $x=\frac{y}{1/\beta-t}=\frac{\beta y}{1-t\beta}$, $dx=\frac{\beta}{1-t\beta}dy$.
> > Then, $M(t)=\int_0^\infty \frac{(\beta y)^{\alpha-1}e^{-y}}{\beta^\alpha\Gamma(\alpha)}\frac{\beta}{1-t\beta}dy=\frac{\beta^{\alpha-1}}{\Gamma(\alpha)(1-t\beta)}\int_0^\infty y^{\alpha-1}e^{-y}dy=\frac{\beta^{\alpha-1}}{\Gamma(\alpha)(1-t\beta)}\Gamma(\alpha)=\frac{\beta^{\alpha-1}\Gamma(\alpha)}{\Gamma(\alpha)(1-t\beta)}=\frac{\beta^{\alpha-1}}{1-t\beta}$.

> Example (Poisson distribution):
> Suppose $X\sim\mathrm{Poisson}(\mu)$. Find the mgf of X.
>
> > Solution: $M(t)=E(e^{tX})=\sum_{x=0}^\infty e^{tx}\frac{\mu^x}{x!}e^{-\mu}=e^{-\mu}\sum_{x=0}^\infty\frac{(\mu e^t)^x}{x!}=e^{-\mu}e^{\mu e^t}\sum_{x=0}^\infty\frac{(\mu e^t)^x}{x!}e^{-e^t\mu}=e^{\mu(e^t-1)}$.

> Example (Normal distribution):
> Suppose $X\sim N(0,1)$. Find the mgf of X.
>
> > Solution: $M(t)=E(e^{tX})=\int_{-\infty}^\infty e^{tx}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx=\int_{-\infty}^\infty\frac{1}{\sqrt{2\pi}}e^{tx-\frac{x^2}{2}}dx=\int_{-\infty}^\infty\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x^2-2tx+t^2)+\frac{1}{2}t^2}dx=$
> > $e^{\frac{1}{2}t^2}\int_{-\infty}^\infty\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-t)^2}dx=e^{\frac{1}{2}t^2}$.

> Question: How to find the mgf of $N(\mu,\sigma^2)$?

- Three important properties of mgf
  i. Suppose the mgf of X is $M(t)$. If $Y=aX+b$, where a and b are constants, then the mgf of Y is $M_Y(t)=e^{bt}M(at)$.
     If $Y\sim N(\mu,\sigma^2)$, then $X=\frac{Y-\mu}{\sigma}\sim N(0,1)$.
     $\implies Y=\mu+\sigma X$, where $X\sim N(0,1)$.
     $M_Y(t)=e^{\mu t}M_X(\sigma t)=e^{\mu t}e^{\frac{1}{2}\sigma^2t^2}$.
  ii. Find the kth moment of X about 0 from $M(t)$:
     $E(X^k)=M^{(k)}(0)=\frac{d^k}{dt^k}M(t)\Big|_{t=0}$.
     $M(t)=E(e^{tX})$, $M'(t)=E(Xe^{tX})$.
     In particular, $E(X)=M'(0)$, $E(X^2)=M''(0)$. Then, $Var(X)=E(X^2)-[E(X)]^2=M''(0)-[M'(0)]^2$.
     > Example (Gamma distribution):
     > If $X\sim\mathrm{Gamma}(\alpha,\beta)$, $M(t)=\left(\frac{1}{1-t\beta}\right)^2$, where $t<\frac{1}{\beta}$.
     > Find $E(X)$ and $Var(X)$.
     >
     > > Solution: $M'(t)=\alpha\beta(1-\beta t)^{-\alpha-1}$, $M''(t)=\alpha(\alpha+1)\beta^2(1-\beta t)^{-\alpha-2}$.
     > > Then, $E(X)=M'(0)=\alpha\beta$, $E(X^2)=M''(0)=\alpha(\alpha+1)\beta^2$.
  iii. Uniqueness of mgf.
     Namely, X and Y have the same distribution iff X and Y have the same mgf.
     > Example: X has mgf $M(t)=e^{t^2/2}$
     >
     > a. Find mgf of $Y=2X-1$.
     > > Solution: $M_Y(t)=e^{-t}M_X(2t)=e^{-t}e^{2t^2}$.
     >
     > b. Find $E(Y)$ and $Var(Y)$.
     > > Solution: $M_Y'(t)=(4t-1)e^{2t^2-t}$. $E(X)=M'Y(0)=-1$.
     > > $M_Y''(t)=4e^{2t^2-t}+(4t-1)^2e^{2t^2-t}$. $E(Y^2)=M_Y''(0)=1+4=5$.
     > > $Var(Y)=E(Y^2)-[E(Y)]^2=5-(-1)^2=4$.
     >
     > c. What is the distribution of Y?
     > > Solution: $Y\sim N(-1,4)$, since $M_Y(t)=e^{-t}e^{2t^2}$.

3 Joint distribution

3.1 Joint and Marginal cdfs

- Definition of joint cdf
  Suppose that X and Y are two r.v.s. The joint cdf of X and Y is defined by $F(x,y) = P(X \leq x, Y \leq y)$ for $x, y \in \mathbb{R}$.

  > Remark: This definition can be extended to n r.v.s. $X_1, X_2, \ldots, X_n$.
  > Joint cdf is $F(x_1, x_2, \ldots, x_n) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n)$.
  > However, we will focus on the case of $n = 2$.

- Properties of joint cdf
  i. Fix y, $F(x,y)$ is monotone increasing function of x, i.e., $F(x_1,y) \leq F(x_2,y)$ if $x_1 < x_2$.

  > Proof: $F(x_1,y) = P(X \leq x_1, Y \leq y)$, since $\{X \leq x_1, Y \leq y\} \subset \{X \leq x_2, Y \leq y\}$, $F(x_1,y) \leq F(x_2,y)$.

  ii. Fix x, $F(x,y)$ is monotone increasing function of y, i.e., $F(x,y_1) \leq F(x,y_2)$ if $y_1 < y_2$.
  iii. $\lim_{x \to -\infty} F(x,y) = 0 = \lim_{y \to -\infty} F(x,y)$.

  > Proof: $F(x,y) = P(X \leq x, Y \leq y) \leq P(X \leq x)$, and consider $\lim_{x \to -\infty} P(X \leq x) = 0$, additionally, by property of joint cdf, $F(x,y) \geq 0$, then by squeeze theorem, $\lim_{x \to -\infty} F(x,y) = 0$.

  iv. $\lim_{x \to \infty, y \to \infty} F(x,y) = 1$.

  > Proof: Consider set $Axy = \{X \leq x\} \cup \{Y \leq y\}$, then as $x, y \to \infty$, $P(\overline{Axy}) \to 0$, then $F(x,y) = P(Axy) \to 1$.

  v. How to find marginal cdf from the joint one?
  $F_1(x) = P(X \leq x) = \lim_{y \to \infty} F(x,y)$.
  Define $Ax = \{X \leq x\}, By = \{Y \leq y\}$.
  As $y \to \infty$, $Ax \cup By \to Ax$.
  $F_2(y) = P(Y \leq y) = \lim_{x \to \infty} F(x,y)$.

3.2 Joint Discrete r.v.s

- Definition: If both X and Y are discrete r.v.s, then as a pair, $X \& Y_{(X,Y)}$ are joint discrete r.v.s X and Y.
- Definition of joint p.f.:
  The joint p.f. of X and Y is given by $f(x,y) = P(X = x, Y = y)$ for any $x, y \in \mathbb{R}$.
- Definition of join support: The support of (X,Y) is the set $A = \{(x,y) \in \mathbb{R}^2 : f(x,y) > 0\}$.
- Basic properties of joint p.f.:
  i. $f(x,y) \geq 0$ for any $(x,y) \in \mathbb{R}^2$.
  ii. $\sum_{(x,y) \in A} f(x,y) = 1$.

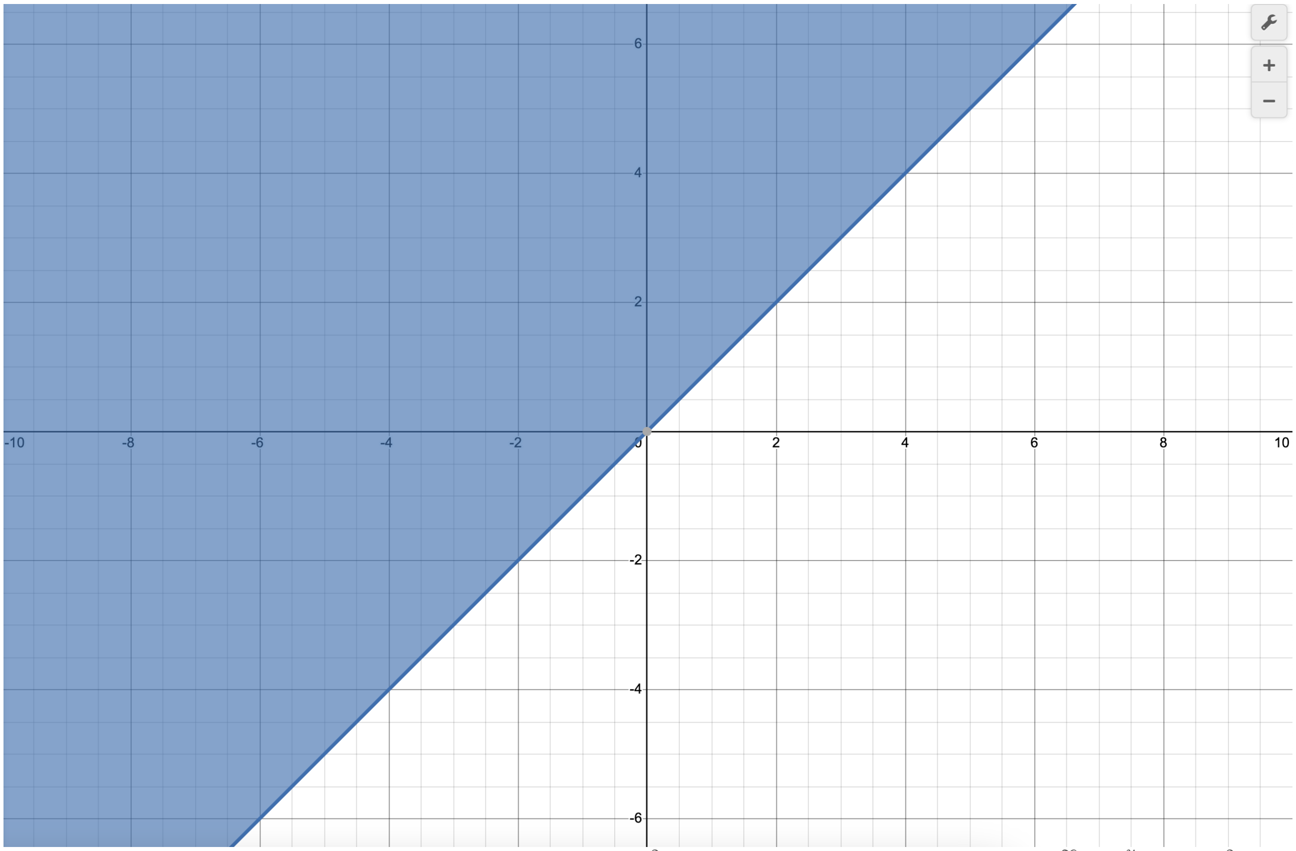
  > Question: How to find probability over a region $C \subseteq \mathbb{R}^2$?

  iii. $P((X,Y) \in C) = \sum_{(x,y) \in C} f(x,y)$.

  > Question: How to find marginal p.f. from the joint one?

  iv. $f_1(x) = P(X = x) = P(X = xY < \infty) = \sum_{y \in \mathbb{R}} f(x,y)$.

  > E.g. Suppose X and Y are independent discrete r.v.s with joint p.f. $f(x,y) = kq^2p^{x+y}$ for $x = 0, 1, ...$ and $y = 0, 1, ...$, and 0 elsewhere. Here $p \in (0,1)$ is a constant, $q = 1 - p$.
  >
  > a. Find k.
  >
  > > Solution: Since $f(x,y) \geq 0$ for any $(x,y) \in \mathbb{R}^2$, $k > 0$.Since $\sum_{x=0}^{\infty} f(x,y) = 1$, Then,
  > >
  > > $$k\left(\sum_{x=0}^{\infty} p^{x+y}q^2\right) = kq^2\left(\sum_{x=0}^{\infty} p^x\right)\left(\sum_{x=0}^{\infty} p^y\right) = kq^2\left(\frac{1}{1-p}\right)\left(\frac{1}{1-p}\right) = k$$
  > >
  > > Therefore, $k = 1$
  >
  > b. Find the marginal p.f. of X and find marginal p.f. of Y.
  >
  > > Solution: The support of X is $Ax = \{0, 1, 2, ...\}$.
  > > Here, $f_1(x) = \sum_{y \in \mathbb{R}} f(x,y) = 0$ if $x \notin Ax$
  > > Given $X \in Ax$, then $f_1(x) = \sum_{y \in \mathbb{R}} f(x,y) = \sum_{y=0}^{\infty} f(x,y) = \sum_{y=0}^{\infty} p^{x+y}q^2 = q^2p^x \sum_{y=0}^{\infty} p^y = q^2p^x \frac{1}{1-p} = qp^x$.
  >
  > c. $P(X \leq Y)$



> Solution: $P(X \le Y) = \sum_{(x,y)\in C} f(x,y)$ where $C = \{(x,y) \in \mathbb{R}^2 = x \le y\}$, therefore, $P(X \le Y) = \sum_{y=0}^{\infty}\sum_{x=0}^{y} p^{x+y}q^2 = \sum_{x=0}^{\infty} p^x q^2 \sum_{y=x}^{\infty} p^y = \sum_{x=0}^{\infty} p^x q^2 \frac{p^x}{1-p} = q\sum_{x=0}^{\infty} p^{2x} = q\frac{1}{1-p^2} = \frac{1}{1+p}$.

3.3 Joint Continuous r.v.s

- Definition: If joint cdf of (X,Y) can be written as $F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)dudv$ then X and Y are joint continuous r.v.s with joint pdf $f(x,y)$.

  Namely, $f(x,y) = \begin{cases} \frac{\partial^2}{\partial x \partial y}F(x,y) & \text{if exists} \\ 0 & \text{o.w.} \end{cases}$.
- Definition of joint support: $A = \{(x,y) \in \mathbb{R}^2 : f(x,y) > 0\}$.
- Properties of joint pdf:
  i. $f(x,y) \ge 0$ for any $(x,y) \in \mathbb{R}^2$.
  ii. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$.

  > Question: How to find probability over a region $C \subseteq \mathbb{R}^2$?

  iii. $P((X,Y) \in C) = \iint_{(x,y)\in C} f(x,y)dxdy$.

  > Question: How to find marginal pdf from the joint one?

  iv. $f_1(x) = \int_{-\infty}^{\infty} f(x,y)dy$ and $f_2(y) = \int_{-\infty}^{\infty} f(x,y)dx$.

  > E.g. X and Y are joint continuous r.v.s with joint pdf $f(x,y) = \begin{cases} x+y & \text{if } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$.
  >
  > a. Show f is a joint pdf.
  >
  > > Solution: $f(x,y) \ge 0$ for any $(x,y) \in \mathbb{R}^2$.
  > >
  > > $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \int_0^1\int_0^1 (x+y)dxdy = \int_0^1 \left(\frac{x^2}{2} + xy\right)\Bigg|_{x=0}^{x=1} dy = \int_0^1 \left(\frac{1}{2} + y\right)dy = \frac{1}{2} + \frac{1}{2} = 1.$
  >
  > b. Find
  >
  > a. $P(X \le 1/3, Y \le 1/2)$
  >
  > > Solution: $P(X \le 1/3, Y \le 1/2) = \int_0^{1/3}\int_0^{1/2}(x+y)dydx = \int_0^{1/3}\left(xy + \frac{y^2}{2}\right)\Bigg|_{y=0}^{y=1/2} dx = \int_0^{1/3}\left(\frac{x}{2} + \frac{1}{8}\right)dx = \frac{1}{36} + \frac{1}{24} = \frac{5}{72}$.

b. $P(X \leq Y)$

Solution: $P(X \leq Y) = \iint_C f(x,y)dxdy = \int_0^1 dx \int_x^1 (x+y)dy = \int_0^1 dy \int_0^y (x+y)dx = \int_0^1 \left(\frac{x^2}{2} + xy\right)\Bigg|_{x=0}^{x=y} dy =$
$\int_0^1 \left(\frac{y^2}{2} + y^2\right) dy = \frac{1}{2}$.

c. $P(X+Y \leq 1/2)$

Solution: Let $C = \{(x,y)|x+y \leq \frac{1}{2}, 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

Then, $P(X+Y \leq 1/2) = \iint_C f(x,y)dxdy = \int_0^{1/2}\int_0^{1/2-x}(x+y)dydx = \int_0^{1/2}\left(xy + \frac{y^2}{2}\right)\Bigg|_{y=0}^{y=1/2-x} dx =$
$\int_0^{1/2}\left(\frac{x}{2} - \frac{x^2}{+2}\frac{1}{2}(x^2 - x + \frac{1}{4}\right) dx = \int_0^{1/2}\left(-\frac{x^2}{2} + \frac{1}{8}\right) dx = \left(-\frac{x^3}{16} + \frac{x}{8}\right)\Big|_0^{1/2} = \frac{1}{24}$.

d. $P(XY \leq 1/2)$

Solution: Find $P(XY > 1/2)$ first.

$P(XY > 1/2) = \int_0^{1/2}\int_0^{1/2/x}(x+y)dydx = \int_0^{1/2}\left(xy + \frac{y^2}{2}\right)\Bigg|_{y=0}^{y=1/2x} dx = \int_0^{1/2}\left(x - \frac{1}{8x^2}\right) dx = \left(\frac{x^2}{2} + \frac{1}{8x}\right)\Big|_0^{1/2} = \frac{1}{4}$.

Therefore, $P(XY \leq 1/2) = 1 - P(XY > 1/2) = 1 - \frac{1}{4} = \frac{3}{4}$

c. Find marginal pdf of X and Y.

Solution: The support of X is $[0,1]$.

Given $x \in [0,1]$, $f_1(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 (x+y)dy = \left(xy + \frac{y^2}{2}\right)\Bigg|_{y=0}^{y=1} = x + \frac{1}{2}$.

E.g. Suppose $f(x) = \begin{cases} ke^{-x-y} & 0 < x < y < \infty \\ 0 & \text{o.w.} \end{cases}$ is the joint pdf of (X,Y).

a. Find k.

Solution: $f(x,y) \geq 0$ for any $(x,y) \in \mathbb{R}^2$, therefore, $k \geq 0$.
Now, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \int_0^{\infty}\int_x^{\infty} ke^{-x-y}dydx = \int_0^{\infty} ke^{-x}\left(-e^{-}y\right)\big|_x^{\infty} dx = \int_0^{\infty} ke^{-2x}dx = -\frac{k}{2}e^{-2x}\big|_0^{\infty} = \frac{k}{2} = 1$,
therefore, $k = 2$.

b. Find:

a. $P(X \leq \frac{1}{3}, Y \leq \frac{1}{2})$

Solution: Let $C = \{(x,y)|x \leq 1/3, y \leq 1/2, 0 < x < y\}$. Then, $P(X \leq \frac{1}{3}, Y \leq \frac{1}{2}) = \iint_C f(x,y)dxdy =$
$\int_0^{1/3}\int_x^{1/2} 2e^{-x-y}dydx = \int_0^{1/3} 2e^{-x}\left(-e^{-y}\right)\Big|_x^{1/2} dx = \int_0^{1/3} 2e^{-x}\left(-e^{-1/2} + e^{-x}\right) dx = \int_0^{1/3} 2e^{-x}\left(e^{-x} - e^{-1/2}\right) dx =$
$\int_0^{1/3} 2e^{-2x} - 2e^{-1/2}e^{-x}dx = -e^{-2x} + 2e^{-1/2}e^{-x}\big|_0^{1/3} = 1 - 2e^{-1/2} - e^{-2/3} - e^{-5/6}$.

b. $P(X \leq Y)$

Solution: $P(X \leq Y) = 1$

c. $P(X+Y \geq 1)$

Solution: Let $C = \{(x,y)|x+y \geq 1, 0 < x < y\}$
Let's find $P(X+Y < 1)$ first.

$P(X+Y < 1) = \iint_{x,y \in \mathbb{R}} 2e^{-x-y}dydx = \int_0^{1/2}\int_x^{1-x} 2e^{-x-y}dydx = \int_0^{1/2} 2e^{-x}\left(-e^{-y}\right)\Big|_x^{1-x} dx =$
$\int_0^{1/2} 2e^{-x}\left(-e^{x-1} + e^{-x}\right) dx = \int_0^{1/2} 2e^{-2x} - 2e^{-1}dx = -e^{-2x} - 2e^{-1}x\big|_0^{1/2} = 1 - 2e^{-1}$.
Hence, $P(X+Y \geq 1) = 1 - P(X+Y < 1) = 2e^{-1}$.

c. Find marginal pdf of X and Y.

Joint support is $A = \{(x,y)|0 < x < y < \infty\}$. The support of X is $A_X = \{0 < x < \infty\}$.
Given $x \in (0,\infty)$, $f_1(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_x^{\infty} 2e^{-x-y}dy = 2e^{-x}\left(-e^{-y}\right)\big|_x^{\infty} = 2e^{-2x}$.
The support of Y is $A_Y = \{0 < y < \infty\}$.
Given $y \in (0,\infty)$, $f_2(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^y 2e^{-x-y}dx = 2e^{-y}\left(-e^{-x}\right)\big|_0^y = 2e^{-y} - 2e^{-2y}$.

d. Find the distribution of $T = X + Y$.

Solution: The support of T is $A_T = \{0 < t < \infty\}$.

a. If $t \leq 0$, $P(T \leq t) = 0$.

> > b. If $t > 0$, $F_T(t) = P(T \le t) = P(X+Y \le t) = \iint_{(x,y)\in C} 2e^{-x-y}dxdy = \int_0^{t/2}\int_x^{t-x} 2e^{-x-y}dydx =$
> > $\int_0^{t/2} (-2e^{-x}e^{-y})|_x^{t-x} = -e^{-2x} - 2e^{-t}x\Big|_0^{t/2} = 1 - e^{-t} - te^{-t}$.
> > The pdf of T is $f_T(t) = \frac{d}{dt}F_T(t) = e^{-t} + te^{-t} = e^{-t} - e^{-t} + te^{-t} = te^{-t}$ for $t > 0$ and 0 otherwise.

3.4 Independent of random variables

- Definition: For any two r.v.s X and Y, we say X and Y are independent if and only if $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ for any $A, B \subseteq \mathbb{R}$.
  Here, $X \in A$ is an event, meaning $\{\omega \in \Omega : X(\omega) \in A\}$.
  > e.g. Let $A = (-\infty, x)$, $B = (-\infty, y)$, $x, y \in \mathbb{R}$.
  > Therefore, if X and Y are independent, $P(X \le x, Y \le y) = P(X \le x)P(Y \le y) = F_1(x)F_2(y)$ for any $x, y \in \mathbb{R}$.
  > Conclusion: X and Y are independent if and only if $F(x,y) = F_1(x)F_2(y)$ for any $x, y \in \mathbb{R}$. (Above shows this is a necessary condition, proof of this is a sufficient condition is beyond the scope of this course.)

  Suppose X and Y has joint p.f. or joint p.d.f, which is denoted by $f(x,y)$, and marginal p.f. or marginal p.d.f, denoted by $f_1(x)$ and $f_2(y)$, then X and Y are independent iff $f(x,y) = f_1(x)f_2(y)$ for every $x, y \in \mathbb{R}$.
  Remark: If X and Y are independent, then $g(X)$ and $h(Y)$ must be independent for any real functions g and h.
  > e.g. If X is independent of Y, then X^2 is independent of Y^2. But X^2 is independent of Y^2, we cannot conclude X is independent of Y.
  > Suppose $P(X=1) = P(X=-1) = \frac{1}{2}$. Let $Y = X$. $P(X=1, Y=1) = P(X=1) = \frac{1}{2}$, but $P(X=1)P(Y=1) = \frac{1}{4}$.
  > $P(Y^2 = 1) = P(X^2 = 1) = 1$.

  > Example: (Joint Discrete r.v.s)
  > Consider the joint p.f. of X and Y is $f(x,y) = q^2p^{x+y}$ for $x = 0, 1, \ldots$ and $y = 0, 1, \ldots$, and 0 elsewhere. Here $p \in (0,1)$ is a constant, $q = 1 - p$.
  > Marginal p.f. of X is $f_1(x) = qp^x$ for $x = 0, 1, \ldots$ and 0 elsewhere.
  > Marginal p.f. of Y is $f_2(y) = qp^y$ for $y = 0, 1, \ldots$ and 0 elsewhere.
  > Thus, $f(x,y) = f_1(x)f_2(y)$ for every $x, y \in \mathbb{R}$ therefore, X and Y are independent.

  > Example (Joint Continuous r.v.s)
  > Suppose the joint pdf of X and Y is $f(x,y) = \begin{cases} x+y & \text{if } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$.
  > The marginal pdf of X is $f_1(x) = x + \frac{1}{2}$ for $x \in [0,1]$ and 0 otherwise.
  > The marginal pdf of Y is $f_2(y) = y + \frac{1}{2}$ for $y \in [0,1]$ and 0 otherwise.
  > Hence, $f(x,y) \ne f_1(x)f_2(y)$ fpr $x \in (0,1)$ and $y \in (0,1)$, therefore, X and Y are not independent.

- Factorization theorem for independence
  Condition 1: $f(x,y) = g(x)h(y)$ for every $x, y \in \mathbb{R}$ for some function g and h where f(x,y) denotes the joint p.f. or joint p.d.f. of X and Y.
  Condition 2: Let A be the joint support of X and Y, and let A_1 be the marginal support of X and A_2 be the marginal support of Y. Then, $A = A_1 \times A_2 = \{(x,y) \in \mathbb{R}^2 : x \in A_1, y \in A_2\}$. (Interpretation: A is a ractangle or the range of X and Y are independent.)
  Conditions 1 and 2 are satisfied if and only if X and Y are independent.
  > Example: If the joint p.f. of X and Y is $f(x,y) = \frac{\mu^{x+y}e^{-2\mu}}{x!y!}$ for $x = 0, 1, \ldots$ and $y = 0, 1, \ldots$ and 0 elsewhere.
  >
  > i. Is X independent of Y?
  > > Solution: Condition 1: $f(x,y) = \frac{\mu^{x+y}e^{-2\mu}}{x!y!} = \frac{\mu^x e^{-\mu}}{x!}\frac{\mu^y e^{-\mu}}{y!}$. If we take $g(x) = \begin{cases} \frac{\mu^x e^{-\mu}}{x!} & \text{if } x = 0, 1, \ldots \\ 0 & \text{o.w.} \end{cases}$ and $h(y) = \begin{cases} \frac{\mu^y e^{-\mu}}{y!} & \text{if } y = 0, 1, \ldots \\ 0 & \text{o.w.} \end{cases}$, then $f(x,y) = g(x)h(y)$ for every $x, y \in \mathbb{R}$.
  > > Condition 2: $A = \{(x,y) \in \mathbb{R}^2 : x \in A_1, y \in A_2\}$, where $A_1 = \{0, 1, \ldots\}$ and $A_2 = \{0, 1, \ldots\}$.
  > > Therefore, by factorization theorem, X and Y are independent.
  >
  > ii. Find the marginal p.f. of X and Y.
  > > Solution: A shortcut: $f_1(x) = C \cdot g(x)$ for some constant C.
  > > Property 1: $f_1(x) \ge 0$ for any $x \in \mathbb{R}$. Here $g(x) = \begin{cases} \frac{\mu^x e^{-\mu}}{x!} & \text{if } x = 0, 1, \ldots \\ 0 & \text{o.w.} \end{cases}$, therefore, $C \ge 0$.
  > > Property 2: The support of X is $A_1 = \{0, 1, \ldots\}$. Therefore, $\sum_0^\infty f_1(x) = \sum_0^\infty C\frac{\mu^x e^{-\mu}}{x!} = C\sum_0^\infty \frac{\mu^x e^{-\mu}}{x!}$, then $C = 1$.

Therefore, $f_1(x) = \begin{cases} \frac{\mu^x e^{-\mu}}{x!} & \text{if } x = 0, 1, \ldots \\ 0 & \text{o.w.} \end{cases}$.

Similarly, $f_2(y) = \begin{cases} \frac{\mu^y e^{-\mu}}{y!} & \text{if } y = 0, 1, \ldots \\ 0 & \text{o.w.} \end{cases}$.

Example (Joint Continuous r.v.s)

Suppose the joint pdf of X and Y is $f(x, y) = \begin{cases} \frac{3}{2}y(1-x^2) & -1 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$.

i. Is X independent of Y?

Solution: Condition 1: $f(x, y) = \left(\frac{3}{2}y\right)(1-x^2)$, then $g(x) = \begin{cases} 1-x^2 & \text{if } -1 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$ and $h(y) = \begin{cases} \frac{3}{2}y & \text{if } 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$.

Then $f(x, y) = g(x)h(y)$ for every $x, y \in \mathbb{R}$.
Condition 2: $A = \{(x, y) \in \mathbb{R}^2 : x \in A_1, y \in A_2\}$, where $A_1 = [-1, 1]$ and $A_2 = [0, 1]$.
Therefore, by factorization theorem, X and Y are independent.

ii. Find the marginal pdf of X and Y.

Solution: A shortcut: $f_1(x) = C \cdot g(x)$ for some constant C, the support of X is $A_1 = [-1, 1]$.

Property 1: $f_1(x) \ge 0$ for any $x \in \mathbb{R}$. Here $g(x) = \begin{cases} 1-x^2 & \text{if } -1 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$, therefore, $C \ge 0$.

Property 2: $\int_{-\infty}^{\infty} f_1(x)dx = \int_{-1}^{1} C(1-x^2)dx = C\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1} = 2C\left(1 - \frac{1}{3}\right) = 1$, therefore, $C = \frac{3}{4}$.

Therefore, $f_1(x) = \begin{cases} \frac{3}{4}(1-x^2) & \text{if } -1 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$.

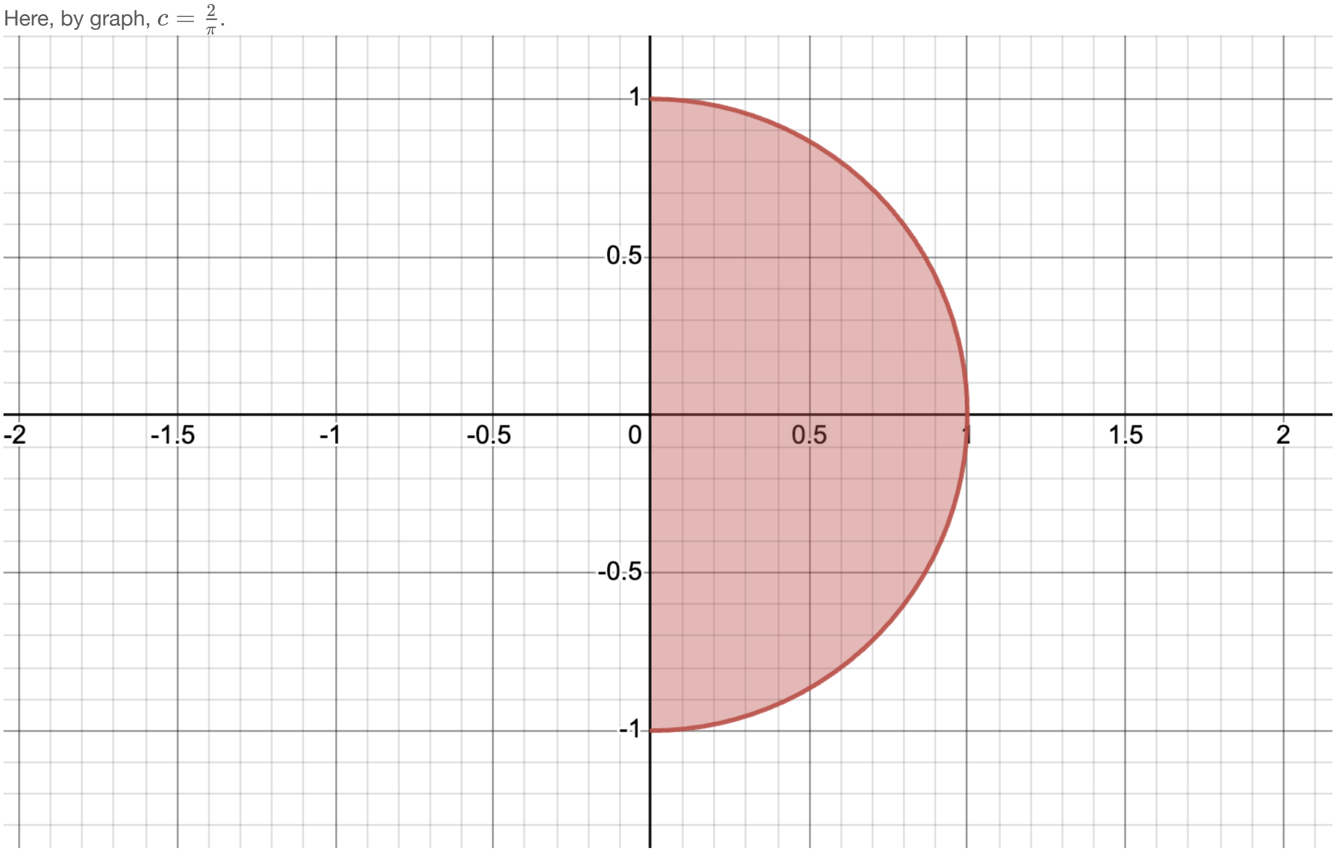
Support of Y is $A_2 = [0, 1]$, given $y \in [0, 1]$, $f_2(y) = \frac{f(x,y)}{f_1(x)} = \frac{\frac{3}{2}y(1-x^2)}{\frac{3}{4}(1-x^2)} = 2y$. Therefore, $f_2(y) = \begin{cases} 2y & \text{if } 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$.

Example (Uniform distribution over a region)

Suppose (X, Y) follows a uniform distribution over $C = \{(x, y) | x \ge 0, x^2 + y^2 \le 1\}$.

Namely, $f(x, y) = \begin{cases} c & \text{if } (x, y) \in C \\ 0 & \text{o.w.} \end{cases}$.

Here, by graph, $c = \frac{2}{\pi}$.



i. Is X independent of Y?

> Solution: Given $x \in [0,1]$, Y can take value in $[-\sqrt{1-x^2}, \sqrt{1-x^2}]$, therefore, X and Y are not independent.

ii. Find the marginal pdf of X and Y.

> Solution: The support of X is $A_1 = [0,1]$, given $x \in [0,1]$, $f_1(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{\pi} dy = \frac{4}{\pi}\sqrt{1-x^2}$.
> The support of Y is $A_2 = [-1,1]$, given $y \in [-1,1]$, $f_2(y) = \int_0^{\sqrt{1-y^2}} \frac{2}{\pi} dx = \frac{2}{\pi}\sqrt{1-y^2}$.

3.5 Joint expectation

- Definition: Suppose $h(x,y)$ is a bivariate function, then $E[h(x,y)] = \begin{cases} \sum_x \sum_y h(x,y)f(x,y) & \text{if } X \text{ and } Y \text{ are joint discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)f(x,y)dxdy & \text{if } X \text{ and } Y \text{ are joint continuous} \end{cases}$, provided $E[|h(x,y)|] < \infty$.

  > e.g. $E[XY] = \begin{cases} \sum_x \sum_y (xy)f(x,y) & \text{if } X \text{ and } Y \text{ are joint discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (xy)f(x,y)dxdy & \text{if } X \text{ and } Y \text{ are joint continuous} \end{cases}$, provided $E[|XY|] < \infty$.

  > e.g. E[X] (i.e. h(x,y) = x))
  >
  > - Method 1:
  >
  > $$E(X) = \begin{cases} \sum_x \sum_y xf(x,y) & \text{joint discrete} \\ \iint_{\mathbb{R}^2} xf(x,y)dxdy & \text{joint continuous} \end{cases}$$
  >
  > - Method 2: find the marginal distribution, i.e., the marginal p.f. or marginal p.d.f. of X first, denoted by $f_1(x)$, then
  >
  > $$E(X) = \begin{cases} \sum_x xf_1(x) & \text{joint discrete} \\ \int_{\mathbb{R}^2} xf_1(x)dx & \text{joint continuous} \end{cases}$$

- Properties of joint expectation:
  i. linearity: $E[ag(X,Y) + bh(X,Y)] = aE[g(X,Y)] + bE[h(X,Y)]$ where a, b are constants, g, h are bivariate functions.
  ii. Under independence assumption (X is independent of Y), $E(XY) = E(X)E(Y)$ and $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$. Further, if $X_1, ..., X_n$ are independent, then $E\left[\prod_{i=1}^n h_i(X_i)\right] = \prod_{i=1}^n E[h_i(X_i)]$.
- Covariance of X and Y
  Definition: Covariance of X and Y is defined as $Cov(X,Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y)$.
  If X and Y are independent, then $Cov(X,Y) = 0$.

  > An example where X and Y are uncorrlated, but not independent.
  > Let $X \sim N(0,1)$ and $Y = X^2$, then $E(X) = 0, E(XY) = E(X^3), Cov(X,Y) = 0$.
  > Now, we find a pair of a and b such that $P(X \le a, Y \le b) \ne P(X \le a)P(Y \le b)$. Consider $a = -2, b = 1$, then $P(X \le a) = P(X \le -2) > 0, P(Y \le b) = P(X^2 \le 1) = P(-1 \le X \le 1) > 0$, but $P(X \le a, Y \le b) = P(X \le -2, Y \le 1) = 0$.

- Results for covariance
  i. $Cov(X,X) = E[(X - \mu_X)(X - \mu_X)] = E[(X - \mu_X)^2] = Var(X)$.
  ii. $Cov(X+Y, Z) = Cov(X,Z) + Cov(Y,Z)$.
- Variance formula

  i.
  $$Var(aX + bY) = Cov(aX + bY, aX + bY)$$
  $$Cov(aX,aX) + Cov(aX,bY) + Cov(bY,aX) + Cov(bY,bY) = Var(aX) + 2abCov(X,Y) + Var(bY) = a^2Var(X) + 2abCov$$

  ii.
  $$Var\left(\sum_{i=1}^n\right) = \sum_{i=1}^n Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)$$

  iii. If $X_1, ..., X_n$ are independent,
  $$Var\left(\sum_{i=1}^n\right) = \sum_{i=1}^n Var(X_i)$$

  > Example 1: Suppose the joint p.f. of X and Y is $f(x,y) = \begin{cases} \frac{\mu^{x+y}e^{-2\mu}}{x!y!} & \text{if } x = 0,1,... \text{ and } y = 0,1,... \\ 0 & \text{o.w.} \end{cases}$. Find $Var(2X + 3Y) = 4Var(X) + 12Cov(X,Y) + 9Var(Y)$.

> > Solution: Since X and Y are independent, $Cov(X,Y)=0$, therefore, $Var(2X+3Y)=4Var(X)+9Var(Y)$.
> > Previously, we find $X\sim Poisson(\mu), Y\sim Poisson(\mu)$, therefore $Var(X)=\mu, Var(Y)=\mu$.
> > Hence, $Var(2X+3Y)=4\mu+9\mu=13\mu$.

> Example 2: Suppose the joint p.f. of X and Y is $f(x,y)=\begin{cases} x+y & \text{if } 0\le x\le 1, 0\le y\le 1 \\ 0 & \text{o.w.} \end{cases}$. Find $Var(X+Y)$.
>
> > Solution:
> >
> > $$\begin{aligned} Var(X+Y) &= Var(X)+2Cov(X,Y)+Var(Y) \\ &= 2Var(X)+2Cov(X,y) \end{aligned}$$
> >
> > the marginal pdf of X is $f_1(x)=\begin{cases} x+1/2 & \text{if } 0\le x\le 1 \\ 0 & \text{o.w.} \end{cases}$.
> > then, $E(X)=\int_0^1 x\left(x+\frac{1}{2}\right)dx=\int_0^1\left(x^2+\frac{x}{2}\right)dx=\left(\frac{x^3}{3}+\frac{x^2}{4}\right)\Big|_0^1=\frac{7}{12}$.
> > $E(X^2)=\int_0^1 x^2\left(x+\frac{1}{2}\right)dx=\int_0^1\left(x^3+\frac{x^2}{2}\right)dx=\left(\frac{x^4}{4}+\frac{x^3}{6}\right)\Big|_0^1=\frac{5}{12}$.
> > $Var(X)=E(X^2)-(E(X))^2=\frac{5}{12}-\left(\frac{7}{12}\right)^2=\frac{11}{144}$.
> > $Cov(X,Y)=E(XY)-E(X)E(Y)$, where $E(X)E(Y)=\left(\frac{7}{12}\right)^2=\frac{49}{144}$.
> >
> > $$\begin{aligned} E(XY) &= \int_0^1\int_0^1 (xy)(x+y)dxdy \\ &= \int_0^1\int_0^1 (x^2y+xy^2)dxdy \\ &= \int_0^1\left(\frac{x^3y}{3}+\frac{x^2y^2}{2}\right)\Big|_{x=0}^{x=1} dy \\ &= \int_0^1\left(\frac{y}{3}+\frac{y^2}{2}\right)dy \\ &= \left(\frac{y^2}{6}+\frac{y^3}{6}\right)\Big|_{y=0}^{y=1} \\ &= \frac{1}{3} \end{aligned}$$
> >
> > $Cov(X,Y)=1/3-49/144=-1/144$.
> > $Var(X+Y)=2Var(X)+2Cov(X,Y)=2\frac{11}{144}+2\frac{-1}{144}=\frac{20}{144}$.
> > Alternatively: Let $T=X+Y$, we can calculate the moment generating function: $E(e^{t(X+Y)})$.

- Corrlation coefficient
  Definition: Correlation coefficient of X and Y is defined as $\rho(X,Y)=\frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$.
  i. Used to describe linear association between X and Y.
  ii. Unit free
  iii. $-1\le\rho(X,Y)\le 1$.
  > (not required): Use the fact $|E(XY)|\le\sqrt{E(X^2)}\sqrt{E(Y^2)}$ to prove $-1\le\rho(X,Y)\le 1$.

- Properties of corrlation corfficient:
  i. $\rho(X,Y)=1 \implies Y=aX+b$ for some constants $a>0$ and b.
  ii. $\rho(X,Y)=-1 \implies Y=aX+b$ for some constants $a<0$ and b.

> Example: Suppose (X,Y) has joint pdf $f(x,y)=\begin{cases} x+y & 0\le x\le y, 0\le y\le 1 \\ 0 & \text{o.w.} \end{cases}$. Find $\rho(X,Y)$.
>
> > Solution: $Cov(X,Y)=-\frac{1}{144}$, $Var(X)=Var(Y)=\frac{11}{144}$, therefore, $\rho(X,Y)=\frac{-1/144}{\sqrt{11/144}\sqrt{11/144}}=-\frac{1}{11}$.

3.6 Conditional distribution

- Definition (Joint Discrete Case)
  Suppose X and Y are joint discrete random variable with joint p.f. denoted by $f(x,y)$. Then, conditional p.f. of X given $Y=y$ is $f_1(x|y)=$

$\frac{f(x,y)}{f_2(y)}$, provided that $f_2(y) > 0$.

> Idea: Let event $A = \{X = x\}, B = \{Y = y\}$, then $f_1(x|y) = P(X = x|Y = y) = \frac{P(A \cap B)}{P(B)} = \frac{f(x,y)}{f_2(y)}$.
> Similarly, the conditional p.f. of Y given $X = x$ is $f_2(y|x) = \frac{f(x,y)}{f_1(x)}$, provided that $f_1(x) > 0$.

- Property: Conditional p.f.s $f_1(x|y)$ and $f_2(x|y)$ are probability functions, i.e.:
  a. $f_1(x|y) \ge 0$ for any $x \in \mathbb{R}$, and y is fixed. Additionally, $\sum_{x \in \mathbb{R}} f_1(x|y) = 1$ for any y, where R is the conditional support of x and may depend on y.
  b. $f_2(y|x) \ge 0$ for any $y \in \mathbb{R}$, and x is fixed. Additionally, $\sum_{y \in \mathbb{R}} f_2(y|x) = 1$ for any x.

- Definition (Joint Continuous Case)
  Suppose X and Y are joint continuous random variable with joint p.d.f. denoted by $f(x,y)$. Then, conditional p.d.f. of X given $Y = y$ is $f_1(x|y) = \frac{f(x,y)}{f_2(y)}$, provided that $f_2(y) > 0$.
  Similarly, the conditional p.d.f. of Y given $X = x$ is $f_2(y|x) = \frac{f(x,y)}{f_1(x)}$, provided that $f_1(x) > 0$.
  - Property: Conditional p.d.f.s $f_1(x|y)$ and $f_2(x|y)$ are probability density functions, i.e.:
    a. $f_1(x|y) \ge 0$ for any $x \in \mathbb{R}$, and y is fixed. Additionally, $\int_{-\infty}^{\infty} f_1(x|y)dx = 1$ for any y.
    b. $f_2(y|x) \ge 0$ for any $y \in \mathbb{R}$, and x is fixed. Additionally, $\int_{-\infty}^{\infty} f_2(y|x)dy = 1$ for any x.

> Example 1: Let $f(x,y) = \begin{cases} 8xy & 0 < y < x < 1 \\ 0 & \text{o.w.} \end{cases}$.
>
> Find:
>
> 1. $f_1(x|y)$
>
> > Solution: $f_1(x|y) = \frac{f(x,y)}{f_2(y)}$.
> > The support of Y is $A_2 = (0,1)$, given $y \in (0,1)$, $f_2(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_y^1 8xydx = 4x^2y\big|_y^1 = 4y - 4y^3$.
> > Therefore, $f_1(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{8xy}{4y-4y^3}$ for $0 < y < x < 1$ and 0 otherwise.
>
> 2. $f_2(y|x)$
>
> > Solution: $f_2(y|x) = \frac{f(x,y)}{f_1(x)}$.
> > The support of X is $A_1 = (0,1)$, given $x \in (0,1)$, $f_1(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^x 8xydy = 4xy^2\big|_0^x = 4x^3$.
> > Therefore, $f_2(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{8xy}{4x^3}$ for $0 < y < x < 1$ and 0 otherwise.

> Example 2: The joint pdf is $f(x,y) = \begin{cases} x + y & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$.
>
> Find $f_1(x|y)$ and $f_2(y|x)$.
>
> > Solution: The marginal pdf of Y is $f_2(y) = \begin{cases} \frac{1}{2} + y & 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$.
> > Given $y \in [0,1]$ $f_1(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{x+y}{\frac{1}{2}+y}$ for $0 \le x \le 1$ and 0 otherwise.
> > The marginal pdf of X is $f_1(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$.
> > Given $x \in [0,1]$ $f_2(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{x+y}{x+\frac{1}{2}}$ for $0 \le y \le 1$ and 0 otherwise.

> Example 3: The joint p.f. of X and Y is $f(x,y) = \begin{cases} q^2p^{x+y} & x = 0,1,\ldots \text{ and } y = 0,1,\ldots \\ 0 & \text{o.w.} \end{cases}$, where $p \in (0,1)$ is a constant, $q = 1 - p$.
>
> Find $f_1(x|y)$ and $f_2(y|x)$.
>
> > Solution: The marginal p.f. of Y is $f_2(y) = \begin{cases} qp^y & y = 0,1,\ldots \\ 0 & \text{o.w.} \end{cases}$.
> > Given $y \in \{0,1,\ldots\}$, $f_1(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{q^2p^{x+y}}{qp^y} = qp^x$ for $x = 0,1,\ldots$ and 0 otherwise.
> > The marginal p.f. of X is $f_1(x) = \begin{cases} qp^x & x = 0,1,\ldots \\ 0 & \text{o.w.} \end{cases}$.
> > Given $x \in \{0,1,\ldots\}$, $f_2(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{q^2p^{x+y}}{qp^x} = qp^y$ for $y = 0,1,\ldots$ and 0 otherwise.

- Applications of conditional distribution:
  i. Check independence:
  X and Y are independent if and only if $f_1(x|y) = f_1(x)$ for any $x \in \mathbb{R}$, or $f_2(y|x) = f_2(y)$ for any $y \in \mathbb{R}$.

> Proof sketch: X and Y are independent $\iff$ $f(x,y)=f_1(x)f_2(y)$ for any $x,y\in\mathbb{R}$. Then, $f_1(x|y)=\frac{f(x,y)}{f_2(y)}=\frac{f_1(x)f_2(y)}{f_2(y)}=f_1(x)$ for any $x,y\in\mathbb{R}$.

ii. Use ocnditional distribution to find joint disteibution:
$f(x,y)=f_1(x|y)f_2(y)=f_2(y|x)f_1(x)$ as $f_1(x|y)=\frac{f(x,y)}{f_2(y)}$ and $f_2(y|x)=\frac{f(x,y)}{f_1(x)}$.

> Example 1: $Y\sim\text{Poisson}(\mu)$. $X|Y=y\sim\text{Binomial}(y,p)$, where $p\in(0,1)$ is a constant. Find the marginal p.f. of X.
>
>> Solution: The joint pf of (X,Y) is \$$f(x,y)=f_2(y)f_1(x|y)=\frac{\mu^y e^{-\mu}}{y!}\binom{y}{x}p^x(1-p)^{y-x}$ **for $x=0,1,...,y$ and $y=0,1,...$.**
>> The support of X is $A=\{0,1,...\}$, given $x\in\{0,1,...\}$, $f_1(x)=\sum_{y=x}^{\infty}f(x,y)=\sum_{y=x}^{\infty}\frac{\mu^y e^{-\mu}}{y!}\binom{y}{x}p^x(1-p)^{y-x}=\sum_{y=x}^{\infty}\frac{\mu^y e^{-\mu}}{y!}\frac{y!}{x!(y-x)!}p^x(1-p)^{y-x}=\frac{(\mu p)^x}{x!}e^{-\mu p}\sum_{y=x}^{\infty}\frac{(\mu(1-p))^{y-x}}{(y-x)!}$. Let $t=y-x$, then, $f_1(x)=\frac{(\mu p)^x}{x!}e^{-\mu p}\sum_{t=0}^{\infty}\frac{(\mu(1-p))^t}{t!}=\frac{(\mu p)^x}{x!}e^{-\mu p}e^{\mu(1-p)}=\frac{(\mu p)^x}{x!}e^{-\mu p}$. Then, $X\sim\text{Poisson}(\mu p)$.

> Example 2: Suppose Y has pdf $f_2(y)=\frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}$ for $y>0$, i.e. $Y\sim\text{Gamma}(\alpha,1)$, and the conditional pdf of X given $Y=y$ is $f_1(x|y)=ye^{-xy}$ for $x>0$, i.e. $X|Y=y\sim\text{Gamma}(1,1/y)$. Find the marginal pdf of X.
>
>> Solution: $f(x,y)=f_2(y)f_1(x|y)=\frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}ye^{-xy}$ for $x>0$ and $y>0$. The support of X is $(0,\infty)$
>> Given $x>0$, $f_1(x)=\int_{-\infty}^{\infty}f(x,y)dy=\int_0^{\infty}\frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}ye^{-xy}dy=\int_0^{\infty}\frac{y^{(\alpha+1)-1}e^{-(x+1)y}}{\Gamma(\alpha)}$. Aside: If $Y\sim\text{Gamma}(\alpha,\beta)$, then $f(x)=\frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}}$ for $x>0$.
>> Let $\bar{\alpha}=\alpha+1,\beta=\frac{1}{x+1}$, then, $f_1(x)=\int_0^{\infty}\frac{y^{\bar{\alpha}-1}e^{-y/\beta}}{\Gamma(\bar{\alpha})\beta^{\bar{\alpha}}}=\frac{\beta^{\bar{\alpha}}}{\Gamma(\bar{\alpha})}\int_0^{\infty}\frac{y^{\bar{\alpha}-1}e^{-y/\beta}}{\beta^{\bar{\alpha}}}=\frac{(\frac{1}{x+1})^{\alpha+1}\Gamma(\alpha+1)}{\Gamma(\alpha)}=\frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)}\frac{1}{(x+1)^{\alpha+1}}=\frac{\alpha}{(x+1)^{\alpha+1}},x>0$.

3.7 Conditional expectation

Since $f_2(y|x)$ is a probability function (if X and Y are joint discrete) or probability density function (if X and Y are joint continuous). We can define expectation with respect to $f_2(y|x)$.

- Definition of conditional expectation (mean):

  The conditional expectation of $g(y)$ given $X=x$ is defined as $E[g(Y)|X=x]=\begin{cases}\sum_y g(y)f_2(y|x) & \text{if } X \text{ and } Y \text{ are joint discrete}\\ \int_{-\infty}^{\infty}g(y)f_2(y|x)dy & \text{if } X \text{ and } Y \text{ are joint continuous}\end{cases}$

  .

  In particular, we are particularly intrested in :
  i. $E[Y|X=x](g(y)=y)$
  ii. $Var(Y|X=x)=E[Y^2|X=x]-(E[Y|X=x])^2$.
  iii. $E(e^{tY}|X=x)(g(y)=e^{ty})$.

  > Example: The joint pdf of X and Y is $f(x,y)=\begin{cases}8xy & 0<y<x<1\\ 0 & \text{o.w.}\end{cases}$. Find $E[X|Y=y]$ and $Var(X|Y=y)$.
  >
  >> Solution: The conditional pdf of X given $Y=y$ is $f_1(x|y)=\frac{2x}{1-y^2},0<y<x<1$.
  >> Given $y\in(0,1)$, $E(X|Y=y)=\int_{-\infty}^{\infty}x\cdot f_1(x|y)dx=\int_y^1 x\cdot\frac{2x}{1-y^2}dx=\frac{2}{1-y^2}\int_y^1 x^2dx=\frac{1}{1-y^2}\left(\frac{2x^3}{3}\right)\Big|_y^1=\frac{2(1-y^3)}{3(1-y^2)}$.
  >> Given $y\in(0,1)$, $E(X^2|Y=y)=\int_{-\infty}^{\infty}x^2\cdot f_1(x|y)dx=\int_y^1 x^2\cdot\frac{2x}{1-y^2}dx=\frac{2}{1-y^2}\int_y^1 x^3dx=\frac{1}{1-y^2}\left(\frac{2x^4}{4}\right)\Big|_y^1=\frac{2(1-y^4)}{4(1-y^2)}=\frac{1+y^2}{2}$.
  >> $Var(X|Y=y)=E(X^2|Y=y)-(E(X|Y=y))^2=\frac{1+y^2}{2}-\left(\frac{2(1-y^3)}{3(1-y^2)}\right)^2=\frac{1+y^2}{2}-\left(\frac{2(1-y^3)}{3(1-y^2)}\right)^2$

- Some useful results regarding conditional expectation
  i. If X and Y are independent, then $E[g(Y)|X=x]=E[g(Y)]$ and $E[h(X)|Y=y]=E[h(X)]$.
  ii. Substitution rule: $E[h(X,Y)|X=x]=E[h(x,Y)|X=x]=h(x,Y)$.

  > e.g. $E[X+Y|X=x]=E[x+Y|X=x]=E[x|X=x]+E[Y|X=x]=x+E[Y|X=x]$.

  > e.g. $E(XY|X=x)=E(xY|X=x)=xE(Y|X=x)$.

  iii. Double Expectation Theorem: $E[E[g(Y)|X]]=E[g(Y)]$.
  Note: $E[g(Y)|X]\neq E[g(Y)|X=x]$.
  Two step method to find $E[g(Y)|X]$:
  Step 1: For any x taken from the support of X, calculate $E[g(Y)|X=x]$, denoted by $h(x)$.
  i.e. $h(x)=E[g(Y)|X=x]=\begin{cases}\sum_y g(y)f_2(y|x) & \text{if } X \text{ and } Y \text{ are joint discrete}\\ \int_{-\infty}^{\infty}g(y)f_2(y|x)dy & \text{if } X \text{ and } Y \text{ are joint continuous}\end{cases}$.

Step 2: $E[g(Y)|X] = h(X)$.
Hence, $E[g(y)|X]$ is a function of X, that is why it is a random variable.

> Example 1: Suppose $Y \sim \text{Poisson}(\mu)$, $X|Y = y \sim \text{Binomial}(y, p)$, where $p \in (0, 1)$ is a constant.
>
> a. Find $E[X]$.
>
> > Method 1: We've found $X \sim \text{Poisson}(\mu p)$, therefore, $E[X] = \mu p$. It is computationally intensive.
> > Method 2: $E[X] = E[E[X|Y]]$.
> > Apply the two step method:
> > Step 1: Given $y \in \{0, 1, ...\}$, $E[X|Y = y] = yp$.
> > Step 2: $E[X|Y] = Yp$.
> > Therefore, $E[X] = E[E[X|Y]] = E[Yp] = pE[Y] = p\mu$.
> > Method 3: $E(e^{tX}) = E[E(e^{tX}|Y)]$.
> > Apply the two step method:
> > Step 1: Given $y \in \{0, 1, ...\}$, $E(e^{tX}|Y = y) = [pe^t + (1 - p)]^y$.
> > Step 2: $E(e^{tX}|Y) = [pe^t + (1 - p)]^Y$.
>
> b. Find $Var(X)$.
>
> > Method 1: We've found $X \sim \text{Poisson}(\mu p)$, therefore, $Var(X) = \mu p$.
> > Method 2: By double expectation theorem, $Var(X) = E[Var(X|Y)] + Var[E(X|Y)]$.
> > As $E(X|Y) = Yp$, $Var[E(X|Y)] = Var(Yp) = p^2 Var(Y) = p^2\mu$. ($Y \sim \text{Poisson}(\mu)$)
> > For $E(Var(X|Y))$, apply the two step method:
> > Step 1: Given $y \in \{0, 1, ...\}$, $Var(X|Y = y) = yp(1 - p)$.
> > Step 2: $Var(X|Y) = Yp(1 - p)$.
> > Therefore, $E[Var(X|Y)] = E[Yp(1 - p)] = p(1 - p)E[Y] = p(1 - p)\mu$.
> > $Var(X) = E[Var(X|Y)] + Var[E(X|Y)] = p(1 - p)\mu + p^2\mu = p\mu$.

> Example 2 (Random variables of different types):
> Suppose $X \sim \text{Unif}[0, 1]$, $Y|X = x \sim \text{Binomial}(10, x)$, find E(Y) and Var(Y).
>
> > Solution: By double expectation theorem, $E(Y) = E[E(Y|X)]$.
> > Step 1: Given $x \in [0, 1]$, $E(Y|X = x) = 10x$.
> > Step 2: $E(Y|X) = 10X$.
> > Therefore, $E(Y) = E[E(Y|X)] = E(10X) = 10E(X) = 10 \cdot \frac{1}{2} = 5$.
> > Var(Y) = E[Var(Y|X)] + Var[E(Y|X)].
> > $Var[E(Y|X)] = Var(10X) = 100Var(X)$
> > For any $x \in [0, 1]$
> > Step 1: $Var(Y|X = x) = 10x(1 - x)$.
> > Step 2: $Var(Y|X) = 10X(1 - X)$.
> > Therefore, $E[Var(Y|X)] = E[10X(1 - X)] = E(10X) - 10E(X^2) = 10E(X) - 10(Var(X) + (E(X))^2) = 10 \cdot \frac{1}{2} - 10\left(\frac{1}{12} + \frac{1}{4}\right) = 5 - 10 \cdot \frac{1}{3}$.
> > $Var(Y) = E[Var(Y|X)] + Var[E(Y|X)] = 5 - 10 \cdot \frac{1}{3} + 100 \cdot \frac{1}{12} = \frac{5}{3}$.

3.8 Joint Moment Generating Function

- Definition: If X and Y are two r.v.s, then $M(t_1, t_2) = E(e^{t_1 X + t_2 Y})$ is called the joint moment generating function (mgf) of X and Y, if M(t_1,t_2) exists($M(t_1, t_2) < \infty$) for $|t_1| < h_1$, $|t_2| < h_2$, where $h_1, h_2 > 0$.
- Application of joint mgf
  i. Find marginal mgf from joint mgf.
  Given $M(t_1, t_2) < \infty$ for $|t_1| < h_1$ and $|t_2| < h_2$. Then, $M_X(t_1) = E(e^{t_1 X}) = M(t_1, 0)$ for $|t_1| < h_1$ and $M_Y(t_2) = E(e^{t_2 Y}) = M(0, t_2)$ for $|t_2| < h_2$.
  ii. Independence of r.v.s
  X and Y are independent if and only if $M(t_1, t_2) = M_X(t_1) M_Y(t_2)$ for $|t_1| < h_1$ and $|t_2| < h_2$.

> Example 1 (Joint mgf):
> Suppose the joint pdf of X and Y is given by $f(x, y) = \begin{cases} e^{-y} & 0 < x < y \\ 0 & \text{o.w.} \end{cases}$.
>
> i. Find the joint mgf of X and Y.

> Solution: $M(t_1,t_2)=E(e^{t_1X+t_2Y})=\iint_{\mathbb{R}} e^{t_1x+t_2y}f(x,y)dxdy=\int_0^\infty\int_x^\infty e^{t_1x+t_2y}e^{-y}dydx=\int_0^\infty e^{t_1x}\int_x^\infty e^{(t_2-1)y}dydx=\int_0^\infty e^{t_1x}\left(\frac{e^{(t_2-1)y}}{t_2-1}\right)\Bigg|_x^\infty dx=\int_0^\infty e^{t_1x}\left(\frac{e^{(t_2-1)x}}{t_2-1}\right)dx=\frac{1}{t_2-1}\int_0^\infty e^{(t_1+t_2-1)x}dx=\frac{1}{t_2-1}\left(\frac{e^{(t_1+t_2-1)x}}{t_1+t_2-1}\right)\Bigg|_0^\infty=\frac{1}{1-t_2}\left(\frac{1}{1-(t_1+t_2)}\right)$.

ii. Are they independent?

> Solution: $M_X(t_1)=M(t_1,0)=\frac{1}{1-t_1}$, $M_Y(t_2)=M(0,t_2)=\frac{1}{1-t_2}$. Therefore, $M_X(t_1)M_Y(t_2)=\frac{1}{(1-t_1)(1-t_2)}\neq M(t_1,t_2)$, therefore, X and Y are not independent.

> Example 2 (Additivity of Poisson r.v.s):
> Suppose $X\sim\text{Poisson}(\mu_1), Y\sim\text{Poisson}(\mu_2)$, X is independent of Y.
> Prove $X+Y\sim\text{Poisson}(\mu_1+\mu_2)$.
>
> > Solution: We first find the mgf of $X+Y$.
> > Let $Z=X+Y$, then the mgf of Z is $M_Z(t)=E(e^{tZ})=E(e^{t(X+Y)})=E(e^{tX}e^{tY})=E(e^{tX})E(e^{tY})=e^{(\mu_1(e^t-1)+\mu_2(e^t-1+)}=e^{(\mu_1+\mu_2)(e^t-1)}$, which is the mgf of $\text{Poisson}(\mu_1+\mu_2)$.
> > By the uniqueness property of mgf, $X+Y\sim\text{Poisson}(\mu_1+\mu_2)$.

3.9 Multinomial Distribution

- Definition: $(X_1,...,X_n)$ are joint discrete r.v.s with joint p.f. $f(x_1,..,x_k)=P(X_1=x_1,...,X_k=x_k)=\frac{n!}{x_1!...x_k!}p_1^{x_1}...p_k^{x_k}$, where $x_i=0,1,...,n$ for $i=1,...,k$. $\sum_i=1^kx_i=n$, $0<p_i<1$ and $\sum_i=1^kp_i=1$. Then, $(X_1,...,X_k)$ follows multinomial distribution, with notation $(X_1,...,X_k)\sim\text{Mult}(n,p_1,...,p_k)$.
- Properties of $\text{Mult}(n,p_1,...,p_k)$:
  i. Joint mgf
     a. $M(t_1,...,t_k)=E(e^{t_1X_1+...+t_kX_k})$
     b. $M(t_1,...,t_{k-1})=E(e^{t_1X_1+...+t_{k-1}X_{k-1}})=(p_1e^{t_1}+...+p_{k-1}e^{t_{k-1}}+p_k)^n$
        > e.g. $k=2$, $M(t_1)=E(e^{t_1X_1})=(p_1e^{t_1}+p_2)^n$, where $p_1+p_2=1$.

  ii. Marginal distribution
     $X_i\sim\text{Binomial}(n,p_i)$ for $i=1,...,k$.
  iii. Let $T=X_i+x_j, i\neq j$. Then, $T\sim\text{Binomial}(n,p_i+p_j)$.
     > e.g. Suppose $i=1,j=2$, set $t_1=t_2=t, t_3=...=t_k=0$ in the joint mgf of $\text{Mult}(n,p_1,...,p_k)$, then, $M_T(t)=[(p_1+p_2)e^t+(1-p_1-p_2)]^n$.

  iv. Joint Moment
     $E(X_i)=np_i$ and $Var(X_i)=np_i(1-p_i)$ for $i=1,...,k$.
     > Question: What is $Cov(X_i,X_j)$ for $i\neq j$?
     >
     > > $Var(X_i+X_j)=Var(X_i)+Var(X_j)+2Cov(X_i,X_j)$.
     > > We know $Var(X_i=np_i(1-p_i), Var(X_j)=np_j(1-p_j), Var(X_i+X_j)=n(p_i+p_j)[1-(p_i+p_j)]$.
     > > Therefore, $Cov(X_i,X_j)=-np_ip_j$.

  v. Conditional distribution
     $X_i|X_i+X_j=t\sim\text{Binomial}(t,p_i/(p_i+p_j))$.
  vi. $X_i|X_j=t\sim\text{Binomial}(n-t,p_i/(1-p_j))$.

3.10 Bivariate Normal Distribution

- Definition:
  Suppose that X_1 and X_2 are joint continuous r.v.s with joint pdf $f(x_1,x_2)=\frac{1}{2\pi|\Sigma|^{\frac{1}{2}}}\exp\{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\}$, where $x=\begin{pmatrix}x_1\\x_2\end{pmatrix}$, $\mu=\begin{pmatrix}\mu_1\\\mu_2\end{pmatrix}$, $\Sigma=\begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\\rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}$, $\rho\in(-1,1)$, and $|\Sigma|$ denotes the determinant of Σ, i,.e. $|\Sigma|=\sigma_1^2\sigma_2^2(1-\rho^2)$.
  Then, (X_1,X_2) follows bivariate normal distribution, with notation $X\sim\text{BVN}(\mu,\Sigma)$.
- Properties:
  i. Joint mgf
     $M(t_1,t_2)=E(e^{t_1X_1+t_2X_2})=E(e^{t^TX})=e^{t^T\mu+\frac{1}{2}t^T\Sigma t}$, where $t=\begin{pmatrix}t_1\\t_2\end{pmatrix}$.
  ii. Marginally
     $M_{X_1}(t_1)=M(t_1,t_2=0)=e^{t_1\mu_1+\frac{1}{2}\sigma_1^2t_1^2}$, $M_{X_2}(t_2)=M(t_1=0,t_2)=e^{t_2\mu_2+\frac{1}{2}\sigma_2^2t_2^2}$.
     Then, $X_1\sim\text{N}(\mu_1,\sigma_1^2)$ and $X_2\sim\text{N}(\mu_2,\sigma_2^2)$, $E(X_1)=\mu_1$, $Var(X_1)=\sigma_1^2$, $E(X_2)=\mu_2$, $Var(X_2)=\sigma_2^2$.

$Cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2)$.
What is $E(X_1X_2)$?

iii. We find the conditional distribution of X_1 given X_2, $X_1|X_2 = x_2$.
Conclusion: $X_1|X_2 = x_2$ is normally distributed.
Then, to find $E(X_1|X_2 = x_2)$ and $Var(X_1|X_2 = x_2)$.
$E(X_1|X_2 = x_2) = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2 - \mu_2)$.
$Var(X_1|X_2 = x_2) = \sigma_1^2(1 - \rho^2)$.
Finding $X_2|X_1 = x_1$ is normal.
$E(X_2|X_1 = x_1) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1 - \mu_1)$.
$Var(X_2|X_1 = x_1) = \sigma_2^2(1 - \rho^2)$.

iv. $Cov(X_1, X_2) = \rho\sigma_1\sigma_2$.

> Proof: To find $E(X_1X_2)$, we apply double expectation theorem.
> $E(X_1X_2) = E(E(X_1X_2|X_2))$
> Step 1: $E(X_1X_2|X_1 = x_1) = x_1E(X_2|X_1 = x_1) = x_1(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1 - \mu_1))$
> Step 2: $E(X_1X_2) = E(x_1(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1 - \mu_1))) = \mu_2E(X_1) + \rho\frac{\sigma_2}{\sigma_1}E(X_1^2) - \mu_1E(X_1) - \rho\frac{\sigma_2}{\sigma_1}\mu_1E(X_1) = \mu_2\mu_1 + \rho\frac{\sigma_2}{\sigma_1}(\sigma_1^2 + \mu_1^2) - \mu_1^2 - \rho\frac{\sigma_2}{\sigma_1}\mu_1^2 = \mu_1\mu_2 + \rho\sigma_1\sigma_2$.
> Therefore, $Cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2) = \mu_1\mu_2 + \rho\sigma_1\sigma_2 - \mu_1\mu_2 = \rho\sigma_1\sigma_2$.
> Furthermore, $\rho(X_1, X_1) = \rho = \frac{Cov(X_1,X_2)}{\sqrt{Var(X_1)}\sqrt{Var(X_2)}} = \frac{\rho\sigma_1\sigma_2}{\sigma_1\sigma_2}$.

v. $\rho = 0$ if and only if X_1 and X_2 are independent.

> Common Mistake: If Y_1 and Y_2 are normally distributed, and $Cov(Y_1, Y_2) = 0$, then Y_1 and Y_2 are independent.
> Counter Example: $Y_1 \sim \mathrm{N}(0, 1)$, $Y_2 = RY_1$, where $P(R = 1) = P(R = -1) = 1/2$, R is independent of X.
> Show that $Y_2 \sim \mathrm{N}(0, 1)$ and $Cov(Y_1, Y_2) = 0$.
> If joint distribution (Y_1, Y_2) follows BVN, then $Y_1 + Y_2$ follows normal distribution, then $P(Y_1 + Y_2 = 0) = 0$, however, $P(Y_1 + Y_2 = 0) = P(R = -1) = 1/2$, then the joint distribution of (Y_1, Y_2) is not BVN.

vi. If $X \sim \mathrm{BVN}(\mu, \Sigma)$ and $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is a constant vector, then $C^TX = c_1X_1 + c_2X_2$ is normally distributed with mean $E(C^TX) = c_1\mu_1 + c_2\mu_2 = C^T\mu$ and variance $Var(C^TX) = C^T\Sigma C$.
Here we only consider a single linear combination of X_1 and X_2.
Furthermore, such a fact can be extend, and used to prove normal tests, i.e., if $X_1, \ldots, X_k$ are normally distributed with mean μ and variance σ^2, then $\bar{X} = \frac{1}{k}\sum_{i=1}^k X_i$ is normally distributed with mean μ and variance $\frac{\sigma^2}{k}$.

> Common Mistake: For normally distributed r.v.s Y_1 and Y_2, $c_1Y_1 + c_2Y_2$ is normally distributed.

vii. If $A \in \mathbb{R}^{2\times 2}, b \in \mathbb{R}^{2\times 1}$, then $Y = AX + b \sim \mathrm{BVN}$, with mean vector $E(Y) = AE(X) + b = A\mu + b$, and variance $Var(Y) = Cov(AX + b, AX + b) = A\Sigma A^T$.

viii. $(X - \mu)^T\Sigma^{-1}(X - \mu) \sim \chi_2^2$
We define $\chi_1^2 = Z^2$, where $Z \sim \mathrm{N}(0, 1)$, and $\chi_k^2 = \sum_{i=1}^k Z_i^2$, where $Z_1, \ldots, Z_k$ are independent and identically distributed as $\mathrm{N}(0, 1)$.

> Proof: Since Σ is symmatric, then $\Sigma = Q\Lambda Q^T$, where Q is orthogonal (i.e. $QQ^T = Q^TQ = I$), and $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where λ_1, λ_2 are eigenvalues of Σ.
> Let $\Sigma^{1/2} = Q\Lambda^{1/2}Q^T$, where $\Lambda^{1/2} = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix}$, then $\Sigma^{1/2}\Sigma^{1/2} = \Sigma$, and $\Sigma^{-1/2} = Q\Lambda^{-1/2}Q^T$, where $\Lambda^{-1/2} = \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 \\ 0 & \frac{1}{\sqrt{\lambda_2}} \end{pmatrix}$.
> Now, $(X - \mu)^T\Sigma^{-1}(X - \mu) = (X - \mu)^T\Sigma^{-1/2}\Sigma^{-1/2}(X - \mu)$. Let $Z = \Sigma^{-1/2}(X - \mu)$, then Z is normally distributed with mean $E(Z) = \Sigma^{-1/2}E(X - \mu) = \Sigma^{-1/2}(\mu - \mu) = 0$, and variance $Var(Z) = \Sigma^{-1/2}Var(X - \mu)\Sigma^{-1/2} = \Sigma^{-1/2}\Sigma\Sigma^{-1/2} = I$, so Z_1, Z_2 are independent and identically distributed as $\mathrm{N}(0, 1)$.
> Therefore, $(X - \mu)^T\Sigma^{-1}(X - \mu) = Z^TZ = Z_1^2 + Z_2^2 \sim \chi_2^2$.
> A simple fact: if $X \sim \mathrm{N}(\mu, \sigma^2)$, then $\left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi_1^2$.
> That also means if $X_1, \ldots, X_n$ are iid $\mathrm{N}(\mu, \sigma^2)$, then $\frac{\sum_{i=1}^n (X_i-\mu)^2}{\sigma^2} \sim \chi_n^2$.

Chapter 4: Functions of Random Variables

Problems we want to answer:

- Given $X_1, \ldots, X_n$, which are continuous r.v., and their pdf is known, we are interested in finding the distribution of $Y = h(X_1, \ldots, X_n)$, where h is a function.
  Three main methods to be introduced:

1. cdf technique
2. one-to-one bivariate transformation
3. mgf technique

4.1 CDF Technique

Define $Y = h(X_1, ..., X_n)$, where h is a function.
Main idea:

- Step 1: Find the cdf of Y, $F_Y(y) = P(Y \le y)$.
- Step 2: Find the pdf of Y, $f_Y(y) = \frac{d}{dy} F_Y(y)$.

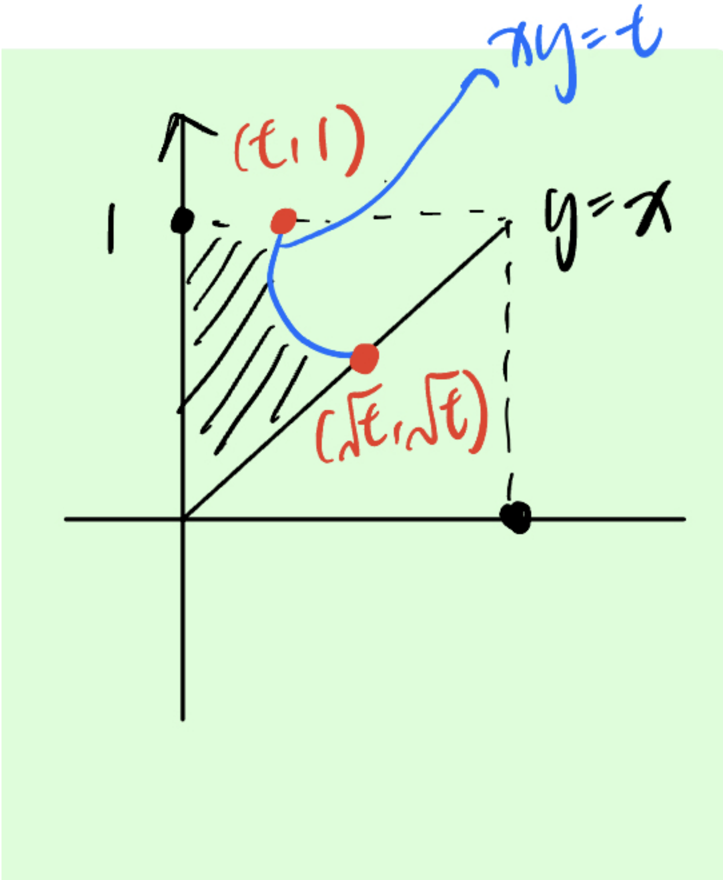
Case 1: Y is a function of one single random variable ($n = 1$), i.e. $Y = h(X)$, where the distribution of X is known.

> Example (χ_1^2): If $X \sim \mathrm{N}(0,1)$, find the distribution of $Y = X^2$.
>
> > Solution: The support of Y is $A_Y = [0, \infty)$.
> >
> > 1. $y \le 0$, $F_Y(y) = P(Y \le y) = 0$.
> > 2. $y > 0$, $F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$
> >   The for $y \to 0$, the pdf of y us $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \frac{1}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \frac{1}{\sqrt{y}}.$
> >   Therefore, $f^Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \frac{1}{\sqrt{y}} & y > 0 \\ 0 & \text{o.w.} \end{cases}$, whcih is the pdf of $\mathrm{Gamma}(\alpha = \frac{1}{2}, \beta = \frac{1}{2})$.

> Example 2: The pdf of X is $f(x) = \frac{\theta}{x^{\theta+1}}$ for $x \ge 1$, where $\theta > 0$ is a constant. Find the distribution of $Y = \log X (\ln X)$.
>
> > Solution: The support of Y is $A_Y = [0, \infty)$.
> >
> > 1. $y \le 0$, $F_Y(y) = P(Y \le y) = 0$.
> > 2. $y > 0$, $F_Y(y) = P(Y \le y) = P(\ln X \le y) = P(X \le e^y) = \int_1^{e^y} \frac{\theta}{x^{\theta+1}} dx = \left(-\frac{1}{x^\theta}\right)\Big|_1^{e^y} = 1 - e^{-\theta y}.$
> >   Therefore, $f_Y(y) = \begin{cases} \theta e^{-\theta y} & y \ge 0 \\ 0 & \text{o.w.} \end{cases}$, which is the pdf of $\mathrm{Exponential}(\lambda = \theta)$.

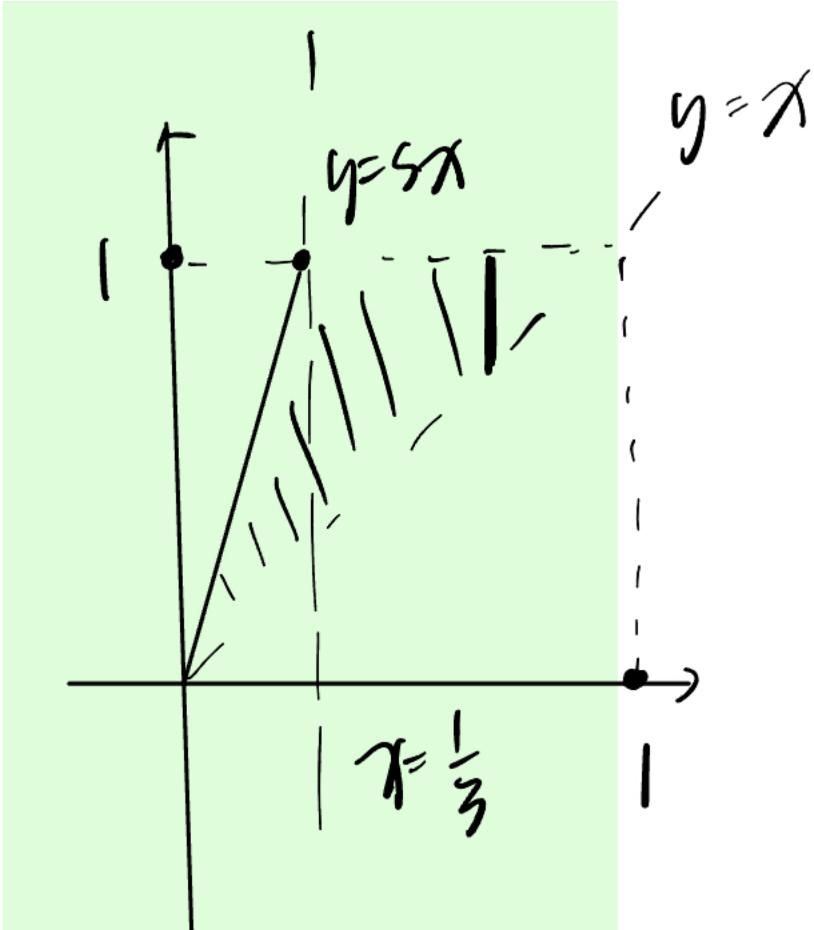
Case 2: Y is a function of more than one random variable ($n > 1$), i.e. $Y = h(X_1, ..., X_n)$, where the distribution of $X_1, ..., X_n$ is known.

- Case 2.1: $n = 2, Y = h(X_1, X_2)$
  > Example: Joint pdf of X and Y is $f(x, y) = 3y$ if $0 \le x \le y \le 1$, and 0 otherwise. Find the distribution of $T = XY$ and $S = Y/X$.
  >
  > > Solution: The support of T is $A_T = [0, 1]$. Now we consider the cdf:
  > >
  > > i. $t \le 0$, $F_T(t) = P(T \le t) = 0$.
  > > ii. $t \ge 1$, $F_T(t) = P(T \le t) = 1$.
  > > iii. $0 < t < 1$, $F_T(t) = P(T \le t) = P(XY \le t)$.
  > > We calculate $P(T > t)$ instead.



$P(T > t) = \int_{\sqrt{t}}^{1}\int_{t/y}^{y} 3y dx dy = \int_{\sqrt{t}}^{1} 3y(y - \frac{t}{y})dy = \int_{\sqrt{t}}^{1} 3y^2 - 3t dy = (1 - 3t) - (t^{3/2} - 3t^{1/2}) = 1 - 3t + 2t^{3/2}$.
$P(T \leq t) = 1 - P(T > t) = 3t - 2t^{3/2}$. Therefore, the p.d.f. of T is $f_T(t) = 3 - 3t^{1/2}$ when $0 < t < 1$, and 0 otherwise.
For S, the support of S is $A_S = [1, \infty)$. Now we consider the cdf:

iv. $s \leq 1$, $F_S(s) = P(S \leq s) = 0$.



v. $s > 1$, $F_S(s) = P(S \le s) = P(Y/X \le s) = P(Y \le sX) = \int_0^1 \int_{y/s}^{y} 3y dx dy = \int_0^1 3y(y - y/s)dy = \int_0^1 (3y^2 - 3y^2/s)dy = (y^3 - 3y^3/2s)\big|_0^1 = 1 - 1/s$.
Hence, the pdf of S is $f_S(s) = \frac{1}{s^2}$ when $s > 1$, and 0 otherwise.

- Case 2.2: $n > 2, Y = h(X_1, ..., X_n)$
In particular, we are interested in the distribution of order statistics. More specifically, assume $X_1, ..., X_n$ are iid r.v.s with pdf $f(x)$. Define the order statistics $Y_1 = \min\{X_1, ..., X_n\}$, denoted as $X(1)$, and $Y_n = \max\{X_1, ..., X_n\}$, denoted as $X(n)$.

> Example (Order Statistics): Suppose $X_1, ..., X_n \overset{iid}{\sim} \text{Unif}[0, \theta]$. Find the distribution of $X(1)$ and $X(n)$.
>
> > Solution: For $X(n), thesupportof$X(n)isA_{X(n)} = [0,\theta]$. Now we consider the cdf:
> >
> > i. $x \le 0$, $F_{X(n)}(x) = P(X(n) \le x) = 0$.
> > ii. $x \ge \theta$, $F_{X(n)}(x) = P(X(n) \le x) = 1$.
> > iii. $0 < x < \theta$, $F_{X(n)}(x) = P(X(n) \le x) = P(\max\{X_1, ..., X_n\} \le x) = P\left(\bigcap_{i=1}^n \{X_i \le x\}\right) = \prod_{i=1}^n P(X_i \le x) = \prod_{i=1}^n \frac{x}{\theta} = \left(\frac{x}{\theta}\right)^n$.
> > Then the pdf of $X(n)$ is $f_{X(n)}(x) = \frac{nx^{n-1}}{\theta^n}$ when $0 < x < \theta$, and 0 otherwise.
> > For $X(1)$, the support of $X(1)$ is $A_{X(1)} = [0, \theta]$. Now we consider the cdf:
> > iv. $x \le 0$, $F_{X(1)}(x) = P(X(1) \le x) = 0$.
> > v. $x \ge \theta$, $F_{X(1)}(x) = P(X(1) \le x) = 1$.
> > vi. $0 < x < \theta$, $F_{X(1)}(x) = P(X(1) \le x) = P(\min\{X_1, ..., X_n\} \le x) = 1 - P(\min\{X_1, ..., X_n\} > x) = 1 - P\left(\bigcap_{i=1}^n \{X_i > x\}\right) = 1 - \prod_{i=1}^n P(X_i > x) = 1 - \prod_{i=1}^n \left(\int_x^\theta \frac{1}{\theta} dx_i\right) = 1 - \left(\frac{\theta - x}{\theta}\right)^n$.
> > Then the pdf of $X(1)$ is $f_{X(1)}(x) = \frac{n(\theta - x)^{n-1}}{\theta^n}$ when $0 < x < \theta$, and 0 otherwise.

4.2 One-to-One Bivariate Transformation

Problem we are going to solve:
Given the joint pdf of (X, Y) denoted by $f(x, y)$, we want to find $U = h_1(X, Y)$ and $V = h_2(X, Y)$.

- Definition of one-to-one function: These two transfomations (h_1 and h_2) is one-to-one bivariate transformation if there exist other two functions (ω_1 and ω_2) such that $x = \omega_1(U, V)$ and $y = \omega_2(U, V)$. Note: $U = h_1(x, y)$ and $V = h_2(x, y)$.
- Notation: Jacobian of $U = h_1(x, y)$ and $V = h_2(x, y)$:

$$\frac{\partial(U,V)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

, determinet of 2×2 matrix.

- Theorem: The p.d.f. of U and V is $f_{U,V}(u,v) = f_{X,Y}(\omega_1(u,v), \omega_2(u,v)) \left|\frac{\partial(U,V)}{\partial(x,y)}\right|$.

> Example 1: $X \sim N(0,1) and Y \sim N(0,1)$, assume X and Y are independent. Find the joint pdf of $U = X+Y$ and $V = X-Y$.
>
> > Solution: Since $U = X+Y$ and $V = X-Y$, then support of U and V is $A_U = (-\infty, \infty)$ and $A_V = (-\infty, \infty)$.
> > then, $x = \frac{U+V}{2}$ and $y = \frac{U-V}{2}$.
> > $\frac{\partial(U,V)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$.
> > Then, the joint pdf of U and V is $g(u,v) = f(x,y) \cdot |J| = f_1(x) \cdot f_2(y) \cdot 1/2 = \frac{1}{2\pi} e^{-\frac{x^2}{2}} \cdot \frac{1}{2\pi} e^{-\frac{y^2}{2}} \cdot \frac{1}{2} = \frac{1}{4\pi} e^{-\frac{u^2+v^2}{4}}$.

> Example 2: Suppose the joint pdf of X and Y is $f(x,y) = e^{-x-y} for$0 < X < \infty$and$0 < Y < \infty$, and 0 elsewhere. Find the pdf of$ U=X+Y$.
>
> > Solution: Define $V = X$, then $U = X+Y$ and $V = X$, therefoer, $x = v$ and $y = u - v$.